Canonical Rules on Neighbourhood Frames

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is a study of logics whose semantics is based on neighbourhood frames. Neighbourhood frames are a generalization of Kripke frames and are generally used as a semantic framework for non-normal modal logics. We study logics with a neighbourhood based semantics by means of canonical rules and formulas. Canonical rules and formulas have been extensively studied in the context of lattices of normal modal logics, in particular for obtaining uniform axiomatizations for these logics.

In this thesis, we develop analoguous methods for lattices of logics with a neighbourhood based semantics. Firstly we define stable canonical rules for neighbourhood frames to axiomatize all classical and monotonic modal logics and multi-conclusion consequence relations. We look at two instances of these rules, namely stable rules and Jankov rules. Modal logics and multi-conclusion consequence relations axiomatized by the former have the finite model property whereas the latter axiomatizes splittings in the lattice $\mathrm{CExt}\mathbf{S_E}$ of all classical modal multi-conclusion consequence relations.

Secondly we look at Instantial Neighbourhood Logic. We define co-stable canonical rules to axiomatize all instantial neighbourhood logics and their corresponding multi-conclusion consequence relations. Moreover, we define co-stable canonical formulas to axiomatize all splittings in the lattice $\mathrm{Ext}\mathbf{INL}$ of instantial neighbourhood logics.

Keywords: Classical modal logics, Instantial Neighbourhood Logic, neighbourhood frames, canonical rules and formulas, splittings

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Contents

Chapter 1

Introduction

In the field of modal logic, the normal modal logics are by far the most-studied. However, there exist many logics below the least normal modal logic **K** that are of interest. Examples of such logics include Deontic Logic as discussed by Chellas [13], Coalition Logic of Pauly [29], Parikh's Game Logic [27], etc. In order to study these logics semantically, we need to extend Kripke semantics. This is achieved by considering neighbourhood frames and models. Whereas in a Kripke frame each world w is associated with a single set of successors, a neighbourhood frame associates w with a collection of sets. Each set in this collection is called a *neighbourhood* of w. Neighbourhood frames and corresponding logics will be the main subject of study in this thesis.

The lattices of normal modal logics have been extensively invsetigated, see e.g. [12]. We are interested in studying the lattices of logics whose semantics is based on neighbourhood frames. We investigate these logics by means of canonical rules and formulas. A canonical formula (rule) is a formula (rule) based on a finite frame or algebra that encodes the structure of this frame or algebra. Extensive study has gone into the importance of these formulas and rules for modal logics. Jankov [21] and de Jongh [14] used them to axiomatize a range of intermediate logics, including all splitting logics. Rautenberg [30] and Fine [17] adapted these results to modal logics. Zakharyaschev extended these methods to axiomatize all intermediate logics and logics above **K4** by canonical formulas [33, 34]. Moreover, Blok [10], building on work by McKenzie [26], employed similar methods to characterize splittings in the lattice of normal modal logics.

More recently, Jeřábek [22] generalized the method of canonical formulas by moving from formulas to rules, proving that any multi-conclusion consequence relation based on **K4** or **IPC** can be axiomatized by these canonical rules. Whereas the technique of canonical formulas and rules had so far depended on the method of selective filtration, Bezhanishvili et al. [5] used (standard) filtration to define stable canonical rules and axiomatize any normal modal multi-conclusion consequence relation and logic by these stable canonical rules.

In Part I of the thesis, we focus our attention on classical and monotonic modal logics, generalizing results from [5]. We prove that any multi-conclusion consequence relation based on a classical modal logic can be axiomatized by so-called stable canonical rules. The same is shown for any multi-conclusion consequence relation based on a monotonic modal logic. We illustrate these results by looking at the class of stable classical modal multi-conclusion consequence relations, showing there are continuously many and proving that they all have the finite model property. We also give a few examples of logics axiomatized by these rules.

In order to make the transition from canonical rules to canonical formulas, we find ourselves limited by the setting of classical and monotonic modal logic. A justification for this lies in coalgebra. We can view the neighbourhood frames for classical and monotonic modal logic as $\check{\mathcal{P}} \circ \check{\mathcal{P}}$- and $\mathcal{U}p\mathcal{P}$-coalgebras respectively. Their functors lack certain preservation properties. We therefore consider a logic recently introduced by van Benthem et al.: Instantial Neighbourhood Logic (INL) [3]. This logic is interpreted on neighbourhood frames that correspond to coalgebras for the double covariant powerset functor $\mathcal{P} \circ \mathcal{P}$, which is much better behaved. The language of INL comes with a new more expressive $n+1$-ary modal operator $\Box(\psi_1, \ldots, \psi_n; \phi)$.

Part II is dedicated to results on logics and consequence relations extending INL. In this setting, we can define canonical formulas. We show that all logics and multi-conclusion consequence relations extending INL can be axiomatized by co-stable canonical rules. Moreover, we prove an analogue of Blok's splitting theorem by using the canonical formulas to give a characterization of all splitting logics in the lattice of logics extending INL.

The main contributions of this thesis are therefore: axiomatization results for each classical and monotonic modal logic and multi-conclusion consequence relation in terms of stable canonical rules, similar axiomatization results for each logic and multi-conclusion consequence relation extending INL in terms of co-stable canonical rules and a splitting theorem for the lattice of all logics extending INL in terms of co-stable canonical formulas.

Part I

Classical Modal Logic

Chapter 2

Syntax and Semantics

2.1 Syntax of Modal Logic

The notation and terminology presented in this chapter is standard in modal logic, see e.g. [9, 13]. In Part I of this thesis, we will be working in the basic modal language $\mathcal{L}_\Box$ with only one unary modal operator $\Box$. We fix a countable set of propositional letters Prop. All well-defined formulas of $\mathcal{L}_\Box$ are defined as follows:

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi \text{ where } p \in \mathsf{Prop}.$$

We use the usual abbreviations for $\top, \wedge, \to$ and $\leftrightarrow$. The modal operator $\Diamond$ will be an abbreviation for $\neg\Box\neg$. We refer to a well-defined formula in the language $\mathcal{L}_\Box$ as a *modal formula*. Let **Form** denote the set of modal formulas. We write $\mathbf{Form}(\Phi)$ when using a specific set of propositional letters Φ. For such a set Φ, we define a *substitution* to be a map $\sigma : \Phi \to \mathbf{Form}(\Phi)$. A substition σ can be extended to a map $(\cdot)^\sigma : \mathbf{Form}(\Phi) \to \mathbf{Form}(\Phi)$ in the usual way. Abusing notation we write $\sigma(\phi)$ for $(\phi)^\sigma$. This allows us to define the logics we will be concerned with in the first part of this thesis.

Definition 2.1 (**Modal Logic**). A *modal logic* is a set of modal formulas Λ containing all propositional tautologies closed under the rules of *modus ponens* $\phi, \phi \to \psi / \psi$ and *uniform substitution* $\phi/\sigma(\phi)$ with σ a substitution. ⊣

For the modal logics we will consider, the following inference rules are of importance.

$$(\mathrm{RE}_\Box) \quad \frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

$$(\mathrm{RM}_\Box) \quad \frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

$$(\mathrm{Nec}) \quad \frac{\phi}{\Box\phi}$$

We say that a modal logic Λ is *classical* whenever it is closed under the rule $\mathrm{RE}_\Box$. The smallest classical modal logic we denote by **E**. We call a logic Λ a *classical extension* of **E** whenever Λ contains **E** and is closed under $\mathrm{RE}_\Box$. A modal logic Λ is called *monotonic* if Λ is closed under the rule $\mathrm{RM}_\Box$. We denote the smallest monotonic modal logic by **M**. Any extension of **M** closed under $\mathrm{RM}_\Box$ we call a *monotonic extension* of **M**. A modal logic Λ is called *normal* if Λ is closed under the rule of Necessitation (Nec) and contains the Kripke axiom $K : \Box(p \to q) \to (\Box p \to \Box q)$.

Example 2.2. As an example of non-normal modal logics, we look at Deontic Logic (DL) as described by Chellas [13]. We let **DL** denote the smallest normal modal logic containing the axiom P: $\neg\Box\bot$. In DL, we interpret $\Box\phi$ as "it ought to be the case that ϕ" or "it is obligatory that ϕ".

The equivalence $\neg\Box\bot \leftrightarrow \neg(\Box\phi) \wedge \Box\neg\phi)$ is provable in **DL**. Whereas the lefthand side expresses that there do not exist impossible obligations, the righthand side says that there exist no conflicting obligations. Some may object to adopting the righthand side as an axiom, but accept the axiom P.

One way to solve this issue is by restricting to a monotonic modal logic. Let **MD** denote the smallest monotonic modal logic containing P (denoted by **D** in [13]). However, an argument known as Ross's Paradox argues why forcing a deontic logic to be closed under the monotonicity rule $\mathrm{RM}_\Box$ might be too strong of a condition. Indeed, consider the following two sentences:

(1) It is obligatory that the letter is mailed.
(2) It is obligatory that the letter is mailed or the letter is burned.

These two sentences can be written as $\Box m$ and $\Box(m \vee b)$ respectively. In **MD**, the second sentence follows from the first. Yet linguistically, this seems strange. We therefore also consider **D**, defined as the least classical modal logic containing P. In **D**, Ross's Paradox no longer occurs.

For a set of modal formulas Σ we define $\mathbf{E}.\Sigma$ to be the least classical modal logic containing all formulas $\phi \in \Sigma$. When Σ is finite, i.e. $\Sigma = \{\sigma_1, \dots, \sigma_n\}$, we sometimes write $\mathbf{E}.\sigma_1 \dots \sigma_n$ or $\mathbf{E} + \sigma_1 + \dots + \sigma_n$. Likewise we let $\mathbf{M}.\Sigma$ denote the least monotonic modal logic containing all formulas from Σ.

The lattice Next**K** of all normal modal logics extending the smallest normal modal logic **K** has been extensively studied, see [12] for an overview. We will study the lattices for classical and monotonic modal logics. The classical extensions of **E** form a lattice where the meet is set-theoretic intersection $\cap$ and the join operation is $+$. Here $\Lambda + \mathcal{M}$ is the least classical modal logic containing both classical modal logics Λ and $\mathcal{M}$. We denote the lattice by CExt**E**. Similarly, we let MExt**M** denote the lattice of all monotonic modal logics with operations $\cap$ and $+$ where $\Lambda + \mathcal{M}$ denotes the smallest monotonic modal logic containing both monotonic modal logics Λ and $\mathcal{M}$. Both CExt**E** and MExt**M** are complete bounded lattices, with **E** and **M** as their least elements respectively and the inconsistent logic as their greatest element.

For a modal logic Λ, we say that ϕ is a *theorem* of Λ if $\phi \in \Lambda$. We denote this by $\vdash_\Lambda \phi$, or $\vdash \phi$ whenever Λ is clear from the context. Derivations will be Hilbert-style proofs and we define deducibility as usual: for a set of modal formulas $\Sigma \cup \{\phi\}$ we say that *ϕ is deducible from Σ in Λ* (denoted by $\Sigma \vdash_\Lambda \phi$) if there are $\sigma_1, \dots, \sigma_n \in \Sigma$ such that all ϕ_i are either axioms in Λ or follow from the previous σ_j by the rules of modus ponens, $\mathrm{RE}_\Box$ or uniform substitution. If ϕ is not deducible from Σ, we write $\Sigma \nvdash_\Lambda \phi$. Such a set Σ of modal formulas is Λ-*consistent* if $\Sigma \nvdash_\Lambda \bot$ and Λ-*inconsistent* otherwise. A modal logic Λ is *consistent* if $\nvdash_\Lambda \bot$ and *inconsistent* otherwise.

In this thesis we will not only be concerned with axioms but also with rules.

Definition 2.3 (**Multi-Conclusion Modal Rule**)**.** Let Γ and Δ be finite sets of modal formulas. The expression Γ/Δ is called a *multi-conclusion modal rule*. Whenever

Δ is a singleton $\{\phi\}$, we call Γ/Δ a *single-conclusion modal rule* and write Γ/ϕ. If $\Gamma = \emptyset$, we call Γ/Δ an *assumption-free modal rule* and write $/\Delta$. $\dashv$

We can associate each assumption-free single-conclusion modal rule $/\phi$ with a modal formula ϕ. In this way, rules are a generalization of formulas.

Similar to normal modal multi-conclusion consequence relations in [22, 5], we define a consequence relation that we associate with the classical modal logic **E**.

Definition 2.4 (**Classical Modal Multi-Conclusion Consequence Relation**)**.** A *classical modal multi-conclusion consequence relation* is a set $\mathcal{S}$ of modal rules such that

(i) $\phi/\phi \in \mathcal{S}$;
(ii) $\phi, \phi \to \psi/\psi \in \mathcal{S}$;
(iii) $\phi \leftrightarrow \psi/\Box\phi \leftrightarrow \Box\psi \in \mathcal{S}$;
(iv) $/\phi \in \mathcal{S}$ for each theorem $\phi \in \mathbf{E}$;
(v) if $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$;
(vi) if $\Gamma/\Delta, \phi \in \mathcal{S}$ and $\Gamma, \phi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$;
(vii) if $\Gamma/\Delta \in \mathcal{S}$ and σ is a substitution, then $\sigma[\Gamma]/\sigma[\Delta] \in \mathcal{S}$. $\dashv$

We let $\mathbf{S_E}$ denote the least classical modal multi-conclusion consequence relation. For every set Ξ of multi-conclusion modal rules, we let $\mathbf{S_E} + \Xi$ be the least classical modal multi-conclusion consequence relation containing Ξ. If $\mathcal{S} = \mathbf{S_E} + \Xi$, we say that $\mathcal{S}$ is *axiomatizable by* Ξ. If Ξ is finite, $\mathcal{S}$ is *finitely axiomatizable*. For a rule $\rho \in \mathcal{S}$, we say that the classical modal multi-conclusion consequence relation $\mathcal{S}$ *entails* or *derives* ρ.

To link logics and consequence relations, we define two maps. For a classical modal multi-consequence relation $\mathcal{S}$, we let $\Lambda(\mathcal{S}) = \{\phi \mid /\phi \in \mathcal{S}\}$ be the modal logic corresponding to $\mathcal{S}$. For a classical modal logic Λ, we let $\mathcal{S}(\Lambda) = \mathbf{S_E} + \{/\phi \mid \phi \in \Lambda\}$. A classical modal logic Λ is *axiomatized by* a set Ξ of multi-conclusion modal rules if $\Lambda = \Lambda(\mathbf{S_E} + \Xi)$.

The classical modal multi-conclusion consequence relations form a lattice, denoted by $\mathrm{CExt}\mathbf{S_E}$, where the meet operator is set-theoretic intersection $\cap$ and the join, denoted by $+$, of two consequence relations $\mathcal{S}$ and $\mathcal{T}$ is the least classical modal multi-conclusion consequence relation containing $\mathcal{S}$ and $\mathcal{T}$. The lattice is complete and bounded, with $\mathbf{S_E}$ as its bottom element and the set of all multi-conclusion modal rules **Rules** as its top element.

As we will also consider monotonic modal logics, we define their corresponding consequence relations, again inspired by [22, 5]. A *monotonic modal multi-conclusion consequence relation* is a set $\mathcal{S}$ of modal rules satisying (i), (ii) and (v) - (vii) from Definition 2.4 and the following two alternatives to (iii) and (iv):

(iii)′ $\phi \to \psi/\Box\phi \to \Box\psi \in \mathcal{S}$
(iv)′ $/\phi \in \mathcal{S}$ for each theorem $\phi \in \mathbf{M}$.

The smallest monotonic modal multi-conclusion consequence relation we denote by $\mathbf{S_M}$. The lattice of all monotonic modal multi-conclusion consequence relations will be written as $\mathrm{MExt}\mathbf{S_M}$. For a monotonic modal multi-conclusion consequence relation $\mathcal{S}$, we say $\mathcal{S}$ is *axiomatizable by* a set of multi-conclusion modal rules Ξ if $\mathcal{S} = \mathbf{S_M} + \Xi$. For multi-conclusion modal rule $\rho \in \mathcal{S}$, we say $\mathcal{S}$ *entails* or *derives* ρ.

2.2 Neighbourhood Frames and models

In this section we recall the definition of neighbourhood frames and models [15, 20, 13]. These frames and models will be used to study non-normal modal logics semantically. Whereas Kripke frames associate each world w with a single set of successors, neighbourhood frames associate each world w with multiple sets called neighbourhoods. A function $\nu : W \to \mathcal{P}\mathcal{P}W$ mapping each world to its neighbourhoods we call a *neighbourhood function.* Here $\mathcal{P}X$ is the power set of X. For $w \in W$, we call each $X \in \nu(w)$ a *neighbourhood of* w.

Definition 2.5 (**Neighbourhood Frame**)**.** A *neighbourhood frame* is a pair $\mathbb{F} = \langle W, \nu \rangle$, where W is a set of worlds and $\nu : W \to \mathcal{P}\mathcal{P}W$ a neighbourhood function. A *neighbourhood model* is a pair $\mathbb{M} = \langle \mathbb{F}, V \rangle$ where $\mathbb{F}$ is a neighbourhood frame and $V : \mathsf{Prop} \to \mathcal{P}W$ a valuation function. ⊣

As we will mostly be concerned with neighbourhood frames, we will simply refer to them as frames and we refer to neighbourhood models simply as models.

We define the semantics of any classical modal logic as follows.

Definition 2.6 (**Neighbourhood Semantics**)**.** Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a neighbourhood model. The truth of a modal formula is defined inductively as follows:

$$
\begin{array}{lll}
\mathbb{M}, w \vDash \bot & & \text{never;} \\
\mathbb{M}, w \vDash p & \text{if} & w \in V(p) \text{ for } p \in \mathsf{Prop}; \\
\mathbb{M}, w \vDash \neg\phi & \text{if} & \text{not } \mathbb{M}, w \vDash \phi; \\
\mathbb{M}, w \vDash \phi \vee \psi & \text{if} & \mathbb{M}, w \vDash \phi \text{ or } \mathbb{M}, w \vDash \psi; \\
\mathbb{M}, w \vDash \Box\phi & \text{if} & V(\phi) \in \nu(w), \text{ where } V(\phi) = \{w \in W \mid \mathbb{M}, w \vDash \phi\}.
\end{array}
$$
⊣

Note that this definition implies that $\mathbb{M}, w \vDash \Diamond\phi$ if and only if $W \setminus V(\phi) \notin \nu(w)$, i.e. $V(\neg\phi) \notin \nu(w)$. The fact that this definition is slightly more involved justifies the choice of $\Box$ as our base operator.

The neighbourhood function ν defines a map $m_\nu : \mathcal{P}W \to \mathcal{P}W$, defined as follows:

$$m_\nu(X) = \{w \in W \mid X \in \nu(w)\}.$$

For a neighbourhood model $\mathbb{M} = \langle W, \nu, V \rangle$ and modal formula ϕ, we note the following equality:

$$m_\nu(V(\phi)) = \{w \in W \mid V(\phi) \in \nu(w)\} = \{w \in W \mid w \vDash \Box\phi\} = V(\Box\phi).$$

We will use this equality to great extent in the next chapter. We will sometimes write $\Box_\nu X$ for $m_\nu X$.

Remark 2.7. For a neighbourhood frame $\langle W, \nu \rangle$, the maps ν and m_ν are closely related, expressed by the following equivalence:

$$w \in m_\nu(X) \text{ iff } X \in \nu(w).$$

The neighbourhood function ν can be derived from m_ν and vice versa. This implies that we can define our neighbourhood frame in terms of m_ν instead of ν, similarly to Došen [15]. ⊣

We define global truth, satisfiability and validity in the usual way. For a neighbourhood model $\mathbb{M} = \langle W, \nu, V \rangle$ and modal formula ϕ, we say that ϕ is *globally true on* $\mathbb{M}$ if for all $w \in W$, $\mathbb{M}, w \vDash \phi$. We write $\mathbb{M} \vDash \phi$. We say that ϕ is *satisfiable on* $\mathbb{M}$ if there is $w \in W$ such that $\mathbb{M}, w \vDash \phi$. For a neighbourhood frame $\mathbb{F} = \langle W, \nu \rangle$, we say that ϕ is *satisfiable of* $\mathbb{F}$ if there exists a valuation V on $\mathbb{F}$ and world $w \in W$ such that $\langle \mathbb{F}, V \rangle, w \vDash \phi$. We say *valid on* $\mathbb{F}$ if for all valuations V on $\mathbb{F}$, ϕ is globally true on $\langle \mathbb{F}, V \rangle$. We write $\mathbb{F} \vDash \phi$.

If $\mathcal{K}$ is a class of neighbourhood frames, we say that ϕ is *satisfiable on* $\mathcal{K}$ whenever there is a frame $\mathbb{F} \in \mathcal{K}$ such that ϕ is satisfiable on $\mathbb{F}$ and ϕ is *valid on* $\mathcal{K}$ if it is valid on all $\mathbb{F} \in \mathcal{K}$. We write $\Lambda(\mathcal{K})$ for the set of modal formulas that are valid on $\mathcal{K}$. For a single neighbourhood frame $\mathbb{F}$, we define $\Lambda(\mathbb{F}) = \{\phi \in \mathbf{Form} \mid \mathbb{F} \vDash \phi\}$. We call $\Lambda(\mathbb{F})$ the (modal) logic *corresponding to* (*generated by*) $\mathbb{F}$.

For an interpretation of multi-conclusion consequence relations on frames and models, we define the following terminology. A model $\mathbb{M} = \langle W, \nu, V \rangle$ *validates* a multi-conclusion modal rule Γ/Δ if $V(\gamma) = W$ for every $\gamma \in \Gamma$ implies $V(\delta) = W$ for some $\delta \in \Delta$. We write $\mathbb{M} \vDash \Gamma/\Delta$. A modal rule Γ/Δ is *valid* on a frame $\mathbb{F}$ if every model based on $\mathbb{F}$ validates Γ/Δ, written as $\mathbb{F} \vDash \Gamma/\Delta$. We define the consequence relation *corresponding to a frame* $\mathbb{F}$ or *generated by a frame* $\mathbb{F}$ as $\mathcal{S}(\mathbb{F}) = \{\Gamma/\Delta \mid \mathbb{F} \vDash \Gamma/\Delta\}$.

To illustrate the example of Deontic Logic, we briefly state some correspondence results. For proofs, we refer to Hansen [20].

Proposition 2.8. *Let $\mathbb{F} = \langle W, \nu \rangle$ be a neighbourhood frame. We have the following correspondences:*

$$\mathbb{F} \vDash \mathrm{RM}_\Box \quad \textit{iff} \quad \forall w \in W : \nu(w) \textit{ is upwards closed};$$
$$\mathbb{F} \vDash \neg\Box\bot \quad \textit{iff} \quad \forall w \in W : \emptyset \notin \nu(w).$$

We will call the frames validating $\mathrm{RM}_\Box$ *monotonic frames.* We call a neighbourhood model *monotonic* if its underlying frame is monotonic. Consequently, the monotonic frames will correspond to the monotonic modal logics. Note that for a monotonic neighbourhood model, we have the following equivalence that we will use in the example below:

$$\mathbb{M}, w \vDash \Box\phi \text{ iff } \exists X \in \nu(w) \text{ s.t. } X \subseteq V(\phi).$$

Example 2.9. We return to the example of deontic logics **MD** and **D**. By the correspondence results above, the class of neighbourhood frames corresponding to **D** is the class of all frames such that for each world, the empty set is not a neighbourhood. The frames for **MD** are a subset of this class, where all $\nu(w)$ are also upwards closed.

In a neighbourhood model $\mathbb{M}$, each neighbourhood of a world w is now interpreted as a standard of obligation for w. For **D**, a formula ϕ is obligatory at a world w only if ϕ is a standard of obligation at w. For **MD**, a formula ϕ is obligatory at w only if ϕ is entailed by one the standards [13].

As both monotonic and classical modal logics are a generalization of normal modal logics, it is instructive to see how the neighbourhood frames are a generalization of Kripke frames. For this, we call a neighbourhood frame $\mathbb{F} = \langle W, \nu \rangle$ *augmented* if it is both monotonic and $\bigcap \nu(w) \in \nu(w)$ or, equivalently, each $\nu(w)$ is both upwards closed and closed under arbitrary intersections.

Now for any augmented neighbourhood frame $\mathbb{F} = \langle W, \nu \rangle$, we can define its corresponding Kripke frame $\mathbb{F}' = \langle W, R_\nu \rangle$ where $R[w] = \bigcap \nu(w)$. For a Kripke frame $\mathbb{G} = \langle W, R \rangle$ we let $\mathbb{G}' = \langle W, \nu_R \rangle$ be the corresponding augmented neighbourhood frame where $\nu(w) = \{X \subseteq W \mid R[w] \subseteq X\}$. An easy induction on the complexity of the formula gives the following equivalence for valuations V on $\mathbb{F}$ and V' on $\mathbb{G}$:

- $\langle \mathbb{F}, V \rangle, w \vDash \phi$ iff $\langle \mathbb{F}', V \rangle, w \vDash \phi$;
- $\langle \mathbb{G}, V' \rangle, w \vDash \phi$ iff $\langle \mathbb{G}', V' \rangle, w \vDash \phi$.

2.3 Operations on Neighbourhood Frames

In this section we discuss a number of operations and constructions for neighbourhood frames and models. We start by discussing bounded morphisms. Their definitions stem from the literature on classical modal logic, e.g. [20, 15]. We then recall the definition of generated submodels that can be largely accredited to Hansen [20]. We give a slightly more coalgebraic spin to it, which will become apparent in the next section. We will use the generated submodels when discussing generated submodels in Part II of the thesis (Section 5.6.1). Lastly we discuss filtrations. We provide a definition different from [20, 13] that more closely resembles filtrations for Kripke frames and compare these different notions.

2.3.1 Bounded Morphisms

We start by discussing bounded morphisms. We adopt the definition from Došen [15].

Definition 2.10 (Bounded Morphism). Let $\mathbb{M}_1 = \langle W_1, \nu_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, \nu_2 \rangle$ be two neighbourhood frames. We call a function $f : W_1 \to W_2$ a *bounded frame morphism* whenever the following condition holds:

(i) for each $w_1 \in W_1$ and $X_2 \subseteq W_2$:

$$f^{-1}[X_2] \in \nu_1(w_1) \Leftrightarrow X_2 \in \nu_2(f(w_1)).$$

When regarding f as a function between models $\mathbb{M}_1 = \langle \mathbb{F}_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle \mathbb{F}_2, V_2 \rangle$, we call f a *bounded morphism* if it satisfies condition (i) as well as the following condition:

(ii) w and $f(w)$ satisfy the same propositional letters for all $w \in W_1$. ⊣

Proposition 2.11 (Invariance under Bounded Morphism). *Let $\mathbb{M}_1 = \langle W_1, \nu_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, \nu_2, V_2 \rangle$ be two neighbourhood models. If $f : W_1 \to W_2$ is a bounded morphism, then for each modal formula ϕ and each $w \in W_1$, we have:*

$$\mathbb{M}_1, w \vDash \phi \Leftrightarrow \mathbb{M}_2, f(w) \vDash \phi.$$

Proof. The proof is by induction on the complexity of the formula. The Boolean cases are easy. We will only consider the case for $\phi = \Box\psi$. Note that the induction hypothesis can be written as $V_1(\psi) = f^{-1}[V_2(\psi)]$. We show that $\mathbb{M}_1, w \vDash \Box\psi$ iff $\mathbb{M}_2, f(w) \vDash \Box\psi$.

$$\begin{aligned} \mathbb{M}_1, w \vDash \Box\psi &\Leftrightarrow V_1(\psi) \in \nu_1(w) \\ &\Leftrightarrow f^{-1}[V_2(\psi)] \in \nu_1(w) \text{ (by I.H.)} \\ &\Leftrightarrow V_2(\psi) \in \nu_2(f(w)) \text{ (by property (i) of bounded morphism } f) \\ &\Leftrightarrow \mathbb{M}_2, f(w) \vDash \Box\psi \end{aligned}$$

□

In order to define a bounded morphism f for neighbourhood frames, we explicitly make use of the inverse image map f^{-1}. We will see that this inverse map f^{-1} will have a prominent role in all operations on neighbourhood frames. An explanation for this will be given in Section 2.4 when we discuss coalgebra.

2.3.2 Submodels

We will now discuss the notion of a generated submodel. In the case of Kripke semantics, a generated submodel is a set that satisfies the hereditary condition of being closed under the accessibility relation. In a neighbourhood setting a similar construction is not so obvious. Hansen [20] introduced generated submodels for neighbourhood frames. We will recall these definitions and reformulate some in terms of the notion of support. In the next section we look at this notion from a coalgebraic angle.

We define a submodel simply to be the restriction of the original model to the carrier set of the submodel.

Definition 2.12 (Submodel). Let $\mathbb{M} = \langle W, \nu, V\rangle$ and $\mathbb{M}_0 = \langle W_0, \nu_0, V_0\rangle$ be neighbourhood models. Then we say that $\mathbb{M}_0$ is a *submodel of* $\mathbb{M}$ if:

- $W_0 \subseteq W$;
- $\nu_0 = \nu \cap (W_0 \times \mathcal{P}W_0)$, i.e. $\forall w \in W_0 : \nu_0(w) = \{X \subseteq W_0 \mid X \in \nu(w)\}$;
- $V_0(p) = V(p) \cap W_0$. ⊣

When given a neighbourhood model $\mathbb{M} = \langle W, \nu, V\rangle$ and a subset $W_0 \subseteq W$, we can construct the submodel by restricting ν and V to W_0. We write $\mathbb{M}{\restriction_{W_0}} = \langle W_0, \nu{\restriction_{W_0}}, V{\restriction_{W_0}}\rangle$.

Definition 2.13 (Generated Submodel). Let $\mathbb{M} = \langle W, \nu, V\rangle$ be a neighbourhood model and $\mathbb{M}_0 = \langle W_0, \nu_0, V_0\rangle$ a submodel of $\mathbb{M}$. We say that $\mathbb{M}_0$ is a *generated submodel of* $\mathbb{M}$ if the identity map $i : W_0 \to W$ is a bounded morphism, i.e. for all $w_0 \in W_0$ and all $X \subseteq W$:

$$i^{-1}[X] = X \cap W_0 \in \nu_0(w_0) \text{ iff } X \in \nu(w_0).$$

Given neighbourhood model $\mathbb{M} = \langle W, \nu, V\rangle$ and subset $W_0 \subseteq W$, we define the *submodel generated by* W_0 *in* $\mathbb{M}$ as the submodel $\mathbb{M}{\restriction_{W'}}$ where W' is the intersection of all sets Y such that $W_0 \subseteq Y$ and $\mathbb{M}{\restriction_Y}$ is a generated submodel of M. ⊣

Remark 2.14. It is important to note that even though the submodel generated by W_0 in $\mathbb{M}$ always exists, it is generally not a generated submodel. We will see an example of when this is the case in Example 2.19. ⊣

From the inclusion map being a bounded morphism, the following proposition is immediate.

Proposition 2.15 (Invariance under Generated Submodels). *Let* $\mathbb{M}_0 = \langle W_0, \nu_0, V_0\rangle$ *be a generated submodel of* $\mathbb{M} = \langle W, \nu, V\rangle$*. Then for all modal formulas* ϕ *and all* $w_0 \in W_0$*:*

$$\mathbb{M}_0, w_0 \models \phi \Leftrightarrow \mathbb{M}, w_0 \models \phi.$$

Remark 2.16. We can define the notion of generated submodel differently, more closely resembling the case of Kripke semantics: a submodel $\mathbb{M}_0 = \langle W_0, \nu_0, V_0\rangle$ of neighbourhood model $\mathbb{M} = \langle W, \nu, V\rangle$ is a generated submodel if it satisfies the following hereditary condition:

$$w \in W_0 \text{ and } X \in \nu(w) \text{ implies } X \subseteq W_0.$$

This however preserves 'too much structure'. We will see in Part II of this thesis that it preserves structure beyond the scope of the basic modal language.

Moreover, this notion of generated submodel does not translate well when making the switch to monotonic neighbourhood frames. When a neighbourhood model is monotonic, the notion trivializes to two extreme cases. When every $w \in W_0$ has $\nu(w) = \emptyset$, we have W_0 to be the generated submodel. Otherwise W_0 contains a world that has some neighbourhood and so monotonicity implies it has the set W as its neighbourhood. Therefore its generated submodel is the whole original model. ⊣

We will now focus on monotonic frames. Here we can give a slightly different formulation of the notion of generated submodel. Intuitively, in a monotonic neighbourhood frame the smallest neighbourhoods carry the essence of the information, if they exist. Any other neighbourhood springs from some smallest one. A way to capture these smallest neighbourhoods is with the notion of a support. The next section will touch upon the coalgebraic origin of this notion.

Definition 2.17 (Support). For a monotonic neighbourhood frame $\mathbb{F} = \langle W, \nu \rangle$, a set $S \subseteq W$ is called a *support for* $w \in W$ *in* $\mathbb{F}$ if $X \in \nu(w)$ implies $X \cap S \in \nu(w)$. We also say that S *supports* w. A set $Z \subseteq W$ is called a *support-closed set* if for each $w \in Z$ we have that Z is a support for w.

For a monotonic neighbourhood model $\mathbb{M} = \langle W, \nu, V \rangle$, a set $S \subseteq W$ is called a *support for* $w \in W$ *in* $\mathbb{M}$ if it is a support for $w \in W$ in the underlying frame $\langle W, \nu \rangle$. ⊣

The following lemma shows that the notion of support-closed set precisely captures our notion of generated submodel.

Lemma 2.18. *Let* $\mathbb{M} = \langle W, \nu, V \rangle$ *be a monotonic neighbourhood model and* $\mathbb{M}_0 = \langle W_0, \nu_0, V_0 \rangle$ *a submodel of* $\mathbb{M}$. *Then* $\mathbb{M}_0$ *is a generated submodel of* $\mathbb{M}$ *iff* W_0 *is a support-closed set in* $\mathbb{M}$.

Proof. For the direction from left to right, we prove that $X \cap W_0 \in \nu_0(w)$ iff $X \in \nu(w)$. From $X \in \nu(w)$, we obtain $X \cap W_0 \in \nu(w)$ by W_0 being support-closed. As $X \cap W_0 \subseteq W_0$, we obtain $X \cap W_0 \in \nu_0(w)$ by $\mathbb{M}_0$ being a submodel. Now suppose $X \cap W_0 \in \nu_0(w)$. Then by $\mathbb{M}_0$ being a submodel, we get $X \cap W_0 \in \nu(w)$. From monotonicity it follows that $X \in \nu(w)$.

For the other direction, take $X \in \nu(w)$. As $\mathbb{M}_0$ is a generated submodel, we obtain $X \cap W_0 \in \nu_0(w)$. It follows from $\mathbb{M}_0$ being a submodel that $X \cap W_0 \in \nu(w)$. □

We can now reformulate the definition of a generated submodel when we are dealing with a monotonic model. Namely, for a monotonic neighbourhood model $\mathbb{M} = \langle W, \nu, V \rangle$ and subset $W_0 \subseteq W$, the submodel generated by W_0 is the submodel $\mathbb{M}{\upharpoonright_S}$ where S is the smallest support-closed set in $\mathbb{M}$ such that $W_0 \subseteq S$. The following counterexample shows why such a smallest support-closed set in general does not exist.

Example 2.19. Consider the monotonic neighbourhood frame $\mathbb{F} = \langle W, \nu \rangle$ with $W = \omega$ and $\nu(n) = \{Z \subseteq \omega \mid Z \text{ a cofinite subset of } \omega \text{ such that } n \in Z\}$ for each $n \in \omega$. A finite set $X \subseteq \omega$ cannot be support-closed, as for any $n \in X$ and $Z \in \nu(n)$, $Z \cap X$ is finite and therefore not a neighbourhood of n. For $X \subseteq \omega$ that is neither finite nor cofinite, we have again that $X \cap Z$ is not cofinite for cofinite Z. An easy check gives us that any cofinite subset of ω is indeed a support-closed set. However, there does not exist a smallest cofinite subset and therefore no smallest support-closed set exists.

2.3.3 Filtrations

In this section we discuss the notion of filtration of neighbourhood models. Filtrations will be a very important tool in showing the existence of stable canonical rules in Chapter 4. Moreover, it is a standard tool to obtain the finite model property. As opposed to the filtrations discussed in the literature for classical and monotonic modal logics [13, 20], we will take an approach that more closely resembles the filtrations for Kripke semantics. We will remark on this difference in Remark 2.23. At the end of the section, we put our definition of filtration in an algebraic perspective.

Firstly we define the equivalence relation on a neighbourhood model with respect to a set of modal formulas Σ.

Definition 2.20 (**Equivalence Relation over Neighbourhood Model**). Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a neighbourhood model and Σ a set of modal formulas closed under subformulas. We let $\equiv_\Sigma$ be the equivalence relation induced by Σ on W, defined as follows for all $w, v \in W$:

$$w \equiv_\Sigma v \text{ iff } \forall \phi \in \Sigma : \mathbb{M}, w \vDash \phi \Leftrightarrow \mathbb{M}, v \vDash \phi$$

$\dashv$

We will let W_Σ denote $\{|w| \mid w \in W\}$, the set of equivalence classes induced by $\sim_\Sigma$ on W. For $X \subseteq W$, we define $|X| = \{|w| \mid w \in X\}$ and for $Y \subseteq W_\Sigma$, we define $\nmid Y \nmid = \{w \in W \mid |w| \in Y\}$. We summarize some useful properties of these operators in the following proposition. The proofs are simple set-theoretic manipulations. We will use the results without reference.

Proposition 2.21. *Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a neighbourhood model and Σ a set of modal formulas closed under subformulas. Then the following properties hold for the operators $|\cdot|$ and $\nmid \cdot \nmid$, where $\phi, \psi \in \Sigma$ and $X, Y \subseteq W_\Sigma$:*

(1) $\nmid |V(\phi)| \nmid = V(\phi)$;

(2) $|\nmid X \nmid| = X$;

(3) $\nmid X \cap Y \nmid = \nmid X \nmid \cap \nmid Y \nmid$;

(4) $W_\Sigma \setminus |V(\phi)| = |V(\neg\phi)|$;

(5) $W \setminus \nmid X \nmid = \nmid W_\Sigma \setminus X \nmid$.

We can now define filtration of a neighbourhood model.

