SUBFRAMIZATION AND STABILIZATION FOR SUPERINTUITIONISTIC LOGICS

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Abstract. With each superinuitionistic logic (si-logic), we associate its downward and upward subframizations, and characterize them by means of Zakharsyachev's canonical formulas, as well as by embedding si-logics into the extensions of the propositional lax logic **PLL**. In an analogous fashion, with each si-logic, we associate its downward and upward stabilizations, and characterize them by means of stable canonical formulas, as well as by embedding si-logics into extensions of the intuitionistic **S4**.

1. Introduction

Subframe logics were introduced by Fine [14] as modal logics that are characterized by a class of frames closed under subframes. They turned out to be a rather well-behaved class of modal logics (see, e.g., [14, 21, 22, 11]). In particular, there are continuum many subframe logics, all transitive subframe logics have the finite model property, and a transitive modal logic is a subframe logic iff it is axiomatizable by subframe formulas. From the point of view of epistemic logic, subframe logics are exactly the logics admitting epistemic updates (cf. [1, Ch. 2], [12, Sec. 7.4–7.5]).

Since subframe logics form a complete sublattice of the lattice of all modal logics, for each modal logic L, there is a greatest subframe logic underneath L, and a least subframe logic above L, called the *downward* and *upward subframizations* of L. They were studied by Wolter [21, 22] who characterized the downward and upward subframizations in terms of relativizations. Zakharyschev [24] studied superintuitionsitic subframe logics (subframe si-logics), and proved that they are exactly the si-logics axiomatized by $(\wedge, \to)$-formulas. In the intuitionistic case, unlike the modal case, subframes no longer correspond to relativizations (cf. [11, Sec. 9.1]), and hence characterizing downward and upward subframizations requires different technique. We will show that they can be characterized by means of the canonical formulas of Zakharyaschev.

As was shown in [9], subframes correspond to nuclei on Heyting algebras. This provides an interesting link to Grothendieck topology and geometric modality [16], which give rise to the propositional lax logic **PLL** of Fairtlough and Mendler [13]. We show that there are two natural embeddings of si-logics into extensions of **PLL**, which yield a new characterization of subframe si-logics, as well as a more convenient characterization of the downward subframization of a si-logic.

Alongside subframe logics, another well-behaved class of si-logics is that of stable si-logics of [5]. Like subframe si-logics, stable si-logics also form a complete sublattice of the lattice of all si-logics, leading to the notions of *downward* and *upward stabilizations*. We characterize

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downward and upward stabilizations of si-logics by means of stable canonical formulas of [5], which are an alternative to Zakharyaschev's canonical formulas. We observe that the **PLL**-counterpart for stable si-logics is the intuitionistic **S4** (**IS4**) studied by Ono [20]. Since stability requires to work with rooted frames, which are captured by the multiple-conclusion rule $p \vee q/p, q$ (cf. [7, Thm. 8.6]), we will embed stable si-logics into multiple-conclusion consequence relations extending $\mathbf{IS4} + p \vee q/p, q$. We show that there are two natural embeddings of si-logics into extensions of $\mathbf{IS4} + p \vee q/p, q$, which yield a new characterization of stable si-logics, as well as a more convenient characterization of the downward stabilization of a si-logic.

2. Subframes, nuclei, and the lax logic

We assume the reader's familiarity with Esakia duality between Heyting algebras and descriptive Kripke frames (see, e.g., [4]). We will view descriptive Kripke frames as tuples $\mathfrak{F} = (X, \leq)$, where X is a Stone space (zero-dimensional compact Hausdorff space) and $\leq$ is a partial order on X such that ${\uparrow}x := \{y \in X \mid x \leq y\}$ is closed and $U \subseteq X$ clopen (closed and open) implies ${\downarrow}U := \{x \in X \mid \exists u \in U \text{ with } x \leq u\}$ is clopen. We will refer to descriptive Kripke frames as *Esakia frames*. Since the trivial Heyting algebra dually corresponds to the empty Esakia frame, we will allow Esakia frames to be empty.

Let $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (X', \leq')$ be Esakia frames. We recall [24, 9] that $\mathfrak{G}$ is a *subframe* of $\mathfrak{F}$ if X' is a closed subspace of X, $\leq'$ is the restriction of $\leq$, and for each clopen U of X', the set ${\downarrow}U$ is clopen in X.

As was observed in [9, Sec. 5], subframes on an Esakia frame $\mathfrak{F}$ correspond to nuclei on the dual Heyting algebra A of clopen upsets of $\mathfrak{F}$, where we recall that a nucleus on a Heyting algebra is a unary function j satisfying $a \leq ja$, $jja \leq ja$, and $j(a \wedge b) = ja \wedge jb$. Indeed, if $\mathfrak{G} = (S, \leq)$ is a subframe of $\mathfrak{F} = (X, \leq)$, then j given by

$$jU = X \setminus {\downarrow}(S \setminus U) \tag{1}$$

is a nucleus on A, and every nucleus on A is obtained this way. Moreover, if

$$A_j := \{a \in A \mid a = ja\}$$

is the Heyting algebra of fixpoints of j, then the dual Esakia frame of A_j is isomorphic to $\mathfrak{G}$. This motivates the following definition.

Definition 2.1.
(i) A *nuclear algebra* is a pair (A, j) consisting of a Heyting algebra A and a nucleus j on A.
(ii) An *S-frame*[1] is a pair $(\mathfrak{F}, \mathfrak{G})$ consisting of an Esakia frame $\mathfrak{F}$ and a subframe $\mathfrak{G}$ of $\mathfrak{F}$.

Throughout the paper we will use the following notational convention.

Notation 2.2. For an S-frame $(\mathfrak{F}, \mathfrak{G})$, we always assume that $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (S, \leq)$.

Esakia duality coupled with the 1-1 correspondence between nuclei on Heyting algebras and subframes of Esakia frames yields a 1-1 correspondence between nuclear algebras and S-frames. This allows us to interpret the lax modality $\bigcirc$ of Fairtlough and Mendler [13] in S-frames as follows. Suppose $(\mathfrak{F}, \mathfrak{G})$ is an S-frame. As with the intuitionistic propositional calculus **IPC**, we interpret propositional letters as clopen upsets of $\mathfrak{F}$ and intuitionistic

[1]The "S" in S-frame stands for subframe.

connectives as the corresponding operations in the Heyting algebra of clopen upsets of $\mathfrak{F}$. In addition, the lax modality $\bigcirc$ is interpreted as the nucleus j given by (1). Therefore, if v is a valuation on $(\mathfrak{F}, \mathfrak{G})$ and $x \in X$, then

$$x \models_v \bigcirc\varphi \text{ iff } y \models_v \varphi \text{ for all } y \in {\uparrow}x \cap S. \tag{2}$$

Since the defining axioms of $\bigcirc$ match the defining axioms of nuclei, we obtain that the propositional lax logic **PLL** is sound and complete with respect to such interpretation.

This semantics is closely related to the semantics of **PLL** developed by Goldblatt [16] and Fairtlough and Mendler [13] (see also [10]). We recall that a *Goldblatt frame* is a tuple $\mathfrak{F} = (X, R)$, where X is a partially ordered set and R is a binary relation on X such that $x \le yRz$ implies xRz, xRy implies $x \le y$, and xRy implies $xRzRy$ for some $z \in X$. The language of **PLL** is interpreted in a Goldblatt frame $\mathfrak{F}$ by interpreting propositional letters as upsets of $\mathfrak{F}$, intuitionistic connectives as the corresponding operations in the Heyting algebra of upsets of $\mathfrak{F}$, and $\bigcirc$ as the nucleus j_R given by

$$j_R U = X \setminus R^{-1}(X \setminus U). \tag{3}$$

If $(\mathfrak{F}, \mathfrak{G})$ is an S-frame, then let $\mathfrak{F}_{\mathfrak{G}} = (X, R)$, where R is defined by xRy iff $x \le s \le y$ for some $s \in S$. As follows from [9, Rem. 24], $\mathfrak{F}_{\mathfrak{G}}$ is a Goldblatt frame that in addition satisfies xRy iff $(\exists z \in X)(zRz$ and $x \le z \le y)$. Moreover, since $R[x] = {\uparrow}({\uparrow}x \cap S)$, we see that $j_R U = jU$ for each upset U of $\mathfrak{F}$.

We also recall that an *FM-frame* (Fairtlough-Mendler frame) is a tuple $\mathfrak{F} = (X, \le, \preceq, F)$ such that $\le, \preceq$ are partial orders on X, $x \preceq y$ implies $x \le y$, and F is an $\le$-upset of X. The language of **PLL** is interpreted in an FM-frame slightly differently than in a Goldblatt frame. Instead of working with the Heyting algebra of all upsets of $\mathfrak{F}$, we work with the Heyting algebra of the upsets of $\mathfrak{F}$ containing F. Therefore, propositional letters are interpreted as upsets of $\mathfrak{F}$ containing F, intuitionistic connectives as the corresponding operations in this relativized Heyting algebra, and $\bigcirc$ is interpreted as the nucleus $j_{\le\preceq}$ given by

$$j_{\le\preceq} U = \{x \in X \mid \forall y(x \le y \Rightarrow \exists z(y \preceq z \text{ and } z \in U))\}. \tag{4}$$

If $(\mathfrak{F}, \mathfrak{G})$ is an S-frame, then define $\mathfrak{F}^*_{\mathfrak{G}} = (X^*, \le^*, \preceq^*, F^*)$ as follows. Set $X^* = X \cup \{m\}$, where $m \notin X$. Let $\le^*$ extend $\le$ so that m is the maximum of X^*. Set $F^* = \{m\}$ and define $x \preceq^* y$ iff $x = y$ or $x \in X \setminus S$ and $y = m$. It is straightforward to verify that $\mathfrak{F}^*_{\mathfrak{G}}$ is an FM-frame. Moreover, if for an upset U of $\mathfrak{F}$, we let $U^* = U \cup \{m\}$, then U^* is an upset of $\mathfrak{F}^*_{\mathfrak{G}}$ and $j_{\le\preceq}(U^*) = (jU)^*$.

3. Subframe logics and subframization

Let L be a si-logic. We say that an Esakia frame $\mathfrak{F}$ is an L-frame provided $\mathfrak{F} \models L$. Let $\mathsf{Fr}(L)$ be the class of all L-frames. We recall that L is a *subframe logic* provided $\mathsf{Fr}(L)$ is closed under subframes. The class of subframe logics is a well-behaved subclass of the class of all si-logics. There are many characterizations of subframe logics (see, e.g., [14, 24, 25, 21, 22, 11, 9]). We gather some of them below.

We recall that a frame $\mathfrak{F} = (X, \le)$ is *rooted* provided there is $r \in X$, called the *root* of $\mathfrak{F}$, such that $X = {\uparrow}x$. As was shown in [14, Sec. 3], with each finite rooted frame $\mathfrak{F}$ we may associate a formula $\beta(\mathfrak{F})$, called the *subframe formula* of $\mathfrak{F}$, such that for any Esakia frame $\mathfrak{G} = (Y, \le)$,

$$\mathfrak{G} \not\models \beta(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{G}. \tag{5}$$

Proposition 3.1. *For a si-logic L, the following are equivalent:*

(i) *L is a subframe logic.*
(ii) *L is the logic of a class of Esakia frames closed under subframes.*
(iii) *L is axiomatizable by subframe formulas.*
(iv) *L is axiomatizable by $(\wedge, \to)$-formulas.*

Proof. See, e.g., [11, Sec. 11.3]. □

Let Λ_{Subf} be the class of subframe logics. It is well known that Λ_{Subf} is a complete sublattice of the lattice of all si-logics. Therefore, every si-logic L has a greatest subframe neighbor below it and a least subframe neighbor above it (cf. [21, 22]).

Definition 3.2. For a si-logic L, define the *downward subframization of L* as

$$\mathsf{Subf}_{\downarrow}(L) := \bigvee\{L' \in \Lambda_{\mathsf{Subf}} \mid L' \subseteq L\}$$

and the *upward subframization of L* as

$$\mathsf{Subf}_{\uparrow}(L) := \bigwedge\{L' \in \Lambda_{\mathsf{Subf}} \mid L \subseteq L'\}.$$

We summarize some rather obvious facts about the downward and upward subframizations that we will use throughout the paper.

Lemma 3.3.

(i) $\mathsf{Subf}_{\downarrow}$ *is an interior operator and* $\mathsf{Subf}_{\uparrow}$ *is a closure operator on the lattice of si-logics.*
(ii) $\mathsf{Sub}_{\downarrow}(L) = \mathbf{IPC} + \{\varphi \mid \varphi$ *is a* $(\wedge, \to)$*-formula and* $L \vdash \varphi\}$.
(iii) $\mathsf{Subf}_{\downarrow}(L) = \mathbf{IPC}$ *iff for every* $(\wedge, \to)$*-formula* φ*,* $L \vdash \varphi$ *iff* $\mathbf{IPC} \vdash \varphi$.

Proof. (i). Straightforward from the definition.

(ii). By Proposition 3.1, every subframe logic is axiomatizable by $(\wedge, \to)$-formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of $(\wedge, \to)$-formulas that are provable in L. Thus, the set $\{\varphi \mid \varphi$ is a $(\wedge, \to)$-formula and $L \vdash \varphi\}$ axiomatizes the largest subframe logic contained in L.

(iii). Apply (ii). □

We next give a semantic characterization of the downward and upward subframizations of a si-logic L. For a class K of Esakia frames, we write $K \models \varphi$ provided $\mathfrak{F} \models \varphi$ for each $\mathfrak{F} \in K$. Let $\mathsf{Log}(K) = \{\varphi \mid K \models \varphi\}$ be the si-logic of K, and write $\mathsf{Log}(\mathfrak{F})$ if $K = \{\mathfrak{F}\}$.

Proposition 3.4. *Suppose L is a si-logic and $L = \mathsf{Log}(K)$ for some class K of Esakia frames.*

(i) $\mathsf{Subf}_{\downarrow}(L) = \mathsf{Log}(\{\mathfrak{G} \mid \mathfrak{G}$ *is a subframe of some* $\mathfrak{F} \in K\})$.
(ii) $\mathsf{Subf}_{\uparrow}(L) = \mathsf{Log}(\{\mathfrak{F} \mid \mathfrak{G} \models L$ *for all subframes* $\mathfrak{G}$ *of* $\mathfrak{F}\})$.

