Possibility Spaces, Q-Completions and Rasiowa-Sikorski Lemmas for Non-Classical Logics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis, we study various generalizations and weakenings of the Rasiowa-Sikorski Lemma (Rasiowa-Sikorski [55]) for Boolean algebras. Building on previous work from Goldblatt [30], we extend the Rasiowa-Sikorski Lemma to co-Heyting algebras and modal algebras, and show how this yields completeness results for the corresponding non-classical first-order logics. Moreover, working without the full power of the Axiom of Choice, we generalize the framework of possibility semantics from Humberstone [40], and more recently Holliday [39], in order to provide choice-free representation theorems for distributive lattice, Heyting algebras and co-Heyting algebras. We also generalize a weaker version of the Rasiowa-Sikorski Lemma for Boolean algebras, known as Tarski's Lemma, to distributive lattices, HA's and co-HA's, and use these results to define a new semantics for first-order intuitionisitic logic.

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Contents

Chapter 1

Introduction

1.1 The Rasiowa-Sikorski Lemma

In Rasiowa-Sikorski [55], Helena Rasiowa and Roman Sikorski developed some of the most powerful algebraic methods in mathematical logic of their time. Shortly after Gödel [28], Rasiowa and Sikorski provided an algebraic proof of the completeness of Classical Predicate Logic (CPL) with respect to Tarskian models. Their result relied on the celebrated Rasiowa-Sikorski Lemma:

Lemma 1.1.1. [1] *Let B be a Boolean algebra, and Q a countable set of subsets of B. Then for any $a \in B$, if $a \neq 0$, then there exists an ultrafilter U over B such that $a \in U$ and U preserves all meets in Q, i.e., for any $A \in Q$, if $\bigwedge A$ exists in B and $A \subseteq U$, then $\bigwedge A \in U$.*

Rasiowa and Sikorski's original proof involved an application of the Baire Category Theorem (Baire [2]) to the dual Stone space of a Boolean algebra. This was one of the first applications of Stone's Representation Theorem for Boolean algebras (Stone [65]) to the study of first-order logic. In Henkin [36], Leon Henkin developed a proof-theoretic counterpart to Rasiowa and Sikorski's algebraic result. The main ideas of the *Henkin method*, and in particular the use of term models, have been used since then to prove the Compactness and Completeness Theorems of first-order logic, as well as the Omitting Types Theorem, a central result in model theory, and is still a standard way of proving completeness results for non-classical logics.

Since Rasiowa-Sikorski [55], important connections have been observed between several results and principles across various areas of mathematical logic and pure mathematics, including the Baire Category Theorem for Compact Hausdorff Spaces, an abstract version of the Henkin method, the Axiom of Dependent Choices, the existence of generic filters on forcing posets, the Omitting Types Theorem, and the Rasiowa-Sikorski Lemma. In particular, Goldblatt [29] proved that the first five statements of the previous list are equivalent to a weaker form of the Rasiowa-Sikorski Lemma, which Goldblatt calls *Tarski's Lemma*. In fact, Goldblatt [29] shows that the Rasiowa-Sikorski Lemma is equivalent to the conjunction of Tarski's Lemma and the Boolean Prime Ideal theorem (BPI), a weaker form of Zorn's Lemma which plays a crucial role in the Stone Representation Theorem for Boolean algebras.

[1] Throughout the rest of this thesis, we will use a slightly different formulation for the Rasiowa-Sikorski Lemma and related results. For the sake of simplicity, we will only consider sets Q containing only subsets of B whose meets (or joins depending on cases) do exist in B. We will also refer to such sets as "sets of meets" or "sets of joins", although this formulation in itself is not extremely precise.

Recently, generalizations of the Rasiowa-Sikorski Lemma have also been proposed. In particular, in keeping with Rasiowa and Sikorski's original proof, Goldblatt [30] has used the Baire Category Theorem for Compact Hausdorff Spaces to prove a version of the Rasiowa-Sikorski Lemma for distributive lattices. Goldblatt's proof relies on Priestley's Representation Theorem [52], a generalization of Stone's Representation Theorem to distributive lattices. Unlike in Stone's original representation theorem for distributive lattices ([66]), Priestley spaces are compact Hausdorff, and this allowed Goldblatt to provide a simple and elegant proof of the Rasiowa-Sikorski Lemma for distributive lattices. However, Priestley's Representation Theorem itself relies on the Prime Filter Theorem, a non-constructive principle that is equivalent to the Boolean Prime Ideal Theorem. One could therefore wonder if, similarly to the case of Boolean algebras, there exists a statement ϕ that is equivalent to Tarski's Lemma and is such that the Rasiowa-Sikorski Lemma for distributive lattices is equivalent to the conjunction of the Prime Filter Theorem and ϕ. One of the main results of this thesis is that there is such a statement, which we named the Q-Lemma for distributive lattices. We therefore have the following facts about the Rasiowa-Sikorski Lemma and its connections with some other non-constructive principles:

- the Axiom of Choice (AC) implies the Boolean Prime Ideal Theorem (BPI) and the Axiom of Dependent Choices (DC)[2];
- Tarski's Lemma (TL), the Axiom of Dependent Choices, the Baire Category Theorem for Compact Hausdorff Spaces (BCT) are all equivalent;
- The Rasiowa-Sikorski Lemma for Boolean algebras (RS(BA)) is equivalent to the conjunction of Tarski's Lemma and the Boolean Prime Ideal Theorem;
- The Boolean Prime Ideal Theorem is equivalent to the Prime Filter Theorem (PFT);
- the Axiom of Dependent Choices and the Boolean Prime Ideal Theorem are mutually independent;

As we prove in this thesis, the Q-Lemma for Distributive lattices (QDL) is a counterpart to Tarski's Lemma in the following sense:

Theorem 1.1.2.

- *The Rasiowa-Sikorski Lemma for DL (RS(DL)) is equivalent to the conjunction of the Prime Filter Theorem and the Q-Lemma for distributive lattices.*
- *The Q-Lemma is equivalent to Tarski's Lemma.*

For the sake of clarity, we have gathered all these results in a diagram representing the entailment relations between the various choice principles mentioned so far. In this diagram, double lines represent an equivalence over ZF, while an arrow represents a strict implication.

[2]For more details on the Axiom of Choice and some of its weaker versions, Jech [41] and Moore [49] are standard references

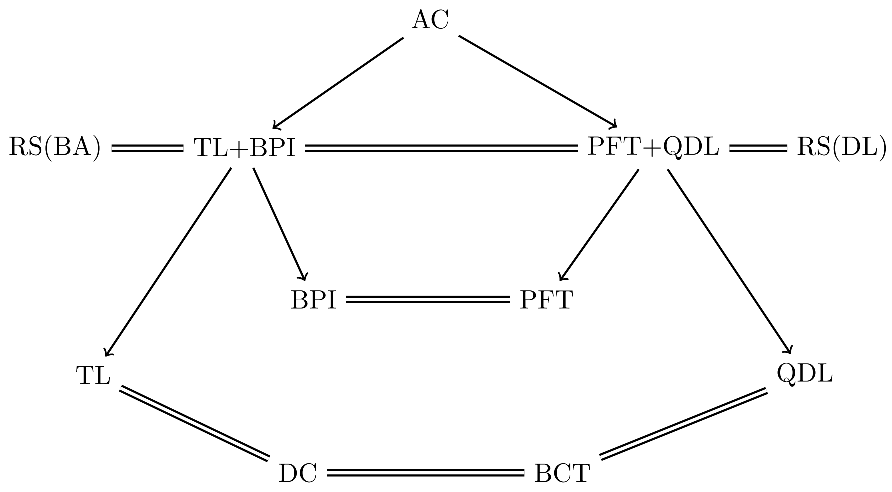

Figure 1.1: Relations between Non-Constructive Principles

1.2 Goals

The first goal of this thesis is therefore to contribute to a deeper understanding of the relationship between the original Rasiowa-Sikorski Lemma, some of its generalizations, and several other non-constructive principles that play an important role in various mathematical areas.

Our second, related goal is to explore alternative methods in lattice theory and mathematical logic, in particular in a choice-free setting. As we noted above, both Stone's and Priestley's representation theorems rely on equivalent versions of the Prime Filter Theorem. By contrast, in this thesis, we study constructive versions of Stone's and Priestley's representation theorem for Boolean algebras and distributive lattices, by working with sets of filters rather than sets of ultrafilters of prime filters. We also show how the results obtained extend to Boolean Algebras with Operators (BAO's), Heyting algebras and co-Heyting algebras in an essentially straightforward way.

Constructive representation theorems for Boolean algebras and BAO's were already known in the literature, and were thoroughly studied in particular in Holliday [39]. These results rely in an essential way on the well-known topological fact that the set of regular open sets of any topological space form a complete Boolean algebra. Our main contribution to the topic is to prove a generalization of this fact in the setting of bi-topological spaces. In particular, we define *refined bi-topological spaces* and *refined regular open sets*, and show that the refined regular open sets of any refined bi-topological spaces form a complete Heyting algebra. Bi-topological approaches to distributive lattices and Heyting algebras were proposed in [6] and [33].

Moreover, this approach provides a straightforward way of studying completions of Heyting algebras and distributive lattices. The study of completion of lattices is a very rich area (see for example [16], [48], [33] and [34]), that goes back to Dedekind's construction of the reals as cuts on the rationals. In particular, in connection with the Q-Lemma for distributive lattices, we define the notion of a Q-completion, and use several version of the Q-Lemma to prove that, assuming the Axiom of Dependent Choices, every distributive lattice, every Heyting algebra, and every co-Heyting algebra has a Q-completion.

The final goal of this thesis is to draw some consequences for logic from the results we obtain in lattice theory. In particular, we explore the connection between constructive representation theorems for varieties of lattices and a recent semantics that has been proposed for classical logic, possibility semantics.

Possibility semantics for classical logic was first proposed by Humberstone [40], and has received an increasing amount of attention in recent years, as a prominent framework in the literature on alternative semantics (see for example [1], [5]). In particular, Holliday [39], [38] has shown how the framework can be used to provide a new semantics for classical propositional logic and propositional modal logic. In this thesis, we define first-order possibility models in a natural way, and show how Tarski's Lemma yields a straightforward proof of the completeness of Classical Predicate Logic with respect to first-order possibility models. Moreover, we extend some of the ideas behind possibility semantics to the intuitionistic setting, and introduce a new semantics for first-order intuitionistic logic. It is worth noting that, in line with our work on constructive representation theorems, the completeness proofs obtained rely on the sole assumption of the Axiom of Dependent Choices[3].

1.3 Outline of the thesis

- In the next chapter, we go through the preliminary results that will be needed throughout this thesis. In particular, we recall and sketch the proof of Stone's, Priestley's and Esakia's Representation Theorems. We also fix propositional, modal and first-order calculi for classical, intuitionistic and co-intuitionistic logic, and recall the standard semantics for all these logics. Finally, we introduce notions of order-theory such as closure operators, MacNeille completions and canonical extensions, as well as some of the choice principles and non-constructive theorems that will play a role in the following chapters.

- In Chapter 3, we first review the general definition of Lindenbaum-Tarski algebras and term models, and recall how they play a key role in algebraic completeness proofs. The remainder of the chapter is devoted to the Rasiowa-Sikorski Lemma for Boolean algebras, and various generalizations to non-classical logics. In particular, we present Goldblatt's [30] topological proof for distributive lattices and Heyting algebras, and show how to generalize the main components of his proof to co-Heyting algebras, modal algebras and BAO's.

- In Chapter 4, we introduce possibility semantics for classical logic, and focus on the features of this semantics that we want to generalize to intuitionistic logic. In particular, we state and prove Tarski's Lemma, and show how this lemma is instrumental in the completeness proof of Classical Predicate Logic CPL with respect to first-order possibility models. In section 3, we show how possibility semantics provides topological representations of various completions of Boolean algebras. Finally, we state and prove a strengthening of Tarski's Lemma in the context of BAO's, and show how this version of the lemma yields an extension of Theorem 4.2.7 to first-order modal logic.

- In Chapter 5, we prove our main results. We first define a generalization of topological spaces, namely refined bi-topological spaces, and show how this yields a representation theorem for distributive lattices and Heyting algebras that mirrors the representation theorem for Boolean algebras at the center of Chapter 4. In the third section, we state and

[3]In this respect, the Q-lemma and its equivalent forms allows from completeness proofs that are strictly more constructive than those obtained by the Rasiowa-Sikorski Lemma, which requires both (BPI) and (DC).

prove a generalization of Tarski's Lemma, the Q-Lemma (QDL), and prove a version of the Q-Lemma for Heyting algebras. We also define Q-completions for distributive lattices and Heyting algebras, and use the Q-Lemma in both cases to prove that every distributive lattice and every Heyting algebra has a Q-completion. We conclude the chapter by defining a new semantics for first-order intuitionistic logic (IPL) based on intuitionistic possibility frames, and we prove soundness and completeness of IPL with respect to IP-models.

- In the last chapter, we first adapt the proofs and methods of Chapter 5 to the setting of co-Heyting algebras. In the second section, we slightly generalize the notion of intuitionistic possibility space, and give topological representations of various completions of distributive lattices and Boolean algebras, thus generalizing the results obtained in chapter 4 regarding completions of Boolean algebras. Finally, we show how intuitionistic possibility spaces generalize several existing frameworks, and compare our semantics to recent related work.

Chapter 2

Preliminaries

In this first chapter, we introduce the basic concepts and results that will be used throughout the thesis. Most of the theorems introduced are either well-known results in duality theory or key results in basic mathematical logic, and their proofs are therefore omitted. Among notable exceptions are the Stone Representation Theorem for Boolean algebras, the Priestley Representation Theorem for distributive lattices, and the Esakia Representation Theorem for Heyting algebras; for those, short proofs are included since many key facts from these proofs will play an important role in the next chapters.

In the first section, we introduce the varieties of lattices and the classes of topological spaces that will play an important role in the following chapters. We also recall important results from duality theory. In section 2, we introduce various propositional and first-order logics and standard models for those logics. Finally, section 3 is concerned with some important notions of lattice theory. We also introduce several non-constructive principles and theorems that will play a role in the following chapters.

2.1 Lattices, Topological Spaces and Dualities

In this section, we review notions of lattice theory and topology, and recall results from the representation theory of lattices. The standard reference for notions of universal algebra and lattice theory is [11]. For topological notions and results in duality theory, see [42] and [14].

2.1.1 Lattices

Definition 2.1.1 (Distributive Lattices)**.** A *(bounded distributive) lattice* is a poset $(L, \leq)$ that satisfies the following requirements:

- For any $a, b \in L$, there exists a greatest lower bound (a meet) $a \wedge b$ and a smallest upper bound (a join) $a \vee b$ of the set $\{a, b\}$.
- Finite meets and joins distribute over one another: for any $a, b, c \in L$, $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (distributive lattice)
- L has a greatest element, noted 1, and a smallest element, noted 0. (bounded distributive lattice)

Throughout this thesis, we will always assume that the distributive lattices we consider are bounded unless otherwise specified. We will also often to the following specific property of lattices:

Definition 2.1.2 (Complete lattice)**.**

Let L be a lattice. Then L is *complete* if for any $A \subseteq L$, A has a greatest lower bound $\bigwedge A$ and a smallest upper bound $\bigvee A$ in L.

The structures defined by the following three definitions will be the main focus of this work:

Definition 2.1.3 (Heyting and co-Heyting Algebras)**.** Let $(L, \wedge, \vee, 0, 1)$ be a distributive lattice. L is a *Heyting algebra* iff there exists an operation $\rightarrow: L \times L \rightarrow L$ such that for any $a, b, c \in L$,

$$a \wedge b \leq c \Leftrightarrow a \leq b \rightarrow c \tag{1}$$

Dually, L is a *co-Heyting algebra* iff there exists a function $\prec\!\!-\!: L \times L \rightarrow L$ such that for any $a, b, c \in L$,

$$a \prec\!\!- b \leq c \Leftrightarrow a \leq b \vee c \tag{2}$$

Both (1) and (2) are called *residuation* properties; if (1) holds, then $\rightarrow$ is the *right residuum* of $\wedge$, and if (2) holds, then $\prec\!\!-$ is the *left residuum* of $\vee$.
In particular, if L is a Heyting algebra, we write $\neg a$ for $a \rightarrow 0$ for any $a \in L$, and if L is a co-Heyting algebra, we write $\sim a$ for $1 \prec\!\!- a$.

Finally, a *bi-Heyting algebra* is a structure $(L, \wedge, \vee, \rightarrow, \prec\!\!-, 0, 1)$ such that $(L, \wedge, \vee, \rightarrow, 0, 1)$ is a Heyting algebra and $(L, \wedge, \vee, \prec\!\!-, 0, 1)$ is a co-Heyting algebra.

Definition 2.1.4 (Boolean Algebras)**.** Let L be a distributive lattice. Then L is a *Boolean algebra* if L is a bi-Heyting algebra and for any $a \in L$, $\neg a = \sim a$. Equivalently, a Boolean algebra is a distributive lattice with an additional unary operation $\neg : L \rightarrow L$ such that for any $a \in L$, $\neg a$ is the unique element of L such that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.

Finally, we will occasionally refer to the following extensions of distributive lattices:

Definition 2.1.5 (Modal Heyting and co-Heyting Algebras)**.** Let L be a Heyting algebra, and $\Box$ a unary operation on L. Then $(L, \Box)$ is a *modal Heyting algebra* if the following conditions hold for any $a, b \in L$:

- $\Box 1 = 1$
- $\Box(a \wedge b) = \Box a \wedge \Box b$

Dually, if M is a co-Heyting algebra and $\Diamond$ is a unary operator on L, then $(M, \Diamond)$ is a *modal co-Heyting algebra* if the following two conditions hold for any $a, b \in L$:

- $\Diamond 0 = 0$
- $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

Definition 2.1.6 (Boolean Algebra with Operators)**.** [1] Let L be a Boolean algebra, and $\Box$ a unary operation on L. Then $(L, \Box)$ is a *Boolean Algebra with Operators* if it is a modal Heyting algebra. Equivalently, $(L, \Box)$ is a Boolean Algebra with Operators if $(L, \Diamond)$ is a modal co-Heyting algebra, where $\Diamond := \neg\Box\neg$.

[1] Throughout this thesis, we will always consider Boolean algebras with only one modal operator, even though the usual definition of BAO's allows for many modalities at once. Since none of the results about BAO's that we will discuss depend on the number of modal operators involved in any meaningful way, we restrict ourselves to the simplest case for the sake of clarity.

Lemma 2.1.7 (Basic properties)**.** *The following are well-known properties of the various algebras described above that we will use repeatedly.*

1. *Let L be a Heyting algebra. Then for any $a, b, c \in L$, $A \subseteq L$:*
   - $a \wedge a \to b \leq b$, $b \leq a \to b$
   - *if* $a \leq b$, *then* $c \to a \leq c \to b$ *and* $b \to c \leq a \to c$.
   - $a \leq b$ *iff* $a \to b = 1$
   - $(a \to c) \wedge (b \to c) = (a \vee b) \to c$, $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$
   - *if* $\bigvee A$ *exists, then* $\bigvee A \wedge b = \bigvee\{a \wedge b \ ; \ a \in A\}$
   - *if* $\bigvee A$ *exists,* $\bigvee A \to b = \bigwedge\{a \to b \ ; \ a \in A\}$, *and* $b \to \bigvee A = \bigvee\{b \to a \ ; \ a \in A\}$
   - *if* $\bigwedge A$ *exists,* $b \to \bigwedge A = \bigwedge\{b \to a \ ; \ a \in A\}$
   - $a \leq \neg\neg a$
2. *Dually, the following properties hold for L a co-Heyting algebra and $a, b, c \in L$, $B \subseteq L$*[2]
   - $a \leq a {-\!\!<}\, b \vee b$, $a {-\!\!<}\, b \leq a$
   - *if* $a \leq b$, *then* $a {-\!\!<}\, c \leq b {-\!\!<}\, c$ *and* $c {-\!\!<}\, b \leq c {-\!\!<}\, a$
   - $a \leq b$ *iff* $a {-\!\!<}\, b = 0$
   - $(c {-\!\!<}\, a) \vee (c {-\!\!<}\, b) = c {-\!\!<}\, (a \wedge b)$, $(b {-\!\!<}\, a) \wedge (c {-\!\!<}\, a) = (b \vee c) {-\!\!<}\, a$
   - *if* $\bigwedge B$ *exists, then* $\bigwedge B \vee a = \bigwedge\{b \vee a \ ; \ b \in B\}$
   - *if* $\bigwedge B$ *exists,* $a {-\!\!<} \bigwedge B = \bigvee\{a {-\!\!<}\, b \ ; \ b \in B\}$, *and* $\bigwedge B {-\!\!<}\, a = \bigwedge\{b {-\!\!<}\, a \ ; \ b \in B\}$
   - *if* $\bigvee B$ *exists,* $\bigvee B {-\!\!<}\, a = \bigvee\{b {-\!\!<}\, a \ ; \ b \in B\}$
   - ${\sim\sim}\, a \leq a$
3. *Additionally, the following are true if L is a Boolean algebra:*
   - $\neg a \vee \neg b = \neg(a \wedge b)$, $\neg a \wedge \neg b = \neg(a \vee b)$
   - $\neg\neg a = a$
4. *Finally, the following equations hold in the case of modal algebras:*
   - *if* $(L, \Box)$ *is a modal Heyting algebra and* $a, b \in L$, *then* $\Box(a \to b) \leq \Box a \to \Box b$, *and if* $a \leq b$, *then* $\Box a \leq \Box b$

[2] As it will become apparent throughout this thesis, many results that hold for Heyting algebras and modal Heyting algebras also hold for co-Heyting algebras and modal co-Heyting algebras. This of course a consequence of the fact that given any Heyting algebra L, the lattice L^{op} obtained by reversing the order on L is a co-Heyting algebra, and conversely and co-Heyting algebra M can be turned into a Heyting algebra M^{op} by reversing the order. As a consequence, every inequation that holds on all Heyting algebras can be translated into a dual inequation that holds on all co-Heyting algebras. A formal presentation of this translation amounts to considering a map $-^{\delta}$ from the language of the signature of Heyting algebras into the language of the signature of co-Heyting defined recursively as follows:

- $0^{\delta} = 1$, $1^{\delta} = 0$,
- $(a \wedge b)^{\delta} := a^{\delta} \vee b^{\delta}$, $(a \vee b)^{\delta} := a^{\delta} \wedge b^{\delta}$,
- $(a \to b)^{\delta} = (b^{\delta} {-\!\!<}\, a^{\delta})$

It is then straightforward to check that that an inequation $\phi \leq \psi$ holds for all Heyting algebras if and only if $\phi^{\delta} \leq \phi^{\delta}$ holds for all co-Heyting algebras. Note that the translation map $-^{\delta}$ also extends to the signature of modal Heyting algebras by setting $(\Box a)^{\delta} := \Diamond a^{\delta}$

- *if $(M, \Diamond)$ is a modal co-Heyting algebra and $a, b \in M$, then $\Diamond a \prec \Diamond b \leq \Diamond(a \prec b)$, and if $a \leq b$, then $\Diamond a \leq \Diamond b$*

We end this brief introduction to the algebraic structures that will play a role throughout the thesis with the following well-known definitions.

Definition 2.1.8 (Filters and Ideals)**.** Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. A *filter* over L is a set $F \subseteq L$ such that for any $a, b \in L$:

- $1 \in F$
- F is upward-closed: if $a \in F$ and $a \leq b$, then $b \in F$
- F is downward-directed: if $a, b \in F$, then $a \wedge b \in F$.

An *ideal* I over L is defined dually as follows:

- $0 \in I$
- I is downward-closed: if $b \in I$ and $a \leq b$, then $a \in I$
- I is upward-directed: if $a, b \in I$, then $a \vee b \in I$.

Note that a filter F (resp. and ideal I) is proper iff $0 \notin F$ (resp. $1 \notin I$).

Definition 2.1.9 (Prime filters)**.** Let L be a bounded lattice. A filter F over L is *prime* if F is proper and for any $a, b \in F$, if $a \vee b \in F$, then $a \in F$ or $b \in F$. Dually, an ideal I over L is prime if I is proper and for any $a, b \in I$, if $a \wedge b \in I$, then $a \in I$ or $b \in I$.

The following definition is important in the setting of Boolean algebras:

Definition 2.1.10. Let L be a Boolean algebra, and F a filter over L. Then F is an *ultrafilter* if for any $a \in L$, either $a \in F$ or $\neg a \in F$.

Finally, we will often refer to special maps between lattices:

Definition 2.1.11 (Homomorphisms)**.** Let L, M be two distributive lattices, and f a map from L to M. Then f is a *DL-homomorphism* if $f(0_L) = 0_M$, $f(1_L) = 1_M$, and for any $a, b \in L$, $f(a \wedge_L b) = f(a) \wedge_M f(b)$ and $f(a \vee_L b) = f(a) \vee_M f(b)$.

Additionnally, if both L and M are Heyting algebras (resp. co-Heyting algebras), then f is a *HA-homomorphism* (resp. *co-HA homomorphism*) if for any $a, b \in L$, $f(a \to_L b) = f(a) \to_M f(b)$ (resp. $f(a -\!\!<_L b) = f(a) -\!\!<_M f(b)$), and if L, M are Boolean algebras, then f is a *BA-homomorphism* if $f(\neg_L a) = \neg_M f(a)$ for any $a \in L$. Finally, if f is said to preserve the operator $\Box$ (resp. $\Diamond$) in L if for any $a \in L$, $f(\Box_L a) = \Box_M f(a)$ (resp. $f(\Diamond_L a) = \Diamond_M f(a)$).
A bijective homomorphism is called an *isomorphism.*

Note that homomorphisms allow one to define the following notion:

Definition 2.1.12 (Sublattice)**.** Let L be a distributive lattice and $M \subseteq L$. Then M is a *DL-sublattice* of L if the inclusion map $\iota : M \to L$ is a DL-homomorphism. The same definition applies for Heyting and co-Heyting algebras, Boolean algebras and modal algebras.

2.1.2 Topological Spaces

Definition 2.1.13 (Topological Space)**.** Let X be a set. A *topology* on X is a set $\tau \subseteq \mathscr{P}(X)$ that contains $\emptyset$, X and is closed under finite intersections and arbitrary unions. For any $U \subseteq X$, U is *open* if $U \in \tau$, and it is *closed* if $U = -V$, i.e. the complement of V, for some $V \in \tau$. We call *clopen* a subset of X that is both open and closed. Given a topological space (X, τ), we write σ for the set of all closed sets in X. A *basis* for τ is a collection of sets $\beta \subseteq \tau$ such that every $U \in \tau$ is a union of elements from β. A *subbasis* of τ is a collection $\gamma \subseteq \tau$ such that the closure of γ under finite intersections is a basis for τ.

Definition 2.1.14 (Interior and Closure operators)**.** For any topological space (X, τ), there exist two maps $I : \mathscr{P}(X) \to \tau$ and $C : \mathscr{P}(X) \to \sigma$ such that for any $U \subseteq X$, IU (the *interior* of U) is the largest open set contained in U, and CU (the *closure* of U) is the smallest closed set contained in U.

The following are well-known properties of interior and closure operators that we will use repeatedly:

Proposition 2.1.15 (Basic Properties of Interior and Closure)**.** *Let (X, τ) be a topological space, and let I and C be the interior and closure operators associated with τ respectively. Then for any $U, V \subseteq X$:*

- $IX = CX = X$, $I\emptyset = C\emptyset = \emptyset$
- $I(U \cap V) = IU \cap IV$, $C(U \cup V) = CU \cup CV$
- *If $U \subseteq V$, then $IU \subseteq IV$ and $CU \subseteq CV$*
- *If U is open, then $IU = U$, and if V is closed, then $CV = V$.*
- *If β is a basis for τ and $\delta = \{-V ; V \in \beta\}$, then $IU = \bigcup\{V \in \beta ; V \subseteq U\}$, and $CU = \bigcap\{V \in \delta ; U \subseteq V\}$.*
- $IU = -C - U$

As a consequence of the previous facts about interior and closure operators, we have the following fact:

Lemma 2.1.16. *Let (X, τ) be a topological space. Then :*

- $\mathsf{O}(X) = \{\tau, \cap, \cup, \Rightarrow, \bigwedge, \bigcup, \emptyset, X\}$ *is a complete Heyting algebra, where for any $A, B \in \tau$, $A \Rightarrow B = I(-A \cup B)$, and for any family $\{A_i\}_{i \in I}$ of sets in τ, $\bigwedge_{i \in I}(A_i) = I(\bigcap_{i \in I} A_i)$;*
- $\mathsf{C}(X) = \{\sigma, \cap, \cup, \Leftarrow, \bigcap, \bigvee, \emptyset, X\}$ *is a complete co-Heyting algebra, where for any $A, B \in \sigma$, $A \Leftarrow B = C(A - B)$, and for any family $\{A_i\}_{i \in I}$ of sets in τ, $\bigvee_{i \in I}(A_i) = C(\bigcup_{i \in I} A_i)$.*

Finally, we will be particularly interested in subsets of a topological space with the following property:

Definition 2.1.17 (Regular open and Regular closed sets)**.** Let (X, τ) be a topological space. A set $U \subseteq X$ is *regular open* if $ICU = U$. It is *regular closed* if $CIU = U$.

We conclude with the definition of some important properties of topological spaces.

Definition 2.1.18 (T_0 Space)**.** Let (X, τ) be a topological space. Two points $x, y \in X$ are *topologically distinguishable* if there exists $U \in \tau$ that contains exactly one of x, y. The space (X, τ) is T_0 if any two points in X are topologically distinguishable.

Definition 2.1.19 (Alexandroff Space)**.** Let (X, τ) be a topological space. Then $(X\tau)$ is an *Alexandroff space* on X if for any collection $\{U_i\}_i \in I$ of open sets, $\bigcap_{i\in I} U_i \in \tau$. Equivalently, $I(\bigcap_{i\in I})U_i = \bigcap_{i\in I} IU_i$ for any family $\{U_i\}_{i\in I}$ of subsets of X.

The following definition plays an important role in the relationship between posets and topological spaces.

Definition 2.1.20 (Specialization preorder)**.** Let (X, τ) be a topological space. The *specialization preorder* $\leq$ of (X, τ) is defined by $x \leq y$ iff $x \in C\{y\}$.

Note that it is straightforward to see that if a topological space is T_0, then its assiociated specialization preorder is a poset. Conversely, we have the following important fact:

Proposition 2.1.21 (Upset topology)**.** *Let $(X, \leq)$ be a preorder. For any $U \subseteq X$, let $\uparrow U = \{a \in X \ ; \ b \leq a$ for some $b \in U\}$. U is called an upset if $\uparrow U = U$. Now let τ be the collection of all upsets in X. Then (X, τ) is an Alexandroff space.*

Proof. It is straightforward to check that $\emptyset, X$ are upsets and that upsets are closed under arbitrary unions and intersections. □

We therefore have a way of going back and forth between preordered sets and topological spaces. Note that, although the specialization preorder of a topological space is always unique, there are in general several way of defining a topology τ on a preordered set $(X, \leq)$ in such a way that τ is compatible with $\leq$, i.e. that the specialization preorder of the topological space (X, τ) is exactly $\leq$. In that respect, the Alexandroff topology or upset topology on a preordered set has the following important property:

Proposition 2.1.22. *Let $(X, \leq)$ be a poset, and let (X, τ) be the upset topology on $(X, \leq)$. Then τ is the finest topology on X compatible with $\leq$ i.e. for any topology τ' on X, if τ' is compatible with $\leq$, then $\tau' \subseteq \tau$.*

Finally, the following classes of topological spaces will be of special interest to us:

Definition 2.1.23 (Compact Hausdorff Space)**.** Let (X, τ) be a topological space. (X, τ) is *compact* if for any collection $\{U_i\}_{i\in I}$ of open sets such that $X = \bigcup_{i\in I} U_i$, there exists $J \subseteq I$ finite such that $X = \bigcup_{j\in J} U_j$. (X, τ) is *Hausdorff* if for any $x, y \in X$, there exists $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

The following proposition is well-known:

Proposition 2.1.24. *Let (X, τ) be a topological space. A subspace of (X, τ) is a topological space (U, τ_U) where $U \subseteq X$ and $\tau_U = \{V \cap U \ ; \ V \in \tau\}$. Then:*

- *If (X, τ) is Hausdorff, so is (U, τ_U);*
- *If (X, τ) is compact and U is closed, then (U, τ_U) is compact.*

We can now recall the most important results regarding the relationship between lattices and topological spaces that we will use throughout this thesis.

2.1.3 Topological Representations of Lattices

The study of the relationship between varieties of distributive lattices and classes of topological spaces is a rich and fruitful area, where results are usually presented as categorical dualities. [14], [42] are standard references on duality theory for distributive lattices and Boolan algebras. In this section, we briefly review some of the results and techniques that are inspired from these dualities and that we will use in the incoming chapters. Note that most of the results in this section rely on the following non-constructive theorem:

Theorem 2.1.25 (Prime Filter Theorem)**.** *Let L be a distributive lattice, and F and I a filter and an ideal over L respectively. Then if $F \cap I = \emptyset$, there exists a prime filter F' and an ideal F' such that $F \subseteq F'$, $I \subseteq I'$ and $F' \cap I' = \emptyset$.*

One of the important result of this thesis is that we will show that most of the results below can be carried out without the Prime Filter Theorem. In particular, we discuss in more detail this theorem in connection with other non-constructive principles in section 3 below.

We now recall the basic elements of Priestley's topological representation for distributive lattices.

Definition 2.1.26 (Priestley Space, Esakia Space, co-Esakia Space)**.** An *ordered topological space* is a tuple $(X, \tau, \leq)$ such that (X, τ) is a topological space and $(X, \leq)$ is a poset. A *Priestley space* is a compact ordered topological space $(X, \tau, \leq)$ that satisfies the Priestley Separation Axiom:

(PSA) for any $x, y \in X$ such that $x \not\leq y$, there exists $U \subseteq X$ such that U is a clopen upset, $X \in U$ and $y \notin U$.

Moreover, $(X, \tau, \leq)$ is an *Esakia space* if for any clopen set $U \subseteq X$, $\downarrow U = \{x \in X \ ; \ \exists y \in U : x \leq y\}$ is also clopen, and it is a *co-Esakia space* if for any clopen set U, $\uparrow U = \{x \in X \ ; \ \exists y \in U : y \leq x\}$ is also clopen.

Definition 2.1.27 (Dual Priestley Space of a DL)**.** Let L be a distributive lattice. Then the *dual Priestley space* of L is a Priestley space $(X_L, \tau, \leq)$ where:

- X_L is the set of all prime filters over L;
- τ is the topology generated by the subbasis $\{|a| \ ; \ a \in L\} \cup \{-|b| \ ; \ b \in L\}$, where for any $a \in L$, $|a| = \{p \in X_L \ ; \ a \in p\}$. Equivalently, τ is generated by the basis $\beta := \{|a| - |b| \ ; \ a, b \in L\}$;
- For any $p, q \in X_L$, $p \leq q$ iff $p \subseteq q$.

Lemma 2.1.28. *Let $(P, \leq)$ be a poset. Then $Up(P)$, the set of all upsets of P, gives rise to a bi-Heyting algebra.*

Proof. It is easy to see that upsets are closed under arbitrary unions and intersections, and hence that $(Up(P), \cap, \cup, \emptyset, P)$ is a complete distributive lattice. To see that it is a Heyting algebra, consider for any $A, B \in Up(P)$ the set $A \Rightarrow B = \bigcup\{C \in Up(P) \ ; \ A \cap C \subseteq B\}$. It is straightforward to see that $\Rightarrow$ is the right-residuum of $\cap$. A moment's reflection shows moreover that for any $A, B \in Up(P)$, $A \Rightarrow B = -\downarrow(A - B)$. To see that $Up(P)$ is a co-Heyting algebra, define $A \Leftarrow B = \bigcap\{C \in Up(P) \ ; \ A \subseteq B \cup C\}$ for any $A, B \in Up(P)$. Once again, it is straightforward to check that $\Leftarrow$ is the left-residuum of $\cup$, and that $A \Leftarrow B = \uparrow(A - B)$. $\square$

Theorem 2.1.29 (Priestley Representation Theorem)**.** *Let L be a distributive lattice with dual space $(X_L, \tau, \leq)$. Then L is isomorphic to a subalgebra of $Up(X_L)$, i.e. the complete distributive lattice induced by the upsets of X_L.*

Proof. Note first that by Lemma 2.1.28 $(Up(X_L), \cap, \cup, \emptyset, X_L)$ is a complete distributive lattice. Moreover, for any $a \in L$, $|a|$ is an upset. Consider now $L^* = (|L|, \cap, \cup, |0|, |1|)$, where $|L| = \{|a| \ ; \ a \in L\}$. By basic properties of prime we have the following:

- $|0| = \emptyset$, since prime filters are proper, and $|1| = X$, since prime filters are non-empty;
- $|a \wedge b| = |a| \cap |b|$, since prime filters are upward closed and downward directed;
- $|a \vee b| = |a| \cup |b|$: the left-to-right direction follows from the fact that all filters in X_L are prime, and the converse follows from the fact that filters are upward closed.