Definition 2.22 (**Filtration**). Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a neighbourhood model and Σ a set of modal formulas closed under subformulas. We call $\mathbb{M}^f = \langle W^f, \nu^f, V^f \rangle$ a *filtration of* $\mathbb{M}$ *through* Σ if the following holds:

(i) $W^f = W_\Sigma$;

(ii) $X \in \nu^f(|w|)$ implies $\nmid X \nmid \in \nu(w)$;

(iii) for all $\Box\phi \in \Sigma$: $V(\phi) \in \nu(w)$ implies $|V(\phi)| \in \nu^f(|w|)$;

(iv) $V^f(p) = |V(p)|$ for all propositional letters p. $\dashv$

Remark 2.23. This definition slightly differs from the definitions known from the literature. In both Hansen [20] and Chellas [13], both conditions (ii) and (iii) are restricted to only the modal formulas in Σ. Our definition more resembles that of filtration for Kripke frames [9, 13, 12]. Like filtration on Kripke models, it has a condition forcing a property globally, condition (ii), and a condition forcing a property restricted to the modal formulas in Σ namely condition (iii). This still leaves open the question why we chose condition (ii) to be the global condition and (iii) the restricted condition. The answer to this lies in the correspondence with algebra. We will hint at this at the end of this section and fully fletch this out when covering the duality result in Chapter 3. $\dashv$

Lemma 2.24 (Filtration Lemma). *Let $\mathbb{M} = \langle W, \nu, V\rangle$ be a neighbourhood model, Σ a subformula closed set of formulas and $\mathbb{M}^f = \langle W^f, \nu^f, V^f\rangle$ a filtration of $\mathbb{M}$ through Σ. Then for each formula $\phi \in \Sigma$ and each $w \in W$:*

$$\mathbb{M}, w \vDash \phi \Leftrightarrow \mathbb{M}^f, |w| \vDash \phi.$$

Proof. We prove by induction on the complexity of ϕ. The Boolean cases are easy and are left out. We will only consider the case when $\phi = \Box\psi$. Note that the induction hypothesis boils down to $V(\psi) = \nmid V^f(\psi)\nmid$ or, equivalently, $V^f(\psi) = |V(\psi)|$ for all $\psi \in \Sigma$ of complexity lower than ϕ.

($\Rightarrow$) Suppose $\mathbb{M}, w \vDash \Box\psi$. Then $V(\psi) \in \nu(w)$. Now by condition (iii) of Definition 2.22, we obtain $|V(\psi)| \in \nu^f(|w|)$. The induction hypothesis now implies $V^f(\psi) \in \nu^f(|w|)$ and therefore, $\mathbb{M}^f, |w| \vDash \Box\psi$.

($\Leftarrow$) Supose $\mathbb{M}^f, |w| \vDash \Box\psi$. Then $V^f(\psi) \in \nu^f(|w|)$. By the induction hypothesis, we obtain $|V(\psi)| \in \nu^f(|w|)$. Applying condition (ii) of Definition 2.22 gives $\nmid|V(\psi)|\nmid = V(\psi) \in \nu(w)$ and thus $\mathbb{M}, w \vDash \Box\psi$. $\square$

We next define smallest and largest filtrations, called ν^s and ν^l respectively.

- $X \in \nu^s(|w|) \Leftrightarrow \exists\, \Box\phi \in \Sigma$ such that $\nmid X\nmid = V(\phi)$ and $V(\phi) \in \nu(w)$;
- $X \in \nu^l(|w|) \Leftrightarrow \forall v \in |w| :\ \nmid X\nmid \in \nu(v)$.

The following lemma shows that neighbourhood models with these neighbourhood functions are indeed correctly defined filtrations and therefore, filtrations for neighbourhood models exist.

Lemma 2.25. *Let $\mathbb{M} = \langle W, \nu, V\rangle$ be a neighbourhood model and Σ a subformula closed set of formulas. Then $\mathbb{M}^s = \langle W_\Sigma, \nu^s, V^f\rangle$ and $\mathbb{M}^l = \langle W_\Sigma, \nu^l, V^f\rangle$ are filtrations of $\mathbb{M}$ through Σ. Moreover, ν^s is the smallest such filtration and ν^l is the greatest.*

Proof. We start with ν^s. Firstly note that ν^s is well-defined, i.e. for $w \sim_\Sigma v$, we have that $V(\phi) \in \nu(w)$ iff $V(\phi) \in \nu(v)$ for $\Box\phi \in \Sigma$. This is taken care of by the definition of $\sim_\Sigma$. Condition (ii) from Definition 2.22 follows directly. To show that ν^s satisfies condition (iii), suppose $V(\phi) \in \nu(w)$ for $\Box\phi \in \Sigma$. Now as $V(\phi) = \nmid|V(\phi)|\nmid$, we obtain $|V(\phi)| \in \nu^s(|w|)$.

To show that ν^s is indeed the smallest filtration, consider $X \in \nu^s(|w|)$. Then there exists $\Box\phi \in \Sigma$ such that $\nmid X\nmid = V(\phi)$ and $V(\phi) \in \nu(w)$. Condition (iii) tells us that $|V(\phi)| \in \nu^f(|w|)$. By $\nmid X\nmid = V(\phi)$ we obtain that $|\nmid X\nmid| = |V(\phi)|$, i.e. $X = |V(\phi)|$. So indeed we have $X \in \nu^f(|w|)$.

For ν^l, condition (ii) follows by definition. For condition (iii), suppose $V(\phi) \in \nu(w)$ for $\Box\phi \in \Sigma$. By definition of $\sim_\Sigma$, this implies that $V(\phi) \in \nu(v)$ for all $v \in |w|$. Now again as $\nmid|V(\phi)|\nmid = V(\phi)$, we obtain $|V(\phi)| \in \nu^l(|w|)$ by definition of ν^l.

To show that it is indeed the largest possible filtration, we take $X \in \nu^f(|w|)$ for any filtration $\mathbb{M}^f$. By condition (ii), we obtain $\nmid X\nmid \in \nu(v)$ for each $v \in |w|$ and then immediately by definition of ν^l, we have $X \in \nu^l(|w|)$. $\square$

To connect the given definitions of bounded morphism and filtration to the algebras we will introduce in Chapter 3, we define a number of properties of maps between neighbourhood frames. As will become apparent in the next chapter, these properties of maps will correspond to their algebraic counterparts.

Definition 2.26 (Stability and Co-Stable Domain Condition). For two neighbourhood models $\mathbb{M}_1 = \langle W_1, \nu_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, \nu_2, V_2 \rangle$, we call a function $f : \mathbb{M}_1 \to \mathbb{M}_2$ a *stable morphism* whenever, for all $w_1 \in W_1$ and $X_2 \subseteq W_2$:

$$X_2 \in \nu_2(f(w_1)) \Rightarrow f^{-1}[X_2] \in \nu_1(w_1).$$

For a subset $D \subseteq \mathcal{P}W_2$, we say that f satisfies the *Co-Stable Domain Condition* (CoSDC) *for* D whenever, for all $w_1 \in W_1$ and $X_2 \in D_2$:

$$f^{-1}[X_2] \in \nu_1(w_1) \Rightarrow X_2 \in \nu_2(f(w_1)).$$

We let these definitions carry over to maps between neighbourhood frames. ⊣

These definitions allow us to reformulate the defined filtrations and bounded morphisms, as summarized in the following proposition.

Proposition 2.27. *Let* $\mathbb{M} = \langle W, \nu, V \rangle$, $\mathbb{M}' = \langle W', \nu', V' \rangle$ *and* $\mathbb{M}^f = \langle W_\Sigma, \nu^f, V^f \rangle$ *be neighbourhood models with* Σ *a subformula closed set of modal formulas. Then:*

(1) $\mathbb{M}^f$ *is a filtration of* $\mathbb{M}$ *through* Σ *iff* $|\cdot|$ *is a stable map satisfying* (CoSDC) *for* $\{V^f(\phi) \mid \Box\phi \in \Sigma\}$.

(2) *A map* $f : W \to W'$ *is a bounded morphism iff* f *is stable and satisfies* (CoSDC) *for* $\mathcal{P}W'$.

Proof. (1) It is obvious that condition (ii) of Definition 2.22 is exactly stability of $|\cdot|$. For (CoSDC), the Filtration Lemma tells us that $\wr V^f(\phi) \wr = V(\phi)$ or, equivalently, $V^f(\phi) = |V(\phi)|$ for all $\phi \in \Sigma$. Therefore condition (iii) implies (CoSDC) for $\{V^f(\phi) \mid \Box\phi \in \Sigma\}$. The other direction of the equivalence can be shown by replicating the Filtration Lemma with (CoSDC) for $\{V^f(\phi) \mid \Box\phi \in \Sigma\}$ replacing condition (iii). Hence condition (iii) is equivalent to $|\cdot|$ satisfying (CoSDC) for $\{V^f(\phi) \mid \Box\phi \in \Sigma\}$.

(2) This is obvious from the definitions. □

2.4 Coalgebra

A prominent role in the background of this thesis is reserved for coalgebra. It is well known that neighbourhood frames as well as Kripke frames can be seen as coalgebras [32, 28]. For this reason we introduce coalgebras briefly in this section. We use it throughout the thesis to remark on correspondences and motivations.

As coalgebras are structures based on categories, we require some basic knowledge on category theory. For the basic concepts, see [1, 25]. We expect the reader to be familiar with the concepts of categories, (endo)functors and their basic operations. We will be working in the category **Sets** that has sets as objects and functions as morphisms.

We start by defining what a coalgebra is. We define them only for endofunctors T : **Sets** $\to$ **Sets** on **Sets**.

Definition 2.28 (**Coalgebra**)**.** Let T : **Sets** $\to$ **Sets** be an endofunctor on the category of **Sets**. A T*-coalgebra* is a pair $\mathcal{C} = \langle X, \gamma \rangle$ where X is a set and $\gamma : X \to TX$ a function. $\dashv$

We will be mainly interested in three different endofunctors. The first is the *covariant powerset functor* $\mathcal{P}$: **Sets** $\to$ **Sets**, defined as follows:

$$\begin{aligned} X &\mapsto \mathcal{P}X \\ f : X \to Y &\mapsto \mathcal{P}f = f[\cdot] : \mathcal{P}X \to \mathcal{P}Y \text{ i.e. } \mathcal{P}f(X') = f[X'] \text{ for } X' \subseteq X. \end{aligned}$$

The $\mathcal{P}$-coalgebras are known to correspond to Kripke-frames, see for example [32, 24, 28]. The composition of this functor with itself $\mathcal{P} \circ \mathcal{P}$: **Sets** $\to$ **Sets** will play a role when we discuss INL in Part II of the thesis.

The second important endofunctor is the *contravariant powerset functor* $\check{\mathcal{P}}$: **Sets** $\to$ **Sets**, defined below.

$$\begin{aligned} X &\mapsto \mathcal{P}X \\ f : X \to Y &\mapsto \check{\mathcal{P}}f = f^{-1}[\cdot] : \mathcal{P}Y \to \mathcal{P}X \text{ i.e. } \check{\mathcal{P}}f(Y') = f^{-1}[Y'] \text{ for } Y' \subseteq Y \end{aligned}$$

The coalgebras for this functor composed with itself $\check{\mathcal{P}} \circ \check{\mathcal{P}}$: **Sets** $\to$ **Sets** are known to correspond to neighbourhood frames [32, 20].

Finally, we introduce the endofunctor whose coalgebras correspond to monotonic neighbourhood frames. It is a restriction of $\check{\mathcal{P}} \circ \check{\mathcal{P}}$ to upwards closed families of sets. The *monotonic neighbourhood functor* $\mathcal{U}p\mathcal{P}$: **Sets** $\to$ **Sets** is defined as follows:

$$\begin{aligned} X &\mapsto \{U \in \mathcal{P}\mathcal{P}X \mid U \text{ is upwards closed }\} \\ f : X \to Y &\mapsto \mathcal{U}p\mathcal{P}f : \mathcal{U}p\mathcal{P}X \to \mathcal{U}p\mathcal{P}Y \text{ where} \\ &\qquad \mathcal{U}p\mathcal{P}f(U) = (\check{\mathcal{P}} \circ \check{\mathcal{P}})(U) = \{D \in \mathcal{P}(Y) \mid f^{-1}[D] \in U\} \text{ for } U \in \mathcal{U}p\mathcal{P}X. \end{aligned}$$

As mentioned, the $\mathcal{U}p\mathcal{P}$-coalgebras correspond to monotonic neighbourhood frames [20].

Another coalgebraic concept we introduce is that of coalgebraic homomorphisms. We use them to justify the definition of morphisms on neighbourhood frames.

Definition 2.29 (**Coalgebraic Homomorphism**)**.** Let T : **Sets** $\to$ **Sets** be an endofunctor and $\mathcal{C} = \langle X, \gamma \rangle$ and $\mathcal{D} = \langle Y, \delta \rangle$ two T-coalgebras. A function $f : X \to Y$ is a *coalgebraic homomorphism* if $\delta \circ f = Tf \circ \gamma$, i.e. if the following diagram commutes: $\dashv$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ TX & \xrightarrow{Tf} & TY \end{array}$$

We can now put some of the concepts discussed in this chapter in a coalgebraic perspective. Firstly, it is known that frame morphisms for neighbourhood frames are exactly coalgebraic homomorphisms on $\check{\mathcal{P}} \circ \check{\mathcal{P}}$-coalgebras [32, 20]. The fact that this contravariant powerset functor maps a function to its inverse image map explains the prominent role that this inverse image map has in the definitions of bounded morphism and filtration. The fact that Kripke frames correspond to $\mathcal{P}$-coalgebras explains why on Kripke frames it is the direct image map adopting the prominent role.

The second important remark to make is the connection of coalgebra with the notion of support. In coalgebraic terms, a support is defined as below [16].

Definition 2.30 (**Coalgebraic Support**)**.** Let $T : \mathbf{Sets} \to \mathbf{Sets}$ be an endofunctor on **Sets**, W a set and and $N \in TW$. A subset $S \subseteq W$ is a (coalgebraic) *support* for N if there exists some $M \in TS$ with $T\iota_{S,W}(M) = N$. $\dashv$

This notion of support for $\mathcal{U}p\mathcal{P}$-coalgebras and the support on monotonic neighbourhood frames now coincide, proven in the following proposition.

Proposition 2.31. *Let $\langle W, \nu \rangle$ be a monotonic neighbourhood frame and $S \subseteq W$ and $\nu(w) \subseteq \mathcal{P}W$ two sets. Then S is a (frame-theorical) support of w iff S is a (coalgebraic) support for $\nu(w)$*

Proof. The coalgebraic definition of support instanced for the monotonic neighbourhood function boils down to the following: for set W and $\nu(w) \in \mathcal{U}p\mathcal{P}W$, a set $S \subseteq W$ is a support for $\nu(w)$ if there exists $M \in \mathcal{U}p\mathcal{P}S$ such that $\nu(w) = \{X \subseteq W \mid X \cap S \in M\}$.

First assume that $X \in \nu(w)$ implies $X \cap S \in \nu(w)$. We need $M \in \mathcal{U}p\mathcal{P}S$ such that $\nu(w) = \{X \subseteq W \mid X \cap S \in M\}$. Let M the set $\{X \cap S \mid X \in \nu(w)\}$. For the inclusion from left to right, note that by the assumption, $X \in \nu(w)$ implies $X \cap S \in M$. For the other direction, take $X \subseteq W$ such that $X \cap S \in M$. Then there exists $Y \in \nu(w)$ such that $Y \cap S = X \cap S$. By the assumption we obtain $Y \cap S \in \nu(w)$, so also $X \cap S \in \nu(w)$. By $\nu(w)$ being upwards closed, $X \in \nu(w)$.

For the other direction, assume we have $M \subseteq \mathcal{P}S$ such that $\nu(w) = \{X \subseteq W \mid X \cap S \in M\}$. Suppose $X \in \nu(w)$, hence $X \cap S \in M$. But then $(X \cap S) \cap S = X \cap S \in M$, so $X \cap S \in \nu(w)$. $\square$

The notion of support plays a prominent role when relating coalgebras to logics, see for example [16] for its role in constructing tree-like structures in Monadic Second Order Logic. We will rely heavily on the notion of support when discussing INL in Part II. There we will work with neighbourhood structures corresponding to $\mathcal{P} \circ \mathcal{P}$-coalgebras where the support has a useful characterization.

Chapter 3

General Neighbourhood Frames and Modal Algebras

There exists a long-standing tradition in logic to relate the algebraic and frame-theoretical semantics via a duality. Examples include the Stone duality between the categories of Stone spaces with continuous maps and Boolean algebras with Boolean homomorphisms as well as the well-known duality between the category of descriptive Kripke frames with bounded morphisms and the category of modal algebras with modal homomorphisms, see e.g. [9, 12].

In this chapter we discuss the duality between the category of descriptive neighbourhood frames with general frame morphisms and the category of classical modal algebras with modal homomorphisms, as given by Došen [15]. We will define the objects and morphisms of both categories. We assume the reader's familiarity with the categorical notion of duality [1, 25]. We will use this duality to prove a completeness result for any classical modal logic with respect to a class of descriptive neighbourhood frames, as well as a completeness result for any classical modal multi-conclusion consequence relation. We also show that both the duality and completeness results can be easily adapted to the case of monotonic modal logics. We will rely heavily on these completeness results in the next chapter.

3.1 Modal Algebras

In this section we introduce the algebras for which any classical modal logic is complete, together with their morphisms. The algebras will be based on Boolean algebras. For the definitions and properties of Boolean algebras, we refer to e.g. [18, 31]. The modal algebras that correspond to descriptive Kripke frames are Boolean algebras together with a unary modal operator $\Box$ such that $\Box a \wedge \Box b = \Box(a \wedge b)$ and $\Box 1 = 1$. We call a unary modal operator $\Box$ satisfying these properties *normal*. The modal algebras where the modal operator is normal we call *normal modal algebras*. In order to deal with non-normal modal logics, we need to generalize these algebras.

Definition 3.1 (Classical Modal Algebra). A *classical modal algebra* is a tuple $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1, \Box \rangle$ where $\langle A, \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and A is closed under the unary modal operator $\Box : A \to A$. We usually write $\mathbb{A} = \langle A, \Box \rangle$ when the Boolean operators are clear from the context.

We call a classical modal algebra $\mathbb{A} = \langle A, \Box \rangle$ *monotonic* if $\Box(a \wedge b) \leq \Box a \wedge \Box b$ or, equivalently, if $a \leq b$ implies $\Box a \leq \Box b$. ⊣

The morphisms between classical modal algebras will be maps preserving all operators. The preservation of the modal operator will be separated into two separate cases.

Definition 3.2 (Stability and Co-Stable Domain Condition). Let $\mathbb{A} = \langle A, \Box_A \rangle$ and $\mathbb{B} = \langle B, \Box_B \rangle$ be two modal algebras and $h : A \to B$ a function. We call h *stable* if it is a Boolean homomorphism such that the following holds for all $a \in A$:

$$h(\Box_A a) \leq \Box_B h(a).$$

For a subset $D \subseteq A$, we say that h satisfies the *Co-Stable Domain Condition* (CoSDC) for D if it is a Boolean homomorphism such that the following holds for all $a \in D$:

$$h(\Box_A a) \geq \Box_B h(a).$$

A function $h : A \to B$ will be called a *modal homomorphism* if it is both stable and satisfies (CoSDC) for A. It is a *modal isomorphism* if it is a modal homomorphism that is both injective and surjective. ⊣

Remark 3.3. Whereas for neighbourhood frames, the definitions for stability and (CoSDC) (Definition 2.26) might not seem intuitive at first, on the algebraic side they are very clear. Stability ensures $h(\Box a) \leq \Box h(a)$, whereas (CoSDC) for $\{a\}$ ensures $\Box h(a) \leq h(\Box a)$. Together, they ensure that h preserves the modal operator for a. ⊣

3.2 Algebraic Completeness

In this section we show the completeness of any classical modal logic and multi-conclusion consequence relation with respect to a class of classical modal algebras. We first show completeness of the latter by a refinement of the Lindenbaum-Tarski method. The completeness of any classical modal logic will then follow as an easy corollary. An easy adaptation of the argument also gives us completeness of any monotonic modal logic and multi-conclusion consequence relation with respect to a class of monotonic modal algebras.

Firstly we see how a classical modal algebra $\mathbb{A} = \langle A, \Box \rangle$ can be viewed as a model for any modal logic. We can define a valuation $V : \mathsf{Prop} \to A$ on $\mathbb{A}$. We call an algebra $\mathbb{A}$ together with such a valuation V an *algebraic model*. This valuation can be extended to all formulas in the usual sense, covering the modal case with $V(\Box\phi) = \Box V(\phi)$.

We now say that a modal formula ϕ is *true* on $\mathbb{A}$ under valuation V (notation: $\langle \mathbb{A}, V \rangle \vDash \phi$) if $V(\phi) = 1$. We say that ϕ is *valid* on $\mathbb{A}$ if ϕ is true under all valuations. We denote this by $\mathbb{A} \vDash \phi$. The modal rule Γ/Δ is *true* on $\mathbb{A}$ under valuation V if $V(\gamma) = 1$ for all $\gamma \in \Gamma$ implies $V(\delta) = 1$ for some $\delta \in \Delta$, written as $\langle \mathbb{A}, V \rangle \vDash \Gamma/\Delta$. The rule is *valid* on $\mathbb{A}$ if it is true under all valuations (notation: $\mathbb{A} \vDash \Gamma/\Delta$). For a class $\mathcal{K}$ of classical modal algebras, we say that ϕ is satisfiable on $\mathcal{K}$ if there exists $\mathbb{A} \in \mathcal{K}$ and valuation V on $\mathbb{A}$ such that $V(\phi) \neq 0$. A modal formula ϕ is *valid on* $\mathcal{K}$ if ϕ is valid on every $\mathbb{A} \in \mathcal{K}$, written as $\mathcal{K} \vDash \phi$ and similarly for modal rules. For any set of modal formulas Σ, we denote $\mathcal{K}_\Sigma$ to be the class of classical modal algebras $\mathbb{A}$ such that all $\phi \in \Sigma$ are valid on $\mathbb{A}$. Likewise for a classical modal multi-conclusion consequence relation $\mathcal{S}$, we denote $\mathcal{K}_{\mathcal{S}}$ to be the class of classical modal algebras $\mathbb{A}$ such that $\mathbb{A} \models \mathcal{S}$.

To show completeness for any classical modal multi-conclusion consequence relation, we will sketch an adaptation of the completeness proof for normal modal multi-conclusion consequence relations as given in [8]. It uses a proof calculus for modal rules based on *hyperformulas*, which are finite sets S of modal formulas written in the form

$$\alpha_1 \mid \cdots \mid \alpha_n.$$

We let $S \mid S'$ mean the set union $S \cup S'$. The hyperformulas $\alpha \mid S$ and $S \mid \alpha$ stand for $\{\alpha\} \mid S$ and $S \mid \{\alpha\}$ respectively.

Definition 3.4 (**K-derivation**)**.** Let Γ be a finite set of modal formulas and K some set of modal rules. A *K-derivation*, or simply *derivation* if K is clear from the context, under assumptions Γ is a finite list of hyperformulas $S_1, \ldots, S_n$ such that for each S_i one of the following holds:

(i) S_i is of the kind $\alpha \mid S$ such that $\alpha \in \Gamma$ or α is a tautology.

(ii) S_i is obtained from preceding hyperformulas by applying a rule from K or the rule $\mathrm{RE}_\Box$ or modus ponens.

We write $\Gamma \Vdash_K S$ to say that there exists a K-derivation ending in S. ⊣

If we say that S_i is obtained by applying a rule $\gamma_1, \ldots, \gamma_n / \delta_1, \ldots, \delta_m$ from K, we mean that there exists a substitution σ such that S_i is of the kind $S \mid \sigma\delta_1 \mid \cdots \mid \sigma\delta_m$ and there exists $j_1, \ldots, j_n < i$ such that for each $k \leq n$, S_{j_k} is of the form $S \mid \sigma\gamma_k$.

One should think of the expression $\Gamma \Vdash_K S$ as stating that $\Gamma / S \in \mathbf{S_E} + K$. We will show this actual equivalence in the proof of Theorem 3.8.

The following lemmas are easily shown by induction on the length of the derivation.

Lemma 3.5. $\Gamma \Vdash_K S \Rightarrow \Gamma \Vdash_K S \mid S'$.

Lemma 3.6. *If* $\Gamma \cup \{\alpha\} \Vdash_K S$ *and* $\Gamma \Vdash_K \alpha \mid S$*, then* $\Gamma \Vdash_K S$*.*

We will sketch the completeness of the proof system $\Vdash$ with respect to classical modal algebras.

Theorem 3.7. *Let K be a set of modal rules, Γ a finite set of modal formulas and S a hyperformula. Then $\Gamma \Vdash_K S$ iff the rule Γ / S is valid in every classical modal algebra validating K.*

Proof. The direction of soundness is easy and left out. For the direction of completeness, suppose that $\Gamma \nVdash_K S$. By Zorn's lemma, we obtain a set of modal formulas Γ^+ maximal with respect to the properties $\Gamma \subseteq \Gamma^+$ and $\Gamma^+ \nVdash_K S$. We will list a few claims about Γ^+ that we will use. Their proofs are easy and are left out.

Claim 1. For each hyperformula $\alpha_1 \mid \cdots \mid \alpha_n$: $\Gamma^+ \Vdash_K \alpha_1 \mid \cdots \mid \alpha_n \mid S \Rightarrow \exists i$ such that $\Gamma^+ \Vdash_K \alpha_i$.

Claim 2. $\Gamma^+ \Vdash_K \alpha$ iff $\alpha \in \Gamma^+$.

Claim 3. Γ^+ is closed under the rules of moduls ponens and $\mathrm{RE}_\Box$ and contains all propositional tautologies.

We now define a relation $\sim$ as follows:

$$\alpha_1 \sim \alpha_2 \qquad \text{iff} \qquad \alpha_1 \leftrightarrow \alpha_2 \in \Gamma^+.$$

By Claim 3, $\sim$ is a congruence relation. We define the classical modal algebra $\mathbb{A}_K^\Gamma = \langle \mathbf{Form}_{/\sim}, \Box \rangle$ where $\Box|\alpha| = |\Box\alpha|$ and show that this algeba validates K and refutes the rule Γ/S.

To show that it refutes the rule Γ/S, define the valuation V on $\mathbb{A}_K^\Gamma$ by setting $V(p) = |p|$. An easy induction on the complexity of the formula gives us that $V(\phi) = |\phi|$ for each modal formula ϕ. First take $\gamma \in \Gamma$, i.e. $\gamma \in \Gamma^+$. We show that $V(\gamma) = |\top|$. By Claim 2, it suffices to show that $\Gamma^+ \Vdash_K \gamma \leftrightarrow \top$. But as both $\gamma \in \Gamma^+$ and $\top \in \Gamma^+$, Claim 3 gives us exactly this. Therefore, $\langle \mathbb{A}_K^\Gamma, V \rangle \models \Gamma$. Now take $\delta \in S$. We claim that $|\delta| \neq |\top|$. For, suppose for a contradiction that $\delta \leftrightarrow \top \in \Gamma^+$. Then by $\top \in \Gamma^+$ and Claim 3, we obtain $\delta \in \Gamma^+$. But then $\Gamma^+ \Vdash_K \delta$. By weakening this would give $\Gamma^+ \Vdash_K S$, which is a contradiction. Consequently, $\langle \mathbb{A}_K^\Gamma, V \rangle \not\models \Gamma/S$.

Now take any rule $\gamma_1, \ldots, \gamma_n/\delta_1, \ldots, \delta_m \in K$ and any valuation V on $\mathbb{A}_K^\Gamma$. For each $p \in \mathsf{Prop}$, pick a representative formula $\phi_p \in V(p)$. This gives substitution σ defined as $\sigma(p) = \phi_p$. An easy induction on the complexity of ϕ now shows that $V(\phi) = |\sigma(\phi)|$ for each modal formula ϕ. Now suppose $V(\gamma_i) = |\top|$ for each $i \leq n$, i.e. $\sigma(\gamma_i) \leftrightarrow \top \in \Gamma^+$. By Claim 2, we obtain $\Gamma^+ \Vdash_K \sigma(\gamma_i)$ for all $i \leq n$. As $\gamma_1, \ldots, \gamma_n/\delta_1, \ldots, \delta_m$ is a rule in K, we obtain the K-derivation $\Gamma^+ \Vdash_K \sigma(\delta_1) \mid \cdots \mid \sigma(\delta_m)$. Now Claim 1 gives us some i such that $\Gamma^+ \Vdash_K \sigma(\delta_i)$. This again implies $\sigma(\delta_i) \leftrightarrow \top \in \Gamma^+$ as before, giving $\langle \mathbb{A}_K^\Gamma, V \rangle \models \delta_i$. Hence, $\langle \mathbb{A}_K^\Gamma, V \rangle \models K$. □

We can now prove completeness of any classical modal multi-conclusion consequence relation and logic.

Theorem 3.8 (**Algebraic Completeness**).

(1) *Let $\mathcal{S}$ be a classical modal multi-conclusion consequence relation. Then $\mathcal{S}$ is sound and complete with respect to the class $\mathcal{K}_\mathcal{S}$. That is, for all modal rules ρ, we have:*

$$\rho \in \mathcal{S} \text{ if and only if } \mathcal{K}_\mathcal{S} \models \rho.$$

(2) *Let Λ be a classical modal logic. Then Λ is sound and complete with respect to the class $\mathcal{K}_\Lambda$. That is, for all modal formulas ϕ, we have:*

$$\vdash_\Lambda \phi \text{ if and only if } \mathcal{K}_\Lambda \models \phi.$$

Proof. (1) To show this, we first prove a claim. Let K be a set of modal rules and Γ/Δ a modal rule. We claim that for each classical modal multi-conclusion consequence relation $\mathcal{S}$ such that $K \subseteq \mathcal{S}$, we have $\Gamma/\Delta \in \mathcal{S}$ iff $\Gamma \Vdash_K \Delta$. The theorem is now a corollary of this claim together with Theorem 3.7.

To show the claim, the direction from right to left can be easily shown by induction on the length of the derivation of $\Gamma \Vdash_K \Delta$. For the other direction, by Lemmas 3.5 and 3.6 the set $\{\Gamma/\Delta \mid \Gamma \Vdash_K \Delta\}$ is a classical modal multi-conclusion consequence relation.

(2) Soundness is an easy check. For completeness we look at the classical modal multi-conclusion consequence relation $\mathcal{S}(\Lambda)$ corresponding to Λ. An easy induction on the length of the derivation shows that $\Vdash_{\mathcal{S}(\Lambda)} /\phi$ implies $\vdash_\Lambda \phi$. Therefore if $\nvdash_\Lambda \phi$, Theorem 3.7 gives a classical modal algebra $\mathbb{A}$ validating $\mathcal{S}(\Lambda)$ but refuting $/\phi$. Consequently, $\mathbb{A}$ validates Λ and refutes ϕ. □

Remark 3.9. The proof of the completeness result above gives us some insight into the connection between logics and their corresponding consequence relations. By the claim shown in clause (2) of the proof, we have for each classical modal logic Λ that $\Lambda(\mathcal{S}(\Lambda)) = \Lambda$. For a classical modal multi-conclusion consequence relation we have $\mathcal{S}(\Lambda(\mathcal{S})) \subseteq \mathcal{S}$. ⊣

To end this section, we briefly look at monotonic modal logics. The method outlined above will easily translate to monotonic modal logics. We merely need to adapt the definition of a K-derivation to use the rule $\text{RM}_\Box$ instead of $\text{RE}_\Box$. The algebra $\mathbb{A}_K^\Gamma$ is now a monotonic modal algebra. This gives us a completeness result of monotonic modal multi-conclusion consequence relations and monotonic modal logics with respect to monotonic modal logics.

Theorem 3.10 (**Algebraic Monotonic Completeness**).

(1) *Let $\mathcal{S}$ be a monotonic modal multi-conclusion consequence relation. Then $\mathcal{S}$ is sound and complete with respect to the class $\mathcal{K}_\mathcal{S}$ of all monotonic modal algebras validating $\mathcal{S}$. That is, for all modal rules ρ, we have:*

$$\rho \in \mathcal{S} \text{ if and only if } \mathcal{K}_\mathcal{S} \models \rho.$$

(2) *Let Λ be a monotonic modal logic. Then Λ is sound and complete with respect to the class $\mathcal{K}_\Lambda$ of all monotonic modal algebras validating Λ. That is, for all modal formulas ϕ, we have:*

$$\vdash_\Lambda \phi \text{ if and only if } \mathcal{K}_\Lambda \models \phi.$$

We can also look at the validity of modal rules from a slightly different perspective. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and $\Delta = \{\delta_1, \ldots, \delta_m\}$ be sets of modal formulas. Now replacing each propositional variable p_j with a variable x_j gives terms $\gamma_i(\vec{x})$ and $\delta_i(\vec{x})$ in the first-order language of classical modal algebras. For classical modal algebra $\mathbb{A}$, we obtain $\mathbb{A} \models \Gamma/\Delta$ iff $\mathbb{A}$ is a model of the universal sentence $\forall \vec{x}(\bigwedge_{i=1}^n \gamma_i(\vec{x}) = 1 \rightarrow \bigvee_{i=1}^m \delta_i(\vec{x}) = 1)$. For modal rule ρ, we let $\rho^\forall$ be this corresponding universal sentence. Classical modal multi-conclusion consequence relations now correspond to classes of classical modal algebras axiomatized by these universal sentences, in the literature referred to as *universal classes* [11]. We let $\mathcal{U}(\mathcal{S})$ denote the universal class of classical modal multi-conclusion consequence relation $\mathcal{S}$. For universal class $\mathcal{U}$, we let $\mathcal{S}(\mathcal{U}) = \{\rho \mid \mathcal{U} \models \rho\}$ denote the classical modal multi-conclusion consequence relation corresponding to $\mathcal{U}$. From the completeness result in Theorem 3.8 above, we obtain that $\mathcal{S}(\mathcal{U}(\mathcal{S})) = \mathcal{S}$ as well as $\mathcal{U}(\mathcal{S}(\mathcal{U}) = \mathcal{U}$. Moreover, it is well known that a class of classical modal algebras is universal iff it is closed under isomorphisms, subalgebras and ultraproducts, see e.g. [11, Theorem V2.20].

3.3 General Neighbourhood Frames

In this section we introduce the generalized version of neighbourhood frames. Recall that a general Kripke frame is a Kripke frame together with a set $A \subseteq \mathcal{P}W$ of *admissible subsets*, which restricts the allowed valuations on the frame [12, 9]. We will follow a similar line of thought and introduce general neighbourhood frames, as defined by Došen [15]. These frames are a bit different from the ones defined in Hansen [20].

Definition 3.11 (**General Neighbourhood Frame**). A *general neighbourhood frame* is a triple $\mathbb{F} = \langle W, \nu, A \rangle$, where W is a set of worlds and $\nu : W \rightarrow \mathcal{P}A$ a neighbourhood function. The set $A \subseteq \mathcal{P}W$ is a non-empty set of subsets of W satisfying the following closure conditions:

- $X, Y \in A$ implies $X \cup Y \in A$;
- $X \in A$ implies $W \setminus X \in A$;
- $X \in A$ implies $m_\nu(X) \in A$.

A *general neighbourhood model* is a pair $\mathbb{M} = \langle \mathbb{F}, V \rangle$ where $\mathbb{F} = \langle W, \nu, A \rangle$ is a general neighbourhood frame and $V : \mathsf{Prop} \to A$ a valuation function. ⊣

Remark 3.12. Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a general neighbourhood frame. In the given definition, A does not simply restrict the possible valuations on $\mathbb{F}$. It also restricts the possible neighbourhoods of any world. Herein we follow the definition presented by Došen [15]. To motivate this restriction, consider any valuation V on $\mathbb{F}$. The set A forces the extended valuation function V to be of the form $V : \textbf{Form} \to A$. The truth of a modal formula will now only depend on those neighbourhoods that are actually admissible. Therefore, neighbourhoods that are not admissible will not be relevant when expressing truth on the model.

Hansen [20] drops the restriction for general monotonic neighbourhood frames. We will remark on the impact of this difference throughout this chapter. In Section 3.5, we will see specifically how we handle monotonic neighbourhood frames. ⊣

We now introduce the morphisms between general neighbourhood frames. They will bear a strong resemblance to the frame morphisms defined before. We only need to deal with the set of admissible subsets. For this, we introduce the concept of *continuity*. This will ensure that each admissible subset in the target model is reflected in the original model.

Definition 3.13 (General Frame Morphism). For two general neighbourhood frames $\mathbb{F}_1 = \langle W_1, \nu_1, A_1 \rangle$ and $\mathbb{F}_2 = \langle W_2, \nu_2, A_2 \rangle$, a function $f : W_1 \to W_2$ is called *continuous* if for each $X_2 \subseteq W_2$:

$$X_2 \in A_2 \Rightarrow f^{-1}[X_2] \in A_1.$$

A continuous f is called *stable* if for all $X_2 \in A_2$ and all $w_1 \in W_1$:

$$X_2 \in \nu_2(f(w_1)) \Rightarrow f^{-1}[X_2] \in \nu_1(w_1).$$

For a set $D \subseteq A_2$, we say that f *satisfies the Co-Stable Domain Condition* (CoSDC) *for D* if, for all $X_2 \in D$ and all $w_1 \in W_1$:

$$f^{-1}[X_2] \in \nu_1(w_1) \Rightarrow X_2 \in \nu_2(f(w_1)).$$

A *general frame morphism* is a continuous, stable general frame morphism satisfying (CoSDC) for A_2. We call f a *general frame isomorphism* if it is a general frame morphism, it is injective as well as surjective and the inverse function f^{-1} is also a general frame morphism. ⊣

3.4 Duality

In this section we show the duality between descriptive neighbourhood frames and classical modal algebras in detail. We replicate the proof as given by Došen [15] to stress the importance of this duality. Moreover, we show a direct correspondence between the properties of stability and satisfying (CoSDC) on classical modal algebras and general neighbourhood frames.

First we establish a correspondence for the objects and then we establish one for the morphisms. We will establish the duality via ultrafilters on the classical modal algebras, which are simply ultrafilters of the underlying Boolean structure. For background on ultrafilters, we refer to [31, 18]. For classical modal algebra $\mathbb{A}$, we let $\mathrm{Uf}(\mathbb{A})$ denote the set of ultrafilters on $\mathbb{A}$. For future reference, we give the definition of an ultrafilter below.

Definition 3.14 (Ultrafilter). For a Boolean algebra $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$, we call a subset $F \subseteq A$ an *ultrafilter* if the following properties hold, for $a, b \in A$:
(i) $1 \in F$ and $0 \notin F$;
(ii) $a \in F$ and $a \leq b$ implies $b \in F$;
(iii) $a, b \in F$ implies $a \wedge b \in F$;
(iv) for each $a \in A$, $a \in F$ or $\neg a \in F$. ⊣

3.4.1 Correspondence for Objects

In this section we discuss the correspondence between general neighbourhood frames and classical modal algebras. We start with the direction from algebra to frame. Consider a classical modal algebra $\mathbb{A} = \langle A, \Box\rangle$. We define the general neighbourhood frame $\mathbb{A}_* = \langle W^{\mathbb{A}}, \nu_\Box, A^{\mathbb{A}}\rangle$, where:
- $W^{\mathbb{A}} = \mathrm{Uf}(\mathbb{A})$;
- $A^{\mathbb{A}} = \{\beta(a) \mid a \in A\}$, where $\beta : A \to \mathcal{P}\mathcal{P}A$ is defined as $\beta(a) = \{U \in \mathrm{Uf}(\mathbb{A}) \mid a \in U\}$;
- $\nu_\Box : \mathrm{Uf}(\mathbb{A}) \to \mathcal{P}A^{\mathbb{A}}$ is defined as $\nu_\Box(U) = \{\beta(a) \mid \Box a \in U\}$.