Proof. (i) Let $K' = \{\mathfrak{G} \mid \mathfrak{G}$ is a subframe of some $\mathfrak{F} \in K\}$. Then $K \subseteq K'$, so $\mathsf{Log}(K') \subseteq \mathsf{Log}(K) = L$. Since K' is closed under subframes, $\mathsf{Log}(K')$ is a subframe logic by Proposition 3.1. If L' is a subframe logic contained in L, then $K \models L'$, so $K' \models L'$ as L' is a subframe logic. Therefore, $L' \subseteq \mathsf{Log}(K')$. Thus, $\mathsf{Log}(K')$ is the largest subframe logic contained in L, and hence $\mathsf{Subf}_{\downarrow}(L) = \mathsf{Log}(K')$.

(ii) Let $K' = \{\mathfrak{F} \mid \mathfrak{G} \models L$ for all subframes $\mathfrak{G}$ of $\mathfrak{F}\}$. It is clear that K' is closed under subframes, so $\mathsf{Log}(K')$ is a subframe logic by Proposition 3.1. Moreover, $K' \models L$, so $L \subseteq \mathsf{Log}(K')$. Let L' be a subframe logic containing L. If $\mathfrak{F} \models L'$, then since L' is a subframe logic, $\mathfrak{G} \models L'$ for every subframe $\mathfrak{G}$ of $\mathfrak{F}$. But then $\mathfrak{G} \models L$ as $L \subseteq L'$, so $\mathfrak{F} \in K'$. Therefore, every L'-frame is contained in K', and so $\mathsf{Log}(K') \subseteq L'$. Thus, $\mathsf{Log}(K')$ is the smallest subframe logic containing L, and hence $\mathsf{Subf}_\uparrow(L) = \mathsf{Log}(K')$. □

We use Proposition 3.4 and Zakharyaschev's canonical formulas to give a syntactic characterization of the downward and upward subframizations of a si-logic L. Zakharyaschev's canonical formulas generalize subframe formulas by adding additional parameters. By Zakharyaschev's theorem [24] (cf. [11, Sec. 9.3]), every si-logic is axiomatizable by canonical formulas.

Let $\mathfrak{F}$ be a finite rooted frame and $\mathfrak{D}$ be a family of upsets of $\mathfrak{F}$, called *closed domains*. Suppose $\mathfrak{G}$ is an Esakia frame. We say that a p-morphism f from a subframe $\mathfrak{H} = (S, \leq)$ of $\mathfrak{G}$ onto $\mathfrak{F}$ satisfies the *closed domain condition* (CDC) provided

$$x \in {\uparrow}S \text{ and } f({\uparrow}x) \in \mathfrak{D} \text{ imply } x \in S. \tag{6}$$

With a finite rooted $\mathfrak{F}$ and $\mathfrak{D}$ we associate the canonical formula $\beta(\mathfrak{F}, \mathfrak{D})$ such that for any Esakia frame $\mathfrak{G}$,

(7) $\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D})$ iff there is a p-morphism from a subframe of $\mathfrak{G}$ onto $\mathfrak{F}$ satisfying CDC.

Remark 3.5.

(i) Here we follow Jerabek's account of canonical formulas [18, Sec. 3], which is slightly different from Zakharyaschev's approach. Namely our closed domains are upsets rather than antichains. Also, closed domains may be empty, which allows us to work with subframes rather than cofinal subframes (see [18, Rem. 3.7]).

(ii) If $\mathfrak{D} = \varnothing$, then $\beta(\mathfrak{F}, \mathfrak{D})$ is the subframe formula $\beta(\mathfrak{F})$. In the other extreme case, when $\mathfrak{D}$ is the set of all upsets of $\mathfrak{F}$, the canonical formula $\beta(\mathfrak{F}, \mathfrak{D})$ is equivalent to the Jankov formula $\chi(\mathfrak{F})$ [11, Sec. 9.3]. If $\mathfrak{D}$ is the set of all nonempty upsets, then $\beta(\mathfrak{F}, \mathfrak{D})$ is the negation-free Jankov formula of $\mathfrak{F}$, and is denoted by $\beta^\sharp(\mathfrak{F})$ [11, Sec. 9.4].

Theorem 3.6. *Let $L = \mathbf{IPC} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i) \mid i \in I\}$ be a si-logic.*

(i) $\mathsf{Subf}_\downarrow(L) = \mathbf{IPC} + \{\beta(\mathfrak{F}) \mid L \vdash \beta(\mathfrak{F})\}$.

(ii) $\mathsf{Subf}_\uparrow(L) = \mathbf{IPC} + \{\beta(\mathfrak{F}_i) \mid i \in I)\}$.

Proof. (i). By Proposition 3.1, every subframe logic is axiomatizable by subframe formulas. Therefore, every subframe logic contained in L is axiomatizable by a set of subframe formulas that are provable in L. Thus, $\mathbf{IPC}+\{\beta(\mathfrak{F}) \mid L \vdash \beta(\mathfrak{F})\}$ is the largest subframe logic contained in L.

(ii). Let $M = \mathbf{IPC} + \{\beta(\mathfrak{F}_i) \mid i \in I\}$. If $\mathfrak{G}$ is an M-frame, then $\mathfrak{G} \models \beta(\mathfrak{F}_i)$ for all $i \in I$. Therefore, by (5) and (7), $\mathfrak{G} \models \beta(\mathfrak{F}_i, \mathfrak{D}_i)$ for all $i \in I$. Thus, $\mathfrak{G}$ is an L-frame, and so $L \subseteq M$. Since M is axiomatized by subframe formulas, M is a subframe logic by Proposition 3.1. It remains to show that M is the least subframe logic containing L. If not, then there is a subframe logic $L' \supseteq L$ and an L'-frame $\mathfrak{G}$ such that $\mathfrak{G} \not\models M$. Therefore, $\mathfrak{G} \not\models \beta(\mathfrak{F}_i)$ for some $i \in I$. By (5), $\mathfrak{F}_i$ is a p-morphic image of a subframe $\mathfrak{S}$ of $\mathfrak{G}$. Since L' is a subframe logic, $\mathfrak{S}$ is an L'-frame. Thus, $\mathfrak{F}_i$ is also an L'-frame. But $\mathfrak{F}_i \not\models \beta(\mathfrak{F}_i, \mathfrak{D}_i)$ by (7) because the identity map is a p-morphism from $\mathfrak{F}$ onto itself that satisfies CDC for any set of closed domains. Consequently, $\mathfrak{F}_i$ is not an L-frame, which is a contradiction since $L' \supseteq L$. □

Remark 3.7.

(i) It follows from Theorem 3.6(ii) that if L is a si-logic axiomatized by a set of formulas Γ, then the upward subframization $\mathsf{Subf}_{\uparrow}(L)$ of L can be calculated effectively from Γ as follows: First use Zakharyaschev's theorem to transform Γ into an equivalent set of canonical formulas; then delete the additional parameters $\mathfrak{D}_i$ in the resulting canonical formulas; and finally apply Theorem 3.6(ii).
(ii) On the other hand, Theorem 3.6(i) does not provide an effective axiomatization of the downward subframization $\mathsf{Subf}_{\downarrow}(L)$ of L. We will come back to this issue at the end of Section 4.

Remark 3.8. In [21] Wolter studied *describable operations* on varieties of modal algebras. This translates to Esakia frames as follows. A map $\mathbf{C}$ that associates with each Esakia frame $\mathfrak{G}$ a set $\mathbf{C}(\mathfrak{G})$ of Esakia frames is *describable* if there is a map $(\cdot)^c$ on the set of formulas of **IPC** such that for each Esakia frame $\mathfrak{G}$ and each formula φ,

$$\mathfrak{G} \models \varphi^c \text{ iff } \mathbf{C}(\mathfrak{G}) \models \varphi.$$

As follows from [21, p. 23], if L is the logic of a class K of Esakia frames, then the logic of $\mathbf{C}(\mathsf{K})$ is axiomatized by $\{\varphi^c \mid L \vdash \varphi^c\}$, and the logic of $\{\mathfrak{F} \in \mathsf{K} \mid \mathbf{C}(\mathfrak{F}) \subseteq \mathsf{K}\}$ is axiomatized by $\{\varphi^c \mid L \vdash \varphi\}$.

Now let $\mathbf{C}(\mathfrak{G}) = \{\mathfrak{H} \mid \mathfrak{H} \text{ is a subframe of } \mathfrak{G}\}$. Since canonical formulas axiomatize every si-logic, we restrict our attention to the set of canonical formulas. We show that

$$\mathfrak{G} \models \beta(\mathfrak{F}) \text{ iff } \mathbf{C}(\mathfrak{G}) \models \beta(\mathfrak{F}, \mathfrak{D}). \tag{8}$$

The left to right direction is obvious. For the right to left direction, suppose $\mathfrak{G} \not\models \beta(\mathfrak{F})$. Then there is a subframe $\mathfrak{H}$ of $\mathfrak{G}$ which is p-morphically mapped onto $\mathfrak{F}$. Since $\mathfrak{F} \not\models \beta(\mathfrak{F}, \mathfrak{D})$, we have $\mathfrak{H} \not\models \beta(\mathfrak{F}, \mathfrak{D})$. Therefore, we found $\mathfrak{H} \in \mathbf{C}(\mathfrak{G})$ such that $\mathfrak{H} \not\models \beta(\mathfrak{F}, \mathfrak{D})$.

From (8) we deduce that $(\beta(\mathfrak{F}, \mathfrak{D}))^c = \beta(\mathfrak{F})$. Thus, applying Wolter's result to Proposition 3.4 yields an alternative proof of Theorem 3.6.

We conclude this section by providing the upward and downward subframizations of many well-known si-logics. Following [6], we denote by $\mathfrak{L}$ the Rieger-Nishimura ladder (the dual Esakia frame of the free cyclic Heyting algebra, see Figure 1).

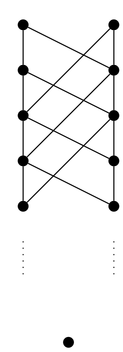


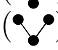
FIGURE 1. The Rieger-Nishimura ladder $\mathfrak{L}$

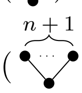
For Esakia frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$, we denote their *ordered sum* by $\bigoplus_{i=1}^n \mathfrak{F}_i$ [6, Sec. 2.2]. We consider the following logics:

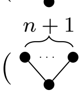
- The Rieger-Nishimura logic **RN**, which is the logic of the Rieger-Nishimura ladder $\mathfrak{L}$.
- The Kuznetsov-Gerciu logic **KG**, which is the logic of $\bigoplus_{i=1}^{n} \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is a generated subframe of $\mathfrak{L}$.
- The Kreisel-Putnam logic $\mathbf{KP} = \mathbf{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$.
- The Gabbay-de Jongh logics $\mathbf{T}_n$, where $\mathbf{T}_n$ is the logic of finite trees of branching $\leq n$.
- The logics $\mathbf{BW}_n$ of finite frames of width $\leq n$. In particular, $\mathbf{BW}_1$ is the Gödel-Dummett logic $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$ of finite linear frames.
- The logics $\mathbf{BTW}_n$ of finite frames of top width $\leq n$. In particular, $\mathbf{BTW}_1$ is the logic **KC** of weak excluded middle, which is the logic of finite directed frames.
- Maksimova's logics $\mathbf{ND}_n = \mathbf{IPC} + (\neg p \to \bigvee_{1\leq i\leq n} \neg q_i) \to \bigvee_{1\leq i\leq n}(\neg p \to \neg q_i)$.

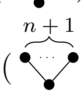
TABLE 1. Axiomatizations in terms of canonical formulas (see [6, Thm. 4.33] for the axiomatization of **RN** and [11, Table 9.7] for the other cases).

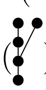
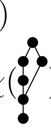
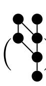
KC $=$ $\mathbf{IPC} + \beta($$, \{\varnothing\})$

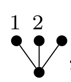
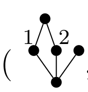
LC $=$ $\mathbf{IPC} + \beta($$)$

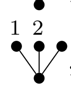
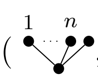
$\mathbf{BTW}_n$ $=$ $\mathbf{IPC} + \beta($$, \{\varnothing\})$

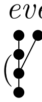
$\mathbf{BW}_n$ $=$ $\mathbf{IPC} + \beta($$)$

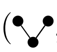
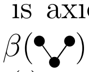
$\mathbf{T}_n$ $=$ $\mathbf{IPC} + \beta^\sharp($$)$

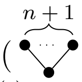
RN $=$ $\mathbf{KG} + \chi($$) + \chi($$) + \chi($$)$

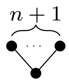
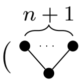
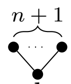
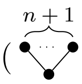
KP $=$ $\mathbf{IPC} + \beta($$, \{\varnothing, \{1,2\}\}) + \beta($$, \{\varnothing, \{1,2\}\})$

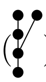
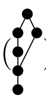
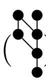
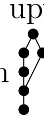
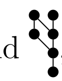
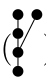
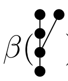
$\mathbf{ND}_n$ $=$ $\mathbf{IPC} + \beta($$, \{\varnothing, \{1,2\}\}) + \cdots + \beta($$, \{\varnothing, \{1,\dots,n\}\})$

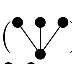
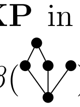
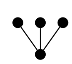
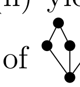
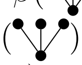
Proposition 3.9.
(i) $\mathsf{Subf}_\downarrow(\mathbf{KC}) = \mathbf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathbf{KC}) = \mathbf{LC}$.
(ii) $\mathsf{Subf}_\downarrow(\mathbf{BTW}_n) = \mathbf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathbf{BTW}_n) = \mathbf{BW}_n$ *for every* $n \geq 2$.
(iii) $\mathsf{Subf}_\downarrow(\mathbf{T}_n) = \mathbf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathbf{T}_n) = \mathbf{BW}_n$ *for every* $n \geq 2$.
(iv) $\mathsf{Subf}_\downarrow(\mathbf{RN}) = \mathbf{KG}$ *and* $\mathsf{Subf}_\uparrow(\mathbf{RN}) = \mathbf{KG} + \beta($$)$.
(v) $\mathsf{Subf}_\downarrow(\mathbf{KP}) = \mathbf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathbf{KP}) = \mathbf{BW}_2$.
(vi) $\mathsf{Subf}_\downarrow(\mathbf{ND}_n) = \mathbf{IPC}$ *and* $\mathsf{Subf}_\uparrow(\mathbf{ND}_n) = \mathbf{BW}_2$ *for every* $n \geq 2$.