From this it follows at once that L^* is a subalgebra of $Up(L)$ and that $|\cdot| : L \to L^*$ is a surjective homomorphism. Hence we only have to prove that it is also injective. To see this, we show that for any $a, b \in L$, $|a| \subseteq |b|$ iff $a \leq b$. The right-to-left direction is immediate, since filters are upward-closed. For the left-to-right direction, note that $a \not\leq b$ implies that $\uparrow a \cap \downarrow b = \emptyset$, where $\uparrow a = \{c \in L \ ; \ a \leq c\}$ and $\downarrow b = \{c \in L \ ; \ c \leq b\}$. By the Prime Filter Theorem, this means that there exists a prime filter p such that $a \in p$ and $b \notin p$. Hence, by contraposition, $|a| \subseteq |b|$ implies that $a \leq b$. ☐

The previous theorem also has the following strengthening:

Theorem 2.1.30 (Esakia Representation Theorem)**.** *Let L be a distributive lattice with dual space $(X_L, \tau, \leq)$. Then:*

1. *if L is a Heyting algebra, then $|\cdot| : L \to L^*$ is an injective Heyting homomorphism*
2. *if L is a co-Heyting algbra, then $|\cdot| : L \to L^*$ is an injective co-Heyting homomorphism*

Proof. 1. In light of Lemma 2.1.28 and Theorem 2.1.29, we only have to show that for any $a, b \in L$, $|a \to b| = - \downarrow(|a| - |b|)$. For the left-to-right direction, note that for any $p \in X_L$, if $a \to b \in p$, then for any $q \supseteq p$, if $a \in q$, then $a \wedge (a \to b) \in q$, and therefore $b \in q$ since q is upward closed. For the converse, assume $a \to b \notin p$ for some $p \in X_L$, and consider the set $p' = \uparrow\{c \wedge a \ ; \ c \in p\}$. It is straightforward to check that p' is a filter, and moreover, we claim that $p' \cap \downarrow b \emptyset$. To see this, assume for a contradiction that there is $c \in p$, $d \in L$ such that $c \wedge a \leq d \leq b$. But then by residuation $c \leq a \to b$, hence $a \to b \in p$, contradicting our assumption. Hence $p' \cap \downarrow b = \emptyset$. By the Prime Filter Theorem, this means that there is $q \in X_L$ such that $p' \subseteq q$ and $b \notin q$. But then $p \in \downarrow(|a| - |b|)$, which completes the proof.

2. Similarly, we only have to prove that for any $a, b \in L$, $|a -\!\!< b| = \uparrow(|a| - |b|)$. For the right-to-left direction, it is enough to note that for any $p \in X_L$, since $a \leq (a -\!\!< b) \vee b$, if $p \in |a| - |b|$, then $a -\!\!< b \in p$. For the converse, recall that a filter p is prime iff its complement p^c is a prime ideal. Now assume $a -\!\!< b \in p$ for some $p \in X_L$. Then consider $p'^c = \{b \vee d \ ; \ d \in p^c\}$. It is straightforward to see that p'^c is an ideal. Moreover, we claim that $\uparrow a \cap p'^c = \emptyset$. If not, then $a \leq b \vee d$ for some $d \in p^c$, and hence by residuation $a -\!\!< b \leq d$, which means that $a -\!\!< b \in p^c$, contradicting our assumption. Hence by the Prime Filter Theorem, there exists a prime ideal q such that $p'^c \subseteq q$ and $a \notin q$. But then q^c is a prime filter in $|a| - |b|$, and moreover, since $p^c \subseteq q$, it follows that $q^c \subseteq p$. Hence $p \in \uparrow(|a| - |b|)$.

☐

Priestley's Representation Theorem is a generalization of the celebrated Stone Representation Theorem.

Definition 2.1.31 (Stone Space). Let (X, τ) be a compact Hausdroff space. Then (X, τ) is a *Stone space* if τ has a clopen basis.

Lemma 2.1.32. *Let L be a Boolean algebra, and F a proper filter over L. Then the following are equivalent:*

1. *F is prime*
2. *F is an ultrafilter : for any $a \in L$, $a \in L$ or $\neg a \in L$.*
3. *F is a maximal filter.*

Proof.

1. $\Rightarrow$ 2. For any $a \in L$, $1 = a \vee \neg a \in F$, and hence, since F is prime, $a \in F$ or $\neg a \in F$.

2. $\Rightarrow$ 3. Assume F is not maximal, i.e. there is a proper filter F' such that $F \subseteq F'$ and there is $a \in L$ such that $a \in F' \cap F^c$. Since F is an ultrafilter, this means that $\neg a \in F$. Hence $a \wedge \neg a \in F'$, a contradiction.

3. $\Rightarrow$ 1. Assume $a, b \notin F$ for some $a, b \in F$. We claim that this means that $\neg a$ and $\neg b \in F$. Consider $F_1 =\uparrow \{a \wedge c \ ; \ c \in F\}$ and $F_2 =\uparrow \{b \wedge c \ ; \ c \in F\}$. Since F is maximal, both F_1 and F_2 must be improper filters, which means that they both contain 0. Hence there exists $c, d \in F$ such that $a \wedge c \leq 0$ and $b \wedge d \leq 0$. But this means by residuation that both $\neg a$ and $\neg b$ are in F. Hence $\neg a \wedge \neg b = \neg(a \vee b) \in F$, which means that $(a \vee b) \notin F$. By contraposition, it follows that F is prime.

$\square$

Definition 2.1.33 (Dual Stone space of a BA). Let L be a Boolean algebra. The *dual Stone space* of L is the space (X_L, τ) where:

- X_L is the set of all ultrafilters over L;
- τ is the topology generated by the clopen basis $\{|a| \ ; \ a \in L\}$.

Theorem 2.1.34 (Stone Representation Theorem). *Let L be a Boolean algebra, and (X_L, τ) its dual Stone space. Then L is isomorphic to a subalgebra of $\mathscr{P}(X_L)$.*

Proof. By Lemma 2.1.32, it is straightforward to see that (X_L, τ) is exactly the dual Priestley space $(X_L, \tau, \leq)$ of L: the order $\leq$ is the identity since all filters in X_L are maximal. Moreover, for any $a \in L$, $-|a| = |\neg a|$ since all filters in X_L are ultrafilters. Hence $|\cdot| : L \to L^*$ is a Boolean isomorphism, and L^* is a Boolean subalgebra of $\mathscr{P}(X_L)$. $\square$

The previous representation theorems show that to every distributive lattice corresponds a Priestley space, and to every Boolean algebra a Stone space. The following theorem also show that one can also go from topological spaces to algebras.

Theorem 2.1.35 (Stone, Priestley, Esakia). *Let $\mathscr{X} = (X, \tau, \leq)$ be a Priestley space, and let $ClopUp(\mathscr{X})$ be the set of all clopen upsets of X. Then:*

- *$(ClopUp(\mathscr{X}), \cap, \cup, \emptyset, X)$ is a distributive lattice;*

- *if $\mathscr{X}$ is an Esakia space, then $(ClopUp(X), \cap, \cup, \Rightarrow, \emptyset, X)$, where for any $A, B \subseteq X$, $A \Rightarrow B = -\downarrow(A - B)$ is a Heyting algebra;*
- *if $\mathscr{X}$ is a co-Esakia space, then $(ClopUp(X), \cap, \cup, \Leftarrow, \emptyset, X)$ is a co-Heyting algebra, where for any $A, B \subseteq X$, $A \Leftarrow B = \uparrow(A - B)$ is a co-Heyting algebra;*
- *if $\mathscr{X}$ is a Stone space, then $Clop(\mathscr{X})$ is a Boolean algebra.*

Finally, we show how the previous representation theorems extend to the modal case.

Definition 2.1.36 (Modal Esakia space, modal co-Esakia space, modal Stone space)**.** Let $\mathscr{X} = (X, \tau, \leq, R)$ be such that $(X, \tau, \leq)$ is a Priestley space and R is a relation on X. Then:

- $(\mathscr{X}, R)$ is a *modal Esakia space* if $(X, \tau, \leq)$ is an Esakia space, $\leq \circ R \subseteq R$, and for any $U \in ClopUp(\mathscr{X})$, $R[U] = \{x \in X \ ; \ \forall y \in X : xRy \Rightarrow y \in U\}$ is also clopen;
- $(\mathscr{X}, R)$ is a *modal co-Esakia space* if $(X, \tau, \leq)$ is a co-Esakia space, $R \circ \geq \subseteq R$, and for any $U \in ClopUp(\mathscr{X})$, $R\langle U\rangle = \{x \in X \ ; \ \exists y \in U : yRx\}$ is also clopen.
- $(\mathscr{X}, R)$ is a *modal Stone space* if $(\mathscr{X}, R)$ is a modal Esakia space and $\leq$ is the identity.

Similarly to the non-modal case above, we have the following two important theorems:

Theorem 2.1.37. *Let $\mathscr{X} = (X, \tau, \leq)$ be a Priestley space and R a relation on X. Then:*

- *if $(\mathscr{X}, R)$ is a modal Esakia space, then $(ClopUp(\mathscr{X}), R[\cdot])$ is a modal Heyting algebra;*
- *if $(\mathscr{X}, R)$ is a modal co-Esakia space, then $(ClopUp(\mathscr{X}), R\langle\cdot\rangle)$ is a modal co-Heyting algebra;*
- *if $(\mathscr{X}, R)$ is a modal Stone space, then $(Clop(\mathscr{X}, R[\cdot])$ is a BAO.*

Theorem 2.1.38.

- *Let $(L, \Box)$ be a modal Heyting algebra with dual Esakia space $(X_L, \tau, \leq)$, and let $R \subseteq X_L \times X_L$ be such that for any $p, q \in X_L$, we have pRq iff $p^{\Box} \subseteq q$, where $p^{\Box} = \{a \in L \ ; \ \Box a \in p\}$. Then $(X_L, \tau, \leq, R)$ is a modal Esakia space, and $(L, \Box)$ is isomorphic to $(ClopUp(X_L), R[\cdot])$.*
- *Let $(M, \Diamond)$ be a modal co-Heyting algebra with dual co-Esakia space $(X_M, \tau, \leq)$, and let $S \subseteq X_L \times X_L$ be such that for any $p, q \in X_M$, we have pRq iff $q \subseteq p_{\Diamond}$, where $p_{\Diamond} = \{a \in L \ ; \ \Diamond a \in p\}$. Then $(X_M, \tau, \leq, R)$ is a modal co-Esakia space, and $(M, \Diamond)$ is isomorphic to $(ClopUp(X_M), R\langle\cdot\rangle)$.*
- *Let $(B, \Box)$ be a BAO with dual Stone space (X_B, τ), and let $R \subseteq X_B \times X_B$ be such that for any $p, q \in X_B$, pSq iff $p^{\Box} \subseteq q$, where $p^{\Box} = \{a \in B \ ; \ \Box a \in p\}$. Then (X_B, τ, S) is a modal Stone space, and $(B, \Box)$ is isomorphic to $(Clop(X_B), S[\cdot])$.*

2.2 Logics and their Models

In this section, we briefly introduce the various logics that we will refer to in the following chapters. We start with a propositional calculus for each of these logics, and then extend each calculi to first-order languages. Throughout this section, we presuppose some familiarity with the basic notions of proof and derivation in a Hilbert-style calculus. Standard references for propositional and first-order classical and intuitionistic logic include [3] and [13], and [9] and [68] for modal classical and intuitionistic logic.

2.2.1 Propositional Logics

We fix a propositional language $\mathfrak{L}$ with a countable set $Prop(\mathfrak{L})$ of propositional variables, two propositional constants $\bot$ and $\top$, and two binary connectives $\wedge$ and $\vee$.

Definition 2.2.1 (IPC and MIPC)**.** Let $\mathfrak{L}_{IPC}$ be the propositional language generated by $\mathfrak{L}$ with an additional connective $\to$. Then the *Intuitionistic Propositional Calculus (IPC)* is determined by the following axioms for any formulas $\phi, \psi, \chi \in \mathfrak{L}_{IPC}$:

(C1) $(\phi \wedge \psi) \to \phi$

(C2) $\phi \to (\phi \wedge \phi)$

(C3) $(\phi \wedge \psi) \to (\psi \wedge \phi)$

(D1) $\phi \to (\phi \vee \psi)$

(D2) $(\phi \vee \phi) \to \phi$

(D3) $(\phi \vee \psi) \to (\psi \vee \phi)$

(B1) $\bot \to \phi$

(T1) $\phi \to \top$

and the following rules:

(MP) from ϕ and $\phi \to \psi$, infer ψ

(I1) from $\phi \to \psi$ and $\psi \to \chi$, infer $\phi \to \chi$

(I2) from $(\phi \wedge \psi) \to \chi$, infer $\phi \to (\psi \to \chi)$

(I3) from $\phi \to (\psi \to \chi)$ infer $(\phi \wedge \psi) \to \chi$

(I4) from $\phi \to \psi$, infer $(\phi \vee \chi) \to (\psi \vee \chi)$

For any $\Gamma \cup \{\phi\} \subseteq Form(\mathfrak{L}_{IPC})$, we write $\Gamma \vdash_{IPC} \phi$ whenever there exists $\psi_0, ..., \psi_n \in \Gamma$, such that $(\psi_0 \wedge ... \wedge \psi_n) \to \phi$ is derivable in IPC. For any formula ϕ, ϕ is a *theorem* of IPC if $\{\top\} \vdash_{IPC} \phi$, in which case we simply write $\vdash_{IPC} \phi$.

Similarly, let $\mathfrak{L}_{MIPC}$ be $\mathfrak{L}_{IPC}$ extended with a unary operator $\Box$. Then the *Modal Intuitionistic Propositional Calculus MIPC* is determined by all the axioms and rules of IPC, with the addition of the two axioms:

(L1) $\Box(\phi \wedge \psi) \to (\Box\phi \wedge \Box\psi)$

(L2) $(\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi)$

and the rule

(N) from ϕ, infer $\Box\phi$.

Theorems of MIPC and the relation $\vdash_{MIPC}$ are defined analogously.

Definition 2.2.2 (cIPC and cMIPC)**.** Let $\mathfrak{L}_{cIPC}$ be the propositional language generated by adding a connective $-\!\!<$ to $\mathfrak{L}$. Then the *co-Intuitionistic Propositional Calculus (cIPC)* is determined by the following axioms for any formulas $\phi, \psi, \chi \in \mathfrak{L}_{cIPC}$[3]:

(cC1) $\phi -\!\!< (\phi \vee \psi)$

(cC2) $(\phi \vee \phi) -\!\!< \phi$

(cC3) $(\phi \vee \psi) -\!\!< (\psi \vee \phi)$

(cD1) $(\phi \wedge \psi) -\!\!< (\phi)$

(cD2) $\phi -\!\!< (\phi \wedge \phi)$

(cD3) $(\phi \wedge \psi) -\!\!< (\psi \wedge \phi)$

(cB1) $\phi -\!\!< \top$

(cB1) $\bot -\!\!< \phi$

and the following rules:

(cMP) from ψ and $\phi -\!\!< \psi$, infer ϕ

(cI1) from $\psi -\!\!< \phi$ and $\chi -\!\!< \psi$, infer $\chi -\!\!< \phi$

(cI2) from $\chi -\!\!< (\phi \vee \psi)$, infer $(\chi -\!\!< \psi) -\!\!< \phi)$

(cI3) from $(\chi -\!\!< \psi) -\!\!< \phi$ infer $\chi -\!\!< (\phi \vee \psi)$

(cI4) from $\psi -\!\!< \phi$, infer $(\psi \wedge \chi) -\!\!< (\phi \wedge \chi)$

For any $\Delta \cup \{\phi\} \subseteq Form(\mathfrak{L}_{cIPC})$, we write $\Delta \vdash_{cIPC} \phi$ whenever there exists $\psi_0, ..., \psi_n \in \Delta$, such that $(\psi_0 \wedge ... \wedge \psi_n) -\!\!< \phi$ is derivable in cIPC. For any formula ϕ, ϕ is a *theorem* of cIPC if $\{\top\} \vdash_{cIPC} \phi$, in which case we simply write $\vdash_{cIPC} \phi$.

Similarly, let $\mathfrak{L}_{cMIPC}$ be $\mathfrak{L}_{cIPC}$ extended with a unary operator $\Diamond$. Then the *Modal Intuitionistic Propositional Calculus cMIPC* is determined by all the axioms and rules of cIPC, with the addition of the two axioms:

(M1) $(\Diamond\phi \vee \Diamond\psi) -\!\!< \Diamond(\phi \vee \psi)$

(M2) $\Diamond(\phi \vee \psi) -\!\!< (\Diamond\phi \vee \Diamond\psi)$

and the rule

(cN) from ϕ, infer $\Diamond\phi$.

Theorems of cMIPC and the relation $\vdash_{cMIPC}$ are defined analogously.

[3] A standard issue with Hilbert-style system for co-intuitionistic logic is that the logic does not have an implication connective, and therefore no *modus ponens*. Here, we avoid this problem by defining a Hiblert-style calculus where contradictions, rather than tautologies, are deducible. Moreover, all rules and axioms of the system are the dual of one rule of axiom of the system IPC.References on co-intuitionistic logic include [56], [60], [61], and [67]

Definition 2.2.3 (CPC)**.** Let $\mathfrak{L}_{CPC} = \mathfrak{L}_{IPC}$ as in Definition 2.2.1. Then the *Classical Propositional Calculus (CPC)* is the calculus determined by all the rules and axioms of IPC, with the extra axiom

(N1) $((\phi \to \bot) \to \bot) \to \phi$

The relation $\vdash_{CPC}$ and theorems of CPC are defined as in Definition 2.2.1.

Similarly, for $\mathfrak{L}_K = \mathfrak{L}_{MIPC}$, the *Modal Classical Propositional Calculus* K is determined by all the axioms and rules of MIPC with the additional axiom $(N1)$.

The definitions of the relations $\vdash_{CPC}$ and $\vdash_K$ and of theorems of CPC and K completely match those of $\vdash_{IPC}$, $\vdash_{MIPC}$, theorems of IPC and theorems of K respectively.

For any propositional calculus C defined in this section, C is sound and complete with respect to a class of lattices defined in the previous section. In order to phrase those results, however, we need to define algebraic semantics for these logics.

Definition 2.2.4 (Valuation)**.** A IPC*-valuation* is a function $V : \mathfrak{L}_{IPC} \to L$ for some Heyting algebra L such that $V(\bot) = 0_L$, $V(\top) = 1_L$, and for any $\phi, \psi \in \mathfrak{L}_{IPC}$:

- $V(\phi \wedge \psi) = V(\phi) \wedge_L V(\psi)$
- $V(\phi \vee \psi) = V(\phi) \vee_L V(\psi)$
- $V(\phi \to \psi) = V(\phi) \to_L V(\psi)$

This definition generalizes in the obvious way to the case of *cIPC-valuations* (into co-Heyting algebras), *CPC-valuations* (into Boolean algebras), *MIPC-valuations* (into modal Heyting algebras), *cMIPC-valuations* (into modal co-Heyting algebras) and *K-valuations* (into modal algebras).

Definition 2.2.5 (Validity)**.** Let $\phi \in \mathfrak{L}_{IPC}$ and L be a Heyting algebra. Then ϕ is *valid* on A if for any valuation $V : \mathfrak{L}_{IPC} \to L$, $V(\phi) = 1_L$. A formula ϕ is valid on the class of all Heyting algebras if ϕ is valid on every Heyting algebra L.

Once again, this definition generalizes in an obvious way to $cIPC$, $MIPC$, $cMIPC$, CPC and K. Finally, we recall the well-known definitions of soundness and completeness:

Definition 2.2.6 (Soundness and Completeness)**.** Let C be a propositional calculus and $\mathfrak{K}$ be a class of algebras. C is *sound* with respect to $\mathfrak{K}$ if any theorem of C is valid on $\mathfrak{K}$. Conversely, C is *complete* with respect to $\mathfrak{L}$ if any formula valid on $\mathfrak{K}$ is a theorem of C.

We can now formulate the completeness theorems mentioned above:

Theorem 2.2.7 (Algebraic completeness)**.**

1. *IPC is sound and complete with respect to the class of all Heyting algebras;*
2. *cIPC is sound and complete with respect to the class of all co-Heyting algebras;*
3. *MIPC is sound and complete with respect to the class of all modal Heyting-algebras;*
4. *cMIPC is sound and complete with respect to the class of all modal co-Heyting algebras;*
5. *CPC is sound and compelte with respect to the class of all Boolean algebras;*
6. *K is sound and complete with espect to the class of all modal algebras.*

2.2.2 Adding Quantifiers

In this section, we fix a basic first-order language $\mathfrak{L}$ and define the first-order counterpart to the propositional calculi defined above. We presuppose some familiarity with the basic notions of a first-order language such as terms, formulas, free and bound variables.

Definition 2.2.8 (First-Order Language). A *basic first-order language* $\mathfrak{L}$ consists of a countable set of variables $Var(\mathfrak{L})$, a countable set of terms $Term(\mathfrak{L})$, relation symbols $Rel(\mathfrak{L})$ which include the relation symbol $=$ (equality), two propositional constants $\bot$ and $\top$, two binary connectives $\wedge$ and $\vee$, and two quantifiers $\forall$ and $\exists$. A *first-order language* $\mathfrak{L}'$ is any basic first-order language that may contain additional connectives or operators.

In order to define first-order calculi, we need to add axioms and rules for quantifiers, as well as axioms for the equality symbol:

Definition 2.2.9 (Standard fist-order axioms). Let $\mathfrak{L}$ be a first-order language that contains connectives $\to$ and $\prec$. We define the following axioms and rules for any formula $\phi(x), \psi \in \mathfrak{L}$:

(U1) $\forall x\phi(x) \to \phi(t)$ for any $t \in Term(\mathfrak{L})$

(U2) from $\psi \to \phi(t)$ for all $t \in Term(\mathfrak{L})$, infer $\psi \to \forall x\phi(x)$

(E1) $\phi(t) \to \exists x\phi(x)$ for any $t \in Term(\mathfrak{L})$

(E2) from $\phi(t) \to \psi$ for all $t \in Term(\mathfrak{L})$, infer $\exists x\phi(x) \to \psi$.

(cU1) $\phi(t) \prec \exists x\phi(x)$ for any $t \in Term(\mathfrak{L})$

(cU2) from $\psi \prec \phi(t)$ for all $t \in Term(\mathfrak{L})$, infer $\psi \prec \exists x\phi(x)$

(cE1) $\forall x\phi(x) \prec \phi(t)$ for any $t \in Term(\mathfrak{L})$

(cE2) from $\phi(t) \prec \psi$ for all $t \in Term(\mathfrak{L})$, infer $\forall x\phi(x) \prec \psi$

As a matter of convention, we call (U1) and (cE1) Universal instantiation axioms, (U2) and (cE2) Universal generalization rules, (E1) and (cU1) Existential instantiation axioms, and (E2) and (cU2) existential generaliation rules.

Additionally, we define the following axioms and rules for equality for any $\phi \in \mathfrak{L}$ and any $t, u, v \in Term(\mathfrak{L})$:

(=1) $\forall x(x = x)$

(=2) from $t = u$, infer $u = t$

(=3) from $t = u$ and $u = v$, infer $u = v$

(=4) from $\phi(t)$ and $t = u$, infer $\phi(u)$

In addition to the standard axioms and rules for quantifiers and equality given in the previous definition, the first-order logics we will consider will also satisfy some additional axioms which correspond both to distributivity conditions on the quantifiers and to constant domain conditions in the standard semantics for those logics. For this reason, we call such axioms Constant Domain Axioms[4].

[4] On the similarities between the various Constant Domain Axioms, see for example Makkai and Reyes [47] on their important role in categorical logic

Definition 2.2.10 (Constant Domain Axioms)**.** Let $\mathfrak{L}$ be a first-order language that contains connectives $\to$ and $-\!\!<$ and operators $\Box$ and $\Diamond$. We define the following Constant Domain Axioms for any $\phi(x), \psi \in \mathfrak{L}$ such that x does not appear freely in ψ:

(CD1) $\forall x(\phi(x) \vee \psi) \to (\forall x\phi(x) \vee \psi)$

(CD2) $(\exists x\phi(x) \wedge \psi) -\!\!< \exists x(\phi(x) \wedge \psi)$

(CD3) $(\forall x\Box\phi(x) \to \Box\forall x\phi(x)) \wedge (\Box\forall x\phi(x) \to \forall x\Box\phi(x))$[5]

(CD4) $(\exists x\Diamond\phi(x) -\!\!< \Diamond\exists x\phi(x)) \vee (\Diamond\exists x\phi(x) -\!\!< \exists x\Diamond\phi(x))$

Definition 2.2.11 (IPL, cIPL, CPL, MIPL, cMIPL, KL)**.** Let $\mathfrak{L}$ be a fixed basic first-order language. Then $\mathfrak{L}_{IPL}$, $\mathfrak{L}_{cIPL}$, $\mathfrak{L}_{MIPL}$ and $\mathfrak{L}_{cMIPL}$ are the first-order languages obtained by adding $\{\to\}$, $\{-\!\!<\}$, $\{\to, \Box\}$ and $\{-\!\!<, \Diamond\}$ to $\mathfrak{L}$ respectively. Then:

- *Intuitionistic Predicate Logic (IPL)* is the first-order calculus determined by all the axioms and rules of IPC, plus (U1), (U2), (E1), (E2), (CD1), and all the axioms and rules for equality;
- *co-Intuitionisitic Predicate Logic (cIPL)* is the first-order calculus determined by all the axioms and rules of $cIPC$, plus (cU1), (cU2), (cE1), (cE2), (CD2), and all the axioms and rules for equality;
- *Classical Predicate Logic (CPL)* is the first-order calculus determined by all the axioms and rules of IPL plus (N1);
- *Modal Intutionistic Predicate Logic (MIPL)* is the first-order calculus determined by all the axioms and rules of IPL plus (L1), (L2), (N) and (CD3);
- *Modal co-Intuitionistic Predicate Logic (cMIPL)* is the first-order calculus determined by all the axioms and rules of $cIPL$ plus (M1), (M2), (cN) and (CD4);
- *Modal Classical Predicate Logic (KL)* is the first-order calculus determined by all the axioms and rules of $MIPL$ plus (N1).

2.2.3 Relational Models

We conclude this section by recalling the standard semantics for some of the logics defined above. We first deal with propositional models.

Definition 2.2.12 (Propositional frames)**.** Let $(X, \tau, \leq)$ be Priestley space. Then:

- $(X, \leq)$ is an *intuitionistic Kripke frame* if $(X, \tau, \leq)$ is an Esakia space and τ is the discrete topology;
- $(X, \leq)$ is a *co-intuitionistic Kripke frame* if $(X, \tau, \leq)$ is a co-Esakia space and τ is the discrete topology;
- X is a *classical propositional frame* if X is a singleton.

Based on this definition, we can now define in a natural way models for IPC, $cIPC$ and CPC:

[5]Note that this axiom is also known in the literature on first-order modal logic as the conjunction of the Barcan and converse Barcan formulas.

Definition 2.2.13 (Propositional models)**.** Let $\mathscr{X} = (X, \tau, \leq)$ be Priestley space. Then:

- $(X, \leq, V)$ is an *intuitionistic Kripke model* if $(X, \leq)$ is an intuitionistic Kripke frame and $V : \mathfrak{L}_{IPC} \to ClopUp(\mathscr{X})$ is an IPC-valuation;
- $(X, \leq, V)$ is an *co-intuitionistic Kripke model* if $(X, \leq)$ is a co-intuitionistic Kripke frame and $V : \mathfrak{L}_{cIPC} \to ClopUp(\mathscr{X})$ is a $cIPC$-valuation;
- (X, V) is a *classical propositional model* if X is a classical propositional frame and $V : \mathfrak{L}_{CPC} \to \mathscr{P}(X)$ is a CPC-valuation.

All the notions above generalize to the modal case in the following way:

Definition 2.2.14 (Modal frames)**.** Let $\mathscr{X} = (X, \tau, \leq, R)$ be a modal Priestley space. Then:

- $(X, \leq, R)$ is a *modal intuitionistic Kripke frame* if $(X, \leq)$ is an intuitionistic Kripke frame;
- $(X, \leq, R)$ is a *modal co-intuitionistic Kripke frame* if $(X, \leq)$ is a co-intuitionistic Kripke frame;
- (X, R) is a *modal Kripke frame* if (X, τ, R) is a modal Stone space.

Definition 2.2.15 (Modal propositional models)**.** Let $\mathscr{X} = (X, \tau, \leq, R)$ be a modal Priestley space. Then:

- $(X, \leq, R, V)$ is a *modal intuitionistic Kripke model* if $(X, \leq, R)$ is a modal intuitionistic Kripke frame and $V : \mathfrak{L}_{MIPC} \to ClopUp(\mathscr{X})$ is a $MIPC$-valuation.
- $(X, \leq, R, V)$ is a *modal co-intuitionistic Kripke model* if $(X, \leq, R)$ is a modal co-intuitionistic Kripke frame and $V : \mathfrak{L}_{cMIPC} \to ClopUp(\mathscr{X})$ is a $cMIPC$-valuation;
- (X, R, V) is a *modal Kripke model* if (X, R) is a modal Kripke frame and $V : \mathfrak{L}_K \to \mathscr{P}(X)$ is a K-valuation.

Finally, we conclude by defining first-order counterparts of the various classes of models defined above. We start with the simplest case, i.e. Tarskian models for CPL:

Definition 2.2.16 (Tarskian Model)**.** Let $\mathfrak{L}$ be a first-order classical language. A *Tarskian model* is a structure $\mathscr{M} = (D, J, \alpha)$ such that D is a set (the domain of $\mathscr{M}$), J maps every n-ary relation symbol R in $\mathfrak{L}$ to a subset of D^n, and $\alpha : Var(\mathfrak{L}) \to D$ maps every variable to an element of D. Every Tarskian model $\mathscr{M} = (D, J, \alpha)$ induces a valuation $V_{\mathscr{M},\alpha} : Form(\mathfrak{L}) \to \{0, 1\}$ defined recursively for any $\phi, \psi \in \mathfrak{L}$, $R \in Rel(\mathfrak{L})$ and $x_1, ..., x_n \in Var(\mathfrak{L})$ as follows:

- If $\phi := R(x_1, ..., x_n)$, then $V_{\mathscr{M},\alpha}(\phi) = 1$ iff $(\alpha(x_1), ..., \alpha(x_n)) \in J(R)$;
- $V_{\mathscr{M},\alpha}(\top) = 1$, $V_{\mathscr{M},\alpha}(\bot) = 0$;
- $V_{\mathscr{M},\alpha}(\phi \wedge \psi) = 1$ iff $V_{\mathscr{M},\alpha}(\phi) = V_{\mathscr{M},\alpha}(\psi) = 1$;
- $V_{\mathscr{M},\alpha}(\phi \vee \psi) = 1$ iff $V_{\mathscr{M},\alpha}(\phi) = 1$ or $V_{\mathscr{M},\alpha}(\psi) = 1$;
- $V_{\mathscr{M},\alpha}(\phi \to \psi) = 1$ iff $V_{\mathscr{M},\alpha}(\phi) = 0$ or $V_{\mathscr{M},\alpha}(\psi) = 1$
- $V_{\mathscr{M},\alpha}(\forall x \phi(x)) = 1$ iff $V_{\mathscr{M},\beta}(\phi(y)) = 1$ for any $\beta : Var(\mathfrak{L}) \to \{0, 1\}$ such that for any $y \in Var(\mathfrak{L})$, $\beta(y) \neq \alpha(y)$ only if $y = x$;

- $V_{\mathscr{M},\alpha}(\exists x\phi(x)) = 1$ iff $V_{\mathscr{M},\beta}(\phi(y)) = 1$ for some $\beta : Var(\mathfrak{L}) \to \{0,1\}$ such that for any $y \in Var(\mathfrak{L})$, $\beta(y) \neq \alpha(y)$ only if $y = x$;

Tarskian models are the standard semantics for CPL. In the following chapter, we will recall how the original Rasiowa-Sikorski Lemma allows for an algebraic proof of the completeness of CPL with respect to Tarskian models. For now, we simply show how the definition of Tarskian models generalizes to the logics IPL, $cIPL$, $MIPL$, $cMIPL$ and KL.

Definition 2.2.17 (Complex model)**.** Let $\mathfrak{L}$ be a first-order language. A *complex model* $\mathscr{M}$ is a tuple $(M, \leq, R, \{f_{ij}\}_{i,j\in M})$ such that:

- M is a set of Tarskian models, i.e. for every $i \in M$, i is a structure (D_i, J_i, α_i)
- $\leq$ is a partial order on M
- R is a relation on $M \times M$
- For any $i, j \in M$, $f_{ij} : D_i \to D_j$ is a function such that for any $i, j, k \in M$:
  - f_{ii} is the identity map;
  - $f_{jk} \circ f_{ij} = f_{ik}$;
  - if $i \leq j$ or iRj, then f_{ij} is surjective;
  - for any $x \in Var(\mathfrak{L})$, $f_{ij}(\alpha_i(x)) = \alpha_j(x)$
  - if $i \leq j$, then for any n-ary $R \in Rel(\mathfrak{L})$ and any $a_1, ..., a_n \in D_i$, if $(a_1, ..., a_n) \in J_i(R)$, then $(f_{ij}(a_1), ..., f_{ij}(a_n)) \in J_j(R)$.

Definition 2.2.18. Let $\mathscr{M} = (M, \leq, R, \{f_{ij}\}_{i,j\in M})$ be a complex model. Then:

- $\mathscr{M}$ is an *IPL-model* if $(M, \leq)$ is an intuitionistic Kripke frame and R is the diagonal relation;
- $\mathscr{M}$ is a *cIPL-model* if $(M, \leq)$ is a co-intuitionistic Kripke frame and R is the diagonal relation;
- $\mathscr{M}$ is a *MIPL-model* if $(M, \leq, R)$ is a modal intuitionistic Kripke frame;
- $\mathscr{M}$ is a *cMIPL-model* if $(M, \leq, R)$ is a modal co-intuitionistic Kripke frame;
- $\mathscr{M}$ is a *KL-model* if (M, R) is a modal Kripke frame and $\leq$ is the identity.

Finally, we define truth in a complex model as follows:

Definition 2.2.19. Let $L \in \{IPC, cIPC, MIPC, cMIPC, KC\}$, L' the first-order logic corresponding to L, $\mathfrak{L}$ its associated first-order language, and let $\mathscr{M} = (M, \leq, R, \{f_{ij}\}_{i,j\in M})$ be a L'-model. We write α for the map that sends every $i \in M$ to the assignment α_i. Given a map β that sends every $i \in M$ to some other assignment of variables β_i, we say that β is a *system of assignments* if $f_{ij}(\beta_i(x)) = \beta_j(x))$ for any $i, j \in M$. Given a system of assignments β different from α we write $\mathscr{M}_\beta$ for the L-model obtained by replacing every assignment α_i by β_i. A L'-valuation on $\mathscr{M}$ is a function $V_{\mathscr{M},\alpha} : Form(\mathfrak{L}) \to Up(\mathscr{M})$ such that:

- $V_{\mathscr{M},\alpha}$ is a L-valuation into $Up(\mathscr{M})$
- For any atomic formula $R(x_1, ..., x_n)$, $V_{\mathscr{M},\alpha}(R(x_1, ..., x_n)) = \{i \in M \ ; \ (\alpha_i(x_1), ..., \alpha_n(x_n)) \in J_i(R)\}$

- For any formula $\phi(x)$ and any $i \in M$, $i \in V_{\mathscr{M},\alpha}(\forall x\phi(x))$ iff $i \in V_{\mathscr{M}'_\beta,\beta}(\phi(x))$ for any $\mathscr{M}'_\beta$ such that for any $i \in M$ and any $y \in Var(\mathfrak{L})$, $\alpha_i(y) \neq \beta_i(y)$ only if $y = x$.
- For any formula $\phi(x)$ and any $i \in M$, $i \in V_{\mathscr{M},\alpha}(\exists x\phi(x))$ iff $i \in V_{\mathscr{M}'_\beta,\beta}(\phi(x))$ for some $\mathscr{M}'_\beta$ such that for any $i \in M$ and any $y \in Var(\mathfrak{L})$, $\alpha_i(y) \neq \beta_i(y)$ only if $y = x$.

A formula ϕ is *true* on $\mathscr{M}$ if $V_{\mathscr{M},\alpha}(\phi) = Up(\mathscr{M})$.

2.3 Algebraic Notions and Choice Principles

In this last section, we first introduce some key concepts that are related to complete lattices. We start we some important facts about closure operators on posets. References for closure operators and nuclei on complete lattices include [26] and [24]. For the literature on canonical extensions and MacNeille completions, see for example [16], [48] and [23].

2.3.1 Closure Operators on Complete Lattices

Definition 2.3.1 (Closure operator on a poset)**.** Let $(P, \leq)$ be a poset. A *closure operator* on $(P, \leq)$ is a map $K : P \to P$ such that for any $a, b \in P$:

- K is monotone : if $a \leq b$, then $K(a) \leq K(b)$;
- K is increasing : $a \leq K(a)$;
- K is idempotent : $KK(a) = K(a)$.

Dually, a *kernel operator* on $(P, \leq)$ is a monotone, decreasing and idempotent map from P to P.

In the setting of lattices, an important strengthening of the definition of a closure operator is the following:

Definition 2.3.2 (Nucleus)**.** Let $(P, \wedge, \vee)$ be a lattice. A *nucleus* on P is a map $j : P \to P$ such that j is a closure operator, and for any $a, b \in P$, $j(a \wedge b) = j(a) \wedge j(b)$. Dually, a *co-nucleus* k on P is a kernel operator such that for any $a, b \in P$, $k(a \vee b) = k(a) \vee k(b)$.