The following lemma ensures that $\mathbb{A}_*$ is indeed a general frame.

Lemma 3.15. *Let $\mathbb{A} = \langle A, \Box\rangle$ be a classical modal algebra. Then $\mathbb{A}_*$ is a general neighbourhood frame.*

Proof. First of all, note that $\nu_\Box$ and $A^{\mathbb{A}}$ are both well-defined by definition. We need but to show that $A^{\mathbb{A}}$ satisfies the closure conditions. First of all, $A^{\mathbb{A}}$ is closed under finite intersection, because ultrafilters are upsets as well as closed under finite meets. This gives $\beta(a \wedge b) = \beta(a) \cap \beta(b)$.

Secondly, $\beta(\neg a) = \mathrm{Uf}(\mathbb{A}) \setminus \beta(a)$, because for any $U \in \mathrm{Uf}(\mathbb{A})$ and for all $a \in A$, either $a \in U$ or $\neg a \in U$.

Lastly, we need that $A^{\mathbb{A}}$ is closed under the operation $m_{\nu_\Box}$. For this, we take $\beta(a) \in A^{\mathbb{A}}$ and look at the following derivation.

$$\begin{aligned} m_{\nu_\Box}(\beta(a)) &= \{U \in \mathrm{Uf}(\mathbb{A}) \mid \beta(a) \in \nu_\Box(U)\} \\ &= \{U \in \mathrm{Uf}(\mathbb{A}) \mid \beta(a) \in \{\beta(b) \mid \Box b \in U\}\} \\ &= \{U \in \mathrm{Uf}(\mathbb{A}) \mid \Box a \in U\} \\ &= \beta(\Box a) \end{aligned}$$ □

Conversely, let $\mathbb{F} = \langle W, \nu, A\rangle$ be a general neighbourhood frame. We define its algebraic counterpart as $\mathbb{F}^* = \langle A, \cap, \cup, \setminus, \emptyset, W, m_\nu\rangle$. Differently stated, it is a powerset algebra with set A as its carrier and m_ν as its modal operator.

Lemma 3.16. *Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a general neighbourhood frame. Then $\mathbb{F}^*$ is a classical modal algebra.*

Proof. Note that $\cap, \cup, \setminus$ and m_ν are well-defined operations on $\mathbb{F}^*$ by the closure properties of A. To show that $\emptyset$ and W are elements of A, we start with any $X \in A$, which exists by non-emptiness. Then $W \setminus X \in A$ and thus $X \cap (W \setminus X) = \emptyset \in A$ and hence also $W \setminus \emptyset = W \in A$. □

To show duality between the structures, we need isomorphisms between them. The isomorphism between modal algebras $\mathbb{A}$ and $(\mathbb{A}_*)^*$ will be the previously defined function β.

Theorem 3.17. *Let $\mathbb{A}$ be a classical modal algebra. Then the map $\beta : A \to A^{\mathbb{A}}$ is a modal isomorphism from $\mathbb{A}$ to $(\mathbb{A}_*)^*$.*

Proof. For a modal algebra $\mathbb{A} = \langle A, \Box \rangle$, we have $(\mathbb{A}_*)^* = \langle \beta[A], \cap, \cup, \setminus, \emptyset, \mathrm{Uf}(\mathbb{A}), m_{\nu_\Box} \rangle$.

First of all, we need that β is a correct modal homomorphism, i.e. preserves all operations and constants. However, the preservation of all operations has been shown in the proof of Lemma 3.16. For the constants, note that 1 belongs to every ultrafilter on $\mathbb{A}$ and 0 belongs to none and hence $\beta(0) = \emptyset$ and $\beta(1) = \mathrm{Uf}(\mathbb{A})$.

We now show that β is a bijection. To show injectivity, take $a, b \in A$ such that $a \neq b$. If $a \leq b$, we construct an ultrafilter U from the element $\neg a \wedge b \neq 0$. If $a \not\leq b$ we use [18, Corollary 3 (p. 173)] to find an ultrafilter U on $\mathbb{A}$ containing a but not b. In both cases $U \in \beta(a)$ and $U \notin \beta(b)$, so $\beta(a) \neq \beta(b)$. Surjectivity follows by definition, as if we take some $X \in \beta[A]$, there is some $a \in A$ such that $X = \beta(a)$. □

We also require an isomorphism between a frame $\mathbb{F}$ and its corresponding frame $(\mathbb{F}^*)_*$. For a general neighbourhood frame $\mathbb{F} = \langle W, \nu, A \rangle$ we define a function $\alpha : W \to \mathcal{P}A$ as $\alpha(w) = \{X \in A \mid w \in X\}$. The following proposition will show that this function is a well-defined function from $\mathbb{F}$ to $(\mathbb{F}^*)_*$.

Proposition 3.18. *Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a general neighbourhood frame and $w \in W$ a world. Then $\alpha(w)$ is an ultrafilter of $\mathbb{F}^*$.*

Proof. We take $w \in W$ and prove all the properties of an ultrafilter for $\alpha(w)$, as stated in Definition 3.14. As $W \in A$ and $w \notin \emptyset$, we have $W \in \alpha(w)$ and $\emptyset \notin \alpha(w)$. To show that $\alpha(w)$ is an upset, consider $X \in \alpha(w)$ such that $X \subseteq Y$. As $w \in X \subseteq Y$, we have $Y \in \alpha(w)$. For closure under finite intersections, consider $X, Y \in \alpha(w)$. Then $w \in X \cap Y$, so $X \cap Y \in \alpha(w)$. For maximality, consider any $X \in A$. If $w \in X$, we obtain $X \in \alpha(w)$. If not, we have $w \in W \setminus X$, so $W \setminus X \in \alpha(w)$. □

We now would like to show that α is a frame isomorphism between $\mathbb{F}$ to $(\mathbb{F}^*)_*$. However, this does not hold in general. We define the following properties of a general neighbourhood frame. We will see that these properties suffice for α to be a frame isomorphism.

Definition 3.19 (**Descriptive Neighbourhood Frame**). Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a general neighbourhood frame. We define the following properties:

- $\mathbb{F}$ is *differentiated* when for all $X \in A$: $(w_1 \in X \Leftrightarrow w_2 \in X)$ implies $w_1 = w_2$.

- $\mathbb{F}$ is *compact* if every subset of A with the finite intersection property has a non-empty intersection. Here a subset $X \subseteq A$ has the finite intersection property (FIP) if the intersection over any finite collection of subsets of X has a non-empty intersection.
- $\mathbb{F}$ is *descriptive* if it is differentiated and compact.
- $\mathbb{F}$ is *full* when $A = \mathcal{P}W$. ⊣

Remark 3.20. Descriptiveness for Kripke frames requires a frame to satisfy differentiatedness, compactness and a property called tightness, forcing a condition on the accessibility relation [9, 12]. Similarly, Hansen [20] requires descriptive monotonic neighbourhood frames to satisfy a form of tightness. We do not force such a property on the descriptive neighbourhood frames. The requirement that all neighbourhoods are admissible suffices. We will see this in the proof of Theorem 3.23. ⊣

We first show a small proposition regarding differentiatedness that will be very useful in the future. By this proposition, we can denote any finite differentiated neighbourhood frame as $\mathbb{F} = \langle W, \nu \rangle$, omitting $A = \mathcal{P}W$.

Proposition 3.21. *Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a finite differentiated neighbourhood frame. Then $A = \mathcal{P}W$.*

Proof. We show that for any $w \in W$, $\{w\} \in A$. By closure under finite union of A, this completes the argument. Consider $w \in W$. By $\mathbb{F}$ being differentiated, there exists $X_v \in A$ such that $w \in X_v$ and $v \notin X_v$ for every $v \neq w$. Now note that $\bigcap_{v \neq w} X_v = \{w\}$, which is an element of A by closure under finite intersections. □

To see that differentiatedness and compactness suffice for α to be a frame isomorphism, we show two claims. First we show that these two properties imply that α is a bijection.

Lemma 3.22. *Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a descriptive neighbourhood frame. Then α is a bijection from $\mathbb{F}$ to $(\mathbb{F}^*)_*$.*

Proof. For injectivity, consider $\alpha(w_1) = \alpha(w_2)$, i.e. $\{X \in A \mid w_1 \in X\} = \{X \in A \mid w_2 \in X\}$. This means we exactly have $w_1 \in X$ if and only if $w_2 \in X$ for every $X \in A$. As $\mathbb{F}$ is differentiated, we obtain $w_1 = w_2$.

For surjectivity, consider $U \in \mathrm{Uf}(\mathbb{A}_{\mathbb{F}})$. Note that every filter on $\mathbb{A}_{\mathbb{F}}$ satisfies the finite intersection property by being closed under finite intersections and being proper. $\mathbb{F}$ being compact now gives $\bigcap U \neq \emptyset$ so there exists some $w \in \bigcap U$. We claim that $\alpha(w) = U$. The inclusion from right to left follows straight from the definition. For the inclusion from left to right, take $X \in \alpha(w)$, i.e. $w \in X$. Suppose for a contradiction that $X \notin U$. By maximality of U, $W \setminus X \in U$. But then $w \notin W \setminus X$. Therefore w cannot be in the intersection of U, which is a contradiction. □

So indeed, α is a bijection whenever the frame is descriptive. The following theorem establishes that it is indeed a frame isomorphism.

Theorem 3.23. *Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a descriptive neighbourhood frame. Then α is a frame isomorphism between $\mathbb{F}$ and $(\mathbb{F}^*)_*$.*

Proof. Note that for $\mathbb{F} = \langle W, \nu, A\rangle$, we have $(\mathbb{F}^*)_* = \langle \mathrm{Uf}(\mathbb{F}^*), \nu_{m_\nu}, \beta[A]\rangle$. We know by Lemma 3.22 that α is a bijection.

We first show that α is a general frame morphism, i.e. that $\alpha^{-1}[Y] \in \nu(w) \Leftrightarrow Y \in \nu_{m_\nu}(\alpha(w))$ for all $Y \in \beta[A]$. To do so, we consider $Y \in \beta[A]$, i.e. $Y = \beta(X)$ for some $X \in A$. We note the following equalities.

$$\begin{aligned}
\alpha^{-1}[\beta(X)] &= \{w \in W \mid \alpha(w) \in \beta(X)\} \\
&= \{w \in W \mid X \in \alpha(w)\} \\
&= \{w \in W \mid w \in X\} = X. \\
\nu_{m_\nu}(\alpha(w)) &= \{\beta(X) \mid m_\nu(X) \in \alpha(w)\} \\
&= \{\beta(X) \mid w \in m_\nu(X)\} \\
&= \{\beta(X) \mid X \in \nu(w)\}.
\end{aligned}$$

From these equalities, the required equivalence easily follows. Secondly, we need that for all $Y \in \beta[A]$, $\alpha^{-1}[Y] \in A$. For this, we can also use the equality of $\alpha^{-1}[\beta(X)] = X$. As before, $Y = \beta(X)$ for some $X \in A$, hence $\alpha^{-1}[Y] = X \in A$.

The inverse of α also needs to be a frame morphism. We show that $(\alpha^{-1})^{-1}[X] \in \nu_{m_\nu}(U)$ iff $X \in \nu(\alpha^{-1}U)$ for each $X \in A$ and $U \in \mathrm{Uf}(\mathbb{F}^*)$. Note that by α being a bijection, we can rewrite this as $\alpha[X] \in \nu_{m_\nu}(\alpha(w))$ iff $X \in \nu(w)$ for all $X \in A$ and $w \in W$.

To show this, we argue that $\alpha[X] = \beta(X)$ for $X \in A$. For the left-to-right inclusion, take $\alpha(w) \in \alpha[X]$. Then $w \in X$ and thus $X \in \alpha(w) \in \beta(X)$. For the inclusion from right to left, take an ultrafilter $U \in \beta(X)$, i.e. $X \in U$. Now by α being a bijection, there is unique $w \in W$ such that $\alpha(w) = U$. This means that $w \in \bigcap U$ and as $X \in U$, $w \in X$. So we have $U = \alpha(w) \in \alpha[X]$. With this taken care of, we easily obtain $\alpha[X] \in \nu_{m_\nu}(\alpha(w)) = \{\beta(X) \mid X \in \nu(w)\}$ iff $X \in \nu(w)$. □

Remark 3.24. In both the algebraic duality for descriptive Kripke frames [9, 12] and the duality result for descriptive monotonic neighbourhood frames as presented by Hansen [20], the notion of tightness is used when showing that $\mathbb{F} \cong (\mathbb{F}^*)_*$. When we show this in the theorem above, the proof is easy as we only need to deal with neighbourhoods $X \in \nu(w)$ that are admissible, i.e. $X \in A$. Difficulties would arise when we would also need to take care of neighbourhoods that are not admissible. By forcing all neighbourhoods to be admissible, we exclude this possibility. ⊣

Lastly, we need to check that our construction that transforms a classical modal algebra to a general neighbourhood frame, actually gives us a descriptive frame.

Lemma 3.25. *For classical modal algebra $\mathbb{A} = \langle A, \Box\rangle$, $\mathbb{A}_*$ is a descriptive neighbourhood frame.*

Proof. From Lemma 3.25, we know that $\mathbb{A}_*$ is a general neighbourhood frame. To show that $\mathbb{A}_*$ is differentiated, take $U, V \in \mathrm{Uf}(\mathbb{A})$ such that $U \neq V$. Then without loss of generality, there exists $a \in U$ such that $a \notin V$. But then $U \in \beta(a)$ and $V \notin \beta(a)$.

To show that $\mathbb{A}_*$ is compact, take $X \subseteq A^{\mathbb{A}} = \beta[A]$ satisfying (FIP). Note that $X = \beta[B]$ for some $B \subseteq A$. From X satisfying (FIP), it easily follows that B has the finite meet property. Such a set B can be expanded to a filter, which in turn can be extended to un ultrafilter U containing B by [18, Theorem 12 (p. 172)]. Then $U \in \bigcap \beta[B]$, hence $\bigcap X \neq \emptyset$. □

This establishes a correspondence between classical modal algebras and descriptive neighbourhood frames.

3.4.2 Correspondence for Maps

In this section we will review the correspondence between the morphisms of classical modal algebras and descriptive neighbourhood frames, i.e. between modal homomorphisms and general frame morphisms. More specifically, we will obtain a correspondence between the notions of stability and the Co-Stable Domain Condition. From this, the correspondence between the morphisms on both sides follows easily. We will end this section by stating the duality between the categories of descriptive neighbourhood frames with frame morphisms and classical modal algebras with modal homomorphisms.

We will establish the correspondence using inverses. For a map f between frames $\mathbb{F}_1 = \langle W_1, \nu_1, A_1\rangle$ and $\mathbb{F}_2 = \langle W_2, \nu_2, A_2\rangle$, we define f^* to be the inverse of f, i.e. a map from A_2 to A_1 defined as $f^*(X_2) = \{w \in W_1 \mid f(w) \in X_2\}$. Likewise, for a map h from algebra $\mathbb{A} = \langle A, \Box_A\rangle$ to $\mathbb{B} = \langle B, \Box_B\rangle$, we define h_* to be the map $h_* : \mathrm{Uf}(\mathbb{B}) \to \mathrm{Uf}(\mathbb{A})$ defined as $h_*(U) = \{a \in A \mid h(a) \in U\}$.

Theorem 3.26. *Let $\mathbb{F}_1 = \langle W_1, \nu_1, A_1\rangle$ and $\mathbb{F}_2 = \langle W_2, \nu_2, A_2\rangle$ be two descriptive neighbourhood frames and $f : W_1 \to W_2$ a map. Then we have the following equivalences, for $D \subseteq A_2$:*

(1) *f is a stable morphism iff $f^* : A_2 \to A_1$ is stable.*
(2) *f satisfies* (CoSDC) *for D iff $f^* : A_2 \to A_1$ satisfies* (CoSDC) *for D.*
(3) *f is a general frame morphism iff $f^* : A_2 \to A_1$ is a modal homomorphism.*

Proof. For descriptive neighbourhood frames $\mathbb{F}_1 = \langle W_1, \nu_1, A_1\rangle$ and $\mathbb{F}_2 = \langle W_2, \nu_2, A_2\rangle$, we have $\mathbb{F}_1^* = \langle A_1, \cap, \cup, \setminus, \emptyset, W_1, m_{\nu_1}\rangle$ and $\mathbb{F}_2^* = \langle A_2, \cap, \cup, \setminus, \emptyset, W_2, m_{\nu_2}\rangle$. Note that for continuous f, $f^* : A_2 \to A_1$ is a well-defined function.

(1) For the direction from left to right, suppose that f is stable. It follows straight from the definition that f^* preserves all set-theoretic operations so f^* is a Boolean homomorphism. To show that f^* is stable, we want that $f^*(m_{\nu_2}X) \subseteq m_{\nu_1}(f^*(X))$. Take $w \in f^*(m_{\nu_2}X)$. Then $f(w) \in m_{\nu_2}X$, so $X \in \nu_2(f(w))$. By f being stable, we get $f^*(X) \in \nu_1(w)$, so $w \in m_{\nu_1}(f^*(X))$.

For the other direction, suppose that f^* is stable. We consider $X \in \nu_2(f(w))$. Then $f(w) \in m_{\nu_2}(X)$, hence $w \in f^*(m_{\nu_2}(X))$. From the stability of f^*, we obtain $w \in m_{\nu_1}f^*(X))$. Therefore $f^*(X) \in \nu_1(w)$.

(2) For the direction from left to right, suppose that f satisfies (CoSDC) for D. It again follows from the definition that f^* preserves all set-theoretic operations. For the modality case, we show that $m_{\nu_1}(f^*(X)) \subseteq f^*(m_{\nu_2}(X))$ for all $X \in D$. Consider $w \in m_{\nu_1}(f^*(X))$ for $X \in D$. Then $f^*(X) \in \nu_1(w)$. By f satisfying (CoSDC) for D, we obtain $X \in \nu_2(f(w))$. Then $f(w) \in m_{\nu_2}(X)$ and thus $w \in f^*(m_{\nu_2}(X))$.

Conersely, suppose that f^* satisfies (CoSDC) for D. We consider $f^*(X) \in \nu_1(w)$ for $X \in D$. Then $w \in m_{\nu_1}(f^*(X))$. By f^* satisfying (CoSDC) for D, we have $w \in f^*(m_{\nu_2}(X))$. Then $f(w) \in m_{\nu_2}(X)$, therefore $X \in \nu_2(f(w))$.

(3) By (1) and (2) of this theorem, it easily follows that f is a general frame morphism if and only if f^* is a modal homomorphism. □

Theorem 3.27. *Let $\mathbb{A} = \langle A, \Box_A\rangle$ and $\mathbb{B} = \langle B, \Box_B\rangle$ be two classical modal algebras and $h : A \to B$ a map. Then we have the following two equivalences, for $D \subseteq A$:*

(1) *h is a stable morphism iff $h_* : Uf(\mathbb{B}) \to Uf(\mathbb{A})$ is stable.*
(2) *h satisfies* (CoSDC) *for D iff $h_* : Uf(\mathbb{B}) \to Uf(\mathbb{A})$ satisfies* (CoSDC) *for $\beta[D]$.*
(3) *h is a modal homomorphism iff $h_* : Uf(\mathbb{B}) \to Uf(\mathbb{A})$ is a general frame morphism.*

Proof. For classical modal algebras $\mathbb{A} = \langle A, \Box_A\rangle$ and $\mathbb{B} = \langle B, \Box_B\rangle$, we have $\mathbb{A}_* = \langle \mathrm{Uf}(\mathbb{A}), \nu_{\Box_A}, \beta[A]\rangle$ and $\mathbb{B}_* = \langle \mathrm{Uf}(\mathbb{B}), \nu_{\Box_B}, \beta[B]\rangle$. It is an easy check that for each $V \in \mathrm{Uf}(\mathbb{B})$, $h_*(V) \in \mathrm{Uf}(\mathbb{A})$ and so h_* is well-defined. To show that h_* is continuous, we consider $a \in A$ and argue as follows:

$$\begin{aligned} h_*^{-1}[\beta(a)] &= \{V \in \mathrm{Uf}(\mathbb{B}) \mid h_*(V) \in \beta(a)\} \\ &= \{V \in \mathrm{Uf}(\mathbb{B}) \mid h(a) \in V\} = \beta(h(a)) \in \beta[B]. \end{aligned}$$

(1) For the direction from left to right, suppose that h is stable. We consider $X \in \nu_{\Box_A}(h_*(V))$ for $V \in \mathrm{Uf}(\mathbb{B})$. By definition of $\nu_{\Box_A}$, $X = \beta(a)$ for some $a \in A$ such that $h(\Box_A a) \in V$. As h is stable and V an upset, $\Box_B h(a) \in V$. Then $\beta(h(a)) \in \nu_{\Box_B}(V)$, hence $h_*^{-1}(X) \in \nu_{\Box_B}(V)$.

Conversely, suppose that h_* is stable. To show that $h(\Box_A a) \leq \Box_B h(a)$, we show that $\beta(h(\Box_A a)) \subseteq \beta(\Box_B h(a))$, as β is an isomorphism between $\mathbb{A}$ and $(\mathbb{A}_*)^*$. We consider $V \in \beta(h(\Box_A a))$, giving $h(\Box_A a) \in V$. Then $\Box_A a \in h_*(V)$, hence $\beta(a) \in \nu_{\Box_A}(h_*(V))$. By stability of h_*, we obtain $h_*^{-1}[\beta(a)] = \beta(h(a)) \in \nu_{\Box_B}(V)$, hence $\Box_B h(a) \in V$ and therefore $V \in \beta(\Box_B h(a))$.

(2) For the direction from left to right, suppose that h satisfies (CoSDC) for D. We consider $\beta(a) \in \beta[D]$ such that $h_*^{-1}[\beta(a)] = \beta(h(a)) \in \nu_{\Box_B}(V)$. Then $\Box_B h(a) \in V$, which implies $h(\Box_A a) \in V$ by h satisfying (CoSDC) for a. Then we obtain $\Box_A a \in h_*(V)$, which implies $\beta(a) \in \nu_{\Box_A}(h_*(V))$.

For the other direction, suppose that h_* satisfies (CoSDC) for $\beta[D]$. We want that $\Box_B h(a) \leq h(\Box_A a)$, for which we show that $\beta(\Box_B h(a)) \subseteq \beta(h(\Box_A a))$. We consider $V \in \beta(\Box_B h(a))$, which implies $\Box_B h(a) \in V$. Then we have $\beta(h(a)) \in \nu_{\Box_B}(V)$, so by h_* satisfying (CoSDC) for $\beta[D]$, we obtain $\beta(a) \in \nu_{\Box_A}(h_*(V))$. Then we obtain $\Box_A a \in h_*(V)$, which implies $h(\Box_A a) \in V$, hence $V \in \beta(h(\Box_A a))$.

(3) From (1) and (2), we get that h is a modal homomorphism if and only if h_* is a general frame morphism. $\Box$

To finalize the duality, we need to prove that the functors $(\cdot)^*$ and $(\cdot)_*$ we defined give rise to natural transformations. We let $\alpha_{\mathbb{F}}$ denote the previously defined function $\alpha : \mathbb{F} \to (\mathbb{F}^*)_*$ and write $\beta_{\mathbb{A}}$ for the function $\beta : \mathbb{A} \to (\mathbb{A}_*)^*$.

Proposition 3.28. *Let $f : \mathbb{F} \to \mathbb{G}$ be a frame morphism between descriptive neighbourhood frames $\mathbb{F}$ and $\mathbb{G}$ and $h : \mathbb{A} \to \mathbb{B}$ a modal homomorphism between classical modal algebras $\mathbb{A}$ and $\mathbb{B}$. Then the following two diagrams commute:*

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{\alpha_{\mathbb{F}}} & (\mathbb{F}^*)_* \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (f^*)_*} \\ \mathbb{G} & \xrightarrow{\alpha_{\mathbb{G}}} & (\mathbb{G}^*)_* \end{array} \qquad \begin{array}{ccc} \mathbb{A} & \xrightarrow{\beta_{\mathbb{A}}} & (\mathbb{A}_*)^* \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (h_*)^*} \\ \mathbb{B} & \xrightarrow{\beta_{\mathbb{B}}} & (\mathbb{B}_*)^*. \end{array}$$

Proof. For the first diagram, let $\mathbb{F} = \langle W, \nu, A \rangle$ and $\mathbb{G} = \langle V, \mu, B \rangle$ be two descriptive neighbourhood frames. Now consider $w \in W$.

$$\begin{aligned}((f^*)_*) \circ \alpha_{\mathbb{F}})(w) &= (f^*)_*(\alpha_{\mathbb{F}}(w)) \\ &= \{U \subseteq V \mid f^{-1}[U] \in \alpha_{\mathbb{F}}(w)\} \\ &= \{U \subseteq V \mid w \in f^{-1}[U]\} \\ &= \alpha_{\mathbb{G}}(f(w)) \\ &= (\alpha_{\mathbb{G}} \circ f)(w).\end{aligned}$$

For the second diagram, let $\mathbb{A} = \langle A, \Box_A \rangle$ and $\mathbb{B} = \langle B, \Box_B \rangle$ be two classical modal algebras. Consider $a \in A$.

$$\begin{aligned}((h_*)^*) \circ \beta_{\mathbb{A}})(a) &= (h_*)^*(\beta_{\mathbb{A}}(a)) \\ &= \{V \in \mathrm{Uf}(\mathbb{B}) \mid h^{-1}[V] \in \beta_{\mathbb{A}}(a)\} \\ &= \{V \in \mathrm{Uf}(\mathbb{B}) \mid a \in h^{-1}[V]\} \\ &= \{V \in \mathrm{Uf}(\mathbb{B}) \mid h(a) \in V\} \\ &= \beta_{\mathbb{B}}(h(a)) \\ &= (\beta_{\mathbb{B}} \circ h)(a).\end{aligned}$$

□

We can now claim the following duality result.

Theorem 3.29. *The category of descriptive neighbourhood frames with general frame morphisms is dually equivalent to the category of classical modal algebras with modal homomorphisms.*

The duality allows us to carry over the algebraic completeness results from Theorem 3.8. For this, note that validity is preserved between the two dual structures. For descriptive neighbourhood frame $\mathbb{F} = \langle W, \nu, A \rangle$ with admissible valuation $V : \mathsf{Prop} \to A$, the same V is also a valuation on the classical modal algebra $\mathbb{F}^*$. For classical modal algebra $\mathbb{A} = \langle A, \Box \rangle$ with valuation $V : \mathsf{Prop} \to A$ we can define valuation $V_* : \mathsf{Prop} \to \beta[A]$ as $V_*(p) = \beta(V(p))$. An easy induction on the complexity of ϕ now gives $\langle \mathbb{F}, V \rangle \vDash \phi$ iff $\langle \mathbb{F}^*, V \rangle \vDash \phi$ and $\langle \mathbb{A}, V \rangle \vDash \phi$ iff $\langle \mathbb{A}_*, V_* \rangle \vDash \phi$.

Theorem 3.30.

(1) *Let Λ be a classical modal logic. Then Λ is sound and complete with respect to the class of all descriptive neighbourhood frames on which Λ is valid.*

(2) *Let $\mathcal{S}$ be a classical modal multi-conclusion consequence relation. Then $\mathcal{S}$ is sound and complete with respect to the class of all descriptive neighbourhood frames on which $\mathcal{S}$ is valid.*

3.5 Monotonic Duality

With the duality established for classical modal logic, this section will briefly show how to adapt this duality to the setting of monotonic modal logic. A duality for monotonic modal algebras is extensively studied in Hansen [20], but as remarked before, Hansen does not assume neighbourhoods to be admissible. For this reason, a restriction of tightness needs to be put on the general neighbourhood frames. We give a duality in which tightness is again captured in the admissibility of the neighbourhoods.

We define monotonic general neighbourhood frames as done by Kracht and Wolter in [23].

Definition 3.31 (General Monotonic Neighbourhood Frame). A *general monotonic neighbourhood frame* is a tuple $\mathbb{F} = \langle W, \nu, A\rangle$ that is a general neighbourhood frame satisfying the following condition:

$$\forall X, Y \in A : \big[X \in \nu(w) \text{ and } X \subseteq Y\big] \Rightarrow Y \in \nu(w). \qquad \dashv$$

By our requirement of the neighbourhoods to be admissible, we need to restrict the upwards closure of the neighbourhoods to upwards closure within A. Now a general monotonic neighbourhood frame is *descriptive* if it is differentiated and compact. Modal homomorphisms between monotonic modal algebras will simply be modal homomorphisms between classical modal algebras. This means we can follow the exact same procedure as followed in the proof for classical modal algebras and descriptive neighbourhood frames. We need to check only two claims, namely that for descriptive monotonic neighbourhood frame $\mathbb{F}$, its corresponding algebra $\mathbb{F}^*$ is monotonic and for monotonic modal algebra $\mathbb{A}$, the frame $\mathbb{A}_*$ is monotonic. The following lemma establishes this.

Lemma 3.32. *Let $\mathbb{F} = \langle W, \nu, A\rangle$ be a descriptive monotonic neighbourhood frame and $\mathbb{A} = \langle A, \Box\rangle$ a monotonic modal algebra. Then $\mathbb{F}^*$ is a monotonic modal algebra and $\mathbb{A}_*$ a descriptive monotonic neighbourhood frame.*

Proof. Firstly, we have $\mathbb{F}^* = \langle A, m_\nu\rangle$. By Lemma 3.16, $\mathbb{F}^*$ is a classical modal algebra. We show that $a \subseteq b$ implies $m_\nu a \subseteq m_\nu b$ for $a, b \in A$. Take $w \in m_\nu a$, meaning $a \in \nu(w)$. By $\mathbb{F}$ being monotonic we obtain $b \in \nu(w)$ implying $w \in m_\nu b$.

Secondly, note that $\mathbb{A}_* = \langle \mathrm{Uf}(\mathbb{A}), \nu_\Box, \beta[A]\rangle$. By Lemma 3.25 $\mathbb{A}_*$ is a descriptive neighbourhood frame. To show it is monotonic, consider $\beta(a), \beta(b) \in \beta[A]$ such that $\beta(a) \in \nu_\Box(U)$ and $\beta(a) \subseteq \beta(b)$. By β being an isomorphism, we obtain $a \leq b$. As $\mathbb{A}$ is monotonic this implies $\Box a \leq \Box b$. From $\beta(a) \in \nu_\Box(U)$ we obtain $\Box a \in U$ which by U being an upset implies $\Box b \in U$, i.e. $\beta(b) \in \nu_\Box(U)$. $\Box$

Theorem 3.33. *The categories of descriptive monotonic neighbourhood frames with general frame morphisms and of monotonic modal algebras with modal homomorphisms are dually equivalent.*

Proof. Using the fact that the morphisms on the frames and algebras do not change, we can refer to the proofs of Theorems 3.26 and 3.27 to show the duality for morphisms. For the duality for objects, we refer to the proofs of Theorems 3.23 and 3.17 together with Lemma 3.32 above. $\Box$

Remark 3.34. It is important to note the difference with the duality result given by Hansen in [20]. The duality we present is of a different nature. Hansen enforces a tightness on the descriptive frames. The approach is tailored to resemble the case of descriptive Kripke frames as closely as possible, where such a tightness condition is also present. For our purposes, simply forcing the neighbourhoods to be admissible suffices. $\dashv$

We conclude this section by using the algebraic completeness results from Theorem 3.10 to obtain a completeness of monotonic modal multi-conclusion consequence relations and logics with respect to descriptive monotonic neighbourhood frames.

Theorem 3.35.

(1) *Let Λ be a monotonic modal logic. Then Λ is sound and complete with respect to the class of all descriptive monotonic neighbourhood frames on which Λ is valid.*

(2) *Let $\mathcal{S}$ be a monotonic modal multi-conclusion consequence relation. Then $\mathcal{S}$ is sound and complete with respect to the class of all descriptive monotonic neighbourhood frames on which $\mathcal{S}$ is valid.*

3.6 Filtrations Revisited

From this point forward, we will be working with descriptive neighbourhood frames. We have already introduced their morphisms in Definition 3.13, which turn out to be simply morphisms for non-general neighbourhood frames with the added condition of continuity. In this section, we discuss filtration of descriptive neighbourhood models. As is the case with morphisms, we merely add the notion of continuity.

Definition 3.36. Let $\mathbb{M} = \langle W, \nu, A, V\rangle$ be a descriptive neighbourhood model and Σ a subformula closed set of modal formulas. A descriptive neighbourhood model $\mathbb{M}^f = \langle W^f, \nu^f, A^f, V^f\rangle$ is a *filtration of* $\mathbb{M}$ *through* Σ if $\langle W^f, \nu^f, V^f\rangle$ is a filtration of $\langle W, \nu, V\rangle$ through Σ and the equivalence map $|\cdot|$ is continuous. ⊣

The Filtration Lemma for this definition of filtration is a direct consequence of the Filtration Lemma for filtrations for non-general neighbourhood models (Lemma 2.24). We now only need to prove the existence of these filtrations. We restrict ourselves to the case when Σ is finite, as this is the case we will be interested in. The following proposition expresses that we can always pick A^f to be $\mathcal{P}W^f$ when Σ is finite to obtain a filtration.

Proposition 3.37. *Let $\mathbb{M} = \langle W, \nu, A, V\rangle$ be a general neighbourhood model and Σ a finite subformula closed set of modal formulas. If $\langle W^f, \nu^f, V^f\rangle$ is a filtration of $\langle W, \nu, V\rangle$ through Σ, then $\mathbb{M}^f = \langle W^f, \nu^f, \mathcal{P}W^f, V^f\rangle$ is a filtration of $\mathbb{M}$ through Σ.*

Moreover, $\mathbb{M}^f$ is descriptive.

Proof. It is easy to see that $\mathbb{M}^f$ is a well-defined general neighbourhood model. We now need to show that the equivalence map $|\cdot|$ is continuous. For this, take any singleton $\{|w|\} \in \mathcal{P}W^f$. We show that $\nmid\{|w|\}\nmid \in A$ for each such singleton. By A being closed under finite union and $\mathbb{M}^f$ being finite, this finishes the argument. Now note that $\nmid\{|w|\}\nmid = |w|$. By definition of $\sim_\Sigma$ we have that:

$$|w| = V(\bigwedge\{\phi \in \Sigma \mid \mathbb{M}, w \vDash \phi\} \wedge \neg \bigvee\{\phi \in \Sigma \mid \mathbb{M}, w \vDash \neg\phi\}).$$

As $V(\phi) \in A$ for each modal formula ϕ, we obtain $|w| \in A$.

Now $\mathbb{M}^f$ is descriptive simply because it is finite, where compactness follows trivially and differentiatedness follows from Proposition 3.21. □

As filtrations for non-general neighbourhood models exist by Lemma 2.25, the proposition above shows that filtrations for general neighbourhood models exist as well when Σ is finite. Moreover, the filtrated model is descriptive. In the next chapter, we explicitly use the filtration of a descriptive neighbourhood frame to define finite refutation patterns for modal rules.

Chapter 4

Stable Canonical Rules

In this chapter we introduce stable canonical rules. We study logics characterized by these stable canonical rules and show that any classical modal multi-conclusion consequence relation as well as any classical modal logic can be axiomatized by these stable canonical rules. For this, we first introduce finite refutation patterns for any modal rule. The stable canonical rule is constructed by encoding these patterns. In Section 4.5 we replicate the results for monotonic modal multi-conclusion consequence relations and logics.

We will be generalizing results from [5], where stable canonical rules and finite refutation patterns are introduced for the case of Kripke semantics. We will adapt the proofs to our case of classical modal logics. Unlike [5] where an algebraic point of view is adopted, we will prove the results fully frame-theoretically.

4.1 Finite Refutation Patterns

In this section we introduce finite refutation patterns for each modal rule ρ. Intuitively, a finite refutation pattern is a (finite) collection of finite neighbourhood frames that characterize exactly when a descriptive neighbourhood frame $\mathbb{G}$ refutes the rule ρ. Frame $\mathbb{G}$ will refute ρ when there exists a stable map from $\mathbb{G}$ onto one of the frames in the refutation pattern satisfying (CoSDC) for some subset D. The existence of such a map we will capture by a stable canonical rule. This allows us to say that rule ρ is refuted if and only if the stable canonical rules for its finite refutation pattern are refuted.

Theorem 4.1.

(1) *For each multi-conclusion modal rule* Γ/Δ*, there exist pairs* $\langle \mathbb{F}_1, D_1\rangle, \dots, \langle \mathbb{F}_n, D_n\rangle$ *such that each* $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ *is a finite neighbourhood frame refuting* Γ/Δ*,* $D_i \subseteq \mathcal{P}W_i$*, and for each descriptive neighbourhood frame* $\mathbb{G} = \langle W, \nu, A\rangle$*, we have* $\mathbb{G} \not\models \Gamma/\Delta$ *iff there exists* $i \leq n$ *and a stable surjective map* $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$ *satisfying* (CoSDC) *for* D_i*.*

(2) *For each modal formula* ϕ*, there exist pairs* $\langle \mathbb{F}_1, D_1\rangle, \dots, \langle \mathbb{F}_n, D_n\rangle$ *such that each* $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ *is a finite descriptive neighbourhood frame refuting* ϕ*,* $D_i \subseteq \mathcal{P}W_i$*, and for each descriptive neighbourhood frame* $\mathbb{G} = \langle W, \nu, A\rangle$*, we have* $\mathbb{G} \not\models \phi$ *iff there exists* $i \leq n$ *and stable surjective map* $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$ *satisfying* (CoSDC) *for* D_i*.*

Proof. (1) If $\mathbf{S_E} \vdash \Gamma/\Delta$, we take $n = 0$. Now assume $\mathbf{S_E} \nvdash \Gamma/\Delta$. We let Θ be the set of all subformulas of the formulas in $\Gamma \cup \Delta$. Let m be the cardinality of Θ. Now note that there exist finitely many pairs $\langle \mathbb{F}, D\rangle$ such that:

- $\mathbb{F}$ is a finite neighbourhood frame of size $\leq 2^m$ and $\mathbb{F} \nvDash \Gamma/\Delta$.
- $D = \{V(\psi) \mid \Box\psi \in \Theta\}$ where V is a valuation on $\mathbb{F}$ witnessing $\mathbb{F} \nvDash \Gamma/\Delta$.