Proof. (i). Since **KC** is axiomatized by $\beta($$, \{\varnothing\})$, it follows from Theorem 3.6(ii) that $\mathsf{Subf}_\uparrow(\mathbf{KC}) = \mathbf{IPC} + \beta($$) = \mathbf{LC}$. To calculate the downward subframization of **KC**, we utilize Proposition 3.4(i). It is well known that **IPC** is the logic of all finite frames and that **KC** is the logic of all finite directed frames. Moreover, adding a new top to a finite frame $\mathfrak{F}$ results in a finite directed frame $\mathfrak{G}$ containing $\mathfrak{F}$ as a subframe. Therefore, by Proposition 3.4(i), $\mathsf{Subf}_{\downarrow}(\mathbf{KC}) = \mathbf{IPC}$.

(ii). From the axiomatization of $\mathbf{BTW}_n$ in Table 1 and Theorem 3.6(ii) it follows that $\mathsf{Subf}_{\uparrow}(\mathbf{BTW}_n) = \mathbf{IPC} + \beta($  $) = \mathbf{BW}_n$. To see that $\mathsf{Subf}_{\downarrow}(\mathbf{BTW}_n) = \mathbf{IPC}$ observe that $\mathbf{BTW}_n \subseteq \mathbf{KC}$ and apply (i) and Lemma 3.3(i).

(iii). It follows from Table 1 that $\mathbf{T}_n$ is axiomatized by the negation-free Jankov formula $\beta^\sharp($  $)$, which we view as the canonical formula $\beta($  $, \mathfrak{D})$, where $\mathfrak{D}$ is the set of all nonempty upsets of  (see Remark 3.5(ii)). Therefore, $\mathsf{Subf}_{\uparrow}(\mathbf{T}_n) = \mathbf{IPC} + \beta($  $) = \mathbf{BW}_n$. To determine the downward subframization, since $\mathbf{T}_n$ has the disjunction property [15] and every si-logic with the disjunction property proves the same disjunction-free formulas as $\mathbf{IPC}$ [19, 23], we conclude that $\mathbf{T}_n$ proves the same $(\wedge, \to)$-formulas as $\mathbf{IPC}$. Thus, by Lemma 3.3(iii), $\mathsf{Subf}_{\downarrow}(\mathbf{T}_n) = \mathbf{IPC}$.

(iv). Since $\mathbf{KG}$ is a subframe logic contained in $\mathbf{RN}$ (cf. [6, Sec. 3]), it follows from the axiomatization of $\mathbf{RN}$ in Table 1 and Theorem 3.6(ii) that the upward subframization of $\mathbf{RN}$ is $\mathbf{KG} + \beta($  $) + \beta($  $) + \beta($  $)$. Since  is a subframe of both  and , the latter logic is equal to $\mathbf{KG} + \beta($  $)$. Therefore, $\mathsf{Subf}_{\uparrow}(\mathbf{RN}) = \mathbf{KG} + \beta($  $)$. To determine the downward subframization, $\mathbf{KG} \subseteq \mathsf{Subf}_{\downarrow}(\mathbf{RN})$ since $\mathbf{KG}$ is a subframe logic contained in $\mathbf{RN}$. For the reverse inclusion, since $\mathbf{KG}$ is the logic of its finite rooted frames, by Proposition 3.4(i), it is sufficient to show that every finite rooted $\mathbf{KG}$-frame is a subframe of the Rieger-Nishimura ladder $\mathfrak{L}$. First note that the subframe of $\mathfrak{L}$ obtained by deleting the first k layers of $\mathfrak{L}$ is isomorphic to $\mathfrak{L}$. Using this it is easy to see that every finite generated subframe of $\mathfrak{L}$ can be realized as a subframe of $\mathfrak{L}$ at an arbitrary depth, i.e., as a subframe of $\mathfrak{L}$ that does not contain the first k-layers of $\mathfrak{L}$ for any $k \in \mathbb{N}$. Therefore, a finite rooted $\mathbf{KG}$-frame $\bigoplus_{i=1}^n \mathfrak{F}_i$ can be realized as a subframe of $\mathfrak{L}$ by embedding $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ below each other so that the two subsequent points in $\mathfrak{L}$ between the embeddings of $\mathfrak{F}_i$ and $\mathfrak{F}_{i+1}$ are skipped.

(v). The axiomatization of $\mathbf{KP}$ in Table 1 and Theorem 3.6(ii) yield that $\mathsf{Subf}_{\uparrow}(\mathbf{KP})$ is axiomatized by $\beta($  $)$ and $\beta($  $)$. But  is a subframe of , so $\mathsf{Subf}_{\uparrow}(\mathbf{KP})$ is axiomatized by $\beta($  $)$, and hence $\mathsf{Subf}_{\uparrow}(\mathbf{KP}) = \mathbf{BW}_2$. Since $\mathbf{KP}$ has the disjunction property, $\mathsf{Subf}_{\downarrow}(\mathbf{KP}) = \mathbf{IPC}$ by the same argument as in (iii).

(vi). Since the 3-fork is a subframe of the n-fork for $n \geq 3$, it follows from the axiomatization of $\mathbf{ND}_n$ in Table 1 and Theorem 3.6(ii) that $\mathsf{Subf}_{\uparrow}(\mathbf{ND}_n) = \mathbf{BW}_2$ for $n \geq 2$. Since $\mathbf{ND}_n$ has the disjunction property, $\mathsf{Subf}_{\downarrow}(\mathbf{ND}_n) = \mathbf{IPC}$ by the same argument as in (iii). $\square$

4. Superintuitionistic logics and lax logics

As we saw in Remark 3.7, the upward subframization of a si-logic $L = \mathbf{IPC} + \Gamma$ can be calculated effectively from Γ. In this section we show how to calculate the downward subframization of L by utilizing the translation of $\mathbf{IPC}$ into $\mathbf{PLL}$. Let $\mathcal{L}_{\mathbf{IPC}}$ be the language of $\mathbf{IPC}$ and $\mathcal{L}_{\mathbf{PLL}}$ be the language of $\mathbf{PLL}$.

Definition 4.1. Define a translation $\tau : \mathcal{L}_{\mathbf{IPC}} \to \mathcal{L}_{\mathbf{PLL}}$ by

- $\tau(p) = \bigcirc p$ for a propositional letter p,
- $\tau(\bot) = \bigcirc\bot$,
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$,
- $\tau(\varphi \to \psi) = \tau(\varphi) \to \tau(\psi)$,
- $\tau(\varphi \vee \psi) = \bigcirc(\tau(\varphi) \vee \tau(\psi))$.

Recall that by Notation 2.2, given an S-frame $(\mathfrak{F}, \mathfrak{G})$, we always assume that $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (S, \leq)$.

Lemma 4.2. *Let v be a valuation on an S-frame $(\mathfrak{F}, \mathfrak{G})$. Define a valuation $v_{\mathfrak{G}}$ on $\mathfrak{G}$ by $v_{\mathfrak{G}}(p) = v(p) \cap S$. Then for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ and $x \in X$,*

$$x \models_v \tau(\varphi) \textit{ iff } y \models_{v_{\mathfrak{G}}} \varphi \textit{ for all } y \in {\uparrow}x \cap S.$$

Proof. The proof is by induction on the complexity of $\varphi \in \mathcal{L}_{\mathbf{IPC}}$.

If $\varphi = p$, then $\tau(\varphi) = \bigcirc p$. Therefore, by (2) and the definition of $v_{\mathfrak{G}}$,

$$\begin{aligned} x \models_v \bigcirc p &\text{ iff } y \models_v p \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } y \models_{v_{\mathfrak{G}}} p \text{ for all } y \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \bot$, then $\tau(\varphi) = \bigcirc\bot$. Therefore, $x \models_v \bigcirc\bot$ iff ${\uparrow}x \cap S = \varnothing$. Thus, $x \models_v \bigcirc\bot$ iff $y \models_{v_{\mathfrak{G}}} \bot$ for all $y \in {\uparrow}x \cap S$.

If $\varphi = \psi \wedge \chi$, then $\tau(\psi \wedge \chi) = \tau(\psi) \wedge \tau(\chi)$. Therefore,

$$\begin{aligned} x \models_v \tau(\psi \wedge \chi) &\text{ iff } x \models_v \tau(\psi) \text{ and } x \models_v \tau(\chi) \\ &\text{ iff } y \models_{v_{\mathfrak{G}}} \psi \text{ and } y \models_{v_{\mathfrak{G}}} \chi \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } y \models_{v_{\mathfrak{G}}} \psi \wedge \chi \text{ for all } y \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \psi \to \chi$, then $\tau(\psi \to \chi) = \tau(\psi) \to \tau(\chi)$. Therefore,

$$\begin{aligned} x \models_v \tau(\psi) \to \tau(\chi) &\text{ iff } z \models_v \tau(\psi) \text{ implies } z \models_v \tau(\chi) \text{ for all } z \geq x \\ &\text{ iff } (w \models_{v_{\mathfrak{G}}} \psi \text{ implies } w \models_{v_{\mathfrak{G}}} \chi \text{ for all } w \in {\uparrow}z \cap S) \text{ for all } z \geq x \\ &\text{ iff } (w \models_{v_{\mathfrak{G}}} \psi \text{ implies } w \models_{v_{\mathfrak{G}}} \chi) \text{ for all } w \in {\uparrow}x \cap S. \end{aligned}$$

If $\varphi = \psi \vee \chi$, then $\tau(\psi \vee \chi) = \bigcirc(\tau(\psi) \vee \tau(\chi))$. Therefore,

$$\begin{aligned} x \models_v \bigcirc(\tau(\psi) \vee \tau(\chi)) &\text{ iff } y \models_v \tau(\psi) \vee \tau(\chi) \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } y \models_v \tau(\psi) \text{ or } y \models_v \tau(\chi) \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } (z \models_{v_{\mathfrak{G}}} \psi \text{ or } z \models_{v_{\mathfrak{G}}} \chi \text{ for all } z \in {\uparrow}y \cap S) \text{ for all } y \in {\uparrow}x \cap S \\ &\text{ iff } z \models_{v_{\mathfrak{G}}} \psi \vee \chi \text{ for all } z \in {\uparrow}x \cap S. \end{aligned}$$

□

Lemma 4.3. *Let $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ and $(\mathfrak{F}, \mathfrak{G})$ be an S-frame.*

(i) *$(\mathfrak{F}, \mathfrak{G}) \models \varphi$ iff $\mathfrak{F} \models \varphi$.*
(ii) *$(\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi)$ iff $\mathfrak{G} \models \varphi$.*

Proof. (i). This is obvious since φ contains no occurrences of $\bigcirc$.

(ii). For the right to left direction, suppose v is a valuation on $(\mathfrak{F}, \mathfrak{G})$ that refutes $\tau(\varphi)$. Define a valuation v' on $\mathfrak{G}$ by $v'(p) = v(p) \cap S$. By Lemma 4.2, v' refutes φ on $\mathfrak{G}$. For the left to right direction, suppose v' is a valuation on $\mathfrak{G}$ that refutes φ. Define a valuation v on $\mathfrak{F}$ by $v(p) = X \setminus {\downarrow}(S \setminus v'(p))$. Then $v'(p) = v(p) \cap S$ for every propositional letter p. Applying Lemma 4.2 again yields that v refutes $\tau(\varphi)$ on $(\mathfrak{F}, \mathfrak{G})$. □

Remark 4.4. An algebraic reformulation of Lemma 4.3 is as follows. If $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ and (A, j) is a nuclear Heyting algebra, then

(i) $(A, j) \models \varphi$ iff $A \models \varphi$.
(ii) $(A, j) \models \tau(\varphi)$ iff $A_j \models \varphi$.

Let $\Lambda(\mathbf{IPC})$ be the lattice of all si-logics and let $\Lambda(\mathbf{PLL})$ be the lattice of all extensions of $\mathbf{PLL}$.

Definition 4.5. Let $L \in \Lambda(\mathbf{IPC})$ and $M \in \Lambda(\mathbf{PLL})$.

(i) We say that $L \in \Lambda(\mathbf{IPC})$ is the *intuitionistic fragment* of M if for all $\varphi \in \mathcal{L}_{\mathbf{IPC}}$,
$$\varphi \in L \text{ iff } \varphi \in M.$$
(ii) We say that $L \in \Lambda(\mathbf{IPC})$ is the *lax fragment* of M if for all $\varphi \in \mathcal{L}_{\mathbf{IPC}}$,
$$\varphi \in L \text{ iff } \tau(\varphi) \in M.$$

Definition 4.6. For $M \in \Lambda(\mathbf{PLL})$, we define
$$\rho_1(M) = \{\varphi \in \mathcal{L}_{\mathbf{IPC}} \mid \varphi \in M\}$$
$$\rho_2(M) = \{\varphi \in \mathcal{L}_{\mathbf{IPC}} \mid \tau(\varphi) \in M\}$$

Lemma 4.7. *Let $M \in \Lambda(\mathbf{PLL})$.*

(i) *$\rho_1(M)$ is the intuitionistic fragment of M and*
$$\rho_1(M) = \mathsf{Log}\left(\{\mathfrak{F} \mid (\mathfrak{F}, \mathfrak{G}) \models M \textit{ for some subframe } \mathfrak{G} \textit{ of } \mathfrak{F}\}\right).$$
(ii) *$\rho_2(M)$ is the lax fragment of M and*
$$\rho_2(M) = \mathsf{Log}\left(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models M\}\right).$$

Proof. We first show (ii). For $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, using Lemma 4.3(ii), we have
$$\begin{aligned} \varphi \in \mathsf{Log}\left(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models M\}\right) &\Leftrightarrow \mathfrak{G} \models \varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models M \\ &\Leftrightarrow (\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi) \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models M \\ &\Leftrightarrow \tau(\varphi) \in M. \end{aligned}$$
Therefore, $\rho_2(M) = \mathsf{Log}\left(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models M\}\right)$. Thus, $\rho_2(M)$ is a si-logic, and hence is the lax fragment of M.