Closure operators and nuclei are related to complete lattices and complete Heyting algebra by the following theorems:

Theorem 2.3.3 (Fixpoints of a closure operator on a complete lattice)**.** *Let $(L, \wedge, \vee, \bigwedge, \bigvee, 0, 1)$ be a complete lattice, and $K : L \to L$ a closure operator on L. Let $L_K = \{A \subseteq X \ ; \ K(A) = A\}$ be the set of all fixpoints of K in L. Then $(L_K, \wedge, \vee_K, \bigwedge_K, \bigvee_K, K(0), 1)$ is a complete lattice, where $A \vee_K B = K(A \vee B)$ for any $A, B \in L_K$, and for any family $\{X_i\}_{i\in I}$ of elements of L_K, $\bigwedge_K\{X_i \ ; \ i \in I\} = K(\bigwedge_{i\in I} X_i$, and $\bigvee_K\{X_i \ ; \ i \in I\} = K(\bigvee_{i\in I} X_i)$.*

Theorem 2.3.4 (Fixpoints of a nucleus on a complete HA)**.** *Let $(L, \wedge, \vee, \to, \bigwedge, \bigvee, 0, 1)$ be a complete Heyting algebra, $j : L \to L$ a nucleus on L, and let $L_j = \{a \in L \ ; \ j(a) = a\}$. Then $(L_j, \wedge, \vee_j, \to, \bigwedge_j, \bigvee_j, 0_j, 1)$ is a complete Heyting algebra, where $0_j = j(0)$, $a \vee_j b = j(a \vee b)$ for any $a, b \in L_j$, and for any family $\{a_i\}_{i\in I}$ of elements of L_j, $\bigwedge_j\{a_i \ ; \ i \in I\} = j(\bigwedge_{i\in I} a_i)$, and $\bigvee_j\{a_i \ ; \ i \in I\} = j(\bigvee_{i\in I} a_i)$.*

Note that dual statements also hold. Given a kernel operator J on a complete lattice $(L, \wedge, \vee, 0, 1)$, the structure $(L_J, \wedge_J, \vee, \bigwedge_J, \bigvee_J, 0, J(1))$, is also a complete lattice, where L_J is the set of fixpoints of J in L, and for any $A, B \in L_J$, $A \wedge_J B = J(A \wedge B)$, and for any family $\{X_i\}_{i \in I}$ of elements of L_J, $\bigwedge_J \{X_i \; ; \; i \in I\} = J(\bigwedge_{i \in I} X_i$, and $\bigvee_J \{X_i \; ; \; i \in I\} = J(\bigvee_{i \in I} X_i)$.

Similarly, given a complete co-Heyting algebra $(M, \wedge, \vee, -\!<, \bigwedge, \bigvee, 0, 1)$ and a co-nucleus $k : M \to M$, the set of fixpoints of k, $M_k = \{a \in M \; ; \; k(a) = a\}$, induces a complete co-Heyting algebra $(M_k, \wedge_k, \vee, -\!<, \bigwedge_k, \bigvee_k, 0, 1_k)$, where $1_k = k(1)$, $a \wedge_k b = k(a \wedge b)$ for any $a, b \in M_k$, and for any family $\{a_i\}_{i \in I}$ of elements of M_k, $\bigwedge_k \{a_i \; ; \; i \in I\} = k(\bigwedge_{i \in I} a_i)$, and $\bigvee_k \{a_i \; ; \; i \in I\} = k(\bigvee_{i \in I} a_i)$.

Finally, we define the following nucleus:

Definition 2.3.5. Let L be a Heyting algebra. Then the *double negation nucleus* on L is the operation $\neg\neg : L \to L$ that sends every $a \in L$ to $(a \to \bot) \to \bot = \neg\neg a$.

The double negation nucleus will play an important role because of the following fact:

Lemma 2.3.6. *Let L be a Heyting algebra. Then the lattice of fixpoints of the double negation nucleus $\neg\neg$ on L is a Boolean algebra.*

2.3.2 Completions of Lattices

We now recall the definition of a completion of a lattice, and introduce two important completions, namely MacNeille completions and canonical extension.

Definition 2.3.7 (Completion of a lattice)**.** Let L be lattice. A completion of L is pair (L', ϕ) such that L' is a complete lattice and $\phi : L \to L'$ is an injective homomorphism.

Generalizing Dedekind's famous construction of the reals as cuts on the set of rationals, MacNeille [46] developed a general method for constructing a completion of an arbitrary poset. Here however, we will only be interested in the case of lattices. MacNeille's method requires first the following definition:

Definition 2.3.8 (Normal ideal)**.** Let L be a poset. For any $A \subseteq L$, let $A^u = \{b \in L \; ; \; \forall a \in A : a \leq b\}$ and $A^l = \{b \in L \; ; \; \forall a \in A : b \leq a\}$. Then $I \subseteq L$ is a normal ideal if $(I^u)^l = I$.

Definition 2.3.9 (MacNeille completion)**.** Let L be a lattice, and let N_L be the set of all normal ideals of L. The MacNeille completion of L is the lattice $N = (N_L, \cap, \vee)$, where $A \vee B = ((A \cup B)^u)^l$ for any normal ideals $A, B \in N_L$.

The following theorem recalls important properties of MacNeille completions of lattices.

Theorem 2.3.10. *Let L be a lattice, N its MacNeille completion, and let $\phi : L \to N$ be such that $\phi(a) = \downarrow a$ for any $a \in L$. Then:*

- *(N, ϕ) is a completion of L.*
- *L is dense in N, i.e. for any $b \in N$, $b = \bigwedge_N \{\phi(a) \; ; \; a \in L, b \leq \phi(a)\}$ and $b = \bigvee_N \{\phi(a) \; ; \; a \in L, \phi(a) \leq b\}$*
- *If L is a complete lattice, then ϕ is an isomorphism between L and N.*
- *If (M, ψ) is a dense completion of L, then N is isomorphic to L.*

- *If L is a Heyting algebra (resp. co-Heyting algebra, Boolean algebra), then N is also a Heyting algebra (resp. co-Heyting algebra, Boolean algebra), and ϕ is a HA (resp. coHA, BA)-homomorphism.*

The second important type of completions that we will refer to is the canonical extension of a poset. Although canonical extensions were originally introduced as a purely algebraic counterpart to the powerset of the dual Stone space of a Boolean algebra, the definition has recently been extended to arbitrary posets. Here, we follow the definition given in [16].

Definition 2.3.11. Let L be a lattice and (C, α) a completion of L. Then C is a *doubly-dense extension* of L if for any $X \subseteq C$, $X = \bigwedge_C \{\bigvee_C \{\alpha(a) \ ; \ a \in A_i\} \ ; \ i \in I\}$ and $X = \bigvee_C \{\bigwedge_C \{\alpha(b) \ ; \ b \in B_j\} \ ; \ j \in J\}$ for some families $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ of subsets of L.

Equivalently, if (C, α) is a doubly-dense extension of a lattice L, then every element in C is both a meet of joins and a join of meets of images of elements from L. The canonical extension of a lattice is defined as a specific completion:

Definition 2.3.12. Let L be a lattice and (C, α) a completion of L. Then (C, α) is *compact* if for any $X, Y \subseteq L$, if $\bigwedge_C \{\alpha(x) \ ; \ x \in X\} \subseteq \bigvee_C \{\alpha(y) \ ; \ y \in Y\}$, then there exist finite $X' \subseteq X$ and $Y' \subseteq Y$ such that $\bigwedge_L X' \leq \bigvee_L Y'$.

Definition 2.3.13. Let L be a lattice. Then the canonical extension of L is the unique up to isomorphism doubly-dense compact extension of L.

2.3.3 Choice Principles

We conclude this chapter by presenting some non-constructive principles and theorems that will play a key role in the following chapters. The first one, the Prime Filter Theorem (PFT) has already be mentioned as instrumental for the proof of Priestley, Stone and Esakia Representation Theorems.

Theorem 2.3.14. *[Prime Filter Theorem] Let L be a distributive lattice and $F, I \subseteq L$ such that F is a filter, I is an ideal, and $F \cap I = \emptyset$. Then there exists a prime filter F' over L such that $F \subseteq F'$ and $I \subseteq F'^c$.*

The restriction of the previous theorem to Boolean algebras also exists in the literature under the name of Ultrafilter Theorem or Boolean Prime Ideal Theorem (BPI):

Theorem 2.3.15 (Boolean Prime Ideal Theorem)**.** *Let L be a Boolean algebra and $F, I \subseteq L$ such that F is a filter, I is an ideal and $F \cap I = \emptyset$. Then there exists an ultrafilter F' over L such that $F \subseteq F'$ and $I \subseteq F'^c$.*

Both theorem are equivalent over ZF and both are strictly implied by Zorn's Lemma.[6] On the other hand, the Prime Filter Theorem is mutually independent with the Axiom of Dependent Choice (DC)[7], which will play a role in chapter 4:

Definition 2.3.16 (Axiom of Dependent Choice)**.** Let A be a set and R a relation on A. If for every $a \in A$, there exists $b \in A$ such that aRb, then for any $a \in A$ there exists a sequence $f : \omega \to A$ such that $f(0) = a$ and $f(n)Rf(n+1)$ for all $n \in \omega$.

Finally, the Baire Category Theorem for compact Hausdorff spaces (BCT) will play an important role in the next chapter:

[6] see [14].

[7] Feferman[20] proved that (DC) does not imply (BPI), while the converse was shown by Halpern and Lévy[32]

Definition 2.3.17 (Baire Category Theorem)**.** Let (X, τ) be a compact Hausdorff space. Then every countable intersection of open and dense sets in (X, τ) is also dense in (X, τ).

Important results linking all the non-constructive principles above have been proved by Goldblatt[29], in particular in connection with the Rasiowa-Sikorski Lemma and Tarski's Lemma, which will be introduced in chapters 2 and 3 respectively.

Theorem 2.3.18 (Goldblatt)**.**

1. *(BCT), (DC) and Tarski's Lemma are equivalent to one another.*
2. *the Rasiowa-Sikorski Lemma is equivalent to the conjunction of (BPI) and Tarski's Lemma.*

As it will become clear in the following chapters, one of the goal of this thesis is to generalize Tarski's Lemma in such a way that an analogue of this theorem can be proved for some of the generalizations of the Rasiowa-Sirkorski Lemma that we introduce in the next chapter.

Chapter 3

Generalizations of the Rasiowa-Sikorski Lemma

The goal of this chapter is to present several variations on the celebrated Rasiowa-Sikorski Lemma, and to study in more detail the fundamentals of its proof and its main consequences for logic. In section 1, we present in an abstract fashion a well-known method to prove the completeness of a given first-order logic with respect to a certain class of structures. Sometimes called the Henkin method, in reference to Henkin [36], it amounts to defining a "term model", i.e. a model based exclusively on the set of terms of a given first-order language. In section 2, we recall the original proof of the Rasiowa-Sikorski Lemma, and show its instrumental role in proving the completeness theorem for first-order logic with respect to Tarskian models. Section 3 is concerned with Goldblatt's recent proof of the Rasiowa-Sikorski Lemma for distributive lattices and Heyting algebras which generalizes the original method of Rasiowa and Sikorski in an essential way. Additionally, we show how the completeness of first-order intuitionistic logic follows as a direct consequence of the Rasiowa-Sikorski Lemma for Heyting algebras and the general construction of term models exposed in section 1. Finally, in section 4, we extend Goldblatt's method in a completely straightforward way and prove analogues of the Rasiowa-Sikorski Lemma for co-Heyting algebras, bi-Heyting algebras and modal algebras.

3.1 Lindenbaum-Tarski Algebras and Term Models

In this section, we present in an abstract setting a general method for proving the completeness of a logic with respect to some class of models. An important milestone for the systematic study of algebraic completeness proofs through Lindenbaum-Tarski algebras is [10].

3.1.1 Lindenbaum-Tarski Algebras

Throughout this section, we fix a countable first-order language $\mathfrak{L}$. A logic $\mathfrak{C}$ is a pair $(\Gamma, \vdash)$, where $\Gamma \subseteq \mathfrak{L}$ is the collection of theorems of $\mathfrak{C}$, and $\vdash$ is a reflexive and transitive relation over $\mathscr{P}(L) \times \mathscr{P}(L)$ that determines a deducibility relation between sets of formulas of $\mathfrak{L}$. The following construction is a well-known and powerful tool in algebraic logic:

Definition 3.1.1. Let $\mathfrak{C}$ be a logic. The *Lindenbaum-Tarski algebra* $LT^{\mathfrak{L}}_{\mathfrak{C}}$ of $\mathfrak{C}$ in language $\mathfrak{L}$ is defined as follows:

- Let $\preccurlyeq$ be a relation on $\mathfrak{L} \times \mathfrak{L}$ such that for any $\phi, \psi \in \mathfrak{L}$, we have $\phi \preccurlyeq \psi$ iff $\{\phi\} \vdash \{\psi\}$.
- Since $\vdash$ is reflexive and transitive, $\preccurlyeq$ is a preorder, and therefore the relation $\sim_\vdash$ defined such that $\phi \sim_\vdash \psi$ iff $\phi \preccurlyeq \psi$ and $\psi \preccurlyeq \phi$ is an equivalence relation over $\mathfrak{L}$.
- Let $LT^{\mathfrak{L}}_{\mathfrak{C}}$ be the set of all equivalence classes determined by $\sim_\vdash$. For any $\phi \in \mathfrak{L}$, we write $\phi^\vdash$ for the equivalence class of ϕ.
- Finally, $LT^{\mathfrak{L}}_{\mathfrak{C}}$ is given the following natural ordering: for any $\phi, \psi \in \mathfrak{L}$, $\phi^\vdash \leq \psi^\vdash$ iff $\phi \preccurlyeq \psi$. It is immediate to see that $(LT^{\mathfrak{L}}_{\mathfrak{C}}, \leq)$ is a poset.

The following facts are well-known results about Lindenbaum-Tarski algebras:

Proposition 3.1.2.

1. *$LT^{\mathfrak{L}}_{CPL}$ is a Boolean algebra;*
2. *$LT^{\mathfrak{L}}_{IPL}$ is a Heyting algebra;*
3. *$LT^{\mathfrak{L}}_{cIPL}$ is a co-Heyting algebra;*
4. *$LT^{\mathfrak{L}}_{KL}$ is a BAO;*
5. *$LT^{\mathfrak{L}}_{MIPL}$ is a modal Heyting algebra;*
6. *$LT^{\mathfrak{L}}_{cMIPL}$ is a modal co-Heyting algebra.*

The next fact, however, is of particular interest for us in this chapter, and was originally due to Mostowski according to Scott [62]:

Lemma 3.1.3. *Let $\mathfrak{C}$ be any logic in $\{CPL, IPL, cIPL, KL, MIPL, cMIPL\}$, and let $LT^{\mathfrak{L}}_{\mathfrak{C}}$ be its Lindenbaum-Tarski algebra. Then for any formula $\phi \in \mathfrak{L}$ and any variable $x \in Var(\mathfrak{L})$:*

- $(\forall x \phi(x))^\vdash = \bigwedge \{\phi(y)^\vdash \; ; \; y \in Var(\mathfrak{L})\}$
- $(\exists x \phi(x))^\vdash = \bigvee \{\phi(y)^\vdash \; ; \; y \in Var(\mathfrak{L})\}$

Proof.

- For the left-to-right direction, note that, by Universal instantiation, $\forall x \phi(x) \vdash \phi(y)$. Conversely, by the Universal introduction rule, for any formula $\psi \in \mathfrak{L}$, if $\psi \vdash \phi(y)$ for all $y \in Var(\mathfrak{L})$, then $\psi \vdash \forall x \phi(x)$.
- For the left-to-right direction, notice that by Existential generalization, $\phi(y) \vdash \exists x \phi(x)$ for all $y \in Var(\mathfrak{L})$. For the converse, by the Existential introduction rule, if $\phi(y) \vdash \psi$ for any $y \in Var(\mathfrak{L}), \psi \in \mathfrak{L}$, then $\exists x \phi(x) \vdash \psi$.

□

3.1.2 General Models and Term Models

In the last part of this section, we show how the Lindenbaum-Tarski construction allows one to characterize models of a logic $\mathfrak{C}$ in algebraic terms, and reduce completeness proofs to the construction of a specific model.

Definition 3.1.4. Let $\mathfrak{C}$ be a logic such that $\mathfrak{C}$ is sound and complete with respect to a variety of algebras $\mathscr{V}$ such that $LT^{\mathfrak{L}}_{\mathfrak{C}} \in \mathscr{V}$, and such that the universal and existential quantifiers correspond to infinitary meet and join in the sense of Lemma 3.1.3. Then a *general first-order model* for $\mathfrak{C}$ is a tuple $(M, S(M), \{D_i, J_i\}_{i \in M}, \{f_{ij}\}_{(i,j) \in R \subseteq M \times M}, \{\alpha_i\}_{i \in M}, V)$ such that:

- M is a set, $S(M) \subseteq \mathscr{P}(M)$;
- for any $i \in M$, D_i is a set, and J_i sends every n-ary symbol P in $\mathfrak{L}$ to a subset of D_i^n;
- for any $(i,j) \in R$, f_{ij} is a map from $D_i \to D_j$ such that $if(i,i) \in R$, then f_{ii} is the identity, and for any $i,j,k \in M$ such that (i,j), (j,k) and (i,k) are in R, $f_{ik} = f_{jk} \circ f_{ij}$;
- for any $i \in M$, α_i is a map from $Var(\mathfrak{L})$ to D_i such that for any $(i,j) \in R$, $\alpha_j(x) = f_{ij}(\alpha_i(x))$;
- Finally, $V : LT^{\mathfrak{L}}_{\mathfrak{C}} \to S(M)$ is a $\mathscr{V}$-homomorphism such that for any $i \in M$, any n-ary symbol P and any variables $(x_1, ..., x_n)$, $i \in V(R(x_1, ..., x_n)^{\vdash})$ iff $(\alpha_i(x_1), ..., \alpha_i(x_n)) \in J_i(P)$;
- V preserves all quantifiers, i.e. for any formula $\phi \in \mathfrak{L}$, $V((\forall x \phi(x))^{\vdash}) = \bigwedge_{S(M)}\{V(\phi(t)^{\vdash}) \ ; \ t \in Term(\mathfrak{L})\}$ and $V((\exists x \phi(x))^{\vdash}) = \bigvee_{S(M)}\{V(\phi(t)^{\vdash}) \ ; \ t \in Term(\mathfrak{L})\}$.

A formula ϕ is *true on a general model* $(M, S(M), \{D_i, J_i\}_{i \in M}, \{f_{ij}\}_{(i,j) \in R \subseteq M \times M}, \{\alpha_i\}_{i \in M}, V)$ if $V(\phi) = S(M)$. It is valid on a class of general model $\mathfrak{K}$ if it is true on any general model in $\mathfrak{K}$.

Despite the generality of Definition 3.1.4 and its apparent complexity, a moment's reflection shows that many well-known examples of models for standard logics are subsumed under the definition of a general model above. For example, Tarskian models are special instances were M is a singleton, and $S(M) = \{\emptyset, \{M\}\}$. For Kripke models, on the other hand, M is a poset, and $S(M) = Up(M)$, i.e. the set of all upsets of M. Throughout this work, we will be particularly interested in some specific models, often called term models or Henkin models.

Definition 3.1.5. Let $LT^{\mathfrak{L}}_{\mathfrak{C}}$ be a logic as in Definition 3.1.4. For any filter F over $LT^{\mathfrak{L}}_{\mathfrak{C}}$, let the relation $\sim_F$ be defined such that for any $x, y \in Term(\mathfrak{L})$, $x \sim_F y$ iff $(x = y)^{\vdash} \in F$. A *term model* for $\mathfrak{C}$ is a general model $(M, S(M), \{D_i, J_i\}_{i \in M}, \{f_{ij}\}_{(i,j) \in R \subseteq M \times M}, \{\alpha_i\}_{i \in M}, V)$ such that for any $i \in M$:

- i is a filter over $LT^{\mathfrak{L}}_{\mathfrak{C}}$;
- $D_i = Term(\mathfrak{L})_{/\sim_i}$ i.e. equivalence classes t^i for every $t \in Term(\mathfrak{L})$
- for any n-ary symbol P and any terms $(t_1, ..., t_2)$, $(t^i_1, ..., t^i_n) \in J_i(P)$ iff $P(t_1, ..., t_n)^{\vdash} \in i$.
- for any $(i,j) \in R$ and any $t \in Term(\mathfrak{L})$, $f_{ij}(t^i) = t^j$
- $\alpha_i(x) = x^i$ for any $x \in Var(\mathfrak{L})$
- for any $\phi \in \mathfrak{L}$, $i \in V(\phi^{\vdash})$ iff $\phi^{\vdash} \in i$.

It is straightforward to check that the definition above indeed yields a general model in the sense of Definition 3.1.4. Moreover, it is easy to see that it subsumes well-known term models like the Henkin model construction for CPL. In particular, a straightforward argument shows that there is a one-to-one correspondance between Tarskian term models for CPL and ultrafilters on $LT^{\mathfrak{L}}_{CPL}$ that preserve all universal quantifiers.

Lemma 3.1.6. *Let $\mathfrak{L}$ be a first-order language containing $\mathfrak{L}_{CPL}$. Then there is a one-to-one correspondence between Tarskian $\mathfrak{L}$-models and ultrafilters on $LT^{\mathfrak{L}}_{CPL}$ that preserve all universal quantifiers.*

Proof. Let $\mathfrak{M} = (D, J, \alpha)$ be a Tarskian model, and let $\mathfrak{M}_\phi$ be the set of all equivalence classes of true formulas in $\mathfrak{M}$. It is straightforward to check that $\mathfrak{M}_\phi$ is an ultrafilter of $LT^{\mathfrak{L}}_{CPL}$ that preserves all universal quantifiers. Conversely, if F is an ultrafilter on $LT^{\mathfrak{L}}_{CPL}$, then define $D_F := Term(\mathfrak{L})_{/\sim_F}$, $\alpha_F : Var(\mathfrak{L}) \to D_F$ such that $a_F(x) = x^F$ for any $x \in Var(\mathfrak{L})$, and J_F such that for any n-ary relation symbol P, $J_F(P) = \{(a_1^F, ..., a_n^F) \ ; \ P(a_1, ..., a_n) \in F\}$. Then it is straightforward to check that $\mathfrak{M}_F = (D_F, J_F, \alpha_F)$ is a Tarskian $\mathfrak{L}$-model. □

This fact is exactly what allowed Rasiowa-Sikorski to provide an algebraic proof of the completeness of CPL with respect to Tarskian models, as we will see in the next section. However, before introducing the Rasiowa-Sikorski lemma for Boolean algebras, we conclude with a simple but far-reaching observation, that will allow us to use analogues of the Rasiowa-Sikorski lemma to obtain completeness results for logics other than CPL.

Lemma 3.1.7. *Let $\mathfrak{C}$ be a logic such that $\mathfrak{C}$ is sound and complete with respect to a variety of algebras $\mathscr{V}$ such that $LT^{\mathfrak{L}}_{\mathfrak{C}} \in \mathscr{V}$, and such that the universal and existential quantifiers correspond to infinitary meet and join in the sense of Lemma 3.1.3. Then for any class $\mathfrak{K}$ of general models of $\mathfrak{C}$, $\mathfrak{C}$ is sound and complete with respect to $\mathfrak{K}$ if there is a term model $\mathfrak{H} \in \mathfrak{K}$ such that the valuation V associated to $\mathfrak{K}$ is injective.*

Proof. Soundness is immediate, since if $\vdash_{\mathfrak{C}} \phi$, then $\phi^{\vdash} = \top^{\vdash}$, which means that for any general model $(M, S(M), ..., V)$, $V(\phi^{\vdash}) = S(M)$. For completeness, it is enough to note that if $\mathfrak{H} = (M, S(M), ..., V)$ is a term model in $\mathfrak{K}$ and V is injective, then for any formula ϕ, if $\phi^{\vdash} \neq \top^{\vdash}$, then $V(\phi^{\vdash}) \neq S(M)$, and hence if $\nvdash_{\mathfrak{C}} \phi$, then there exists a general model $\mathfrak{H}$ such that ϕ is not true on $\mathfrak{H}$, and hence ϕ is not valid on $\mathfrak{K}$.

□

3.2 The Rasiowa-Sikorski Lemma for Boolean Algebras and Completeness of CPL

In this section, we quickly recall the original proof of the Rasiowa-Sikorski Lemma [1] using the Stone representation Theorem. As it will soon become clear, all the generalizations of the lemma presented throughout this chapter are straightforward variations and refinements of this proof. The simplicity of the original proof, however, should not overshadow its originality not the important consequences that it has had for the study of first-order logic.

Lemma 3.2.1 (Rasiowa-Sikorski Lemma)**.** [2] *Let B be a Boolean algebra, and Q a countable set of existing meets in B. Then for any $a \in B$, if $a \neq 0$, then there exists an ultrafilter U over B such that $a \in U$ and U preserves all meets in Q, i.e., for any $\bigwedge A \in Q$, if $A \subseteq U$, then $\bigwedge A \in U$.*

Proof. Let (X_B, τ) be the dual Stone space of B. Recall that for any $a \in B$, $|a| = \{U \in X_B \ ; \ a \in U\}$, and moreover that, by the ultrafilter theorem, for any $a, b \in B$, $a \leq b$ iff $|a| \subseteq |b|$. For any $\bigwedge A \in Q$, let $S_A = \bigcup_{a \in A} -|a| \cup |\bigwedge A|$.

[1] [55]
[2] For the way the lemma is formulated here, we refer to footnote 1

We first claim that S_A is open and dense in (X_B, τ). It is immediate to see that S_A is open, since $|b|$ is always clopen for every $b \in B$. For density, we show that

$$I(-S_A) = I(\bigcap_{a \in A} |a| \cap -|\bigwedge A|)$$

is empty. To see this, recall first that

$$I(\bigcap_{a \in A} |a| \cap -|\bigwedge A|) = I(\bigcap_{a \in A} |a|) \cap I(-|\bigwedge A|).$$

Now assume that there is some basic open set $|c|$ for some $c \in B$ such that $|c| \subseteq \bigcap_{a \in A} |a|$. This means that $|c| \subseteq |a|$ and hence that $c \leq a$ for any $a \in A$. But this clearly implies that $c \leq \bigwedge A$ since c is a lower bound of A, from which it follows that $|c| \subseteq |\bigwedge A|$. Hence for any basic open set $|c| \subseteq \bigcap_{a \in A} |a|$, we have that $|c| \subseteq |\bigwedge A|$. This in turn implies by definition of the interior operator that

$$I(\bigcap_{a \in A} |a|) \subseteq |\bigwedge A|.$$

Hence

$$I(-S_A) = I(\bigcap_{a \in A} |a|) \cap I(-|\bigwedge A) \subseteq |\bigwedge A| \cap -|\bigwedge A| = \emptyset.$$

Therefore S_A is dense in (X_B, τ), since its complement is nowhere dense.
Moreover, recall that since (X_B, τ) is a compact Hausdorff space, it satisfies the Baire Category Theorem. Now let

$$S_Q = \bigcap_{\bigwedge A \in Q} S_A.$$

By the previous result, S_Q is a countable intersection of open dense sets, and hence, by the Baire Category Theorem, it is also dense in (X_B, τ).
The conclusion of the lemma follows at once. For any $a \in X_B$, if $a \neq 0$, then $|a| \neq |0| = \emptyset$. But then, since $|a|$ is always open and S_Q is dense, $|a| \cap S_Q$ is nonempty. Clearly, any $U \in |a| \cap S_Q$ is an ultrafilter that preserves all meets in Q and contains a. □

The main consequence of the previous lemma is the following completeness proof:

Theorem 3.2.2 (Rasiowa and Sikorski)**.** *CPL is complete with respect to the class of all Tarskian models.*

Proof. Let $\mathfrak{L}$ be a countable first-order language. By Lemma 3.1.6, any ultrafilter on $LT^{\mathfrak{L}}_{CPL}$ that preserves all meets in

$$Q_\forall = \{(\forall x \phi(x))^\vdash \; ; \; \phi \in \mathfrak{L}\}$$

gives rise to a Tarskian term model of CPL. Now for any $\phi \in \mathfrak{L}$, if $\nvdash_{CPL} \phi$, then $(\neg\phi)^\vdash \neq \bot^\vdash$. But then, by the Rasiowa-Sikorski lemma, since $Q_\forall$ is countable, there exists an ultrafilter U over $LT^{\mathfrak{L}}_{CPL}$ that preserves all meets in $Q_\forall$ and does not contain ϕ. Hence the Tarskian model $\mathfrak{M}_U$ corresponding to U is such that ϕ is not true on $\mathfrak{M}_U$, which means that ϕ is not valid on the class of all Tarskian model. By contraposition, it follows that CPL is complete with respect to the class of all Tarskian models. □

3.3 Goldblatt's Proof of the Rasiowa-Sikorski Lemma for DL and HA

In this section, we review the two proofs of the Rasiowa-Sikorski Lemma for distributive lattices and for Heyting algebras given in Goldblatt [30]. A generalization of the Rasiowa-Sikorski Lemma to distributive lattices was also proved in [RausSabalski], and a proof of a similar statement for Heyting algebras was given by [31]. The specificity of Goldblatt's proofs, however, it that they rely on representation theorems for DL and HA, and involve straightforward applications of the Baire Category Theorem. For the sake of clarity, we present the proof in an equivalent but slightly different fashion from Goldblatt [30].

3.3.1 The Rasiowa-Sikorski Lemma for DL

Recall that for any distributive lattice L, the dual Priestley space of L $(X_L, \tau, \leq)$ is defined as:

- X_L is the space of all prime filters over L;
- τ is the topology generated by the basis $\beta = \{|a| - |b| \ ; \ a, b \in L\}$, where $|a| = \{p \in X_L \ ; \ a \in p\}$ for any $a \in L$;
- $\leq$ is the inclusion ordering on X_L.

We first fix the following definitions.

Definition 3.3.1. Let L be a distributive lattice, and $A, B \subseteq L$ such that $\bigwedge A$ and $\bigvee B$ exist in L. $\bigwedge A$ is *distributive* if for any $c \in L$, $\bigwedge A \vee c = \bigwedge(A \vee c)$, where $\bigwedge(A \vee c) = \bigwedge\{a \vee c \ ; \ a \in A\}$. Dually, $\bigvee B$ is distributive if for any $c \in L$, $\bigvee B \wedge c = \bigvee(B \wedge c)$, where $\bigvee(B \wedge c) = \bigvee\{b \wedge c \ ; \ b \in B\}$. A filter p over L *preserves* $\bigwedge A$ if $A \subseteq p$ implies $\bigwedge A \in p$. Dually, p preserves $\bigvee B$ if $\bigvee B \in p$ implies $B \cap p \neq \emptyset$.

Goldblatt's topological proof of the Rasiowa-Sikorski Lemma for distributive lattices relies on the following well-known fact:

Proposition 3.3.2. *Let (X, τ) be a topological space. For any $U \subseteq X$, $-U \cap CU$ is nowhere dense, i.e. $I(-U \cap CU) = \emptyset$.*

Proof.
$$I(-U \cap CU) = I(-U) \cap ICU = -CU \cap ICU \subseteq -CU \cap CU = \emptyset.$$
□

The following lemma is of crucial relevance for the rest of the proof.

Lemma 3.3.3. *Let L be a distributive lattice, $(X_L, \tau, \leq)$ be its dual Priestley space, and $A, B \subseteq L$ be such that $\bigwedge A$ and $\bigvee B$ exist and are distributive. Then:*

1. $|\bigwedge A| = I(\bigcap_{a \in A} |a|)$
2. $|\bigvee B| = C(\bigcup_{b \in B} |b|)$

Proof. 1. Let $A \subseteq L$ be such that $\bigwedge A$ exists and is distributive. Clearly, since filters are upward-closed and $|\bigwedge A|$ is open, we have

$$|\bigwedge A| \subseteq I(\bigcap_{a \in A} |a|).$$

For the converse, assume there is some basic open set $|c| - |d|$ such that

$$|c| - |d| \subseteq \bigcap_{a \in A} |a|.$$

Then for any $a \in A$, $|c| - |d| \subseteq |a|$, which entails that

$$|c| \subseteq |a| \cup |d| = |a \vee d|.$$

But this means that $c \leq a \vee d$ for any $a \in A$, and hence

$$c \leq \bigwedge (A \vee d) = \bigwedge A \vee d$$

since, by assumption, $\bigwedge A$ is distributive. Hence

$$|c| \subseteq |\bigwedge A \vee d| = |\bigwedge A| \cup |d|,$$

which entails that $|c| - |d| \subseteq |\bigwedge A|$. Therefore, we have

$$I(\bigcap_{a \in A} \subseteq |\bigwedge A|),$$

which completes the proof.

2. Let $B \subseteq L$ be such that $\bigvee B$ exists and is distributive. Since $|\bigvee B|$ is closed and filters are upward-closed, it follows that

$$C(\bigcup_{b \in B} |b|) \subseteq |\bigvee B|.$$

For the converse, assume there is some basic closed set $-|c| \cup |d|$ such that

$$\bigcup_{b \in B} |b| \subseteq -|c| \cup |d|.$$

Then for any $b \in B$, we have

$$|b| \subseteq -|c| \cup |d|,$$

which means that

$$|b \wedge c| = |b| \cap |c| \subseteq |d|.$$

This implies that for any $b \in B$, we have $b \wedge c \leq d$, from which it follows that

$$\bigvee B \wedge c = \bigvee (B \wedge c) \leq d$$

since by assumption $\bigvee B$ is distributive. Hence

$$|\bigvee B \wedge c| = |\bigvee B| \cap |c| \subseteq |d|,$$

which entails that

$$|\bigvee B| \subseteq -|c| \cup |d|.$$

Therefore we have

$$|\bigvee B| \subseteq C(\bigcup_{b \in B} |b|),$$

which completes the proof.

□

Combining two previous results, we obtain the following:

Lemma 3.3.4. *Let L be a distributive lattice, $(X_L, \tau, \leq)$ be its dual Priestley space, and $A, B \subseteq L$ such that $\bigwedge A$ and $\bigvee B$ exist and are distributive. Then $S_{\bigwedge A} = \bigcup_{a \in A} -|a| \cup |\bigwedge A|$ and $S_{\bigvee B} = \bigcup_{b \in B} |b| \cup -|\bigvee B|$ are open dense sets in X_L.*

Proof. Notice first that since $\bigcup_{a \in A} -|a|$ and $\bigcup_{b \in B} |b|$ are both open, so are $S_{\bigwedge A}$ and $S_{\bigvee B}$. To see that they are also dense, we show that $-S_{\bigwedge A}$ and $-S_{\bigvee B}$ are both nowhere dense sets. In the former case, notice that, by Lemma 3.3.3 1., we have

$$-|\bigwedge A| = -I(\bigcap_{a \in A} |a|) = C(\bigcup_{a \in A} -|a|.$$

Hence

$$-S_{\bigwedge A} = -(\bigcup_{a \in A} -|a|) \cap \ C(\bigcup_{a \in A} -|a|),$$

which by proposition 3.3.2 entails that $-S_{\bigwedge A}$ is nowhere dense in X_L. In the latter case, recall that, by Lemma 3.3.3 2., we have

$$|\bigvee B| = C(\bigcup_{b \in B} |b|)$$

and therefore

$$-S_{\bigvee B} = -(\bigcup_{b \in B} |b|) \cap C(\bigcup_{b \in B} |b|).$$

By proposition 3.3.2 again, this implies that $-S_{\bigvee B}$ is nowhere dense in X_L. □

We can finally state and prove the Rasiowa-Sikorski Lemma for distributive lattices.

Lemma 3.3.5 (Rasiowa-Sikorski Lemma for distributive lattices)**.** *Let L be a distributive lattice, and let Q_M and Q_J be two countable sets of existing distributive meets and joins in L respectively. Then for any $a \nleq b \in L$, there exists a prime filter p over L such that $a \in p$, $b \notin p$, and p preserves all meets in Q_M and all joins in Q_J.*

Proof. Let X_L be the dual Priestely space, and let

$$S_Q = \bigcap_{\bigwedge A \in Q_M} S_{\bigwedge A} \cap \bigcap_{\bigvee B \in Q_J} S_{\bigvee B}.$$

By Lemma 3.3.4, $S_{\bigwedge A}$ and $S_{\bigvee B}$ are both dense open sets in X_L. Hence S_Q is a countable intersection of open and dense sets. Moreover, since X_L is compact Hausdorff, by the Baire Category Theorem, S_Q is dense in X_L. Now let $a, b \in L$ such that $a \nleq b$. This means that $|a| - |b| \neq \emptyset$, and hence

$$S_Q \cap |a| - |b| \neq \emptyset$$

But clearly any prime filter in S_Q preserves all meets in Q_M and all joins in Q_J, and hence any $p \in S_Q \cap |a| - |b|$ satisfies all requirements of the lemma. □

Similarly to the case of Boolean algebras, the Rasiowa-Sikorski Lemma has the following important consequence:

Theorem 3.3.6. *Let L be a distributive lattice, and Q_M, Q_J two countable sets of existing meets and joins in L respectively. Then there exists a poset $(X, \leq)$ and a map $|\cdot| : L \to Up(X)$ such that $|\cdot|$ is an injective DL-homomorphism that preserves all meets in Q_M and all joins in Q_J.*

Proof. Let $(X_Q, \leq)$ be the set of all prime filters over L that preserve all meets in Q_M and all joins in Q_J with the inclusion ordering, and let $|\cdot| : L \to Up(X_Q)$ such that $|a| = \{p \in X_Q \ ; \ a \in p\}$. Then the following hold:

1. $|1| = X_Q$ and $|0| = \emptyset$ since any $p \in X_Q$ is non-empty and proper.
2. $|a \wedge b| = |a| \cap |b|$ since filters are upward closed and downward-directed
3. $|a \vee b| = |a| \cup |b|$ since any $p \in X_Q$ is prime and upward-closed
4. $|\bigwedge A| = \bigcap_{a \in A} |a|$ for any $\bigwedge A \in Q_M$: the left-to-right direction is immediate since filters are upward-closed. For the right-to-left direction, notice that for any $p \in X_Q$, $A \subseteq p$ implies $\bigwedge A \in p$ since p preserves $\bigwedge A$.
5. $|\bigvee B| = \bigcup_{b \in B} |b|$ for any $\bigvee B \in Q_J$: For the left-to-right direction, notice that $\bigvee B \in p$ implies $B \cap p \neq \emptyset$ for any $p \in X_Q$, since p preserves $\bigvee B$. The right-to-left direction follows immediately from the fact that filters are upward-closed.