Let $\langle \mathbb{F}_1, D_1\rangle, \ldots, \langle \mathbb{F}_n, D_n\rangle$ be the enumeration of such pairs. Now take any descriptive neighbourhood frame $\mathbb{G} = \langle W, \nu, A\rangle$. We show both directions of the equivalence.

($\Rightarrow$) Suppose $\mathbb{G} \nvDash \Gamma/\Delta$. This means there exists valuation V on $\mathbb{G}$ witnessing that $\mathbb{G} \nvDash \Gamma/\Delta$. We let $\langle \mathbb{G}', V'\rangle$ be a filtration of $\langle \mathbb{G}, V\rangle$ through Θ. Note that $|\mathbb{G}'| \leq 2^m$. By the Filtration Lemma and V witnessing $\mathbb{G} \nvDash \Gamma/\Delta$, it follows that V' is a witness of $\mathbb{G}' \nvDash \Gamma/\Delta$. This means that $\mathbb{G}' = \mathbb{F}_i$ for some $i \leq n$. Note that the surjective filtration map $|\cdot| : \mathbb{G} \to \mathbb{F}_i$ is stable and satisfies (CoSDC) for $D_i = \{V'(\psi) \mid \Box\psi \in \Theta\}$ by Proposition 2.27.

($\Leftarrow$) Suppose there exists $i \leq n$ and map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$ that is stable and satisfies (CoSDC) for D_i. We define a valuation V_G on $\mathbb{G}$ as $V_G(p) = f^{-1}[V_i(p)]$ for each $p \in \mathsf{Prop}$. By f being continuous, this valuation is well defined. We can now show by induction on the complexity of the formula, similar to the proof of the Filtration Lemma (Lemma 2.24), that $\langle \mathbb{F}_i, V_i\rangle, f(w) \vDash \phi$ if and only if $\langle \mathbb{G}, V_G\rangle, w \vDash \phi$ for all $\phi \in \Theta$ and $w \in W$. As $\mathbb{F}_i \nvDash \Gamma/\Delta$, we obtain $\mathbb{G} \nvDash \Gamma/\Delta$.

(2) If $\mathbf{E} \vdash \phi$, then we take $n = 0$. Otherwise, for a descriptive neighbourhood frame $\mathbb{G}$ we have $\mathbb{G} \vDash \phi$ iff $\mathbb{G} \vDash /\phi$ and therefore $\mathbf{E} \nvdash \phi$ iff $\mathbf{S_E} \nvdash /\phi$ by the completeness results for descriptive neighbourhood frames (Theorem 3.30). We then apply (1). $\square$

Now that we have shown the existence of the finite refutation patterns, we will make the connection with rules. We will define a *stable canonical rule* $\sigma(\mathbb{F}, D)$ dependent on a finite neighbourhood frame $\mathbb{F} = \langle W, \nu\rangle$ and a subset $D \subseteq \mathcal{P}W$. We define it so that a descriptive neighbourhood frame $\mathbb{G}$ validates $\sigma(\mathbb{F}, D)$ iff there exists a stable onto map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}$ satisfying (CoSDC) for D.

Definition 4.2 (**Stable Canonical Rules**)**.** Let $\mathbb{F} = \langle W, \nu\rangle$ be a finite descriptive neighbourhood frame. We introduce a propositional letter p_w for each $w \in W$ and a letter s_X for each subset $X \subseteq W$. For each $D \subseteq \mathcal{P}W$, we define the *stable canonical rule* $\sigma(\mathbb{F}, D)$ to be the rule Γ/Δ, where:

$$
\begin{aligned}
\Gamma \quad &= \quad \{\bigvee_{w\in W} p_w\} \cup \{p_w \to \neg p_v \mid w, v \in W, w \neq v\} \cup \\
&\quad\quad \{s_X \leftrightarrow \bigvee_{w\in X} p_w \mid X \subseteq W\} \cup \\
&\quad\quad \{s_{\Box X} \to \Box s_X \mid X \subseteq W\} \cup \\
&\quad\quad \{\Box s_X \to s_{\Box X} \mid X \in D\} \\
\Delta \quad &= \quad \{\neg p_w \mid w \in W\}.
\end{aligned}
$$

$\dashv$

Remark 4.3. An important difference with the way canonical rules are defined in [22, 5] is the addition of propositional letters for each subset $X \subseteq W$. This addition is not necessary however. As the formula $s_X \leftrightarrow \bigvee_{w\in X} p_w$ is in Γ and $\mathbf{E}$ is closed under $\mathrm{RE}_\Box$, we could replace each propositional letter s_X with the disjunction $\bigvee_{w\in X} p_w$. For notational convenience and to more easily capture the neighbourhoods, we do use propositional letters s_X. $\dashv$

Lemma 4.4. *Let $\mathbb{F} = \langle W, \nu \rangle$ be a finite descriptive neighbourhood frame and $D \subseteq \mathcal{P}W$ a subset. Then we have $\mathbb{F} \not\models \sigma(\mathbb{F}, D)$.*

Proof. We define a valuation V on $\mathbb{F}$ where $V(p_w) = \{w\}$ and $V(s_X) = X$. We first show that $\langle \mathbb{F}, V \rangle, w \models \gamma$ for each $\gamma \in \Gamma$ and every $w \in W$. We take any $w \in W$ and $X \subseteq W$ and make a case distinction on the $\gamma \in \Gamma$.

- $\langle \mathbb{F}, V \rangle, w \models p_w$, so we have $\langle \mathbb{F}, V \rangle, w \models \bigvee_{w \in W} p_w$.
- Consider $v, v' \in W$ such that $v \neq v'$. If $v \neq w$, we have $\langle \mathbb{F}, V \rangle, w \not\models p_v$ thus $\langle \mathbb{F}, V \rangle, w \models p_v \to \neg p_{v'}$. If $v = w$, we have $\langle \mathbb{F}, V \rangle, w \models p_v$ and $\langle \mathbb{F}, V \rangle, w \not\models p_{v'}$, hence $\langle \mathbb{F}, V \rangle, w \models p_v \to \neg p_{v'}$.
- We have $\langle \mathbb{F}, V \rangle, w \models s_X$ if and only if $w \in X$ if and only if $\langle \mathbb{F}, V \rangle, w \models \bigvee_{w \in X} p_w$. So $\langle \mathbb{F}, V \rangle, w \models s_X \leftrightarrow \bigvee_{w \in X} p_w$.
- Suppose $\langle \mathbb{F}, V \rangle, w \models s_{\Box X}$. Then $w \in \Box X$, i.e. $X \in \nu(w)$. As $X = V(s_X)$, we have $\langle \mathbb{F}, V \rangle, w \models \Box s_X$.
- Suppose $\langle \mathbb{F}, V \rangle, w \models \Box s_X$. Then $V(s_X) = X \in \nu(w)$, which implies that $w \in \Box X$, hence $\langle \mathbb{F}, V \rangle, w \models s_{\Box X}$. This means that actually, $\langle \mathbb{F}, V \rangle, w \models \Box s_X \to s_{\Box X}$ for each $X \subseteq W$, so certainly for each $X \in D$.

Now consider $\delta \in \Delta$, i.e. $\delta = \neg p_w$ for some $w \in W$. Note that $\langle \mathbb{F}, V \rangle, w \models p_w$, thus $\langle \mathbb{F}, V \rangle, w \not\models \neg p_w$ and thus $\langle \mathbb{F}, V \rangle, w \not\models \delta$. This means $\mathbb{F} \not\models \sigma(\mathbb{F}, D)$. ☐

Theorem 4.5. *Let $\mathbb{F}_0 = \langle W_0, \nu_0 \rangle$ be a finite descriptive neighbourhood frame and $\mathbb{F} = \langle W, \nu, A \rangle$ an arbitrary descriptive neighbourhood frame. Let $D \subseteq \mathcal{P}W_0$. Then $\mathbb{F} \not\models \sigma(\mathbb{F}_0, D)$ if and only if there exists a stable surjective map $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$ satisfying* (CoSDC) *for D.*

Proof. ($\Rightarrow$) Suppose $\mathbb{F} \not\models \sigma(\mathbb{F}_0, D)$. This means there exists valuation V on $\mathbb{F}$ witnessing this. We define a map $f : \mathbb{F} \to \mathbb{F}_0$ by $f(w) = v$ iff $w \in V(p_v)$.

To show this is well-defined, we show that the collection $\{V(p_v)\}_{v \in W'}$ partitions W. Consider any $w \in W$. As $\bigvee_{v \in W_0} p_v \in \Gamma$, we have $\langle \mathbb{F}, V \rangle, w \models \bigvee_{v \in W_0} p_v$. This means there exists $v \in W_0$ such that $w \in V(p_v)$. Suppose for a contradiction that there exists $v' \in W_0$ such that $v \neq v'$ and $w \in V(p_{v'})$. As $p_v \to \neg p_{v'} \in \Gamma$, we obtain a contradiction between $\langle \mathbb{F}, V \rangle, w \models p_v \to \neg p_{v'}$ and $\langle \mathbb{F}, V \rangle, w \models p_v$ together with $\langle \mathbb{F}, V \rangle, w \models p_{v'}$.

By definition we have $f^{-1}[\{v\}] = V(p_v) \in A$, which gives $f^{-1}[X] \in A$ for each $X \in \mathcal{P}W_0$. So f is continuous. For stability, we consider $X \in \mathcal{P}W_0$ such that $X \in \nu_0(f(w))$. This means we have $f(w) \in \Box_{\nu_0} X$. As $\langle \mathbb{F}, V \rangle, w \models p_{f(w)}$, we have $\langle \mathbb{F}, V \rangle, w \models \bigvee_{v \in \Box_{\nu_0} X} p_v$, so $\langle \mathbb{F}, V \rangle, w \models s_{\Box_{\nu_0} X}$ from $s_{\Box_{\nu_0} X} \leftrightarrow \bigvee_{v \in \Box_{\nu_0} X} p_v \in \Gamma$. As we also have $s_{\Box_{\nu_0} X} \to \Box s_X \in \Gamma$, we obtain $\langle \mathbb{F}, V \rangle, w \models \Box s_X$. This means $V(s_X) \in \nu(w)$. Note that $V(s_X) = V(\bigvee_{v \in X} p_v) = f^{-1}[X]$, from $s_X \leftrightarrow \bigvee_{v \in X} p_v \in \Gamma$ and the definition of f. Hence, $f^{-1}[X] \in \nu(w)$.

For the Co-Stable Domain Condition, we follow the same reasoning as above, but we use $\Box s_X \to s_{\Box_{\nu_0} X} \in \Gamma$ for each $X \in D$. For the surjectiveness of f, we consider $v \in W_0$. As $\neg p_v \in \Delta$, there exists $w \in W$ such that $\langle \mathbb{F}, V \rangle, w \not\models \neg p_v$, thus $\langle \mathbb{F}, V \rangle, w \models p_v$, so $w \in V(p_v)$. This means that $f(w) = v$.

($\Leftarrow$) Suppose we have stable surjection $f : \mathbb{F} \to \mathbb{F}_0$ satisfying (CoSDC) for D. By the previous lemma, we know that the valuation V_0 on $\mathbb{F}_0$ with $V_0(p_w) = \{w\}$ and $V_0(s_X) = X$ witnesses that $\mathbb{F}' \not\models \sigma(\mathbb{F}_0, D)$. We define valuation V on $\mathbb{F}$ by setting $V(p_v) = f^{-1}[V_0(p_v)] = f^{-1}[\{v\}]$ and $V(s_X) = f^{-1}[V_0(s_X)] = f^{-1}[X]$, which is well-defined by f being continuous. We show that this valuation is a witness of $\mathbb{F} \not\models \sigma(\mathbb{F}_0, D)$.

First, we show that $V(\gamma) = W$ for all $\gamma \in \Gamma$. We consider $v, v' \in W_0$ such that $v \neq v'$ and $X \subseteq W_0$.

$$
\begin{aligned}
V(\bigvee_{v \in W_0} p_v) &= \bigcup_{v \in W_0} V(p_v) = \bigcup_{v \in W_0} f^{-1}[\{v\}] \\
&= f^{-1}[\bigcup_{v \in W_0} \{v\}] = f^{-1}[W_0] = W \\
V(p_v \to \neg p_{v'}) &= V(\neg p_v) \cup V(\neg p_{v'}) = [W \setminus V(p_v)] \cup [W \setminus V(p_{v'})] \\
&= \big[W \setminus f^{-1}[\{v\}]\big] \cup \big[W \setminus f^{-1}[\{v'\}]\big] \\
&= f^{-1}[W_0 \setminus \{v\}] \cup f^{-1}[W_0 \setminus \{v'\}] \\
&= f^{-1}[W_0] \text{ (from } v \neq v') \\
&= W \\
V(s_X \leftrightarrow \bigvee_{v \in X} p_v) &= V(s_X \to \bigvee_{v \in X} p_v) \cap V(\bigvee_{v \in X} p_v \to s_X) \\
&= \big[V(\neg s_X) \cup V(\bigvee_{v \in X} p_v)\big] \cap \big[V(\neg \bigvee_{v \in X} p_v) \cup V(s_X)\big] \\
&= \big[[W \setminus f^{-1}[X]] \cup \bigcup_{v \in X} f^{-1}[\{v\}]\big] \cap \big[[W \setminus \bigcup_{v \in X} f^{-1}[\{v\}]] \cup f^{-1}[X]\big] \\
&= \big[[W \setminus f^{-1}[X]] \cup f^{-1}[X]\big] \cap \big[[W \setminus f^{-1}[X]] \cup f^{-1}[X]\big] = W \\
V(s_{\Box X} \to \Box s_X) &= V(\neg s_{\Box X}) \cup V(\Box s_X) \\
&= [W \setminus V(s_{\Box X})] \cup \{w \in W \mid V(s_X) \in \nu(w)\} \\
&= [W \setminus f^{-1}[\Box X]] \cup \{w \in W \mid f^{-1}[X] \in \nu(w)\} \\
&\supseteq [W \setminus f^{-1}[\Box X]] \cup \{w \in W \mid X \in \nu'(f(w))\} \text{ (by stability of } f) \\
&= [W \setminus f^{-1}[\Box X]] \cup \{w \in W \mid f(w) \in \Box X\} = W
\end{aligned}
$$

Now we take $X \in D$.

$$
\begin{aligned}
V(\Box s_X \to s_{\Box X}) &= V(\neg \Box s_X) \cup V(s_{\Box X}) = [W \setminus V(\Box s_X)] \cup f^{-1}[\Box X] \\
&= [W \setminus \{w \in W \mid V(s_X) \in \nu(w)\}] \cup f^{-1}[\Box X] \\
&= [W \setminus \{w \in W \mid f^{-1}[X] \in \nu(w)\}] \cup f^{-1}[\Box X] \\
&\supseteq [W \setminus \{w \in W \mid X \in \nu_0(f(w))\}] \cup f^{-1}[\Box X] \\
&\qquad \text{(by } f \text{ satisfying (CoSDC) for } D) \\
&= [W \setminus \{w \in W \mid f(w) \in \Box X\}] \cup f^{-1}[\Box X] = W
\end{aligned}
$$

We also need that $V(\delta) \neq W$ for each $\delta \in \Delta$. Consider $\neg p_v \in \Delta$ for $v \in W'$. Note that $V(\neg p_v) = W \setminus V(p_v) = W \setminus f^{-1}[\{v\}]$. As f is surjective, $f^{-1}[\{v\}] \neq \emptyset$ and thus $V(\neq p_v) \neq W$. $\Box$

Combining the results above gives us the tools to characterize each modal rule by a finite collection of finite neighbourhood frames, using the defined stable canonical rules.

Theorem 4.6.

(1) *For each modal rule* Γ/Δ*, there exist pairs* $\langle \mathbb{F}_1, D_1\rangle, \dots, \langle \mathbb{F}_n, D_n\rangle$ *such that each* $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ *is a finite neighbourhood frame refuting* Γ/Δ*,* $D_i \subseteq \mathcal{P}W_i$*, and for each descriptive neighbourhood frame* $\mathbb{G} = \langle W, \nu, A\rangle$ *we have:*

$$\mathbb{G} \models \Gamma/\Delta \textit{ iff } \mathbb{G} \models \sigma(\mathbb{F}_1, D_1), \dots, \sigma(\mathbb{F}_n, D_n).$$

(2) *For each modal formula* ϕ*, there exist pairs* $\langle \mathbb{F}_1, D_1\rangle, \dots, \langle \mathbb{F}_n, D_n\rangle$ *such that each* $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ *is a finite neighbourhood frame refuting* ϕ*,* $D_i \subseteq \mathcal{P}W_i$*, and for each descriptive neighbourhood frame* $\mathbb{G} = \langle W, \nu, A\rangle$ *we have:*

$$\mathbb{G} \models \phi \textit{ iff } \mathbb{G} \models \sigma(\mathbb{F}_1, D_1), \dots, \sigma(\mathbb{F}_n, D_n).$$

Proof. (1) The result is a combination of the two previous theorems. By Theorem 4.1(1), we have pairs $\langle \mathbb{F}_1, D_1\rangle, \dots, \langle \mathbb{F}_n, D_n\rangle$ such that $\mathbb{G} \not\models \Gamma/\Delta$ iff there exists $i \leq n$ and onto stable map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$ satisfying (CoSDC) for D_i. By Theorem 4.5, we know that the existence of such a map is equivalent to $\mathbb{G} \not\models \sigma(\mathbb{F}_i, D_i)$. This gives that $\mathbb{G} \models \Gamma/\Delta$ is equivalent to $\mathbb{G} \models \sigma(\mathbb{F}_i, D_i)$ for each i.

(2) This is proved similarly to (1), now using Theorem 4.1(2). □

We can now prove the main result of this section.

Theorem 4.7.

(1) *Each classical modal multi-conclusion consequence relation* $\mathcal{S}$ *is axiomatizable by stable canonical rules. Moreover, if* $\mathcal{S}$ *is finitely axiomatizable, then* $\mathcal{S}$ *is axiomatizable by finitely many stable canonical rules.*

(2) *Each classical modal logic* Λ *is axiomatizable by stable canonical rules. Moreover, if* Λ *is finitely axiomatizable, then* Λ *is axiomatizable by finitely many stable canonical rules.*

Proof. (1) Let $\mathcal{S}$ be a classical modal multi-conclusion consequence relation. Then there is a family $\{\rho_i\}_{i\in I}$ of modal rules such that $\mathcal{S} = \mathbf{S_E} + \{\rho_i\}_{i\in I}$. By Theorem 4.6, for each $i \in I$ there exists $\langle \mathbb{F}_{i1}, D_{i1}\rangle, \dots, \langle \mathbb{F}_{in_i}, D_{in_i}\rangle$ such that each $\mathbb{F}_{ij} = \langle W_{ij}, \nu_{ij}\rangle$ is a finite neighbourhood frame, $D_{ij} \subseteq \mathcal{P}W_{ij}$, and for each descriptive neighbourhood frame $\mathbb{G} = \langle W, \nu, A\rangle$, we have:

$$\mathbb{G} \models \rho_i \text{ iff } \mathbb{G} \models \sigma(\mathbb{F}_{i1}, D_{i1}), \dots, \sigma(\mathbb{F}_{in_i}, D_{in_i}).$$

This gives $\mathbb{G} \models \mathcal{S}$ iff $\mathbb{G} \models \{\rho_i\}_{i\in I}$ iff $\mathbb{G} \models \sigma(\mathbb{F}_{i1}, D_{i1}), \dots, \sigma(\mathbb{F}_{in_i}, D_{in_i})$ for each $i \in I$. By completeness with respect to descriptive neighbourhood frames (Theorem 3.30), $\mathcal{S} = \mathbf{S_E} + \bigcup_{i\in I}\{\sigma(\mathbb{F}_{i1}, D_{i1}), \dots, \sigma(\mathbb{F}_{in_i}, D_{in_i})\}$ and therefore, $\mathcal{S}$ is axiomatizable by stable canonical rules. In particular, if $\mathcal{S}$ is finitely axiomatizable, then $\mathcal{S}$ is axiomatizable by finitely many stable canonical rules.

(2) Let Λ be a classical modal logic. Then $\mathcal{S}(\Lambda) = \mathbf{S_E} + \{/\phi\}_{\phi\in\Lambda}$ is a classical modal multi-conclusion consequence relation. By (1), we then get $\mathcal{S}(\Lambda) = \mathbf{S_E} + \{\sigma(\mathbb{F}_i, D_i)\}_{i\in I}$. Hence, $\Lambda = \Lambda(\mathcal{S}(\Lambda)) = \Lambda(\mathbf{S_E} + \{\sigma(\mathbb{F}_i, D_i)\}_{i\in I})$. In particular, if Λ is finitely axiomatizable, then Λ is axiomatizable by finitely many stable canonical rules.

□

Remark 4.8. By the duality between classical modal algebras and descriptive neighbourhood frames, we can rephrase all results in this section in terms of classical modal algebras. Let $\mathbb{A} = \langle A, \Box \rangle$ be a finite classical modal algebra and $D \subseteq A$. We introduce propositional letters p_a for each $a \in A$. Now we define its stable canonical rule $\rho(\mathbb{A}, D)$ as the rule Γ/Δ with Γ and Δ as follows:

$$\begin{aligned}
\Gamma &= \{p_{a \vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\
&\quad \{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\
&\quad \{p_{\Box a} \to \Box p_a \mid a \in A\} \cup \\
&\quad \{\Box p_a \to p_{\Box a} \mid a \in D\}, \\
\Delta &= \{p_a \mid a \in A, a \neq 1\}.
\end{aligned}$$

With this definition, a classical modal algebra $\mathbb{B} = \langle B, \Box_B \rangle$ refutes $\rho(\mathbb{A}, D)$ if and only if there exists a stable embedding $h : \mathbb{A} \rightarrowtail \mathbb{B}$ satisfying (CoSDC) for D. This allows us to prove all results on stable canonical rules presented here completely algebraically. We can even show that the two notions of stable canonical rules coincide. Indeed, let $\mathbb{F} = \langle W, \nu \rangle$ and $\mathbb{G}$ be two descriptive neighbourhood frames with $\mathbb{F}$ finite and $\mathbb{A}$ and $\mathbb{B}$ their respective dual classical modal algebras. Then for $D \subseteq \mathcal{P}W$, we have $\mathbb{G} \models \sigma(\mathbb{F}, D)$ iff $\mathbb{B} \models \rho(\mathbb{A}, D)$.

One might wonder why, if the two definitions of stable canonical rules coincide, the definition for the classical modal algebras looks much cleaner. As each element a on an algebra corresponds to a subset on its dual frame, we could indeed define the stable canonical rule for frames using only propositional letters for subsets, using a similar definition as given for algebras above. We however choose to write the rule using propositional letters for each world, as this more closely express the world-oriented view we take when looking at a neighbourhood frame. $\dashv$

4.2 Finite Model Property

For a stable canonical rule $\sigma(\mathbb{F}, D)$ for finite descriptive neighbourhood frame $\mathbb{F} = \langle W, \nu \rangle$ and $D \subseteq A$, there are two extreme cases that deserve a closer look. When $D = \emptyset$, we call the stable canonical rule $\sigma(\mathbb{F}, \emptyset)$ simply a *stable rule* and we denote it by $\sigma(\mathbb{F})$. When $D = \mathcal{P}W$, we call the rule $\sigma(\mathbb{F}, \mathcal{P}W)$ a *Jankov rule* and denote it by $\chi(\mathbb{F})$. In this section, we will look at the consequence relations and modal logics axiomatized by stable rules and prove that they have the finite model property. In the next section we show that the Jankov rules characterize special elements in the lattice $\mathrm{CExt}\mathbf{S_E}$ called splittings.

As a consequence of Theorem 4.5, we have the following properties for stable rules and Jankov rules.

Corollary 4.9. *Let $\mathbb{F}_0 = \langle W_0, \nu_0 \rangle$ and $\mathbb{F} = \langle W, \nu, A \rangle$ be descriptive neighbourhood frames with W_0 finite.*

(1) $\mathbb{F} \not\models \sigma(\mathbb{F}_0)$ *if and only if there exists an onto stable map* $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$.

(2) $\mathbb{F} \not\models \chi(\mathbb{F}_0)$ *if and only if there exists an onto bounded frame morphism* $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$.

To characterize the classes axiomatized by stable rules, we define the following *stable* class, analogue of the definition in [5]. For a classical modal multi-conclusion consequence relation, we let $\mathcal{K}_\mathcal{S} = \{\mathbb{F} \mid \mathbb{F} \models \mathcal{S}\}$ denote the class of all descriptive neighbourhood frames validating $\mathcal{S}$.

Definition 4.10 (**Stable Class and Stable Logic**).

(1) A class $\mathcal{K}$ of frames is called *stable* if for frames $\mathbb{F}$ and $\mathbb{G}$, if $\mathbb{F} \in \mathcal{K}$ and there exists an onto stable map $f : \mathbb{F} \twoheadrightarrow \mathbb{G}$, then $\mathbb{G} \in \mathcal{K}$.

(2) A classical modal multi-conclusion consequence relation $\mathcal{S}$ is called *stable* if the corresponding class $\mathcal{K}_\mathcal{S}$ is stable.

(3) We call a classical modal logic Λ *stable* if there is a stable classical modal multi-conclusion consequence relation $\mathcal{S}$ such that $\Lambda = \Lambda(\mathcal{S})$. ⊣

Remark 4.11. Dually, we can define a class $\mathcal{K}$ of algebras to be stable provided that for modal algebras $\mathbb{A}$ and $\mathbb{B}$, if $\mathbb{B} \in \mathcal{K}$ and there exists a stable embedding $h : \mathbb{A} \rightarrowtail \mathbb{B}$, then $\mathbb{A} \in \mathcal{K}$. Similarly, a classical modal multi-conclusion consequence relation $\mathcal{S}$ is stable if the corresponding universal class $\mathcal{U}(\mathcal{S})$ of classical modal algebras is stable. ⊣

We can now show that, indeed, the choice of terminology for stable classes is justified, analogue to the proof in [5].

Theorem 4.12.

(1) *A classical modal multi-conclusion consequence relation $\mathcal{S}$ is stable if and only if $\mathcal{S}$ is axiomatizable by stable rules.*

(2) *A classical modal logic Λ is stable if and only if Λ is axiomatizable by stable rules.*

Proof. (1) For one direction, suppose that $\mathcal{S}$ is stable. We let $\mathcal{A}_\mathcal{S}$ be the set of all non-isomorphic finite neighbourhood frames refuting $\mathcal{S}$. We show that $\mathcal{S} = \mathbf{S_E} + \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}_\mathcal{S}}$. By completeness of $\mathcal{S}$ with respect to descriptive neighbourhood frames (Theorem 3.30) it suffices to show the two consequence relations coincide on any descriptive neighbourhood frame. Consider a descriptive neighbourhood frame $\mathbb{G} = \langle W, \nu, A\rangle$. If $\mathbb{G} \not\models \mathcal{S}$, there exists rule $\rho \in \mathcal{S}$ such that $\mathbb{G} \not\models \rho$. By the proof of Theorem 4.1, we obtain finite frame $\mathbb{G}_0$ refuting ρ and onto stable map $f : \mathbb{G} \twoheadrightarrow \mathbb{G}_0$. Then $\mathbb{G}_0 \in \mathcal{A}_\mathcal{S}$. By Corollary 4.9 we have $\mathbb{G} \not\models \sigma(\mathbb{G}_0)$. This implies $\mathbb{G} \not\models \mathbf{S_E} + \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}_\mathcal{S}}$. Conversely, suppose that $\mathbb{G} \not\models \mathbf{S_E} + \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}_\mathcal{S}}$. Then there exists $\mathbb{F} \in \mathcal{A}_\mathcal{S}$ such that $\mathbb{G} \not\models \sigma(\mathbb{F})$. By Corollary 4.9 again we obtain onto stable map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}$. If $\mathbb{G} \models \mathcal{S}$ we obtain $\mathbb{F} \models \mathcal{S}$ by $\mathcal{S}$ being stable, which is a contradiction. So $\mathbb{G} \not\models \mathcal{S}$ and thus, $\mathcal{S} = \mathbf{S_E} + \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}_\mathcal{S}}$.

For the other direction, suppose that $\mathcal{S}$ is axiomatizable by stable rules, i.e. $\mathcal{S} = \mathbf{S_E} + \{\sigma(\mathbb{F}_i)\}_{i\in I}$ for some family of descriptive neighbourhood frames $\{\mathbb{F}_i\}_{i\in I}$. Now take descriptive neighbourhood frame $\mathbb{F}$ such that $\mathbb{F} \models \mathcal{S}$ and there exists onto stable map $f : \mathbb{F} \twoheadrightarrow \mathbb{G}$. If $\mathbb{G} \not\models \mathcal{S}$, we have $i \in I$ such that $\mathbb{G} \not\models \sigma(\mathbb{F}_i)$. By Corollary 4.9 there exists an onto stable map $g : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$. Therefore there exists onto stable map $g \circ f : \mathbb{F} \twoheadrightarrow \mathbb{F}_i$. By Corollary 4.9 again, we obtain $\mathbb{F} \not\models \sigma(\mathbb{F}_i)$. This contradiction gives us $\mathbb{G} \models \mathcal{S}$ and hence $\mathcal{S}$ is stable.

(2) A classical modal logic Λ is stable iff $\Lambda = \Lambda(\mathcal{S})$ for a stable classical modal multi-conclusion consequence relation $\mathcal{S}$, which by (1) is axiomatizable by stable canonical rules. Consequently, Λ is axiomatizable by stable canonical rules. □

We will show a small lemma on stable classical modal multi-conclusion consequence relation that we will use in Section 4.4. For classical modal multi-conclusion consequence relation $\mathcal{S}$, we say that $\mathcal{K}_\mathcal{S}$ is *generated by a class* $\mathcal{K}$ if $\mathcal{S}(\mathcal{K}) = \mathcal{S}$.

Lemma 4.13. *Let $\mathcal{S}$ be a classical modal multi-conclusion consequence relation. Then $\mathcal{S}$ is stable iff $\mathcal{K}_{\mathcal{S}}$ is generated by some stable class $\mathcal{K}$.*

Proof. If $\mathcal{S}$ is stable, the class $\mathcal{K}_{\mathcal{S}}$ is stable by definition. So suppose $\mathcal{K}_{\mathcal{S}}$ is generated by some stable class $\mathcal{K}$. We let $\mathcal{A}$ be the set of finite non-isomorphic descriptive neighbourhood frames $\mathbb{F}$ such that $\mathbb{F} \notin \mathcal{K}$. We show that $\mathcal{S}$ is axiomatized by the rules $\{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}}$. By the completeness theorem from Theorem 3.30, it suffices to show that $\mathcal{S}$ and $\mathbf{S_E} + \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}}$ coincide on descriptive neighbourhood frames, i.e. $\mathcal{K}_{\mathcal{S}} = \{\mathbb{G} \mid \mathbb{G} \vDash \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}}\}$.

For the direction from left to right, we show that $\mathcal{K} \vDash \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}}$, which implies $\mathcal{K}_{\mathcal{S}} \vDash \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}}$. Suppose for a contradiction that there exists a descriptive neighbourhood frames $\mathbb{G} \in \mathcal{K}$ and $\mathbb{F} \in \mathcal{A}$ such that $\mathbb{G} \nvDash \sigma(\mathbb{F})$. Consequently, there exists a stable onto map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}$. As $\mathcal{K}$ is stable, we have $\mathbb{F} \in \mathcal{K}$, contradicting $\mathbb{F} \in \mathcal{A}$.

For the direction from right to left, consider a descriptive neighbourhood frame $\mathbb{G}$ such that $\mathbb{G} \vDash \{\sigma(\mathbb{F})\}_{\mathbb{F}\in\mathcal{A}}$. Suppose for a contradiction that $\mathbb{G} \notin \mathcal{K}_{\mathcal{S}}$. This gives $\Gamma/\Delta \in \mathcal{S}$ such that $\mathbb{G} \nvDash \Gamma/\Delta$. Let V be a valuation on $\mathbb{G}$ witnessing this. Now filtrating $\langle \mathbb{G}, V\rangle$ through $\Gamma \cup \Delta$ gives finite stable image $\mathbb{G}'$ of $\mathbb{G}$ such that $\mathbb{G}' \nvDash \Gamma/\Delta$. This gives $\mathbb{G} \nvDash \sigma(\mathbb{G}')$, implying that $\mathbb{G}' \notin \mathcal{A}$ by the assumption. But then $\mathbb{G}' \in \mathcal{K}$, implying $\mathbb{G}' \in \mathcal{K}_{\mathcal{S}}$ which contradicts $\mathbb{G}' \nvDash \Gamma/\Delta$. Therefore $\mathbb{G} \in \mathcal{K}_{\mathcal{S}}$. □

The most straightforward application of stable rules is in proving the finite model property. The stable classes will have the finite model property, as shown in the following theorem, again analogue to the proof in [5]. We first recall the definition of the finite model property.

Definition 4.14 (Finite Model Property). Let Λ be a classical modal logic and $\mathcal{S}$ a classical modal multi-conclusion consequence relation. We say that Λ has the *finite model property* if $\Lambda \nvdash \phi$ implies that there exists a finite descriptive neighbourhood frame $\mathbb{F}$ such that $\mathbb{F} \vDash \Lambda$ and $\mathbb{F} \nvDash \phi$. Similarly $\mathcal{S}$ has the *finite model property* if $\rho \notin \mathcal{S}$ implies that there exists finite descriptive neighbourhood frame $\mathbb{F}$ such that $\mathbb{F} \vDash \mathcal{S}$ and $\mathbb{F} \nvDash \rho$. ⊣

Theorem 4.15.

(1) *Every stable classical modal multi-conclusion consequence relation $\mathcal{S}$ has the finite model property.*

(2) *Every stable classical modal logic Λ has the finite model property.*

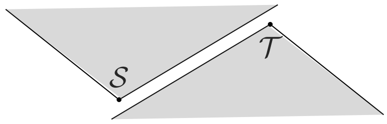
Proof. (1) Let $\mathcal{S}$ be a stable classical modal multi-conclusion consequence relation and ρ a rule such that $\rho \notin \mathcal{S}$. By Theorem 3.30, we have completeness of $\mathcal{S}$ with respect to descriptive neighbourhood frames so there exists a descriptive neighbourhood frame $\mathbb{F} \in \mathcal{S}$ such that $\mathbb{F} \nvDash \rho$. By the proof of Theorem 4.1, we obtain a finite neighbourhood frame $\mathbb{F}_0$ such that $\mathbb{F}_0 \nvDash \rho$ and an onto stable map $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$. As $\mathcal{S}$ is stable, $\mathbb{F}_0 \vDash \mathcal{S}$ and hence $\mathcal{S}$ has the finite model property.

(2) When looking at a classical modal logic Λ as its corresponding consequence relation $\mathcal{S}(\Lambda)$, this is a direct consequence of (1). □

Corollary 4.16. $\mathbf{E}$ *and* $\mathbf{S_E}$ *have the finite model property.*

4.3 Splittings

In this section we discuss properties of Jankov rules. They will characterize splittings and join splittings. Intuitively, a splitting in a lattice is a pair of elements such that every other element is either above the first or below the second arguments. It 'splits' the lattice in two parts, as illustrated in the picture below.



Splitting pair $\mathcal{S}$ and $\mathcal{T}$

Definition 4.17 (**Splittings and Join Splittings**). A classical modal multi-conclusion consequence relation $\mathcal{S}$ is called *splitting* if there exists another classical modal multi-conclusion consequence relation $\mathcal{T}$ such that $\mathcal{S} \not\subseteq \mathcal{T}$ and for every other classical modal multi-conclusion consequence relation $\mathcal{U}$, we have either $\mathcal{S} \subseteq \mathcal{U}$ or $\mathcal{U} \subseteq \mathcal{T}$. We call $(\mathcal{S}, \mathcal{T})$ a *splitting pair*.

$\mathcal{S}$ is *join splitting* if it is a join of splitting classical modal multi-conclusion consequence relations in the lattice $\mathrm{CExt}\mathbf{S_E}$ of classical extensions of $\mathbf{S_E}$. ⊣

It is important to note that splitting pairs are unique in the sense that if $(\mathcal{S}_1, \mathcal{T})$ and $(\mathcal{S}_2, \mathcal{T})$ are splitting pairs, then $\mathcal{S}_1 = \mathcal{S}_2$. We will use this in the proof of the next theorem.

As seen in Corollary 4.9, for frames $\mathbb{F}$ and $\mathbb{G}$ we have $\mathbb{G} \not\models \chi(\mathbb{F})$ if and only if there exists an onto bounded morphism from $\mathbb{F}$ to $\mathbb{G}$. We use this fact to show that consequence relations axiomatized by Jankov rules exactly characterize splittings. We follow reasoning as used by Jeřábek [22], but first show a well-known proposition regarding splittings.

Proposition 4.18. *For a splitting pair $(\mathcal{S}, \mathcal{T})$, $\mathcal{T}$ is completely meet-prime, meaning that $\bigcap_{i \in I} \mathcal{T}_i \subseteq \mathcal{T}$ implies $\mathcal{T}_i \subseteq \mathcal{T}$ for some $i \in I$. Moreover, $\mathcal{T}$ is completely meet-irreducible, meaning that $\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$ implies $\mathcal{T} = \mathcal{T}_i$ for some $i \in I$.*

Proof. Suppose $\bigcap_{i \in I} \mathcal{T}_i \subseteq \mathcal{T}$. For contradiction, suppose we have $\mathcal{T}_i \not\subseteq \mathcal{T}$ for all $i \in I$. By $(\mathcal{S}, \mathcal{T})$ being a splitting pair we obtain $\mathcal{S} \subseteq \mathcal{T}_i$ for each $i \in I$, from which it follows that $\mathcal{S} \subseteq \bigcap_{i \in I} \mathcal{T}_i \subseteq \mathcal{T}$, contradicting $\mathcal{S} \not\subseteq \mathcal{T}$.

For the second statement, note that being completely meet-prime implies completely meet-irreducibility by simply writing out the definition. □

Theorem 4.19. *Let $\mathcal{S}$ be a classical modal multi-conclusion consequence relation.*
(1) *$\mathcal{S}$ is splitting in* $\mathrm{CExt}\mathbf{S_E}$ *iff $\mathcal{S}$ is axiomatized by a single Jankov rule.*
(2) *$\mathcal{S}$ is join splitting* $\mathrm{CExt}\mathbf{S_E}$ *iff $\mathcal{S}$ is axiomatized by Jankov rules.*

Proof. (1) For the direction from right to left, suppose that $\mathcal{S}$ is axiomatized by the Jankov rule $\chi(\mathbb{F})$. We show that $(\mathcal{S}, \mathcal{S}(\mathbb{F}))$ is a splitting pair in $\mathrm{CExt}\mathbf{S_E}$. Note that $\mathbb{F} \not\models \chi(\mathbb{F})$ by Lemma 4.4, so we have $\mathcal{S} \not\subseteq \mathcal{S}(\mathbb{F})$. Now consider another $\mathcal{U} \in \mathrm{CExt}\mathbf{S_E}$ such that $\mathcal{S} \not\subseteq \mathcal{U}$. By classical modal multi-conclusion consequence relations being complete with respect to descriptive neighbourhood frames (Theorem 3.30), there exists a frame

$\mathbb{G}$ such that $\mathbb{G} \models \mathcal{U}$ but $\mathbb{G} \not\models \mathcal{S}$ and thus $\mathbb{G} \not\models \chi(\mathbb{F})$. This means there exists an onto bounded morphism from $\mathbb{G}$ to $\mathbb{F}$. Therefore, $\mathcal{U} \subseteq \mathcal{S}(\mathbb{G}) \subseteq \mathcal{S}(\mathbb{F})$.