(i). This is proved similarly but uses Lemma 4.3(i) instead. □

Remark 4.8. An algebraic reformulation of Lemma 4.7 is as follows:

(i) $\rho_1(L) = \mathsf{Log}\left(\{A \mid (A, j) \models M \text{ for some nucleus } j \text{ on } A\}\right)$.
(ii) $\rho_2(L) = \mathsf{Log}\left(\{A_j \mid (A, j) \models M\}\right)$.

Definition 4.9. For a si-logic L, we define
$$\sigma_1(L) = \mathbf{PLL} + \{\varphi \mid \varphi \in L\}$$
$$\sigma_2(L) = \mathbf{PLL} + \{\tau(\varphi) \mid \varphi \in L\}$$

Lemma 4.10. *Let L be a si-logic.*

(i) $\sigma_1(L) = \mathsf{Log}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{F} \models L\}\right)$.
(ii) $\sigma_2(L) = \mathsf{Log}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G} \models L\}\right)$.

Proof. We first show (ii). Suppose $(\mathfrak{F}, \mathfrak{G})$ is an S-frame. By Lemma 4.3(ii), $\mathfrak{G} \models L$ iff $(\mathfrak{F}, \mathfrak{G}) \models \{\tau(\varphi) \mid \varphi \in L\}$. Thus, $\sigma_2(L) = \mathsf{Log}\,(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G} \models L\})$.

(i). This is proved similarly but uses Lemma 4.3(i) instead. □

Remark 4.11. In algebraic terms, Lemma 4.10 can be expressed as follows:

(i) $\sigma_1(L) = \mathsf{Log}\,(\{(A, j) \mid A \models L\})$.
(ii) $\sigma_2(L) = \mathsf{Log}\,(\{(A, j) \mid A_j \models L\})$.

Lemma 4.12. *Let L be a si-logic.*

(i) *$L = \rho_1\sigma_1(L)$. In fact, $\sigma_1(L)$ is the least element of $\rho_1^{-1}(L)$.*
(ii) *$L = \rho_2\sigma_2(L)$. In fact, $\sigma_2(L)$ is the least element of $\rho_2^{-1}(L)$.*

Proof. (i). Let $\varphi \in \mathcal{L}_{\mathbf{IPC}}$. Then $\varphi \in L$ implies $\varphi \in \sigma_1(L)$, which implies $\varphi \in \rho_1\sigma_1(L)$. Therefore, $L \subseteq \rho_1\sigma_1(L)$. If $\varphi \notin L$, then there is an L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Consider the S-frame $(\mathfrak{F}, \mathfrak{F})$. By Lemma 4.10(i), $(\mathfrak{F}, \mathfrak{F}) \models \sigma_1(L)$, and by Lemma 4.3(i), $(\mathfrak{F}, \mathfrak{F}) \not\models \varphi$. Thus, $\varphi \notin \sigma_1(L)$, and so by Lemma 4.7(i), $\varphi \notin \rho_1\sigma_1(L)$. This shows that $L = \rho_1\sigma_1(L)$. If $M \in \rho_1^{-1}(L)$, then for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, we have $\varphi \in L$ iff $\varphi \in M$. Consequently, $\sigma_1(L) \subseteq M$, and hence $\sigma_1(L)$ is the least element of $\rho_1^{-1}(L)$.

(ii). Let $\varphi \in \mathcal{L}_{\mathbf{IPC}}$. Then $\varphi \in L$ implies $\tau(\varphi) \in \sigma_2(L)$, which implies $\varphi \in \rho_2\sigma_2(L)$. Therefore, $L \subseteq \rho_2\sigma_2(L)$. If $\varphi \notin L$, then there is an L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By Lemma 4.10(ii), the S-frame $(\mathfrak{F}, \mathfrak{F})$ is a $\sigma_2(L)$-frame, and by Lemma 4.3(ii), $(\mathfrak{F}, \mathfrak{F}) \not\models \tau(\varphi)$. Thus, $\varphi \notin \sigma_2(L)$, and so by Lemma 4.7(ii), $\varphi \notin \rho_2\sigma_2(L)$. This shows that $L = \rho_2\sigma_2(L)$. If $M \in \rho_2^{-1}(L)$, then for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, we have $\varphi \in L$ iff $\tau(\varphi) \in M$. Consequently, $\sigma_2(L) \subseteq M$, and hence $\sigma_2(L)$ is the least element of $\rho_2^{-1}(L)$. □

As follows from Lemma 4.12, for a si-logic L, both $\rho_1^{-1}(L)$ and $\rho_2^{-1}(L)$ have least elements, but they may not have largest elements. To see this we require the following lemmas.

Lemma 4.13. *Let $(\mathfrak{F}, \mathfrak{G})$ be an S-frame.*

(i) *$(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p \leftrightarrow p$ iff $\mathfrak{F} = \mathfrak{G}$.*
(ii) *$(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p$ iff $\mathfrak{G} = \varnothing$.*

Proof. (i). First suppose that $\mathfrak{F} = \mathfrak{G}$. Then it is clear that $(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p \leftrightarrow p$. Next suppose that $\mathfrak{F} \neq \mathfrak{G}$. Let $x \in X \setminus S$. Then $x \notin {\uparrow}x \cap S$, so $x \notin {\uparrow}({\uparrow}x \cap S)$. Therefore, since ${\uparrow}({\uparrow}x \cap S)$ is a closed upset of X, there is a clopen upset U of X with ${\uparrow}({\uparrow}x \cap S) \subseteq U$ and $x \notin U$. Let v be a valuation on $(\mathfrak{F}, \mathfrak{G})$ such that $v(p) = U$. Clearly $x \not\models_v p$. On the other hand, $x \models_v \bigcirc p$ by (2). Thus, $(\mathfrak{F}, \mathfrak{G}) \not\models \bigcirc p \leftrightarrow p$.

(ii). If $\mathfrak{G} = \varnothing$, then it is clear that $(\mathfrak{F}, \mathfrak{G}) \models \bigcirc p$. If $\mathfrak{G} \neq \varnothing$, then let v be a valuation on $(\mathfrak{F}, \mathfrak{G})$ such that $v(p) = \varnothing$. For $x \in S$, we then have $x \not\models_v \bigcirc p$, so $(\mathfrak{F}, \mathfrak{G}) \not\models \bigcirc p$. □

For $\psi \in \mathcal{L}_{\mathbf{PLL}}$, let ψ^- be the formula obtained from ψ by deleting all occurrences of the $\bigcirc$ modality and let ψ^* be the formula obtained from ψ by replacing all subformulas of the form $\bigcirc\chi$ with $\top$. Clearly $\psi^-, \psi^* \in \mathcal{L}_{\mathbf{IPC}}$. Both ψ^- and ψ^* were considered in [13, Sec. 3].

Lemma 4.14. *Let $M \in \Lambda(\mathbf{PLL})$.*

(i) *If $\bigcirc p \leftrightarrow p \in M$, then $\psi \in M$ iff $\psi^- \in M$ for every formula $\psi \in \mathcal{L}_{\mathbf{PLL}}$.*
(ii) *If $\bigcirc p \in M$, then $\psi \in M$ iff $\psi^* \in M$ for every formula $\psi \in \mathcal{L}_{\mathbf{PLL}}$.*

Proof. (i). Suppose that $\bigcirc p \leftrightarrow p \in M$ and let $\psi \in \mathcal{L}_{\mathbf{PLL}}$. By Lemma 4.13(i), M is the logic of the class of S-frames of the shape $(\mathfrak{F}, \mathfrak{F})$. For $(\mathfrak{F}, \mathfrak{F})$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$, we have $x \models_v \bigcirc\varphi$ iff $x \models_v \varphi$. Therefore, induction on ψ yields $(\mathfrak{F}, \mathfrak{F}) \models \psi$ iff $(\mathfrak{F}, \mathfrak{F}) \models \psi^-$. Thus, $\psi \in M$ iff $\psi^- \in M$.

(ii). Let $\bigcirc p \leftrightarrow \top \in M$ and let $\psi \in \mathcal{L}_{\mathbf{PLL}}$. By Lemma 4.13(ii), M is the logic of the class of S-frames of the shape $(\mathfrak{F}, \varnothing)$. For $(\mathfrak{F}, \varnothing)$, a valuation v on $\mathfrak{F}$, and $x \in \mathfrak{F}$, we have $x \models_v \bigcirc\varphi$. Therefore, induction on ψ yields $(\mathfrak{F}, \varnothing) \models \psi$ iff $(\mathfrak{F}, \varnothing) \models \psi^*$. Thus, $\psi \in M$ iff $\psi^* \in M$. □

Lemma 4.15. *Let L be a si-logic.*

(i) $\sigma_1(L) + \bigcirc p \leftrightarrow p$ *is a maximal element of both* $\rho_1^{-1}(L)$ *and* $\rho_2^{-1}(L)$.
(ii) $\sigma_1(L) + \bigcirc p$ *is a maximal element of* $\rho_1^{-1}(L)$.

Proof. (i). Let $M = \sigma_1(L) + \bigcirc p \leftrightarrow p$. First we show that M is a maximal element of $\rho_1^{-1}(L)$. By Lemma 4.13(i), an S-frame $(\mathfrak{F}, \mathfrak{G})$ validates M iff $\mathfrak{F}$ is an L-frame and $\mathfrak{F} = \mathfrak{G}$. Therefore, by Lemma 4.7(i), $\rho_1(M) = L$, so $M \in \rho_1^{-1}(L)$. To see that M is maximal in $\rho_1^{-1}(L)$, suppose that $M \subseteq M' \in \rho_1^{-1}(L)$. We show that $M = M'$. Let $\psi \in \mathcal{L}_{\mathbf{PLL}}$. If $\psi \notin M$, then by Lemma 4.14(i), $\psi^- \notin M$, and so $\psi^- \notin L$ as $\psi^- \in \mathcal{L}_{\mathbf{IPC}}$. Since $\rho_1(M') = L$, we see that $\psi^- \notin M'$. Because $M \subseteq M'$, we have $\bigcirc p \leftrightarrow p \in M'$, so $\psi \notin M'$ by Lemma 4.14(i). Thus, $M = M'$, and hence M is maximal in $\rho_1^{-1}(L)$.

Next we show that M is a maximal element of $\rho_2^{-1}(L)$. By Lemma 4.7(ii), $\rho_2(M) = L$, so $M \in \rho_2^{-1}(L)$. Suppose $M \subseteq M' \in \rho_2^{-1}(L)$. We show that $M = M'$. Let $\psi \in \mathcal{L}_{\mathbf{PLL}}$. If $\psi \notin M$, then $\psi^- \notin M$ by Lemma 4.14(i). Therefore, $\tau(\psi^-) \notin M$ because $(\tau(\psi^-))^- = \psi^-$. Thus, $\psi^- \notin L$, and so $\tau(\psi^-) \notin M'$. Since $M \subseteq M'$, we have $\bigcirc p \leftrightarrow p \in M'$, and hence $\psi^- = (\tau(\psi^-))^- \notin M'$ by Lemma 4.14(i). Consequently, $\psi \notin M'$, and so $M = M'$, which yields that M is maximal in $\rho_2^{-1}(L)$.

(ii). Let $M = \sigma_1(L) + \bigcirc p$. By Lemma 4.13(ii), an S-frame $(\mathfrak{F}, \mathfrak{G})$ validates M iff $\mathfrak{F}$ is an L-frame and $\mathfrak{G} = \varnothing$. Therefore, by Lemma 4.7(i), $\rho_1(M) = L$, so $M \in \rho_1^{-1}(L)$. To see that M is maximal in $\rho_1^{-1}(L)$, suppose that $M \subseteq M' \in \rho_1^{-1}(L)$. We show that $M = M'$. Let $\psi \in \mathcal{L}_{\mathbf{PLL}}$. If $\psi \notin M$, then by Lemma 4.14(ii), $\psi^* \notin M$, and so $\psi^* \notin L$ as $\psi^* \in \mathcal{L}_{\mathbf{IPC}}$. Since $\rho_1(M') = L$, we see that $\psi^* \notin M'$. Because $M \subseteq M'$, we have $\bigcirc p \in M'$, so $\psi \notin M'$ by Lemma 4.14(ii). Thus, $M = M'$, and hence M is maximal in $\rho_1^{-1}(L)$. □

Remark 4.16.

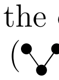
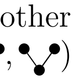
(i) Let L be a consistent si-logic. Then $\sigma_1(L) + \bigcirc p \leftrightarrow p$ and $\sigma_1(L) + \bigcirc p \leftrightarrow \top$ are different. Indeed, the S-frame $(\{x\}, \varnothing)$ validates $\sigma_1(L) + \bigcirc p \leftrightarrow \top$ but refutes $\sigma_1(L) + \bigcirc p \leftrightarrow p$. Therefore, by Lemma 4.15, $\rho_1^{-1}(L)$ need not have a largest element.
(ii) To see that $\rho_2^{-1}(L)$ also does not have a largest element, by Lemma 4.15, $\sigma_1(L) + \bigcirc p \leftrightarrow p$ is a maximal element of $\rho_2^{-1}(L)$. Let $L = \mathbf{BTW}_2$. For simplicity, we denote the canonical formula axiomatizing $\mathbf{BTW}_2$ by β (see Table 1). Set $M = \sigma_1(\mathbf{KC}) + \tau(\beta)$. By Lemmas 4.10(i) and 4.3(ii), an S-frame $(\mathfrak{F}, \mathfrak{G})$ validates M iff $\mathfrak{F}$ is a $\mathbf{KC}$-frame and $\mathfrak{G}$ is a $\mathbf{BTW}_2$-frame. Therefore, by Lemma 4.7(ii), $\mathbf{BTW}_2 \subseteq \rho_2(M)$. To see the reverse inclusion, suppose that $\varphi \notin \mathbf{BTW}_2$. Then there is a finite $\mathbf{BTW}_2$-frame $\mathfrak{G}$ with $\mathfrak{G} \not\models \varphi$. Let $\mathfrak{F}$ be obtained from $\mathfrak{G}$ by adding a new top node. Then $\mathfrak{F}$ is a $\mathbf{KC}$-frame, so $(\mathfrak{F}, \mathfrak{G})$ validates M, but refutes $\tau(\varphi)$ by Lemma 4.3(ii). Thus, $\varphi \notin \rho_2(M)$. Consequently, $\rho_2(M) = L$, and so $M \in \rho_2^{-1}(L)$. On the other hand, M is not contained in $\sigma_1(L) + \bigcirc p \leftrightarrow p$ as for example the S-frame (, ) validates $\sigma_1(L) + \bigcirc p \leftrightarrow p$ but refutes M.