Hence $|\cdot|$ is a DL-homomorphism that preserves all meets in Q_M and all joins in Q_J. Moreover, by Lemma 3.3.5, for any $a, b \in L$ such that $a \not\leq b$, there exists $p \in X_Q$ such that $a \in p$ and $b \notin p$. Hence $a \leq b$ iff $|a| \subseteq |b|$ for any $a, b \in L$, which means that $|\cdot|$ is injective. □

3.3.2 Extension to Heyting Algebras

Theorem 3.3.6 above provides us with almost all we need in order to prove the completeness of IPL with respect to IPL-models. Indeed, given a first-order language $\mathfrak{L}$, Theorem 3.3.6 implies that for $LT^{\mathfrak{L}}_{IPL}$ and countable sets $Q_M = \{(\forall x \phi(x))^{\vdash} \ ; \ \phi(x) \in \mathfrak{L}\}$ and $Q_J = \{(\exists x \phi(x))^{\vdash} \ ; \ \phi(x) \in \mathfrak{L}\}$, the canonical map $|\cdot| : LT^{\mathfrak{L}}_{IPL} \to H^{\mathfrak{L}}_{IPL}$ is a injective DL-homomorphism, where $H^{\mathfrak{L}}_{IPL}$ is the term model where the domain is indexed by the set of all prime filters over $LT^{\mathfrak{L}}_{IPL}$. Hence we also need to prove that it is a HA-homomorphism, i.e. that for any $\phi, \psi \in \mathfrak{L}$,

$$|(\phi \to \psi)^{\vdash}| = |\phi^{\vdash}| \Rightarrow_{H^{\mathfrak{L}}_{IPL}} |\psi^{\vdash}|.$$

This amounts to extending Lemma 3.3.5 to the setting of Heyting algebras. We begin with the following definitions.

Definition 3.3.7. Let L be a HA, $(X_L, \tau, \leq)$ be the dual Priestley space of L, and $p \in X_L$. For any $U \subseteq X_L$, we denote as U_p the set $U \cap \uparrow p = \{q \in U \ ; \ p \subseteq q\}$, and we write $(\uparrow p, \tau_p)$ for the topological space induced by restricting τ to $\uparrow p$. For any $U \subseteq X_L$, it is clear that $(IU)_p = I_p(U_p)$ and $(CU)_p = C_p(U_p)$, where I_p and C_p are the interior and closure operator induced by τ_p respectively.

Definition 3.3.8. Let L be a Heyting algebra and Q_M, Q_J two countable sets of distributive meets and joins existing in L respectively. Then Q_M and Q_J are said to be $(\bigwedge, \to)$-complete if for any $c, d \in L$:

1. if $\bigwedge A \in Q_M$, then $\bigwedge(c \to (A \vee d)) \in Q_M$, where $c \to (A \vee d) = \{c \to (a \vee d) \ ; \ a \in A\}$;
2. if $\bigwedge B \in Q_J$, then $\bigwedge((B \wedge c) \to d) \in Q_J$, where $(B \wedge c) \to d = \{(b \wedge c) \to d \ ; \ b \in B\}$.

We can know prove an analogue of Lemma 3.3.3 above.

Lemma 3.3.9. *Let L be a HA, Q_M and Q_J two $\bigwedge\!\to$-complete countable sets of distributive meets and joins existing in L. Let $(X_L, \tau, \leq)$ be the dual Priestley space of L, and $p \in X_L$ such that p preserves all meets in Q_M and all joins in Q_J. Then for any $\bigwedge A \in Q_M$, $\bigvee B \in Q_J$:*

1. $|\bigwedge A|_p = (I(\bigcap_{a\in A}|a|))_p$;
2. $|\bigvee B|_p = (C(\bigcup_{b\in B}|b|))_p$.

Proof. Recall first that for any $p \in X_L$ and $a, b \in A$

$$|a|_p \subseteq |b|_p \Leftrightarrow a \to b \in p \tag{1}$$

. We use this fact repeatedly in the following two proofs.

1. Note first that the left-to-right direction follows immediately from the fact that

$$|\bigwedge A| \subseteq I(\bigwedge_{a\in A}|a|).$$

For the converse, we prove that for any basic open set in τ_p of the form $(|c| - |d|)_p = |c|_p - |d|_p$, if

$$|c|_p - |d|_p \subseteq (\bigcap_{a\in A}|a|)_p = \bigcap_{a\in A}|a|_p,$$

then

$$|c|_p - |d|_p \subseteq |\bigwedge A|_p.$$

This will suffice to show that

$$(I(\bigcap_{a\in A}|a|))_p = I_p(\bigcap_{a\in A}|a|_p) \subseteq |\bigwedge A|_p.$$

So let $|c|_p - |d|_p \subseteq \bigcap_{a\in A}|a|_p$. This means that $|c|_p - |d|_p \subseteq |a|_p$ for any $a \in A$, and hence

$$|c|_p \subseteq |a|_p \cup |d|_p = |a \vee d|_p.$$

By (1) above, this means that $c \to (a \vee d) \in p$ for all $a \in A$, which implies, since p preserves all meets in Q_M,

$$\bigwedge(c \to (A \vee d)) \in p.$$

Moreover,

$$\bigwedge(c \to (A \vee d)) = c \to \bigwedge(A \vee d) = c \to (\bigwedge A \vee c)$$

since $\bigwedge A$ is distributive and $c \to -$ preserves joins. This implies that

$$c \to \bigwedge A \vee d \in p.$$

By (1) again, it follows that

$$|c|_p \subseteq |\bigwedge A \vee d|_p = |\bigwedge A|_p \cup |d|_p,$$

and therefore

$$|c|_p - |d|_p \subseteq |\bigwedge A|_p.$$

2. The right-to-left direction is immediate from the fact that

$$C(\bigcup_{b\in B}|b|) \subseteq |\bigcup B|.$$

For the converse, we prove that for any basic closed set of the form $(-|c|\cup|d|)_p = -|c|_p\cup|d|_p$, if

$$(\bigcup_{b\in B}|b|)_p = \bigcup_{b\in B}|b|_p \subseteq -|c|_p \cup |d|_p,$$

then

$$|\bigvee B|_p \subseteq -|c|_p \cup |d|_p.$$

This will suffice to show that

$$|\bigvee B|_p \subseteq C_p((\bigcup_{b\in B}|b|)_p) = (C(\bigcap_{b\in B}|b|)_p.$$

Let $-|c|_p \vee |d|_p$ be such that

$$\bigcup_{b\in B}|b|_p \subseteq -|c|_p \vee |d|_p.$$

This means that for any $b \in B$,

$$|b|_p \subseteq -|c|_p \vee |d|_p,$$

hence

$$|b \wedge c|_p = |b|_p \cap |c|_p \subseteq |d|_p.$$

By (1) above, it follows that $(b \wedge c) \to d \in p$ for every $b \in B$, and therefore

$$\bigwedge((B \wedge c) \to d) \in p$$

since p preserves all meets in Q_M. Moreover, we have

$$\bigwedge((B \wedge c) \to d) = \bigvee(B \wedge c) \to d = (\bigvee B \wedge c) \to d$$

since $\bigvee B$ is distributive and $- \to d$ turns meets into joins, and therefore, by (1) again, it follows that

$$|\bigvee B|_p \cap |c|_p = |\bigvee B \wedge c|_p \subseteq |d|_p.$$

Hence $|\bigvee B|_p \subseteq -|c|_p \vee |d|_p$.

□

Lemma 3.3.9 yields the following analogue of Lemma 3.3.4:

Lemma 3.3.10. *Let L be a HA, Q_M and Q_J two $(\bigwedge, \to)$-complete countable sets of distributive meets and joins existing in L, $(X_L, \tau, \leq)$ the dual Priestley space of L, and $p \in X_L$ such that p preserves all meets in Q_M and all joins in Q_J. Then for any $\bigwedge A \in Q_M$ and any $\bigwedge B \in Q_J$, $(S_{\bigwedge A})_p = \bigcup_{a\in A} -|a|_p \cup |\bigwedge A|_p$ and $(S_{\bigvee B})_p = \bigcup_{b\in B}|b|_p \cup -|\bigvee B|_p$ are open dense sets in $(\uparrow p, \tau_p)$.*

Proof. Let $\bigwedge A \in Q_M$ and $\bigvee B \in Q_J$. Notice first that since $S_{\bigwedge A}$ and $S_{\bigvee B}$ are in τ, $(S_{\bigwedge A})_p$ and $(S_{\bigvee B})_p$ are in τ_p. For density, we prove that both $(-S_{\bigwedge A})_p$ and $(-S_{\bigvee B})_p$ are nowhere dense.
In the former case, notice that, by Lemma 3.3.91., we have that

$$(-|\bigwedge A|)_p = -I_p((\bigcap_{a\in A}|a|)_p) = C_p((\bigcup_{a\in A} -|a|)_p).$$

This means that

$$(-S_{\bigwedge A})_p = (-(\bigcup_{a \in A} -|a|)_p) \cap C_p((\bigcup_{a \in A} |a|-)_p),$$

which by proposition 3.3.2 implies that $(-S_{\bigwedge A})_p$ is nowhere dense in $\uparrow p$. In the latter case, recall that, by Lemma 3.3.92., we have that

$$|\bigvee B|_p = C_p((\bigcup_{b \in B} |b|)_p),$$

and therefore

$$(-S_{\bigvee B})_p = (-(\bigcup_{b \in B} |b|))_p \cap C_p((\bigcup_{b \in B} |b|)_p).$$

By proposition 3.3.2 again, this implies that $(-S_{\bigvee B})_p$ is nowhere dense in $\uparrow p$. □

The Rasiowa-Sikorski Lemma for Heyting algebras now follows from the results established above.

Lemma 3.3.11 (Rasiowa-Sikorski Lemma for Heyting algebras)**.** *Let L be a HA, Q_M and Q_J two $(\bigwedge, \rightarrow)$-complete countable sets of distributive meets and joins existing in L, and p a prime filter over L such that p preserves all meets in Q_M and all joins in Q_J. Then for any $a, b \in L$, if $a \rightarrow b \notin p$, then there exists a prime filter q over L such that $p \subseteq q$, $a \in q$, $b \notin q$ and q preserves all meets in Q_M and all joins in Q_J.*

Proof. Let $(X_L, \tau, \leq)$ be the dual Priestley space of L. Since $\uparrow p = \bigcap_{a \in p} |a|$, $\uparrow p$ is closed in (X_L, τ) and $(\uparrow p, \tau_p)$ is a compact Hausdorff subspace of $(X_L, \tau, \leq)$, and hence satisfies the Baire Category Theorem. Moreover, since

$$(S_Q)_p = \bigcap_{\bigwedge A \in Q_M} (S_{\bigwedge A})_p \cap \bigcap_{\bigvee B \in Q_J} (S_{\bigvee B})_p,$$

by Lemma 3.3.10 $(S_Q)_p$ is a countable intersection of open and dense sets in $\uparrow p$, and hence, by the Baire Category Theorem, $(S_Q)_p$ is also dense in $\uparrow p$. Now let $a, b \in L$ such that $a \rightarrow b \notin p$. Then by fact (1) above, $|a|_p - |b|_p \neq \emptyset$, and therefore

$$|a|_p - |b|_p \cap (S_Q)_p \neq \emptyset.$$

Clearly, any prime filter $q \in |a|_p - |b|_p \cap (S_Q)_p$ satisfies exactly the requirement of the lemma. □

Finally, we show how Lemma 3.3.11 is instrumental in the proof of completeness of IPL with respect to IPL-models.

Theorem 3.3.12. *Let $LT^{\mathfrak{L}}_{IPL}$ be the Lindenbaum-Tarski algebra of IPL in some language $\mathfrak{L}$, $Q_\forall = \{(\forall x \phi(x))^\vdash \ ; \ \phi \in \mathfrak{L}\}$ and $Q_\exists = \{(\exists x \phi(x))^\vdash \ ; \ \phi \in \mathfrak{L}\}$ two countable sets of distributive meets and joins respectively, and let $H^{\mathfrak{L}}_{IPL}$ be the term model associated with $LT^{\mathfrak{L}}_{IPL}$. Then $|\cdot| : LT^{\mathfrak{L}}_{IPL} \rightarrow Up(H^{\mathfrak{L}}_{IPL})$ is an injective HA homomorphism that preserves all meets in $Q_\forall$ and all joins in $Q_\exists$.*

Proof. Recall that the domain of $H^{\mathfrak{L}}_{IPL}$ is the set S_Q of all prime filters over $LT^{\mathfrak{L}}_{IPL}$ that preserve all meets in $Q_\forall$ and all joins in $Q_\exists$. By Theorem 3.3.6, $|\cdot|$ is an injective DL homomorphism that preserves all meets in $Q_\forall$ and all joins in $Q_\exists$. Hence we only need to check that for any $\phi, \psi \in \mathfrak{L}$, we have

$$|(\phi \rightarrow \psi)^\vdash| = -\!\downarrow\!(|\phi^\vdash| - |\psi^\vdash|).$$

This amounts to proving that for any $p \in S_Q$, $\phi^{\vdash} \to \psi^{\vdash} \in p$ iff $\phi^{\vdash} \in q$ implies $\psi^{\vdash} \in q$ for all $q \supseteq p$. The left-to-right direction simply follows from basic properties of filters and from the fact that $\phi^{\vdash} \wedge (\phi \to \psi)^{\vdash} \leq \psi^{\vdash}$. The converse is exactly the statement of Lemma 3.3.11, which holds since $Q_\forall$ and $Q_\exists$ are clearly $(\bigwedge, \to)$-complete. $\square$

Corollary 3.3.13. *IPL is complete with respect to the class of IPL-models.*

Proof. Immediate from Theorem 3.3.12 and Lemma 3.1.7. $\square$

3.4 Generalization to co-Heyting Algebras and Modal Algebras

If one reflects upon Goldblatt's method presented in the previous section, the proof of the Rasiowa-Sikorski Lemma for Heyting algebras relies on the following fundamental facts which are true for any Heyting algebra L with dual Priestley space X_L:

- For any $p \in X_L$ and any $a, b \in L$, $a \to b \in p$ iff $|a|_p \subseteq |b|_p$
- For any $(\bigwedge, \to)$ complete sets Q_M, Q_J of distributive meets and joins respectively, any prime filter $p \in X_L$ that preserves all meets in Q_M and all joins in Q_J, and any $\bigwedge A \in Q_M$, $\bigvee B \in Q_J$, the set $(S_{\bigwedge A})_p$ of filters extending p and preserving $\bigwedge A$, and the set $(S_{\bigvee B})_p$ of filters extending p and preserving $\bigvee B$ are both open and dense sets in $(\uparrow p, \tau_p)$.
- $\uparrow p$ is a closed subset of X_L, hence $(\uparrow p, \tau_p)$ is compact Hausdorff and therefore satisfies the conditions of the Baire Category Theorem.

In this section, we show how the exact same method applies to the case of BAO's and co-Heyting algebras. We first deal with the case of co-Heyting algebras.

Definition 3.4.1. Let L be a co-Heyting algebra with dual Priestley space $(X_L, \tau, \leq)$. For any $p \in X_L$, $U \subseteq X_L$, we denote as U^p the set $U \cap \downarrow p = \{q \in U \ ; \ q \subseteq q\}$, and we write τ^p for the subspace topology induced by $\downarrow p$. Clearly, for any $U \subseteq X_L$, $(IU)^p = I^p(U^p)$ and $(CU)^p = C^p(U^p)$, where I^p and C^p are the interior and closure operators on τ^p respectively.

Definition 3.4.2. Let L be a co-Heyting algebra. Two countable sets Q_M, Q_J of distributive meets and joins existing in L are $(\bigvee, -\!\!<)$-complete if for any $A, B \subseteq L$ and $c, d \in L$:

- if $\bigwedge A \in Q_M$, then $\bigvee (c -\!\!< (A \vee d)) \in Q_J$, where $c -\!\!< (A \vee d) = \{c -\!\!< (a \vee d) \ ; \ a \in A\}$
- if $\bigvee B \in Q_J$, then $\bigvee ((B \wedge c) -\!\!< d) \in Q_J$, where $(B \wedge c) -\!\!< d = \{(b \wedge c) -\!\!< d \ ; \ b \in B\}$

Lemma 3.4.3. *Let L be a co-Heyting algebra, $(X_L, \tau, \leq)$ its dual Priestley space, Q_M and Q_J two $(\bigvee, -\!\!<)$-complete countable sets of distributive meets and joins in L, and $p \in X_L$ such that p preserves all meets in Q_M and all joins in Q_J. Then:*

1. *For any $a, b \in L$, $a -\!\!< b \notin p$ iff $|a|^p \subseteq |b|^p$;*
2. *For any $\bigwedge A \in Q_M$, $(I(\bigcap_{a \in A}))^p = |\bigwedge A|^p$;*
3. *For any $\bigvee B \in Q_J$, $(C(\bigcup_{b \in B}))^p = |\bigvee B|^p$;*
4. *For any $\bigwedge A \in Q_M$, $(S_{\bigwedge A})^p$ is dense in $(\downarrow p, \tau^p)$;*
5. *For any $\bigvee B \in Q_J$, $(S_{\bigvee B})^p$ is dense in $(\downarrow p, \tau^p)$;*

6. $\downarrow p$ is closed in (X_L, τ).

Proof.

1. see Theorem 2.1.30.(2.)

2. It is straightforward to see that $|\bigwedge A|^p \subseteq (I(\bigcap_{a\in A}|a|))^p$. For the converse, we show that for any basic open $|c|^p - |d|^p \in \tau_p$, if
$$|c|^p - |d|^p \subseteq \bigcap_{a\in A} |a|^p,$$
then we have that
$$|c|^p - |d|^p \subseteq |\bigwedge A|^p.$$
So let $|c|^p - |d|^p \subseteq \bigcap_{a\in A}|a|^p$. Then $|c|^p - |d|^p \subseteq |a|^p$ for all $a \in A$, which means that
$$|c|^p \subseteq |a|^p \vee |d|^p = |a \vee d|^p.$$
Hence by 1. we have that
$$c \prec (a \vee d) \notin p$$
for any $a \in A$, which implies that
$$c \prec (\bigwedge A \vee d) = c \prec \bigwedge (A \vee d) = \bigvee (c \prec (a \vee d) \notin p,$$
since p preserves all joins in Q_J and $c \prec -$ sends joins to meets. Therefore by 1. again we have that
$$|c|^p \subseteq |\bigwedge A \vee d|^p = |\bigwedge A|^p \cup |d|^p,$$
from which it follows that
$$|c|^p - |d|^p \subseteq |\bigwedge A|^p.$$

3. Once again, it is clear that $(C(|\bigcup_{b\in B}|b|))^p \subseteq |\bigvee B|^p$. For the converse, we show that for any basic closed set $-|c|^p \cup |d|^p$, if
$$\bigcup_{b\in B} \subseteq |b|^p \subseteq -|c|^p \cup |d|^p,$$
then we have that
$$|\bigvee B|^p \subseteq -|c| \cup |d|.$$
So assume $|b|^p \subseteq -|c|^p \cup |d|^p$ for all $b \in B$. Then
$$|b \wedge c|^p = |b| \cap |c|^p \subseteq |d|^p,$$
which means by 1. that $(b \wedge c) \prec d \notin p$. Hence, since p preserves all joins in Q_L and $- \prec d$ preserves joins, we have that
$$(\bigvee B \wedge c) \prec d = \bigvee (B \wedge c) \prec d = \bigvee ((B \wedge c) \prec d) \notin p.$$
By 1. again, this yields
$$|\bigvee B|^p \cap |c|^p = |\bigvee B \wedge c|^p \subseteq |d|^p,$$
and therefore
$$|\bigvee|^p \subseteq -|c|^p \cup |d|^p.$$

4. By 2., we have that
$$(-|\bigwedge A|)^p = (-I(\bigcap_{a\in A} |a|))^p = (C(\bigcup_{a\in A} -|a|))^p.$$
Hence
$$(-S\bigwedge A)^p = (-(\bigcup_{a\in A} -|a|))^p \cap (C(\bigcup_{a\in A} -|a|))^p,$$
which means by proposition 3.3.2 that $(-S_{\bigwedge A})^p$ is nowhere dense in $(\downarrow p, \tau^p)$.

5. By 3., we have that
$$|\bigvee B|^p = (C(\bigcup_{b\in B} |b|))^p.$$
This implies that
$$(-S_{\bigvee B})^p = (\bigcup_{b\in B} |b|)^p \cap (C(\bigcup_{b\in B} |b|))^p$$
which by proposition 3.3.2 implies that $(-S_{\bigvee B})^p)$ is nowhere dense in $(\downarrow p, \tau^p)$.

6. $\downarrow p = \bigcap_{a\notin p} -|a|$, and therefore $\downarrow p$ is closed in (X_L, τ).

□

As expected, the previous lemma implies the following statement of the Rasiowa-Sikorski Lemma for co-HA:

Lemma 3.4.4. *Let L be a co-Heyting algebra, Q_M and Q_J two $(\bigvee, -\!\!<)$-complete countable sets of distributive meets and joins in L, and p a prime filter over L that preserves all meets in Q_M and all joins in Q_J. Then for any $a, b \in L$, if $a -\!\!< b \in p$, then there exists a prime filter $p' \subseteq p$ such that $a \in p'$, $b \notin p'$, and p' preserves all meets in Q_M and all joins in Q_J.*

Proof. Let $(X_L, \tau, \leq)$ be the dual Priestley space of L. By Lemma 3.4.3 6., $(\downarrow p)$ is closed, which means that $(\downarrow p, \tau_p)$ is compact Hausdorff, and hence satisfies the conditions of the Baire Category Theorem. Moreover, by Lemma 3.4.3 4. and 5., $(S_Q)^p$ is a countable intersection of open and dense sets in $\downarrow p$, and hence by the Baire Category Theorem it is also dense in p. Now if $a -\!\!< b \notin p$, then by Lemma 3.4.3 1. $|a|^p - |b|^p \neq \emptyset$. This means that
$$(S_Q)^p \cap (|a|^p - |b|^p)$$
is nonempty. But clearly any prime filter q in $(S_Q)^p \cap (|a|^p - |b|^p)$ satisfies all the requirements of the lemma. □

3.4.1 Modal Heyting Algebras and Modal co-Heyting Algebras

We now apply the same method as above to the case of modal Heyting and co-Heyting algebras. We begin with the following lemma:

Lemma 3.4.5.

1. *Let $(L, \Box)$ be a modal Heyting algebra, $(X_L, \tau, \leq, R)$ its dual space, and p a prime filter over L. For any $U \subseteq X_L$, let $U_{R[p]} = \{q \in U \ ;\ pRq\}$. Then for any $a, b \in L$, $\Box(a \to b) \in p$ iff $|a|_{R[p]} \subseteq |b|_{R[p]}$.*

2. Let $(M, \Diamond)$ be a modal co-Heyting algebra, $(X_M, \tau, \leq, S)$ its dual space, and p a prime filter over L. For any $V \subseteq X_L$, let $V^{S\langle p\rangle} = \{q \in V \; ; \; pSq\}$. Then for any $a, b \in L$, $\Diamond(a \prec b) \notin p$ iff $|a|^{S\langle p\rangle} \subseteq |b|^{S\langle p\rangle}$.

Proof.

1. Recall that for any $p, q \in X_L$, pRq iff $p^{\Box} \subseteq q$, where $p^{\Box} = \{a \in L \; ; \; \Box a \in p\}$. We first prove that for any $a, b \in L$, if $\Box(a \to b) \in p$, then we have that $|a|_{R[p]} \subseteq |b|_{R[p]}$. Clearly, if $\Box(a \to b) \in p$, then $a \to b \in q$ for any q such that pRq. But then, if $a \in q$, it follows that $b \in q$ since $a \wedge (a \to b) \leq b$. Conversely, assume $\Box(a \to b) \notin p$. Then there exists q such that pRq and $a \to b \notin q$. But this means that there exists q' such that $q \subseteq q'$ and $q \in |a| - |b|$. But then since $p^{\Box} \subseteq q \subseteq q'$, it follows that $q' \in |a|_{R[p]} - |b|_{R[p]}$.
2. Recall that for any $p, q \in X_M$, pSq iff $q \subseteq p_{\Diamond}$, where $p_{\Diamond} = \{a \in M \; ; \; \Diamond a \in p\}$. Now assume that $\Diamond(a \prec b) \in p$ for some $a, b \in M$. This means that $a \prec b \in q$ for some q such that pRq. But then, there exists $q' \subseteq q$ such that $q' \in |a| - |b|$. Moreover, since $q' \subseteq q \subseteq p_{\Diamond}$, this means that $q' \in |a|^{S\langle p\rangle} - |b|^{S\langle p\rangle}$. Hence if $|a|^{S\langle p\rangle} \subseteq |b|^{S\langle p\rangle}$, then $\Diamond(a \prec b) \notin p$. For the converse, assume there exists $q \in |a|^{S\langle p\rangle} - |b|^{S\langle p\rangle}$. Then since $a \leq (a \prec b) \vee b$ and q is prime, it follows that $a \prec b \in q$. But then $\Diamond(a \prec b) \in p$.

$\Box$

We now define the conditions on the sets Q_M and Q_J that we will need for our generalization of the Rasiowa-Sikorski Lemma:

Definition 3.4.6. Let $(L, \Box)$ be a modal Heyting algebra. A meet $\bigwedge A$ existing in L is *Barcan* if $\Box \bigwedge A = \bigwedge\{\Box a \; ; \; a \in A\}$. Dually, if $(M, \Diamond)$ is a modal co-Heyting algebra, then a join $\bigvee B$ existing in M is *Barcan* if $\Diamond \bigvee B = \bigvee\{\Diamond b \; ; \; b \in B\}$

Definition 3.4.7.

1. Let $(L, \Box)$ be a modal Heyting algebra, and let Q_M, Q_J be two countable sets of Barcan meets and joins in L. Q_M, Q_J are $(\Box, \to)$-*complete* if for any $A, B \subseteq L$ and $c, d \in L$:
   - if $\bigwedge A \in Q_M$, then $\bigwedge(\Box(c \to (A \vee d))) \in Q_M$;
   - if $\bigvee B \in Q_J$, then $\bigwedge(\Box((B \wedge c) \to d)) \in Q_M$.
2. Dually, if $(M, \Diamond)$ is a modal Heyting algebra, and Q_M, Q_J are two countable sets of Barcan meets and joins in L, then Q_M, Q_J are $(\Diamond, \prec)$-*complete* if for any $A, B \subseteq L$ and $c, d \in L$:
   - if $\bigwedge A \in Q_M$, then $\bigvee(\Diamond(c \prec (A \vee d))) \in Q_J$;
   - if $\bigvee B \in Q_J$, then $\bigvee(\Diamond((B \wedge c) \prec d)) \in Q_J$.

Lemma 3.4.8.

1. *Let $(L, \Box)$ be a modal Heyting algebra, and let Q_M, Q_J be two $(\Box, \to)$- complete countable sets of Barcan meets and joins in L. Then for any $A, B \in L$:*

   i) if $\bigwedge A \in Q_M$, then $|\bigwedge A|_{R[p]} = (I(\bigcap_{a \in A} |a|))_{R[p]}$

   ii) if $\bigvee B \in Q_J$, then $|\bigvee B|_{R[p]} = (C(\bigcup_{b \in B} |b|))_{R[p]}$

2. *Dually, if $(M, \Diamond)$ is a modal Heyting algebra, and Q_M, Q_J are two $(\Diamond, -<)$- complete countable sets of Barcan meets and joins in M, then for any $A, B \in M$:*

   iii) if $\bigwedge A \in Q_M$, then $|\bigwedge A|^{S\langle p\rangle} = (I(\bigcap_{a \in A} |a|))^{S\langle p\rangle}$

iv) if $\bigvee B \in Q_J$*, then* $|\bigvee B|^{S\langle p\rangle} = (C(\bigcup_{b\in B}|b|))^{S\langle p\rangle}$

Proof.

i) We only prove the non-immediate direction, i.e. the left-to-right direction. As usual, assume $|c|_{R[p]} - |d|_{R[p]} \subseteq |a|_{R[p]}$ for all $a \in A$. Then $|c|_{R[p]} \subseteq |a \vee d|_{R[p]}$, which means by Lemma 3.4.5 (1.) that $\Box(c \to (a \vee d)) \in p$ for any $a \in A$. Hence since p preserves all meets in Q_M, it follows that

$$\bigwedge \Box(c \to (A \vee d)) = \Box \bigwedge (c \to (A \vee d) = \Box(c \to (\bigwedge A \vee d) \in p.$$

By Lemma 3.4.5(1.) again, this implies that

$$|c|_{R[p]} \subseteq |\bigwedge A \vee d|_{R[p]} = |\bigwedge A|_{R[p]} \cup |d|_{R[p]},$$

which in turn implies that

$$|c|_{R[p]} - |d|_{R[p]} \subseteq |\bigwedge A|_{R[p]}.$$

ii) We only prove the right-to-left direction. Assume $|b|_{R[p]} \subseteq (-|c|)_{R[p]} \cup |d|_{R[p]}$ for all $b \in B$. Then

$$|b \wedge c|_{R[p]} = |b|_{R[p]} \cap |c|_{R[p]} \subseteq |d|_{R[p]},$$

which by Lemma 3.4.5 (1.) implies that $\Box((b \wedge c) \to d) \in p$ for all $b \in B$. This implies that

$$\bigwedge \Box((B \wedge c) \to d) = \Box \bigwedge ((B \wedge c) \to d) = \Box((\bigvee B \wedge c) \to) \in p.$$

Therefore, by Lemma 3.4.5 (1.) again, we have that

$$|\bigvee B|_{R[p]} \cap |c|_{R[p]} = |\bigvee B \cap c|_{R[p]} \subseteq |d|_{R[p]},$$

which implies that $|\bigvee b| \subseteq (-|c|)_{R[p]} \cup |d|_{R[p]}$.

iii) The proof of the left-to-right direction is similar to the proof in i) above. Assume that

$$|c|^{S\langle p\rangle} - |d|^{S\langle p\rangle} \subseteq |a|^{S\langle p\rangle}$$

for all $a \in A$. This implies that $|c|^{S\langle p\rangle} \subseteq |a \vee d|^{S\langle p\rangle}$, which means by Lemma 3.4.5 (2.) that $\Diamond(c \prec (a \vee d) \notin p)$ for any $a \in A$. Hence since p preserves all joins in Q_J, we have that

$$\bigvee \Box(c \prec (A \vee d)) = \Box \bigvee (c \prec (A \vee d)) = \Box(c \prec (\bigvee A \vee d)) \notin p.$$

By Lemma 3.4.5(2.) again, this implies that

$$|c|^{S\langle p\rangle} \subseteq |\bigwedge A \vee d|^{S\langle p\rangle} = |\bigwedge A|^{S\langle p\rangle} \cup |d|^{S\langle p\rangle},$$

which in turn implies that

$$|c|^{S\langle p\rangle} - |d|^{S\langle p\rangle} \subseteq |\bigwedge A|^{S\langle p\rangle}.$$

iv) Once again, the proof of the left-to-right direction is similar to the proof in ii) above. Assume that

$$|b|^{S\langle p\rangle} \subseteq (-|c|)^{S\langle p\rangle} \cup |d|^{S\langle p\rangle}$$

for all $b \in B$. Then

$$|b \wedge c|^{S\langle p\rangle} = |b|^{S\langle p\rangle} \cap |c|^{S\langle p\rangle} \subseteq |d|^{S\langle p\rangle},$$

which by Lemma 3.4.5 (2.) implies that $\Diamond((b \wedge c) \prec d) \notin p$ for all $b \in B$. Hence

$$\bigvee \Diamond((B \wedge c) \prec d) = \Diamond \bigvee ((B \wedge c) \prec d) = \Diamond((\bigvee B \wedge c) \prec) \in p.$$

Hence by Lemma 3.4.5 (2.) again, we have that

$$|\bigvee B|^{S\langle p\rangle} \cap |c|^{S\langle p\rangle} = |\bigvee B \cap c|^{S\langle p\rangle} \subseteq |d|^{S\langle p\rangle},$$

which yields that

$$|\bigvee B| \subseteq (-|c|)^{S\langle p\rangle} \cup |d|^{S\langle p\rangle}.$$

$\square$

As in the case for Heyting and co-Heyting algebras, the last step required before proving the Rasiowa-Sikorski lemma is the following:

Lemma 3.4.9. *1. Let $(L, \Box)$ be a modal Heyting algebra, $(X_L, \tau, \leq, R)$ be the dual modal Priestley space of L, and let Q_M and Q_J be two countable $(\Box, \to)$-complete sets of Barcan meets and joins respectively. Then for any $p \in X_L$ which preserves all meets in Q_M and all joins in Q_J and any $A, B \subseteq L$:*

i) $R[p]$ is closed in (X_L, τ)

ii) if $\bigwedge A \in Q_M$, then $(S_{\bigwedge A})_{R[p]}$ is dense in $(R[p], \tau_{R[p]})$, and if $\bigvee B \in Q_J$, then $(S_{\bigvee B})_{R[p]}$ is dense in $(R[p], \tau_{R[p]})$.

2. Dually, if $(M, \Diamond)$ is a modal co-Heyting algebra, $(X_M, \sigma, \leq, S)$ is the dual modal Priestley space of M, and Q_M and Q_J are two countable $(\Diamond, \prec)$-complete sets of Barcan meets and joins respectively, then for any $p \in X_M$ which preserves all meets in Q_M and all joins in Q_J and any $A, B \subseteq M$:

iii) $S\langle p\rangle$ is closed in (X_L, τ)

iv) if $\bigwedge A \in Q_M$, then $(S_{\bigwedge A})^{S\langle p\rangle}$ is dense in $(S\langle p\rangle, \tau^{S\langle p\rangle})$, and if $\bigvee B \in Q_J$, then $(S_{\bigvee B})^{S\langle p\rangle}$ is dense in $(S\langle p\rangle, \tau^{S\langle p\rangle})$.

Proof.

i) This is a consequence of the fact that $R[p] = \bigcap_{\Box a \in p} |a|$

ii) This is an immediate consequence of Lemma 3.3.3 and Lemma 3.4.8 i) and ii).

iii) This follows from the fact that $S\langle p\rangle = \bigcap_{\Diamond b \notin p} -|b|$

iv) Similarly, this is an immediate consequence of Lemma 3.3.3 and Lemma 3.4.8 iii) and iv).

$\square$

We can finally state and prove the Rasiowa-Sikorski Lemma for modal Heyting and co-Heyting algebras:

Lemma 3.4.10.

1. *Let $(L, \Box)$ be a modal Heyting algebra, and let Q_M and Q_J be two countable $(\Box, \to)$-complete sets of Barcan meets and joins respectively. Then for any prime filter p over L that preserves all meets in Q_M and all joins in Q_J, and any $a \in L$, if $\Box a \notin p$, then there exists a prime filter q such that $p^{\Box} \subseteq q$, $a \notin q$, and q preserves all meets in Q_M and all joins in Q_J.*

2. *Dually, if $(M, \Diamond)$ is a modal co-Heyting algebra, and Q_M and Q_J are two countable $(\Diamond, -\!\!<)$-complete sets of Barcan meets and joins respectively, then for any prime filter p over M which preserves all meets in Q_M and all joins in Q_J and any $b \in M$, if $\Diamond b \in p$, then there exists a prime filter q such that $q \subseteq p_{\Diamond}$, $b \in q$, and q preserves all meets in Q_M and all joins in Q_J.*

Proof.

1. Let $(X_L, \tau, \leq, R)$ be the dual modal Priestley space of L, and let $p \in X_L$ such that p preserves all meets in Q_M and all joins in Q_J. Note that by Lemma 3.4.9 i), $(R[p], \tau_{R[p]})$ is compact Hausdorff, and therefore satisfies the conditions of the Baire Category Theorem. Moreover, by Lemma 3.4.9 ii), $(S_Q)_{R[p]}$ is a countable intersection of open dense sets in $R[p]$, and is therefore dense by the Baire Category Theorem. Now assume $\Box a \notin p$ for some $a \in L$. This means that $|a|_{R[p]}$ is non-empty, and hence $|a|_{R[p]} \cap (S_Q)_{R[p]} \neq \emptyset$. But any q in this set satisfies all the requirements of the lemma.