For the other direction, suppose $\mathcal{S}$ is splitting. This gives another classical modal multi-conclusion consequence relation $\mathcal{T}$ such that $(\mathcal{S}, \mathcal{T})$ is a splitting pair. As $\mathbf{S_E}$ has the finite model property (Corollary 4.16), we have $\mathbf{S_E} = \bigcap\{\mathcal{S}(\mathbb{F}) \mid \mathbb{F} \text{ finite}\}$. Now by $\mathcal{T}$ being completely meet-prime (Proposition 4.18) and $\mathbf{S_E} \subseteq \mathcal{T}$, we obtain a finite frame $\mathbb{F}$ such that $\mathcal{S}(\mathbb{F}) \subseteq \mathcal{T}$. If we let n denote the cardinality of $\mathbb{F}$, we obtain $\mathbb{F}$ validating the following rule Size_n expressing that $\mathbb{F}$ has at most size n:

$$(\mathsf{Size}_n)\frac{}{p_0, p_0 \to p_1, p_0 \wedge p_1 \to p_2, \ldots, \bigwedge_{i<n} p_i \to p_n}.$$

But this means that this rule is also in $\mathcal{T}$. This gives $\mathcal{T} = \bigcap\{\mathcal{S}(\mathbb{F}) \mid \mathbb{F} \models \mathcal{T} \text{ and } |\mathbb{F}| \leq n\}$. As $\mathcal{T}$ is completely meet-irreducible by Proposition 4.18, we obtain $\mathcal{T} = \mathcal{S}(\mathbb{F})$ for such finite $\mathbb{F}$. This means $(\mathcal{S}, \mathcal{S}(\mathbb{F}))$ is a splitting pair. By reasoning as before, $(\mathbf{S_E} + \chi(\mathbb{F}), \mathcal{S}(\mathbb{F}))$ is a splitting pair so by uniqueness, we get $\mathcal{S} = \mathbf{S_E} + \chi(\mathbb{F})$.

(2) Consider a family $\{\mathcal{S}_i\}_{i\in I}$ of classical modal multi-conclusion consequence relations such that for each $i \in I$, $\mathcal{S}_i = \mathbf{S_E} + \Xi_i$ for a set of modal rules Ξ_i. Then $\Sigma_{i\in I}\mathcal{S}_i = \mathbf{S_E} + \bigcup_{i\in I} \Xi_i$. Now the statement easily follows from (1). □

4.4 Examples

This section will be devoted to examples of classes of frames and consequence relations that can be characterized by stable rules. To start off, we will show that these stable classes actually do exist. In fact, there are continuously many of them. Afterwards we show a few concrete examples of consequence relations axiomatized by stable rules.

Before we show this, we need to establish a notation for depicting finite descriptive neighbourhood frames. When drawing a neighbourhood frame, a connection $\sim$ from a world w to a set of worlds X will indicate that X is a neighbourhood of w. Sometimes we will have to depict a neighbourhood frame in regular text. Then we will simply give the neighbourhood function, referred to as μ. From this, one can deduce the set of worlds W. Moreover, as we are working with finite descriptive neighbourhood frames, the set of admissible subsets A is the full powerset. As an example, the frame $[u \mapsto \{\{u\}\}, v \mapsto \emptyset]$ is the frame $\langle W, \mu, A\rangle$ where $W = \{u, v\}$, $\mu(u) = \{\{u\}\}$, $\mu(v) = \emptyset$ and $A = \mathcal{P}W$.

Proposition 4.20. *There is a continuum of stable classical modal multi-conclusion consequence relations.*

Proof. For $n \geq 1$, we let $\mathbb{F}_n$ denote the frame $[w_1 \mapsto \{w_1, \ldots, w_n\}, w_2 \mapsto \{w_2, \ldots, w_n\}, \ldots, w_n \mapsto \{w_n\}]$, as can be seen in Figure 4.1. Let $\omega_{\geq 1} := \{n \in \omega \mid n \geq 1\}$. For a subset $I \subseteq \omega_{\geq 1}$, we set $\mathcal{K}_I = \{\mathbb{G} \mid \exists n \in I \text{ such that there exists stable onto map } f : \mathbb{F}_n \twoheadrightarrow \mathbb{G}\}$. Clearly, $\mathcal{K}_I$ is a stable class. Let $\mathcal{S}_I = \{\rho \mid \mathcal{K}_I \models \rho\}$ be the classical modal multi-conclusion consequence relation corresponding to $\mathcal{K}_I$. As $\mathcal{K}_I$ is stable, so is $\mathcal{S}_I$, by Lemma 4.13. We show that if $I \neq J$, then $\mathcal{S}_I \neq \mathcal{S}_J$, giving continuously many unique stable classical modal multi-conclusion consequence relations. For this, we first show that $n \in I$ iff $\sigma(\mathbb{F}_n) \notin \mathcal{S}_I$.

If $n \in I$, then $\mathbb{F}_n \in \mathcal{K}_I$, so $\mathbb{F}_n \models \mathcal{S}_I$. But from Lemma 4.4, we obtain $\mathbb{F}_n \not\models \sigma(\mathbb{F}_n)$, hence $\sigma(\mathbb{F}_n) \notin \mathcal{S}_I$. For the other direction, suppose that $\sigma(\mathbb{F}_n) \notin \mathcal{S}_I$. That means there

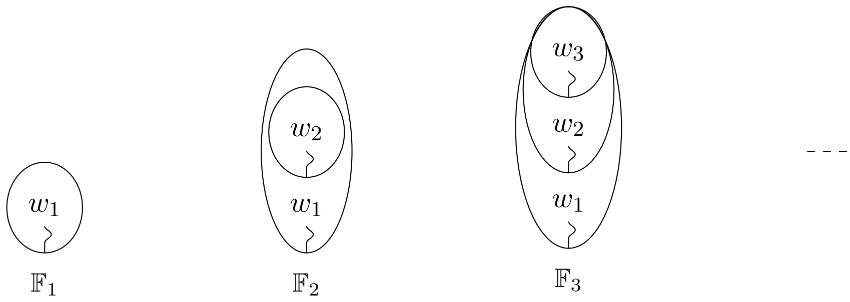


Figure 4.1

exists frame $\mathbb{G} \in \mathcal{K}_I$ such that $\mathbb{G} \not\models \sigma(\mathbb{F}_n)$. But then there exists a stable onto map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_n$. As $\mathcal{K}_I$ is closed under stable images, we have that $\mathbb{F}_n \in \mathcal{K}_I$. If $n \notin I$, there is $m \in I$ and onto stable map $f : \mathbb{F}_m \twoheadrightarrow \mathbb{F}_n$. By definition, $|\mathbb{F}_m| = m > n = |\mathbb{F}_n|$ and thus f must identify at least two distinct points w_k and w_l of $\mathbb{F}_m$, i.e. $f(w_k) = f(w_l) = w_p$ for some w_p in $\mathbb{F}_n$. Stabilitly of f forces that $f^{-1}[\{w_p, \dots, w_n\}] \in \mu(w_l) \cap \mu(w_k)$. But by definition, $\mu(w_l)$ and $\mu(w_k)$ are disjoint. So we indeed get $n \in I$. Hence, $n \in I$ iff $\sigma(\mathbb{F}_n) \notin \mathcal{S}_I$.

If $I \neq J$, we obtain without loss of generality some $n \in I \setminus J$. Hence, $\sigma(\mathbb{F}_n) \in \mathcal{S}_J \setminus \mathcal{S}_I$ and therefore, $\mathcal{S}_I \neq \mathcal{S}_J$. □

To illustrate the results in the previous section, we will give a few examples of classes of frames that can be (finitely) axiomatized by stable rules. Let $\mathbf{S_{Form}} = \mathcal{S}(\mathbf{Form})$ be the least classical modal multi-conclusion consequence relation containing $/\phi$ for every modal formula ϕ. Then $\mathbf{S_{Form}}$ corresponds to the class consisting of just the empty frame. The set **Rules** of all modal rules corresponds to the empty class of frames. Moreover, we will show how the classical modal multi-conclusion consequence relation based on the deontic logic **D**, $\mathbf{S_E} + /\mathrm{P}$, can be axiomatized by stable canonical rules. We let $\sigma()$ denote the stable rule of the empty frame.

Theorem 4.21.
(1) $\mathbf{S_{Form}} = \mathbf{S_E} + \sigma([u \mapsto \emptyset])$.
(2) $\mathbf{Rules} = \mathbf{S_E} + \sigma() + \sigma([u \mapsto \emptyset])$.
(3) $\mathbf{S_E} + /\mathrm{P} = \mathbf{S_E} + \sigma([u \mapsto \{\emptyset\}]) + \sigma([u \mapsto \{\emptyset\}, v \mapsto \emptyset])$.

Proof. By the completeness result in Theorem 3.30, it suffices to show that the consequence relations are validated on the same descriptive neighbourhood frames.

(1) A descriptive neighbourhood frame $\mathbb{F}$ can be stably mapped onto $[u \mapsto \emptyset]$ iff $\mathbb{F}$ is non-empty. Therefore the class of frames corresponding to $\mathbf{S_E} + \sigma([u \mapsto \emptyset])$ is that of the empty frames. As this is also the class of frames corresponding to $\mathbf{S_{Form}}$, we obtain $\mathbf{S_{Form}} = \mathbf{S_E} + \sigma([u \mapsto \emptyset])$.

(2) A descriptive neighbourhood frame $\mathbb{F}$ can be mapped stably onto $[u \mapsto \emptyset]$ if it is non-empty and if it is empty, it can be mapped stably onto the empty frame. This means that the class corresponding to $\mathbf{S_E} + \sigma() + \sigma([u \mapsto \emptyset])$ is the empty class. Therefore, $\mathbf{Rules} = \mathbf{S_E} + \sigma() + \sigma([u \mapsto \emptyset])$ as the class of frames corresponding to **Rules** is also empty.

(3) We need to show that both consequence relations coincide on descriptive neighbourhood frames. We first show that both consequence relations have the finite model

property and hence it suffices to show that the consequence relations coincide on all finite descriptive neighbourhood frames. Firstly note that $\mathbf{S_E} + \sigma([u \mapsto \{\emptyset\}]) + \sigma([u \mapsto \{\emptyset\}, v \mapsto \emptyset])$ has the finite model property by Theorem 4.15. Secondly we can show that $\mathbf{S_E} + /\mathrm{P}$ has the finite model property by proving it is stable. Note that by Proposition 2.8, the class of frames corresponding to $\mathbf{S_E} + /\mathrm{P}$ consists of all descriptive neighbourhood frames $\mathbb{F} = \langle W, \nu, A \rangle$ such that $\emptyset \notin \nu(w)$ for each $w \in W$. Let $f : \mathbb{F} \twoheadrightarrow \mathbb{G}$ be a stable onto map with $\mathbb{F} \models \mathbf{S_E} + /\mathrm{P}$ and suppose for a contradiction that there exists $v \in \mathbb{G}$ such that $\emptyset \in \mu(v)$. By f being onto, we obtain $w \in \mathbb{F}$ such that $f(w) = v$. Now by stability, we have $\emptyset \in \nu(w)$, contradicting $\mathbb{F} \models /\mathrm{P}$. By Theorem 4.12 we obtain that $\mathbf{S_E} + /\mathrm{P}$ is stable. By Theorem 4.15, $\mathbf{S_E} + /\mathrm{P}$ has the finite model property.

We can now show that the two consequence relations are validated by exactly the same finite descriptive neighbourhood frames. Let $\mathbb{F} = \langle W, \nu \rangle$ be a finite descriptive neighbourhood frame. For the direction from right to left, suppose there exists $w \in W$ with $\emptyset \in \nu(w)$. If $|W| = 1$, we can map w stably onto the frame $[u \mapsto \{\emptyset\}]$. If $|W| > 1$, we define a map from $\mathbb{F}$ to $[u \mapsto \{\emptyset\}, v \mapsto \emptyset]$ by sending w to u and the rest to v. This again gives a stable onto map. For the direction from left to right, firstly suppose that $\mathbb{F} \not\models \sigma([u \mapsto \{\emptyset\}])$, i.e. we have a stable map from $\mathbb{F}$ onto this frame. But as $\emptyset \in \mu(u)$, we have that $\emptyset \in \nu(w)$ for every $w \in W$. At least one such w exists by surjectivity. Secondly, suppose that $\mathbb{F} \not\models \sigma([u \mapsto \{\emptyset\}, v \mapsto \emptyset])$, implying the existence of a stable map f onto $[u \mapsto \{\emptyset\}, v \mapsto \emptyset]$. Then there exists $w \in W$ such that $f(w) = u$, hence $\emptyset \in \mu(f(w))$. By stability of f, we obtain $\emptyset \in \nu(w)$.

We have now established that both consequence relations correspond on finite descriptive neighbourhood frames. As they both have the finite model property, we obtain $\mathbf{S_E} + /\mathrm{P} = \mathbf{S_E} + \sigma([u \mapsto \{\emptyset\}]) + \sigma([u \mapsto \{\emptyset\}, v \mapsto \emptyset])$. □

4.5 Monotonic Modal Logic

The results discussed in this chapter so far all relate to classical modal logics. In this section we discuss their restriction to monotonic modal logics. Most proofs can be carried over with some minor tweaks. This section is devoted to pointing out these differences and stating the main results for monotonic modal logics.

We firstly need a filtration that preserves monotonicity. It turns out we have already found such a filtration, namely the greatest filtration ν^l.

Lemma 4.22. *Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a descriptive monotonic neighbourhood frame and Σ a finite set closed under subformulas. Then the model $\mathbb{M}^f = \langle W^f, \nu^l, \mathcal{P}W^f, V^f \rangle$ is a filtration of $\mathbb{M}$ through Σ such that $\mathbb{M}^f$ is monotonic.*

Proof. By Proposition 3.37, we know that $\mathbb{M}^f$ is a filtration of $\mathbb{M}$ through Σ. We need to show that $\mathbb{M}^f$ is monotonic. Consider $X, Y \in \mathcal{P}W^f$ such that $X \in \nu^l(|w|)$ and $X \subseteq Y$. By $X \in \nu^l(|w|)$, we get $\nmid X \nmid \in \nu(v)$ for all $v \in |w|$. As $X \subseteq Y$, $\nmid X \nmid \subseteq \nmid Y \nmid$. So as $\mathbb{M}$ is monotonic, we obtain $\nmid Y \nmid \in \nu(v)$ for all $v \in |w|$. Consequently, $Y \in \nu^l(|w|)$. □

We are now ready to replicate the proofs in this chapter adapted to monotonic neighbourhood frames. With the Lemma above, we can replicate the proof of Theorem 4.1 to obtain finite refutation patterns consisting of descriptive monotonic neighbourhood frames. For finite descriptive monotonic neighbourhood frame $\mathbb{F}$ and $D \subseteq \mathbb{F}$, we call $\sigma(\mathbb{F}, D)$ a *monotonic stable canonical rule.* As Theorem 4.5 also holds for monotonic neighbourhood frames, we can reproduce every major result proved in this chapter. Firstly, we get the equivalent of the main result of Section 4.1.

Theorem 4.23.
(1) *Each monotonic modal multi-conclusion consequence relation $\mathcal{S}$ is axiomatizable by monotonic stable canonical rules. Moreover, if $\mathcal{S}$ is finitely axiomatizable, then $\mathcal{S}$ is axiomatizable by finitely many monotonic stable canonical rules.*
(2) *Each monotonic modal logic Λ is axiomatizable by monotonic stable canonical rules. Moreover, if Λ is finitely axiomatizable, then Λ is axiomatizable by finitely many monotonic stable canonical rules.*

We call a stable rule $\sigma(\mathbb{F})$ *monotonic* if $\mathbb{F}$ is monotonic. Likewise, we say that a Jankov rule $\chi(\mathbb{F})$ is *monotonic* if $\mathbb{F}$ is monotonic. We now obtain the analogue of Theorem 4.15 as well as a splitting theorem for the lattice $\mathrm{MExt}\mathbf{S_M}$ of monotonic modal multi-conclusion consequence relations.

Theorem 4.24.
(1) *Every stable monotonic modal multi-conclusion consequence relation $\mathcal{S}$ has the finite model property.*
(2) *Every stable monotonic modal logic Λ has the finite model property.*

Theorem 4.25. *Let $\mathcal{S}$ be a monotonic modal multi-conclusion consequence relation.*
(1) *$\mathcal{S}$ is splitting in* $\mathrm{MExt}\mathbf{S_M}$ *iff $\mathcal{S}$ is axiomatized by a single monotonic Jankov rule.*
(2) *$\mathcal{S}$ is join splitting* $\mathrm{MExt}\mathbf{S_M}$ *iff $\mathcal{S}$ is axiomatized by monotonic Jankov rules.*

As an example of a stable monotonic modal multi-conclusion consequence relation, we can again look at deontic logic. By the same reasoning as Theorem 4.21(3), we obtain the following characterization of the monotonic modal multi-conclusion consequence relation based on the logic **MD**.

Theorem 4.26. $\mathbf{S_M} + /\mathrm{P} = \mathbf{S_M} + \sigma([u \mapsto \{\emptyset, \{u\}\}]) + \sigma([u \mapsto \{\emptyset, \{u\}\}, v \mapsto \emptyset])$.

4.6 Master Modality and Canonical Formulas

We have now shown that any classical modal logic or multi-conclusion consequence relation can be axiomatized by stable canonical rules. A stronger result would however be to have an axiomatization in terms of canonical formulas. In the case of Kripke semantics, such axiomatizations exist when one restricts to the transitive normal modal logics above **K4**, as shown by for example Zakharyaschev [34] or alternatively proven for stable canonical formulas in [5]. The reason we are able to define such formulas above **K4** is that we have the "master modality" $\Box^+$ at our disposal. If a world w makes $\Box^+\phi$ true, ϕ is true in any successor in $R^*[w]$. On rooted models, this means that satisfiability in the model is reduced to satisfiability in the root. This allows us to replace canonical rules with canonical formulas. For the details, see for example [5].

This begs the question of whether we can mimic this behaviour in classical or monotonic modal logic. Trivially, this can be done by simply restricting ourselves to normal modal logics above **K4** but this is not very interesting. Non-trivially, coalgebra may assist us in finding an answer. The functors $\check{\mathcal{P}} \circ \check{\mathcal{P}}$ and $\mathcal{U}p\mathcal{P}$ corresponding to non-monotonic and monotonic neighbourhood frames respectively do not preserve weak preimages, implying that they do not preserve weak pullbacks. This has a number of consequences.

Most importantly in this context, it complicates the existence of a master modality. It is known that any expressible modality corresponds to a *predicate lifting* [32]. Here, a predicate lifting for a functor $T : \mathbf{Sets} \to \mathbf{Sets}$ is a natural transformation $\alpha : \check{\mathcal{P}} \to \check{\mathcal{P}} \circ T$. We would be interested in a master modality on generated submodels. We have seen in Section 2.3.2 that there exists an intimate connection between generated submodels and the notion of support. One could now look at a master modality $\heartsuit\phi$ on a generated submodel as expressing that ϕ is valid in that generated submodel, i.e. in that support. A result by Gumm [19] allows us to put these things together.

Gumm defines a map μ assigning the family of all supports to a set of neighbourhoods. He then connects the failure of $\check{\mathcal{P}} \circ \check{\mathcal{P}}$ and $\mathcal{U}p\mathcal{P}$ to preserve weak preimages with the map μ not being a natural transformation. As μ is not a natural transformation, the predicate lifting corresponding to the desired master modality can also not be natural and therefore cannot exist.

A solution to these issues is to consider a different functor that does preserve weak pullbacks (and therefore weak preimages). This is exactly what we do in Part II. We will look at the recently introduced Instantial Neighbourhood Logic (INL) [3], whose frames correspond to coalgebras for the double covariant powerset functor $\mathcal{P} \circ \mathcal{P}$. INL comes with a new modality that is more expressive than the basic modal operator. Moreover, we will see that $\mathcal{P} \circ \mathcal{P}$-coalgebras have a nice characterization of support. In INL, we will be able to define canonical formulas.

Part II

Instantial Neighbourhood Logic

Chapter 5

Syntax and Semantics of INL

As discussed in Section 4.6, the functors $\check{\mathcal{P}} \circ \check{\mathcal{P}}$ and $\mathcal{U}p\mathcal{P}$ do not preserve weak pullbacks, complicating the existence of a master modality and therefore the ability to define canonical formuals. In this part of the thesis, we look at coalgebras whose functor does preserve weak pullbacks, namely the double covariant powerset functor $\mathcal{P} \circ \mathcal{P}$. As $\mathcal{P} \circ \mathcal{P}$ acts the same on objects as $\check{\mathcal{P}} \circ \check{\mathcal{P}}$, the $\mathcal{P} \circ \mathcal{P}$-coalgebras are again neighbourhood frames. On the logical side, we will work with Instantial Neighbourhood Logic (INL), recently introduced in [3]. Its language comes with a new $n+1$-ary modal operator, more expressive than the basic modal operator.

We will explore the properties of extensions of INL. We start by introducing its language, its models and some operations and constructions on these models. We then continue by defining canonical rules for INL and proving axiomatization and characterization results similar to the ones proven in Part I. Afterwards we will illustrate the extra expressive power that INL possesses when we show the existence of canonical formulas and prove a splitting theorem for logics based on INL.

5.1 Syntax of INL

In this section we introduce the syntax of INL. Its language will contain $\neg$, $\vee$ and $\bot$ as primitive connectives together with an $n+1$-ary modal operator $\Box$ for each $n \geq 0$. We let Prop again denote the set of propositional variables. The formulas are defined as follows:

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \Box(\phi, \ldots, \phi; \phi) \text{ where } p \in \mathsf{Prop}.$$

For the other logical connectives $\top, \wedge, \to$ and $\leftrightarrow$, we use the usual abbreviations. A formula in this language we call an *instantial neighbourhood formula* and the set of all such formulas we denote by **INForm**. For a set of propositional variables Φ, we let **INForm**(Φ) denote the set of formulas with variables from Φ. For the modal operator $\Box$, we will sometimes refer to the first n arguments simply as the first argument and the last $n+1$-th argument as the second argument. We do not define a dual operator, as we will see in the next section that the modal operator $\Box$ already carries both universal and existential information.

For a set Φ of propositional variables, we call a map $\sigma : \Phi \to \mathbf{INForm}(\Phi)$ a *substitution* and as usual we can extend it to a map $(\cdot)^\sigma : \mathbf{INForm}(\Phi) \to \mathbf{INForm}(\Phi)$, where the modal case is taken care of by $\Box(\psi_1, \ldots, \psi_n; \phi)^\sigma = \Box(\psi_1^\sigma, \ldots, \psi_n^\sigma; \phi^\sigma)$. As before, we abuse notation by writing $\sigma(\phi)$ for ϕ^σ. We also define the modal depth of a formula, as it will be useful when discussing completeness of INL.

Definition 5.1 (**Modal Depth**). The *modal depth* $\mathrm{md}(\phi)$ of an instantial neighbourhood formula ϕ is defined by induction as follows:

$$\begin{aligned} \mathrm{md}(\bot) &= 0 \\ \mathrm{md}(\neg\phi) &= \mathrm{md}(\phi) \\ \mathrm{md}(\phi \vee \psi) &= \max\{\mathrm{md}(\phi), \mathrm{md}(\psi)\} \\ \mathrm{md}(\Box(\psi_1, \ldots, \psi_n; \phi)) &= \max\{\mathrm{md}(\psi_1), \ldots, \mathrm{md}(\psi_n), \mathrm{md}(\phi)\} \end{aligned}$$

We say that ϕ is *of modal depth* $\leq n$ if $\mathrm{md}(\phi) \leq n$. ⊣

The proof system for INL will be a Hilbert-style calculus. We define the logics belonging to it in the following definition.

Definition 5.2. A logic Λ in the language of INL is called an *instantial neighbourhood logic* if it contains all propositional tautologies as well as the following axioms:

(NW) $\Box p_1, \ldots, p_n; q) \to \Box(p_1, \ldots, p_n; q \vee r)$;
(SW) $\Box(p_1, \ldots, p_n, r; q) \to \Box(p_1, \ldots, p_n, r \vee s; q)$;
(SR) $\Box(p_1, \ldots, p_n, r; q) \to \Box(p_1, \ldots, p_n, r \wedge q; q)$;
(SC) $\neg\Box(\bot; p)$;
(NT) $\Box(p_1, \ldots, p_n; q) \to \Box(p_1, \ldots, p_n, r; q) \vee \Box(p_1, \ldots, p_n; q \wedge \neg r)$;
(AD) $\Box(p_1, \ldots, p_n, s, r_1, \ldots, r_m; q) \to \Box(p_1, \ldots, p_n, r_1, \ldots, r_m; q)$;
(AI) $\Box(p_1, \ldots, p_n, r_1, \ldots, r_m; q) \to \Box(p_1, \ldots, p_n, s, r_1, \ldots, r_m; q)$, provided $s \in \{p_1, \ldots, p_n, r_1, \ldots, r_m\}$

and it is closed under the rules of *modus ponens*, *uniform substitution* and the following modal rule:

$$(\mathrm{RE_{INL}}) \qquad \frac{\psi_1 \leftrightarrow \psi_1', \ldots, \psi_n \leftrightarrow \psi_n', \phi \leftrightarrow \phi'}{\Box(\psi_1, \ldots, \psi_n; \phi) \leftrightarrow \Box(\psi_1', \ldots, \psi_n', \phi')}$$

The smallest instantial neighbourhood logic we denote by **INL**. ⊣

Remark 5.3. In [3] a different axiomatization is given, based on axiom schemes and the rule for substitution of equivalents. It is however easily checked that the axiomatization given here and the one given in [3] are equivalent. We choose this axiomatization over the one presented in [3] as it more closely resembles the axiomatizations of classical and monotonic modal logics. This makes it easier to translate results from classical and monotonic modal logics over to instantial neighbourhood logics. ⊣

It is shown in [3] that a formula $\Box(\psi_1, \ldots, \psi_n; \phi)$ is provably equivalent to $\Box(\psi_{i_1}, \ldots, \psi_{i_n}; \phi)$, where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$. Moreover, adding and removing duplicates in the first argument gives a provably equivalent formula as well. Consequently, we can write $\Box(\{\psi_1, \ldots, \psi_n\}; \phi)$ for $\Box(\psi_1, \ldots, \psi_n; \phi)$.

The usual definitions of theorems, deducibility and consistency apply. The (complete bounded) lattice of all instantial neighbourhood logics is denoted by Ext**INL**, defined with the same operations as CExt**E**.

We are also interested in the multi-conclusion consequence relations based on **INL**. For two finite sets Γ and Δ of instantial neighbourhood formulas, we call Γ/Δ an *instantial neighbourhood rule*. We let **INRules** denote the set of all instantial neighbourhood rules.

Definition 5.4 (**Instantial Neighbourhood Multi-conclusion Consequence Relation**)**.** A *instantial neighbourhood multi-conclusion consequence relation* is a set $\mathcal{S}$ of instantial neighbourhood rules such that

(i) $\phi/\phi \in \mathcal{S}$;
(ii) $\phi, \phi \to \psi/\psi \in \mathcal{S}$;
(iii) $\psi_1 \leftrightarrow \psi_1', \ldots, \psi_n \leftrightarrow \psi_n', \phi \leftrightarrow \phi'/\ \Box(\psi_1, \ldots, \psi_n; \phi) \leftrightarrow \Box(\psi_1', \ldots, \psi_n', \phi') \in \mathcal{S}$;
(iv) $/\phi \in \mathcal{S}$ for each theorem $\phi \in$ **INL**;
(v) if $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$;
(vi) if $\Gamma/\Delta, \phi \in \mathcal{S}$ and $\Gamma, \phi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$;
(vii) if $\Gamma/\Delta \in \mathcal{S}$ and s is a substitution, then $s[\Gamma]/s[\Delta] \in \mathcal{S}$. ⊣

We define $\Lambda(\mathcal{S}) = \{\phi \mid /\phi \in \mathcal{S}\}$ to be the logic corresponding to instantial neighbourhood multi-conclusion consequence relation $\mathcal{S}$, as well as $\mathcal{S}(\Lambda) = \mathbf{S_{INL}} + \{/\phi \mid \phi \in \Lambda\}$ to be the consequence relation corresponding to an instantial neighbourhood logic Λ. The complete bounded lattice of all instantial neighbourhood multi-conclusion consequence relations will be denoted by Ext$\mathbf{S_{INL}}$, with the usual operations.

5.2 Semantics of INL

This section discusses the semantics of INL. As already mentioned, the models for instantial neighbourhood logics can be viewed as $\mathcal{P} \circ \mathcal{P}$-coalgebras. As the functor $\mathcal{P} \circ \mathcal{P}$ acts the same on objects as $\check{\mathcal{P}} \circ \check{\mathcal{P}}$ does, their coalgebras are the same. Therefore the models for INL are again neighbourhood models as defined in Part I (Definition 2.5). The following semantics was introduced in [3].

Definition 5.5. Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a neighbourhood model. The truth of an instantial neighbourhood formula is defined inductively as follows:

$\mathbb{M}, w \vDash \bot$		never;
$\mathbb{M}, w \vDash p$	if	$w \in V(p), p \in \mathsf{Prop}$;
$\mathbb{M}, w \vDash \neg\phi$	if	not $\mathbb{M}, w \vDash \phi$;
$\mathbb{M}, w \vDash \phi \vee \psi$	if	$\mathbb{M}, w \vDash \phi$ or $\mathbb{M}, w \vDash \psi$;
$\mathbb{M}, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$	if	$\exists X \in \nu(w)$ s.t. $\mathbb{M}, x \vDash \phi$ for all $x \in X$ and $\forall i \leq n\ \exists x \in X$ s.t. $\mathbb{M}, x \vDash \psi_i$. ⊣

Differently phrased, $\mathbb{M}, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$ if there exists $X \in \nu(w)$ such that $X \subseteq V(\phi)$ and $X \cap V(\psi_i) \neq \emptyset$ for each $i \leq n$. If $X \in \nu(w)$ satisfies these properties, we will say that X *witnesses* or *is a witness of* $\Box(\psi_1, \ldots, \psi_n; \phi)$.

Remark 5.6. As one can see, the modal operator $\Box(\psi_1, \dots, \psi_n; \phi)$ carries existential information in the first argument and universal information in the second argument. This combination of existential and universal information justifies that we do not define a dual operator of $\Box$. ⊣

Remark 5.7. If we write $\Box\phi$ for $\Box(\emptyset; \phi)$, we have that $\mathbb{M}, w \vDash \Box\phi$ iff there exists $X \in \nu(w)$ such that $X \subseteq V(\phi)$. Consequently, $\Box\phi$ corresponds to the modal operator for monotonic modal logics. In this sense, instantial neighbourhood logics are a generalization of monotonic modal logics.

This deserves the additional remark that the monotonicity lies in the semantics, as opposed to the frames being monotonic. When we do add monotonicity to the frames, the semantics of INL trivialize in the sense that $\mathbb{M}, w \vDash \Box(\psi_1, \dots, \psi_n; \phi)$ iff $V(\phi) \in \nu(w)$ and $V(\psi_i) \cap V(\phi) \neq \emptyset$ for all $i \leq n$, i.e. $V(\phi) \in \nu(w)$ and each ψ_i is consistent with ϕ in $\mathbb{M}$. This implies that on a world w for which $\nu(w) \neq \emptyset$, we are able to define universal modalities in the following sense:

$$\mathbb{M}, w \vDash \Box(\psi; \top) \quad \text{iff} \quad \exists v \in W : \mathbb{M}, v \vDash \psi;$$
$$\mathbb{M}, w \vDash \neg\Box(\neg\psi; \top) \quad \text{iff} \quad \forall v \in W : \mathbb{M}, v \vDash \psi.$$

⊣

Example 5.8. One of the ways one can interpret INL is as an extension of Evidence Logic, as presented in [4]. We can look at the formula $\Box(\psi_1, \dots, \psi_n; \phi)$ as expressing "the agent has evidence for ϕ that is consistent with each of the $\psi_1, \dots, \psi_n$". The neighbourhoods can be viewed as evidence sets for the agent from different sources. By Remark 5.7 above, indeed $\Box(\emptyset; \phi)$ behaves like the $\Box$-operator presented in [4].

With the new semantics we associate a new map $m_\nu : (\mathcal{P}W)^{<\omega} \times \mathcal{P}W \to \mathcal{P}W$. Here $X^{<\omega}$ is the set of all finite tuples of elements of X.

$$m_\nu(X_1, \dots, X_n; Y) = \{w \in W \mid \exists Z \in \nu(w) \text{ s.t. } Z \subseteq Y \;\&\; Z \cap X_i \neq \emptyset \text{ for each } i \leq n\}.$$

If there exists $Z \in \nu(w)$ such that $Z \subseteq Y$ and $Z \cap X_i \neq \emptyset$ for each $i \leq n$, we say that *Z is a witness of* or *witnesses* $w \in m_\nu(X_1, \dots, X_n; Y)$. The following easy unfolding of definitions shows that $V(\Box(\psi_1, \dots, \psi_n; \phi)) = m_\nu(V(\psi_1), \dots, V(\psi_n); V(\phi))$.

$$\begin{aligned} m_\nu(V(\psi_1), \dots, V(\psi_n); V(\phi)) &= \{w \in W \mid \exists Z \in \nu(w) \text{ s.t. } Z \subseteq V(\phi) \text{ and} \\ &\qquad Z \cap V(\psi_i) \neq \emptyset \text{ for each } i \leq n\} \\ &= \{w \in W \mid w \vDash \Box(\psi_1, \dots, \psi_n; \phi)\} \\ &= V(\Box(\psi_1, \dots, \psi_n; \phi)). \end{aligned}$$

Remark 5.9. As opposed to the case of classical modal logic, there is no direct equivalence between the map m_ν and the neighbourhood function ν, stemming from the monotonic nature of INL. ⊣

Global truth, satisfiability and validity of instantial neighbourhood formulas will be defined in the usual way, similarly to how it is done for modal formulas. The same applies to instantial neighbourhood rules. In Part II of the thesis, we will let $\Lambda(\mathbb{F}) = \{\phi \in \textbf{INForm} \mid \mathbb{F} \vDash \phi\}$ denote the (instantial neighbourhood) logic *corresponding to* or *generated by* neighbourhood frame $\mathbb{F}$. Similarly, we let $\mathcal{S}(\mathbb{F}) = \{\Gamma/\Delta$ an instantial neighbourhood rule $\mid \mathbb{F} \vDash \Gamma/\Delta\}$ denote the instantial neighbourhood multi-conclusion consequence relation corresponding to $\mathbb{F}$.

5.3 Completeness of Instantial Neighbourhood logics

In this section we prove completeness of any instantial neighbourhood logic. A completeness result for **INL** that heavily relied on a normal form representation theorem was proven in [3]. We present a more traditional canonical model construction, as was laid out in an early draft of [2]. We will use it to show completeness for any instantial neighbourhood logic.

Just as there exist normal modal logics that are not complete with respect to a class of Kripke frames, there may exist instantial neighbourhood logics that are not complete with respect to a class of neighbourhood frames. No such logics have been found yet, mainly due to how recent this topic is, but as to already encompass these possibly incomplete logics, we generalize neighbourhood frames once again. This time we tailor them to instantial neighbourhood logics. We refer to them as general INL-frames to avoid confusion with the general neighbourhood frames in Part I.

Definition 5.10 (**General INL-frame**)**.** A *general INL-frame* is a triple $\mathbb{F} = \langle W, \nu, A \rangle$ with W a set of worlds, $\nu : W \to \mathcal{PP}W$ a neighbourhood function and $A \subseteq \mathcal{P}W$ a set of admissible subsets satisfying the following closure properties:

- $X, Y \in A$ implies $X \cup Y \in A$;
- $X \in A$ implies $W \setminus X \in A$;
- $X_1, \dots, X_n, Y \in A$ implies $m_\nu(X_1, \dots, X_n; Y) \in A$.

A *general INL-model* is an INL-frame $\mathbb{F} = \langle W, \nu, A \rangle$ together with a valuation $V : \mathsf{Prop} \to A$. ⊣

Remark 5.11. There is a subtle difference between general neighbourhood frames and general INL-frames. In general neighbourhood frames, the neighbourhood function ν is forced to be of the form $\nu : W \to \mathcal{P}A$. On general INL-frames, we allow ν to be of the form $\nu : W \to \mathcal{PP}W$. We require this, as we will need it for the canonical model of an instantial neighbourhood logic to be a general INL-model. ⊣

Finding a Jónsson-Tarski style duality for INL-frames is still an open problem. We therefore cannot give a notion of a descriptive INL-frame. However, we will restrict ourselves to differentiated INL-frames, as finite differentiated INL-frames are indeed finite neighbourhood frames. Recall that a general neighbourhood frame $\mathbb{F} = \langle W, \nu, A \rangle$ is called differentiated when $w \neq v$ implies that there exists $X \in A$ such that $w \in X$ and $v \notin X$. We call a general INL-frame *differentiated* exactly when this property holds. A similar argument as the one used in Proposition 3.21 shows that finite differentiated INL-frames indeed are full. We therefore write $\mathbb{F} = \langle W, \nu \rangle$ for a finite differentiated INL-frame $\mathbb{F}$.

We now define our canonical model $\mathbb{M}_\Lambda$ for any instantial neighbourhood logic Λ. Let MCS_Λ denote the set of maximally Λ-consistent sets of formulas.

Definition 5.12 (**Canonical Modal**)**.** Let Λ be an instantial neighbourhood logic. The *canonical frame for* Λ is the triple $\mathbb{F}_\Lambda = \langle W_\Lambda, \nu_\Lambda, A_\Lambda \rangle$ such that:

(i) $W_\Lambda = \mathsf{MCS}_\Lambda$;

(ii) $Z \in \nu_\Lambda(\Gamma)$ iff $\forall \psi_1, \dots, \psi_n, \phi$: $\phi \in \bigcap Z$ and $\psi_i \in \bigcup Z$ for each i implies $\Box(\psi_1, \dots, \psi_n; \phi) \in \Gamma$;

(iii) $A_\Lambda = \{\hat{\phi} \mid \phi \text{ an instantial neighbourhood formula }\}$ where $\hat{\phi} = \{\Gamma \in \mathsf{MCS}_\Lambda \mid \phi \in \Gamma\}$;

The *canonical valuation* V_Λ on $\mathbb{F}_\Lambda$ is defined as $V_\Lambda(p) = \hat{p}$. The model $\mathbb{M}_\Lambda = \langle \mathbb{F}_\Lambda, V_\Lambda \rangle$ we call the *canonical model.* ⊣

At the basis of the proof of the truth lemma lies a normal form theorem, proved in [3]. We will restate a version of this theorem here.