Figure 2 illustrates the situation for σ_1 and ρ_1, where $\mathbf{CPC}$ denotes the classical propositional logic and $\mathbf{Fml}$ the inconsistent logic. The situation is similar for σ_2 and ρ_2. Note that in general the maximum of both $\rho_1^{-1}(L)$ and $\rho_2^{-1}(L)$ is rather complicated.

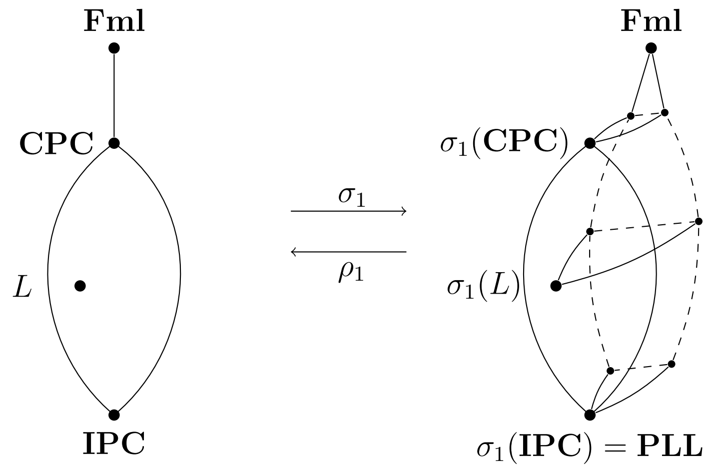


Figure 2

We are ready to obtain a new characterization of subframe si-logics.

Theorem 4.17. *For a si-logic L, the following are equivalent:*

(i) *L is a subframe logic.*
(ii) $\sigma_2(L) \subseteq \sigma_1(L)$.
(iii) $\sigma_2(L) + \{\varphi \mid \varphi \in L\} = \sigma_1(L)$.
(iv) $\rho_2\sigma_1(L) = L$.
(v) *$\sigma_1(L)$ is closed under the rule $\varphi/\tau(\varphi)$ for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$.*

Proof. (i)⇒(ii). Suppose $(\mathfrak{F}, \mathfrak{G})$ is an S-frame such that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)$. By Lemma 4.10(i), $\mathfrak{F} \models L$. Since L is a subframe logic, $\mathfrak{G} \models L$. Therefore, by Lemma 4.10(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)$. Thus, $\sigma_2(L) \subseteq \sigma_1(L)$.

(ii)⇒(iii). This is obvious.

(iii)⇒(iv). By Lemmas 4.12(ii) and 4.7(ii), $L = \rho_2\sigma_2(L) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)\})$ and $\rho_2\sigma_1(L) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)\})$. Therefore, it is sufficient to show that $\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)\} = \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)\}$. The inclusion $\supseteq$ is immediate from (iii). For the reverse inclusion, suppose that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)$. By Lemma 4.10(ii), $\mathfrak{G} \models L$, so $(\mathfrak{G}, \mathfrak{G}) \models \sigma_1(L)$ by Lemma 4.10(i). Thus, $\mathfrak{G} \in \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)\}$.

(iv)⇒(v). Suppose that there is $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ such that $\varphi \in \sigma_1(L)$ but $\tau(\varphi) \notin \sigma_1(L)$. Then there is an S-frame $(\mathfrak{F}, \mathfrak{G})$ with $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)$ and $(\mathfrak{F}, \mathfrak{G}) \not\models \tau(\varphi)$. By Lemma 4.7(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)$ implies $\mathfrak{G} \models \rho_2\sigma_1(L) = L$, and by Lemma 4.3(ii), $(\mathfrak{F}, \mathfrak{G}) \not\models \tau(\varphi)$ implies $\mathfrak{G} \not\models \varphi$. Therefore, $\varphi \notin L$, contradicting $\varphi \in \sigma_1(L)$.

(v)⇒(i). Let $\mathfrak{F}$ be an L-frame and $\mathfrak{G}$ be a subframe of $\mathfrak{F}$. By Lemma 4.10(i), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)$. By (v), $(\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi)$ for each $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ such that $\varphi \in \sigma_1(L)$. Therefore, $(\mathfrak{F}, \mathfrak{G}) \models \tau(\varphi)$ for each $\varphi \in L$. Thus, $\mathfrak{G} \models L$ by Lemma 4.3(ii), and we conclude that L is a subframe logic. □

Remark 4.18. In general, $\sigma_1(L) \not\subseteq \sigma_2(L)$. In fact, for any consistent si-logic L, from $\sigma_1(L) \subseteq \sigma_2(L)$ it follows that $L = \mathbf{IPC}$. To see this, suppose $L \neq \mathbf{IPC}$. Then there is a finite frame $\mathfrak{F}$ that refutes L. Pick a point in $\mathfrak{F}$ and let $\mathfrak{G}$ be the subframe of $\mathfrak{F}$ consisting of the point. Clearly $\mathfrak{G}$ is an L-frame. Therefore, by Lemma 4.10(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)$. On the other hand, by Lemma 4.10(i), $(\mathfrak{F}, \mathfrak{G}) \not\models \sigma_1(L)$. Thus, $\sigma_1(L) \not\subseteq \sigma_2(L)$.

As a consequence of Theorem 4.17, we obtain the following characterization of the downward subframization of a si-logic.

Theorem 4.19. *Let L be a si-logic. Then* $\mathsf{Subf}_{\downarrow}(L) = \rho_2\sigma_1(L)$.

Proof. Let $\mathfrak{G}$ be an Esakia frame. By Lemma 4.7(ii), $\mathfrak{G} \models \rho_2\sigma_1(L)$ iff there is an Esakia frame $\mathfrak{F}$ such that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)$. By Lemma 4.10(i), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_1(L)$ iff $\mathfrak{F} \models L$. Therefore, $\mathfrak{G} \models \rho_2\sigma_1(L)$ iff $\mathfrak{G}$ is a subframe of some $\mathfrak{F} \models L$. Thus, by Proposition 3.4(i), $\rho_2\sigma_1(L) = \mathsf{Subf}_{\downarrow}(L)$. □

Remark 4.20.

(i) Let L be a si-logic and $\varphi \in \mathcal{L}_{\mathbf{IPC}}$. By Theorem 4.19, $\varphi \in \mathsf{Subf}_{\downarrow}(L)$ iff $\tau(\varphi) \in \sigma_1(\mathbf{PLL})$. Therefore, if $\sigma_1(\mathbf{PLL})$ is decidable, then so is $\mathsf{Subf}_{\downarrow}(L)$.

(ii) In contrast to Theorem 4.19, for every si-logic L, we have $\rho_1\sigma_2(L) = \mathbf{IPC}$. Indeed, suppose L is a si-logic and $\mathfrak{F}$ is an Esakia frame. By Lemma 4.7(i), $\mathfrak{F} \models \rho_1\sigma_2(L)$ iff there is a subframe $\mathfrak{G}$ of $\mathfrak{F}$ such that $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)$. By Lemma 4.10(ii), $(\mathfrak{F}, \mathfrak{G}) \models \sigma_2(L)$ iff $\mathfrak{G} \models L$. Therefore, $\mathfrak{F} \models \rho_1\sigma_2(L)$ iff $\mathfrak{G} \models L$ for some subframe $\mathfrak{G}$ of $\mathfrak{F}$. Now every frame contains the empty frame as a subframe and since the empty frame is an L-frame, we conclude that every frame validates $\rho_1\sigma_2(L)$. Thus, $\rho_1\sigma_2(L) = \mathbf{IPC}$.

Remark 4.21. We recall that a subframe $\mathfrak{G}$ of an Esakia frame $\mathfrak{F}$ is *cofinal* provided it contains the maximum of $\mathfrak{F}$. Cofinal subframes of an Esakia frame $\mathfrak{F}$ correspond to *dense* nuclei on the dual Heyting algebra A of $\mathfrak{F}$, where we recall that a nucleus j is dense if $j0 = 0$. Since being a dense nucleus can be expressed by adding $\bigcirc\neg\bot$ to $\mathbf{PLL}$, the correspondence between subframe logics and extensions of $\mathbf{PLL}$ discussed in this section extends to the correspondence between cofinal subframe logics and extensions of $\mathbf{PLL} + \bigcirc\neg\bot$.

5. Stable logics and stabilization

Another well-behaved class of si-logics, along with subframe logics, is that of stable logics of [5]. In this section we define upward and downward stabilizations of si-logics, which are stable analogues of upward and downward subframizations of Section 3, and obtain their semantic and syntactic characterizations. For the syntactic characterization, we make use of the stable canonical formulas of [5].

A continuous map $f : \mathfrak{F} \to \mathfrak{G}$ between Esakia frames is called *stable* provided it is order preserving ($x \leq y$ implies $f(x) \leq f(y)$), and $\mathfrak{G}$ is a *stable image* of $\mathfrak{F}$ provided there is an onto stable map $f : \mathfrak{F} \to \mathfrak{G}$. It is easy to see that stable images of rooted frames are rooted.

Definition 5.1. A si-logic L is *stable* provided its rooted frames are closed under stable images (that is, if $\mathfrak{F}$ is an L-frame, then so is every stable image of $\mathfrak{F}$).

Remark 5.2. Definition 5.1 is slightly different from [5, Def. 6.6] but it follows from [8, Thm. 5.3] that the two are equivalent.

In [5, Thm. 6.8] it is shown that every stable logic has the finite model property. We will require the following characterization of stable logics.

Proposition 5.3. *For a si-logic L, the following are equivalent.*

(i) *L is stable.*

(ii) *L is the logic of a class of frames closed under stable images.*

(iii) *The rooted L-frames are closed under finite stable images.*

Proof. The equivalence of (i) and (ii) is proved in [8, Thm. 5.3]. It is clear that (i) implies (iii). It is left to show that (iii) implies (i). Let $\mathfrak{F}$ be a rooted L-frame and let $\mathfrak{G}$ be a stable image of $\mathfrak{F}$. If $\mathfrak{G}$ is not an L-frame, then $\mathfrak{G} \not\models \varphi$ for some $\varphi \in L$. By [5, Lem. 3.6], there is a finite stable image $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{H} \not\models \varphi$. Therefore, $\mathfrak{H}$ is a finite stable image of $\mathfrak{F}$. By (iii), $\mathfrak{H}$ is an L-frame, contradicting $\mathfrak{H} \not\models \varphi$. Thus, $\mathfrak{G}$ is an L-frame, and hence L is stable, yielding (i). ☐

Let Λ_{Stab} be the collection of all stable logics.

Lemma 5.4. Λ_{Stab} *is a complete sublattice of* $\Lambda(\mathbf{IPC})$.

Proof. Let $\{L_i \mid i \in I\}$ be a family of stable logics. Then the classes of rooted L_i-frames are stable. Therefore, so are $\bigcap_i \{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } L_i\text{-frame}\}$ and $\bigcup_i \{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } L_i\text{-frame}\}$. The intersection of all L_i-frames is exactly the class of all $(\bigvee_i L_i)$-frames. Since every logic is characterized by its rooted frames, $\bigvee_i L_i$ is characterized by $\bigcap_i \{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } L_i\text{-frame}\}$. Thus, by Proposition 5.3, $\bigvee_i L_i$ is stable. The logic $\bigwedge_i L_i$ is characterized by $\bigcup_i \{\mathfrak{F} \mid \mathfrak{F} \text{ is a rooted } L_i\text{-frame}\}$ (see, e.g., [11, Sec. 4]). Therefore, $\bigwedge_i L_i$ is also stable. Thus, Λ_{Stab} is a complete sublattice of $\Lambda(\mathbf{IPC})$. ☐

Lemma 5.4 allows us to define the least and greatest stable neighbors of a given si-logic.

Definition 5.5. For a si-logic L, define the *downward stabilization of* L as

$$\text{Stab}_{\downarrow}(L) := \bigvee \{L' \in \Lambda_{\text{Stab}} \mid L' \subseteq L\}$$

and the *upward stabilization of* L as

$$\text{Stab}_{\uparrow}(L) := \bigwedge \{L' \in \Lambda_{\text{Stab}} \mid L \subseteq L'\}.$$

The following lemma is obvious.

Lemma 5.6. $\text{Stab}_{\downarrow}$ *is an interior operator and* $\text{Stab}_{\uparrow}$ *is a closure operator on the lattice of si-logics.*

We next give a semantic characterization of upward and downward stabilizations.

Proposition 5.7. *Let L be a si-logic.*

(i) $\text{Stab}_{\downarrow}(L) = \mathsf{Log}\,(\{\mathfrak{G} \mid \mathfrak{G}$ *is a stable image of a rooted L-frame* $\mathfrak{F}\})$.

(ii) $\text{Stab}_{\uparrow}(L) = \mathsf{Log}\,(\{\mathfrak{F} \mid \mathfrak{F}$ *is finite rooted and* $\mathfrak{G} \models L$ *for every stable image* $\mathfrak{G}$ *of* $\mathfrak{F}\})$.