2. Dually, let $(X_M, \tau, \leq, S)$ be the dual modal space of M. A similar argument shows that for any p preserving all meets in Q_M and all joins in Q_J, $(S_Q)^{S\langle p\rangle}$ is dense in $S\langle p\rangle$ by Lemma 3.4.9 iii) and iv). Now for any $b \in M$ such that $\Diamond b \in p$, $|b|^{S\langle p\rangle}$ is non-empty. But this means that $|b|^{S\langle p\rangle} \cap (S_Q)^{S\langle p\rangle} \neq \emptyset$. Since any q in this set satisfies all the requirements, the lemma is proved.

$\Box$

3.4.2 Consequences of the Generalized Rasiowa-Sikorski Lemmas

We can finally reap the fruits on the previous three generalizations of the Rasiowa-Sikorski Lemma.

Lemma 3.4.11. *Let L be a distributive lattice, Q_M, Q_J be two countable sets of distributive meets and joins in L respectively, and let $(S_Q, \tau, \leq)$ be the subspace of the dual Priestley space $(X_L, \tau^*, \leq^*)$ of L, where S_Q is the set of all prime filters over L that preserve all meets in Q_M and all joins in Q_J. Then:*

1. *The canonical map : $|\cdot| : L \to Up(S_Q)$ is a, injective DL-homomorphism;*

2. *If $(L, \to)$ is a Heyting algebra, and Q_M, Q_J are $(\bigwedge, \to)$-complete, then $|\cdot|$ is also a Heyting-homomorphism;*

3. *If $(L, -\!\!<)$ is a co-Heyting algebra, and Q_M, Q_J are $(\bigvee, -\!\!<)$-complete then $|\cdot|$ is also a co-Heyting-homomorphism*

4. *If $(L, \to, \Box)$ is a Heyting modal algebra, and Q_M, Q_J are $(\Box, \to)$ complete sets of Barcan meets and joins, then $|\cdot|$ also preserves the $\Box$ operator;*

5. *If $(L, -\!\!<, \Diamond)$ is a co-Heyting modal algebra, and Q_M, Q_J are $(\Diamond, -\!\!<)$ complete sets of Barcan meets and joins, then $|\cdot|$ also preserves the $\Diamond$ operator.*

Proof.

1. This is simply Theorem 3.3.6
2. We need to check that for any $a, b \in L$, $|a \to b| = - \downarrow \{|a| - |b|\}$. The left-to-right direction follows from basic properties of filters, and the converse is exactly the statement of Lemma 3.3.11
3. We need to check that for any $a, b \in L$, $|a \prec b| = \uparrow \{|a| - |b|\}$. But the right-to-left direction follows from basic properties of prime filters, and the converse is precisely the statement of the Rasiowa-Sikorski lemma for co-Heyting algebras, i.e. Lemma 3.4.4.
4. We prove that for any $a \in L$, $|\Box a| = \blacksquare |a|$, where for any $U \subseteq X_Q$, we have that $\blacksquare U = \{q \in S_Q \ ; \ \forall r \in S_Q : q^{\Box} \subseteq r \text{ implies } r \in U\}$. The left-to-right direction is immediate by definition of $\blacksquare$. The converse is precisely the statement of Lemma 3.4.10 (1.).
5. We prove that for any $b \in L$, $|\Diamond b| = \blacklozenge |b|$, where for any $V \subseteq X_Q$, we have that $\blacklozenge Y = \{q \in S_Q \ ; \ \exists r \in Y \text{ s.t. } r \subseteq p_{\Diamond}\}$. The right-to-left direction is immediate from the definition of $\blacklozenge$. Once again, the converse is precisely the statement of Lemma 3.4.10(2.).

□

Corollary 3.4.12. *Let $(L, \Box)$ be a BAO, and Q_M a set of Barcan meets such that for any $A \subseteq L$, $c \in L$, if $\bigwedge A \in Q_M$, then $\bigwedge \Box(A \vee c) \in Q_M$. Then for any $a \neq 0 \in L$, there exists an ultrafilter p over L such that $a \in p$ and p preserves all meets in Q_M.*

Proof. By definition, $(L, \Box)$ is a modal Heyting algebras, and by simple syntactic manipulations, it is easy to see that Q_M satisfies the requirement in Definition 3.4.7 (1.). Hence the Rasiowa-Sikorski applies to $(L, \Box)$ and Q_M, which means that if $a \neq 0 \in L$, then there is a prime filter p over L that preserves all meets in Q_M and contains a. But since L is a Boolean algebra, p is an ultrafilter. □

We finish this section, as expected, with the immediate consequence for various logics of the results above:

Theorem 3.4.13.

1. *cIPL is complete with respect to the class of cIPL-models;*
2. *MIPL is complete with respect to the class of MIPL-models;*
3. *cMIPL is complete with respect to the class of cMIPL-models;*
4. *KL is complete with respect to the class of KL-models.*

Proof. Items $1-3$ are immediate consequences of the term model construction and Lemma 3.4.11 $3-5$ respectively. Similarly, item 4 is a direct consequence of the term model construction and Corollary 3.4.12. □

Chapter 4

Possibility Semantics and Tarski's Lemma for Boolean Algebras

The aim of this chapter is to introduce the methods and ideas that will be used under one form or another in chapters 5 and 6. More precisely, we present the main features of possibility semantics for classical logic that we will seek to generalize to the intuitionistic case. Possibility semantics was first proposed by Humberstone [40] as an alternative to possible worlds semantics for modal logic. Humberstone's idea was to work with partially determined worlds rather than complete possible worlds, and to define satisfaction at a partial world by quantifying over all possible refinements of that world. Holliday [39] offers a systematic study of possibility models and their relationship with standard Kripke models.

In the first section of this chapter, we recall well-known facts about the regular open sets of a topological space (X, τ), and in particular that for any topological space the IC operator corresponds to the double-negation nucleus on $\mathsf{O}(X)$. We use this fact to provide a choice-free representation theorem for Boolean algebras and to prove the completeness of CPC with respect to possibility models for LT_{CPC}. In section 2, we consider the extension of this result to first-order classical logic. The proof relies on an equivalent form of the axiom of dependent choice, known as Tarski's Lemma. A proof of this result was given in [4], although the proof relies on a Henkin-style argument, while our proof is algebraic. In section 3, we relate canonical possibility spaces to important algebraic constructions, and show how to represent the canonical extension and the MacNeille completion of a Boolean algebra as the regular open sets of some possibility space. Finally, section 4 is concerned with generalizations of Tarski's Lemma to BAO's and Heyting algebras. We will show that, although a straightforward adaptation of Tarski's proof for BAO's is possible, the statement of Tarski's Lemma itself for a Heyting algebra is equivalent to a strong form of Kuroda's axiom.

4.1 Possibility Models and Semantics for CPC

4.1.1 The IC Operator on a Topological Space

In this section, we review some well-known facts about the regular open sets of any topological space. The most important result is that for any topological space (X, τ), the regular open sets of (X, τ) form a Boolean algebra. This result will be used extensively in the sections, and a generalization of it to bi-topological spaces will be one of the driving forces of the next chapter.

Lemma 4.1.1. *Let (X, τ) be a topological space, and let $\mathsf{O}(X)$ be the Heyting algebra of open sets of X. For any $U \subseteq X$, let $IC(U)$ be the interior of the closure of U. Then IC is the double-negation nucleus on $\mathsf{O}(X)$, i.e. for any $U \in \mathsf{O}(X)$, $IC(U) = \neg_{\mathsf{O}(X)} \neg_{\mathsf{O}(X)}(U)$.*

Proof. Recall from Lemma 2.1.16 that for any $U \in \mathsf{O}(X)$, $\neg_{\mathsf{O}(X)}(U) = I(-U \cup \emptyset) = -C-(-U) = -C(U)$. Hence $\neg_{\mathsf{O}(X)} \neg_{\mathsf{O}(X)}(U) = -C - C(U) = IC(U)$. □

Theorem 4.1.2. *Let (X, τ) be a topological space. Then $(\mathsf{RO}(X), \cap, \vee, \neg, \bigwedge, \bigvee)$ is a complete Boolean algebra, where $RO(X)$ is the set of all regular open sets of X, for any $A, B \in RO(X)$, $A \vee B = IC(A \cup B)$, $\neg A = -C(A)$, and for any family $\{A_i\}_{i \in I}$ of regular open sets, $\bigwedge_{i \in I} A_i = I(\bigwedge_{i \in I} A_i)$ and $\bigvee_{i \in I} A_i = IC(\bigcup_{i \in I} A_i)$.*

Proof. This is a direct consequence of Lemma 4.1.1, and the general theory of nuclei (Theorem 2.3.4 and Definition 2.3.5) [1]. □

4.1.2 Choice-Free Representation Theorem for Boolean Algebras

In this section, we briefly review how every Boolean algebra embeds into the regular open sets of some topological space. The main interest of this representation theorem is that it is entirely choice-free since, unlike Stone representation, the canonical space of a Boolean algebra will be based on the set of filters of A, rather than on the set of its ultrafilters.

Definition 4.1.3. Let A be a Boolean algebra, and let $(X, \leq)$ be the poset of all (proper) filters of A, where for any $F, F' \in X$, $F \leq F'$ iff $F \subseteq F'$. Then the *canonical possibility space* of A is the topological space (X, τ), where τ is the Alexandroff topology on X induced by $\leq$.

Theorem 4.1.4. *Let A be a Boolean algebra, and (X, τ) its canonical possibility space. Then the map $|\cdot| : A \to \mathscr{P}(X)$ defined by $|a| = \{F \in X \ ; \ a \in F\}$ is an embedding of A into $\mathsf{RO}(X)$, i.e.:*

1. *$|\cdot|$ restricts to a map from A to $\mathsf{RO}(X)$;*
2. *$|\cdot|$ is a Boolean homomorphism;*
3. *$|\cdot|$ is injective.*

Proof.

1. We prove that $IC|a| = |a|$ for any $a \in A$. We claim that for any $a \in A$, $|a| = -C|\neg a|$. Note that since τ is the Alexandroff topology on the inclusion ordering on X, this means that we have to prove that for any proper filter $F \subseteq A$, $a \notin F$ iff there exists $F' \in X$ such that either $F \cup \{\neg a\} \subseteq F'$. For the right-to-left direction, if suffices to notice that if for any $F, F' \in X$, if $F \cup \{\neg a\} \subseteq F'$, then $a \notin F$, for otherwise $a \wedge \neg a = 0 \in F'$, contradicting the fact that F' is proper. For the left-to-right direction, we claim that if $a \notin F$, then $F' = \uparrow\{\neg a \wedge b \ ; \ b \in F\}$ is the required filter. It is routine to check that F' is a filter. To see that F' is proper, assume $\neg a \wedge b \leq 0$ for some $b \in F$. But then $b \leq \neg\neg a = a$, contradicting the assumption that $a \notin F$. Therefore $|a| = -C|\neg a|$ for any $a \in A$. But then we have that
$$IC|a| = -C - C|a| = -C|\neg a| = |\neg\neg a| = |a|.$$
Hence $|\cdot|$ restricts to a map from $A \to \mathsf{RO}(X)$.

[1] For a step by step proof using only the definitions of interior and closure, the reader is directed to [27] chapter 9.

2. The following hold for any $a, b \in A$:
   - $|1| = X$ and $|0| = \emptyset$, since all $F \in X$ are proper filters;
   - $|a \wedge b| = |a| \cap |b|$ since, for any $F \in X$, $a \wedge b \in F$ iff $a \in F$ and $b \in F$;
   - $|\neg a| = -C|a|$ by 1. above;
   - $|a \vee b| = |\neg(\neg a \wedge \neg b)| = -C(-C|a| \cap -C|b|) = I - (-C|a| \cap -C|b|) = I(C|a| \cup C|b|) = = IC(|a| \vee |b|)$, since closure distributes over unions.
3. We prove that for any $a, b \in A$, $a \leq b$ iff $|a| \subseteq |b|$. The left-to-right direction is obvious, since filters are upward-closed. For the converse, it suffices to notice that if $a \nleq b$, then $\uparrow a \in |a| - |b|$.

□

4.1.3 Completeness of CPC with respect to Possibility Semantics

Finally, we show how the results recalled above allow one to define a relational semantics for CPC known as possibility semantics.

Definition 4.1.5. Let A be a Boolean algebra. A *possibility model for A* is a triple $(X, \leq, V)$, such that $(X, \leq)$ is a poset and V is a homomorphism from A to $\mathsf{RO}(X)$, where the (implicit) topology on X is the Alexandroff topology induced by $\leq$. More concretely, V has the following properties for any $a, b \in A$, $x \in X$:

- $x \in V(a)$ iff $\forall y \geq x \; \exists z \geq y$ such that $z \in V(a)$;
- $x \in V(a \wedge b)$ iff $x \in V(a)$ and $x \in V(b)$;
- $x \in V(a \vee b)$ iff $\forall y \geq x \; \exists z \geq y$ such that $z \in V(a) \cup V(b)$;
- $x \in V(\neg a)$ iff $\forall y \geq x$, $y \notin V(a)$.

A *possibility model* is a possibility model for LT_{CPC}, the Lindenbaum-Tarski algebra of CPC. Finally, a formula ϕ of CPC is valid iff for any possibility model $(X, \leq, V)$, $V(\phi) = X$.

Theorem 4.1.6. *CPC is sound and complete with respect to possibility models, i.e. for any formula ϕ, ϕ is a theorem of CPC iff ϕ is valid in the sense of Definition 4.1.5.*

Proof. Soundness is a direct consequence of Definition 4.1.5, since, for any possibility model (X, τ, V), if $V : LT_{CPC} \to \mathsf{RO}(X)$ is a homomorphism, then if ϕ is a theorem of CPC, $V(\phi^{\vdash}) = V(\top^{\vdash}) = X$. For the converse, note that if ϕ is not a theorem of CPC, then $\top^{\vdash} \nleq \phi^{\vdash}$. Moreover, the canonical possibility space of LT_{CPC} with the embedding $|\cdot|$ from Theorem 4.1.4 form a possibility model $(X, \tau, |\cdot|)$, and, since $|\cdot|$ is injective, $|\phi^{\vdash}| \neq |\top^{\vdash}| = X$. Hence ϕ is not valid. □

4.2 First-Order Possibility Models, Tarski's Lemma and the Completeness of CPL

In this section, we extend the result of the previous section to the case of first-order classical logic, i.e. we define first-order possibility models and prove that CPL is sound and complete with respect to the set of all first-order possibility models. In order to achieve this result, we will adopt the same strategy as above, that is, we will construct a canonical first-order possibility model for the Lindebaum-Tarski algebra of CPL, denoted as LT_{CPL}. Note that, for any Boolean algebra

B, the canonical possibility model for B defined in Theorem 4.1.4 above will not in general preserve the infinitary structure on B, and in particular infinitary meets. In order to overcome this issue, we will restrict the set of filters upon which the canonical possibility model of B is based upon, and prove that this yields a completion of B that preserves a countable number of meets.

Definition 4.2.1 (Q-filter)**.** Let B be a Boolean algebra, and $Y \subseteq B$ such that $\bigwedge Y$ exists. A filter $F \subseteq B$ is set to *decide* $\bigwedge Y$ if either $\bigwedge Y \in F$ or there exists $\neg y \in F$ for some $y \in Y$. Note that if F *decides* $\bigwedge Y$, then F also *preserves* $\bigwedge Y$, i.e. if $Y \subseteq F$, then $\bigwedge Y \in F$. Given a countable set Q of meets in B, F is a *Q-filter* if F decides $\bigwedge Y$ for every $\bigwedge Y \in Q$.

Lemma 4.2.2. *Let B be a Boolean algebra, Q a countable set of meets in B, and (X_Q, τ) be a topological space such that X_Q is the set of all Q-filters and τ is the Alexandroff topology induced by the filter-ordering of X. Then $|\cdot| : B \to \mathsf{RO}(X_Q)$ is a Boolean homomorphism that preserves all meets in Q, i.e. for any $\bigwedge Y \in Q$, $|\bigwedge Y| = \bigwedge_{\mathsf{RO}_{X_Q}} \{|y| \ ; \ y \in Y\}$.*

Proof. Similarly to the proof of Theorem 4.1.4, the key fact for proving that $|\cdot|$ is a well-defined Boolean homomorphism is that $-C|\neg a| = |a|$ for all $a \in B$. Since any $F \in X_Q$ is a filter, it follows that $-C|\neg a| \subseteq |a|$. For the converse, it suffices to notice that if F is a Q-filter, then any F' such that $F \subseteq F'$ is also a Q-filter. Hence if $a \notin F$, for some Q-filter F, $F' = \uparrow\{\neg a \wedge b \ ; \ b \in F\}$ is also a Q-filter, and $F \cup \{\neg a\}$. Hence $-C|\neg a| = |a|$, which implies that $|\cdot|$ is a Boolean homomorphism.
To see that $|\cdot|$ also preserves all meets in Q, recall first that for any $U \subseteq \mathsf{RO}(X_Q)$, we have that

$$\bigwedge_{\mathsf{RO}(X_Q)} U = I(\bigcap U) = \bigcap U,$$

since arbitrary intersections of open sets are also open in the Alexandroff topology. Now let $Y \subseteq B$ be such that $\bigwedge Y \in Q$. Then for any $F \in X_Q$, since F preserves $\bigwedge Y$, we have that $\bigwedge Y \in F$ iff $Y \subseteq F$. Hence $|\bigwedge Y| = \bigcap_{y \in Y} |y|$. ☐

The next lemma will be of crucial importance in order to prove that the embedding $|\cdot|$ defined above is injective:

Lemma 4.2.3 (Tarski's Lemma)**.** [2] *Let B be a Boolean algebra, and Q a countable set of meets in B. For any $a \in B$, if $a \neq 0$, then there exists a Q-filter F over B that contains a.*

Proof. Let a be a non-zero element of B. Notice first that for any $\bigwedge Y \in Q$, either $a \leq \bigwedge Y$ or $a \nleq y$ for some $y \in Y$. Now let $\{\bigwedge Y_i\}_{i \in \mathbb{N}}$ be an enumeration of all the meets in Q. We build inductively a sequence $\{a_i\}_{i \in \mathbb{N}}$ as follows:

- At stage 0, set: $a_0 = a$
- At stage $i+1$, note that $a_i \leq \bigwedge Y_i$ or $a_i \nleq y_i$ for some $y_i \in Y_i$. In the first case, set $a_{i+1} = a_i$, and in the latter case, set $a_{i+1} = a_i \wedge y_i$

Now let $F = \uparrow\{c \in B \ ; \ a_i \leq c \text{ for some } a_i \in \{a_i\}_{i \in \mathbb{N}}\}$. It is easy to check that F is a proper filter: by construction, $a_i \nleq 0$ for all $a_i \in \{a_i\}_{i \in \mathbb{N}}$, and, moreover, since $\{a_i\}_{i \in \mathbb{N}}$ is a descending chain of elements of B, for any $b, c \in B$ and $a_i, a_j \in \{a_i\}_{i \in \mathbb{N}}$, if $a_i \leq b$ and $a_j \leq b$, then $a_{max(i,j)} \leq b \wedge c$, which proves that F is closed under finite meets. Finally, it is easy to see that $a \in F$, and that F is a Q-filter by construction. ☐

[2] According to [Goldblatt], this lemma appears for the first time in [Tarski]

We can now use the two previous lemmas to obtain the main result of this section.

Definition 4.2.4. (Q-completion of a BA) Let B be a Boolean algebra, and Q a countable set of meets in B. A *Q-completion* of B is a pair (C, α), where C is a complete Boolean algebra and $\alpha : B \to C$ is an injective Boolean homomorphism such that $\alpha(\bigwedge_B Y) = \bigwedge_C \{\alpha(y) \ ; \ y \in Y\}$ for any $\bigwedge_B Y \in Q$.

Theorem 4.2.5. *Let B be a Boolean algebra, Q a countable set of meets in B, and (X_Q, τ) and $|\cdot|$ as in Lemma 4.2.2. Then $(\mathsf{RO}(X_Q), |\cdot|)$ is a Q-completion of B.*

Proof. By Lemma 4.2.2 above, $|\cdot| : B \to \mathsf{RO}(X_Q)$ is a Boolean homomorphism that preserves all meets in Q. To see that $|\cdot|$ is injective, we show that for any $a, b \in B$, if $a \nleq b$, then $|a| \nsubseteq |b|$. If $a \nleq b$, then $a \wedge \neg b \neq 0$, and hence by Lemma 4.2.3 there exists $F \in X_Q$ such that $a \wedge \neg b \in F$. But this means that $a \in F$ and $b \notin F$, and hence $|a| \nsubseteq |b|$. $\square$

We conclude this section by showing how we can use Theorem 4.2.5 to define a semantics for first-order logic based on possibility models.

Definition 4.2.6. Let $\mathfrak{L}$ be a first-order language, $LT^{\mathfrak{L}}_{CPL}$ the Lindenbaum-Tarski algebra of CPL over the language $\mathfrak{L}$, and let $Q^{\mathfrak{L}}_{\forall} = \{(\forall x \phi(x))^{\vdash} \ ; \ \phi \in Form(\mathfrak{L})\}$. *A first-order possibility $\mathfrak{L}$-model* is a tuple $\mathfrak{X} = (X, \leq, \{(D_i, J_i\}_{i \in X}, \{f_{ij}\}_{i \leq j \in X}, \{a_i\}_{i \in X}, V_\alpha)$ such that $(X, \leq)$ is a poset, V_α is a Boolean homomorphism from $LT_T \to \mathsf{RO}(X)$, and the following holds for every $i, j, k \in X$:

- D_i is a set, called the domain of i
- J_i maps every n-ary symbol R of $\mathfrak{L}$ to a subset of D_i^n
- if $i \leq j$, then f_{ij} is a surjective map from D_i to D_j
- f_{ii} is the identity map on D_i
- $f_{jk} \circ f_{ij} = f_{ik}$
- $\{\alpha_i\}_{i \in X}$ is a system of assignments: for any $i \in X$, α_i is a map from $Var(\mathfrak{L})$ into D_i that is coherent with all f_{ij}, i.e. for any $i, j \in X$ such that $i \leq j$ and any $x \in Var(\mathfrak{L})$, $f_{ij}(\alpha_i(x)) = \alpha_j(x)$
- for any $R(x_1, ..., x_n) \in At(\mathfrak{L})$, $i \in V_\alpha(R(x_1, ..., x_n)^{\vdash})$ iff $(\alpha_i(x_1), ..., \alpha_i(x_n) \in J_i(R)$.
- for any $\forall x \phi(x) \in Q^{\mathfrak{L}}_{\alpha}$, $i \in V((\forall x \phi(x))^{\vdash})$ iff for any system of assignments $\{\beta_i\}_{i \in X}$ such that for all $j \in X$, $\beta_j(y) = \alpha_j(y)$ for every $y \neq x \in Var(\mathfrak{L})$, $i \in V_\beta(\phi(x)^{\vdash})$

A formula ϕ is *true on* a first-order possibility model $\mathfrak{X}$ if $V_\alpha(\phi^{\vdash}) = X$. ϕ is valid on $\mathfrak{X}$ if for any system of assignemnt $\{\beta_i\}_{i \in X}$, $V_\beta(\phi^{\vdash}) = X$.

Note that for any first-order possibility model $\mathfrak{X} = (X, \leq, \{(D_i, J_i\}_{i \in X}, \{f_{ij}\}_{i \leq j \in X}, \{a_i\}_{i \in X}, V_\alpha)$, an equivalent characterization of the valuation V can be given using two conditions on the collection $\{D_i, J_i\}_{i \in X}$ which hold for any $i, j \in X$:

- if $i \leq j$, then for any n-ary relation symbol R, if $(a_1, ..., a_n) \in J_i(R)$, then $(f_{ij}(a_1), ..., f_{ij}(a_n)) \in J_j(R)$ (persistence condition)
- for any n-ary relation symbol R of $\mathfrak{L}$, if $(a_1, ..., a_n) \notin J_i(R)$, then there exists $j \in X$ such that $i \leq j$ and for all $k \in X$, if $j \leq k$, then $(f_{ik}(a_1), ..., f_{ik}(a_n)) \notin J_k(R)$ (refinement condition);

and a satisfaction relation defined inductively as follows for any $i \in X, \phi, \psi \in Form(\mathfrak{L})$:

- if $\phi := R(x_1, ...x_n)$ for some n-ary relation symbol R, then $i \Vdash_\phi (a_1, ..., a_n)$ iff $(a_1, ..., a_n) \in J_i(R)$
- $i \Vdash \phi \wedge \psi$ iff $i \Vdash \phi$ and $i \Vdash \psi$
- $i \Vdash \phi \vee \psi$ iff for all $j \in X$, if $i \leq j$, then there exists $k \in X$ such that $j \leq k$, and $k \Vdash \phi$ or $k \Vdash \psi$
- $i \Vdash \neg\phi$ iff for all $j \in X$ such that $i \leq j$, $j \nVdash \phi$
- $i \Vdash \forall x \phi(x)$ iff for all $a \in D_i$, $i \Vdash \phi(a)$
- $i \Vdash \exists x \phi(x)$ iff for all $j \in X$, if $i \leq j$, then there exists $k \in X$ such that $j \leq k$, and $k \Vdash_\phi (a)$ for some $a \in D_k$.

Theorem 4.2.7. *CPL is sound and complete with respect to first-order possibility models.*

Proof. We use essentially the same argument as in Lemma 3.1.7. Soundness of CPL with respect to first-order possibility models is a direct consequence of Definition 4.2.6, since V_α is always assumed to be a Boolean homomorphism from $LT^{\mathfrak{L}}_{CPL}$ to $\mathsf{RO}(X)$.
For completeness, we construct a term model $\mathfrak{X} = (X, \leq, \{(D_i, J_i)\}_{i \in X}, \{f_{ij}\}_{i \leq j \in X}, \{a_i\}_{i \in X}, V_\alpha)$ as follows:

- $(X, \leq)$ is the set of all $Q^{\mathfrak{L}}_\forall$-filters over $LT^{\mathfrak{L}}_{CPL}$, ordered by inclusion;
- for every filter $F \in X$, let D_F be the set of all terms of $\mathfrak{L}$ quotiented by the equivalence relation $\sim_F$ defined as $t \sim_F u$ iff $(t = u)^\vdash \in F$;
- for any filter $F \in X$, any n-ary relation symbol R in $\mathfrak{L}$ and any terms $t_1, ..., t_n$, $(t_1^{\sim_F}, ..., t_n^{\sim_F}) \in I_F(R)$ iff $R(t_1, ..., t_n)^\vdash \in F$;
- for any $F, F' \in X$, set $f_{FF'}(t^{\sim_F}) = t^{\sim_{F'}}$ for any term t;
- Finally, for any $F \in X$ and any variable x, set $\alpha_F(x) = x^{\sim_F}$.

It is then straightforward to verify that for any formula $\phi \in \mathfrak{L}$ and any $F \in X$, we have $F \in V_\alpha(\phi^\vdash)$ iff $\phi^\vdash \in F$. In particular, the $\forall$-step of the proof goes through because all filters in X decide and hence also preserve all meets in $Q^{\mathfrak{L}}_\forall$. But then, for any formula ϕ, if ϕ is not a theorem of CPL, then by Lemma 4.2.3, there exists $F \in X$ such that $\phi^\vdash \notin F$, and hence $V_\alpha(\phi^\vdash) \neq X$. Hence ϕ is not valid on the set of all first-order possibility frames. □

4.3 Possibility Spaces and Completions of Boolean Algebras

In this section, we draw a connection between the possibility spaces defined in section 1 and well known completions of Boolean algebras, namely canonical extensions and MacNeille completions. Most of the results appeared already in Holliday [39], but we present them in a somewhat more general fashion, as this will allow us to transfer some of the main ideas to the setting of Heyting algebras and distributive lattices in the next chapter. Our goal is to show how the MacNeille completion and the canonical extension of a Boolean algebra can be given a natural representation as the algebra of regular open sets of topological spaces that are mere variations on the canonical possibility space defined in section 1. In particular, we show that well-known facts about the dual space of a Boolean algebra arise as special cases of the more general phenomena described here. We first fix some definitions for the rest of the section.

Definition 4.3.1. Let B be a Boolean algebra. A set $\mathscr{C}$ of filters over B is called *separative* if for any $a, b \in B$ such that $a \nleq b$, there exists $F \in \mathscr{C}$ such that $a \in F$ and $b \notin F$. $\mathscr{C}$ is called *rich* if for every $F \in \mathscr{C}$ and any $a \in B$, if $a \notin F$, then there exists $F' \in \mathscr{C}$ such that $\neg a \in F'$.

Definition 4.3.2. Let B be a Boolean algebra, and $\mathscr{C}$ a set of filters over B. We define two topologies on the poset $(\mathscr{C}, \leq)$, where $\leq$ is the inclusion ordering on $\mathscr{C}$.

- The *Alexandroff topology* $\tau_{\leq}$ is the smallest topology such that all upsets with respect to $\subseteq$ are open.
- The *Stone topology* τ_S is given by the basis $\beta = \{|a| \; ; \; a \in B\}$, where $|a| = \{F \in \mathscr{C} \; ; \; a \in F\}$.

Lemma 4.3.3 (Representation for MacNeille completions)**.** *Let B be a Boolean algebra, and let $(\mathscr{C}, \tau_S)$ be the topological space given by the Stone topology on some separative set $\mathscr{C}$. Then $(\mathsf{RO}_{\tau_S}(\mathscr{C}))$, the algebra of regular open sets of $(\mathscr{C}, \tau_S)$, is isomorphic to the MacNeille completion of B.*

Proof. We prove that the map $|\cdot|_{\mathscr{C}} : B \to (\mathsf{RO}_{\tau_S}(\mathscr{C}))$ is an injective homomorphism, and that every element of $(\mathsf{RO}_{\tau_S}(\mathscr{C}))$ is both a meet and a join of images of B. To see first that $|\cdot|$ is well-defined and is a Boolean homomorphism, we apply the same reasoning as in Theorem 4.1.4, which means that we only have to prove that $|\neg a| = -C|a|$ for all $a \in B$. So let $a \in B$, and notice first that

$$C|a| = \bigcap_{c \in C_{|a|}} -|c|,$$

where $C_{|a|} = \{c \in B; |a| \subseteq -|c|\}$. Since all filers in $\mathscr{C}$ are proper, we have that for any $F \in \mathscr{C}$, if $a \in F$, then $\neg a \notin F$, which means that $|a| \subseteq -|\neg a|$, and hence $\neg a \in C_{|a|}$. But this at once implies that $C|a| \subseteq -|\neg a|$, and therefore, by taking complements, this yields

$$|\neg a| \subseteq -C|a|.$$

For the converse, assume that $F \notin C|a|$. Then there is $c \in F$ such that $|a| \subseteq -|c|$. But this last fact implies that for any $F' \in \mathscr{C}$, $a \wedge c \notin F'$, and hence, by separativeness, $a \wedge c \leq 0$. This is in turn implies that $c \leq \neg a$, we thus have that $\neg a \in F$, which completes the proof that

$$-C|a| \subseteq |\neg a|.$$

As in Theorem 4.1.4, simple properties of filters and algebraic manipulations ensure that $|\cdot|$ is a well-defined Boolean homomorphism, and injectivity follows immediately from separativeness of $\mathscr{C}$.

Finally, we prove that every regular open sets is both a meet and join of images of elements of B under $|\cdot|$. Let $X \in (\mathsf{RO}_{\tau_S}(\mathscr{C}))$. Notice first that since X is open, $X = \bigcup_{a \in A} |a|$ for some $A \subseteq B$. Hence

$$X = IC(X) = IC(\bigcup_{a \in A} |a|),$$

which means that B is join-dense in $(\mathsf{RO}_{\tau_S}(\mathscr{C}))$. For meet-density, define for any $X \in RO_{\tau_S}(\mathscr{C})$ the set $A_X = \{\neg a \in B \; ; \; X \subseteq -|a|\}$. We claim that

$$X = I(\bigwedge_{\neg a \in A_X} |\neg a|).$$

For the left-to-right direction, notice that if $X \subseteq -|a|$, then

$$X = I(X) \subseteq I - |a| = -C|a| = |\neg a|.$$

Hence for every $\neg a \in A_X$, $X \subseteq |\neg a|$, which means that

$$X \subseteq I(\bigcap_{\neg a \in A_X} |\neg a|).$$

For the converse, note that for any $\neg a \in A_X$, $|\neg a| \subseteq -|a|$. Hence

$$\bigcap_{\neg a \in A_X} |\neg a| \subseteq \bigcap_{a \in A_X} -|a| = C(X),$$

and therefore we have

$$I(\bigcap_{\neg a \in A_X} |\neg a|) \subseteq IC(X) = X.$$

Hence B is both meet-dense and join-dense in $(\mathsf{RO}_{\tau_S}(\mathscr{C}))$, which means that $(\mathsf{RO}_{\tau_S}(\mathscr{C}))$ is isomorphic to the MacNeille completion of B. $\square$

The previous lemma shows the tight connection that exists between MacNeille completions of Boolean algebras and Stone topologies on sets of filters over those. It also has as a straightforward consequence the following well-known fact:

Corollary 4.3.4. *Let B be a Boolean algebra. Then the algebra of regular open sets of the dual space of B is isomorphic to its MacNeille completion.*

Proof. Let $\mathscr{C}^{\mathfrak{U}}$ the set of all ultrafilters over B. Then $(\mathsf{RO}_{\tau_S}(\mathscr{C}^{\mathfrak{U}}))$ is exactly the algebra of regular open sets of the Stone space of B. Moreover, by the ultrafilter theorem, $\mathscr{C}^{\mathfrak{U}}$ is separative. Hence by Lemma 4.3.3, $(\mathsf{RO}_{\tau_S}(\mathscr{C}^{\mathfrak{U}}))$ is isomorphic to the MacNeille completion of B. $\square$

Moreover, the following two lemmas show that a similar study of the algebraic constructions induced by the Alexandroff topology is possible.

Lemma 4.3.5. *Let B be a Boolean algebra, and $\mathscr{C}$ a separative and rich set of filters over B. Then $(\mathsf{RO}_{\tau_\leq}(\mathscr{C}), |\cdot|_{\mathscr{C}})$ is a doubly-dense completion of B i.e. for every $X \in \mathsf{RO}_{\tau_\leq}(\mathscr{C})$, X is both a meet of joins and a join of meets of images of B.*

Proof. Once again, in light of Theorem 4.1.4, we merely have to prove that for any $a \in B$, $C|a| = -|\neg a|$ in order to prove that $|\cdot| : B \to \mathsf{RO}_{\leq}(\mathscr{C})$ is a well-defined Boolean homomorphism. Let $a \in B$. By standard properties of filters, we have that $|\neg a| \subseteq -|a|$, and since $-|a|$ is closed, we also have that $C|\neg a| \subseteq -|a|$. For the converse, note that, since $\mathscr{C}$ is rich, for any $F \in \mathscr{C}$ such that $a \notin F$, there exists $F' \in \mathscr{C}$ such that $F \cup \{\neg a\} \subseteq F'$. But since $\tau_\leq$ is the Alexandroff topology induced by the inclusion ordering on $\mathscr{C}$, it follows at once that $-|a| \subseteq C|\neg a|$. Moreover, the fact that $\mathscr{C}$ is separative implies that $|\cdot|$ is injective.

Finally, we show that for any $X \in \mathsf{RO}_{\tau_\leq}(\mathscr{C})$, we have that X is both a meet of joins and a join of meets of images of elements from B. For any $X \in \mathsf{RO}_{\tau_\leq}(\mathscr{C})$, let $G_X = \{F \in \mathscr{C} \,;\, F \in X\}$. Since X is an upset, it is straightforward to see that

$$X = \bigcup_{F \in \mathscr{C}} \bigcap_{a \in F} |a|,$$

and since X is regular open, we have that

$$X = IC(X) = IC(\bigcup_{F\in\mathscr{C}} \bigcap_{a\in F} |a|),$$

which means that X is a join of meets of images of elements of B. Moreover, let $J_X = \{F \ ; \ F \in -C(X)\}$. We claim that

$$X = \bigcap_{F\in J_X} IC(\bigcup_{a\in F} |\neg a|)^3.$$

For the left-to-right direction, let $F \in X$ and $G \in J_X$. Since X is an upset, for any $F' \supseteq F$, $F' \in X$. But since $G \in -C(X)$, this means that $G \nsubseteq F'$ for every $F' \supseteq F$. Hence for any $F' \supseteq F$, there is $a \in G$ such that $a \notin F'$, which means, since $\mathscr{C}$ is rich, that there exists $F^* \supseteq F'$ such that $\neg a \in F^*$. Hence for any $G \in J_X$ and any $F' \supseteq F$, there exists $F^* \supseteq F'$ and $a \in G$ such that $\neg a \in F^*$. It follows that

$$X \subseteq \bigcap_{G\in J_X} IC(\bigcup_{a\in G} |\neg a|).$$

For the converse, assume $F \notin X$. Then since X is regular open, there exist $F' \supseteq F$ such that for all F^* such that $F' \subseteq F^*$, we have that $F^* \notin X$. Hence $F' \in J_X$. But, clearly, for any $F^* \supseteq F'$, $\neg a \notin F^*$ for any $a \in F'$. This implies that

$$F \in \bigcup_{G\in J_X} CI(\bigcap_{a\in G} -|\neg a|),$$

and therefore

$$-X \subseteq -(\bigcap_{G\in J_X} IC(\bigcup_{a\in G} |\neg a|)).$$

This completes the proof that X is a meet of joins of images of elements of B. □

Hence given any rich and separative set of filters $\mathscr{C}$ over a Boolean algebra B, the Alexandroff topology on $(\mathscr{C}, \subseteq)$ gives rise to a doubly-dense extension of B. However, this completion may not be in general the canonical extension of B. Indeed, in order to obtain a completion isomorphic to the canonical extension of B, we need to strengthen the assumptions on $\mathscr{C}$:

Definition 4.3.6. Let B a Boolean algebra. A set $\mathscr{C}$ of filters over B is called *cofinal* if for every filter $F \subseteq B$, there is a filter $F' \in \mathscr{C}$ such that $F \subseteq F'$.

