Prehistoric Phenomena and Self-referentiality in Realization Procedure

YU Junhua*

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Abstract

By terms-allowed-in-types capacity, the Logic of Proofs **LP** includes formulas of the form $t : \phi(t)$, which have self-referential meanings. In this paper, "prehistoric phenomena" in a Gentzen-style formulation of modal logic **S4** are defined. A special phenomenon, i.e., "left prehistoric loop", is then shown to be necessary for self-referentiality in **S4-LP** realization.

1 Introduction

The Logic of Proofs **LP** is introduced systematically in [1] by Prof. Sergei Artëmov, where it is shown to be the explicit counterpart of modal logic **S4** by verifying the realization theorem. With terms being allowed in types, **LP** has its polynomials as advanced combinatory terms, and hence, extends the idea of propositions-as-types in proof theory. By this new capacity, types of the form $t{:}\phi(t)$ are also included. This sort of types, however, has self-referential meanings, and hence, may indicate some essential properties of this capacity. As [5] says, by any arithmetical semantic $*$, $t : \phi(t)$ is interpreted to be the arithmetical sentence $Proof(t^*, \ulcorner(\phi(t))^*\urcorner)$, which is not true in Peano Arithmetic with the standard Gödel numbering, since the Gödel number of a proof can not be smaller than that of its conclusion.

Dr. Roman Kuznets has scrutinized this issue and verified the following meaningful result: *there is an **S4**−theorem, $\neg\Box\neg(p \to \Box p)$, with **any** realizations of it calling for self-referential constant specifications* (see Result 13, also [6] and [4]). In Kuznets's papers, self-referentiality was scrutinized at a "logic-level", i.e., whether or not a modal logic can be realized non-self-referentially.

Correspondingly, it is also interesting to consider this topic at a "theorem-level", i.e., self-referentiality in realizations of specified theorems. That is, which **S4**−theorems have to call for self-referential constant specifications to prove their realized forms in **LP**? Are there some easy criteria for this? Roughly speaking, if we can fix the class of non-self-referential-realizable **S4**−theorems, then we may find some **S4** (and hence, intuitionistic) measure of self-referentiality introduced by the terms-allowed-in-types capacity.

In this paper, we define and consider "prehistoric phenomena" in **G3s**, a Gentzen-style formulation of **S4**. This notion is then used to scrutinize self-referentiality in realization procedure. A special prehistoric phenomenon, i.e., left-prehistoric-loop, is then shown to be necessary for self-referentiality.

At beginning, we enumerate some preliminary notions and results which will be referred directly in this paper. In [7] (see also [6]), a Gentzen-style formulation of **S4** was presented.

Definition 1 (A Gentzen-style Formulation of S4: G3s [7] [6]) **G3s** *has the following axioms and rules:*

*DF7G5036@hotmail.com, Department of Philosophy, Tsinghua University, Beijing, 100084, China.

$Ax.$	$\dfrac{}{p, \Gamma \Rightarrow \Delta, p}$	$L\bot.$	$\dfrac{}{\bot, \Gamma \Rightarrow \Delta}$
$L\neg.$	$\dfrac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta}$	$R\neg.$	$\dfrac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}$
$L\wedge.$	$\dfrac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}$	$R\wedge.$	$\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$
$L\vee.$	$\dfrac{\phi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}$	$R\vee.$	$\dfrac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$
$L\rightarrow.$	$\dfrac{\Gamma \Rightarrow \Delta, \phi \quad \psi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \psi, \Gamma \Rightarrow \Delta}$	$R\rightarrow.$	$\dfrac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \psi}$
$L\Box.$	$\dfrac{\phi, \Box\phi, \Gamma \Rightarrow \Delta}{\Box\phi, \Gamma \Rightarrow \Delta}$	$R\Box.$	$\dfrac{\Box\phi_1, \cdots, \Box\phi_n \Rightarrow \psi}{\Box\phi_1, \cdots, \Box\phi_n, \Gamma \Rightarrow \Delta, \Box\psi}$

*Besides, cut-elimination holds for **G3s** [7].*

The notion of "family (of □'s)" was defined in [1].

Definition 2 (Family of □'s [1] [6]) *In a **G3s**−rule: (1) Each occurrence of □ in a side formula ϕ in a premise is related only to the corresponding occurrence of □ in ϕ in the conclusion; (2) Each occurrence of □ in an active formula in a premise is related only to the corresponding occurrence of □ in the principal formula of the conclusion. A* family *of □'s is an equivalence class w.r.t. the reflexive transitive closure of the relation above. We denote families by $f_0, f_1, \cdots$. Since **G3s** enjoys cut-elimination, all rules of **G3s** respect the polarity of formulas, and hence, each family consists of □'s of a same polarity. A family is* positive (negative) *if it consists of positive (negative) □'s. If a positive family has at least one of its □'s correspond to a principal □ in an R□ rule, this family is* principal *(or* essential*). Positive families which are not principal are* non-principal *families.*

Result 3 (Equivalence of S4 and G3s [7]) $\vdash_{G3s} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{S4} \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Although we may have been familiar with **LP**, we present the definition again, for the unification of terminologies.

Definition 4 (The Logic LP [1] [6] [5]) *The* language *is defined by:* $\phi ::= \bot \,|\, p \,|\, \phi \rightarrow \phi \,|\, t\!:\!\phi$*, where t is a* proof polynomial *(or* term*), which is defined by:* $t ::= x \,|\, c \,|\, !t \,|\, t \cdot t \,|\, t + t$*, where x and c are called* proof variable *and* proof constant*, respectively.*

Axioms and rules:

$A0.$	$Propositional\ tautologies$
$A1.$	$\vdash t\!:\!\phi \rightarrow \phi$
$A2.$	$\vdash t\!:\!(\phi \rightarrow \psi) \rightarrow (s\!:\!\phi \rightarrow t \cdot s\!:\!\psi)$
$A3.$	$\vdash t\!:\!\phi \rightarrow !t\!:\!t\!:\!\phi$
$A4.$	$\vdash t\!:\!\phi \rightarrow (s + t\!:\!\phi \wedge t + s\!:\!\phi)$
$R1(MP).$	$\phi, \phi \rightarrow \psi \vdash \psi$
$R2(AN).$	$\vdash c\!:\!A,$ $where\ A\ is\ an\ axiom\ A0 - A4,$ $and\ c\ is\ a\ proof\ constant.$

$R2$ is called axiom necessitation*(AN) or* axiom internalization*. For $k \in \{0, 1, 2, 3, 4\}$, we take $AN(Ak)$ to denote an instance of AN rule, which introduces $c\!:\!A$ in, with A being an Ak axiom.*

Result 5 (Lifting Lemma [1] [6]) *If* $x_1 : \phi_1, \cdots, x_n : \phi_n \vdash_{LP} \psi$*, then there is a proof polynomial* $t = t(x_1, \cdots, x_n)$ *s.t.* $x_1\!:\!\phi_1, \cdots, x_n\!:\!\phi_n \vdash_{LP} t(x_1, \cdots, x_n)\!:\!\psi$.