Proof. (i). Let $K = \{\mathfrak{G} \mid \mathfrak{G} \text{ is a stable image of a rooted } L\text{-frame } \mathfrak{F}\}$. Then K is closed under stable images, so $\mathsf{Log}(\mathsf{K})$ is a stable logic. Since K contains the class of rooted L-frames, $\mathsf{Log}(\mathsf{K}) \subseteq L$. Let L' be a stable logic contained in L. Then the class K' of rooted L'-frames contains the class of rooted L-frames and is closed under stable images. Therefore, $K \subseteq K'$, and so $L' \subseteq \mathsf{Log}(\mathsf{K})$. Thus, $\mathsf{Log}(\mathsf{K})$ is the largest stable logic contained in L.

(ii). Let $K = \{\mathfrak{F} \mid \mathfrak{F} \text{ is finite rooted and } \mathfrak{G} \models L \text{ for every stable image } \mathfrak{G} \text{ of } \mathfrak{F}\}$. Then K is closed under stable images, so $\mathsf{Log}(K)$ is a stable logic. Since K is contained in the class of rooted L-frames, $L \subseteq \mathsf{Log}(K)$. Let L' be a stable logic extending L, and let $\mathfrak{F}$ be a finite rooted L'-frame. Since L' is stable, all stable images of $\mathfrak{F}$ are L'-frames, and hence also L-frames. Therefore, $\mathfrak{F} \in K$. Since L' is stable, L' is the logic of its finite rooted frames. Thus, $\mathsf{Log}(K) \subseteq L'$, and so $\mathsf{Log}(K)$ is the least stable extension of L. ☐

For a syntactic characterization of $\mathrm{Stab}_\downarrow(L)$ and $\mathrm{Stab}_\uparrow(L)$, we briefly recall the definition and main properties of the stable canonical formulas of [5]. Let $\mathfrak{F}$ be a finite rooted frame and $\mathfrak{D}$ be a family of upsets of $\mathfrak{F}$, called *closed domains*. Suppose $\mathfrak{G}$ is an Esakia frame. We say that a stable map f from $\mathfrak{G}$ onto $\mathfrak{F}$ satisfies the *closed domain condition* (CDC) for $\mathfrak{D}$ provided

$$\uparrow f(x) \cap d \neq \emptyset \ \Rightarrow \ f[\uparrow x] \cap d \neq \emptyset \text{ for all } d \in \mathfrak{D}.$$

As was shown in [5], each such pair $(\mathfrak{F}, \mathfrak{D})$ gives rise to the *stable canonical formula* $\gamma(\mathfrak{F}, \mathfrak{D})$ such that for each Esakia frame $\mathfrak{G}$,

$$\mathfrak{G} \not\models \gamma(\mathfrak{F}, \mathfrak{D}) \quad \text{iff} \quad \begin{array}{l}\text{there are a point-generated subframe } \mathfrak{H} \text{ of } \mathfrak{G} \text{ and a}\\ \text{stable onto map } f : \mathfrak{H} \to \mathfrak{F} \text{ satisfying CDC for } \mathfrak{D}.\end{array}$$

Moreover, stable canonical formulas axiomatize all si-logics. Stable canonical formulas of the form $\gamma(\mathfrak{F}, \varnothing)$ are called *stable formulas*, and are denoted $\gamma(\mathfrak{F})$. As follows from the above,

$$\mathfrak{G} \not\models \gamma(\mathfrak{F}) \quad \text{iff} \quad \begin{array}{l}\text{there are a point-generated subframe } \mathfrak{H} \text{ of } \mathfrak{G} \text{ and a}\\ \text{stable onto map } f : \mathfrak{H} \to \mathfrak{F}.\end{array}$$

By [5, Thm. 6.11], a si-logic L is stable iff L is axiomatizable by stable formulas.

Proposition 5.8. *Let $L = \mathbf{IPC} + \{\gamma(\mathfrak{F}_i, \mathfrak{D}_i) \mid i \in I\}$ be a si-logic.*

(i) $\mathrm{Stab}_\downarrow(L) = \mathbf{IPC} + \{\gamma(\mathfrak{F}) \mid L \vdash \gamma(\mathfrak{F})\}$.
(ii) $\mathrm{Stab}_\uparrow(L) = \mathbf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$.

Proof. (i). By [5, Thm. 6.11], $\mathbf{IPC} + \{\gamma(\mathfrak{F}) \mid L \vdash \gamma(\mathfrak{F})\}$ is the largest stable logic contained in L. Therefore, $\mathrm{Stab}_\downarrow(L) = \mathbf{IPC} + \{\gamma(\mathfrak{F}) \mid L \vdash \gamma(\mathfrak{F})\}$.

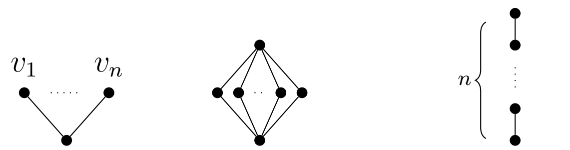
(ii). Let $M = \mathbf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, and let $\mathfrak{G}$ be a rooted M-frame. Then $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$. Thus, $\mathfrak{G} \models \gamma(\mathfrak{F}_i, \mathfrak{D}_i)$ for all $i \in I$ as can easily be seen by the semantic description of the formulas. Therefore, $\mathfrak{G}$ is an L-frame, and so $L \subseteq M$. Since M is axiomatized by stable formulas, M is a stable logic. Suppose L' is a stable extension of L, and $\mathfrak{G}$ is a rooted L'-frame. If $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$ for some $i \in I$, then $\mathfrak{F}_i$ is a stable image of some point-generated subframe $\mathfrak{H}$ of $\mathfrak{G}$. Therefore, $\mathfrak{F}_i$ is an L'-frame. But $\mathfrak{F}_i$ is not an L-frame, which contradicts to L' being an extension of L. Thus, $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$, and so $M \subseteq L'$. Consequently, M is the least stable extension of L, and hence $\mathrm{Stab}_\uparrow(L) = M$. □

Remark 5.9. If a si-logic L is axiomatized by a set of formulas Γ, then $\mathrm{Stab}_\uparrow(L)$ can be calculated effectively as follows: First use [5, Thm. 3.7] to transform Γ into an equivalent set of stable canonical formulas; then delete the additional parameters $\mathfrak{D}_i$ in the resulting canonical formulas; and finally apply Proposition 5.8(ii). We will come back to this issue at the end of Section 6.

Remark 5.10. By restricting Wolter's describable operations (cf. Remark 3.8) to the class of rooted Esakia frames, we can obtain an alternative proof of Theorem 5.8. For a rooted Esakia frame $\mathfrak{G}$, let $\mathbf{C}(\mathfrak{G}) = \{\mathfrak{H} \mid \mathfrak{H} \text{ is a stable image of } \mathfrak{G}\}$. We show that

$$\mathfrak{G} \models \gamma(\mathfrak{F}) \text{ iff } \mathbf{C}(\mathfrak{G}) \models \gamma(\mathfrak{F}, \mathfrak{D}). \tag{9}$$

The left to right direction is obvious. For the right to left direction, suppose $\mathfrak{G} \not\models \gamma(\mathfrak{F})$. Since $\mathfrak{G}$ is rooted, it follows from [8, Prop. 5.1] that $\mathfrak{F}$ is a stable image of $\mathfrak{G}$. Therefore, $\mathfrak{F} \in \mathbf{C}(\mathfrak{G})$. Thus, since $\mathfrak{F} \not\models \gamma(\mathfrak{F}, \mathfrak{D})$, we conclude that $\mathbf{C}(\mathfrak{G}) \not\models \gamma(\mathfrak{F}, \mathfrak{D})$. Set $(\gamma(\mathfrak{F}, \mathfrak{D}))^c = \gamma(\mathfrak{F})$. Because every logic is characterized by its rooted Esakia frames, Wolter's result applied to Proposition 5.7 yields an alternative proof of Theorem 5.8.



$\mathfrak{F}_n$ n-fork $\mathfrak{G}_n$ n-fork with top $\mathfrak{C}_n$ n-chain

FIGURE 3

We conclude this section by giving several examples of upward and downward stabilizations of si-logics. In addition to the si-logics from Section 3, we consider the following si-logics.

- The logics $\mathbf{BD}_n$ of finite rooted frames of depth $\leq n$.
- The logics $\mathbf{BC}_n$ of finite rooted frames of cardinality $\leq n$.

Proposition 5.11.
(i) $\mathrm{Stab}_{\downarrow}(\mathbf{BD}_n) = \mathbf{IPC}$ *and* $\mathrm{Stab}_{\uparrow}(\mathbf{BD}_n) = \mathbf{BC}_n$ *for all* $n \geq 2$.
(ii) *If L is consistent and has the disjunction property, then* $\mathsf{Stab}_{\downarrow}(L) = \mathbf{IPC}$.
(iii) $\mathrm{Stab}_{\downarrow}(\mathbf{T}_n) = \mathbf{IPC}$ *and* $\mathrm{Stab}_{\uparrow}(\mathbf{T}_n) = \mathbf{BW}_n$ *for all* $n \geq 2$.

Proof. (i). First we show that $\mathrm{Stab}_{\downarrow}(\mathbf{BD}_n) = \mathbf{IPC}$ for all $n \geq 2$. Since $\mathbf{BD}_n \subseteq \mathbf{BD}_2$ for all $n \geq 2$, it suffices to show that $\mathrm{Stab}_{\downarrow}(\mathbf{BD}_2) = \mathbf{IPC}$. Let $\mathfrak{F}$ be a finite rooted frame. Suppose $\mathfrak{F}$ has at most $n+1$ elements, and $\mathfrak{F}_n$ is the n-fork shown in Figure 3. Mapping the root of $\mathfrak{F}_n$ to the root of $\mathfrak{F}$ and the top nodes of $\mathfrak{F}_n$ surjectively onto the other nodes of $\mathfrak{F}$ defines a stable map from $\mathfrak{F}_n$ onto $\mathfrak{F}$. Since $\mathfrak{F}_n$ is a $\mathbf{BD}_2$-frame, by Proposition 5.7(i), $\mathfrak{F} \models \mathrm{Stab}_{\downarrow}(\mathbf{BD}_2)$ for every finite rooted frame $\mathfrak{F}$. Thus, $\mathrm{Stab}_{\downarrow}(\mathbf{BD}_2) = \mathbf{IPC}$.

Next we show that $\mathrm{Stab}_{\uparrow}(\mathbf{BD}_n) = \mathbf{BC}_n$ for all $n \geq 2$. Suppose $\mathfrak{F}$ is a finite rooted frame. If $\mathfrak{F}$ has no more than n elements, then every stable image of $\mathfrak{F}$ also has no more than n elements. Therefore, every stable image of $\mathfrak{F}$ is a $\mathbf{BC}_n$-frame. On the other hand, if $\mathfrak{F}$ has at least $n+1$ elements, then we can define a stable map from $\mathfrak{F}$ on the $(n+1)$-chain $\mathfrak{C}_{n+1}$ (see Figure 3) as follows: Map the root r of $\mathfrak{F}$ to the root of $\mathfrak{C}_{n+1}$; map the immediate successors of r on top of each other; continue this process with the immediate successors of the immediate successors of r, and so on; if you run out of points in $\mathfrak{C}_{n+1}$, then map the remaining points to the top node of $\mathfrak{C}_{n+1}$. Since $\mathfrak{C}_{n+1}$ is not a $\mathbf{BD}_n$-frame, $\mathfrak{F}$ has a stable image refuting $\mathbf{BD}_n$. Thus, by Proposition 5.7(ii), $\mathrm{Stab}_{\uparrow}(\mathbf{BD}_n) = \mathbf{BC}_n$.

(ii). Suppose L is consistent and has the disjunction property. By [11, Thm. 15.5], if $\mathfrak{F}_1, \mathfrak{F}_2$ are rooted L-frames, then their disjoint union $\mathfrak{F}_1 \sqcup \mathfrak{F}_2$ is a generated subframe of some rooted L-frame. This implies that for every n, there is a rooted L-frame $\mathfrak{F}$ containing at least n maximal points. To see this, since L is consistent, the one-point frame $\mathfrak{F}_1$ is an L-frame. Therefore, $\mathfrak{F}_1 \sqcup \mathfrak{F}_1$ is a generated subframe of some rooted L-frame $\mathfrak{F}_2$. Clearly $\mathfrak{F}_2$ has at least 2 maximal points. By the same argument, $\mathfrak{F}_2 \sqcup \mathfrak{F}_2$ is a generated subframe of some rooted L-frame $\mathfrak{F}_3$ that has at least 4 maximal points. Continuing this process yields a rooted L-frame $\mathfrak{F}$ with at least n maximal points, say $\{x_1, x_2, \ldots x_n\}$. We show that the n-fork $\mathfrak{F}_n$ is a stable image of $\mathfrak{F}$. Separate $x_1, \ldots, x_n$ by disjoint clopen upsets $U_1, \ldots, U_n$ with $x_i \in U_i$ for $1 \leq i \leq n$, and define a map $f : \mathfrak{F} \to \mathfrak{F}_n$ by

$$f(x) = \begin{cases} v_i & \text{if } x \in U_i \text{ for some } i \in I, \\ r & \text{otherwise,} \end{cases}$$

where r is the root of $\mathfrak{F}_n$. It is straightforward to see that f is an onto stable map. Thus, $\text{Stab}_\downarrow(L) \subseteq \mathbf{BD}_2$. Now apply (i) to conclude that $\text{Stab}_\downarrow(L) = \mathbf{IPC}$.

(iii). Since $\mathbf{T}_n$ is consistent and has the disjunction property for all $n \geq 2$, by (ii), $\text{Stab}_\downarrow(\mathbf{T}_n) = \mathbf{IPC}$ for all $n \geq 2$.