Definition 5.13. Let Λ be an instantial neighbourhood logic and ϕ an instantial neighbourhood formula. We say that ϕ is *k-complete in* Λ if for each formula ψ of modal depth $\leq k$, we have $\phi \vdash_\Lambda \psi$ or $\phi \vdash_\Lambda \neg\psi$. If Λ is clear from the context, we simply say that ϕ is k-complete. If ϕ is k-complete in Λ and of modal depth $\leq k$, we call ϕ a *state description for* Λ *of depth* k, again omitting Λ if it is clear from the context. ⊣

Theorem 5.14. *Let Λ be an instantial neighbourhood logic and $\psi_1, \ldots, \psi_n, \phi$ instantial neighbourhood formulas of modal depth $\leq k$. Then $\Box(\psi_1, \ldots, \psi_n; \phi)$ is provably equivalent in Λ to a disjunction of the form:*

$$\Box(\Psi_1; \bigvee \Psi_1) \vee \cdots \vee \Box(\Psi_m; \bigvee \Psi_m)$$

where each Ψ_i is a set of state descriptions for Λ of depth k such that for each i, we have:

(i) $\bigvee \Psi_i \vdash \phi$
(ii) *for each $j \in \{1, \ldots, n\}$, there exists some $\theta \in \Psi_i$ such that $\theta \vdash \psi_j$.*

We now prove the truth lemma for the canonical model.

Lemma 5.15 (**Truth Lemma**)**.** *Let Λ be an instantial neighbourhood logic and ϕ an instantial neighbourhood formula. Then we have for any $\Gamma \in \mathsf{MCS}_\Lambda$:*

$$\mathbb{M}_\Lambda, \Gamma \vDash \phi \textit{ iff } \phi \in \Gamma.$$

Proof. The proof is by induction on complexity of ϕ. The only interesting case is the modal one.

($\Rightarrow$) Suppose $\mathbb{M}_\Lambda, \Gamma \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$. Then there is $Z \in \nu_\Lambda(\Gamma)$ such that $Z \subseteq V_\Lambda(\phi)$ and $Z \cap V_\Lambda(\psi_i) \neq \emptyset$ for each $i \leq n$. This is equivalent to $\phi \in \bigcap Z$ and $\psi \in \bigcup Z$ for each $i \leq n$. By definition of ν_Λ, we obtain $\Box(\psi_1, \ldots, \psi_n; \phi) \in \Gamma$.

($\Leftarrow$) Suppose $\Box(\psi_1, \ldots, \psi_n; \phi) \in \Gamma$, where all $\psi_1, \ldots, \psi_n, \phi$ are of modal depth $\leq k$. By Theorem 5.14, we obtain the following provable equivalence:

$$\vdash_\Lambda \Box(\psi_1, \ldots, \psi_n; \phi) \leftrightarrow \Box(\Psi_1; \bigvee \Psi_1) \vee \cdots \vee \Box(\Psi_m; \bigvee \Psi_m) \tag{5.1}$$

such that each Ψ_i is a set of state descriptions for Λ of depth k such that for each i, we have:

(i) $\bigvee \Psi_i \vdash \phi$
(ii) for each $j \in \{1, \ldots, n\}$, there exists some $\theta \in \Psi_i$ such that $\theta \vdash \psi_j$.

The remainder of the proof we divide into three claims. Together they will lead us to the result that $\mathbb{M}_\Lambda, \Gamma \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$. We first construct an infinite sequence of sets of formulas.

<u>Claim 1.</u> There exists a sequence $\Phi_0, \Phi_1, \ldots$ of finite sets of formulas such that each Φ_m consists of consistent state descriptions of depth $m + k$ such that $\bigvee \Phi_0 \vdash \phi$ and for each $j \in \{1, \ldots, n\}$ there exists $\gamma \in \Phi_0$ such that $\gamma \in \psi_j$ as well as for any $m \in \omega$:

(i) $\Box(\Phi_m; \bigvee \Phi_m) \in \Gamma$;
(ii) $\bigvee \Phi_{m+1} \vdash \bigvee \Phi_m$;
(iii) for each $\gamma' \in \Phi_m$, there exists $\gamma \in \Phi_{m+1}$ such that $\gamma \vdash \gamma'$.

Proof of Claim 1. For Φ_0, we take some Ψ_i such that $\Box(\Psi_i; \bigvee \Psi_i) \in \Gamma$. This Ψ_i exists by equivalence 5.1 and Γ being a maximally consistent set.

Now suppose Φ_m is constructed. All formulas in Φ_m are of depth $\leq k+m$ and therefore of depth $\leq k+m+1$. We write $\Box(\Phi_m; \bigvee \Phi_m)$ using Theorem 5.14 as the provable equivalent of:

$$\Box(\Theta_1; \bigvee \Theta_1) \vee \cdots \vee \Box(\Theta_l; \bigvee \Theta_l).$$

Here each Θ_i is a set of state descriptions of depth $m+k+1$ such that again $\bigvee \Theta_i \vdash \bigvee \Phi_m$ and for all $\gamma' \in \Phi_m$ there exists $\gamma \in \Theta_i$ such that $\gamma \vdash \gamma'$. Again by the equivalence there exists Θ_i such that $\Box(\Theta_i; \bigvee \Theta_i) \in \Gamma$. We set $\Phi_{m+1} = \Theta_i$. ■

With the constructed sequence of Φ_m from the claim above, we define the following set:

$$Z = \{\Delta \in W_\Lambda \mid \forall m \in \omega : \Delta \cap \Phi_m \neq \emptyset\}.$$

We show that this set is a neighbourhood of Γ and that $\phi \in \bigcap Z$ and $\psi_i \in \bigcup Z$ for each $i \leq n$. This will give us the desired result.

Claim 2. $Z \in \nu_\Lambda(\Gamma)$.

Proof of Claim 2. Let $\beta_1, \dots, \beta_l, \alpha$ be instantial neighbourhood formulas all of modal depth $\leq m+k$ such that $\alpha \in \bigcap Z$ and for each $i \leq l$, $\beta_l \in \bigcup Z$. Now look at $\Box(\Phi_m; \bigvee \Phi_m) \in \Gamma$. We show that $\vdash \Box(\Phi_m; \bigvee \Phi_m) \to \Box(\alpha_1, \dots, \alpha_n; \beta)$.

We first show that every $\gamma \in \Phi_m$ is in some $\Delta \in Z$. Take $\gamma \in \Phi_m$. We construct a chain of formulas $\gamma_0, \gamma_1, \dots$ with $\gamma_0 = \gamma$ and for each $p \in \omega$, $\gamma_q \in \Phi_{m+q}$ and $\gamma_{q+1} \vdash \gamma_q$. Such a sequence exists by property (iii) of Claim 1. All these γ_q are now consistent and can be extended to a maximally Λ-consistent set Δ, giving $\Phi_q \cap \Delta \neq \emptyset$ for all $q \geq m$. For $q < m$, note that $\gamma \vdash \bigvee \Phi_q$ for all $q < m$ by property (ii) of Claim 1. This implies that also $\Phi_q \cap \Delta \neq \emptyset$ for each $q < m$. Therefore, $\Delta \in Z$.

Secondly, we show that for each $\gamma \in \Phi_m$, $\gamma \vdash \alpha$. As each γ is a state description of depth $m+k$ and α is of modal depth $\leq m+k$, we have either $\gamma \vdash \alpha$ or $\gamma \vdash \neg\alpha$. But $\alpha \in \bigcap Z$ and γ is in some $\Delta \in Z$, giving a maximally Λ-consistent set Δ such that $\alpha \in \Delta$ and $\gamma \in \Delta$. Therefore $\gamma \vdash \alpha$.

Lastly, we show that for each β_i, there exists $\gamma \in \Phi_m$ such that $\gamma \vdash \beta_i$. Consider such a β_i. Now there exists $\Delta \in Z$ such that $\beta_i \in \Delta$. As $\Phi_m \cap \Delta \neq \emptyset$, there exists $\gamma \in \Phi_m$ such that $\{\gamma, \beta_i\}$ is consistent. But as γ is a state description of depth $m+k$ and β_i is of modal depth $\leq m+k$, we obtain $\gamma \vdash \beta_i$.

Now we have $\bigvee \Phi_m \vdash \alpha$ as well as each β_i being implied by some $\gamma \in \Phi_m$, we obtain that $\vdash \Box(\Phi_m; \bigvee \Phi_m) \to \Box(\beta_1, \dots, \beta_n; \alpha)$ implying $\Box(\beta_1, \dots, \beta_n; \alpha) \in \Gamma$. ■

Claim 3. $\phi \in \bigcap Z$ and for all $i \leq n$, $\psi_i \in \bigcup Z$.

Proof of Claim 3. An easy induction on m using property (ii) of Claim 1 shows that for each $m \in \omega$, $\bigvee \Phi_m \vdash \phi$ and thus $\gamma \vdash \phi$ for each $\gamma \in \Phi_m$. This implies $\phi \in \bigcap Z$.

Moreover, look at any ψ_i. By repeated use of property (iii) of Claim 1 we create a sequence $\gamma_0, \gamma_1, \dots$ of formulas such that each $\gamma_m \in \Phi_m$ and we have $\psi_i \dashv \gamma_0 \dashv \gamma_1 \dashv \dots$. This sequence of γ_m can be extended to an MCS Δ that is an element of Z. Therefore

$\psi_i \in \bigcup Z$. ■

With $Z \in \nu_\Lambda(\Gamma)$, $\phi \in \bigcap Z$ as well as $\psi_i \in \bigcup Z$ for each $i \leq n$, we obtain $\mathbb{M}_\Lambda, \Gamma \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$. □

Remark 5.16. Usually, the truth lemma for the canonical model provides an intuition on how to establish an algebraic duality. This is not the case here however. The proof of this truth lemma heavily relies on a normal form theorem (Theorem 5.14), making it a syntactic proof. Consequently, it does not translate directly to the construction of a duality. ⊣

We now prove the sought-after completeness result.

Theorem 5.17. *Any instantial neighbourhood logic Λ is sound and complete with respect to the class of differentiated INL-frames validating Λ.*

Proof. We get soundness by definition. For completeness, suppose that $\nvdash_\Lambda \phi$. We look at canonical frame $\mathbb{F}_\Lambda$. It is easy to see that $\mathbb{F}_\Lambda$ is a differentiated INL-frame. Now by a replication of the Truth Lemma for arbitrary valuations on $\mathbb{F}_\Lambda$, we obtain that $\mathbb{F}_\Lambda \vDash \Lambda$. We extend the consistent set $\{\neg\phi\}$ to a maximally consistent set Γ by the standard Lindenbaum construction. Finally by the Truth Lemma, we obtain $\mathbb{M}_\Lambda, \Gamma \nvDash \phi$. This gives $\mathbb{F}_\Lambda \nvDash \phi$. □

We would also like to prove a completeness result for any instantial neighbourhood multi-conclusion consequence relation. We however do not have the required tools yet to show this, as we need the notion of a generated submodel. We will return to this issue in Section 5.7.

5.4 Bounded Morphisms

We will now focus our attention on the operations on general INL-frames. In this section we discuss the concept of bounded morphisms. We define two types of morphisms. Firstly we define those corresponding to the coalgebraic morphisms of $\mathcal{P} \circ \mathcal{P}$-coalgebras. Secondly, we define a weaker variant based on the function m_ν. We will use a combination of both when defining filtrations in the next section. At the end of this section, we show that the two approaces coincide when the frame in the range of the function is finite.

Before we define the bounded moprhisms, we define some properties for maps between general INL-frames.

Definition 5.18. Let $\mathbb{F}_1 = \langle W_1, \nu_1, A_1 \rangle$ and $\mathbb{F}_2 = \langle W_2, \nu_2, A_2 \rangle$ be two general INL-frames, $D \subseteq A_2^{<\omega}$ and $f : \mathbb{F}_1 \to \mathbb{F}_2$ a map. We say f is *continuous* if for each $X' \subseteq W_2$:

$$X' \in A_2 \Rightarrow f^{-1}[X'] \in A_1.$$

We call a continuous f *stable* if for all $X' \subseteq W_2, w \in W_1$:

$$X' \in \nu_2(f(w)) \Rightarrow \exists X \in \nu_1(w) \text{ such that } f[X] = X'.$$

We say that a continuous f is *co-stable* if for all $X \subseteq W_1, w \in W$:

$$X \in \nu_1(w) \Rightarrow f[X] \in \nu_2(f(w));$$

A continuous f satisfies the *Stable Domain Condition* (SDC) for D if for all $(X_1, \ldots, X_n, Y) \in D$:

$$f^{-1}[m_{\nu_2}(X_1, \ldots, X_n; Y)] \subseteq m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y]).$$

$\dashv$

We can now introduce the coalgebraic notion of bounded morphism. We adopt the definition from [3].

Definition 5.19 (**Bounded Morphism**). Let $\mathbb{F} = \langle W, \nu, A\rangle$ and $\mathbb{F}' = \langle W', \nu', A'\rangle$ be two general INL-frames. Then a map $f : W \to W'$ is called a *frame morphism* if, for all $w \in W$:

(i) f is continuous;
(ii) f is stable;
(iii) f is co-stable.

If the map f is between two general INL-models $\mathbb{M} = \langle \mathbb{F}, V\rangle$ and $\mathbb{M}' = \langle \mathbb{F}', V'\rangle$, then we call f a *bounded morphism* if it is a frame morphism that also satisfies the following condition:

(iv) $w \in V(p)$ iff $f(w) \in V'(p)$ for every propositional letter p; $\dashv$

Properties (ii) and (iii) taken together are equivalent to $\nu'(w) = \{f[X] \mid X \in \nu(w)\}$. This makes a frame morphism exactly a coalgebraic morphism for $\mathcal{P} \circ \mathcal{P}$-coalgebras.

Proposition 5.20 (**Invariance under Bounded Morphism**). *Let $\mathbb{M} = \langle W, \nu, A, V\rangle$ and $\mathbb{M}' = \langle W', \nu', A', V'\rangle$ be two general INL-models and $f : W \to W'$ a bounded morphism. Then for all instantial neighbourhood formulas ϕ and $w \in W$, we have:*

$$\mathbb{M}, w \vDash \phi \textit{ iff } \mathbb{M}', f(w) \vDash \phi.$$

Proof. The proof is by induction on the complexity of the formula. We only show the modal case. For this, suppose $\mathbb{M}, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$, i.e. there exists $X \in \nu(w)$ such that $X \subseteq V(\phi)$ and $X \cap V(\psi_i) \neq \emptyset$. Then $f[X] \subseteq f[V(\phi)] = f[f^{-1}[V'(\phi)]] \subseteq V'(\phi)$, where the equality follows from the induction hypothesis. We also have some $x \in X \cap V(\psi_i)$, so $f(x) \in f[X] \cap V'(\psi_i)$ by induction hypothesis again. Condition (iii) of direct bounded morphism gives $f[X] \in \nu'(f(w))$, so we obtain $\mathbb{M}', f(w) \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$.

For the other direction, suppose that we have $X' \in \nu'(f(w))$ such that X' witnesses $\Box(\psi_1, \ldots, \psi_n; \phi)$. By condition (ii) of direct bounded morphism, $X' = f[X]$ for $X \in \nu(w)$. Now $X \subseteq V(\phi)$, for take $x \in X$. Then $f(x) \in f[X]$, which gives $f(x) \in V'(\phi)$, so $x \in f^{-1}[V'(\phi)] = V(\phi)$, where the equality follows from the induction hypothesis. We also get $f(x) \in f[X] \cap V'(\psi_i)$ for $x \in X$, so $x \in f^{-1}[V'(\psi_i)] = V(\psi_i)$ and thus $X \cap V(\psi_i) \neq \emptyset$. Therefore X will also be a witness for $\Box(\psi_1, \ldots, \psi_n; \phi)$. $\Box$

The condition that $\nu'(w) = \{f[X] \mid X \in \nu(w)\}$ is a strong one. It does not leave any question on how the neighbourhood function of the image model should look. The other notion of bounded morphism we discuss is a weaker one, based on the map m_ν.

Definition 5.21 (**Weak Bounded Morphism**). Let $\mathbb{F} = \langle W, \nu, A\rangle$ and $\mathbb{F}' = \langle W', \nu', A'\rangle$ be two general INL-frames. Then a map $f : W \to W'$ is called a *weak frame morphism* if:

(i) f is continuous;

(ii) $f^{-1}[m_{\nu_2}(X_1, \ldots, X_n; Y)] = m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])$ for all $X_1, \ldots, X_n, Y \in A_2$;

For two general INL-models $\mathbb{M} = \langle \mathbb{F}, V \rangle$ and $\mathbb{M}' = \langle \mathbb{F}', V' \rangle$, the map f is called a *weak bounded morphism* if it is a weak frame morphism and also satisfies the following condition:

(iii) $w \in V(p)$ iff $f(w) \in V'(p)$ for every propositional letter p.

$\dashv$

Proposition 5.22 (**Invariance under Weak Bounded Morphism**). *Let* $\mathbb{M} = \langle W, \nu, A, V \rangle$ *and* $\mathbb{M}' = \langle W', \nu', A', V' \rangle$ *be two general INL-models and* $f : W \to W'$ *a weak bounded morphism. Then for all instantial neighbourhood formulas* ϕ *and* $w \in W$ *we have:*

$$\mathbb{M}, w \vDash \phi \textit{ iff } \mathbb{M}', f(w) \vDash \phi.$$

Proof. The proof is an induction on the complexity of ϕ. We only prove the modal case. Suppose that $\mathbb{M}, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$, which is equivalent to $w \in m_\nu(V(\psi_1), \ldots, V(\psi_n); V(\phi)) = m_\nu(f^{-1}[V'(\psi_1)], \ldots, f^{-1}[V'(\psi_n)]; f^{-1}[V'(\phi)])$, where the equality follows from the induction hypothesis. By condition (ii) of a weak bounded morphism, this happens if and only if $f(w) \in m_{\nu'}(V'(\psi_1), \ldots, V'(\psi_n); V'(\phi))$ which equates to $\mathbb{M}', f(w) \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$. $\square$

Any frame morphism is also a weak frame morphism, which will become apparent from the proof of next proposition, showing that when the frame in the range of the function is finite, the two notions even coincide.

Proposition 5.23. *Let* $\mathbb{F}_1 = \langle W_1, \nu_1, A_1 \rangle$ *and* $\mathbb{F}_2 = \langle W_2, \nu_2 \rangle$ *be two general INL-frames with* $\mathbb{F}_2$ *finite and full and* $f : \mathbb{F}_1 \to \mathbb{F}_2$ *a continuous map. Then* f *is a bounded morphism iff* f *is a weak bounded morphism.*

Proof. We divide the proof into two parts. Firstly we show that f is stable iff f satisfies (SDC) for $A_2^{<\omega}$.

($\Rightarrow$) Suppose f is stable. Take $f(w) \in m_{\nu_2}(X_1, \ldots, X_n; Y)$, so there exists $X' \in \nu_2(w)$ such that $X' \subseteq Y$ and for each X_i, there exists $x'_i \in X'$ such that $x'_i \in X' \cap X_i$. By stability, there exists $X \in \nu_1(w)$ such that $X' = f[X]$. Note that from $X' \subseteq Y$ and $f[X] = X'$, we get $X \subseteq f^{-1}[Y]$. Also, from $x'_i \in X' \cap X_i$ and $f[X] = X'$, we get $x'_i = f(x_i)$ for some $x_i \in X$ and thus $x_i \in X \cap f^{-1}[X_i]$. Hence, $w \in m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])$ with X as its witness.

($\Leftarrow$) Suppose that $f^{-1}(m_{\nu_2}(X_1, \ldots, X_n; Y)) \subseteq m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])$ for all $X_1, \ldots, X_n, Y \subseteq W_2$. We consider $X' \in \nu_2(f(w))$. We take Y to be X' and write $X' = \{x_1, \ldots, x_n\}$, which we can do as $\mathbb{F}_2$ is finite. We define $X_i := \{x_i\}$ for each $1 \leq i \leq n$. Note that then, $f(w) \in m_{\nu_2}(X_1, \ldots, X_n; Y)$ and thus, by our assumption, $w \in m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])$. This means there exists $X \in \nu_1(w)$ such that $X \subseteq f^{-1}[X']$ and $X \cap f^{-1}X_i \neq \emptyset$ for each X_i. We claim that $X' = f[X]$. Note that we have $f[X] \subseteq f[f^{-1}[X']] \subseteq X'$. For the other direction of the inclusion, take $x' \in X'$. Then there exists i such that $X_i = \{x'\}$. As $X \cap f^{-1}(\{x'\}) \neq \emptyset$, there must be some $x \in X$ such that $f(x) = x'$ and thus $x' \in f[X]$.

Secondly we show that f is co-stable iff f satisfies $m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y]) \subseteq f^{-1}[m_{\nu_2}(X_1, \ldots, X_n; Y)]$ for all $(X_1, \ldots, X_n, Y) \in A_2^{<\omega}$.

(⇒) Suppose f is co-stable. Take $w \in m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])$ for $X_1, \ldots, X_n, Y \subseteq \mathcal{P}W_2$, so there exists $Z \in \nu_1(w)$ such that $Z \subseteq f^{-1}[Y]$ and for each X_i, there exists x_i such that $x_i \in Z \cap f^{-1}X_i$. By stability, $f[Z] \in \nu_2(f(w))$. Note that we have that $f[Z] \subseteq f[f^{-1}[Y]] \subseteq Y$ and $f(x_i) \in f[Z] \cap X_i$. Therefore, $f(w) \in (m_{\nu_2}(X_1, \ldots, X_n; Y))$ and thus $w \in f^{-1}[m_{\nu_2}(X_1, \ldots, X_n; Y)]$.

(⇐) Suppose $m_{\nu_1}(f^{-1}X_1, \ldots, f^{-1}X_n; f^{-1}Y) \subseteq f^{-1}(m_{\nu_2}(X_1, \ldots, X_n; Y))$ for all $X_1, \ldots, X_n, Y \subseteq W_2$. We consider $X \in \nu_1(w)$. Now let $X_i := \{x_i\}$ for each $x_i \in f[X]$. As $\mathbb{F}_2$ is finite, we have $1 \leq i \leq n$ where $n = |f[X]|$. Note that $w \in m_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[f[X]])$, because $X \subseteq f^{-1}[f[X]]$ and for each i, $X \cap f^{-1}X_i \neq \emptyset$. By our assumption, we obtain $w \in f^{-1}[m_{\nu_2}(X_1, \ldots, X_n; f[X])]$, so $f(w) \in m_{\nu_2}(X_1, \ldots, X_n; f[X])$, i.e. there exists $Z \in \nu_2(f(w))$ such that $Z \subseteq f[X]$ and $Z \cap X_i \neq \emptyset$ for each X_i. We argue that $Z = f[X]$. For, consider $f(x) \in f[X]$. Then there is some i such that $X_i = \{f(x)\}$ and as $Z \cap X_i$ is non-empty, it must be that $f(x) \in Z$. So indeed, $f[X] \in \nu_2(f(w))$.

The two proven equivalences taken together now give us that f is a frame morphism iff f is a weak frame morphism. □

Remark 5.24. The direction from left to right in the proposition above can also be proven from a coalgebraic perspective. For this, note that the $n+1$-ary modal operator m_ν arises from an $n+1$-ary predicate lifting λ_ν of the functor $\mathcal{P} \circ \mathcal{P}$. Condition (ii) of Definition 5.21 of a weak bounded morphism can now be rewritten as follows:

$$\nu_2(f(w)) \in \lambda_{\nu_2}(X_1, \ldots, X_n, Y) \Leftrightarrow \nu(w) \in \lambda_{\nu_1}(f^{-1}[X_1], \ldots, f^{-1}[X_n], f^{-1}[Y]).$$

If we assume f to be a coalgebraic morphism, this equivalence can be easily derived making use of the fact that λ_{ν_1} and λ_{ν_2} are predicate liftings. ⊣

Finite differentiated INL-frames are full, by Proposition 3.21. Therefore, the proposition above restricted to differentiated INL-frames only requires the frame in the range of f to be finite.

5.5 Filtrations

In this section we introduce a notion of filtration of general INL-models. We define filtrations, show they indeed preserve the satisfiability of formulas and prove their existence. Recall that for Kripke models, a filtration map is like a bounded morphism but with one of the conditions weakened. We will take a similar approach here. The non-weakened condition will come from the definition of bounded morphism, whereas the weakened condition will stem from the definition of a weak bounded morphism.

In the same fashion as Section 2.3.3, we define an equivalence relation $\sim_\Sigma$ on the set of worlds in a model $\mathbb{M}$ where $w \sim_\Sigma v$ iff w and v satisfy exactly the same formulas $\phi \in \Sigma$. Here Σ is a subformula closed set of instantial neighbourhood formulas.

Definition 5.25. Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a general INL-model and Σ a finite set of instantial neighbourhood formulas closed under subformulas. Then a model $\mathbb{M}^f = \langle W^f, \nu^f, A^f, V^f \rangle$ is called a *filtration of* $\mathbb{F}$ *through* Σ if it satisfies the following conditions:

(i) $W^f = W_\Sigma$;
(ii) $X \in \nu(w)$ implies $|X| \in \nu^f(|w|)$;

(iii) for all $\Box(\psi_1, \ldots, \psi_n; \phi) \in \Sigma$: $|w| \in m_{\nu^f}(|V(\psi_1)|, \ldots, |V(\psi_n)|; |V(\phi)|)$ implies $w \in m_\nu(V(\psi_1), \ldots, V(\psi_n); V(\phi))$;

(iv) $V^f(p) = |V(p)|$ for all propositional letters p;

(v) $|\cdot|$ is a continuous map. $\dashv$

Remark 5.26. As opposed to the filtration defined for general neighbourhood frames, here the full condition (ii) is one from the original model to the filtrated model. The restricted condition (iii) holds for the other direction. In this sense, this definition more closely resembles the definition of filtrations of Kripke models. The explanation for this can be found in the fact that we are dealing with $\mathcal{P} \circ \mathcal{P}$-coalgebras, where the direct image map plays a crucial role. $\dashv$

Lemma 5.27 (Filtration Lemma). *Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a general INL-model and Σ a finite subformula closed set of instantial neighbourhood formulas. Let $\mathbb{M}^f = \langle W^f, \nu^f, V^f \rangle$ be a filtration of $\mathbb{M}$ through Σ. Then for each formula $\phi \in \Sigma$ and each $w \in W$:*

$$\mathbb{M}, w \vDash \phi \Leftrightarrow \mathbb{M}^f, |w| \vDash \phi.$$

Proof. The proof is an easy induction on the complexity of ϕ. We only consider the modal case. From left to right, suppose $\mathbb{M}, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$ giving $X \in \nu(w)$ such that $X \subseteq V(\phi) = \nmid V^f(\phi) \nmid$ and $X \cap V(\psi_i) = X \cap \nmid V^f(\psi_i) \nmid \neq \emptyset$. The equalities follow from the induction hypothesis. By condition (ii) of Definition 5.25 of filtrations, we obtain $|X| \in \nu^f(|w|)$. Some easy set-theoretic manipulations give $|X| \subseteq V^f(\phi)$ and $|X| \cap V^f(\psi_i) \neq \emptyset$, implying that $\mathbb{M}^f, |w| \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$.

For the other direction, suppose we have $X' \in \nu^f(|w|)$ such that $X' \subseteq V^f(\phi) = |V(\phi)|$ and $X' \cap V^f(\psi_i) = X' \cap |V(\psi_i)| \neq \emptyset$ for each i, where the equalities again follow from the induction hypothesis. Condition (iii) gives $X \in \nu(w)$ such that $X \subseteq V(\phi)$ and $X \cap V(\psi_i) \neq \emptyset$. Hence, $\mathbb{M}, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$. $\Box$

We now define the following filtration, which turns out to be the smallest one.

$$X' \in \nu^s(|w|) \text{ iff } \exists v \in |w| \; \exists X \in \nu(v) \text{ such that } X' = |X|.$$

We show that this defined neighbourhood function indeed gives rise to a filtration. However, as we force continuity on the equivalence map $|\cdot|$, we restrict ourselves to a finite Σ.

Proposition 5.28 (Existence of Filtrations). *Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a general INL-model and Σ a finite subformula closed set of formulas. Then the model $\mathbb{M}^s = \langle W_\Sigma, \nu^s, \mathcal{P}W^f, V^f \rangle$ is a filtration of $\mathbb{M}$ through Σ. Moreover, ν^s is the smallest filtration.*

Proof. Firstly note that condition (ii) from Definition 5.25 of filtration is satisfied by definition of ν^s. To show condition (iii), let $\Box(\psi_1, \ldots, \psi_n; \phi) \in \Sigma$. Now consider $X' \in \nu^s(|w|)$ such that $X' \subseteq |V(\phi)|$ and $X' \cap |V(\psi_i)| \neq \emptyset$ for each $i \leq n$. Then $X' = |X|$ for some $X \in \nu(v)$ with $v \in |w|$. As $\nmid |V(\chi)| \nmid = V(\chi)$ for each $\chi \in \Sigma$, we obtain that $X \subseteq V(\phi)$ and $X \cap V(\psi_i) \neq \emptyset$ for each $i \leq n$. Consequently $\mathbb{M}^s, v \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$. By definition of $\sim_\Sigma$ and $v \in |w|$, we obtain that also $\mathbb{M}^s, w \vDash \Box(\psi_1, \ldots, \psi_n; \phi)$, i.e. $w \in m_\nu(V(\psi_1), \ldots, V(\psi_n); V(\phi))$.

To show that ν^s is the smallest filtration, suppose $X' \in \nu^s(|w|)$ meaning there exist $v \in |w|$ and $X \in \nu(v)$ such that $|X| = X'$. Therefore, by condition (ii) we obtain $X' \in \nu^f(|w|)$ for any filtration. $\Box$

We can capture the filtration map in terms of the previously defined notions of co-stability and Stable Domain Condition. The following proposition expresses this fact.

Proposition 5.29. *Let $\mathbb{M} = \langle W, \nu, A, V\rangle$ and $\mathbb{M}^f = \langle W_\Sigma, \nu^f, A^f, V^f\rangle$ be two general INL-models with Σ a subformula closed set of instantial neighbourhood formulas and $V^f(p) = |V(p)|$. Then $|\cdot|$ is a co-stable map satisfying* (SDC) *for $\{(V^f(\psi_1), \ldots, V^f(\psi_n)) \mid \Box(\psi_1, \ldots, \psi_n; \phi) \in \Sigma\}$ iff $\mathbb{M}^f$ is a filtration of $\mathbb{M}$ through Σ.*

Proof. Firstly note that condition (ii) of Definition 5.25 of filtration expresses exactly that $|\cdot|$ is co-stable. Moreover, by the Filtration Lemma we obtain that $|V(\phi)| = V^f(\phi)$ and $V(\phi) = \{\!\!\{V^f(\phi)\}\!\!\}$ for each $\phi \in \Sigma$. Therefore condition (iii) implies that $|\cdot|$ satisfies (SDC) for $\{(V^f(\psi_1), \ldots, V^f(\psi_n)) \mid \Box(\psi_1, \ldots, \psi_n; \phi) \in \Sigma\}$.

We can also replicate the proof of the Filtration Lemma when replacing condition (iii) of a filtration with the condition that $|\cdot|$ satisfies (SDC) for $\{(V^f(\psi_1), \ldots, V^f(\psi_n)) \mid \Box(\psi_1, \ldots, \psi_n; \phi) \in \Sigma\}$. Consequently, condition (iii) and $|\cdot|$ satisfying (SDC) for $\{(V^f(\psi_1), \ldots, V^f(\psi_n)) \mid \Box(\psi_1, \ldots, \psi_n; \phi) \in \Sigma\}$ are equivalent. □

5.6 Support Relation

In this section we discuss the *support relation*, a relation underlying a neighbourhood frame, together with all operations based on it. This relation corresponds to the coalgebraic support of a set of neighbourhoods on $\mathcal{P}\circ\mathcal{P}$-coalgebras. It will play an important role in the remainder of this thesis. We will use it to construct generated submodels as well as unravellings. In a certain sense we are reducing the neighbourhood structure to a relational structure. This allows us to use some well-known methods for Kripke frames.

Definition 5.30 (Support Relation). Let $\mathbb{F} = \langle W, \nu, A\rangle$ be a general INL-frame. We define the *support relation* $S_\mathbb{F} \subseteq W \times W$ by setting, for $u, w \in W$:

$$(w, u) \in S_\mathbb{F} \text{ if and only if } u \in \bigcup \nu(w)$$

We let $S_\mathbb{F}^\star$ denote the reflexive and transitive closure of $S_\mathbb{F}$. Moreover, we inductively define $S_\mathbb{F}^0[w] = \{w\}$ and $S_\mathbb{F}^{n+1}[w] = \{v \in W \mid (u, v) \in S_\mathbb{F} \text{ for some } u \in S_\mathbb{F}^n\}$. ⊣

Remark 5.31. We have seen this relation defined on neighbourhood frames before in Remark 2.16 where we considered it to define generated submodels in the context of monotonic neighbourhood frames. In that context, the support relation trivializes in the sense that there exist only two generated submodels: either the full set W or the empty set. For INL-frames, it does not trivialize. ⊣

For a general INL-model $\mathbb{M}$, we sometimes write $S_\mathbb{M}$ meaning the support relation $S_\mathbb{F}$ for the underlying frame $\mathbb{F}$.

As mentioned before, the support relation stems from the notion of support on $\mathcal{P}\circ\mathcal{P}$-coalgebras. It is instructive to see how the two notions correspond. Firstly note that $\mathcal{P}\circ\mathcal{P}$ preserves inclusions, meaning that $\mathcal{P}\circ\mathcal{P}\iota_{S,W} = \iota_{\mathcal{P}\circ\mathcal{P}S,\mathcal{P}\circ\mathcal{P}W}$. Now for a $\mathcal{P}\circ\mathcal{P}$-coalgebra $\langle W, \nu\rangle$, a subset $S \subseteq W$ is a support for some $N \in \mathcal{P}\mathcal{P}W$ if there exists $M \in \mathcal{P}\mathcal{P}S$ such that $\mathcal{P}\circ\mathcal{P}\iota_{S,W}(M) = N$, i.e. $M = N$. Consequently, $S \subseteq W$ is a support for $N \in \mathcal{P}\mathcal{P}W$ if $Z \subseteq S$ for each $Z \in N$. The smallest such support now exists, namely $\bigcup N$. This is exactly the image of the support relation. We summarize this in the proposition below.

Proposition 5.32. *Let $\mathbb{F} = \langle W, \nu, A \rangle$ a general INL-frame. Then $S_{\mathbb{F}}[w]$ is the smallest (coalgebraic) support for $\nu(w)$, when interpreting $\langle W, \nu \rangle$ as a $\mathcal{P} \circ \mathcal{P}$-coalgebra.*

Looking at a neighbourhood structure in terms of this support relation provides us with a new perspective. It allows us to borrow well-known constructions from Kripke frames. We will discuss the notion of generated submodels as well as unravellings in terms of this support relation. This will be used to prove a convenenient completeness result for **INL**. We will then briefly discuss the expressibility of the support relation in the language of INL and end with the promised completeness result for instantial neighbourhood multi-conclusion consequence relations.

5.6.1 Generated Submodels

We now look at generated submodels. As we saw in Section 2.3.2, submodels for neighbourhood frames are not very intuitive. When using the support relation however, we can draw inspiration from the construction for Kripke frames.

We first define submodels of a general INL-model.

Definition 5.33 (Submodel). Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a general INL-model and $W_0 \subseteq W$ a subset. The *submodel of* $\mathbb{M}$ *restricted to* W_0, denoted by $\mathbb{M}{\restriction}_{W_0}$ is defined as the structure $\langle W_0, \nu_0, A_0, V_0 \rangle$ where:

- $\nu_0(w) = \{X \cap W_0 \mid X \in \nu(w)\}$ for each $w \in W_0$;
- $A_0 = \{X \cap W_0 \mid X \in A\}$;
- $V_0(p) = V(p) \cap W_0$.

For a general INL-frame $\mathbb{F}$, the *subframe* $\mathbb{F}{\restriction}_{W_0}$ *of* $\mathbb{F}$ *restricted to* W_0 is obtained via the same reasoning leaving out the valuation. $\dashv$

Generally, a submodel $\mathbb{M}{\restriction}_{W_0}$ will not be a well-defined general INL-model as A' might not be closed under m_{ν_0}. However, when the set W_0 is admissible in the original model, we do have a well-defined model.

Lemma 5.34. *Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a general INL-model and $W_0 \subseteq W$ a subset. If $W_0 \in A$, then $\mathbb{M}{\restriction}_{W_0}$ is a well-defined general INL-model.*

Proof. The tricky case is A_0 being closed under m_{ν_0}. For this, take $X_1', \ldots, X_n', Y' \in A'$, so $Y' = Y \cap W_0$ and $X_i' = X_i \cap W_0$ for $X_1, \ldots, X_n, Y \in A$. We show that $m_{\nu_0}(X_1', \ldots, X_n'; Y') \in A$. By $W_0 \in A$, we have $X_i \cap W_0 \in A$ for each i as well as $W_0 \to Y \in A$. Via some easy set-theoretic manipulations, the following equality can be shown:

$$m_{\nu_0}(X_1', \ldots, X_n'; Y') = m_\nu(X_1 \cap W_0, \ldots, X_n \cap W_0; W_0 \to Y) \cap W_0.$$

Now as $m_\nu(X_1 \cap W_0, \ldots, X_n \cap W_0; W_0 \to Y) \in A$, we obtain $m_{\nu_0}(X_1', \ldots, X_n'; Y') \in A_0$. $\square$

We now look at generated submodels. We adopt the definition from [3], but use our above-defined notion of submodel.

Definition 5.35 (**Generated Submodel**). Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a general INL-model and $\mathbb{M}_0 = \langle W_0, \nu_0, A_0, V_0 \rangle$ a submodel of $\mathbb{M}$. Then $\mathbb{M}_0$ is called a *generated submodel of* $\mathbb{M}$ if the following hereditary condition holds for all $w, v \in W$:

$$w \in W_0 \text{ and } wS_{\mathbb{M}}v \Rightarrow v \in W_0. \tag{5.2}$$

Given general INL-model $\mathbb{M} = \langle W, \nu, V \rangle$ and subset $W_0 \subseteq W$, we define the *submodel generated by* W_0 *in* $\mathbb{M}$ as the submodel $\mathbb{M}{\restriction_{W'}}$ where W' is the smallest set satisfying the hereditary condition above.