Definition 6 (Constant Specification, Self-referentiality and LP($\mathcal{CS}$) [1] [5]) *A* constant specification $\mathcal{CS}$ *is a set of **LP**−formulas $c_1\!:\!A_1, c_2\!:\!A_2, \cdots$, where c_i's are proof constants and A_i's are instances of axioms $A0 - A4$. By $\mathcal{TCS}$, we mean the* total constant specification*, which is the union of all constant specifications.*

$\mathcal{CS}$ *is* injective *if for each c there is at most one formula* $c : A$ *in* $\mathcal{CS}$. $\mathcal{CS}$ *is* non-direct-self-referential *(or* non-self-referential, *respectively) if* $\mathcal{CS}$ *does not contain any formulas (or subsets) of the form* $c : A(c)$ *(or* $\{c_1 : A_1(c_2), \cdots, c_{n-1} : A_{n-1}(c_n), c_n : A_n(c_1)\}$*). A non-direct-self-referential (or non-self-referential) constant specification is denoted by* $\mathcal{CS}^*$ *(or* $\mathcal{CS}^{\circledast}$*).*

By $\mathbf{LP}(\mathcal{CS})$*, we mean the system, which enjoys the same language,* $A0 - A4$ *and* $R1$ *with* ***LP****, while taking the following rule instead of* $R2$*:*

$$R2_{\mathcal{CS}}.\ \vdash \phi, \text{ where } \phi \in \mathcal{CS}.$$

For instance, $\boldsymbol{LP}(\emptyset)$ *is the system obtained by dropping* $R2$ *from* ***LP****.*

Result 7 (Deduction Theorem of LP [1]) *If* $\Gamma, \phi \vdash_{\boldsymbol{LP}(\mathcal{CS})} \psi$*, then* $\Gamma \vdash_{\boldsymbol{LP}(\mathcal{CS})} \phi \to \psi$*.*

Result 8 (Substitution Lemma of LP [1]) *For* $\mathcal{CS} \in \{\emptyset, \mathcal{TCS}\}$*, if* $\Gamma(x, p) \vdash_{\boldsymbol{LP}(\mathcal{CS})} \psi(x, p)$ *for some variable x and propositional letter p, then for any term t and any formula* ϕ*, we have:* $\Gamma(x/t, p/\phi) \vdash_{\boldsymbol{LP}(\mathcal{CS})} \psi(x/t, p/\phi)$*.*

A Gentzen-style formulation of $\mathbf{LP}(\emptyset)$, i.e., $\mathbf{LPG}_0$, is presented in [1] on the propositional base **G2c** from [7]. For convenience, we take **G3c** from [7] as our propositional base of $\mathbf{LPG}_0$.

Definition 9 (A Gentzen-style Formulation of $\mathbf{LP}(\emptyset)$: $\mathbf{LPG}_0$ [1]) $\mathbf{LPG}_0$ *has the following axioms and rules:*

$Ax, L\bot, L\neg, R\neg, L\wedge, R\wedge, L\vee, R\vee, L\to, R\to$ *as the same form of those in Definition 1, while formulas in them being* ***LP****−formulas now.*

$$L+_L.\quad \frac{\Gamma \Rightarrow \Delta, t : \phi}{\Gamma \Rightarrow \Delta, s + t : \phi} \qquad\qquad L+_R.\quad \frac{\Gamma \Rightarrow \Delta, t : \phi}{\Gamma \Rightarrow \Delta, t + s : \phi}$$

$$L:.\quad \frac{\phi, t : \phi, \Gamma \Rightarrow \Delta}{t : \phi, \Gamma \Rightarrow \Delta} \qquad\qquad R!.\quad \frac{\Gamma \Rightarrow \Delta, t : \phi}{\Gamma \Rightarrow \Delta, !t : t : \phi}$$

$$L\cdot.\quad \frac{\Gamma \Rightarrow \Delta, s : (\phi \to \psi) \quad \Gamma \Rightarrow \Delta, t : \phi}{\Gamma \Rightarrow \Delta, s \cdot t : \psi}$$

Result 10 (Equivalence of $\mathbf{LP}(\emptyset)$ and $\mathbf{LPG}_0$ [1]) *(1)* $\vdash_{\boldsymbol{LP}(\emptyset)} \bigwedge\Gamma \to \bigvee\Delta$ *iff* $\vdash_{\boldsymbol{LPG}_0} \Gamma \Rightarrow \Delta$*;*
(2) $\vdash_{\boldsymbol{LP}(\mathcal{CS})} \phi$ *iff* $\vdash_{\boldsymbol{LPG}_0} \mathcal{CS}_0 \Rightarrow \phi$*, where* $\mathcal{CS}_0$ *is a finite subset of* $\mathcal{CS}$*.*

LP and **S4** are linked by the following definition and result in [1].

Definition 11 (Forgetful Projection and Realization [1] [5]) *The* forgetful projection $\circ$ *is a function from the language of* ***LP*** *to the language of* ***S4****, which meets the following clauses:*

$$p^\circ = p \qquad \bot^\circ = \bot \qquad (\phi \to \psi)^\circ = \phi^\circ \to \psi^\circ \qquad (t : \phi)^\circ = \Box\phi^\circ$$

An ***LP****−formula* ϕ *is a* realization[1] *of an* ***S4****−formula* ψ *(notation:* ψ^r*), provided* $\phi^\circ = \psi$*.*

Result 12 (Realization Theorem of S4 [1]) $\vdash_{\boldsymbol{LP}} \phi$ *iff* $\vdash_{\boldsymbol{S4}} \phi^\circ$*. Or equally,* $\vdash_{\boldsymbol{S4}} \phi$ *iff* $\vdash_{\boldsymbol{LP}} \phi^r$ *for some* ϕ^r*.*

If we generalize Definition 11 to sets of formulas, i.e., $\Gamma^\circ = \{\phi^\circ | \phi \in \Gamma\}$, then Result 12 can be stated as:

$$\mathbf{LP}^\circ = \mathbf{S4}.$$

In [1], the result above is showed by offering a realization procedure, which can mechanically calculate a suitable ϕ^r, if a proof of the **S4**−theorem ϕ is given. The procedure can be displayed by Figure 1 in general. Note that the constant specification $\mathcal{CS}$ of the resulting **LP**−proof is determined only by the left branch, and hence, only by employing Lifting Lemma (i.e., Result 5) while dealing with instances of $R\Box$ rules. Unfortunately, we can not include a complete instruction of this procedure here. For any details not included in Figure 1, we refer to [1] and [6].

[1] Note that $^\circ$ is a function while ^{r} being not, i.e., for each **LP**−formula ϕ, ϕ° is uniquely determined, while an **S4**−formula may have lots of realizations.