Next we show that $\text{Stab}_\uparrow(\mathbf{T}_n) = \mathbf{BW}_n$ for all $n \geq 2$. Let $K = \{\mathfrak{F} \mid \mathfrak{F}$ is finite rooted and $\mathfrak{G} \models \mathbf{T}_n$ for every stable image $\mathfrak{G}$ of $\mathfrak{F}\}$. By Proposition 5.7(ii), $\text{Stab}_\uparrow(\mathbf{T}_n) = \mathsf{Log}(K)$. Let K' be the class of finite rooted frames of width $\leq n$. We show that $K = K'$. Let $\mathfrak{F}$ be finite and rooted. If $\mathfrak{F}$ is of width $\leq n$, then so are all its stable images (see [5, Thm. 7.3(2)]). Therefore, $K' \subseteq K$. Conversely, if $\mathfrak{F}$ has width greater than n, then by [5, Thm. 7.5(3)], either the $(n+1)$-fork or the $(n+1)$-fork with top (see Figure 3) is a stable image of $\mathfrak{F}$. Since neither of these is a $\mathbf{T}_n$-frame, $\mathfrak{F} \notin K$. Thus, $K = K'$, and as $\mathbf{BW}_n$ is the logic of K', we conclude that $\text{Stab}_\uparrow(\mathbf{T}_n) = \mathbf{BW}_n$. □

6. Stable logics and intuitionistic S4

As we saw in Section 4, there is a close connection between subframe logics and the propositional lax logic **PLL**. In this section we show that there is a close connection between stable logics and intuitionistic **S4** [20]. Intuitionistic **S4** is the least set of formulas of the propositional modal language containing **IPC**, the axioms $\Box p \to p$, $\Box p \to \Box\Box p$, $\Box(p \to q) \to (\Box p \to \Box q)$, and closed under substitution, modus ponens, and necessitation. We denote it by **IS4**.

As was observed in [20], the algebraic semantics of **IS4** is provided by *interior Heyting algebras*, which are pairs $(A, \Box)$, where A is a Heyting algebra and $\Box$ is an *interior operator* on A; that is, $\Box$ is a unary function on A satisfying $\Box a \leq a$, $\Box a \leq \Box\Box a$, $\Box(a \wedge b) = \Box a \wedge \Box b$, and $\Box 1 = 1$. The fixpoints $A_\Box := \{a \in A \mid \Box a = a\}$ form a bounded sublattice of A, which is also a Heyting algebra, where $a \to_\Box b = \Box(a \to b)$. In fact, interior Heyting algebras correspond to pairs (A, A_0) of Heyting algebras such that A_0 is a bounded sublattice of A and the embedding $A_0 \rightarrowtail A$ has a right adjoint (cf. [2, Sec. 3]).

Given such a pair (A, A_0), let $\mathfrak{F} = (X, \leq)$ be the Esakia frame of A and $\mathfrak{G} = (Y, \leq)$ be the Esakia frame of A_0. Since the embedding $A_0 \rightarrowtail A$ is a bounded lattice morphism, the dual map $\pi : X \to Y$ is an onto stable map. Moreover, the right adjoint $\Box : A \to A_0$ of the embedding $A_0 \rightarrowtail A$ is dually described as follows: if U is a clopen upset of $\mathfrak{F}$, then $\Box U = Y \setminus {\downarrow}\pi(X \setminus U)$. Therefore, for each clopen U in X, we have that ${\downarrow}\pi(U)$ is a clopen subset of Y. Thus, interior Heyting algebras correspond to pairs of Esakia frames $(\mathfrak{F}, \mathfrak{G})$ and an onto stable map between them satisfying ${\downarrow}\pi(U)$ is clopen in $\mathfrak{G}$ for each clopen U in $\mathfrak{F}$ (cf. [3]). This yields the following definition.

Definition 6.1. An *St-frame* (stable frame) is a pair $(\mathfrak{F}, \mathfrak{G})$ such that $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (Y, \leq)$ are Esakia frames and $\pi : X \to Y$ is an onto stable map satisfying ${\downarrow}\pi(U)$ is clopen in Y for each clopen U in X.

The correspondence between interior Heyting algebras and St-frames allows us to interpret formulas of **IS4** in St-frames. Let $(\mathfrak{F}, \mathfrak{G})$ be an St-frame, where $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (Y, \leq)$. We interpret propositional letters as clopen upsets of $\mathfrak{F}$ and intuitionistic connectives as the corresponding operations in the Heyting algebra of clopen upsets of $\mathfrak{F}$. In addition, $\Box$ is interpreted as the corresponding unary function on the clopen upsets of $\mathfrak{F}$; that is, $\Box U = \pi^{-1}(Y \setminus {\downarrow}\pi(X \setminus U))$. Therefore, if v is a valuation on $(\mathfrak{F}, \mathfrak{G})$ and $x \in X$, then $x \notin \Box v(\varphi)$ iff $\pi(x) \in {\downarrow}\pi(X \setminus v(\varphi))$, which happens iff there is $z \in X \setminus v(\varphi)$ with $\pi(x) \leq \pi(z)$.

Thus,

$$x \models_v \Box\varphi \text{ iff } z \models_v \varphi \text{ for all } z \in X \text{ with } \pi(x) \leq \pi(z).$$

We utilize the Gödel-McKinsey-Tarski translation to translate a formula φ of **IPC** into the formula $t(\varphi)$ of **IS4** as follows:

- $t(p) = \Box p$ for a propositional letter p,
- $t(\bot) = \Box\bot$,
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$,
- $t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$,
- $t(\varphi \to \psi) = \Box(t(\varphi) \to t(\psi))$.

It is clear that for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ and every interior Heyting algebra $(A, \Box)$, we have:

(i) $(A, \Box) \models \varphi$ iff $A \models \varphi$.
(ii) $(A, \Box) \models t(\varphi)$ iff $A_\Box \models \varphi$.

In dual terms we have:

Lemma 6.2. *For every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ and every St-frame $(\mathfrak{F}, \mathfrak{G})$,*

(i) $(\mathfrak{F}, \mathfrak{G}) \models \varphi$ *iff* $\mathfrak{F} \models \varphi$.
(ii) $(\mathfrak{F}, \mathfrak{G}) \models t(\varphi)$ *iff* $\mathfrak{G} \models \varphi$.

As we saw in the previous section, a si-logic L is stable iff rooted L-frames are closed under stable images. It is known (see, e.g., [7, Thm. 8.6]) that rooted frames are characterized by the multi-conclusion disjunction rule $p \vee q/p, q$. Therefore, instead of working with logics above **IS4**, we will work with multi-conclusion consequence relations above **IS4**.

We recall (see, e.g., [18, 17, 8]) that a *multi-conclusion rule* is an expression of the form Γ/Δ, where Γ and Δ are finite sets of formulas. A *multi-conclusion consequence relation over* **IS4** is a set $\mathcal{S}$ of multi-conclusion rules such that

- $\varphi/\varphi \in \mathcal{S}$.
- $\varphi, \varphi \to \psi/\psi \in \mathcal{S}$.
- $\varphi/\Box\varphi \in \mathcal{S}$.
- $/\varphi \in \mathcal{S}$ for each theorem φ of **IS4**.
- If $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$.
- If $\Gamma/\Delta, \varphi \in \mathcal{S}$ and $\Gamma, \varphi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$.
- If $\Gamma/\Delta \in \mathcal{S}$ and s is a substitution, then $s(\Gamma)/s(\Delta) \in \mathcal{S}$.

Let $\mathcal{S}_{\mathbf{IS4}}$ be the multi-conclusion consequence relation over **IS4** that in addition contains the disjunction rule $p \vee q/p, q$. Let also $\Sigma(\mathcal{S}_{\mathbf{IS4}})$ be the complete lattice of all multi-conclusion consequence relations extending $\mathcal{S}_{\mathbf{IS4}}$.

A multi-conclusion rule Γ/Δ is *valid* on an interior Heyting algebra $(A, \Box)$ if for every valuation v on A, from $v(\gamma) = 1$ for every $\gamma \in \Gamma$ it follows that $v(\delta) = 1$ for some $\delta \in \Delta$. The validity of Γ/Δ on an St-frame $(\mathfrak{F}, \mathfrak{G})$ is defined similarly.

Consequence relations in $\Sigma(\mathcal{S}_{\mathbf{IS4}})$ correspond to universal classes of interior Heyting algebras whose underlying Heyting algebras are *well-connected* ($a \vee b = 1$ implies $a = 1$ or $b = 1$). Dually they are characterized by classes of St-frames $(\mathfrak{F}, \mathfrak{G})$ such that $\mathfrak{F}$ is rooted. We call such St-frames *rooted*. For a class $\mathcal{K}$ of rooted St-frames, let $\mathsf{Con}(\mathcal{K})$ be the set of multi-conclusion rules that are valid in $\mathcal{K}$. Then $\mathsf{Con}(\mathcal{K}) \in \Sigma(\mathcal{S}_{\mathbf{IS4}})$.

Notation 6.3. From now on all St-frames are assumed to be rooted.

Definition 6.4. Let $L \in \Lambda(\mathbf{IPC})$ and $\mathcal{S} \in \Sigma(\mathcal{S}_{\mathbf{IS4}})$.

(i) We say that L is the *intuitionistic fragment* of $\mathcal{S}$ if for all formulas $\varphi \in \mathcal{L}_{\mathbf{IPC}}$,

$$\varphi \in L \text{ iff } /\varphi \in \mathcal{S}.$$

(ii) We say that L is the *stable fragment* of $\mathcal{S}$ if for all formulas $\varphi \in \mathcal{L}_{\mathbf{IPC}}$,

$$\varphi \in L \text{ iff } /t(\varphi) \in \mathcal{S}.$$

For $\mathcal{S} \in \Sigma(\mathcal{S}_{\mathbf{IS4}})$, we define

$$\zeta_1(\mathcal{S}) = \{\varphi \in \mathcal{L}_{\mathbf{IPC}} \mid /\varphi \in \mathcal{S}\},$$
$$\zeta_2(\mathcal{S}) = \{\varphi \in \mathcal{L}_{\mathbf{IPC}} \mid /t(\varphi) \in \mathcal{S}\}.$$

Lemma 6.5. *Let $\mathcal{S} \in \Sigma(\mathcal{S}_{\mathbf{IS4}})$.*

(i) *$\zeta_1(\mathcal{S})$ is the intuitionistic fragment of $\mathcal{S}$ and*

$$\zeta_1(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{F} \mid \exists \mathfrak{G} : (\mathfrak{F}, \mathfrak{G}) \textit{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right).$$

(ii) *$\zeta_2(\mathcal{S})$ is the stable fragment of $\mathcal{S}$ and*

$$\zeta_2(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \textit{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right).$$

Proof. (i). For $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, we have

$$\begin{aligned}
& \varphi \in \mathsf{Log}\left(\{\mathfrak{F} \mid \exists \mathfrak{G} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right) \\
\Leftrightarrow \quad & \mathfrak{F} \models \varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\
\Leftrightarrow \quad & \mathfrak{F} \models /\varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\
\Leftrightarrow \quad & (\mathfrak{F}, \mathfrak{G}) \models /\varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\
\Leftrightarrow \quad & /\varphi \in \mathcal{S} \\
\Leftrightarrow \quad & \varphi \in \zeta_1(\mathcal{S}).
\end{aligned}$$

Therefore, $\zeta_1(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{F} \mid \exists \mathfrak{G} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right)$. Thus, $\zeta_1(\mathcal{S})$ is a si-logic, and so it is the intuitionistic fragment of $\mathcal{S}$.

(ii). For $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, we have

$$\begin{aligned}
& \varphi \in \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right) \\
\Leftrightarrow \quad & \mathfrak{G} \models \varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\
\Leftrightarrow \quad & \mathfrak{G} \models /\varphi \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\
\Leftrightarrow \quad & (\mathfrak{F}, \mathfrak{G}) \models /t(\varphi) \text{ for all } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S} \\
\Leftrightarrow \quad & /t(\varphi) \in \mathcal{S} \\
\Leftrightarrow \quad & \varphi \in \zeta_2(\mathcal{S}).
\end{aligned}$$

Therefore, $\zeta_2(\mathcal{S}) = \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } (\mathfrak{F}, \mathfrak{G}) \models \mathcal{S}\}\right)$. Thus, $\zeta_1(\mathcal{S})$ is a si-logic, and so it is the stable fragment of $\mathcal{S}$. $\square$

Conversely, for a si-logic L, define:

$$\eta_1(L) = \mathcal{S}_{\mathbf{IS4}} + \{/\varphi \mid \varphi \in L\},$$
$$\eta_2(L) = \mathcal{S}_{\mathbf{IS4}} + \{/t(\varphi) \mid \varphi \in L\}.$$

Lemma 6.6. *For every si-logic L, we have:*

(i) $\eta_1(L) = \mathsf{Con}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{F} \textit{ is an } L\textit{-frame}\}\right)$,

(ii) $\eta_2(L) = \mathsf{Con}\left(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G} \textit{ is an } L\textit{-frame})\}\right.$.

Proof. We prove (ii), the proof of (i) is similar. For an St-frame $(\mathfrak{F}, \mathfrak{G})$ we have $\mathfrak{G} \models L$ iff $(\mathfrak{F}, \mathfrak{G}) \models \{t(\varphi) \mid \varphi \in L\}$, which happens iff $(\mathfrak{F}, \mathfrak{G}) \models \{/t(\varphi) \mid \varphi \in L\}$. Thus, $\eta_2(L) = \mathsf{Con}\,(\{(\mathfrak{F}, \mathfrak{G}) \mid \mathfrak{G}$ is an L-frame$\})$. □

Lemma 6.7. *Let L be a si-logic.*

(i) *$L = \zeta_1\eta_1(L)$, and $\eta_1(L)$ is the least multi-conclusion consequence relation in $\zeta_1^{-1}(L)$.*
(ii) *$L = \zeta_2\eta_2(L)$, and $\eta_2(L)$ is the least multi-conclusion consequence relation in $\zeta_2^{-1}(L)$.*

Proof. (i). Let $\varphi \in \mathcal{L}_{\mathbf{IPC}}$. Then $\varphi \in L$ implies $/\varphi \in \eta_1(L)$, which implies $\varphi \in \zeta_1\eta_1(L)$. Therefore, $L \subseteq \rho_1\eta_1(L)$. If $\varphi \notin L$, then there is a rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Consider the St-frame $(\mathfrak{F}, \mathfrak{F})$, where π is the identity map. Then $(\mathfrak{F}, \mathfrak{F}) \not\models /\varphi$, and $(\mathfrak{F}, \mathfrak{F}) \models \eta_1(L)$ by Lemma 6.6(i). Therefore, by Lemma 6.5(i), $\varphi \notin \zeta_1\eta_1(L)$. This shows that $L = \zeta_1\eta_1(L)$. If $\mathcal{S} \in \zeta_1^{-1}(L)$, then for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, we have $\varphi \in L$ iff $/\varphi \in \mathcal{S}$. Thus, $\eta_1(L) \subseteq \mathcal{S}$, and hence $\eta_1(L)$ is the least element of $\zeta_1^{-1}(L)$.