When W_0 is a singleton $\{w\}$, we call $\mathbb{M}[\{w\}]$ a *point-generated submodel* and denote it as $\mathbb{M}[w]$. A subframe $\mathbb{F}_0$ is called a *generated subframe* if it satisfies the hereditary condition above. ⊣

There are a few things to remark on this definition. Firstly, for a generated submodel $\mathbb{M}_0$ of $\mathbb{M}$ we have $\nu_0 = \nu{\restriction_{W_0}}$ as $\bigcup \nu(w) \subseteq W_0$ for each $w \in W_0$. This holds because W_0 is closed under $S_{\mathbb{M}}$. From this, we also obtain that $\mathbb{M}_0$ is always well-defined, even if $W_0 \notin A$, by the following equality:

$$m_{\nu'}(X_1 \cap W', \dots, X_n \cap W'; Y \cap W') = m_\nu(X_1, \dots, X_n; Y) \cap W'.$$

Secondly, for any general INL-model $\mathbb{M}$ and subset $W_0 \subseteq W$, the submodel $\mathbb{M}[W_0]$ generated by W_0 is always a generated submodel, as opposed to generated submodels on monotonic neighbourhood frames. For this, we take $W' = \{v \in W \mid \exists w \in W_0$ such that $(w, v) \in S^\star_{\mathbb{M}}\}$. This set is the smallest set that satisfies the hereditary condition 5.2. Then submodel $\mathbb{M}{\restriction_{W'}}$ is the submodel generated by W_0.

Remark 5.36. Coalgebraically, the hereditary condition 5.2 expresses exactly that W_0 is a support for $\nu(w)$ for all $w \in W_0$. This gives us a similar correspondence as the one we have for monotonic neighbourhood frames in Lemma 2.18. However, the smallest support for $\mathcal{P} \circ \mathcal{P}$-coalgebras now always exists, whereas for $\mathcal{U}p\mathcal{P}$-coalgebras it does not. ⊣

The invariance of any formula under the operation of generated submodel is an easy check.

Proposition 5.37 (**Invariance under Generated Submodel**). *Let* $\mathbb{M} = \langle W, \nu, A, V \rangle$ *be a general INL-model and* $\mathbb{M}[W_0] = \langle W', \nu', A', V' \rangle$ *a generated submodel generated by subset* $W_0 \subseteq W$*. Then the inclusion map* $\iota : W' \to W$ *is a bounded morphism from* $\mathbb{M}[W_0]$ *to* $\mathbb{M}$*. Therefore, for all instantial neighbourhood formulas* ϕ *and all* $w \in W'$*, we have:*

$$\mathbb{M}[W_0], w \models \phi \textit{ iff } \mathbb{M}, w \models \phi.$$

In particular, for a point-generated submodel $\mathbb{M}[w_0]$*, we have:*

$$\mathbb{M}[w_0], w_0 \models \phi \textit{ iff } \mathbb{M}, w_0 \models \phi.$$

This notion of point-generated submodel immediately gives us an idea of how to define a rooted neighbourhood frame, again drawing inspiration from the constructions on Kripke frames.

Definition 5.38 (Support-Rooted). A general INL-frame $\mathbb{F} = \langle W, \nu, A \rangle$ is called *support-rooted* if there exists some world $r \in W$ such that $S^*_{\mathbb{F}}[r] = W$. Such a world r is called a *root of* $\mathbb{F}$.

A general INL-model $\mathbb{M}$ is called *support-rooted* if its underlying frame structure is support-rooted. $\dashv$

One of the goals of this section is to construct support-rooted frames that are not too large. This size can be bounded in two different ways: depth and width. For this reason, we introduce the height and the branching degree of a world.

Definition 5.39 (Height and Branching Degree). Let $\mathbb{F} = \langle W, \nu, A \rangle$ be a general INL-frame. We say a world $w \in W$ is *of height* $\leq n$ if all $S_{\mathbb{F}}$-paths from w are of length $\leq n$. The frame $\mathbb{F}$ is of height $\leq n$ if all worlds in $\mathbb{F}$ are of height $\leq n$. $\mathbb{F}$ is *of finite height* if it is of height $\leq n$ for some $n < \omega$.

We say $w \in W$ has *branching degree* $\leq n$ if w has at most n distinct $S_{\mathbb{F}}$-successors. The frame $\mathbb{F}$ has branching degree $\leq n$ if all worlds have branching degree $\leq n$ and $\mathbb{F}$ has *finite branching degree* if it has branching degree $\leq n$ for some $n < \omega$. $\dashv$

To restrict the height of a support-rooted frame, we introduce the notion of a depth k point-generated submodel. This will be a generated submodel based on $S^k_{\mathbb{M}}$ instead of $S^*_{\mathbb{M}}$, i.e. we cut off the point-generated submodel after k steps. The definition is adopted from [3].

Definition 5.40 (Depth k Point-Generated Submodel). Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be any INL-model, $w_0 \in W$ and k any integer. The *depth* k *point-generated submodel of* $\mathbb{M}$ *generated by* w_0, denoted by $\mathbb{M}[w_0, k]$, is defined as the submodel $\mathbb{M}\restriction_{S^k_{\mathbb{M}}[w_0]}$. $\dashv$

As is the case with submodels, the depth k point-generated submodel may not be well-defined, as the set of admissible subsets might not be closed under the modal operator. By Lemma 5.34, it is well-defined when $S^k_{\mathbb{M}}[w_0] \in A$. When it is well-defined, a root of a depth k point-generated submodel preserves instantial neighbourhood formulas up to modal depth k.

Proposition 5.41 (Invariance under Depth k Point-Generated Submodel). *Let $\mathbb{M} = \langle W, \nu, A, V \rangle$ be a general INL-model such that $S^k_{\mathbb{M}} \in A$, $w_0 \in W$ and ϕ an instantial neighbourhood formula of modal depth $\leq k$. Then:*

$$\mathbb{M}, w_0 \vDash \phi \text{ iff } \mathbb{M}[w_0, k], w_0 \vDash \phi.$$

Proof. This proposition follows as an easy corollary of a slightly stronger result: for all formulas ϕ with depth $\leq m$ and all worlds v reachable with a path from w_0 of length n, we have $\mathbb{M}, v \vDash \phi$ iff $\mathbb{M}[w_0, m+n], v \vDash \phi$. This proof is an easy induction on the complexity of ϕ, that we leave to the reader. $\square$

It is important to note that a depth k point-generated submodel is in general not of height $\leq k$. There might exist cycles in the support relation, creating a depth k point-generated submodel of infinite height. When the support-relation of a general INL-frame does not contain cycles, we call the frame *cycle-free*. The next section deals with creating such cycle-free frames.

5.6.2 Unravellings

In this section we look at a method of removing cycles from a general INL-model and creating tree-like structures. We will use the unravelling construction from [3]. We only define the unravelling of a general INL-model whenever the original model is full. Defining such a construction for any general INL-model is far from a trivial matter and for our purposes, unravellings for full INL-models suffice.

Definition 5.42 (Unravelling). Let $\mathbb{M} = \langle W, \nu, V \rangle$ be a full INL-model and $w_0 \in W$. The *tree-unravelling of* $\mathbb{M}$ *at* w_0 is defined to be the structure $\mathbb{M}^U_{w_0} = \langle W', \nu', A', V' \rangle$, where:

- W' is the set of all finite sequences $(w_0, \dots, w_n)$ such that $(w_i, w_{i+1}) \in S_{\mathbb{M}}$ for each $0 \leq i < n$;
- $\nu'((w_0, \dots, w_n)) = \{\{(w_0, \dots, w_n, v) \mid v \in Z\} \mid Z \in \nu(w_n)\}$;
- $A' = \mathcal{P}W'$;
- $V'(p) = \{(w_0, \dots, w_n) \in V' \mid w_n \in V(p)\}$. ⊣

The invariance under this operation is an easy consequence of the map $(w_0, \dots, w_n) \mapsto w_n$ being a bounded morphism.

Proposition 5.43 (Invariance under Unravellings). *Let* $\mathbb{M} = \langle W, \nu, A, V \rangle$ *be a full INL-model and* $w_0 \in W$*. Then for any instantial neighbourhood formula* ϕ*:*

$$\mathbb{M}, w_0 \vDash \phi \text{ iff } \mathbb{M}^U_{w_0}, (w_0) \vDash \phi.$$

It is not hard to see that a tree-unravelling is a cycle-free INL-model. With the notion of unravelling together with depth k point-generated submodels and filtrations, we prove a useful completeness result for **INL**.

Theorem 5.44 (Completeness). *The logic* **INL** *is sound and complete with respect to the class of finite support-rooted INL-frames of finite height.*

Proof. The soundness follows from **INL** being sound with respect to differentiated INL-frames. For completeness, we suppose $\nvdash \phi$ for instantial neighbourhood formula ϕ of modal depth $\leq n$. By completeness with respect to differentiated INL-frames (Theorem 5.17), we obtain differentiated INL-model and world w such that $\mathbb{M}, w \nvDash \phi$. We now filtrate $\mathbb{M}$ through the set of subformulas of ϕ to obtain a finite differentiated INL-model $\mathbb{M}^f$. This model $\mathbb{M}^f$ is now full. We take the tree-unravelling of $\mathbb{M}^f$ at $|w|$, giving a full cycle-free INL-model $(\mathbb{M}^f)^U_{|w|}$. As $\mathbb{M}^f$ was finite, this unravelled model has a finite branching degree. We now finish by taking the depth n point-generated submodel generated by $(|w|)$. As the unravelled model had a finite branching degree, the depth n point-generated submodel is finite. Moreover, as the unravelled model was cycle-free, our point-generated submodel is of finite height. By the invariance results of all these constructions, the formula ϕ is refuted at the root of the point-generated submodel. This gives a finite support-rooted differentiated INL-frame of finite height refuting ϕ. □

5.6.3 Expressibility

In this section we see that the support relation is actually expressible in the language of INL. We define the modal operator $\diamondsuit\phi := \Box(\phi; \top)$ with its dual $\boxdot\phi := \neg\Box(\neg\phi; \top)$.

These operators can be easily shown to be exactly the operators expressing reachability in terms of the support relation, i.e.:

$$\begin{array}{lll} \mathbb{M}, w \vDash \boxdot\phi & \text{iff} & \forall v \in S_{\mathbb{M}}[w] : \mathbb{M}, v \vDash \phi; \\ \mathbb{M}, w \vDash \Diamond\!\!\!\cdot\,\phi & \text{iff} & \exists v \in S_{\mathbb{M}}[w] : \mathbb{M}, v \vDash \phi. \end{array}$$

We iterate the operators by defining $\boxdot^0\phi = \phi$ and $\boxdot^{n+1}\phi = \boxdot\boxdot^n\phi$ and $\Diamond\!\!\!\cdot\,^n$ similarly. Moreover, we define $\blacksquare_n\phi = \bigwedge_{m\leq n}\boxdot^m\phi$.

Now consider a finite support-rooted differentiated INL-model $\mathbb{M}$ of height $\leq n$ with root r. For an instantial neighbourhood formula ϕ, satisfiability of ϕ on $\mathbb{M}$ is now reduced to truth in r, i.e.

$$\mathbb{M}, r \vDash \blacksquare_n\phi \qquad \text{iff} \qquad \mathbb{M} \vDash \phi.$$

This means we have found the master modality that we lacked in the case of classical and monotonic modal logic. As we will see in Section 6.2.2, this will allow us to prove a splitting theorem for all instantial neighbourhood logics.

In fact, $\boxdot$ is nothing more than a modality for the underlying support relation structure of a neighbourhood frame. It behaves exactly like the modality $\Box$ does for the accessibility relation in Kripke frames. The following proposition establishes a syntactic proof for this: $\boxdot$ is a normal operator.

Proposition 5.45. *The operator $\boxdot$ satisfies Necessitation and the Kripke axiom.*

Proof. For Necessitation, we make the following derivation.

$$\begin{array}{lr} \vdash \phi & \text{(assumption)} \\ \vdash \neg\phi \to \bot & \text{(by classical logic)} \\ \vdash \Box(\neg\phi; \top) \to \Box(\bot; \top) & \text{(rule (SW}')\text{ [3, Lemma 4.2])} \\ \vdash \neg\Box(\bot; \top) \to \neg\Box(\neg\phi; \top) & \text{(by classical logic)} \\ \vdash \neg\Box(\neg\phi; \top) & \text{(by (SC) and modus ponens)} \\ \vdash \boxdot\phi & \text{(by definition of } \boxdot) \end{array}$$

For the Kripke axiom, we look at the following derivation.

$$\begin{array}{lr} \vdash \neg\psi \to \big[\neg(\phi \to \psi) \vee \neg\phi\big] & \text{(classical tautology)} \\ \vdash \Box(\neg\psi; \top) \to \Box(\neg(\phi \to \psi); \top) \vee \Box(\neg\phi; \top) & \text{(by (SS) [3, Lemma 4.2])} \\ \vdash \neg\Box(\neg(\phi \to \psi); \top) \wedge \neg\Box(\neg\phi; \top) \to \neg\Box(\neg\psi; \top) & \text{(by classical logic)} \\ \vdash \boxdot(\phi \to \psi) \wedge \boxdot\phi \to \boxdot\psi & \text{(by definition of } \boxdot) \\ \vdash \boxdot(\phi \to \psi) \to (\boxdot\phi \to \boxdot\psi) & \text{(by classical logic)} \end{array}$$

$\square$

5.7 Completeness for Instantial Neighbourhood Consequence Relations

In this section we prove the promised completeness result for any instantial neighbourhood multi-conclusion consequence relation with respect to a class of differentiated INL-frames. We will prove this by taking a specific generated submodel of the canonical model discussed in Section 5.3, as done by Jeřábek [22] for normal modal multi-conclusion consequence relations.

Theorem 5.46. *Let $\mathcal{S}$ be an instantial neighbourhood multi-conclusion consequence relation. Then $\mathcal{S}$ is sound and complete with respect to the class of differentiated INL-frames validating $\mathcal{S}$.*

Proof. Soundness is easy and is left out. For completeness, we take instantial neighbourhood rule Γ/Δ such that $\mathcal{S} \not\vdash \Gamma/\Delta$. We can look at the set P of all pairs $\langle x, y\rangle$ of sets of instantial neighbourhood formulas satisfying the following two properties:

(i) $\Gamma \subseteq x$ and $\Delta \subseteq y$;

(ii) for all finite $\Gamma' \subseteq x$ and $\Delta' \subseteq y$, $\Gamma'/\Delta' \notin \mathcal{S}$.

This set P can be viewed as a poset with the pairwise subset relation as an order. Each chain in this poset has a maximal element, so by Zorn's Lemma, we obtain a pair $\langle x, y\rangle$ that is maximal with respect to conditions (i) and (ii) above.

Firstly note that $x \cap y = \emptyset$, as if not, there exists $\phi \in x \cap y$ contradicting $\phi/\phi \in \mathcal{S}$. Moreover, $x \cup y$ cover all instantial neighbourhood formulas. To show this, suppose for a contradiction there exists $\phi \notin x \cup y$. As $\phi \notin x$, by x and y being maximal there exists finite $\Gamma_x \subseteq x$ and $\Delta_x \subseteq y$ such that $\Gamma_x, \phi/\Delta_x \in \mathcal{S}$. As $\phi \notin y$, we obtain finite $\Gamma_y \subseteq x$ and $\Delta_y \subseteq y$ such that $\Gamma_y/\Delta_y, \phi \in \mathcal{S}$. This gives $\Gamma_x \cup \Gamma_y, \phi/\Delta_x \cup \Delta_y \in \mathcal{S}$ as well as $\Gamma_x \cup \Gamma_y/\Delta_x \cup \Delta_y, \phi \in \mathcal{S}$. By $\mathcal{S}$ being closed under the Cut rule, we obtain $\Gamma_x \cup \Gamma_y/\Delta_x \cup \Delta_y \in \mathcal{S}$ contradicting condition (ii) of x and y.

Now note that x is actually closed under rules in $\mathcal{S}$, i.e. for $\Gamma'/\Delta' \in \mathcal{S}$, we have that $\Gamma' \subseteq x$ implies $x \cap \Delta' \neq \emptyset$. For suppose not. Then $\Delta' \cap x = \emptyset$, which by $x \cap y = \emptyset$ and $x \cup y$ being the set of all instantial neighbourhood formulas gives $\Delta' \subseteq y$. This however contradicts property (ii) of x and y.

We now look at the canonical frame $\mathbb{F}_{\mathbf{INL}}$. We define $W = \{\Gamma' \in W_{\mathbf{INL}} \mid x \subseteq \Gamma'\}$ and we show that $\mathbb{F}_{\mathbf{INL}}{\restriction}_W$ is a generated subframe, i.e. it satisfies the following hereditary condition:

$$\Gamma' \in W \text{ and } \Gamma' S_{\mathbb{F}_{\mathbf{INL}}} \Delta' \Rightarrow \Delta' \in W.$$

We use the following claim to show this.

<u>**Claim 1.**</u> $\Gamma' S_{\mathbb{F}_{\mathbf{INL}}} \Delta' \Rightarrow \left[\exists Z \in \nu_{\mathbf{INL}}(\Gamma')\ \forall \phi \in \mathbf{INForm} : \boxdot\phi \in \Gamma' \Rightarrow \phi \in \bigcap Z\right]$.
Proof of Claim 1. Suppose $\Gamma' S_{\mathbb{F}_{\mathbf{INL}}} \Delta'$. Then there exists $Z \in \nu_{\mathbf{INL}}(\Gamma')$ such that for all instantial neighbourhood formulas $\psi_1, \dots, \psi_n, \phi$, we have that $\phi \in \bigcap Z$ and $\psi_i \in \bigcup Z$ for each i implies $\Box(\psi_1, \dots, \psi_n; \phi) \in \Gamma'$. In particular, we have that $\top \in \bigcap Z$ and $\neg\phi \in \bigcup Z$ implies $\Box(\neg\phi; \top) \in \Gamma'$. Contraposition now gives that $\neg\Box(\neg\phi; \top) \in \Gamma'$ implies $\top \notin \bigcup Z$, which is never the case, or $\neg\phi \notin \bigcup Z$, i.e. $\phi \in \bigcap Z$. This means that $\boxdot\phi \in \Gamma'$ implies $\phi \in \bigcap Z$. ∎

Now suppose $\Gamma' \in W$ and $\Gamma' S_{\mathbb{F}_{\mathbf{INL}}} \Delta'$. We show $x \subseteq \Delta'$. Consider $\phi \in x$. As $\phi/\boxdot\phi \in \mathcal{S}$, we obtain $\boxdot\phi \in x$. Then as $\Gamma' \in W$, we have $\boxdot\phi \in \Gamma'$ by x being closed under rules in $\mathcal{S}$. The claim above now gives $\phi \in \Delta'$. So $\mathbb{F}_{\mathbf{INL}}{\restriction}_W$ is indeed a generated subframe. Note that as $\mathbb{F}_{\mathbf{INL}}$ is differentiated, so is $\mathbb{F}_{\mathbf{INL}}{\restriction}_W$. The following claim now shows that our restricted frame $\mathbb{F}_{\mathbf{INL}}{\restriction}_W$ indeed validates $\mathcal{S}$.

<u>**Claim 2.**</u> $\mathbb{F}_{\mathbf{INL}}{\restriction}_W \vDash \mathcal{S}$.

Proof of Claim 2. Take an admissible valuation V on $\mathbb{F}_{\mathbf{INL}}\restriction_W$. This valuation V can be viewed as a substitution σ if we set $\sigma(p) = \phi$ where $V(p) = \hat{\phi}$. We first show that $\langle \mathbb{F}_{\mathbf{INL}}\restriction_W, V\rangle \models \phi$ iff $\sigma(\phi) \in x$. For the direction from right to left, $\sigma(\phi) \in x$ gives $\sigma(\phi) \in \Gamma'$, implying $\langle \mathbb{F}_{\mathbf{INL}}, V\rangle, \Gamma' \models \phi$ for each $\Gamma' \in W$. Conversely, suppose that $\sigma(\phi) \notin x$. This means $x \cup \{\neg\sigma(\phi)\}$ is a consistent set and can therefore be extended to maximally **INL**-consistent set $\mathcal{E} \supseteq x$ such that $\sigma(\phi) \notin \mathcal{E}$. Therefore $\mathcal{E} \in W$. Now from $\langle \mathbb{F}_{\mathbf{INL}}, V\rangle, \mathcal{E} \not\models \phi$ we obtain $\langle \mathbb{F}_{\mathbf{INL}}\restriction_W, V\rangle \not\models \phi$.

Now take any rule $\Gamma'/\Delta' \in \mathcal{S}$ and suppose $\langle W, V\rangle \models \Gamma'$. This implies $\sigma[\Gamma'] \subseteq x$. As $\mathcal{S}$ is closed under substitution, we also have $\sigma[\Gamma']/\sigma[\Delta'] \in \mathcal{S}$. By x being closed under rules in $\mathcal{S}$, we obtain $\sigma[\Delta'] \cap x \neq \emptyset$, giving $\sigma(\delta) \in \sigma[\Delta']$ such that $\sigma(\delta) \in x$. Consequently, $\langle W, V\rangle \models \sigma(\delta)$ and therefore $\langle W, V\rangle \models \Gamma'/\Delta'$. ■

The canonical valuation restricted to $\mathbb{F}_{\mathbf{INL}}\restriction_W$ now refutes Γ/Δ. This gives us a differentiated INL-frame validating $\mathcal{S}$ but refuting Γ/Δ. □

Remark 5.47. The implication we show in Claim 1 in the proof above should look familiar to the reader familiar with the canonical model construction for Kripke frames. In the canonical model for **K**, the canonical relation R^c is defined as:

$$\Gamma R^c \Delta \text{ iff for all modal formulas } \phi\text{: } \Box\phi \in \Gamma \text{ implies } \phi \in \Delta.$$

Claim 1 now states that the support relation on the canonical neighbourhood frame implies a similar condition imposed on the canonical relation, now for the normal modal operator $\boxdot$. This justifies the choice of this particular canonical model. ⊣

We will make great use of the proven completeness results as well as the notions of filtration and generated submodel when we define co-stable canonical rules and formulas in the next chapter.

Chapter 6

Canonical Rules and Formulas for INL

This chapter is divided into two parts. In the first part we introduce refutation patters for each instantial neighbourhood rule, introduce co-stable canonical rules and prove that any instantial neighbourhood multi-conclusion consequence relation and any instantial neighbourhood logic is axiomatized by these co-stable canonical rules. We define co-stable classes of frames and show they have the finite model property as well as give a characterization theorem for splittings of the lattice Ext$\mathbf{S_{INL}}$. The results will largely be analoguous to results of Chapter 4 and will therefore not be discussed in detail. We only address where their proofs differ from the ones given before.

In the second part of this chapter we exceed the limitations that classical and monotonic modal logic posed and define co-stable canonical formulas. We use these formulas to prove the main result of this chapter: a characterization theorem for splittings in the lattice Ext**INL**. After that, we will touch upon the subject of transitivity for INL-frames and discuss what is needed to obtain axiomatization results for all instantial neighbourhood logics in terms of canonical formulas. We end with a few examples of consequence relations that are axiomatized by canonical rules.

6.1 Co-Stable Canonical Rules for INL

In this section we look at canonical rules for INL. We start by showing the existence of finite refutation patterns for instantial neighbourhood rules. Afterwards we define co-stable canonical rules $\sigma(\mathbb{F}, D)$, refuted only when there exists a co-stable map onto $\mathbb{F}$ satisfying (SDC) for D. We then combine these results to obtain the main theorem of this section, namely that each instantial neighbourhood multi-conclusion consequence relation and every instantial neighbourhood logic is axiomatized by co-stable canonical rules.

6.1.1 Finite Refutation Patterns for INL

We start by defining finite refutation patterns. We define the patterns with the same intuition as for classical modal logics. They are a finite collection of finite INL-frames that characterize exactly when a differentiated INL-frame refutes an instantial neighbourhood rule ρ.

Theorem 6.1 (**Finite Refutation Patterns**).

(1) *For each instantial neighbourhood rule* Γ/Δ*, there exist pairs* $\langle \mathbb{F}_1, D_1\rangle, \ldots, \langle \mathbb{F}_n, D_n\rangle$ *such that each* $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ *is a finite differentiated INL-frame refuting* Γ/Δ*,* $D_i \subseteq (\mathcal{P}W_i)^{<\omega}$*, and for each differentiated INL-frame* $\mathbb{G} = \langle W, \nu, A\rangle$*, we have* $\mathbb{G} \not\models \Gamma/\Delta$ *iff there exists* $i \leq n$ *and a co-stable surjective map* $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$ *satisfying* (SDC) *for* D_i*.*

(2) *For each instantial neighbourhood formula* ϕ*, there exist pairs* $\langle \mathbb{F}_1, D_1\rangle, \ldots, \langle \mathbb{F}_n, D_n\rangle$ *such that each* $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ *is a finite differentiated INL-frame refuting* ϕ*,* $D_i \subseteq (\mathcal{P}W_i)^{<\omega}$*, and for each differentiated INL-frame* $\mathbb{G} = \langle W, \nu, A\rangle$*, we have* $\mathbb{G} \not\models \phi$ *iff there exists* $i \leq n$ *and co-stable surjective map* $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_i$ *satisfying* (SDC) *for* D_i*.*

Proof. The proof of this theorem is an analogue of the proof of Theorem 4.1. The refutation patterns will look as follows: let Γ/Δ be an instantial neighbourhood rule such that $\mathbf{S_{INL}} \not\vdash \Gamma/\Delta$ and let Θ be the set of all subformulas of formulas in $\Gamma \cup \Delta$. Let m denote the cardinality of Θ. The refutation pattern for Γ/Δ consists of all pairs $\langle \mathbb{F}, D\rangle$ such that:

- $\mathbb{F}$ is a finite differentiated INL-frame of size $\leq 2^m$ and $\mathbb{F} \not\models \Gamma/\Delta$.
- $D = \{(V(\psi_1), \ldots, V(\psi_n), V(\phi)) \mid \Box(\psi_1, \ldots, \psi_n; \phi) \in \Theta\}$ where V is a valuation on $\mathbb{F}$ witnessing $\mathbb{F} \not\models \Gamma/\Delta$.

For the required equivalence we merely note that a filtration of a differentiated INL-frame through a finite Σ is again a differentiated INL-frame. Moreover, such a filtration is a co-stable onto map satisfying (SDC) for $\{(V^f(\psi_1), \ldots, V^f(\psi_n), V^f(\phi)) \mid \Box(\psi_1, \ldots, \psi_n; \phi) \in \Theta\}$ by Proposition 5.29. □

We now define co-stable canonical rules. Their definition will bear close resemblance to that of stable canonical rules (Defintion 4.2). We only change the clauses of the rule containing the modal operator, which we now tailor to the extended modal operator of INL.

Definition 6.2 (**Co-stable Canonical Rules**). Let $\mathbb{F} = \langle W, \nu\rangle$ be a finite differentiated INL-frame. We introduce a propositional letter p_w for each $w \in W$ and a letter s_X for each subset $X \subseteq W$. For each $D \subseteq (\mathcal{P}W)^{<\omega}$, we define the *co-stable canonical rule* $\sigma(\mathbb{F}, D)$ to be the rule Γ/Δ, where:

$$\begin{aligned}
\Gamma = \{ & \bigvee_{w\in W} p_w\} \cup \{p_w \to \neg p_v \mid w, v \in W, w \neq v\} \\
& \cup \{s_X \leftrightarrow \bigvee_{w\in X} p_w \mid X \subseteq W\} \\
& \cup \{s_{\Box(X_1,\ldots,X_n;Y)} \to \Box(s_{X_1}, \ldots, s_{X_n}; s_Y) \mid (X_1, \ldots, X_n, Y) \in D\} \\
& \cup \{\Box(s_{X_1}, \ldots, s_{X_n}; s_Y) \to s_{\Box(X_1,\ldots,X_n;Y)} \mid X_1, \ldots, X_n, Y \subseteq W\} \\
\Delta = \{ & \neg p_w \mid w \in W\}.
\end{aligned}$$

⊣

As before, any finite differentiated INL-frame refutes its own co-stable canonical rule.

Lemma 6.3. *Let* $\mathbb{F} = \langle W, \nu\rangle$ *be a finite differentiated INL-frame and* $D \subseteq (\mathcal{P}W)^{<\omega}$ *a subset. Then* $\mathbb{F} \not\models \sigma(\mathbb{F}, D)$*.*

Proof. We define valuation V on $\mathbb{F}$ with $V(p_w) = \{w\}$ and $V(s_X) = X$. We only show that for the two modal formulas ϕ in Γ, we have $V(\phi) = W$. The rest of the proof is an analogue of the proof of Lemma 4.4.

Suppose $\langle \mathbb{F}, V\rangle, w \vDash s_{\Box(X_1,\dots,X_n;Y)}$ for $(X_1, \dots, X_n, Y) \in D$. Then $w \in m_\nu(X_1, \dots, X_n; Y)$, so there exists $Z \in \nu(w)$ such that $Z \subseteq Y = V(s_Y)$ and $Z \cap X_i = Z \cap V(s_{X_i}) \neq \emptyset$ for all i. Then $w \in V(\Box(X_1, \dots, X_n; Y))$ and thus $\langle \mathbb{F}, V\rangle, w \vDash \Box(s_{X_1}, \dots, s_{X_n}; s_Y)$.

Suppose $\langle \mathbb{F}, V\rangle, w \vDash \Box(s_{X_1}, \dots, s_{X_n}; s_Y)$ for $X_1, \dots, X_n, Y \in A$. Then $w \in V(\Box(s_{X_1}, \dots, s_{X_n}; s_Y))$, which by the same reasoning as above equals $m_\nu(X_1, \dots, X_n; Y)$. Therefore, $\langle \mathbb{F}, V\rangle, w \vDash s_{\Box(X_1,\dots,X_n;Y)}$. $\square$

Theorem 6.4. *Let $\mathbb{F}_0 = \langle W_0, \nu_0\rangle$ be a finite differentiated INL-frame, $\mathbb{F} = \langle W, \nu, A\rangle$ an arbitrary differentiated INL-frame and $D \subseteq (\mathcal{P}W_0)^{<\omega}$ a subset. Then $\mathbb{F} \nvDash \sigma(\mathbb{F}_0, D)$ if and only if there's an onto co-stable function $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$ satisfying (SDC) for D.*

Proof. The proof will be an analogue of that of Theorem 4.5.

($\Rightarrow$) Suppose $\mathbb{F} \nvDash \sigma(\mathbb{F}_0, D)$ giving valuation V on $\mathbb{F}$ witnessing this. We again define a map $f : \mathbb{F} \to \mathbb{F}_0$ by $f(w) = v$ iff $w \in V(p_v)$. We only show that f is co-stable and satisfies (SDC) for D. For the remainder, we refer to the proof of Theorem 4.5.

For the co-stability condition, we consider $X \in \nu(w)$. We need that $f[X] \in \nu_0(f(w))$. We write $f[X] = \{v_1, \dots, v_n\}$. Note that $X \subseteq f^{-1}[f[X]]$ and $X \cap f^{-1}[\{v_i\}] \neq \emptyset$ for each i. This gives $\langle \mathbb{F}, V\rangle, w \vDash \Box(s_{\{v_1\}}, \dots, s_{\{v_n\}}; s_{f[X]})$. By the implication $\Box(s_{a_1}, \dots, s_{a_n}; s_b) \to s_{\Box(a_1,\dots,a_n;b)}$ being in Γ for all $a_1, \dots, a_n, b \subseteq W_0$, we obtain $\langle \mathbb{F}, V\rangle, w \vDash s_{\Box(\{v_1\},\dots,\{v_n\};f[X])}$. This gives $f(w) \in m_{\nu_0}(\{v_1\}, \dots, \{v_n\}; f[X])$ and therefore $f[X] \in \nu_0(f(w))$, as required.

For the (SDC) for D, consider $f(w) \in m_{\nu_0}(X_1, \dots, X_n; Y)$. This gives $\langle \mathbb{F}, V\rangle, w \vDash s_{\Box(X_1,\dots,X_n;Y)}$. By the implication $s_{\Box(X_1,\dots,X_n;Y)} \to \Box(s_{X_1}, \dots, s_{X_n}; s_Y)$ being in Γ, we obtain $\langle \mathbb{F}, V\rangle, w \vDash \Box(s_{X_1}, \dots, s_{X_n}; s_Y)$, which means $w \in m_\nu(f^{-1}[X_1], \dots, f^{-1}[X_n]; f^{-1}[Y])$ by $V(s_Z) = f^{-1}[Z]$ for all $Z \subseteq W_0$.

($\Leftarrow$) Suppose there exists a co-stable onto map $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$ satisfying (SDC) for D. By Lemma 6.3, we obtain valuation V_0 on $\mathbb{F}_0$ witnessing that $\mathbb{F}_0 \nvDash \sigma(\mathbb{F}_0, D)$. We define a valuation V on $\mathbb{F}$ by setting $V(p_v) = f^{-1}[V_0(p_v)] = f^{-1}[\{v\}]$ and $V(s_X) = f^{-1}[V_0(s_X)] = f^{-1}[X]$. We show that $V(\phi) = W$ for the two modal clauses $\phi \in \Gamma$. For the remainder of the proof, we refer to the proof of Theorem 4.5. We take $(X_1, \dots, X_n, Y) \in D$.

$$\begin{aligned}
&V(s_{\Box(X_1,\dots,X_n;Y)} \\
&\quad \to \Box(s_{X_1}, \dots, s_{X_n}; s_Y)) = V(\neg s_{\Box(X_1,\dots,X_n;Y)}) \cup V(\Box(s_{X_1}, \dots, s_{X_n}; s_Y)) \\
&\qquad = [W \setminus V(s_{\Box(X_1,\dots,X_n;Y)})] \cup \\
&\qquad\quad m_\nu(V(s_{X_1}), \dots, V(s_{X_n}); V(s_Y)) \\
&\qquad = \big[W \setminus f^{-1}[m_{\nu_0}(X_1, \dots, X_n; Y)]\big] \cup \\
&\qquad\quad m_\nu(f^{-1}[X_1], \dots, f^{-1}[X_n]; f^{-1}[Y]) \\
&\qquad \supseteq \big[W \setminus f^{-1}[m_{\nu_0}(X_1, \dots, X_n; Y)]\big] \cup \\
&\qquad\quad f^{-1}[m_{\nu_0}(X_1, \dots, X_n; Y)] \text{ (by } f \text{ being co-stable)} \\
&\qquad = W
\end{aligned}$$

Now take any $X_1, \ldots, X_n, Y \subseteq W_0$.

$$
\begin{aligned}
& V(\Box(s_{X_1}, \ldots, s_{X_n}; s_Y) \\
& \qquad \to s_{\Box(X_1,\ldots,X_n;Y)}) = [W \setminus V(\Box(s_{X_1}, \ldots, s_{X_n}; s_Y))] \cup \\
& \qquad\qquad V(s_{\Box(X_1,\ldots,X_n;Y)}) \\
& \qquad = [W \setminus m_\nu(V(s_{X_1}), \ldots, V(s_{X_n}); V(s_Y))] \cup \\
& \qquad\qquad f^{-1}[m_{\nu_0}(X_1, \ldots, X_n; Y)] \\
& \qquad = [W \setminus m_\nu(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])] \cup \\
& \qquad\qquad f^{-1}[m_{\nu_0}(X_1, \ldots, X_n; Y)] \\
& \qquad \supseteq [W \setminus m_\nu(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y])] \cup \\
& \qquad\qquad m_\nu(f^{-1}[X_1], \ldots, f^{-1}[X_n]; f^{-1}[Y]) \\
& \qquad\qquad\qquad \text{(by } f \text{ satisfying (SDC) for } D) \\
& \qquad = W
\end{aligned}
$$

□

We now have all required to state the main result of this section.

Theorem 6.5.

(1) *For each instantial neighbourhood rule Γ/Δ, there exist paris $\langle \mathbb{F}_1, D_1\rangle, \ldots, \langle \mathbb{F}_n, D_n\rangle$ such that each $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ is a finite differentiated INL-frame refuting Γ/Δ, $D_i \subseteq (\mathcal{P}W_i)^{<\omega}$, and for each differentiated INL-frame $\mathbb{G} = \langle W, \nu\rangle$ we have:*

$$\mathbb{G} \models \Gamma/\Delta \text{ iff } \mathbb{G} \models \sigma(\mathbb{F}_1, D_1), \ldots, \sigma(\mathbb{F}_n, D_n).$$

(2) *For each instantial neighbourhood formula ϕ, there exist pairs $\langle \mathbb{F}_1, D_1\rangle, \ldots, \langle \mathbb{F}_n, D_n\rangle$ such that each $\mathbb{F}_i = \langle W_i, \nu_i\rangle$ is a finite differentiated INL-frame refuting ϕ, $D_i \subseteq (\mathcal{P}W_i)^{<\omega}$, and for each differentiated INL-frame $\mathbb{G} = \langle W, \nu\rangle$ we have:*

$$\mathbb{G} \models \phi \text{ iff } \mathbb{G} \models \sigma(\mathbb{F}_1, D_1), \ldots, \sigma(\mathbb{F}_n, D_n).$$

Proof. The proof is an analogue of that of Theorem 4.6, now making use of Theorems 6.1 and 6.4 instead of 4.1 and 4.5. □

Theorem 6.6.

(1) *Each instantial neighbourhood multi-conclusion consequence relation $\mathcal{S}$ is axiomatizable by co-stable canonical rules. Moreover, if $\mathcal{S}$ is finitely axiomatizable, then $\mathcal{S}$ is axiomatizable by finitely many co-stable canonical rules.*

(2) *Each instantial neighbourhood logic Λ is axiomatizable by co-stable canonical rules. Moreover, if Λ is finitely axiomatizable, then Λ is axiomatizable by finitely many co-stable canonical rules.*

Proof. The proof of this result is an analogue of that of Theorem 4.7. We now use Theorem 6.5 and the completeness of instantial neighbourhood multi-conclusion consequence relations and logics with respect to differentiated INL-frames, Theorems 5.46 and 5.17. □

6.1.2 Finite Model Property

Similar to the case of classical modal logics, we take a closer look at specific co-stable canonical rules. For rule $\sigma(\mathbb{F}, D)$ for finite differentiated INL-frame $\mathbb{F} = \langle W, \nu \rangle$ and $D \subseteq \mathcal{P}W$, we call the rule a *co-stable rule* whenever $D = \emptyset$ and denote it by $\sigma(\mathbb{F})$. When $D = \mathcal{P}W$, we call the rule a *Jankov rule* and denote it by $\chi(\mathbb{F})$. Note that co-stable rules are refuted exactly when there exists a co-stable onto map, whereas a Jankov rule is refuted whenever there exists an onto bounded morphism.