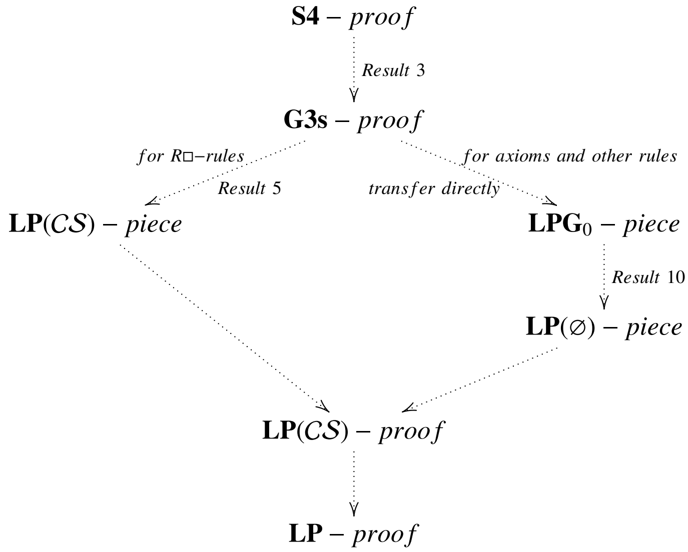

Figure 1: Realization procedure of **S4**

By the realization procedure stated in [1], we can construct an **LP**−proof, and hence, employ a constant specification. It is stated in [1, page 27] that the realization procedure there may lead to constant specifications of the sort $c : \phi(c)$ where $\phi(c)$ contains c. This sort of formulas is interesting since they have self-referential meanings in both *arithmetical semantics* [1] and *Fitting semantics* [3]. The following result shows that self-referentiality is an essential property of **S4**.

Result 13 (Necessity of Direct-self-referentiality in S4 − LP Realization [6]) *the S4−theorem*

$$\neg\Box\neg(p \to \Box p) \tag{1}$$

can not be realized in any $\boldsymbol{LP}(\mathcal{CS}^*)$.

[4] and [5] consider self-referentiality of some other "modal logic-justification logic" pairs[2]. Here we present results from [5] without details.

Result 14 ([5]) *Each* ***K***−*(or* ***D***−*)theorem can be realized with a non-self-referential constant specification, while in* ***T***, ***K4***, ***D4***, *self-referentiality is necessary for realization.*

As we have stated at the beginning, self-referentiality will be considered at a "theorem-level", instead of a "logic-level" in this paper. In Section 2, a series of notations about **G3s** and the standard realization procedure are introduced. Then in Section 3, "prehistoric phenomena" in **G3s** are defined, and some results are verified. In Section 4, we prove the necessity of "left prehistoric loop" for self-referentiality in **S4-LP** realization. In Section 5, we list some relative open problems, which are suggested for further research.

2 Notations and Preliminary Discussions

We introduce a series of notations in this section. Though some of them seem cumbersome, they are employed with a view of denoting notions in realization procedure in detail.

2.1 On Gentzen-style formulation G3s

Observations from [4] and [5] indicate that the behaviors of □−families in Gentzen-style proofs are essential to self-referentiality. In this subsection, we fix some notations down, and then, consider in general the behaviors of □−families in a **G3s**−proof.

[2] Explicit counterparts of **K**, **D**, **T**, **K4**, **D4** were presented in [2]. These counterparts, together with **LP** and some other variants, are called "justification logics" now.

A **G3s**−proof (as a tree) is denoted by $\mathcal{T} = (T, R)$, where the node set $T := \{s_0, s_1, \cdots, s_n\}$ is the set of **occurrences** of sequents, and

$$R := \{ (s_i, s_j) \in T \times T \mid s_i \textit{ is the conclusion of a rule which has } s_j \textit{ as a premise} \}$$

is a binary relation. The only root of $\mathcal{T}$ is denoted by s_r. Since each path in $\mathcal{T}$ from s_r is associated with a unique end-node, we can denote paths by their end-nodes[3]. In what follows, whenever we say "path s_0", we mean the only path from s_r to s_0. $\mathcal{T} \upharpoonright s$ is the subtree of $\mathcal{T}$ with root s. As usual, the transitive closure and reflexive-transitive closure of P is denoted by P^+ and P^*, respectively, for any binary relation P.

Sometimes, we take $\boxminus$ ($\boxdot, \boxplus$) to denote a negative (non-principal-positive, principal-positive, respectively) occurrence of $\Box$ in $\mathcal{T}$. Particularly, we take $\oplus$ to denote a principal-positive occurrence of $\Box$ in the conclusion of an $R\Box$ rule, if this $\Box$ is just introduced principally[4] in this rule.

In $\mathcal{T}$, we have only finitely many principal-positive families, say, $f_1, \cdots, f_m$. An occurrence of $\boxplus$ of family f_i is denoted by $\boxplus_i$. Related $\boxplus$'s may occur in different sequents of $\mathcal{T}$. We take $\boxplus_i^s$ as the notation for an occurrence of $\boxplus_i$ in sequent s. Note that a family can have more than one occurrences in a sequent.

For each family, say f_i, there are only finitely many $R\Box$ rules, denoted by $(R\Box)_{i.1}, \cdots, (R\Box)_{i.m_i}$, which introduce finitely many $\oplus$'s, denoted by $\oplus_{i.1}, \cdots, \oplus_{i.m_i}$, of this family. We also use $(R\Box)_i$ and $\oplus_i$, if we only concern the family it belongs. In $\mathcal{T}$, the premise (conclusion) of $(R\Box)_{i.j}$ are denoted by $I_{i.j}$ ($O_{i.j}$).

We are now ready to present some properties of $\Box$−families in **G3s**−proofs.

Lemma 15 *In a **G3s**−proof $\mathcal{T}$, each family has exactly one occurrence in s_r.*

Proof. *By an easy induction on the height[5] of $\mathcal{T}$. For the inductive step: No matter which rule the last rule is, it does not relate two occurrences of $\Box$'s in the conclusion to a same occurrence in a premise. Hence, any two occurrences of $\Box$'s in s_r had different corresponding occurrences before the application of the last rule. Then by i.h., we know that these two occurrences of $\Box$'s belong to different families.* ⊣

Theorem 16 *In a sequent s in a **G3s**−proof $\mathcal{T}$, any pair of nested $\Box$'s belong to different families.*

Proof. *By an induction on the height of $\mathcal{T}$. For the inductive step, employ the fact that no **G3s**−rule can relate two nested $\Box$'s in a premise to a same one in the conclusion.* ⊣

Theorem 17 *In a **G3s**−proof $\mathcal{T}$, if a $\boxplus_j$ occurs in the scope of a $\boxplus_i$ in a sequent, then for any $\oplus_{j.y}$, there is an $\oplus_{i.x}$ s.t. $I_{i.x}R^*O_{j.y}$.*

Proof. *Suppose a formula of the form $\phi(\boxplus_i\psi(\boxplus_j\chi))$ occurs in $\mathcal{T}$. An easy induction shows that the formula occurs as a subformula in s_r. By Lemma 15, it is the only place where $\boxplus_i$ and $\boxplus_j$ occur in s_r (*).*

Assume for the sake of a contradiction that there is no $(R\Box)_i$ rule in the path $O_{j.y}$[6].