(ii). Let $\varphi \in \mathcal{L}_{\mathbf{IPC}}$. Then $\varphi \in L$ implies $/t(\varphi) \in \eta_2(L)$, which implies $\varphi \in \zeta_2\eta_2(L)$. Therefore, $L \subseteq \zeta_2\eta_2(L)$. If $\varphi \notin L$, then there is a rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Then $(\mathfrak{F}, \mathfrak{F}) \not\models /t(\varphi)$, and $(\mathfrak{F}, \mathfrak{F})$ is a $\eta_2(L)$-frame by Lemma 6.6(ii). Thus, by Lemma 6.5(ii), $\varphi \notin \zeta_2\eta_2(L)$. This shows that $L = \zeta_2\eta_2(L)$. If $\mathcal{S} \in \zeta_2^{-1}(L)$, then for every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, we have $\varphi \in L$ iff $/t(\varphi) \in \mathcal{S}$. Consequently, $\eta_2(L) \subseteq \mathcal{S}$, and hence $\eta_2(L)$ is the least element of $\zeta_2^{-1}(L)$. □

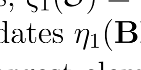
As follows from Lemma 6.7, for a si-logic L, both $\zeta_1^{-1}(L)$ and $\zeta_2^{-1}(L)$ have least elements, but they may not have largest elements. To see this we require the following lemma.

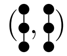
Lemma 6.8. *Let $(\mathfrak{F}, \mathfrak{G})$ be an St-frame. Then $(\mathfrak{F}, \mathfrak{G}) \models /\Box p \leftrightarrow p$ iff π is an isomorphism.*

Proof. Let $\mathfrak{F} = (X, \leq)$ and $\mathfrak{G} = (Y, \leq)$. First suppose that π is an isomorphism. Then it is clear that $(\mathfrak{F}, \mathfrak{G}) \models \Box p \leftrightarrow p$. Next suppose that π is not an isomorphism. Then there are $x \not\leq y$ with $\pi(x) \leq \pi(y)$. Let U be a clopen upset of $\mathfrak{F}$, with $x \in U$ but $y \notin U$. Define a valuation v on $(\mathfrak{F}, \mathfrak{G})$ with $v(p) = U$. Then $x \models_v p$ but $x \not\models_v \Box p$. Thus, $(\mathfrak{F}, \mathfrak{G}) \not\models \Box p \leftrightarrow p$. □

For $\psi \in \mathcal{L}_{\mathbf{IS4}}$, let ψ^- be the formula obtained from ψ by deleting all occurrences of $\Box$. Similarly to Lemmas 4.14 and 4.15, we can show that for every $\mathcal{S} \in \Sigma(\mathcal{S}_{\mathbf{IS4}})$, if $/\Box p \leftrightarrow p \in \mathcal{S}$, then $/\psi \in \mathcal{S}$ iff $/\psi^- \in \mathcal{S}$. From this we can infer that $\eta_1(L) + /\Box p \leftrightarrow p$ is maximal in both $\zeta_1^{-1}(L)$ and $\zeta_2^{-1}(L)$. On the other hand, neither of $\zeta_1^{-1}(L)$ and $\zeta_2^{-1}(L)$ has to have a largest element, as the next example shows.

Example 6.9. Let γ abbreviate $(p \to q) \vee (q \to p)$ and let $\mathcal{S} = \eta_1(\mathbf{BD}_2) + /t(\gamma)$. By Lemma 6.2, an St-frame $(\mathfrak{F}, \mathfrak{G})$ is an $\mathcal{S}$-frame iff $\mathfrak{F}$ is a $\mathbf{BD}_2$-frame and $\mathfrak{G}$ is an $\mathbf{LC}$-frame.

(i) We show that $\zeta_1(\mathcal{S}) = \mathbf{BD}_2$. By Lemma 6.5(i), $\mathbf{BD}_2 \subseteq \zeta_1(\mathcal{S})$. Conversely, suppose $\varphi \notin \mathbf{BD}_2$. Then there is a finite rooted $\mathbf{BD}_2$-frame $\mathfrak{F}$ refuting φ. Let $n = |\mathfrak{F}|$ and let $\mathfrak{G}$ be the n-chain. As we saw in the proof of Proposition 5.11(i), $\mathfrak{G}$ is a stable image of $\mathfrak{F}$. Therefore, $(\mathfrak{F}, \mathfrak{G})$ is an $\mathcal{S}$-frame refuting φ. Thus, $\zeta_1(\mathcal{S}) = \mathbf{BD}_2$. On the other hand, $\mathcal{S} \not\subseteq \eta_1(\mathbf{BD}_2) + /\Box p \leftrightarrow p$ because  validates $\eta_1(\mathbf{BD}_2) + /\Box p \leftrightarrow p$ but refutes $\mathcal{S}$. Consequently, $\zeta_1^{-1}(\mathbf{BD}_2)$ does not have a largest element.
(ii) We show that $\zeta_2(\mathcal{S}) = \mathbf{LC}$. By Lemma 6.5(ii), $\mathbf{LC} \subseteq \zeta_2(\mathcal{S})$. Conversely, suppose $\varphi \notin \mathbf{LC}$. Then there is a finite chain $\mathfrak{G}$ refuting φ. Let $n = |\mathfrak{G}|$. As follows from the proof of Proposition 5.11(i), $\mathfrak{G}$ is a stable image of the $(n-1)$-fork $\mathfrak{F}$. Therefore, $(\mathfrak{F}, \mathfrak{G})$ is an $\mathcal{S}$-frame and $(\mathfrak{F}, \mathfrak{G}) \not\models t(\varphi)$. Thus, $\varphi \notin \zeta_2(\mathcal{S})$. On the other hand,

$\mathcal{S} \not\subseteq \eta_1(\mathbf{LC}) + \Box p \leftrightarrow p$ because  satisfies $\eta_1(\mathbf{LC}) + /\Box p \leftrightarrow p$ but refutes $\mathcal{S}$. Consequently, $\zeta_2^{-1}(\mathbf{LC})$ does not have a largest element.

We will use the above correspondence between si-logics and $\Sigma(\mathcal{S}_{\mathbf{IS4}})$ to provide another characterization of stable logics.

Theorem 6.10. *For a si-logic L, the following are equivalent.*

(i) *L is a stable logic.*
(ii) *$\eta_2(L) \subseteq \eta_1(L)$.*
(iii) *$\eta_2(L) + \{/\varphi \mid \varphi \in L\} = \eta_1(L)$.*
(iv) *$\zeta_2\eta_1(L) = L$.*
(v) *For every $\varphi \in \mathcal{L}_{\mathbf{IPC}}$, from $/\varphi \in \eta_1(L)$ it follows that $/t(\varphi) \in \eta_1(L)$.*

Proof. (i)⇒(ii). Suppose that $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)$. By Lemma 6.6(i), $\mathfrak{F} \models L$. Since L is a stable logic, $\mathfrak{G} \models L$. Therefore, by Lemma 6.6(ii), $(\mathfrak{F}, \mathfrak{G}) \models \eta_2(L)$. Thus, $\eta_2(L) \subseteq \eta_1(L)$.

(ii)⇒(iii). This is obvious.

(iii)⇒(iv): By Lemmas 6.7(ii) and 6.5(ii), $L = \zeta_2\eta_2(L) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_2(L)\})$ and $\zeta_2\eta_1(L) = \mathsf{Log}(\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)\})$. Therefore, it is sufficient to show that $\{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_2(L)\} = \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)\}$. The inclusion $\supseteq$ is immediate from (iii). For the reverse inclusion, suppose that $(\mathfrak{F}, \mathfrak{G}) \models \eta_2(L)$. By Lemma 6.6(ii), $\mathfrak{G} \models L$, so $(\mathfrak{G}, \mathfrak{G}) \models \eta_1(L)$ by Lemma 6.6(i). Thus, $\mathfrak{G} \in \{\mathfrak{G} \mid (\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)\}$.

(iv)⇒(v). Suppose that there is $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ such that $/\varphi \in \eta_1(L)$ but $/t(\varphi) \notin \eta_1(L)$. Then there is an St-frame $(\mathfrak{F}, \mathfrak{G})$ with $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)$ and $(\mathfrak{F}, \mathfrak{G}) \not\models t(\varphi)$. By Lemma 6.5(ii), $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)$ implies $\mathfrak{G} \models \zeta_2\eta_1(L) = L$. Also, $(\mathfrak{F}, \mathfrak{G}) \not\models t(\varphi)$ implies $\mathfrak{G} \not\models \varphi$. Therefore, $\varphi \notin L$, contradicting $/\varphi \in \eta_1(L)$.

(v)⇒(i). Suppose that $\mathfrak{F}$ is a rooted L-frame and $\mathfrak{G}$ is a stable image of $\mathfrak{F}$. Then $(\mathfrak{F}, \mathfrak{G})$ is an St-frame, and by Lemma 6.6(i), $(\mathfrak{F}, \mathfrak{G}) \models \eta_1(L)$. By (v), $(\mathfrak{F}, \mathfrak{G}) \models t(\varphi)$ for each $\varphi \in \mathcal{L}_{\mathbf{IPC}}$ such that $/\varphi \in \eta_1(L)$. Therefore, $(\mathfrak{F}, \mathfrak{G}) \models t(\varphi)$ for each $\varphi \in L$. Thus, $\mathfrak{G} \models L$, and we conclude that L is a stable logic. □

Theorem 6.11. *Let L be a si-logic. Then $\mathsf{Stab}_\downarrow(L) = \zeta_2\eta_1(L)$.*

Proof. By Lemma 6.5(ii),

$$\zeta_2\eta_1(L) = \mathsf{Log}\left(\{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } \mathfrak{F} \models L\}\right).$$

Let

$$\begin{aligned} K &= \{\mathfrak{G} \mid \exists \mathfrak{F} : (\mathfrak{F}, \mathfrak{G}) \text{ is an St-frame and } \mathfrak{F} \models L\}, \\ K' &= \{\mathfrak{G} \mid \mathfrak{G} \text{ is a stable image of a rooted } L\text{-frame } \mathfrak{F}\}. \end{aligned}$$

By Proposition 5.7(i), $\mathsf{Stab}_\downarrow(L) = \mathsf{Log}(K')$. Clearly $K \subseteq K'$, so $\mathsf{Stab}_\downarrow(L) = \mathsf{Log}(K') \subseteq \mathsf{Log}(K) = \zeta_2\eta_1(L)$. Suppose that $\varphi \notin \mathsf{Stab}_\downarrow(L)$. Then there is $\mathfrak{G} \in K'$ refuting φ. Therefore, there is an L-frame $\mathfrak{F}$ such that $\mathfrak{G}$ is a stable image $\mathfrak{F}$. Applying [5, Lem. 3.6] yields a finite stable image $\mathfrak{G}'$ of $\mathfrak{G}$ refuting φ. Since $\mathfrak{G}'$ is finite, $(\mathfrak{F}, \mathfrak{G}')$ is an St-frame (because the topological condition of Definition 6.1 trivializes), so $\mathfrak{G}' \in K$. Thus, $\varphi \notin \zeta_2\eta_1(L)$. □

Remark 6.12.

(i) Let L be a si-logic and $\varphi \in \mathcal{L}_{\mathbf{IPC}}$. By Theorem 6.11, $\varphi \in \mathsf{Stab}_\downarrow(L)$ iff $t(\varphi) \in \mathcal{S}_{\mathbf{IS4}} + \{/\varphi \mid \varphi \in L\}$. In particular, if $\mathcal{S}_{\mathbf{IS4}} + \{/\varphi \mid \varphi \in L\}$ is decidable, then so is $\mathsf{Stab}_\downarrow(L)$.

(ii) In contrast to Theorem 6.11, if L is consistent, then $\zeta_1\eta_2(L) = \mathbf{IPC}$. Indeed, suppose $\mathfrak{F}$ is a nonempty Esakia frame. Let $\mathfrak{G}$ be the one-point frame. Then $(\mathfrak{F}, \mathfrak{G})$ is an St-frame. Since L is consistent, $\mathfrak{G}$ is an L-frame, so $(\mathfrak{F}, \mathfrak{G}) \models \eta_2(L)$ by Lemma 6.6(ii), and hence $\mathfrak{F} \models \zeta_1\eta_2(L)$ by Lemma 6.5(i). Thus, $\zeta_1\eta_2(L) = \mathbf{IPC}$.

Remark 6.13. We recall [8] that a stable map $f : \mathfrak{F} \to \mathfrak{G}$ between Esakia frames is *cofinal stable* provided $\max{\uparrow} f(x) = f(\max{\uparrow} x)$, where $\max U$ is the set of maximal points of U. A si-logic L is *cofinal stable* provided its rooted frames are closed under cofinal stable images (that is, if $\mathfrak{F}$ is an L-frame, then so is every cofinal stable image of $\mathfrak{F}$). It follows from [8] that cofinal stable images of an Esakia frame $\mathfrak{F}$ correspond to pseudocomplemented sublattices (that is, bounded sublattices preserving $\neg$) of the dual Heyting algebra A of $\mathfrak{F}$. Since being a pseudocomplemented sublattice is expressed by adding $\Box\neg\Box p \leftrightarrow \neg\Box p$ to $\mathcal{S}_{\mathbf{IS4}}$, the correspondence between stable logics and the multi-conclusion consequence relations extending $\mathcal{S}_{\mathbf{IS4}}$ discussed in this section extends to the correspondence between cofinal stable logics and the multi-conclusion consequence relations extending $\mathcal{S}_{\mathbf{IS4}} + /\Box\neg\Box p \leftrightarrow \neg\Box p$.

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