Note that it is straightforward to see that if a set $\mathscr{C}$ of filters is cofinal, then it is also rich and separative. This observation motivates the following result:

Lemma 4.3.7 (Representation for canonical extensions)**.** *Let B be a Boolean algebra, and $\mathscr{C}$ a cofinal set of filters over B. Then $(\mathsf{RO}_{\tau_\leq}(\mathscr{C}), |\cdot|)$ is isomorphic to the canonical extension of B.*

Proof. Recall that the canonical extension of B is the unique up to isomorphism doubly-dense and compact completion of B. Now if $\mathscr{C}$ is cofinal, hence separative and rich, then by Lemma 4.3.5, $(\mathsf{RO}_{\tau_\leq}(\mathscr{C}), |\cdot|)$ is a doubly-dense completion of B. Hence we only have to prove that it is also compact in the sense that for every $X, Y \subseteq B$, if

$$\bigcap_{a\in X} |a| \subseteq IC(\bigcup_{b\in Y} |b|,$$

[3]the proof here is a straightforward adaptation of [39]

there exist finite subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that

$$\bigwedge X' \leq \bigvee Y'.$$

To see this, assume that X and Y are such that $\bigwedge X' \nleq \bigvee Y'$ for any $X' \subseteq X, Y' \subseteq Y$ finite. Then let $X^\wedge$ and $Y^\wedge$ be the closures of X and Y under finite meets respectively. We claim that

$$F = \uparrow\{a \wedge \neg b \; ; \; a \in X^\wedge, b \in Y^\wedge\}$$

is a filter. Clearly, F is an upset and is closed under finite meets. To see that $0 \notin F$, assume this is the case. Then there are $x_1, ..., x_n \in X$ and $y_1, ..., y_m \in Y$ such that

$$x_1 \wedge ... \wedge x_n \wedge \neg y_1 \wedge ... \wedge \neg y_m \leq 0.$$

But then we have that

$$x_1 \wedge ... \wedge x_n \wedge \neg(y_1 \vee ... \vee y_m) \leq 0,$$

which means that

$$\bigwedge_{1 \leq i \leq n} x_i \leq \bigvee_{1 \leq j \leq m} y_j,$$

contradicting our assumption. Hence F is a filter, and since $\mathscr{C}$ is cofinal, there exists $F' \in C$ such that $F \subseteq F'$. But then, clearly, $X \subseteq F'$ and for any $y \in Y$, $\neg y \in F'$, which means that $F' \in CI(\bigcap_{b \in Y} -|b|)$. Hence

$$\bigcap_{a \in X} |a| \nsubseteq IC(\bigcup_{b \in Y} |b|),$$

which completes the proof. $\square$

Lemma 4.3.7 shows that cofinality and separativeness are sufficient for a set $\mathscr{C}$ of filters over a Boolean algebra B to give rise to the canonical extension of B. It is also worth noting that the fact that $\mathscr{C}$ is rich is not sufficient for it to give rise to the canonical extension: as an example, the set of Q-filters defined in the previous section is a rich and separative class, but, given a Boolean algebra A and a set Q of meets in B, the algebra of regular opens of the canonical Q-space of A is not in general isomorphic to the canonical extension, since it may not be compact: for example, if there is some $\bigvee X$ such that $\bigwedge X \in Q$, it may be the case that $\bigvee X \nleq \bigvee X'$ for any finite $X' \subseteq X$, and yet we have that

$$|\bigvee X| \subseteq IC(\bigcup_{x \in X} |x|$$

since all filters in $\mathscr{C}$ preserve $\bigvee X$. Moreover, we can also retrieve from Lemma 4.3.7 the standard characterization of the canonical extension of a Boolean:

Corollary 4.3.8. *Let B be a Boolean algebra. Then the powerset of the dual space of B is isomorphic to its canonical extension.*

Proof. Let $\mathscr{C}^{\mathfrak{U}}$ be the set of all ultrafilters over B. Then by the Ultrafilter Theorem, $\mathscr{C}^{\mathfrak{U}}$ is cofinal. Moreover, the powerset of the dual space of B is exactly $(\mathsf{RO}_{\tau_\leq}(\mathscr{C}^{\mathfrak{U}}), |\cdot|)$, since the Alexandroff topology induced by $(\mathscr{C}^{\mathfrak{U}}, \subseteq)$ is discrete. Hence, by Lemma 4.3.7, $(\mathsf{RO}_{\tau_\leq}(\mathscr{C}^{\mathfrak{U}}), |\cdot|)$ is a doubly-dense and compact completion of B. $\square$

We conclude this section by bridging the gap between the Stone topology and the Alexandroff topology, and giving sufficient conditions on $\mathscr{C}$ for the two topologies to coincide on the algebra of regular open sets they induce.

Definition 4.3.9. Let B be a Boolean algebra. A set $\mathscr{C}$ of filters over B is called *normal* if for all $F \in \mathscr{C}$ and every existing meet $\bigwedge X \in B$, F preserves $\bigwedge X$.

Lemma 4.3.10. *Let B be a Boolean algebra, and $\mathscr{C}$ a normal, separative and rich set of filters over B. Then both $\mathsf{RO}_{\tau_S}(\mathscr{C})$ and $\mathsf{RO}_{\tau_\leq}(\mathscr{C})$ are isomorphic to the MacNeille completion of B if one of the two following conditions is met:*

1. *Every filter in $\mathscr{C}$ is principal;*
2. *B is a complete Boolean algebra.*

Proof. By Lemma 4.3.3 above, since $\mathscr{C}$ is separative, it follows that $\mathsf{RO}_{\tau_S}(\mathscr{C})$ is isomorphic to the MacNeille completion of B. Moreover, since $\mathscr{C}$ is also rich, $\mathsf{RO}_{\tau_\leq}(\mathscr{C})$ is also a doubly-dense extension of B. In particular, this means that for any $X \in \mathsf{RO}_{\tau_\leq}(\mathscr{C})$, there exist stwo sets $G_X, J_X \subseteq \mathscr{P}(\mathscr{C})$ such that

$$X = IC(\bigcup_{G \in G_X} \bigcap_{a \in G} |a|)$$

and

$$X = \bigcap_{J \in J_X} IC(\bigcup_{a \in J} |\neg a|).$$

We claim that if condition 1 or 2 holds, then for any $F \in \mathscr{C}$, we have that

$$\bigcap_{a \in F} |a| = |b|$$

and

$$IC(\bigcup_{a \in F} |\neg a|) = |c|$$

for some $b, c \in B$. This is enough to conclude that $\mathsf{RO}_{\tau_\leq}(\mathscr{C})$ is a dense completion of B, and hence is isomorphic to the MacNeille completion of B.

1. Let $F \in C$, and let a_F be the element of B that generates F. Then for any $G \in \mathscr{C}$, $F \subseteq G$ iff $a_F \in G$. Hence
$$\bigcap_{a \in F} |a| = |a_F|.$$
Moreover, we claim that
$$|\neg a_F| = IC(\bigcup_{a \in F} |\neg a|).$$
The left-to-right direction is obvious since $a_F \in F$. For the converse, assume that $\neg a_F \notin G$ for some $G \in \mathscr{C}$. Then, since $\mathscr{C}$ is rich, there exists $G' \in C$ such that $G \cup \{a_F\} \subseteq G'$. But then, since a_F generates F, it follows that $F \subseteq G'$. Hence for any $a \in F$ and any $G^* \supseteq G'$, $\neg a \notin G^*$. Hence $G \in -IC(\bigcup_{a \in F} |\neg a|$, which completes the proof.

2. If B is a complete Boolean algebra, then for any $F \in \mathscr{C}$, $\bigwedge F$ exists. Moreover, since $\mathscr{C}$ is normal, for any $F \in \mathscr{C}$, $\bigwedge F \in F$. Hence every filter in $\mathscr{C}$ is principal, and the claim follows from 1.

□

It is worth noting that Lemma 4.3.10 has the following corollary that we will generalize to the setting of Heyting algebras in chapter 6.

Corollary 4.3.11 (Representation for complete Boolean algebras)**.** *Let B be a complete Boolean algebra. Then B is isomorphic to the algebra of regular open sets of some Alexandroff space.*

Proof. Let $\mathscr{C}$ be any normal, separative and rich set of filters over B, such as, for example, the set $\mathscr{C}^{\mathfrak{P}}$ of all principal filters over B. Then by Lemma 4.3.10, $\mathsf{RO}_{\tau_{\leq}}(\mathscr{C})$ is isomorphic to the MacNeille completion of B. But since B is complete, it is isomorphic to its MacNeille completion. Hence B is isomorphic to $\mathsf{RO}_{\tau_{\leq}}(\mathscr{C})$. □

We conclude with a table that summarizes under which conditions on $\mathscr{C}$ and τ the algebra of regular opens $\mathsf{RO}_{\tau}(\mathscr{C})$ yields a specific type of completion:

	τ is the Alexandroff topology	τ is the Stone topology
$\mathscr{C}$ is separative	-	MacNeille Completion
$\mathscr{C}$ is rich and separative	Doubly-dense extension	MacNeille Completion
$\mathscr{C}$ is cofinal	Canonical Extension	MacNeille Completion
$\mathscr{C}^{\mathfrak{P}}$ (all principal filters)	MacNeille Completion	MacNeille Completion

Table 4.1: Properties of $\mathsf{RO}_{\tau}(\mathscr{C})$

4.4 Generalizations of Tarski's Lemma

In this section, we consider two straightforward generalizations of Tarski's Lemma to BAO's and Heyting algebras. We show that although the proof goes through without restrictions in the first case, Tarski's Lemma for a Heyting algebra A is equivalent to the validity of the Kuroda Axiom on A in some precise sense.

4.4.1 Boolean Algebras with Operators

Since every BAO B is a Boolean algebra, Tarski's lemma trivially holds on B. However, recall that the main interest of Tarski's Lemma is to show that there are 'enough' good filters, or, to phrase things in the terminology of the previous section, that the set of Q-filters over any Boolean algebra is separative. However, in the case of a BAO B, we need some extra condition on a set of filters $\mathscr{C}$ over B in order to define a completion of B that also preserves the operator $\Box$:

Definition 4.4.1. Let B be a BAO, and $\mathscr{C}$ a set of filters over B. Then $\mathscr{C}$ satisfies the $\Box$ condition if for every $a \in B$ and every $F \in \mathscr{C}$, if $\Box a \notin F$, then there exists $F' \in \mathscr{C}$ such that $a \notin F'$ and $F^{\Box} \subseteq F'$, where $F^{\Box} = \{b \in B \,;\, \Box b \in F\}$.

Note that it is straightforward to see that the fact that the set of ultrafilters of a LT_K satisfies the $\Box$-condition plays a key role in the construction of the canonical Kripke frame of K. Here, the generalization of Tarski's lemma we prove will ensure that the set of all Q-filters of a BAO also satisfies the square-condition, provided certain conditions hold on Q. The following lemma makes clear why the $\Box$-condition is an important property of sets of filters over BAO's.

Lemma 4.4.2. *Let B be a BAO and $\mathscr{C}$ an upward-closed separative set of filters over B that satisfies the $\Box$-condition. Then $(\mathscr{C}, \tau_{\leq})$ induces a BAO $(\mathsf{RO}_{\tau_{\leq}}(\mathscr{C}), \blacksquare)$ such that $|\cdot| : B \to (\mathsf{RO}_{\leq_\tau}(\mathscr{C}))$ is an injective BAO homomorphism.*

Proof. Notice first that if $\mathscr{C}$ is upward-closed then it is also rich: if $F \in \mathscr{C}$ and $a \notin F$, then $F' = \uparrow\{b \wedge \neg a \,;\, b \in F\}$ is a proper filter extending F, and hence $F' \in \mathscr{C}$. Hence by Lemma 4.3.5, $(\mathsf{RO}_{\tau_{\leq}}, |\cdot|)$ is a doubly-dense extension of B. Now let $\blacksquare : \mathscr{P}(\mathscr{C}) \to \mathscr{P}(\mathscr{C})$ be defined as $\blacksquare X = \{F \in C \,;\, \forall F' \in \mathscr{C},\, F^{\Box} \subseteq F' \text{ implies } F' \in X\}$. We first check that for any $X \in \mathsf{RO}_{\tau_{\leq}}(\mathscr{C})$, we have that $\blacksquare X \in \mathsf{RO}_{\tau_{\leq}}(\mathscr{C})$. In fact, we claim that for any $X \in \mathsf{RO}_{\tau_{\leq}}(\mathscr{C})$,

$$\blacksquare X = \bigcap_{G \in -C(X)} IC(\bigcup_{a \in G} |\Box \neg a|),$$

which is clearly sufficient.

- For the left-to-right direction, assume $F \in \Box X$ and let $G \in -C(X)$ such that for any $a \in G$, $\Box \neg a \notin F$. Then we claim that

  $$F' = \uparrow\{a \wedge b \,;\, a \in G, b \in F^{\Box}\}$$

  is a proper filter. Clearly, F' is an upset, and since both G and $F^{\Box}$ are closed under finite meets (since if $b, b' \in F^{\Box}$, then so is $b \wedge b'$, because $\Box(b \wedge b') = \Box b \wedge \Box b' \in F$), F' is also closed under finite meets. Now assume that $a \wedge b \leq 0$ for some $a \in G, b \in F^{\Box}$. Then $b \leq \neg a$, which means that $\Box b \leq \Box \neg a$. But then it follows that $\Box(\neg a) \in F$, since $\Box(b) \in F$, contradicting our assumption. Hence F' is a filter. Now since $G \cup F'$ and $\mathscr{C}$ is upward-closed, $F' \in \mathscr{C}$. Moreover, since $F^{\Box} \subseteq F'$, we also have that $F' \in X$. Hence $G \in C(X)$, a contradiction. Hence for any $G \in -C(X)$, we have that

  $$F \in \bigcup_{a \in G} |\Box \neg a| \subseteq IC(\bigcup_{a \in G} |\Box \neg a|),$$

  which means that

  $$\blacksquare X \subseteq \bigcap_{G \in -C(X)} IC(\bigcup_{a \in G} |\Box \neg a|).$$

- For the right-to-left direction, assume $F \notin \blacksquare X$. Then there exists G such that $F^{\Box} \subseteq G$ and $G \notin X$, which means, since X is regular open, that there exists $G' \supseteq G$ such that $G' \in -C(X)$. Now let

  $$F' = \uparrow\{b \wedge \Diamond a \,;\, b \in F, a \in G'\}$$

  . We claim that F' is a filter. To see that F' is a closed under finite meets, assume there are $a, a' \in G'$, $b, b' \in F$, and $c, c' \in B$ such that $b \wedge \Diamond a \leq c$ and $b' \wedge \Diamond a' \leq c'$. Then

  $$(b \wedge b') \wedge (\Diamond a \wedge \Diamond a') \leq c \wedge c'.$$

  Now $b \wedge b' \in F$ and $a \wedge a' \in G'$, and since, by monotonicity of $\Diamond$,

  $$\Diamond(a \wedge a') \leq \Diamond(a) \wedge \Diamond(a'),$$

by monotonicity of $\Diamond$, it follows that

$$(b \wedge b') \wedge \Diamond(a \wedge a') \leq c \wedge c',$$

and hence $c \wedge c' \in F'$. Moreover, assume that $b \wedge \Diamond a \leq 0$ for some $b \in F$, $a \in G'$. Then

$$b \leq \neg\Diamond a = \Box\neg a,$$

which means that $\Box\neg a \in F$. But since $F^{\Box} \subseteq G \subseteq G'$, this implies that $\neg a \in G'$, a contradiction.
Therefore F' is a filter, and since $\mathscr{C}$ is upward-closed and $F \subseteq F'$, we have that $F' \in \mathscr{C}$. But clearly, for any $F^* \supseteq F'$,

$$F^* \in \bigcap_{a \in G'} -|\Box\neg a|.$$

This means that

$$F \in CI(\bigcap_{a \in G'} -|\Box\neg a|)$$

for some $G' \in -C(X)$, and hence

$$-\blacksquare X \subseteq - \bigcap_{G \in -C(X)} IC(\bigcup_{a \in G} |\Box\neg a|),$$

which completes the proof of the claim.

Hence $\mathsf{RO}_{\tau_{\leq}}(\mathscr{C})$ is closed under $\blacksquare$. Moreover, it is easy to see that $(\mathsf{RO}_{\tau_{\leq}}(\mathscr{C}), \blacksquare)$ is a BAO. Notice first that $\blacksquare$ is monotonic, which implies that $\blacksquare(X \cap Y) \subseteq \blacksquare X \cap \blacksquare Y$. For the converse, simply notice that if $F \in \blacksquare X \cap \blacksquare Y$, then $G \in X$ and $G \in Y$ for any $G \in \mathscr{C}$ such that $F^{\Box} \subseteq G$. Hence $F \in \blacksquare(X \cap Y)$. Finally, it follows immediately from the definition of $\blacksquare$ that $\blacksquare\mathscr{C} = \mathscr{C}$.

In order to complete the proof, we therefore only have to check that $|\cdot|$ preserves $\Box$, i.e. that $\blacksquare|a| = |\Box a|$ for any $a \in B$. Clearly, for any $F \in \mathscr{C}$, if $\Box a \in F$, then for any filter F' such that $F^{\Box} \subseteq F'$, $F' \in |a|$. Hence $|\Box a| \subseteq \blacksquare|a|$. For the converse, note that if $\Box a \notin F$, then since $\mathscr{C}$ satisfies the $\Box$-condition, there exists $F' \in C$ such that $a \notin F'$ and $F^{\Box} \subseteq F'$. But this immediately implies that $F \notin \blacksquare|a|$. Hence $|\cdot|$ is an injective BAO-homomorphism from $(B, \Box)$ into $(\mathsf{RO}_{\tau_{\leq}}(\mathscr{C}), \blacksquare)$. $\Box$

The previous lemma makes clear the reasons why a generalization of Tarski's Lemma to BAO's would be of interest. Indeed, from Tarski's Lemma for Boolean algebras, we know already that the set of Q-filters over a Boolean algebra B is upward-closed and separative. Hence if we can also prove that it satisfies the $\Box$-condition, then by Lemma 4.4.2 we would have a general way of embedding B into the algebra of regular open sets of some topological space in a way that preserves both the $\Box$-operator and all the meets in Q. In other words, we could define a semantics for KL based on possibility spaces. In the remainder of this section, we show how this result can be achieved by imposing some natural conditions on Q. We begin by recalling the following definition from the previous chapter:

Definition 4.4.3. Let B be a BAO. A meet $\bigwedge X$ existing in B is a *Barcan* meet if $\Box \bigwedge X = \bigwedge\{\Box x \,;\, x \in X\}$.

We can now state and prove the key lemma that will be used in our generalization of Tarski's lemma.

Lemma 4.4.4. *Let B be a BAO, $a \in B$, $\bigwedge X$ a meet existing in B such that $\bigwedge(a \to X) = \bigwedge\{a \to x\ ;\ x \in X\}$ is a Barcan meet, and F a filter that decides $\bigwedge\{\Box(a \to x)\ ;\ x \in X\}$. Then either $b \wedge a \leq \bigwedge X$ for some $b \in F^{\Box}$, or there exists $x \in X$ such that $b \wedge a \nleq x$ for all $b \in F^{\Box}$.*

Proof. Let a be any element of B, and $\bigwedge X$ a meet existing in B such that $a \to \bigwedge X$ is a Barcan meet decided by F. Assume that for any $x \in X$, there exists $b \in F^{\Box}$ such that $b \wedge a \leq x$. Then for any $x \in X$, there exists $b_x \in F^{\Box}$ such that $b_x \leq a \to x$. This implies that $\Box b_x \leq \Box(a \to x)$, and therefore that $\Box(a \to x) \in F$ for any $x \in X$. Since F decides $\bigwedge \Box(a \to X)$, $\bigwedge \Box(a \to X) \in F$, and hence

$$\Box(a \to \bigwedge X) = \Box \bigwedge (a \to X) = \bigwedge \Box(a \to X) \in F$$

since $\bigwedge(a \to X)$ is a Barcan meet. But this means that $(a \to \bigwedge X) \in F^{\Box}$, and hence there exists $b \in F^{\Box}$ such that $b \wedge a \leq \bigwedge X$.

$\Box$

Lemma 4.4.4 motivates the following definition, which is equivalent to the one in Corollary 3.4.12 and provides the adequate conditions on Q needed to prove Tarski's Lemma for BAO's.

Definition 4.4.5. Let B be a BAO. A set of meets Q existing in B is $(\Box, \to)$-*complete* if for every $a \in B$ and every $\bigwedge X \in Q$, $\bigwedge \Box(a \to X) \in Q$.

Lemma 4.4.6 (Tarski's Lemma for BAO's)**.** *Let B be a BAO and Q a countable $\Box \to$-complete set of Barcan meets in B. For every Q-filter F over B and every $a \in B$, if $\Box a \notin F$, then there exists a Q-filter F' such that $a \notin F'$ and $F^{\Box} \subseteq F'$.*

Proof. Assume F is a Q-filter such that $\Box a \notin F$, and let $\{X_n\}_{n \in \mathbb{N}}$ be an enumeration of Q. We defined a countable descending chain of elements $\{a_n\}_{n \in \mathbb{N}}$ of B as follows:

- $a_0 = \neg a$
- $a_{n+1} = a_n \wedge \bigwedge X_n$ if there exists $b \in F^{\Box}$ such that $b \wedge a_n \leq \bigwedge X$, and $a_{n+1} = a_n \wedge \neg x_n$ for some $x_n \in X_n$ such that $b \wedge a_n \nleq x_n$ for every $b \in F^{\Box}$ otherwise.

Note that Lemma 4.4.4 and the conditions on Q ensure that this sequence is well-defined. We now claim that $F' = \uparrow \{b \wedge a_n\ ;\ b \in F^{\Box} n \in \mathbb{N}\}$ is the required filter. We first check that F' is indeed a filter. F' is obviously an upset. For closure under finite meets, assume there is $b, b' \in F^{\Box}$ and a_n, a_m such that $b \wedge a_n \leq c$ and $b' \wedge a_m \leq c'$ for some $c, c' \in F'$. Then we have that

$$(b \wedge a_n) \wedge (b' \wedge a_m) = (b \wedge b') \wedge (a_n \wedge a_m) \leq c \wedge c'.$$

Recall that if $b, b' \in F^{\Box}$, then so is $b \wedge b'$. Moreover, since $\{a_n\}_{n \in \mathbb{N}}$ is a descending chain, $a_n \wedge a_m = a_{max(m,n)}$. Hence

$$(b \wedge b') \wedge a_{max(m,n)} \leq c \wedge c',$$

which means that $c \wedge c' \in F'$. Finally, we show by induction that for any $n \in \mathbb{N}$, $b \wedge a_n \nleq 0$ for any $b \in F^{\Box}$. Note that the induction step is straightforward by construction of the sequence $\{a_n\}_{n \in \mathbb{N}}$. For $n = 0$, assume that $b \wedge \neg a \leq 0$ for some $b \in F^{\Box}$. This means that $1 \leq b \to a$, and hence $\Box(b \to a) \in F$. But since $\Box(b \to a) \leq \Box b \to \Box a$ and $\Box b \in F$, it follows that $\Box a \in F$, contradicting our assumption.
Hence F' is a proper filter, and, by construction, $F^{\Box} \subseteq F'$, $a \notin F'$, and F' decides every meet in Q. $\Box$

We can now use this generalization of Tarski's Lemma to prove the analogue of Theorem 4.2.5 for BAO's:

Theorem 4.4.7. *Let $(B, \Box)$ be a BAO and Q a countable $(\Box, \to)$-complete set of Barcan meets existing in B. Then $(B, \Box)$ has a Q-completion that preserves $\Box$.*

Proof. Let $(X_Q, \tau_\leq, \blacksquare)$ be the topological space where X_Q is the set of all Q-filters over B, $\tau_\leq$ is the Alexandroff topology induced by the inclusion ordering on X_Q, and $\blacksquare : \mathscr{P}(X_Q) \to \mathscr{P}(X_Q)$ defined as $\blacksquare U = \{F \in X_Q \; ; \; \forall \; G \in X_Q, F^\Box \subseteq G \Rightarrow G \in U\}$. Then by Theorem 4.2.5, $(\mathsf{RO}_{\tau_\leq}(X_Q), |\cdot|)$ is a Q-completion of B. Moreover, by definition, X_Q is upward-closed, and it is also a separative set of filters by Lemma 4.2.3 (Tarski's Lemma). Moreover, by Lemma 4.4.6, X_Q also satisfies the $\Box$-condition. Hence by Lemma 4.4.2, $|\cdot|$ also preserves $\Box$, and hence $((\mathsf{RO}_{\tau_\leq}(X_Q), \blacksquare), |\cdot|)$ is a Q-completion that also preserves $\Box$. $\Box$

Finally, we show how we can define a semantics for first-order modal logic KL as a straightforward consequence of Theorem 4.4.7.

Definition 4.4.8. Let $LT^{\mathfrak{L}'}_{KL}$ be the Lindenbaum-Tarski algebra of KL, where $\mathfrak{L}' = \mathfrak{L} \cup \{\Box\}$ for some first-order language $\mathfrak{L}$. A *first-order modal possibility $\mathfrak{L}'$-model* is a tuple $(\mathfrak{X}, R) = (X, \leq, R, \{(D_i, J_i\}_{i \in X}, \{f_{ij}\}_{i \leq j \in X}, \{a_i\}_{i \in X}, V_\alpha)$ such that $\mathfrak{X}$ is a first-order possibility $\mathfrak{L}$-model, R is a binary relation on X such that for any $U \in RO_{\tau_\leq}(X)$, $U^R = \{y \in X \; ; \; yRx \Rightarrow x \in U\} \in RO_{\tau_\leq}(X)$, and $V_\alpha((\Box\phi)^\vdash) = V^R_\alpha(\phi^\vdash)$ for any $\phi \in \mathfrak{L}'$. A formula $\phi \in \mathfrak{L}'$ is valid on $(\mathfrak{X}, R)$ if $V_\beta(\phi) = X$ for any system of assignments $\{\beta_i\}_{i \in X}$.

Theorem 4.4.9. *KL is sound and complete with respect to the set of all first-order modal possibility models, i.e. for any first-order modal language $\mathfrak{L}$ and any $\phi \in \mathfrak{L}$, $KL \vdash \phi$ iff ϕ is valid on any first-order modal possibility $\mathfrak{L}$-model.*

Proof. Soundness follows from the fact that for any first-order modal possibility $\mathfrak{L}$-model $(\mathfrak{X}, R)$, $V_\alpha : LT^{\mathfrak{L}}_{KL} \to (\mathsf{RO}(X_{\tau_\leq}), \cdot^R)$ is a BAO-homomorphism by Definition 4.4.8. For completeness, consider $LT^{\mathfrak{L}}_{KL}$ and $Q_\forall = \{(\forall x \phi(x))^\vdash \; ; \; \phi \in \mathfrak{L}\}$. Then by Theorem 4.2.7, the first-order possibility term model $\mathfrak{L}$ constructed is such that $|\cdot| : LT^{\mathfrak{L}}_{K_\forall} \to \mathsf{RO}(X)$ is an injective Boolean homomorphism that preserves all meets in $Q_\forall$, since for any $\phi \in \mathfrak{L}'$, $V_\alpha(\phi^\vdash) = |\phi^\vdash|$, where $\mathfrak{L}' = \mathfrak{L} \setminus \Box$. Now let $R \subseteq X \times X$ be defined by FRF' iff $F^\Box \subseteq F'$. Then, since $Q_\forall$ is $(\Box, \to)$-complete, it is straightforward to see that, by Lemma 4.4.2 and Theorem 4.4.7, (X, R) is a first-order modal possibility model, and that for any $\phi \in \mathfrak{L}$, $V_\alpha(\phi^\vdash) = |\phi^\vdash|$. But then, for any such formula ϕ, if $K_\forall \nvdash \phi$, then $\top^\vdash \nleq \phi^\vdash$, and hence $V_\alpha(\phi^\vdash) \neq X$. This completes the proof. $\Box$

4.4.2 Tarski's Lemma and Kuroda's Axiom

As we saw in the previous section, the setting of BAO's allows us to prove a generalization of Tarski's Lemma that ensures that, given certain conditions on Q, the set of all Q-filters of any BAO enjoys a form of completeness in the sense it satisfies the $\Box$ condition. In this section, we show that the situation is very different in the setting of Heyting algebras. Indeed, we prove that Tarski's Lemma for a Heyting algebra A and a set Q of existing meets in A is equivalent to a property of all meets in Q that corresponds to the Kuroda axiom[4] for IPL.

Lemma 4.4.10. *For every Heyting algebra A such that for every $X \subseteq A$, if $\bigwedge X$ exists, then $\bigwedge \neg\neg X = \bigwedge\{\neg\neg x; x \in X\}$ also exists, and every countable set of meets Q, the following are equivalent:*

[4] For references on Kuroda's axiom, see [22] and [63]

1. *Tarski's Lemma holds on A for Q : for every $a \neq 0 \in A$, there exists a Q-filter F over A such that $a \in F$.*

2. *For every $X \in Q$, $\bigwedge \neg\neg X \leq \neg\neg \bigwedge X$.*

Proof.

- $1 \Rightarrow 2$: Assume that $\bigwedge \neg\neg X \nleq \neg\neg \bigwedge X$ for some $X \in Q$. This means by residuation that $\bigwedge \neg\neg X \wedge \neg \bigwedge X \nleq 0$. Hence, if Tarski's Lemma holds, $\bigwedge \neg\neg X \wedge \neg \bigwedge X$ belongs to some proper Q-filter F. Since F is proper, this means that for every $X' \in Q$,

$$\bigwedge \neg\neg X \wedge \neg \bigwedge X \wedge \bigwedge X' \nleq 0,$$

or

$$\bigwedge \neg\neg X \wedge \neg \bigwedge X \wedge \neg x' \nleq 0$$

for some $x' \in X'$. In particular, this holds for X' = X. But this is clearly a contradiction.

- $2 \Rightarrow 1$: Let $a \neq 0 \in A$ and $X \in Q$ In light of Lemma 4.2.3, in order to prove Tarski's Lemma, we only need to prove that $a \wedge \bigwedge X \nleq 0$ or $a \wedge \neg x \nleq 0$ for some $x \in X$. Assume for reductio that $a \wedge \bigwedge X \leq 0$ and that $a \wedge \neg x \leq 0$ for every $x \in X$. From the first assumption we have that $a \leq \neg \bigwedge X$. From the second one, it follows that $a \leq \neg\neg x$ for every $x \in X$, and hence, since $\bigwedge \neg\neg X$ exists by assumption, that $a \leq \bigwedge \neg\neg X$. Hence we have that

$$a \leq \neg \bigwedge X \wedge \bigwedge \neg\neg X$$

. Now since $X \in Q$, we have that

$$\bigwedge \neg\neg X \leq \neg\neg \bigwedge X,$$

which implies that

$$\neg \bigwedge X \wedge \bigwedge \neg\neg X \leq 0.$$

Hence $a = 0$, contradicting our assumption.

□

Theorem 4.4.11. *Let $LT_L^{\mathfrak{L}}$ be the Lindenbaum-Tarski algebra of some logic L stronger than Intuitionisitic Predicate Logic IPL in some language $\mathfrak{L}$, and let $Q = \{(\forall x \phi(x))^{\vdash}\ ;\ \phi \in \mathfrak{L}\}$. Then the following are equivalent:*

1. *Tarski's Lemma holds for $LT_L^{\mathfrak{L}}$, Q: for any $\phi^{\vdash} \in LT_L^{\mathfrak{L}}$, if $\phi^{\vdash} \neq 0$, then there exists a Q-filter over $LT_L^{\mathfrak{L}}$ that contains $\phi^{\vdash}$*

2. *Kuroda's axiom is valid in L: for any $\phi(x) \cdot \in LT_L^{\mathfrak{L}}$, $(\forall(x)\neg\neg\phi(x) \to \neg\neg\forall(x)\phi(x))^{\vdash} = \top^{\vdash}$*

Proof. This is a direct consequence of the previous lemma. □

An immediate consequence of Theorem 4.4.11 is that, by contrast with the case of classical logic, we cannot rely on Tarski's Lemma to prove the completeness of IPL with respect to some set of possibility, since Tarski's Lemma is not true for LT_{IPL}. Moreover, it is easy to see that even in the case of $IPL+$ Kuroda's axiom, a completeness proof for this system does not immediately follow from the work of this chapter for two reasons. First of all, by contrast with CPL, existential and universal quantifiers are not inter-definable in IPL, which means that we need to modify the notion of Q-filter in order to take into account arbitrary joins on top of arbitrary meets. Moreover, note that Tarski's Lemma entails that the set of Q-filters over a Boolean algebra is separative only because, in any Boolean algebra B, we have that for any $a, b \in B$, $a \nleq b$ iff $a \wedge \neg b \neq 0$. However, the left-t-right direction of this bi-conditional does not hold in general for Heyting algebras. This means that, if we are to generalize the ideas from this chapter to the intuitionistic setting, we have to substantially modify the statement of Tarski's Lemma. This is exactly the work we undertake in the next chapter, but this will first require a substantial detour via a generalization of possibility semantics to intuitionistic logic.

4.5 Conclusion of This Chapter

We conclude this chapter with a short summary of the most salient features of possibility semantics that we used in the first two sections in order to prove the completeness of CPC with respect to first-order possibility models. We hope that this list will give the reader a better grasp of the sense in which a generalization of possibility semantics is introduced in the next chapter.

1. For any topological space (X, τ), the map $IC : \mathsf{O}(X) \to \mathsf{RO}(X)$, which sends any open set to the interior of its closure, corresponds to the double negation nucleus on $\mathsf{O}(X)$. As a consequence, the regular open sets $\mathsf{RO}(X)$ of any topological space form a complete Boolean algebra (Theorem 4.1.2).

2. Given a Boolean algebra A, a possibility model $(X, \leq, V)$ for A is based on a poset $(X, \leq)$, and a Boolean homomorphism $V : A \to \mathsf{RO}(X)$, where $\mathsf{RO}(X)$ is the Boolean algebra of regular open sets of $(X, \tau_{\leq})$, the topological space obtained by taking the Alexandroff topology on $(X, \leq)$ (Definition 4.1.5).

3. Any Boolean algebra A is isomorphic to a subalgebra of $\mathsf{RO}(X)$ for some topological space (X, τ). This representation theorem, unlike the Stone representation Theorem, does not require any form of the axiom of choice, but involves the construction of a possibility model for A (Theorem 4.1.4).

4. As a consequence of the three previous items, CPC is sound and complete with respect to the set of all possibility models for LT_{CPC}, the Lindenbaum-Tarski algebra of CPC (Theorem 4.1.6).

5. For any Boolean algebra A, given a countable set Q of existing meets in A, A embeds into the regular open sets of some possibility space for A in such a way that the embedding preserves all meets in Q. This proof relies on Tarski's Lemma (Lemma 4.2.3), and presupposes the Axiom of Dependent Choice (DC)(Theorem 4.2.5).

6. For $\mathfrak{L}$ a first-order language, a first-order possibility model is determined by a Boolean homomorphism $V : LT^{\mathfrak{L}}_{CPL} \to \mathsf{RO}(X)$ which preserves all meets in $Q_{\forall}$. (Definition 4.2.6)

7. As a consequence of the two previous items, CPL is sound and complete with respect to the set of all first-order possibility models. (Theorem 4.2.7)

Chapter 5

Intuitionistic Possibility Spaces

In this chapter, we introduce *intuitionistic possibility spaces* (IP spaces) and *first-order IP spaces*, and give a complete semantics for IPL based on these spaces, thus generalizing the setting of possibility semantics to intuitionistic logic. In section 1, we introduce *refined bi-topological spaces* (IP-spaces) and *refined regular open sets*, which will play the same role as topological spaces and regular open sets for Boolean algebras. In section 2, we define the canonical possibility space of a Heyting algebra, and show how to embed any HA into the complete Heyting algebra of refined regular opens of its canonical possibility space. Section 3 is devoted to a refinement of this framework that allows for an embedding of Heyting algebras that also preserves a countable number of meets and joins. In particular, we propose generalizations of Tarski's Lemma to distributive lattices and Heyting Algebras, which we refer to as the Q-Lemma for DL and HA respectively. Moreover, we show how they yield a proof of the existence of Q-completions for distributive lattices and Heyting algebras. Finally, in section 4, we draw consequences from the main results of section 2 and 3 and define a new semantics for IPL based on these results.

5.1 Refined Bi-Topological Spaces

Recall that possibility semantics for classical logic relies on the fact that the IC (interior-closure) operator in any topological space (X, τ) corresponds to the double-negation nucleus on the Heyting algebra of open sets of X. In this section, we introduce refined bi-topological spaces and refined regular opens as a generalization of this fact.

Definition 5.1.1. A *refined bi-topological space* is a triple (X, τ_1, τ_2) such that τ_1 and τ_2 are topologies on X, and $\tau_1 \subseteq \tau_2$. We denote as I_i and C_i the interior and closure operators associated with τ_i for $1 \leq i \leq 2$. A set $U \subseteq X$ is called *refined regular open* iff $I_1 C_2(U) = U$.