- *Obviously, there is a $\boxplus_j^{O_{j.y}}$, i.e., $\oplus_{j.y}$, which is not in the scope of any $\boxplus_i^{O_{j.y}}$, occurs in $O_{j.y}$.*
- *For s_0Rs_1 in the path above, by i.h., we know that there is a $\boxplus_j^{s_1}$ being out of the scope of any $\boxplus_i^{s_1}$.*
  - *If s_1 and s_0 are linked by a (one-premise or two-premises) Boolean rule, then obviously the related $\boxplus_j^{s_0}$ is not in the scope of any $\boxplus_i^{s_0}$.*
  - *If s_1 and s_0 are linked by an $L\Box$ rule, then:*
    - *If the promised (by i.h.) $\boxplus_j^{s_1}$ is in a side formula, then the case is similar as the Boolean case.*
    - *If the promised $\boxplus_j^{s_1}$ is in an active formula, then we may prefix a $\Box$ in front of this formula. However, the prefixed $\Box$ is a $\boxminus$. Hence the related $\boxplus_j^{s_0}$ is not in the scope of any $\boxplus_i^{s_0}$.*

[3] We have the same notation for a branch and its end-node, which is, more or less, a little inappropriate. However, this convention will not cause any problem in what follows. Since the system of notations employed is cumbersome, we will take this convention, in view of simplifying the notations.

[4] For the $R\Box$ rule stated in Definition 1, only the prefixal $\Box$ in $\Box\psi$ of the conclusion is principally introduced. Any $\Box$'s which are weakened in are not principally introduced.

[5] For the definition of "height", we refer to [7, Definition 1.1.9]

[6] As being stated above, we use a node as the name for the only path from s_r to this node.

- *If s_1 and s_0 are linked by an $R\Box$ rule, then:*
  - *If the promised (by i.h.) $\boxplus_j^{s_1}$ is in a side formula, then the case is similar as the Boolean case.*
  - *If the promised $\boxplus_j^{s_1}$ is in the active formula, then we will prefix a $\boxplus^{s_0}$ in front of this formula. Since this $R\Box$ rule is not an $(R\Box)_i$, the prefixed $\boxplus^{s_0}$ is not a $\boxplus_i^{s_0}$. Hence the related $\boxplus_j^{s_0}$ is not in the scope of any $\boxplus_i^{s_0}$.*

Now we know that there is a $\boxplus_j^{s_r}$ which is not in the scope of any $\boxplus_i^{s_r}$, which contradicts with (*).

Therefore, for any $\oplus_{j.y}$, there is an $(R\Box)_i$ in the path $O_{j.y}$. That is to say, there is an $\oplus_{i.x}$ s.t. $I_{i.x}R^*O_{j.y}$. ⊣

Theorem 18 *In a **G3s**−proof $\mathcal{T}$, if a $\boxplus_j$ occurs in the scope of a $\boxplus_i$ in a sequent, then for any $\boxplus_i$ in any sequent of $\mathcal{T}$, there is a $\boxplus_j$ occurs in the scope of this $\boxplus_i$.*

Proof. *Similar as the proof of Theorem 17.* ⊣

2.2 On S4-LP realization procedure

Here we need to denote notions from the realization procedure. We refer to [1] and [6] for complete descriptions of the procedure without presenting any more here, since including a detailed instruction of it may considerably prolong this paper. A general view of this procedure is given in Figure 1, which may also help.

In the realization procedure, we need to apply two substitutions. The first one is to substitute all occurrences of $\Box$'s in a **G3s**−proof by terms (with provisional variables). Particularly, each $\boxplus_i$ is replaced by the sum of provisional variables of this family, i.e., $u_{i.1} + \cdots + u_{i.m_i}$. After the first substitution described above, the resulting tree is then denoted by $\mathcal{T}' = (T', R')$, while the resulting sequent, set of formulas and formula corresponding to s_i, Γ, ϕ, being denoted by s_i', Γ', ϕ', respectively. $R\Box$ rules of $\mathcal{T}$ are temporally replaced by "Lifting Lemma rules" in $\mathcal{T}'$, while the other rules being automatically transferred to corresponding $\mathbf{LPG}_0$−rules.

During the second substitution (in fact, a series of substitutions applied inductively), all of provisional variables (denoted by u's) are replaced by (provisional-variable-free) **LP** terms (denoted by t's). We apply this from leaf-side-most "Lifting Lemma rules" to the root-side-most one. Hence we have a function ϵ s.t. for any $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, m_i\}$, the "Lifting Lemma rule" corresponding to $(R\Box)_{i.j}$ is dealt as the $\epsilon(i.j)$-th one. It should be emphasized that $O_{i_1.j_1}R^+O_{i_2.j_2}$ implies $\epsilon(i_2.j_2) < \epsilon(i_1.j_1)$. Suppose that $\epsilon(i_0.j_0) = 1$, then we use $\mathcal{T}^{\epsilon(i.j)}$ to denote $\mathcal{T}'(u_{i_0.j_0}/t_{i_0.j_0})\cdots(u_{i.j}/t_{i.j})$, i.e., the result of substituting all provisional variables which have been dealt till $\epsilon(i.j)$ by corresponding **LP**−terms. $s^{\epsilon(i.j)}, \Gamma^{\epsilon(i.j)}, \phi^{\epsilon(i.j)}$ have similar meanings. Particularly, we have $\mathcal{T}^0 = \mathcal{T}'$. To generate $t_{i.j}$, we apply Lifting Lemma (Result 5) on an **LP**−derivation of $I_{i.j}^{\epsilon(i.j)-1}$. We denote this derivation by $d_{i.j}$. During this application, we may need some (finitely many) new constants, which are then denoted by $c_{i.j.1}, \cdots, c_{i.j.m_{i.j}}$. In the standard realization procedure presented in [1], the constant specification employed is injective. That is to say, given a constant, say, $c_{i.j.k}$, the corresponding formula being prefixed, denoted by $A_{i.j.k}$, is determined. The collection of all formulas introduced by AN rules in the promised (by Lifting Lemma) derivation is denoted by $\mathcal{CS}_{i.j}$. After the second substitution, we take notations $\mathcal{T}'' = (T'', R'')$, s_i'', Γ'', ϕ'' to denote the tree, sequent, set of formulas and formula, respectively.

3 Prehistoric Phenomena

In this section, we introduce "prehistoric phenomena", while proving some related results.