We will start with the co-stable rules. As an analogue of the stable classes, we will introduce co-stable classes.

Definition 6.7 (**Co-stable Class**).

(1) A class $\mathcal{K}$ of differentiated INL-frames is called *co-stable* if for frames $\mathbb{F}$ and $\mathbb{G}$, if $\mathbb{F} \in \mathcal{K}$ and there exists an onto co-stable map $f : \mathbb{F} \twoheadrightarrow \mathbb{G}$, then $\mathbb{G} \in \mathcal{K}$.

(2) An instantial neighbourhood multi-conclusion consequence relation $\mathcal{S}$ is called *co-stable* if its corresponding class $\mathcal{K}_{\mathcal{S}} = \{\mathbb{F} \mid \mathbb{F} \vDash \mathcal{S}\}$ of differentiated INL-frames is co-stable.

(3) An instantial neighbourhood logic Λ is called *co-stable* if there exists an instantial neighbourhood multi-conclusion consequence relation $\mathcal{S}$ such that $\Lambda = \Lambda(\mathcal{S})$. ⊣

Again, this terminology can bo shown to be well-chosen by the following theorem.

Theorem 6.8. *An instantial neighbourhood multi-conclusion consequence relation $\mathcal{S}$ is co-stable if and only if $\mathcal{S}$ is axiomatizable by co-stable rules.*

Proof. The proof is an analogue of a similar proof in classical modal logic (Theorem 4.12). We merely note that $\mathcal{S}$ is complete with respect to a class of differentiated INL-frames by Theorem 5.46 and the composition of two co-stable maps is again co-stable by $g[f[X]] = (g \circ f)[X]$. □

We can now go on to prove our main result regarding co-stable rules. The classes axiomatized by these rules will have the finite model property. The definition of the finite model property is similar to Definition 4.14.

Theorem 6.9.

(1) *Every co-stable instantial neighbourhood multi-conclusion consequence relation has the finite model property.*

(2) *Every co-stable instantial neighbourhood logic has the finite model property.*

Proof. The proof is similar to that of Theorem 4.15. We use completeness of instantial neighbourhood multi-conclusion consequence relations with respect to differentiated INL-frames, Theorem 5.46. □

Corollary 6.10. **INL** *and* $\mathbf{S_{INL}}$ *have the finite model property.*

6.1.3 Splittings

We now discuss Jankov rules in more detail. We show that rule systems axiomatized by these rules exactly characterize splittings and join splittings in the lattice $\mathrm{Ext}\mathbf{S_{INL}}$ of instantial neighbourhood multi-conclusion consequence relations. In the next section we will see that we can do the same for instantial neighbourhood logics using canonical formulas.

Splittings and join splittings in $\mathrm{Ext}\mathbf{S_{INL}}$ are defined similarly to Definition 4.17.

Theorem 6.11. *Let $\mathcal{S}$ be an instantial neighbourhood multi-conclusion consequence relation.*

(1) *$\mathcal{S}$ is splitting in* $\mathrm{Ext}\mathbf{S_{INL}}$ *iff $\mathcal{S}$ is axiomatizable by a single Jankov rule.*
(2) *$\mathcal{S}$ is join splitting in* $\mathrm{Ext}\mathbf{S_{INL}}$ *iff $\mathcal{S}$ is axiomatizable by Jankov rules.*

Proof. The proof is an analogue of that of Theorem 4.19. Note that Proposition 4.18 holds in the lattice $\mathrm{Ext}\mathbf{S_{INL}}$ as well. □

6.2 Co-Stable Canonical Formulas

In this section we show that we can define co-stable canonical formulas in the setting of INL. We will use them to show that every splitting in the lattice Ext**INL** of instantial neighbourhood logics is axiomatized by these co-stable canonical formulas. It is at this point that we can go beyond the results we were able to obtain for classical and monotonic modal logics. We will make great use of the support relation and the modality $\boxdot$.

6.2.1 Co-stable Canonical Formulas

In the Kripke case, canonical formulas are defined for rooted transitive frames, where there exists a master modality [22, 5]. We discuss transitivity for INL-frames in Section 6.3. For now, we will focus on the frames we need to prove a splitting theorem. It is here that we use the completeness result with respect to finite support-rooted differentiated INL-frames of finite height (Theorem 5.44). We define the co-stable canonical formulas only for these particular frames.

Definition 6.12 (Co-Stable Canonical Formula). Let $\mathbb{F} = \langle W, \nu \rangle$ be a finite support-rooted differentiated INL-frame of height $\leq n$ and $D \subseteq (\mathcal{P}W)^{<\omega}$ a subset. We introduce propositional variables p_w for each $w \in W$ and s_X for each $X \subseteq W$. Using Γ and Δ as in Definition 6.2, we define the *co-stable canonical formula* $\epsilon(\mathbb{F}, D)$ as follows:

$$\epsilon(\mathbb{F}, D) = \left[\boxdot^{n+1} \bot \wedge \bigwedge\{\blacksquare_n \gamma \mid \gamma \in \Gamma\}\right] \to \bigvee\{\blacksquare_n \delta \mid \delta \in \Delta\} \qquad \dashv$$

We call a co-stable canonical formula $\epsilon(\mathbb{F}, D)$ a *Jankov formula* when $D = \mathcal{P}W$ and denote it by $\epsilon(\mathbb{F})$. Firstly, we show what this formula expresses.

Theorem 6.13. *Let $\mathbb{F}_0 = \langle W_0, \nu_0 \rangle$ be a finite support-rooted differentiated INL-frame of height $\leq n$, $D \subseteq \mathcal{P}W_0$ and $\mathbb{F} = \langle W, \nu, A \rangle$ any differentiated INL-frame. Then $\mathbb{F} \not\models \epsilon(\mathbb{F}_0, D)$ if and only if there exists a point-generated subframe $\mathbb{F}'$ of $\mathbb{F}$ and co-stable onto map $f : \mathbb{F}' \twoheadrightarrow \mathbb{F}_0$ satisfying* (SDC) *for D.*

Proof. ($\Leftarrow$) Supposethat we have point-generated subframe $\mathbb{F}' = \langle W', \nu', A' \rangle$ of $\mathbb{F}$ and co-stable onto map $f : \mathbb{F}' \twoheadrightarrow \mathbb{F}_0$ satisfying (SDC) for D. We define valuation V_0 on $\mathbb{F}_0$ by $V_0(p_w) = \{w\}$ and $V_0(s_X) = X$ for $w \in W$ and $X \subseteq W$. By the proof of Lemma 6.3 we obtain $V_0(\gamma) = W_0$ and $V_0(\delta) \neq W_0$ for all $\gamma \in \Gamma$ and $\delta \in \Delta$. It then easily follows that $V_0(\blacksquare_n \gamma) = W_0$ for $\gamma \in \Gamma$. Moreover, we have that $V_0(\blacksquare_n \delta) \neq W_0$ for each $\delta \in \Delta$ as each such δ can be refuted at a root r_0 of $\mathbb{F}_0$. By $\mathbb{F}_0$ being of height $\leq n$, we also get $V_0(\boxdot^{n+1} \bot) = W_0$. Therefore, we obtain $\mathbb{F}_0 \not\models \epsilon(\mathbb{F}_0, D)$.

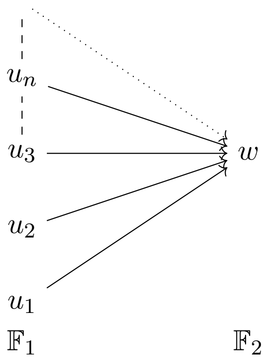
Now define a valuation V' on $\mathbb{F}'$ by setting $V'(p_w) = f^{-1}[V_0(p_w)] = f^{-1}[\{w\}]$ and $V'(s_X) = f^{-1}[V_0(s_X)] = f^{-1}[X]$, well-defined by continuity. By the proof of Theorem 6.4, we obtain $V'(\gamma) = W'$ as well as $V'(\delta) \neq W'$ for $\gamma \in \Gamma$ and $\delta \in \Delta$. It immediately follows that $V'(\blacksquare_n \gamma) = W'$ for all $\gamma \in \Gamma$ and thus, $V'(\bigwedge\{\blacksquare_n \gamma\}_{\gamma \in \Gamma}) = W'$. For $\boxdot^{n+1} \bot$ being true in the whole of $\mathbb{F}'$, suppose for a contradiction we have $v \in W'$ such that $\langle \mathbb{F}', V' \rangle, v \not\vDash \boxdot^{n+1} \bot$. This means there exists a path $vX_0v_1X_1 \dots v_nX_n$ such that $X_0 \in \nu'(v)$ as well as $X_i \in \nu'(v_i)$ and $v_i \in X_{i-1}$ for $1 \leq i \leq n$. Now by f being co-stable, we obtain a path $f(v)f[X_0]f(v_1)f[X_1] \dots f(v_n)f[X_n]$ in $\mathbb{F}_0$. This gives a contradiction with $\mathbb{F}_0$ being of height $\leq n$, so $V'(\boxdot^{n+1} \bot) = W'$. Note that this actually means that $\mathbb{F}'$ is of height $\leq n$ as well. Therefore, for each $\delta \in \Delta$, $\blacksquare_n \delta$ can be refuted at a root, implying that that root also refutes $\bigvee\{\blacksquare_n \delta\}_{\delta \in \Delta}$.

($\Rightarrow$) Suppose $\mathbb{F} \not\vDash \epsilon(\mathbb{F}_0, D)$, so we have valuation V on $\mathbb{F}$ and world w witnessing this. We now take the submodel of $\langle \mathbb{F}, V \rangle$ generated from w and denote it by $\mathbb{M}[w]$. By $\langle \mathbb{F}, V \rangle, w \vDash \boxdot^{n+1} \bot$, we get $S^{\star}_{\mathbb{F}}[w] = S^{n}_{\mathbb{F}}[w]$ and hence, $\mathbb{M}[w]$ is of height $\leq n$. We now need a map $f : \mathbb{M}[w] \twoheadrightarrow \mathbb{F}_0$ that is co-stable and satisfies (SDC) for D. We define it as $f(u) = v$ iff $\mathbb{M}[w], u \vDash p_v$. Note that we still have $\mathbb{M}[w], w \vDash \epsilon(\mathbb{F}_0, D)$ and $\mathbb{M}[w]$ being of height $\leq n$, so we actually have $\mathbb{M}[w] \vDash \gamma$ for all $\gamma \in \Gamma$ as well as $\mathbb{M}[w] \not\vDash \delta$ for each $\delta \in \Delta$. Now by the same reasoning as in the proof of Theorem 6.4, we obtain that f has the correct properties. □

6.2.2 Splittings

We will now prove a splitting theorem for all instantial neighbourhood logics. A splitting theorem for the lattice of normal modal logics has been proved by Blok [10], see also [12, Chapter 10]. We will use similar reasoning here, greatly leaning on the support relation.

In the proofs of the splitting theorems for multi-conclusion consequence relations (Theorems 4.19 and 6.11), we explicitly make use of the rule Size_n, expressing that a frame has cardinality $\leq n$. However, a formula expressing such a property does in general not exist, as is clear from the following rather trivial example. Here all worlds u_i as well as w have empty neighbourhood functions. The arrows depict a frame morphism. Both frames $\mathbb{F}_1$ and $\mathbb{F}_2$ clearly satisfy the same formulas, but one is infinite and the other is not.



When we restrict our attention to support-rooted frames of finite height however, we are able to express finiteness. We look at depth and branching degree of a frame. We use n propositional variables $p_1, \dots, p_n$ and define formulas $\phi_i := p_1 \wedge \dots \wedge p_{i-1} \wedge$

$\neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$. For each $n < \omega$, we define two formulas Height_n and Branch_n as follows:

$$\mathsf{Height}_n := \boxdot^n \bot$$
$$\mathsf{Branch}_n := \neg(\diamonddot\phi_1 \wedge \cdots \wedge \diamonddot\phi_n)$$

It is easy to check that on a differentiated INL-frame $\mathbb{F}$, Height_n is valid whenever $\mathbb{F}$ is of height $\leq n-1$, whereas Branch_n is valid only if $\mathbb{F}$ has branching degree $\leq n-1$. If $\mathbb{F}$ is also support-rooted, satisfying both Height_n and Branch_n for some n is equivalent to $\mathbb{F}$ being finite. We summarize this in the following lemma.

Lemma 6.14. *Let $\mathbb{F}$ be a differentiated INL-frame. Then we have the following equilances:*

$\mathbb{F} \vDash \mathsf{Depth}_n$	*iff*	$\mathbb{F}$ *is of height* $\leq n-1$;
$\mathbb{F} \vDash \mathsf{Branch}_n$	*iff*	$\mathbb{F}$ *has branching degree* $\leq n-1$.

If $\mathbb{F}$ is also support-rooted, we have:

$\mathbb{F} \vDash \mathsf{Depth}_n, \mathsf{Branch}_n$ *for some* n	*iff*	$\mathbb{F}$ *is finite* .

We are now ready to prove the main result of this section.

Theorem 6.15.
(1) *An instantial neighbourhood logic $\Lambda \in \mathrm{Ext}\mathbf{INL}$ is splitting iff Λ is axiomatized by a single Jankov formula.*
(2) *An instantial neighbourhood logic $\Lambda \in \mathrm{Ext}\mathbf{INL}$ is join-splitting iff Λ is axiomatized by Jankov formulas.*

Proof. (1) For the direction from right to left, suppose $\Lambda = \mathbf{INL} + \epsilon(\mathbb{F}_0)$ with $\mathbb{F}_0$ a finite support-rooted differentiated INL-frame of height $\leq n$. We show that $(\Lambda, \Lambda(\mathbb{F}_0))$ is a splitting pair. From the proof of Theorem 6.13, we obtain $\mathbb{F}_0 \nvDash \epsilon(\mathbb{F}_0)$, so we have $\Lambda \nsubseteq \Lambda(\mathbb{F}_0)$. Now consider any logic $\mathcal{M} \in \mathrm{Ext}\mathbf{INL}$ such that $\Lambda \nsubseteq \mathcal{M}$. By completeness (Theorem 5.44), there exists some differentiated INL-frame $\mathbb{F}$ such that $\mathbb{F} \vDash \mathcal{M}$ but $\mathbb{F} \nvDash \Lambda$, so $\mathbb{F} \nvDash \epsilon(\mathbb{F}_0)$. By Theorem 6.13 again, we obtain point-generated subframe $\mathbb{G}$ of $\mathbb{F}$ and onto frame morphism $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_0$. This gives us $M \subseteq \Lambda(\mathbb{F}) \subseteq \Lambda(\mathbb{G}) \subseteq \Lambda(\mathbb{F}_0)$, completing the argument.

For the other direction, suppose Λ is splitting, i.e. there is some $\mathcal{M} \in \mathrm{Ext}\mathbf{INL}$ such that $(\Lambda, \mathcal{M})$ is a splitting pair. As $\mathbf{INL}$ is complete with respect to all finite support-rooted differentiated INL-frames of finite height (Theorem 5.44), we have that $\mathbf{INL} = \bigcap\{\Lambda(\mathbb{F}) \mid \mathbb{F}$ a finite support-rooted differentiated INL-frame of finite height$\}$. By $\mathcal{M}$ being completely meet-prime, we obtain such a frame $\mathbb{F}$ such that $\Lambda(\mathbb{F}) \subseteq \mathcal{M}$. If we let n denote the size of $\mathbb{F}$, we can easily check that $\mathbb{F} \vDash \mathsf{Height}_{n+1} \wedge \mathsf{Branch}_{n+1}$. This also gives $\mathsf{Height}_{n+1} \wedge \mathsf{Branch}_{n+1} \in \mathcal{M}$. We now take a look at the canonical model $\mathbb{M}^c$ of $\mathcal{M}$. As this formula is in $\mathcal{M}$, we can verify that $\mathbb{M}^c_{\mathcal{M}}$ actually is of height $\leq n$ and has branching degree $\leq n$. For $\mathbb{M}^c$ being of height $\leq n$, it suffices to note that any $\Gamma \in \mathbb{M}^c_{\mathcal{M}}$ satisfies $\boxdot^{n+1}\bot$. For the branching, suppose that we have $\Gamma \in \mathbb{M}^c_{\mathcal{M}}$ such that there exists $n+1$ distinct Δ_i such that $\Gamma S_{\mathbb{M}^c_{\mathcal{M}}} \Delta_i$ for each $1 \leq i \leq n+1$. As each Δ_i is distinct, the construction of the canonical model gives ψ_{ij} for each $1 \leq i, j \leq n+1$

with $i \neq j$ such that $\psi_{ij} \in \Delta_i$ but $\psi_{ij} \notin \Delta_j$. We define $\chi_i = \neg \bigwedge_{j \neq i} \psi_{ij}$ such that $\mathbb{M}^c, \Delta_i \vDash \chi_1 \wedge \cdots \wedge \chi_{i-1} \wedge \neg\chi_i \wedge \chi_{i+1} \wedge \cdots \wedge \chi_{n+1}$. This gives $\mathbb{M}^c, \Gamma \vDash \diamonddot\phi'_1 \wedge \cdots \wedge \diamonddot\phi'_{n+1}$, where $\phi'_i = \phi_i[\chi_1/p_1, \ldots, \chi_{n+1}/p_{n+1}]$. This now contradicts $\mathbb{M}^c_{\mathcal{M}} \vDash \mathsf{Branch}_{n+1}$.

This means that any submodel generated from $\mathbb{M}^c_{\mathcal{M}}$ is of height $\leq n$ and has branching degree $\leq n$ and is therefore finite, implying that $\mathcal{M}$ is complete with respect to finite support-rooted differentiated INL-frames of finite height. Consequently, $\mathcal{M}$ is the intersection of all logics of the form $\Lambda(\mathbb{F})$ with $\mathbb{F}$ a finite support-rooted differentiated INL-frame of finite height such that $\mathbb{F} \vDash \mathcal{M}$, which by complete meet-primality gives such a frame $\mathbb{G}$ such that $\mathcal{M} = \Lambda(\mathbb{G})$. The uniqueness of a splitting pair now implies that $\mathcal{M} = \mathbf{INL} + \epsilon(\mathbb{G})$.

(2) Let $\{\Lambda_i\}_{i \in I}$ be a family of instantial neighbourhood logics such that for each $i \in I$, $\Lambda_i = \mathbf{INL} + \Phi_i$ for a set of instantial neighbourhood formulas Φ_i. Then $\Sigma_{i \in I} \Lambda_i = \mathbf{INL} + \bigcup_{i \in I} \Phi_i$. The statement now easily follows from (1). □

6.3 Transitivity

Recall that for modal logics above **K4**, it is possible to show that each normal transitive modal logic is axiomatizable over **K4** by canonical formulas as proven by Zakharyaschev [34] and alternatively proven for stable canonical formulas in [5]. In this section we explore whether we can prove a similar axiomatization result if we restrict ourselves to instantial neighbourhood logics satisfying some form of transitivity.

What allows for the axiomatization results above **K4** is the existence of a master modality. Depending on the notion of transitivity we choose, such a master modality also exists in the setting of INL. We discuss transitivity of the support relation. We call a general INL-frame *support-transitive* if the support relation $S_{\mathbb{F}}$ is transitive. By the well-known correspondence result from normal modal logic, see e.g. [9], this property is expressed by the axiom $\boxdot p \to \boxdot\boxdot p$.

In a support-transitive general INL-frame, it is easy to see that for any integer n and world w, $S^n_{\mathbb{F}}[w] = S_{\mathbb{F}}[w]$. This gives $\mathbb{F}, w \vDash \blacksquare_n \phi$ if and only if $\mathbb{F}, w \vDash \phi \wedge \boxdot\phi$. We define $\boxdot^+\phi := \phi \wedge \boxdot\phi$. Now in a support-rooted support-transitive INL-frame $\mathbb{F}$ with valuation V on $\mathbb{F}$ and root r, we obtain

$$\langle \mathbb{F}, V \rangle, r \vDash \boxdot^+\phi \text{ iff } \langle \mathbb{F}, V \rangle \vDash \phi.$$

We have now once again reduced validity on a model to truth in the root of the model, meaning that in the setting of support-rooted support-transitive INL-frames, we also have a master modality. With this master modality $\boxdot^+$, we can define an alternative version of a co-stable canonical formula. For a finite support-transitive support-rooted differentiated INL-frame $\mathbb{F} = \langle W, \nu \rangle$ and $D \subseteq \mathcal{P}W$, we define the co-stable canonical formula $\tau(\mathbb{F}, D)$ as follows:

$$\tau(\mathbb{F}, D) = \bigwedge\{\boxdot^+\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\boxdot^+\delta \mid \delta \in D\}.$$

Here Γ and Δ are defined as in Definition 6.2.

Theorem 6.16. *Let $\mathbb{F}_0 = \langle W_0, \nu_0 \rangle$ be a finite support-rooted support-transitive differentiated INL-frame, $D \subseteq \mathcal{P}W_0$ and $\mathbb{F} = \langle W, \nu, A \rangle$ any support-transitive differentiated INL-frame. Then $\mathbb{F} \nvDash \tau(\mathbb{F}_0, D)$ if and only if there exists some point-generated subframe $\mathbb{G}$ of $\mathbb{F}$ and co-stable onto map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_0$ satisfying* (SDC) *for D.*

Proof. The proof is an analogue of that of Theorem 6.13. We merely use transitivity instead of the finite height to carry the formulas through the frame. It is important to note here that the point-generated subframe of a support-transitive INL-frame is again support-transitive. $\square$

With the correct characterization in place, one would hope we can extract from this an axiomatization result for any instantial neighbourhood logic containing $\boxdot p \to \boxdot\boxdot p$. However, the existence of refutation patterns is the weak link here. As the method we have been using for constructing these patterns has been based on filtrations, we need a filtration that preserves support-transitivity. As of yet, the existence of such a filtration remains an open problem. If such a filtration does exist however, the methods we have employed in proving the axiomatization results in terms of canonical rules can be used. This gives us the following conditional result.

Theorem 6.17. *Let Λ be an instantial neighbourhood logic above* $\mathbf{INL} + \boxdot p \to \boxdot\boxdot p$*. Suppose that there exists a filtration that preserves support-transitivity. Then Λ can be axiomatized by co-stable canonical formulas $\tau(\mathbb{F}, D)$. Moreover, if Λ is finitely axiomatizable, it can be axiomatized by finitely many co-stable canonical formulas $\tau(\mathbb{F}, D)$.*

Remark 6.18. A remark deserves to be made on why finding the correct filtration proves difficult and consequently, the result above is a conditional result. The reason seems to be a mismatch between the modality a filtration is required to preserve, namely $\Box(\psi_1, \ldots, \psi_n; \phi)$, and the property of support-transitivity we would like to preserve. The support-transitivity is expressed by the sub-operator $\boxdot$ of the $n+1$-ary modality $\Box$, meaning that tailoring the filtration to preserve support-transitivity fails to preserve some instances of $\Box$, whereas tailoring the filtration to preserve all formulas with $\Box$ as its main operator fails to preserve the support-transitivity. A different notion of transitivity might be worth investigating. $\dashv$

6.4 Examples

This section will be devoted to examples of classes of INL-frames axiomatized by co-stable canonical rules. We make a connection between normal modal multi-conclusion consequence relations axiomatized by stable canonical rules and instantial neighbourhood multi-conclusion consequence relations axiomatized by co-stable canonical rules. This allows us to characterize a variety of classes satisfying some property involving the support relation. An example of this is the class of support-rooted frames.

To make the connection between the stable normal modal rule systems and co-stable instantial neighbourhood rule systems, we look at the the similarity between stable canonical rules in the Kripke case and co-stable canonical rules in the case of INL. For a quick recap on descriptive Kripke frames and their maps, see for example [9, 12]. As seen in [5, 6], a stable canonical rule $\sigma^K(\mathbb{F}_0)$ is defined for a finite descriptive Kripke frame $\mathbb{F}_0 = \langle W_0, R_0, A_0\rangle$ to be the rule such that for each descriptive Kripke frame $\mathbb{F} = \langle W, R, A\rangle$, $\mathbb{F} \not\models \sigma^K(\mathbb{F}_0)$ if and only if there exists a continuous onto map $f : \mathbb{F} \twoheadrightarrow \mathbb{F}_0$ such that the following holds:

$$wRv \text{ implies } f(w)R_0f(v).$$

Such a condition on f we call *Kripke-stability.* Co-stability of a map in the setting of INL implies a similar condition related to the support relation $S_{\mathbb{F}}$, expressed in the following lemma. The proof follows straightforwardly from the definition.

Lemma 6.19. *Let $\mathbb{F}$ and $\mathbb{G}$ be differentiated INL-frames and $f : \mathbb{F} \to \mathbb{G}$ a co-stable. Then f satisfies the following property:*

$$wS_{\mathbb{F}}v \Rightarrow f(w)S_{\mathbb{G}}f(v).$$

This implication suggests that co-stability serves a similar purpose for the support relation as Kripke-stability does for the accessibility relation. We will make this idea precise below.

We define a transformation $(\cdot)^\nu$ from a finite descriptive Kripke frame $\mathbb{F} = \langle W, R\rangle$ to a finite differentiated INL-frame by defining $\mathbb{F}^\nu = \langle W^\nu, \nu_R\rangle$ as:

- $W^\nu = W$;
- $\nu_R(w) = \mathcal{P}(R[w])$.

We call $\mathbb{F}^\nu$ the *neighbourhood expansion* of $\mathbb{F}$. Note that we have that $S_{\mathbb{F}^\nu} = R$.

Conversely we define an operation $(\cdot)^K$ transforming a finite differentiated INL-frame $\mathbb{F} = \langle W, \nu\rangle$ to its *Kripke reduct* $\mathbb{F}^K = \langle W, S_{\mathbb{F}}\rangle$.

Lemma 6.20. *Let $\mathbb{F}_0 = \langle W_0, R_0\rangle$ be a finite Kripke frame and $\mathbb{G} = \langle W, \nu\rangle$ a differentiated INL-frame. Then we have:*

$$\mathbb{G}^K \vDash \sigma^K(\mathbb{F}_0) \textit{ iff } \mathbb{G} \vDash \sigma(\mathbb{F}_0^\nu).$$

Proof. For the direction from left to right, suppose that $\mathbb{G}^K \nvDash \sigma^K(\mathbb{F}_0)$, i.e. there exists a Kripke-stable onto map $f : \mathbb{G}^K \twoheadrightarrow \mathbb{F}_0$. We define a map $f^\nu : \mathbb{G} \to \mathbb{F}_0^\nu$ simply as $f^\nu = f$. Obviously f^ν is onto and continuous directly from f being so. To show that f^ν is co-stable, consider $Z \in \nu(w)$. This means $Z \subseteq S_{\mathbb{G}}[w]$. By Kripke-stability of f we obtain $f[Z] \subseteq R_0[f(w)]$. Consequently, the definition of ν_{R_0} gives $f[Z] \in \nu_{R_0}(f(w))$.

For the other direction, suppose that $\mathbb{G} \nvDash \sigma(\mathbb{F}_0^\nu)$ so there exists co-stable onto map $f : \mathbb{G} \twoheadrightarrow \mathbb{F}_0^\nu$. We define a map $f^K : \mathbb{G}^K \to \mathbb{F}_0$ by $f^K = f$. As before, f^K is continuous and surjective because f is. To prove Kripke-stability, suppose $wS_{\mathbb{G}}v$. This gives $Z \in \nu(w)$ such that $v \in Z$. By co-stability of f, $f[Z] \in \nu_0(f(w))$ from which it follows by definition of ν_0 that $f[Z] \subseteq R_0[f(w)]$. Therefore $f(w)R_0f(v)$. ☐

We now have all the machinery necessary to borrow examples from the Kripke case. We will use the usual notation to denote Kripke frames, i.e. $\circ$ denotes a reflexive point and $\bullet$ denotes an irreflexive point. The neighbourhood expansions of these Kripke frames will be used to characterize the corresponding classes. We will use characterizations for Kripke frames given in [5].

For some terminology, let $\mathbb{F} = \langle W, \nu, A\rangle$ be a differentiated INL-frame. For $w, v \in W$, we say there is a *weak $S_{\mathbb{F}}$-path* between w and v if there are $u_0, \dots, u_n$ such that $u_0 = w$, $u_n = v$ and for each $i < n$, $u_iS_{\mathbb{F}}u_{i+1}$ or $u_{i+1}S_{\mathbb{F}}u_i$. We say that $\mathbb{F}$ is *weakly connected* if it is non-empty and there is a weak $S_{\mathbb{F}}$-path between every $w, v \in W$. We let **WCon** denote the class of finite differentiated INL-frames that are weakly connected and $\mathcal{S}(\mathbf{WCon})$ the instantial neighbourhood multi-conclusion consequence relation that corresponds to it. Similarly, we define **SRooted** to be the class of finite support-rooted differentiated INL-frames and $\mathcal{S}(\mathbf{SRooted})$ its corresponding instantial neighbourhood multi-conclusion consequence relation. Furthermore, let $\mathbf{S_{INForm}} := \mathcal{S}(\mathbf{INForm})$ denote the smallest instantial neighbourhood multi-conclusion consequence relation containg $/\phi$ for each $\phi \in \mathbf{INForm}$. We let $\sigma()$ denote the co-stable rule of the empty frame.

Theorem 6.21.
(1) $\mathbf{S_{INForm}} = \mathbf{S_{INL}} + \sigma((\circ)^\nu)$.
(2) $\mathbf{INRules} = \mathbf{S_{INL}} + \rho() + \sigma((\circ)^\nu)$.
(3) $\mathcal{S}(\mathbf{WCon}) = \mathbf{S_{INL}} + \sigma() + \sigma((\circ\ \circ)^\nu)$.
(4) $\mathcal{S}(\mathbf{SRooted}) = \mathbf{S_{INL}} + \sigma() + \sigma((\circ\ \circ)^\nu) + \sigma((\circ \nearrow \overset{\circ}{\ } \nwarrow \circ)^\nu)$.
(5) $\mathbf{Form} = \Lambda(\mathbf{S_{INL}} + \sigma((\circ)^\nu))$.
(6) $\mathbf{INL} + \Diamond\!\!\!\cdot\;\top = \Lambda(\mathbf{S_{INL}} + \sigma((\bullet)^\nu) + \sigma((\circ\!\longrightarrow\!\bullet)^\nu))$.
(7) $\mathbf{INL} + \boxdot p \to p = \Lambda(\mathbf{S_{INL}} + \sigma((\bullet)^\nu) + \sigma((\circ\!\longleftrightarrow\!\bullet)^\nu))$.

Proof. By Theorems 5.17 and 5.46, we have completeness of all instantial neighbourhood multi-conclusion consequence relations as well as all instantial neighbourhood logics with respect to differentiated INL-frames. Therefore, to prove the needed equalities we merely need to show that their corresponding classes of frames coincide.

(1) It is easy to see that a co-stable onto map $f : \mathbb{F} \twoheadrightarrow (\circ)^\nu$ exists iff $\mathbb{F}$ is non-empty. This gives us $\mathbb{F} \not\models \sigma((\circ)^\nu)$ iff $\mathbb{F}$ is nontrivial. Therefore, the class of differentiated INL-frames validating $\mathbf{S_{INL}} + \sigma((\circ)^\nu)$ consists of the trivial INL-frame and thus $\mathbf{S_{INForm}} = \mathbf{S_{INL}} + \sigma((\circ)^\nu)$.
(2) For INL-frame $\mathbb{F}$, if $\mathbb{F}$ is trivial it can be mapped onto the empty frame and if $\mathbb{F}$ is non-trivial it can be mapped onto $(\circ)^\nu$, so the class corresponding to $\mathbf{S_{INL}} + \sigma() + \sigma((\circ)^\nu)$ is empty. Therefore, $\mathbf{INRules} = \mathbf{S_{INL}} + \sigma() + \sigma((\circ)^\nu)$.
(3) From Theorem 6.8 we obtain that $\mathbf{S_{INL}} + \sigma() + \sigma((\circ\ \circ)^\nu)$ is co-stable and by Theorem 6.9, this means that it has the finite model property. We therefore only need to show that the two consequence relations correspond on finite differentiated INL-frames. Take any finite weakly connected differentiated INL-frame $\mathbb{F}$. Then its Kripke reduct $\mathbb{F}^K$ is also weakly connected. Now by [5, Theorem 8.1(3)], we obtain that $\mathbb{F}^K \models \sigma^K(), \sigma^K(\circ\ \circ)$. Lemma 6.20 tells us that this is equivalent to $\mathbb{F} \models \sigma(), \sigma((\circ\ \circ)^\nu)$. The other direction follows similarly. This gives us $\mathcal{S}(\mathbf{WCon}) = \mathbf{S_{INL}} + \sigma() + \sigma((\circ\ \circ)^\nu)$.
(4) This follows from a similar reasonig as (3), now using [5, Theorem 8.1(4)].
(5) By (1), $\mathbf{S_{INL}} + \sigma((\circ)^\nu)$ corresponds to the class of the empty frame, which is exactly the class of differentiated INL-frames validating the inconsistent logic.
(6) It is easy to see that $\Diamond\!\!\!\cdot\;\top$ expresses seriality of the support relation. Moreover, any co-stable map preserves this seriality of the support relation, in particular filtrations. Consequently, $\mathbf{INL} + \Diamond\!\!\!\cdot\;\top$ has the finite model property. Theorems 6.8 and 6.9 imply the finite model property of $\Lambda(\mathbf{S_{INL}} + \sigma((\bullet)^\nu) + \sigma((\circ\!\longrightarrow\!\bullet)^\nu))$. So it suffices to only look at finite differentiated INL-frames here. Take a finite differentiated INL-frame $\mathbb{F}$ such that $S_{\mathbb{F}}$ is serial. Then its Kripke reduct $\mathbb{F}^K$ is serial. By [5, Theorem 8.3(2)] we obtain that $\mathbb{F}^K \models \sigma^K(\bullet), \sigma^K(\circ\!\longrightarrow\!\bullet)$. Lemma 6.20 now implies that $\mathbb{F} \models \sigma((\bullet)^\nu) + \sigma((\circ\!\longrightarrow\!\bullet)^\nu)$. The other direction is similar.
(7) Any filtration preserves reflexivity of the support relation. This allows for similar reasoning as (6), only now using [5, Theorem 8.3(3)]. □

The theorem above illustrates an intimate connection between INL-frames and Kripke frames. This connection is established by the support relation. This relation allows reasoning on neighbourhood frames as if they were Kripke frames. The constructions of generated submodel and unravelling are prime examples of this. Moreover, the support relation can be expressed in the language of INL. We can therefore define canonical formulas leading to results such as the splitting theorem for the lattice Ext**INL** (Theorem 6.15).

Chapter 7

Conclusions and Future Work

The aim of this thesis has been to study logics whose semantics is based on neighbourhood frames. We have done so via the method of canonical rules and formulas. In Part I we took the perspective of classical and monotonic modal logics, whereas in Part II we took that of instantial neighbourhood logics. For each part, we will briefly discuss the obtained results.

In Part I we have looked at results regarding classical and monotonic modal logics. Using a notion of filtration, we defined stable canonical rules that we used to axiomatize any classical and monotonic modal logic and multi-conclusion consequence relation. As two particular cases of canonical rules, we defined stable rules and Jankov rules. Classical and monotonic modal consequence relations and logics axiomatized by the former have the finite model property, whereas the latter exactly axiomatize splittings in the lattice CExt$\mathbf{S_E}$. Making the step to canonical formulas was hindered by the functors $\check{\mathcal{P}} \circ \check{\mathcal{P}}$ and $\mathcal{U}p\mathcal{P}$ failing to preserve weak pullbacks.

In Part II we looked at instantial neighbourhood logics. The frames of this logic correspond to coalgebras of the functor $\mathcal{P} \circ \mathcal{P}$ which does preserve weak pullbacks. We defined a notion of filtration of the corresponding models. This enables us to define co-stable canonical rules and replicate results similar to those obtained in Part I. Using the notion of the support relation, we defined co-stable canonical formulas, with which we showed an analogue of Blok's splitting theorem [10], namely that each instantial neighbourhood logic Λ is splitting in Ext**INL** iff Λ is axiomatized by co-stable canonical formulas.

There is still a large number of open questions and future research topics worthy of a closer look. We highlight a few of them.

- *Algebraic duality for INL-frames.* One of the open questions is the existence of an algebraic duality for INL-frames. One can easily define algebras for which any instantial neighbourhood logic is complete: take a Boolean algebra $\mathbb{A}$ together with an $n+1$-ary operator $\Box_n$ for each $n \in \omega$ such that each $\Box_n$ satisfies all axioms of **INL**. We should now be able to define a construction similar to the algebraic duality between descriptive neighbourhood frames and classical modal algebras. The search for this construction goes hand-in-hand with the search for a canonical model construction for instantial neighbourhood logics that does not rely on a normal form theorem.

- *Transitive filtration for INL-frames.* We would like to transform the conditional result given in Theorem 6.17 into a definitive result. For this, a filtration on INL-frames needs to be found that preserves support-transitivity. A notion of filtration different from the one presented in Section 6.3 might be necessary.

- *Structure and cardinality of lattices* Ext$\mathbf{S_{INL}}$ *and* Ext**INL**. As Instantial Neighbourhood Logic is a very recent topic, the extensions of INL are still an unexplored topic. Although we have shown a splitting theorem for both lattices Ext$\mathbf{S_{INL}}$ and Ext**INL** and looked closely at co-stable logics and consequence relations, many unanswered questions still remain. How many (co-stable) instantial neighbourhood logics or multi-conclusion consequence relations are there? Do there exist such logics or consequence relations that are not complete with respect to neighbourhood frames?

- *Epistemic applications for INL.* As mentioned in Example 5.8, we can interpret INL as an evidence logic similar to the one presented in [4]. We interpet $\Box(\psi_1, \ldots, \psi_n; \phi)$ as saying "the agent has evidence for ϕ consistent with $\psi_1, \ldots, \psi_n$". This gives us an epistemic perspective on INL. With the notion of submodel provided in the thesis (Definition 5.33), one can develop a dynamic epistemic logic based on INL.

- *Proof theory of stable logics.* Bezhanishvili and Ghilardi [8] investigated proof systems for stable normal modal logics, i.e. normal modal logics axiomatized by stable canonical rules based on finite Kripke frames. They showed that for any such stable normal modal logic, a proof system exists that has the bounded proof property. An interesting question would be whether we can mimic these methods to devise proof systems for stable classical and monotonic modal logics and similarly for co-stable instantial neighbourhood logics.

- *Admissibility of rules.* Jeřábek [22] originally introduced canonical rules to show the decidability of the admissibility problem in transitive modal logics. To do so, he showed that any canonical rule is either admissible or equivalent to an assumption-free rule. A similar property was shown for stable canonical rules for transitive modal logics in [7]. An interesting question is whether a similar property holds for stable canonical rules for non-normal modal logics or co-stable canonical rules for instantial neighbourhood logics.

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