The main result of this section is the following theorem:

Theorem 5.1.2. *Let (X, τ_1, τ_2) be a refined bi-topological space, and let O_1 be the Heyting algebra of open sets in τ_1. Then $I_1 C_2$ is a nucleus on O_1.*

Proof. We first prove that $I_1 C_2$ is a closure operator on O_1.

- Monotonicity is obivous, since both I_1 and C_2 are monotone.
- To see that $I_1 C_2$ is increasing, notice that if U is in τ_1, then $U \subseteq I_1(U)$. But $U \subseteq C_2(U)$, and hence by monotonicity of I_1, we have $U \subseteq I_1(U) \subseteq I_1 C_2(U)$.

- Finally, for idempotence, notice first that, for any $U \subseteq X$ since $\tau_1 \subseteq \tau_2$, it follows that $I_1(U) \subseteq I_2(U)$, and $C_2(U) \subseteq C_1(U)$. Moreover, it is well known that for any $U \in \tau_2$, $C_2I_2C_2(U) = C_2(U)$. Since $\tau_1 \subseteq \tau_2$, this also holds for any $U \in \tau_1$. Hence for any $U \in \tau_1$, $I_1C_2I_1C_2(U) \subseteq I_1C_2I_2C_2(U) = I_1C_2(U)$. The converse follows from the fact that I_1C_2 is increasing.

It remains to be proved that for any $U, V \in \tau_1$, $I_1C_2(U \cap V) = I_1C_2(U) \cap I_1C_2(V)$. As usual, the left-to-right direction follows from the monotonicity of I_1C_2. For the converse, note that if U is open in τ_1, it is also in τ_2, and then it is a general topological fact that $U \cap C_2(V) \subseteq C_2(U \cap V)$. Taking complements, C_1 and complements again, we obtain

$$-C_1 - (U \cap C_2(V)) \subseteq I_1C_2(U \cap V).$$

The left-hand-side is equal to $-C_1(-U \cap -C_2(V))$, and since closure distributes over unions and U is open in τ_1, we have

$$U \cap I_1C_2(V) \subseteq I_1C_2(U \cap V).$$

Applying this result twice to $I_1C_2(U) \cap V$, we get that

$$I_1C_2(U) \cap I_1C_2(V) \subseteq I_1C_2(I_1C_2(U) \cap V) \subseteq I_1C_2(I_1C_2(U \cap V))$$

which, by idempotence, yields the required result. □

Because of the general theory of nuclei on Heyting algebras, Theorem 5.1.2 has the following important consequence:

Corollary 5.1.3. *For any IP-space (X, τ_1, τ_2), the refined regular opens of X form a complete Heyting algebra $\mathsf{RO}_{12} = (RO_{12}(X), \cap, \vee, \to, \bigwedge, \bigvee)$, where $A \vee B = I_1C_2(A \cup B)$, $A \to B = I_1(-A \cup B)$, $\bigwedge_{i \in I} B_i = I_1(\bigcap_{i \in I} B_i)$ and $\bigvee_{i \in I} B_i = I_1C_2(\bigcup_{i \in I} B_i)$.*

It is now possible to see in which sense the previous construction generalizes the algebra of regular opens of a topological space. Note first that any topological space (X, τ) can be turned into an IP-space (X, τ_1, τ_2) where $\tau_1 = \tau_2 = \tau$. In this case, the refined regular opens in (X, τ_1, τ_2) are exactly the regular opens in (X, τ). However, it is not true in general that the refined regular opens of an IP-space form a Boolean algebra. Indeed, the regular opens of a topological space X form a Boolean algebra because for any regular open $U \subseteq X$, its complement $\neg U$ in $\mathsf{RO}(X)$ is defined as $-C(U)$. It then follows by proposition 3.3.2 that $U \cup \neg U$ is dense in X, which entails that $C(U \cup \neg U) = X$, and hence $U \vee \neg U = IC(U \cup \neg U) = X$. However, in the algebra of refined regular opens, $\neg U$ is defined as $I_1(-U) = -C_1(U)$. It follows that $U \cup \neg U$ is dense in X relative to τ_1, but not relative to τ_2. Hence it may not be the case in general that $C_2(U \cup \neg U) = X$, and, as a consequence, that $U \vee \neg U = X$. We will see in the next section that this is the reason why refined regular opens can play the same role for Heyting algebras as regular opens for Boolean algebras, as we will show how to embed any HA into the refined regular opens of an IP-space.

5.2 Canonical Intuitionistic Possibility Spaces

The main result of this section is to prove that every Heyting algebra can be given a concrete representation as a subalgebra of the algebra of refined regular opens of some refined bi-topological space. The main interest of this result is that, contrary to some other well-known representation theorems for DL and HA, such as Priestley's and Esakia's, this representation is entirely choice-free. The proof will be carried out for Heyting algebras, but only properties of distributive lattices will be used, and hence the result also holds for distributive lattices. We begin by defining the canonical IP-space of a Heyting algebra, based on pairs of filters and ideals. [1]. We then define a canonical map from a Heyting algebra into the algebra of refined regular opens of its canonical IP-space, and prove that it is an embedding.

Definition 5.2.1. Let A be a Heyting algebra. We denote by $\mathscr{F}$ and $\mathscr{I}$ respectively the set of filters and ideals of A. Let (F, I) be an ordered pair such that $F \in \mathscr{F}$ and $I \in I$. We define the following properties for such pairs:

- (F, I) is a *compatible* pair if $F \cap I = \emptyset$.
- (F, I) has the Right Meet Property (RMP) if for any $c \in A$, if there exists $a \in F$ and $b \in I$ such that $a \wedge c \leq b$, then $c \in I$.
- (F, I) has the Left Join Property (LJP) if for any $c \in A$, if there exists $a \in F$ and $b \in I$ such that $a \leq b \vee c$, then $c \in F$.

A technical motivation for our interest in the RMP and the LJP is that pairs with both properties are a generalization of prime filters in the following sense.

Lemma 5.2.2. *Let L be a distributive lattice and F a prime filter over L. Then the pair (F, F^c), where $F^c = L \setminus F$, has both the RMP and LJP.*

Proof. Recall that if F is a prime filter, then F^c is a prime ideal. Assume $a \in F$, $b \in F^c$ and $a \wedge c \leq b$. Then $a \wedge c \in F^c$, and since $a \in F$, it follows from the fact that F^c is prime that $c \in F^c$. Hence (F, F^c) has the RMP. Dually, assume $a \in F$, $b \in F^c$ and $a \leq b \vee c$. Then $b \vee c \in F$, and since F is prime, and $b \in F^c$ this means that $c \in F$. Hence (F, F^c) has the LJP. $\square$

On a more intuitive level, compatible pairs can be seen as partial information states providing some positive and negative information: for a compatible pair (F, I), every element in F can be interpreted as "what we know to be true" at partial state (F, I), while every element in I can be interpreted as "what we know to be false" at (F, I). A pair that has the RMP can be seen as *negatively complete* in the following sense: if we know that a is true and that b is false, then if $a \wedge c \leq b$, we know already that c cannot be true: for if it were, then b would also be true. Hence we can already conclude that c is false, i.e. $c \in I$. Dually, the LJP can be seen as a kind of positive completeness : if $a \leq b \vee c$ and we know already that a is true and b is false, then we can already conclude that c must be true, and hence c should belong to F. The following lemma makes this idea more precise.

Lemma 5.2.3. *For a filter F, an ideal I and $c \in A$, we write $F \wedge c$ for the set $\uparrow\{a \wedge c; a \in F\}$ and $I \vee c$ for the set $\downarrow\{b \vee c; b \in I\}$. It is well-known that for any $F \in \mathscr{F}$, $I \in I$ and $c \in A$, $F \wedge c$ is also a filter, $I \vee c$ is also an ideal, $F \cup \{c\} \subseteq F \wedge c$, and $I \cup \{c\} \subseteq I \vee c$. Moreover, the following holds for any F, I and c:*

i) If (F, I) is a compatible pair with the RMP, then $c \in I$ iff $(F \wedge c, I)$ is not a compatible pair.

ii) If (F, I) is a compatible pair with the LJP, then $c \in F$ iff $(F, I \vee c)$ is not a compatible pair.

Proof.

[1] Jung and Moshier [43] also work with pairs of positive and negative information, although their general framework is quite different

i) the left-to-right direction is obvious. For the converse, if $(F \wedge c, I)$ is not a pair, then there exists $a \in F, b \in I$ such that $a \wedge c \leq b$. But then, since (F, I) has the RMP by assumption, $c \in I$.

ii) The argument is exactly the same, using the LJP of (F, I) for the right-to-left direction. $\square$

The next lemma is of crucial importance, as it guarantees that compatible pairs that enojy both the RMP and LJP exist in sufficient number.

Lemma 5.2.4. *For every compatible pair (F, I), there exists compatible pairs (F', I') and (F'', I'') such that:*

i) $F \subseteq F'$, $I \subseteq I'$ and (F', I') has the RMP

ii) $F \subseteq F''$, $I \subseteq I''$ and F'', I'' has the LJP property.

Proof.

i) The proof is constructive. Let (F, I) be a compatible pair, and consider $I' = \downarrow\{c \,;\, a \wedge c \leq b$ for some $a \in F, b \in I\}$. We claim that (F, I') is a compatible pair and that $I \subseteq I'$, i.e.:

- I' is an ideal. Since I' is defined as a downset, we only need to check that if $c_1, c_2 \in I'$, then $c_1 \vee c_2 \in I'$. Assume that $c_1, c_2 \in I'$. Then there exists $a, a' \in F$, $b, b' \in I$ and $c, c' \in A$ such that $a \wedge c \leq b$, $a' \wedge c' \leq b'$, $c_1 \leq c$ and $c_2 \leq c'$. From this it follows that $(a \wedge a') \wedge c \leq b$, and $(a \wedge a') \wedge c' \leq b'$. Hence since A is distributive we have that
$$(a \wedge a') \wedge (c \vee c') = ((a \wedge a') \wedge c) \vee ((a \wedge a') \wedge c') \leq b \vee b'.$$
But $a \wedge a' \in F$ since F is a filter, and $b \vee b' \in I$ since I is an ideal, hence $c \vee c' \in I'$. Moreover, since $c_1 \vee c_2 \leq c \vee c'$ and I' is a downset, this yields the required result.
- $F \cap I' = \emptyset$. Assume there exists some $a \in F \cap I'$. Then there exists $c \in I'$, $a' \in F$ and $b \in I$ such that $a \leq c$ and $a' \wedge c \leq b$. But then, since F is filter, $a' \wedge a \in F$, and since $a' \wedge a \leq a' \wedge c \leq b$, it follows that $b \in F$, contradicting the assumption that (F, I) is a compatible pair.
- (F, I') has the RMP. Assume $a \wedge c \leq b$ for some $a \in F, b \in I'$. Then there exists $c' \in A, a' \in F$ and $b' \in I$ such that $b \leq c'$ and $a' \wedge c' \leq b'$. But from this it follows that
$$(a' \wedge a) \wedge c \leq a' \wedge (a \wedge c) \leq a' \wedge c \leq b'$$
Since F is a filter, $a \wedge a' \in F$, and hence $c \in I'$.

ii) Consider the set $F'' = \uparrow\{c \,;\, a \leq b \vee c$ for some $a \in F$, $b \in I\}$. The proof that $F \subseteq F''$ and that (F'', I) is a compatible pair with the LJP property is completely similar to that of Lemma 5.2.4 i) and is therefore left to the reader.

$\square$

The previous lemma guarantees that every compatible pair can be extended to a compatible pair that has the RMP and to a pair that has the LJP. However, we can actually achieve a stronger result, namely that every compatible pair extends to a compatible pair that has both the RMP and LJP. The proof of this result relies on the following important fact about distributive lattices, that will also play a crucial role in the next section.

Lemma 5.2.5. *Let A be a distributive lattice, and let $a, b, c \in A$. If $a \wedge c \leq b$ and $a \leq b \vee c$, then $a \leq b$.*

Proof. Assume that $a \wedge c \leq b$ and $a \leq b \vee c$. Then:

$$a \leq a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c) \leq (a \wedge b) \vee b \leq b.$$

□

We can now prove our main result about compatible pairs:

Lemma 5.2.6. *For any compatible pair (F, I), there exists a compatible pair (F^*, I^*) such that $F \subseteq F^*$, $I \subseteq I^*$ and (F^*, I^*) has both the RMP and LJP.*

Proof. Let (F, I) be a compatible pair. By Lemma 5.2.4 i), we know that (F, I) extends to a compatible pair (F, I') with the RMP, and by Lemma 5.2.4ii), (F, I') extends to a compatible pair (F'', I') with the LJP. We now claim that (F'', I') also has the LMP. To see this, assume that

$$a_1 \wedge c \leq b_1 \tag{1}$$

for some $a_1 \in F''$, $b_1 \in I'$, and $c \in A$. This means that there exists $a'_1 \in A$, $a_2 \in F$ and $b_2 \in I'$ such that $a'_1 \leq a_1$ and $a_2 \leq a'_1 \vee b_2$, and hence

$$a_2 \leq a_1 \vee b_2. \tag{2}$$

(1) and (2) imply that

$$a_1 \wedge (a_2 \wedge c) = a_2 \wedge (a_1 \wedge c) \leq b_1 \vee b_2 \tag{3}$$

and

$$a_2 \wedge c \leq (a_1 \vee b_2) \vee b_1 = a_1 \vee (b_1 \vee b_2). \tag{4}$$

By Lemma 5.2.5, this implies that

$$a_2 \wedge c \leq b_1 \vee b_2 \tag{5}$$

. Moreover, since b_1 and b_2 are in I', this means that $b_1 \vee b_2 \in I'$, and hence there exists $a_3 \in F$, $b_3 \in I$, and $c' \geq c$ such that $a_3 \wedge c' \leq b_3$, and hence

$$a_3 \wedge (b_1 \vee b_2) \leq b_3 \tag{6}$$

But (5) and (6) entail that

$$(a_3 \wedge a_2) \wedge c = a_3 \wedge (a_2 \wedge c) \leq b_3. \tag{7}$$

But both a_3 and a_2 belong to F, hence $a_3 \wedge a_2 \in F$, and $b_3 \in I$. Hence $c \in I'$, which completes the proof. □

In light of Lemma 5.2.6, for every compatible pair (F, I) we can always construct a compatible pair that extends both F and I and has both the RMP and LJP. From now on, we will call such pairs pseudo-complete pairs and, for every compatible pair (F,I), we will denote as $(F, I)^*$ the pseudo-complete pair constructed from (F, I) in Lemma 5.2.6.

For a Heyting algebra A, let X be the set of all pseudo-complete pairs on A. We define two relations $\leq_1$ (the filter inclusion ordering) and $\leq_2$ (the filter-ideal inclusion ordering) on $X \times X$:

- For any (F, I), let $(F', I'), (F, I) \leq_1 (F', I')$ iff $F \subseteq F'$;

- For any (F, I), let $(F', I'), (F, I) \leq_2 (F', I')$ iff $F \subseteq F'$ and $I \subseteq I'$.

It is straightforward to see that both $\leq_1$ and $\leq_2$ are partial orders, and that $\leq_2 \subseteq \leq_1$. Now let τ_1 and τ_2 be the upset topologies on X induced by $\leq_1$ and $\leq_2$ respectively. Notice that since $\leq_2 \subseteq \leq_1$, every upset in $\leq_1$ is also an upset in $\leq_2$, and hence $\tau_1 \subseteq \tau_2$. This allows for the following definition:

Definition 5.2.7. For a Heyting algebra A, the canonical intuitionistic possibility space (IP-space) of A is the refined bi-topological space (X, τ_1, τ_2), where X is the set of all pseudp-complete pairs (F, I), and τ_1 and τ_2 are the upset topologies induced respectively by the filter inclusion ordering and the filter-ideal inclusion ordering on X.

We can now prove the main result of this section, namely that every Heyting algebra A embeds into the algebra of refined regular opens $\mathsf{RO}_{12}(X)$ of its canonical IP-space (X, τ_1, τ_2). The proof proceeds in three steps: we first define a canonical map $|\cdot| : A \to \mathsf{RO}_{12}(X)$, then prove that it is a Heyting homomorphism, and finally that it is injective.

Definition 5.2.8. Let A be a Heyting algebra and (X, τ_1, τ_2) its canonical IP-space. We define the two following maps:

- the positive canonical map $|\cdot| : A \to \mathscr{P}(X)$ such that for $a \in A$, $|a| = \{(F, I) \ ; \ a \in F\}$,
- the negative canonical map $|\cdot|^- : A \to \mathscr{P}(X)$ such that for $a \in A$, $|a|^- = \{(F, I) \ ; \ a \in I\}$.

In order to make sure that $|\cdot|$ is a map from A to RO_{12}, we need the following lemma.

Lemma 5.2.9. *For every $a \in A$, $C_2(|a|) = -|a|^-$ and $C_1(|a|^-) = -|a|$*

Proof. Note first that it is easy to see that for any $a \in A$, $|a|$ is open in τ_1 (hence also in τ_2), and $|a|^-$ is open in τ_2. Moreover, since for every $(F, I) \in X$, we have $F \cap I = \emptyset$, it follows that $|a| \subseteq -|a|^-$ and $|a|^- \subseteq -|a|$. Hence it remains to be proved that $-|a|^- \subseteq C_2(|a|)$ and that $-|a| \subseteq C_1(|a|^-)$.

For the first inclusion, let (F, I) be a pseudo-complete pair such that $a \notin I$. Since (F, I) has the RMP, Lemma 5.2.3 i) implies that $(F \wedge a, I)$ is a compatible pair, and hence $(F \wedge a, I)^*$ is a pseudo-complete pair. But $(F, I) \leq_2 (F \wedge a, I)^*$, and since τ_2 is the upset topology induced by $\leq_2$, this means that $(F, I) \in C_2(|a|)$.

For the second inclusion, let (F, I) be a pseudo-complete pair such that $a \notin F$. Then clearly $(F, \downarrow a)$ is a compatible pair, which means that $(F, \downarrow a)^*$ is a pseudo-complete pair. But $(F, I) \leq_1 (F, \downarrow a)^*$, and since τ_1 is the upset topology induced by $\leq_1$, this means that $(F, I) \in C_1(|a|^-)$. □

Lemma 5.2.10. *For every $a \in A$, we have $I_1 C_2(|a|) = |a|$.*

Proof. For every $a \in A$, we have $I_1 C_2(|a|) = -C_1(-C_2(|a|))$. By Lemma 5.2.9, this means that

$$I_1 C_2(|a|) = -C_1(- - |a|^-) = -C_1(|a|^-) = - - |a| = |a|.$$

□

The previous lemma ensures that the canonical positive map $|\cdot|$ restricts to a map from A to $\mathsf{RO}_{12}(X)$. We now prove that $|\cdot|$ is also a Heyting homomorphism.

Lemma 5.2.11. *For every Heyting algebra A, the canonical map $|\cdot|$ from A into the algebra of refined regular open sets of its canonical IP-space has the following properties for any $a, b \in A$:*

1. $|1| = X, |0| = \emptyset$
2. $|a \wedge b| = |a| \cap |b|$
3. $|a \vee b| = I_1 C_2(|a| \cup |b|)$
4. $|a \to b| = I_1(-|a| \cup |b|)$

Proof.

1. For every $(F, I) \in X$, since F is a filter, we have $1 \in F$, which means that $(F, I) \in |1|$. Moreover, since I is an ideal, $0 \in I$, hence $0 \notin F$ since $F \cap I = \emptyset$. Hence $(F, I) \in -|0|$.

2. For any $(F, I) \in X$, we have $a \wedge b \in F$ iff $a \in F$ and $b \in F$. Hence $|a \wedge b| = |a| \cap |b|$.

3. Note first that for any $(F, I) \in X$, we have $a \vee b \in I$ iff $a \in I$ and $b \in I$. Hence $|a \vee b|^- = |a|^- \cap |b|^-$. Moreover, by Lemma 5.2.9,

   $$I_1 C_2(|a| \cup |b|) = I_1(C_2(|a|) \cup C_2(|b|)) = I_1(-|a|^- \cup -|b|^-) = I_1(-(|a|^- \cap |b|^-)) = I_1(-|a \vee b|^-)$$

   and hence

   $$I_1 C_2(|a| \cup |b|) = I_1(-(|a \vee b|^-)) = -C_1(|a \vee b|^-) = - - |a \vee b| = |a \vee b|.$$

4. Note first that for any pair (F, I), if $a \to b \in F$ and $a \in F$, then since $a \wedge a \to b \leq b$, it follows that $b \in F$. Hence $|a \to b| \subseteq -|a| \cup |b|$. Since $|a \to b|$ is open in τ_1, we only have to show that $I_1(-|a| \cup |b|) \subseteq |a \to b|)$. Let (F, I) be a pair such that $a \to b \notin F$. We claim that $(F \wedge a, \downarrow b)$ is a compatible pair. Indeed, if $F \wedge a \cap \downarrow b \neq \emptyset$, then there is some $c \in F$ such that $c \wedge a \leq b$. But then $c \leq a \to b$, which means that $a \to b \in F$, contradicting our assumption. Hence by Lemma 5.2.6, $(F \wedge a, \downarrow b)$ extends to a pseudo-complete pair $(F \wedge a, \downarrow b)^*$. Clearly, $(F, I) \leq_1 (F \wedge a, \downarrow b)^*$, and, moreover, $(F \wedge a, \downarrow b)^* \in |a| \cap -|b|$. Hence $(F, I) \in C_1(-(|a| \cup |b|)) = -I_1(|a| \cup |b|)$. By contraposition, this means that $I_1(-|a| \cup |b|) \subseteq |a \to b|)$, which completes the proof.

□

Corollary 5.2.12. *For every Heyting algebra A, the canonical map $|\cdot|$ from A into the algebra of refined regular open sets of its canonical IP-space is a Heyting homomorphism.*

Proof. Immediate from Lemma 5.2.11 □

The final step in proving the main result of this section is to prove that the canonical positive embedding is injective:

Lemma 5.2.13. *For every Heyting algebra A, the canonical map $|\cdot|$ from A into the algebra of refined regular open sets of its canonical IP-space is injective, i.e., for any $a, b \in A$, $a \leq b$ iff $|a| \subseteq |b|$.*

Proof. Assume $a \leq b$. Then for any $(F, I) \in X$, if $a \in F$, then also $b \in F$, since F is a filter. Hence $|a| \subseteq |b|$. For the converse, note that if $a \not\leq b$, then $(\uparrow a, \downarrow b)$ is a compatible pair. By Lemma 5.2.3, this means that there exists a pseudo-complete pair (F, I) such that $a \in F$ and $b \in I$. But then $(F, I) \in |a| \cap -|b|$, which means that $|a| \not\subseteq |b|$. □

Theorem 5.2.14 (Representation theorem for Heyting algebras)**.** *Every Heyting algebra is isomorphic to a subalgebra of the algebra of refined regular open sets of its canonical IP-space.*

Proof. By Lemmas 5.2.11 and 5.2.13, for any Heyting algebra A with canonical IP-space X, the positive canonical map $|\cdot| : A \to \mathsf{RO}_{12}(X)$ is an embedding. Hence the image of $|\cdot|$ is a subalgebra of $\mathsf{RO}_{12}(X)$ isomorphic to A. □

As noted above, Theorem 5.2.14 is stated for Heyting algebras, but also holds for distributive lattices, since no specific property of Heyting algebras was used in the proof. Moreover, one of the main interest of Theorem 5.2.14 is that it provides a representation theorem for DL and HA that is entirely choice-free, by contrast with more standard constructions such as the Priestley space of a distributive lattice and the Esakia space of a Heyting algebra. This is an important fact since it will allow us to give a proof that every DL (resp. HA) embeds into a complete DL (resp. HA) that preserves countably many meets and joins using only the Axiom of Dependent Choice, rather than the Prime Filter Theorem.

5.3 Q-Completions for Distributive Lattices and Heyting Algebras

In this section, we introduce *Q-completions* for distributive lattices and Heyting algebras, and prove that every DL (resp. HA) embeds into a Q-completion based on the algebra of refined regular opens of its IP-space. We first prove this result for distributive lattices, and then adapt the proof for Heyting algebras. In each case, the key result is a version of Tarski's Lemma for DL and HA, that we call the Q-Lemma.

5.3.1 Q-Lemma and Q-Completions for Distributive Lattices

Definition 5.3.1 (Q-pair)**.** Let A be a distributive lattice, and let Q_J, Q_M be two countable sets of joins and meets in A such that for every $\bigvee X \in Q_J$, $\bigvee X$ exists and is distributive, and for every $\bigwedge Y \in Q_M$, $\bigwedge Y$ exists and is distributive. A *Q-pair* on A is a pair (F, I) such that:

- F is a filter on A, I is a filter on A and $F \cap I = \emptyset$;
- for every $\bigvee X \in Q_J$, either $\bigvee X \in I$, or $x \in F$ for some $x \in X$;
- for every $\bigwedge Y \in Q_M$, either $\bigwedge Y \in F$, or $y \in I$ for some $y \in Y$.

The main result below for distributive lattices will be that there are "enough" such Q-pairs. In order to prove this, however, we first need the following technical lemma:

Lemma 5.3.2. *Let A be a distributive lattice, $a, b \in A$, and X and Y two subsets of A such that $\bigvee X$ and $\bigwedge Y$ exist and are distributive. If $a \nleq b$, then:*

1. *$a \nleq b \vee \bigvee X$, or $a \wedge x \nleq b$ for some $x \in X$*
2. *$a \wedge \bigwedge Y \nleq b$, or $a \nleq y \wedge b$ for some $y \in Y$*

Proof.

1. Since $\bigvee X$ exists and is distributive, $\bigvee(a \wedge X) = a \wedge \bigvee X$. Now assume $a \wedge x \leq b$ for all $x \in X$. Then b is an upper bound of the set $a \wedge X$, and hence $a \wedge \bigvee X = \bigvee(a \wedge X) \leq b$. But then, if $a \leq b \vee \bigvee X$, then by Lemma 5.2.5, $a \leq b$, contradicting our assumption. Hence either $a \nleq b \vee \bigvee X$, or $a \wedge x \nleq b$ for some $x \in X$.

2. Since $\bigwedge Y$ exists and is distributive, $\bigwedge(b \vee Y) = b \vee \bigwedge Y$. Now assume $a \leq y \vee b$ for all $y \in Y$. Then a is a lower bound of the set $b \vee Y$, and hence $a \leq \bigwedge(b \vee Y) = b \vee \bigwedge Y$. But then, if $a \wedge \bigwedge Y \leq b$, then by Lemma 5.2.5, $a \leq b$, contradicting our assumption. Hence either $a \wedge \bigwedge Y \nleq b$, or $a \nleq y \wedge b$ for some $y \in Y$.

□

Lemma 5.3.3 (Q-Lemma for Distributive Lattices)**.** *Let A be a distributive lattice, and Q_J and Q_M as in Definition 5.3.1. For every $a, b \in A$ such that $a \nleq b$, there exists a Q-pair (F, I) such that $a \in F$ and $b \in I$.*

Proof. We construct the required pair (F, I) as follows. Let $\{Z_i\}_{i \in \mathbb{N}}$ be an enumeration of all subsets Z of A such that $\bigvee Z \in Q_J$ or $\bigwedge Z \in Q_M$. We define inductively a countable sequence of pairs $(a_i, b_i)_{i \in \mathbb{N}}$ as follows:

- $a_0 = a$, $b_0 = b$
- at stage $i+1$, we assume that $a_i \nleq b_i$. Then there are two cases:
  - if $\bigvee Z_{i+1} \in Q_J$, by Lemma 5.3.2, either $a_i \nleq b_i \vee \bigvee Z_{i+1}$, or $a_i \wedge z_{i+1} \nleq b_i$ for some $z_{i+1} \in Z$. In the first case, set $a_{i+1} = a_i$ and $b_{i+1} = b_i \vee \bigvee Z_{i+1}$. In the latter case, set $a_{i+1} = a_i \wedge z_{i+1}$ and $b_{i+1} = b_i$;
  - if $\bigwedge Z_{i+1} \in Q_M$, by Lemma 5.3.2, either $a_i \wedge \bigwedge Z_{i+1} \nleq b_i$, or $a_i \nleq z_{i+1} \wedge b_i$ for some $z_{i+1} \in Z_{i+1}$. In the first case, set $a_{i+1} = a_i \wedge \bigwedge Z_{i+1}$ and $b_{i+1} = b_i$. In the latter case, set $a_{i+1} = a_i$ and $b_{i+1} = b_i \vee z_{i+1}$.

Clearly, since we assume that $a \nleq b$, it follows by construction that for any $i \in \mathbb{N}$, $a_i \nleq b_i$. Now let $F = \uparrow\{a_i\}_{i \in \mathbb{N}}$ and $I = \downarrow\{b_i\}_{i \in \mathbb{N}}$. We claim that (F, I) is a compatible pair:

- We first prove that F is a filter and that I is an ideal. Since F is an upset and I is a downset, we only have to check that they are closed under meets and joins respectively. Assume $c, c' \in F$. Then there are $i, j \in \mathbb{N}$ such that $a_i \leq c$ and $a_j \leq c'$. Without loss of generality, assume $i \leq j$. Then, by construction of $\{a_i\}_{i \in \mathbb{N}}$, we have that $a_j \leq a_i$. But then $a_j \leq c \wedge c'$, and hence $c \wedge c' \in F$. Similarly, if $d, d' \in I$, then there are $i, j \in \mathbb{N}$ such that $d \leq b_i$ and $d' \leq a_j$. Without loss of generality, assume $i \leq j$. Then, by construction of $\{b_i\}_{i \in \mathbb{N}}$, we have that $b_i \leq b_j$. But then $d \vee d' \leq a_j$, and hence $d \vee d' \in I$.
- $F \cap I = \emptyset$. For assume $c \in F \cap I$. Then there are $i, j \in \mathbb{N}$ such that $a_i \leq c \leq b_j$. But then, by construction, $a_{max\{i,j\}} \leq a_i \leq c \leq b_j \leq b_{max\{i,j\}}$, a contradiction.

Therefore (F, I) is a compatible pair, and, by construction, we also have that $a \in F$, $b \in I$, and (F, I) is a Q-pair. This completes the proof. □

The Q-Lemma for distributive lattices can be seen as an analogue of Tarski's Lemma for two reasons. The first is made clear by the following lemma:

Lemma 5.3.4. *The following are equivalent for a distributive lattice A:*

1. *Lemma 5.3.3 holds, and so does the Prime Filter Theorem (Theorem 2.3.14).*
2. *Lemma 3.3.5 (The Rasiowa-Sikorski Lemma for distributive lattices) holds : for any $a, b \in A$ such that $a \nleq b$, and any countable collections Q_J and Q_M of distributive joins and meets in A respectively, there exists a prime filter F such that $a \in F$, $b \notin F$, and F preserves all joins in Q_J and all meets in Q_M.*

Proof.

$1 \Rightarrow 2$ Note first that if a pair (F, I) is a Q-pair, then every compatible pair (F', I') such that $F \subseteq F'$ and $I \subseteq I'$ is also a Q-pair. Moreover, if (F, I) is a Q-pair, then F preserves all meets and joins in Q. For if $\bigvee X \in F$ for some $\bigvee X \in Q_J$, then $\bigvee X \notin I$, and hence, since (F, I) is a Q-pair, there exists $x \in X \cap F$. Similarly, if $Y \subseteq F$ for some $\bigwedge Y \in Q_M$, then $Y \cap I = \emptyset$, which means that $\bigwedge Y \in F$ since (F, I) is a Q-pair. Assume now that $a \nleq b$ for $a, b \in A$. By Lemma 5.3.3, there exists a Q-pair (F, I) such that $a \in F$ and $b \in I$. But then, by the Prime Filter Theorem, (F, I) extends to a pair (F', I') such that $F \subseteq F'$, $I \subseteq I'$ and $I' = A \setminus F'$. Since (F, I) was a Q-pair, so is (F', I'), and hence F' preserves all meets in Q_M and all joins in Q_J.

$2 \Rightarrow 1$ The Rasiowa-Sikorskia Lemma for distributive lattices implies the Rasiowa-Sikorski Lemma for Boolean algebras as a special case. But this lemma in turn implies the ultrafilter theorem for Boolean algebras, as noted by Goldblatt [1985]. Moreover, it is a well-known result that Boolean Prime Ideal Theorem (BPI) is equivalent to the Prime Filter Theorem for DL. Hence the Rasiowa-Sikorski Lemma for DL implies the prime filter theorem for DL. Moreover, it also implies Lemma 5.3.3. Notice first that if F is a prime filter that preserves all meets in Q_M and all joins in Q_J, then (F, I) is a Q-pair for $I = A \setminus F$. For $\bigvee X \in Q_J$, since F preserves $\bigvee X$, either there exists $x \in X \cap F$ or $\bigvee X \in A \setminus F$ and similarly, for $\bigwedge Y \in Q_M$, since F preserves $\bigwedge Y$, either $\bigwedge Y \in F$ or there exists $x \in Y \cap (A \setminus F)$. Now let $a, b \in A$ such that $a \nleq b$. By the Rasiowa-Sikorski Lemma for DL, there exists a prime filter F such that $a \in F$, $b \notin F$ and F preserves all meets in Q_M and all joins in Q_J. But then $(F, A \setminus F)$ is a Q-pair such that $a \in F$ and $b \in A \setminus F$, which completes the proof.

□

Recall that in the Boolean case, Tarski's Lemma could be seen as the core of the Rasiowa-Sikorski Lemma, in the sense that it did not rely the Stone Representation Theorem nor on the Ultrafilter Theorem. As Lemma 5.3.4 shows, the situation is the same for the Q-Lemma in the setting of distributive lattices. Moreover, just like Tarski's Lemma allowed us to prove that every Boolean Algebra embeds into a Q-completion, the Q-Lemma plays the same role for distributive lattices.

Definition 5.3.5. Let A be a distributive lattice, and Q_J, Q_M as in Definition 5.3.1. A *Q-completion* of A is a pair (C, α) such that C is a complete distributive lattice, $\alpha : A \to C$ is an embedding, and for every $\bigvee X \in Q_J$, $\bigwedge Y \in Q_M$, $\alpha(\bigvee_A X) = \bigvee_C \alpha[X]$ and $\alpha(\bigwedge_A Y) = \bigwedge_A \alpha[Y]$.

The following is the main result of this section for distributive lattices:

Theorem 5.3.6. *Let A be a distributive lattice, and Q_J, Q_M as in Definition 5.3.1. Then A has a Q-completion.*

The proof of Theorem 5.3.6 relies on the following definition and lemmas.

Definition 5.3.7. Let A be a distributive lattice, and Q_J, Q_M as in Definition 5.3.1. The *canonical Q-space* of A is the IP-space (X_Q, τ_1, τ_2) where:

- X_Q is the set of all pseudo-complete Q-pairs on A;
- τ_1 is the upset topology induced by the filter inclusion ordering on X;
- τ_2 is the upset topology induced by the filter-ideal inclusion ordering on X.

The canonical positive and negative maps $|\cdot|$ and $|\cdot|^-$ from A to $\mathscr{P}(X_Q)$ are defined as follows for all $a \in A$

- $|a| = \{(F, I)\ ;\ a \in F\}$;
- $|a|^- = \{(F, I)\ ;\ a \in I\}$.

Lemma 5.3.8. *For a distributive lattice A with canonical Q-space (X_Q, τ_1, τ_2), $|\cdot|$ restricts to a map from A to $\mathsf{RO}_{12}(X_Q)$*

Proof. In light of Lemma 5.2.9 and Lemma 5.2.10, we only have to prove that for any $a \in A$, $-|a|^- \subseteq C_2(|a|)$ and $-|a| \subseteq C_1(|a|^-)$.

1. For the first inclusion, assume that (F, I) is a pseudo-complete Q-pair such that $a \notin I$. Since (F, I) is pseudo-complete, this means that $(F \wedge a, I)$ is a compatible pair. Now since (F, I) is a Q-pair, so are $(F \wedge a, I)$ and $(F \wedge a, I)^*$, which means that $(F \wedge a, I)^* \in X_Q$. Hence $(F, I) \in C_2(|a|)$.
2. For the second inclusion, assume (F, I) is a pseudo-complete Q-pair such that $a \notin F$. Then, since (F, I) is pseudo-complete, by Lemma 5.2.3ii), $(F, I \vee a)$ is a compatible pair. But since (F, I) is a Q-pair, so are $(F, I \vee a)$ and $(F, I \vee a)^*$, which means that $(F, I \vee a)^* \in X_Q$. Hence $-|a|^- \in C_1(|a|)$.

As it was shown in Lemma 5.2.10, (1.) and (2.) imply that $|a| = I_1C_2(|a|)$ for every $a \in A$, which completes the proof. □

We can now prove that the canonical positive map is an embedding, and that it preserves all meets in Q_M and all joins in Q_J.

Lemma 5.3.9. *For a distributive lattice A with canonical Q-space (X_Q, τ_1, τ_2), the canonical positive map $|\cdot|$ is an embedding, i.e.*

1. *$|\cdot|$ is injective;*
2. *$|\cdot|$ is a DL homomorphism.*

Proof.

1. We prove that for any $a, b \in A$, $a \leq b$ iff $|a| \subseteq |b|$. the left-to-right direction follows that for every $(F, I) \in X_Q$, F is a filter. For The right-to-left direction, assume $a \not\leq b$. Then by the Q-Lemma for distributive lattices (Lemma 5.3.3), there exists a Q-pair (F, I) such that $a \in F$ and $b \in I$. But then $(F, I)^*$ is also a Q-pair, which means that $(F, I)^* \in X_Q$. Since $(F, I)^* \in |a| \cap -|b|$, it follows that $|a| \not\subseteq |b|$. Hence if $|a| \subseteq |b|$, then $a \leq b$.
2. The proof of this fact is completely similar to that of Lemma 5.2.111.-3., and is therefore left to the reader.