Definition 19 (History) *For any branch s_0 of the form $s_rR^*O_{i.j}RI_{i.j}R^*s_0$ in a **G3s**−proof $\mathcal{T}$, the path $O_{i.j}$, i.e., $s_rR^*O_{i.j}$ is called a* history *of f_i in branch s_0.*

Definition 20 (Prehistoric Relation) *We define* prehistoric relation *w.r.t. branches at first, and w.r.t. a whole proof tree then.*

For any principal positive families f_i and f_h, any branch s of the form $s_r R^* O_{i.j} R I_{i.j} R^* s$:

If $I_{i.j}$ has the form of

$$\boxminus\xi_1, \cdots, \boxminus\xi_k(\boxplus_h^{I_{i.j}}(\cdots)), \cdots, \boxminus\xi_n \Rightarrow \eta,$$

then f_h is a left prehistoric family in s of f_i. *Notation: $h \prec_L^s i$.*

If $I_{i.j}$ has the form of

$$\boxminus\xi_1, \cdots, \boxminus\xi_n \Rightarrow \eta(\boxplus_h^{I_{i.j}}(\cdots)),$$

then f_h is a right prehistoric family in s of f_i. *Notation: $h \prec_R^s i$.*

The relation of prehistoric family in s *is defined by: $\prec^s := \prec_L^s \cup \prec_R^s$.*

In **G3s**−*proof $\mathcal{T}$, binary relations of* left prehistoric*,* right prehistoric*, and* prehistoric *is defined by:*

$\prec_L := \bigcup\{\prec_L^s \mid s \ is \ a \ leaf \ of \ \mathcal{T}\}, \quad \prec_R := \bigcup\{\prec_R^s \mid s \ is \ a \ leaf \ of \ \mathcal{T}\}, \quad \prec := \prec_L \cup \prec_R.$

At a first sight, the notion of prehistoric relation is not built on the notion of history directly. We now present a lemma to indicate the desired connection.

Lemma 21 *The following two statements are equivalent: (1) $h \prec^s i$; (2) In branch s, there is a node s' (which is also a sequent), with an occurrence of $\boxplus_h^{s'}$ in. There is also a history of f_i in s, which does not include s'.*

Proof. *The ($\Rightarrow$) direction is trivial, since $\boxplus_h^{I_{i.j}}$ mentioned in the definition is the $\boxplus_h$ desired. For the ($\Leftarrow$) direction, the following arguments applies. By the assumption, s has the form of $s_r R^* O_{i.j} R I_{i.j} R^* s' R^* s$ for some j. Since* **G3s** *is a cut-free system, we know that no matter what rules are applied from s' to $I_{i.j}$, the $\boxplus_h^{s'}$ will occur in $I_{i.j}$ as a $\boxplus_h^{I_{i.j}}$.* ⊣

We have a few remarks here. (1) Intuitively speaking, a history can be seen as a list of sequents, whose **inverse** starts from the conclusion of an $R\Box$ rule, and ends at the root of the proof tree. Each history in a branch breaks it into two parts, i.e, the “historic period”, which is from the conclusion of the $R\Box$ rule to the root of the proof tree, and the “prehistoric period”, which is from the leaf of the branch to the premise of the $R\Box$ rule. (2) Note that Definition 19 does not care the number of histories of a family in a branch. If a family is principally introduced into a branch for more than one times[7], it may have many different histories in this branch. (3) It is possible that $\prec_L^s \cap \prec_R^s \neq \emptyset$, which is instanced by the following proof:

$$\begin{array}{ll}
\hline
(Ax) & \eta, \boxminus\eta, \boxminus\neg\boxplus_i\boxplus_h\eta \Rightarrow \eta \\
\hline
(L\Box) & \boxminus\eta, \boxminus\neg\boxplus_i\boxplus_h\eta \Rightarrow \eta \\
\hline
(R\Box) & \boxminus\eta, \boxminus\neg\boxplus_i\boxplus_h\eta \Rightarrow \oplus_h\eta \\
\hline
(R\Box) & \boxminus\eta, \boxminus\neg\boxplus_i\boxplus_h\eta \Rightarrow \oplus_i\boxplus_h\eta \\
\hline
(L\neg) & \boxminus\eta, \boxminus\neg\boxplus_i\boxplus_h\eta, \neg\boxplus_i\boxplus_h\eta \Rightarrow \\
\hline
(L\Box) & \boxminus\eta, \boxminus\neg\boxplus_i\boxplus_h\eta \Rightarrow
\end{array}$$

(4) We know by Lemma 21 that $h \prec^s i$ iff $\boxplus_h$ has an occurrence in the “prehistoric period” of f_i in s. That is the reason why the $\prec$ relation is called “prehistoric relation”.

In Section 2, we have gained some properties about **G3s**−proofs. Now in the terminology of prehistoric phenomena, we have the following corollaries:

Corollary 22 *For any principal positive family f_i, $i \nprec_R i$.*

Proof. *Otherwise, we will have a $\boxplus_i$ in the scope of $\oplus_i$ in O_i, which is forbidden by Theorem 16.* ⊣

Corollary 23 *For any principal positive families f_i and f_j, if $j \prec_R i$, then for any branch s, $j \prec^s i$ provided that there is an occurrence of $\oplus_j$ in some sequent of s.*

[7] The author is still not sure whether this is necessary for some theorem. In another word, suppose that we have a **G3s**−proof, in which a family is principally introduced into a branch multi-timely. Then, must there be a **G3s**−proof of the same theorem, in which each family is principally introduced into a branch no more than once?

Proof. *Since $j \prec_R i$, we have a $\boxplus_j$ in the scope of $\oplus_i$ in some O_i. Then by Theorem 17, for any $\oplus_{j.y}$, there is an $\oplus_{i.x}$ s.t. $I_{i.x}R^*O_{j.y}$. That is to say, any occurrences of $\oplus_j$ is in the "prehistoric period" of f_i in the branch, say s, it resides in. Hence, by Lemma 21, we have $j \prec^s i$.* ⊣

Corollary 24 *If $k \prec_R j$ and $j \lhd i$, then $k \lhd i$, where $\lhd$ is one of $\prec, \prec_L, \prec_R, \prec^s, \prec^s_L, \prec^s_R$.*

Proof. *Since $k \prec_R j$, we know that there is a $\boxplus_k$ occurring in the scope of an $\oplus_j$. By Theorem 18, wherever $\boxplus_j$ occurs, there is a $\boxplus_k$ occurring in the scope of it.*

For $\lhd = \prec^s_L$, since $j \prec^s_L i$, the branch s has a form of $s_rR^*O_{i.x}RI_{i.x}R^*s$, where $I_{i.x}$ is

$$\boxminus\xi_1, \cdots, \boxminus\xi_y(\boxplus_j^{I_{i.x}}(\cdots)), \cdots, \boxminus\xi_n \Rightarrow \eta.$$

By the observation above, we know that there is a $\boxplus_k^{I_{i.x}}$ occurring in the scope of $\boxplus_j^{I_{i.x}}$. Therefore, $k \prec^s_L i$.

The case that $\lhd = \prec^s_R$ can be shown similarly, while the case that $\lhd = \prec^s$ being an easy consequence of the previous cases.

In the same way, we can gain the result for $\prec_L$ and $\prec_R$, and then for $\prec$. ⊣

We are now ready to present the notion of "prehistoric loop", which indicates a special structure of principal positive families w.r.t. prehistoric relations.

Definition 25 (Prehistoric Loop) *In a* **G3s**$-$*proof $\mathcal{T}$, the ordered sequent of principal positive families $f_{i_1}, \cdots, f_{i_n}$ are called a* prehistoric loop *or* left prehistoric loop *respectively, if we have:*
$i_1 \prec i_2 \prec \cdots \prec i_n \prec i_1$ or $i_1 \prec_L i_2 \prec_L \cdots \prec_L i_n \prec_L i_1$.

In an $R\Box$ rule, formulas residing in the left or right of $\Rightarrow$ in the premise play different roles. This property allows us to care about differences between $\prec_R$ and $\prec_L$. With Corollary 24, we know that $\prec_L$'s are the only essential steps in a prehistoric loop, as stated in the following theorem.

Theorem 26 *$\mathcal{T}$ has a prehistoric loop iff $\mathcal{T}$ has a left prehistoric loop.*

Proof. *For the ($\Rightarrow$) direction, by assumption, we have $i_1 \prec i_2 \prec \cdots \prec i_n \prec i_1$. We claim that there must be a $\prec_L$ in the prehistoric loop listed above. Otherwise, we have $i_1 \prec_R i_2 \prec_R \cdots \prec_R i_n \prec_R i_1$. By Corollary 24, we will have $i_1 \prec_R i_1$ eventually, which is forbidden by Corollary 22.*

By the observation above, we know that there is a $\prec_L$ in the loop. If there is no $\prec_R$, then we have done since it is already a left prehistoric loop. So it is sufficient to treat the case that there are both $\prec_L$'s and $\prec_R$'s in the loop, which can be displayed roughly in Figure 2.

$$\begin{array}{ccc} & \cdots \prec i_1 \prec_R i_2 \prec \cdots & \\ \vdots & & \vdots \\ \curlyvee & & \curlywedge \\ \vdots & \cdots \succ i_4 \succ_L i_3 \succ \cdots & \vdots \end{array}$$

Figure 2: A prehistoric loop with both $\prec_L$ and $\prec_R$

As being indicated by the figure, there must be an "RL-border", i.e., $\cdots \prec_R \prec_L \cdots$[8]. W.l.o.g., we have $\cdots \prec i_1 \prec_R i_2 \prec_L i_3 \prec \cdots$. Then by Corollary 24, we have $\cdots \prec i_1 \prec_L i_3 \prec \cdots$. While having less $\prec_R$'s, it is still a prehistoric loop. Since there are only finitely many $\prec_R$'s in the original loop, we can, eventually, gain a prehistoric loop with only $\prec_L$'s, which is, a left prehistoric loop.

The ($\Leftarrow$) direction is trivial. ⊣

In the last of this section, we consider the role of prehistoric phenomena in **G3s**. By considering Definition 1, it is obvious that all axioms and most of rules in **G3s** are common devices in $G3-$systems. The only

[8]Otherwise, since a $\prec_R$, say $i_1 \prec_R i_2$, is included, the loop would look like $\cdots i_1 \prec_R i_2 \prec_R i_3 \prec_R \cdots$ (*loop to the left*), and then no $\prec_L$ could ever been included. For a similar reason, there must also be an "LR-border". To gain a left prehistoric loop, it is sufficient (see following observations to see why) to consider one of "RL-case" and "LR-case". With Corollary 24 in hand, we take the "RL-case" in what follows.

two exceptions are $L\Box$ and $R\Box$. $L\Box$ is the only rule, which can relate two occurrences of $\Box$ in **one** sequent together. Roughly speaking, it is $L\Box$ who determines the family-wise-situation of a proof. Correspondingly, $R\Box$ is the only rule, which can found "orders" between (principal positive) families. These "orders" are now defined as prehistoric relations. While the right prehistoric relation behaving so "explicitly" (this relation can be seen from the form of the succedent of an ($R\Box$)'s conclusion), the left prehistoric relation is not very obvious. By the notion of prehistoric relations, the behavior of families in **G3s**−proofs are highlighted.

4 Left Prehistoric Loop and Self-referentiality

In this section, we consider (not necessarily direct) self-referentiality of realization procedures with notions of prehistoric phenomena. The notation system employed is introduced in Section 2. We offer some lemmas at first, to avoid prolix proofs. The first lemma tells, when applying Lifting Lemma (Result 5), which provisional variables may be included in the corresponding constant specification. This is important, since putting a later[9] provisional variable in an earlier $\mathcal{CS}$ is the only way which can force a later constant to occur in the housing axiom of an earlier constant.

Lemma 27 *Any provisional variable $u_{x.y}$, which does not occur in $I_{i.j}^{\epsilon(i.j)-1}$, does not occur in $\mathcal{CS}_{i.j}^{\epsilon(i.j)-1}$.*

Proof. *Firstly, we **claim** that $u_{x.y}$ does not occur in any sequent of $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$.*

Proof of the claim: {Case 1: If $\boxplus_x$ does not occur in $I_{i.j}$, we know that $\boxplus_x$ does not occur in any sequent of $\mathcal{T} \upharpoonright I_{i.j}$, since **G3s** enjoys the subformula property. Therefore, $u_{x.y}$ does not occur in any sequent of $\mathcal{T}^0 \upharpoonright I_{i.j}^0$, and hence, does not occur in any sequent of $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$. Case 2: If $\boxplus_x$ occurs in $I_{i.j}$, then $u_{x.y}$ occurs in $I_{i.j}^0$. In this case, $u_{x.y}$ does not occur in $I_{i.j}^{\epsilon(i.j)-1}$ implies $\epsilon(x.y) < \epsilon(i.j)$. Therefore, $u_{x.y}$ does not occur in any sequent of $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$.}

*Secondly, we **claim** that each sequent $s^{\epsilon(i.j)-1}$ of $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$ has an $u_{x.y}$-free **LP**−derivation.*

Proof of the claim: {We prove this claim by an induction on $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$, which is an "$\mathbf{LPG}_0$ + *Lifting Lemma Rule*"−proof.

(1) For base cases, we take (Ax) as an example, since ($L\bot$) can be treated similarly. Assume that s, which has the form[10] of

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}$$

is introduced by an (Ax). Then we can take

$$p \wedge (\wedge\Gamma), \qquad p \wedge (\wedge\Gamma) \rightarrow (\vee\Delta) \vee p, \qquad (\vee\Delta) \vee p$$

to be the **LP**−derivation desired[11]. Note that all provisional variables in this derivation have occurrences in s, which is $u_{x.y}$-free (by the first claim). Hence, the resulting derivation is also $u_{x.y}$-free.

(2) For Boolean cases, we take ($L \rightarrow$) as an instance, while all other Boolean rules being able to be dealt in similar ways. Assume that s is introduced by an ($L \rightarrow$), i.e.,

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \psi, \Gamma \Rightarrow \Delta}$$

By i.h., we have $u_{x.y}$-free derivations, d_L and d_R of the two premises. We apply the deduction theorem of **LP** (Result 7) to d_R, and denote the resulting derivation ($\wedge\Gamma \vdash \psi \rightarrow \vee\Delta$) by d'_R. Specifically, if we employ the standard method to calculate d'_R, the resulting d'_R is also $u_{x.y}$-free. Now the following is a $u_{x.y}$-free derivation of s.

$$\wedge\Gamma \qquad \overrightarrow{(d_L)} \qquad (\vee\Delta) \vee \phi, \qquad \phi \rightarrow \psi, \qquad ((\vee\Delta) \vee \phi) \rightarrow (\phi \rightarrow \psi) \rightarrow ((\vee\Delta) \vee \psi),$$

[9] Here "earlier" and "later" are used w.r.t. the order indicated by ϵ. See Section 2 for details.

[10] Since s is a sequent in $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$, precisely speaking, we should denote it by $s^{\epsilon(i.j)-1}$, with similar superscripts added to formulas in it. However, this would considerably complicate our notations, and make what follows much harder to read. Therefore, we omit those superscripts. This ephemeral convention does not matter, since we are now living in the scope of a specified tree, instead of a series of trees.

[11] Strictly speaking, we need to specify an order for conjuncts and disjuncts here. Since this issue is not essential for our proof, we can take any reasonable order, e.g., the one employed in [6]. In what follows, we take $\bigwedge$ and $\bigvee$ with an assumed reasonable order without presenting it explicitly.

$$(\phi \to \psi) \to ((\vee\Delta) \vee \psi), \qquad (\vee\Delta) \vee \psi, \qquad \wedge\Gamma \qquad \overrightarrow{(d'_R)} \qquad \psi \to \vee\Delta,$$
$$((\vee\Delta) \vee \psi) \to (\psi \to \vee\Delta) \to \vee\Delta, \qquad (\psi \to \vee\Delta) \to \vee\Delta, \qquad \vee\Delta$$

(3) For non-Boolean $\mathbf{LPG}_0$−rules, we take (L :), which corresponding to ($L\Box$) in **G3s**, for example. Suppose that s is obtained by an (L :), i.e.,

$$\frac{\phi, t:\phi, \Gamma \Rightarrow \Delta}{t:\phi, \Gamma \Rightarrow \Delta}$$

By i.h., we have a $u_{x.y}$-free derivation, say d', of the premise. The desired $u_{x.y}$-free derivation of s is then gained by adding

$$t:\phi, \qquad t:\phi \to \phi$$

at the beginning of d'.

(4) For the "Lifting Lemma Rule". Assume that s is obtained by applying a "Lifting Lemma Rule":

$$\frac{x_1:\xi_1, \cdots, x_n:\xi_n \Rightarrow \eta}{x_1:\xi_1, \cdots, x_n:\xi_n, \Gamma \Rightarrow \Delta, t(x_1, \cdots, x_n):\eta}$$

By i.h., we have a $u_{x.y}$-free derivation of the premise. To construct t, we apply Lifting Lemma (Result 5) on this derivation. Note that the resulting derivation is also $u_{x.y}$-free[12]. Then, we make some Boolean amendment (to allow the weakening) on the resulting derivation, which is similar to the situation in Boolean-rule-cases. This amendment does not introduce occurrences of $u_{x.y}$, since each formula we need to weakened in has an explicit occurrence in s, which is $u_{x.y}$-free (by the first claim).}

Thirdly, as an easy consequence of the second claim, we know that $I_{i.j}^{\epsilon(i.j)-1}$ has a $u_{x.y}$-free derivation, which is constructed inductively on $\mathcal{T}^{\epsilon(i.j)-1} \upharpoonright I_{i.j}^{\epsilon(i.j)-1}$. That is to say, $d_{i.j}$ is $u_{x.y}$-free.

Lastly, we turn to consider $\mathcal{CS}_{i.j}^{\epsilon(i.j)-1}$, which consists of formulas of the from $c : A^{\epsilon(i.j)-1}$. If $c \in \{c_{i.j.1}, \cdots, c_{i.j.m_{i.j}}\}$, then $A^{\epsilon(i.j)-1}$ is an axiom employed in $d_{i.j}$. If $c \notin \{c_{i.j.1}, \cdots, c_{i.j.m_{i.j}}\}$, then $c : A^{\epsilon(i.j)-1}$ is introduced by an AN rule in $d_{i.j}$. In both cases, we have $A^{\epsilon(i.j)-1}$ occurs in $d_{i.j}$. Since $u_{x.y}$ does not occur in $d_{i.j}$, $u_{x.y}$ does not occur in $A^{\epsilon(i.j)-1}$.

Thus, $u_{x.y}$ does not occur in $\mathcal{CS}_{i.j}^{\epsilon(i.j)-1}$. ⊣

When dealing with a "Lifting Lemma Rule", we can **try to** reduce the risk of committing self-referentiality by choosing new constants. Hence, the order ϵ is essential. Having considered this, we present the second lemma, which says, given a prehistoric-loop-free proof, we can arrange ϵ in such a way that the order respects prehistoric relations.

Lemma 28 *If a **G3s**−proof $\mathcal{T}$ is prehistoric-loop-free, then we can realize it in such a way that: If $h_2 \prec h_1$, then $\epsilon(h_2.j_2) < \epsilon(h_1.j_1)$*

Proof. *We **claim** that there is a family f_{i_1} s.t. $h \nprec i_1$ for any principal positive family f_h.*

Proof of the claim: {Otherwise, let $Z = \{f_1\}$. Since $\mathcal{T}$ is prehistoric-loop-free, by Lemma 21, the ⊞ which occur to the leaf-side of any O_1 can not belong to the family in Z. That is to say, it must belong to another family, say, f_2, w.l.o.g.. Then let $Z = \{f_1, f_2\}$. Similarly, the ⊞ which occur to the leaf-side of any O_2 can not belong to any families in Z. Hence it must belong to a new family, say, f_3. So and so on. Since there are only finitely many principal positive families in $\mathcal{T}$, in the way described above, we will add all of these families into Z at sometime. At that time, we will no longer have new families to set the required ⊞. By this observation, we have verified the claim above.}

Let $\epsilon(i_1.j) = j$, for any $j \in \{1, \cdots, m_{i_1}\}$.

*We further **claim** that there is a family f_{i_2} s.t. if $h \prec i_2$, then $f_h \in \{f_{i_1}\}$.*

[12]In applying Lifting Lemma, each axiom ϕ is transferred to $c : \phi$ for some new constant c, each premise $x : \phi$ is transferred to $x:\phi \quad x:\phi \to !x:x:\phi \quad !x:x:\phi$, each result of ($AN$) $a:\phi$ is transferred to $a:\phi \quad a:\phi \to !a:a:\phi \quad !a:a:\phi$, while each result of ($MP$) with i.h. $s:(\phi \to \psi)$ and $t:\phi$ is transferred by applying (MP) on these two, together with $s:(\phi \to \psi) \to (t:\phi \to (s\cdot t):\psi)$. It is easy to see that the whole algorithm does not add any new provisional variables. For further details about Lifting Lemma, we refer to [6].

Proof of the claim: {As we have done for the first one.}

Let $\epsilon(i_2.j) = m_{i_1} + j$, for any $j \in \{1, \cdots, m_{i_2}\}$.

Similarly, we can show that there is a family f_{i_x} s.t. if $h \prec i_x$, then $f_h \in \{f_{i_1}, \cdots, f_{i_{x-1}}\}$, and set

$$\epsilon(i_x.j) = \sum_{w=1}^{x-1} m_{i_w} + j$$

for any $j \in \{1, \cdots, m_{i_x}\}$.

Since there are only finitely many principal positive families in $\mathcal{T}$, in the way above, we can set the order of all these families, and hence, generate an ϵ as being required. ⊣

Given a prehistoric-loop-free proof, we have generated an ϵ which respects prehistoric relations. Now the third lemma below says, with such an ϵ, the way the constants reside is also restricted.

Lemma 29 *Assume the proof tree is prehistoric-loop-free. Taken the ϵ generated in Lemma 28, we have:*

If $\epsilon(i_0.j_0) \geqslant \epsilon(i.j)$, then for any $k_0 \in \{1, \cdots, m_{i_0.j_0}\}$, any $k \in \{1, \cdots, m_{i.j}\}$, $c_{i_0.j_0.k_0}$ does not occur in $A''_{i.j.k}$.

Proof. *We employ the ϵ generated in Lemma 28.*

*When dealing with $(R\Box)_{i.j}$, we need to apply the lifting procedure on $d_{i.j}$, which is an **LP**−derivation of $I_{i.j}^{\epsilon(i.j)-1}$. Since $O_{i.j}RI_{i.j}$, by Lemma 28, we know that: each $\boxplus$ in $I_{i.j}$ belongs to a family, say f_w, s.t. $\epsilon(w.j_w) < \epsilon(i.j)$ for any $j_w \in \{1, \cdots, m_w\}$. That is to say, $I_{i.j}^{\epsilon(i.j)-1}$ is provisional-variable-free. Hence, by Lemma 27, $\mathcal{CS}_{i.j}^{\epsilon(i.j)-1}$ is provisional-variable-free, which implies that $\mathcal{CS}_{i.j}^{\epsilon(i.j)-1} = \mathcal{CS}''_{i.j}$ and*

$$A_{i.j.k}^{\epsilon(i.j)-1} = A''_{i.j.k} \qquad for\ any\ k \in \{1, \cdots, m_{i.j}\}. \tag{2}$$

Since $\epsilon(i_0.j_0) \geqslant \epsilon(i.j)$, $c_{i_0.j_0.k_0}$ had not been introduced by the procedure when we began to apply lifting procedure on $d_{i.j}$. Therefore, $c_{i_0.j_0.k_0}$ does not occur in $d_{i.j}$, and hence, does not occur in any axioms employed in $d_{i.j}$. That is to say, for any $k \in \{1, \cdots, m_{i.j}\}$, $c_{i_0.j_0.k_0}$ does not occur in $A_{i.j.k}^{\epsilon(i.j)-1}$. By (2), we know that $c_{i_0.j_0.k_0}$ does not occur in $A''_{i.j.k}$. ⊣

With the three lemmas above in hand, now we are ready to verify the main theorem.

Theorem 30 (Necessity of Left Prehistoric Loop for Self-referentiality) *If an **S4**−theorem ϕ has a left-prehistoric-loop-free **G3s**−proof, then there is an **LP**−formula ψ s.t. $\psi^\circ = \phi$ and $\vdash_{\boldsymbol{LP}(\mathcal{CS}^{\circledast})} \psi$.*

Proof. *Since the **G3s**−proof is left-prehistoric-loop-free, by Theorem 26, the proof is prehistoric-loop-free. Hence, we can take the ϵ generated in Lemma 28, which implies the result stated in Lemma 29.*

Assume with the hope of a contradiction that the resulting constant specification $\mathcal{CS}$ is self-referential. That is to say, we have:

$$\{c_{i_1.j_1.k_1} : A''_{i_1.j_1.k_1}(c_{i_2.j_2.k_2}), \cdots, c_{i_{n-1}.j_{n-1}.k_{n-1}} : A''_{i_{n-1}.j_{n-1}.k_{n-1}}(c_{i_n.j_n.k_n}), c_{i_n.j_n.k_n} : A''_{i_n.j_n.k_n}(c_{i_1.j_1.k_1})\} \subseteq \mathcal{CS}''$$

By Lemma 29, we then have:

$$\epsilon(i_n.j_n) < \epsilon(i_{n-1}.j_{n-1}) < \cdots < \epsilon(i_2.j_2) < \epsilon(i_1.j_1) < \epsilon(i_n.j_n)$$

which is impossible.

Hence, the resulting constant specification is non-self-referential. ⊣

Having finished the proof, we may notice that what have been done are natural. By the left-prehistoric-loop-free condition, we are given an order of families. The order is then inherited by an ϵ, which indicates the order of lifting procedures. Eventually, the order is echoed by the way in which the constants reside themselves in their housing axioms.

5 Conclusions

In this paper, we define "prehistoric phenomena" in a Gentzen-style formulation (**G3s**) of modal logic **S4**. After presenting some basic results about this notion, a proof of the necessity of left prehistoric loop for self-referentiality is given. The presented work constitutes a small step in the journey of finding a criterion for self-referentiality in realization procedures, while the whole journey having its meaning in offering an **S4** (and hence, intuitionistic) measure of self-referentiality introduced by terms-allowed-in-types capacity.

It should be emphasized that, there are still spaces for this work to be developed. For example, it is unclear whether left prehistoric loop is sufficient for self-referentiality. We conjecture that if all **G3s**−proofs of an **S4**−theorem ϕ have left prehistoric loops, then any realizations of ϕ will necessarily call for self-referential constant specifications. For another example, we have not answered the question that whether there is an **S4**−theorem, the realization of which necessarily calls for self-referentiality, but not for direct self-referentiality. Also, despite the applications shown above, we assume that "prehistoric phenomena" have interests of their own, since they describe some family-wise structures of Gentzen-style modal proof trees.

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