□

Lemma 5.3.10. *For a distributive lattice A, Q_J, Q_M as in Definition 5.3.1 and (X_Q, τ_1, τ_2) the canonical Q-space of A, the following hold:*

1. *For any $\bigwedge Y \in Q_M$, $|\bigwedge Y| = \bigcap_{y \in Y} |y|$;*
2. *For any $\bigvee X \in Q_J$, $|\bigvee X| = I_1C_2(\bigcup_{x \in X} |x|)$.*

Proof.

1. Let $(F, I) \in X_Q$. Clearly, since F is a filter, if $\bigwedge Y \in F$, then for every $y \in Y$, $y \in F$. Hence we have that
$$|\bigwedge Y| \subseteq \bigcap_{y \in Y} |y|.$$
Conversely, if $F \in \bigcap_{y \in Y} |y|$, then $Y \cap I = \emptyset$. But since (F, I) is a Q-pair, this means that $\bigwedge Y \in F$. Therefore
$$\bigcap_{y \in Y} |y| \subseteq |\bigwedge Y|.$$

2. Note that for any $x \in X, |x| \subseteq |\bigvee X|$, and hence $\bigcup_{x \in X} |x| \subseteq |\bigvee X|$. But then it follows that
$$I_1 C_2(\bigcup_{x \in X} |x|) \subseteq I_1 C_2(|\bigvee X|) = |\bigvee X|.$$
For the converse, assume that $\bigvee X \in F$ for some $(F, I) \in X_Q$. Then $\bigvee X \notin I$, and, since (F, I) is a Q-pair, this means that $x \in F$ for some $x \in X$. Hence $|\bigvee X| \in \bigcup_{x \in X} |x|$. But since $|x| \in \tau_1$ for all $x \in X$, it follows that $\bigcup_{x \in X} |x| \in \tau_1$. But then, by Theorem 5.1.2,
$$\bigcup_{x \in X} |x| \subseteq I_1 C_2(\bigcup_{x \in X} |x|).$$
Therefore we have
$$|\bigvee X| \subseteq I_1 C_2(\bigcup_{x \in X} |x|),$$
which completes the proof.

□

Corollary 5.3.11. *For a distributive lattice A, Q_J, Q_M as in Definition 5.3.1 and (X_Q, τ_1, τ_2) the canonical Q-space of A, the positive canonical map $|\cdot|$ preserves all meets in Q_M and all joins in Q_J.*

Proof. By Theorem 5.1.2, $I_1 C_2$ is a nucleus on the Heyting algebra of open sets in τ_1, which means that for any $U \subseteq \mathsf{RO}_{12}(X_Q)$, we have that
$$\bigvee_{\mathsf{RO}_{12}(X_Q)} U = I_1 C_2(\bigcup_{U_i \in U} U_i),$$
and that
$$\bigwedge_{\mathsf{RO}_{12}(X_Q)} U = I_1(\bigcap_{U_i \in U} U_i).$$
Moreover, since τ_1 is an upset topology, it is closed under arbitrary intersections. Hence for any $U \subseteq \mathsf{RO}_{12}(X_Q)$, we have
$$\bigwedge_{\mathsf{RO}_{12}(X_Q)} U = \bigcap_{U_i \in U} U_i.$$
By Lemma 5.3.10, this means that for any $\bigvee X \in Q_J$ and $\bigwedge Y \in Q_M$,
$$|\bigvee X| = \bigvee_{\mathsf{RO}_{12}(X_Q)} X$$
and
$$|\bigwedge Y| = \bigwedge_{RO_{12}(X_Q)} Y.$$

□

Theorem 5.3.12. *For any distributive lattice A, Q_J, Q_M as in Definition 5.3.1 and (X_Q, τ_1, τ_2) the canonical Q-space of A, $(\mathsf{RO}_{12}(X_Q), |\cdot|)$ is a Q-completion of A.*

Proof. By Corollary 5.1.3, $RO_{12}(X_Q)$ is a complete Heyting algebra, hence also a complete distributive lattice. Moreover, by Lemma 5.3.9 and Corollary 5.3.11, $|\cdot| : A \to RO_{12}(X_Q)$ is an embedding that preserves all meets in Q_M and all joins in Q_J. $\square$

Clearly, Theorem 5.3.6 follows as a direct consequence of Theorem 5.3.12

5.3.2 Q-Lemma and Q-Completions for Heyting Algebras

In light of Theorem 5.3.12 and Lemma 5.2.11, in order to extend the previous result to Heyting algebras, we need to make sure that the canonical map from a Heyting algebra into the refined regular open sets of its Q-space is a Heyting homomorphism. This requires imposing some familiar conditions on the sets Q_M and Q_J (i.e. $(\bigwedge, \to)$-completeness) in order to prove a version of the Q-Lemma for Heyting algebras. However, we first need the following technical lemma:

Lemma 5.3.13. *Let A be a Heyting algebra, and $a, b, c \in A$. Then $(a \to (b \vee c)) \wedge ((a \wedge c) \to b) \leq a \to b$.*

Proof. Note first that $a \wedge (a \to (b \vee c)) \leq b \vee c$, and so by distributivity, for any $d \in A$

$$(a \wedge (a \to (b \vee c))) \wedge d \leq (b \vee c) \wedge d \leq (d \wedge c) \vee (d \wedge b) \leq (d \wedge c) \vee b \tag{1}$$

Moreover, $(a \wedge ((a \wedge c) \to b)) \wedge c = (a \wedge c) \wedge ((a \wedge c) \to b) \leq b$. So by (1) above we have that

$$(a \wedge (a \to (b \vee c))) \wedge (a \wedge ((a \wedge c) \to b)) \leq (a \wedge (((a \wedge c) \to b)) \wedge c) \vee b \leq b \vee b = b \tag{2}$$

But since $a \wedge ((a \to (b \vee c)) \wedge ((a \wedge c) \to b))) = (a \wedge (a \to (b \vee c))) \wedge (a \wedge ((a \wedge c) \to b))$, (2) implies that

$$a \wedge ((a \to (b \vee c)) \wedge ((a \wedge c) \to b))) \leq b \tag{3}$$

and hence by residuation

$$(a \to (b \vee c)) \wedge ((a \wedge c) \to b)) \leq a \to b \tag{4}$$

$\square$

Definition 5.3.14. Let A be a Heyting algebra, and Q_M and Q_J be as in Definition 5.3.1. Q_M and Q_J are $(\bigwedge, \to)$-complete if for any $a, b \in A$:

- if $\bigvee X \in Q_J$, then $\bigwedge((a \wedge X) \to b) = \bigwedge\{(a \wedge x) \to b \,;\, x \in X\} \in Q_M$;
- if $\bigwedge Y \in Q_M$, then $\bigwedge(a \to (Y \vee b)) = \bigwedge\{a \to (y \vee b) \,;\, y \in Y\} \in Q_M$.

Note that those conditions on the sets Q_M and Q_J match exactly those of the Rasiowa-Sikorski Lemma for Heyting algebras (Lemma 3.3.11). In the case of the Rasiowa-Sikorski Lemma, they ensured that for any prime filter p, the set $S_Q \cap {\uparrow} p$ of prime Q-filters extending p was dense in ${\uparrow} p$, thus allowing one to apply the Baire Category Theorem. As the following two lemmas will show, they will be needed in our setting to prove a version of the Q-Lemma for Heyting algebras.

Lemma 5.3.15. *Let A be Heyting algebra, Q_M and Q_J as in Definition 5.3.14, and (F, I) a Q-pair such that $a \to b \notin F$ for some $a, b \in A$. Then:*

1. *If $\bigvee X \in Q_J$, then $(a \wedge x) \to b \notin F$ for some $x \in X$, or $a \to (\bigvee X \vee b) \notin F$*

2. *If $\bigwedge Y \in Q_M$, then $(a \wedge \bigwedge Y) \to b \notin F$ or $a \to (y \wedge b) \notin F$ for some $y \in Y$*

Proof. 1. Note first that if $\bigvee X \in Q_J$, then $\bigwedge((a \wedge X) \to b) \in Q_M$ by assumption. Assume now that $(a \wedge x) \to b \in F$ for all $x \in X$. Since (F, I) is a Q-pair, this means that $\bigwedge((a \wedge X) \to b) \in F$. Now recall that in any Heyting algebra A, for any $c \in A$ and $Z \subseteq A$ such that $\bigvee Z$ exists, $\bigwedge(Z \to c) = \bigvee Z \to c$. Hence we have that

$$(a \wedge \bigvee X) \to b = \bigvee(a \wedge X) \to b = \bigwedge((a \wedge X) \to b) \in F.$$

But if $a \to (\bigvee X \vee b) \in F$, then

$$((a \wedge \bigvee X) \to b) \wedge (a \to (\bigvee X \vee b)) \in F,$$

which by Lemma 5.3.13 implies that $a \to b \in F$, contradicting our assumption.

2. Note that if $\bigwedge Y \in Q_M$, then $\bigwedge(a \to (Y \vee b)) \in Q_M$ by assumption. Assume that $a \to (y \vee b) \in F$ for all $y \in Y$. Since (F, I) is a Q-pair, this means that $\bigwedge(a \to (Y \vee b)) \in F$. Again, recall that in any Heyting algebra, for any existing meet $\bigwedge Z$ and any element c, $\bigwedge(c \to Z) = c \to \bigwedge Z$. Since Y is distributive, this means that

$$a \to (\bigwedge Y \vee b) = a \to \bigwedge(Y \vee b) = \bigwedge(a \to (Y \vee b)) \in F.$$

But then if $(a \wedge \bigwedge Y) \to b \in F$, it follows that

$$((a \wedge \bigwedge Y) \to b) \wedge (a \to (\bigwedge Y \vee b)) \in F$$

, and hence by Lemma 5.3.13 $a \to b \in F$, contradicting again our assumption.

□

Lemma 5.3.16 (Q-Lemma for Heyting algebras). *Let A be Heyting algebra, Q_M and Q_J as in Definition 5.3.14, and (F, I) a Q-pair such that $a \to b \notin F$ for some $a, b \in A$. Then there exists a Q-pair (F', I') such that $F \cup \{a\} \subseteq F'$ and $b \in I'$.*

Proof. The proof is an adaptation of that of Lemma 5.3.3. Let $\{Z_i\}_{i \in \mathbb{N}}$ be an enumeration of all the joins and meets in $Q_J \cup Q_M$. We define inductively a countable sequence of pairs $(a_i, b_i)_{i \in \mathbb{N}}$ as follows:

- $a_0 = a$, $b_0 = b$
- at stage $i + 1$, we assume that $a_i \to b_i \notin F$. Then there are two cases:
  - if $\bigvee Z_{i+1} \in Q_J$, by Lemma 5.3.15, either $a_i \to (b_i \vee \bigvee Z_{i+1}) \notin F$, or $(a_i \wedge z_{i+1}) \to b_i \notin F$ for some $z_{i+1} \in Z$. In the first case, set $a_{i+1} = a_i$ and $b_{i+1} = b_i \vee \bigvee Z_{i+1}$. In the latter case, set $a_{i+1} = a_i \wedge z_{i+1}$ and $b_{i+1} = b_i$;
  - if $\bigwedge Z_{i+1} \in Q_M$, by Lemma 5.3.15, either $(a_i \wedge \bigwedge Z_{i+1}) \to b_i \notin F$, or $a_i \to (z_{i+1} \wedge b_i) \notin F$ for some $z_{i+1} \in Z_{i+1}$. In the first case, set $a_{i+1} = a_i \wedge \bigwedge Z_{i+1}$ and $b_{i+1} = b_i$. In the latter case, set $a_{i+1} = a_i$ and $b_{i+1} = b_i \vee z_{i+1}$.

By construction, and since we assumed that $a \to b \notin F$, it follows that for any $i \in \mathbb{N}$, $a_i \to b_i \notin F$. Now let $A' = \uparrow\{a_i\}_{i \in \mathbb{N}}$ and $I' = \downarrow\{b_i\}_{i \in \mathbb{N}}$. As proved already in Lemma 5.3.3, A' is a filter and I' is an ideal. Moreover, this implies that $F \vee A' = \uparrow\{c \wedge a' \ ; \ c \in F, a' \in A'\}$ is also a filter. It remains to be proved that $(F \vee A') \cap I' = \emptyset$. Assume there exist $c \in F, a' \in A', b' \in I'$ and some

$d \in A$ such that $c \wedge a' \leq d \leq b'$. This means that there exist some a_i and b_j such that $a_i \leq a'$ and $b' \leq b_j$, and hence $c \wedge a_i \leq b_j$. But since $\{a_i\}_{i \in \mathbb{N}}$ is a decreasing sequence of elements of A and $\{a_i\}_{i \in \mathbb{N}}$ is increasing, it follows that $c \wedge a_{max(i,j)} \leq b_{max(i,j)}$. By residuation, this implies that $c \leq a_{max(i,j)} \rightarrow b_{max(i,j)}$, and since $c \in F$, this means that $a_{max(i,j)} \rightarrow b_{max(i,j)}$, which is impossible by construction. Hence $((F \vee A'), I')$ is a compatible pair, and, by construction, it is a Q-pair such that $F \cup \{a\} \subseteq (F \vee A')$ and $b \in I'$. □

Just as in the case of the Q-Lemma for distributive lattices, Lemma 5.3.16 can be seen as a generalization of Tarski's Lemma for Heyting algebras in the following sense:

Lemma 5.3.17. *The following are equivalent for a Heyting algebra A and set Q_M, Q_J as in Definition 5.3.14 :*

1. *Lemma 5.3.4* 1. *(the Q-Lemma and the Prime Filter Theorem for distributive lattices) holds, and Lemma 5.3.16 holds for any Q-pair (F, I) with the LJP;*

2. *Lemma 5.3.4* 2. *(the Rasiowa-Sikorski Lemma for distributive lattices) holds, and so does the Rasiowa-Sikorski Lemma for Heyting algebras (Lemma 3.3.11).*

Proof.

$1 \Rightarrow 2$ Recall that by Lemma 5.3.4, The Q-Lemma and the Prime Filter Theorem for DL imply the Rasiowa-Sikorski Lemma for DL. Hence we only have to show that under the assumptions of the Q-Lemma and the Prime Filter Theorem for DL, the Q-Lemma for compatible pairs of HA with the LJP implies the Rasiowa-Sikorski Lemma for HA. Let F be a prime filter that preserves all meets in Q_M and all joins in Q_J, and is such that $a \rightarrow b \notin F$. Then the pair (F, I) where $I = A \setminus F$ is a Q-pair, as noted above. Moreover, it has the LJP, for if $c \leq d \vee d'$ for $c \in F, d \in I$, then $d \vee d' \in F$ and $d \notin F$, which means that $d' \in F$, since F is prime. Hence by assumption, there exists a Q-pair (F', I') such that $F \cup \{a\} \subseteq F'$ and $b \in I'$. Now by the Prime Filter Theorem, (F', I') extends to a pair (F^*, I^*), where $F' \subseteq F^*$, $I' \subseteq I^*$, and $I^* = A \setminus F^*$. But then F^* is the required prime filter.

$2 \Rightarrow 1$ By Lemma 5.3.4, the Rasiowa-Sikorski Lemma for DL implies the Q-Lemma and the Prime Filter Theorem for DL. Hence we only have to show that, under the assumption of the Q-Lemma and the Prime Filter Theorem for DL, the Rasiowa-Sikorski Lemma for HA implies the Q-Lemma for compatible pairs of a HA A with the LJP. So assume that (F, I) is a Q-pair with the LJP that is such that $a \rightarrow b \notin F$ for some $a, b \in A$. Since (Q, I) has the LJP, this means that $(F, (a \rightarrow b) \vee I)$ is a compatible Q-pair. Since the RS-Lemma for DL implies the Prime filter Theorem, we can extend the pair $(F, (a \rightarrow b) \vee I)$ to a pair (F', I') such that $F \subseteq F'$, $(a \rightarrow b) \vee I \subseteq I'$, and $I' = A \setminus F'$. Since (F, I) was a Q-pair, so is (F', I'), which means that F' preserves all meets in Q_M and all joins in Q_J. But then, since $a \rightarrow b \notin F$, by the Rasiowa-Sikorski Lemma for Heyting algebras, there exists a filter F^* that preserves all meets in Q_M and all joins in Q_J, and is such that $F' \cup a \subseteq F^*$ and $b \notin F^*$. But then this means that the pair F^*, I^*, where $I^* = A \setminus F^*$, is the required Q-pair.

□

Now if we wish to extend Theorem 5.3.6 to Heyting algebras, and prove that every Heyting algebra has a Q-completion provided the sets Q_M and Q_J are $(\bigwedge, \rightarrow)$-complete, we first need the following definition:

Definition 5.3.18 (Q-completion of Heyting algebras)**.** Let A be a Heyting algebra, and Q_J, Q_M as in Definition 5.3.14. A Q-completion of A is a pair (C, α) such that C is a complete Heyting algebra, $\alpha : A \to C$ is an embedding, and for every $\bigvee X \in Q_J$, $\bigwedge Y \in Q_M$, $\alpha(\bigvee_A X) = \bigvee_C \alpha[X]$ and $\alpha(\bigwedge_A Y) = \bigwedge_C \alpha[Y]$.

Recall that, by Corollary 5.1.3and Theorem 5.3.12, for any Heyting algebra A, and any two countable sets of distributive meets and joins Q_M and Q_J, the algebra $\mathsf{RO}_{12}(X_Q)$ of refined regular opens of the Q-space of A is a complete Heyting algebra. Moreover, the canonical map $|\cdot| : A \to \mathsf{RO}_{12}(X_Q)$ is a distributive lattice embedding that preserves all meets in Q_M and all joins in Q_J. So if we want to prove that $(\mathsf{RO}_{12}X_Q, |\cdot|)$ is a Q-completion of A, we just need to prove that $|\cdot|$ preserves Heyting implications.

Lemma 5.3.19. *Let A be a Heyting algebra, Q_M and Q_J as in Definition 5.3.14, X_Q the Q-space of A, and $|\cdot|$ the canonical map from A to $\mathsf{RO}_{12}(A)$. Then for any $a, b \in A$, $|a \to b| = |a| \to_{\mathsf{RO}_{12}(X_Q)} |b|$.*

Proof. Recall that for any $U, V \in \mathsf{RO}_{12}(X_Q)$, $U \to_{\mathsf{RO}_{12}(X_Q)} V = I_1(-U \cup V)$. Clearly, for any pseudo-complete Q-pair $(F, I) \in X_Q$, if $a, a \to b \in F$, then $b \in F$, and hence $|a \to b| \subseteq -|a| \cup |b|$. Moreover, since $|a \to b| \in \tau_1$, we only have to prove that $I_1(-|a| \cup |b|) \subseteq |a \to b|$. So let (F, I) be a pseudo-complete Q-pair such that $a \to b \notin F$. Then by Lemma 5.3.16, there exists a Q-pair (F', I') such that $\{a\} \cup F \subseteq F'$ and $b \in I'$. But then this means that $(F', I')^*$ is a pseudo-complete Q-pair that belongs to $|a| \cap -|b|$ hence does not belong to $-|a| \cup |b|$, and since $(F, I) \leq_1 (F', I')^*$, it follows that $(F, I) \notin I_1(-|a| \cup |b|)$. Hence, by contraposition, $I_1(-|a| \cup |b|) \subseteq |a \to b|$. This completes the proof. □

We therefore get as a direct consequence the main result about Heyting algebras of this section:

Theorem 5.3.20. *Let A be a Heyting algebra, Q_M and Q_J as in* 5.3.14*, X_Q the canonical Q-space of A. Then $(\mathsf{RO}_{12}(X_Q), |\cdot|)$ is a Q-completion of A.*

Proof. By Theorem 5.3.12, $\mathsf{RO}_{12}(X_Q)$ is a complete Heyting algebra, and $|\cdot|$ is an injective DL-homomorphism that preserves all meets in Q_M and all joins in Q_J. Moreover, by Lemma 5.3.19, $|\cdot|$ also preserves all Heyting implications, and is therefore an injective Heyting homomorphism. Hence $(\mathsf{RO}_{12}(X_Q), |\cdot|)$ is a Q-completion of A. □

We conclude with a few remarks regarding the non-constructive principles used in this section. It was noted by Goldblatt [29] that, in the Boolean setting, the Rasiowa-Sikorski is equivalent to the conjunction of the Boolean Prime Ideal Theorem (BPI) and Tarski's Lemma, which is itself equivalent to the axiom of Dependent Choice (DC). Both statements are strictly weaker than the axiom of choice (AC), and neither implies the other. In order to see which picture emerges from the results of this section, we recall first the statement of (DC):

Definition 5.3.21 (Axiom of Dependent Choice)**.** Let A be a set and R a relation on A. If for every $a \in A$, there exists $b \in A$ such that aRb, then for any $a \in A$ there exists a sequence $f : \omega \to A$ such that $f(0) = a$ and $f(n)Rf(n+1)$ for all $n \in \omega$.

It is quite straightforward to see that (DC) was implicitly used in the proof of Lemmas 5.3.3 and 5.3.16 in order to construct the sequences $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$. We will prove in the next chapter that the Q-Lemma for distributive lattices implies Tarski's Lemma. By Goldblatt's result, this will in turn imply that the Q-Lemma for DL is equivalent to (DC).

5.4 Possibility Semantics for First-Order Intuitionistic Logic

In this section, we use the results of the previous sections to build a new semantics for first-order intuitionistic logic which generalizes the ideas from possibility semantics for first-order classical logic. The main interest of this new semantics is mostly technical rather than philosophical or intuitive, as we will mostly translate the results of the previous sections about algebras and topological spaces into the language of syntax and semantics. However, we invite the reader who would like to have an intuitive grasp of the semantics we're giving to consider the following picture.

One way of giving an intuitive motivation for standard Kripke semantics for intuitionistic logic is to point out that Kripke models, contrary to classical Tarskian models, seek to capture the process of proving propositions (or discovering true propositions) rather than the mere fact that some are true and some are false. According to this interpretation of standard Kripke semantics, the elements of a Kripke frame $(K, \leq)$ are stages at which some propositions are proved, and some remain unproved, and for any two stages $i, j \in K$, $i \leq j$ means that j is a stage that belongs to the 'future' of i, i.e. that if I, as a mathematician, have proved all and only those propositions that are forced at stage i, then I could go on and prove all and only those propositions that are forced at stage j. In particular, this means that, if I am at an arbitrary stage i, there may be many propositions that I haven't proved nor disproved. However, at that same stage, I would always know exactly which propositions I have proved and which I haven't: if a proposition is forced at stage i, then I have proved it, and if it is not forced, then I haven't proved it. In other words, we could say that standard Kripke semantics formalize the process of discovering new truths from a first-person perspective.

It is this very last feature of standard Kripke semantics which the semantics we develop here doesn't necessarily share. Rather, this new semantics could be seen as a shift from Kripke semantics to a *third-person* viewpoint. Imagine that instead of modeling the way I, as a mathematician, try to prove more and more propositions, I were to model the way another mathematician, say Eloise, works, based on the partial information she gives me. In this case, at any given stage, there may not only be propositions that are neither proved nor disproved, but also propositions of which I don't know if they have already been proved or not by Eloise. In particular, if she only gives me partial information about what she has proved and what she hasn't proved, it might happen that she tells me that she has proved a disjunction $\phi \vee \psi$, without telling me which one of ϕ or ψ she has proved. If we're both serious constructive mathematicians, I know that she wouldn't claim to have a proof of $\phi \vee \psi$ unless she already had a proof of one of the two disjuncts. But that doesn't mean that I can know with certainty which one of the two she has proved. If such cases arise, I would like to be able to make the most of the partial information that Eloise gives me.

For example, I would like to be able to determine as much as possible what she has already proved, and what I know she can't have proved already, but also what she may or may not prove in the future. In order to do so, I could consider all possible sets of positive and negative information that she could give me, and order them in two different ways via two relations $\leq_1$ and $\leq_2$. The first ordering represents a possible increase in positive information about what Eloise has proved. That is, for any two possible information states $i = (i^+, i^-)$ and $j = (j^+, j^-)$, where the first component of each pair is a set of formulas that I know Eloise has proved, and the second component is a set of formulas that I know she hasn't proved, we have that $i \leq_1 j$ if and only if, given that I know that Eloise has proved all the formulas in i^+, if I were to ask her what else she has proved, she could reply that she has proved all formulas in j^+. Of course, she could reply by telling me about propositions that she had already proved at stage i without letting me know, or she could tell me about new propositions that she has proved since then. The relation $\leq_1$ would therefore be very close to the standard ordering in intuitionistic Kripke models, where one state $i \leq j$ if j represents a future development of i, one where more propositions are proved.

However, since Eloise may not always give me complete information about what she has already proved and not proved, I may not know exactly what she has already proved and what she hasn't proved at any given time. Therefore, if I want to have a clearer picture of what exactly she has proved and not proved at a given stage i, I should also reason about all the possible stages she could be at based on the information she gave me. This means that I should also consider a second relation between stages, which represents an increase in both positive and negative information about what Eloise has proved. In other words, $i \leq_2 j$ means that, based on the information she gave me at stage i, j contains more information which she could give me if I were to press her with more questions about what she has already proved *and* what she hasn't proved yet.

Of course, any state j that represents a possible increase in both positive and negative information is also state that represents a possible increase in positive information. However, the converse may not be true in general. For example, it could be that, at a certain state i, Eloise has told me that she hasn't proved a certain proposition ϕ, while if I were to ask her at a future stage j, she would answer that she now has a proof of ϕ. This means that the two orders between partial states of information should be such that $i \leq_2 j$ should always entail $i \leq_1 j$, but not the other way around.

With this intuitive interpretation in mind, we can proceed to define our semantics. For the remainder of this chapter, we fix a first-order language $\mathfrak{L}$.

Definition 5.4.1. An intuitionistic possibility frame is a triple $(X, \leq_1, \leq_2)$ such that $\leq_1$ and $\leq_2$ are both partial orders on X, and $\leq_2 \subseteq \leq_1$. A first-order intuitionistic possiblity frame (IP frame in what follows) is a tuple $(X, \{D_i, J_i^+, J_i^-\}_{i\in X}, \leq_1, \leq_2, \{f_{ij}\}_{i\leq_1 j \in X})$ such that $(X, \leq_1, \leq_2)$ is an intuitionistic possibility frame, and for every $i, j, k \in X$, the following holds:

- D_i is a set, and J_i^+ and J_i^- both send every n-ary symbol P in $\mathfrak{L}$ to a subset of D_i^n;
- if $i \leq_1 j$, then f_{ij} is a surjective map from D_i to D_j;
- f_{ii} is the identity map on D_i;
- $f_{jk} \circ f_{ij} = f_{ik}$.

Moreover, the following holds for any $i, j \in X$, any n-ary relation symbol R, and any $(a_1, ...a_n) \in D_i^n$:

- disjointness: $J_i^+(R) \cap J_i^-(R) = \emptyset$;
- positive persistence: if $(a_1, ...a_n) \in J_i^+(R)$ and $i \leq_1 j$, then $(f_{ij}(a_1), ...f_ij(a_n)) \in J_i^+(R)$;
- negative persistence: if $i \in J_i^-(R)$ and $i \leq_2 j$, then $(f_{ij}(a_1), ...f_ij(a_n)) \in J_i^-(R)$;
- positive refinement: if $(a_1, ..., a_n) \notin J_i^-(R)$, then there exists j such that $i \leq_2 j$ and $(f_{ij}(a_1), ...f_ij(a_n)) \in J_i^+(R)$;
- negative refinement: if $(a_1, ..., a_n) \notin J_i^+(R)$, then there exists j such that $i \leq_1 j$ and $(f_{ij}(a_1), ...f_ij(a_n)) \in J_i^-(R)$.

The previous definition corresponds to the picture sketched above in the following sense: given an intuitionistic possibility frame $(X, \leq_1, \leq_2)$, elements in X can be seen as partial states of information both about what I know Eloise has proved and what I know she has not proved yet. The first partial order (the proof-order) represents an increase in the number of propositions proved by Eloise, while the second order (the information-order) represents an increase in information about what she has already proved or not. In an IP frame, every state of information i comes with a domain of individuals D_i, and a positive and negative interpretation for all relation symbols, which, intuitively, represents the fact that we are working with partial states of information, where the extension of predicates is only partially defined. The conditions on the maps f_{ij} match exactly those on standard IPL-models.

Definition 5.4.2. Let $\mathfrak{L}$ be the language of first-order intuitionistic logic, with $Var\mathfrak{L}$ and $At(\mathfrak{L})$ the set of variables and atomic formulas of $\mathfrak{L}$ respectively. Let $\mathbf{X} = (X, \{D_i, J_i^+, J_i^-\}_{i\in X}, \leq_1, \leq_2$ $, \{f_{ij}\}_{i\leq_1 j\in X})$ be an IP frame. An intuitionistic possibility model over $\mathbf{X}$ is a triple $(\{\alpha_i\}_{i\in X}, V^+, V^-)$ such that:

- For every $i \in X$, $\alpha_i : Var\mathfrak{L} \to D_i$ assigns an individual in D_i to every variable in $Var\mathfrak{L}$
- the assignments are coherent with the maps between information states, i.e. for every $i \leq_1 j \in X$ and every $x \in Var\mathfrak{L}$, $\alpha_j(x) = f_{ij}(\alpha_i(x))$
- V^+ (the positive valuation), and V^- (the negative valuation) are maps from the set of atomic formulas $At(\mathfrak{L})$ of $\mathfrak{L}$ into $\mathscr{P}(X)$ that satisfy the following conditions for any atomic formula $R(x_1, ..., x_n)$ and any $i \in X$:
  - $i \in V^+(R(x_1, ..., x_n))$ iff $(\alpha_i(x_1), ..., \alpha_i(x_n)) \in J_i^+(R)$;
  - $i \in V^-(R(x_1, ..., x_n))$ iff $(\alpha_i(x_1), ..., \alpha_i(x_n)) \in J_i^-(R)$.

Note that the previous definitions entail that the for any intuitionistic possibility model $(\mathbf{X}, \{\alpha_i\}_{i\in X}, V^+, V^-)$, the positive and negative valuations have the following properties:

- disjointness: $V^+(\phi) \cap V^-(\phi) = \emptyset$;
- positive persistence: if $i \in V^+(\phi)$ and $i \leq_1 j$, then $j \in V^+(\phi)$;
- negative persistence: if $i \in V^-(\phi)$ and $i \leq_2 j$, then $j \in V^-(\phi)$;
- positive refinement: if $i \notin V^-(\phi)$, then there exists j such that $i \leq_2 j$ and $j \in V^+(\phi)$;
- negative refinement: if $i \notin V^+(\phi)$, then there exists j such that $i \leq_1 j$ and $j \in V^-(\phi)$.

Again, the intuitive reading of Definition 5.4.2 is that the positive and negative valuations determine which atomic formulas are "known to be proved by Eloise" and "known to be not proved yet by Eloise" at some information state. This means that those valuations should satisfy some natural requirements. First of all, the information states should be consistent: Eloise can't both have proved and not have proved something at the same time. Second, proofs and information should be preserved: if a state i is lower in the proof-order than a state j, then everything that she has proved at i is also proved at j. Similarly, if a state i is less informative than a state j about what she has already proved and not proved, then all the things I know she's proved or not proved at i should still be known to have been proved or not proved at j. Finally, information states should be complete in the following sense: if, at a state i, I don't know that Eloise has already proved some proposition ϕ, then this must mean that there is a future or simultaneous stage j that can be reached and at which I know that she hasn't proved

ϕ, and similarly, if I don't know that she hasn't proved ϕ, it means that there is a stage higher in the information-order at which I know that she has proved ϕ. Similarly to standard Kripke semantics, we can then inductively define satisfaction in an IP model and at a state i as follows:

Definition 5.4.3. Let $(\mathbf{X}, \{\alpha_i\}_{i\in X}, V^+, V^-)$ be an IP model. We define inductively two relations $i \Vdash^+_{\alpha_i} \phi$ (satisfaction) and $i \Vdash^-_{\alpha_i} \phi$ (refutation) on $X \times \mathfrak{L}$ as follows for any $\phi, \psi \in \mathfrak{L}$, $i \in X$:

- If $\phi \in At(\mathfrak{L})$, $i \Vdash^+_{\alpha_i} \phi$ iff $i \in V^+(\phi)$, and $i \Vdash^-_{\alpha_i} \phi$ iff $i \in V^-(\phi)$
- $i \Vdash^+_{\alpha_i} \top$ and $i \Vdash^-_{\alpha_i} \bot$ always
- $i \Vdash^+_{\alpha_i} \bot$ and $i \Vdash^-_{\alpha_i} \top$ never
- $i \Vdash^+_{\alpha_i} \phi \wedge \psi$ iff $i \Vdash^+_{\alpha_i} \phi$ and $i \Vdash^+_{\alpha_i} \psi$
- $i \Vdash^-_{\alpha_i} \phi \wedge \psi$ iff for all j such that $i \leq_2 j$, $j \nVdash^+_{\alpha_j} \phi$ or $j \nVdash^+_{\alpha_j} \psi$
- $i \Vdash^+_{\alpha_i} \phi \vee \psi$ iff for all j such that $i \leq_1 j$, $j \nVdash^-_{\alpha_j} \phi$ or $j \nVdash^-_{\alpha_j} \psi$
- $i \Vdash^-_{\alpha_i} \phi \vee \psi$ iff $i \Vdash^-_{\alpha_i} \phi$ and $i \Vdash^-_{\alpha_i} \psi$
- $i \Vdash^+_{\alpha_i} \phi \to \psi$ iff for all j such that $i \leq_1 j$, if $j \Vdash^+_{\alpha_j} \phi$, then $j \Vdash^+_{\alpha_j} \psi$
- $i \Vdash^-_{\alpha_i} \phi \to \psi$ iff for all j such that $i \leq_2 j$ there exist k such that $j \leq_1 k$, $k \Vdash^+_{\alpha_j} \phi$ and $k \Vdash^-_{\alpha_j} \psi$
- $i \Vdash^+_{\alpha_i} \forall x\phi(x)$ iff $i \Vdash^+_{\beta_i} \phi(x)$ for any system of assignment $\{\beta_i\}_{i\in X}$ such that for any $i \in X$, $y \in Var\mathfrak{L}$, $\beta_i(y) \neq \alpha_i(y)$ only if $y = x$.
- $i \Vdash^-_{\alpha_i} \forall x\phi(x)$ iff for all j such that $i \leq_2 j$, $j \nVdash^+_{\beta_i} \phi(x)$ for some system of assignment $\{\beta_i\}_{i\in X}$ such that for any $i \in X$, $y \in Var\mathfrak{L}$, $\beta_i(y) \neq \alpha_i(y)$ only if $y = x$.
- $i \Vdash^+_{\alpha_i} \exists x\phi(x)$ iff for all j such that $i \leq_1 j$, $j \nVdash^-_{\beta_i} \phi(x)$ for some system of assignment $\{\beta_i\}_{i\in X}$ such that for any $i \in X$, $y \in Var\mathfrak{L}$, $\beta_i(y) \neq \alpha_i(y)$ only if $y = x$.
- $i \Vdash^-_{\alpha_i} \exists x\phi(x)$ iff $i \Vdash^-_{\beta_i} \phi(x)$ for any system of assignment $\{\beta_i\}_{i\in X}$ such that for any $i \in X$, $y \in Var\mathfrak{L}$, $\beta_i(y) \neq \alpha_i(y)$ only if $y = x$.

The reader should once again appeal to the interpretation of IP models in terms of information states in order to make sense of the previous definition. For a given information state i, tautologies are always known to be true, and contradictions always known to be false. Moreover, I know at i that Eloise has proved a conjunction if and only if I know she has proved both at i, while I know that she has not proved a disjunction at i if and only if I know she has not proved either disjunct yet. However, I might know that she proved a particular disjunction at some state i, without knowing at i already which disjunct she proved, but simply because, regardless of any new information I could get about what she has proved or what she may prove, I can never reach a stage at which I know that she hasn't proved either disjunct. Similarly, I can know that she hasn't proved a conjunction at i without knowing which conjunct she hasn't proved yet, but simply because I can never reach a stage j that is more informative about what she has and hasn't proved at i and is such that at j I know that she has proved both conjuncts.

As for implication, the definition is in line with standard Kripke semantics: I know that Eloise has proved $\phi \to \psi$ at a given information state i if and only if I may never come to learn that ϕ is true without also knowing that ψ is true. On the other hand, I know that she hasn't proved

$\phi \to \psi$ yet if, regardless of any more information I could obtain about what she has and hasn't proved yet, I can reach a stage at which I know that she has proved ϕ but not ψ.

Finally, the semantics for the universal and existential quantifiers is justified by their interpretation as generalized conjunction and disjunction respectively: I know that Eloise has proved $\forall x\phi(x)$ at i iff I know at i that she has proved ϕ for all individuals, and I know that she hasn't proved $\exists x\phi(x)$ at i yet if I know that she hasn't proved ϕ at i for any individual. On the other hand, I can know that she hasn't proved $\forall x\phi(x)$ at i without knowing already for which individual a she hasn't proved $\phi(a)$, if I know that, whichever extra information I may acquire about what she has currently proved or not proved, she will never have proved ϕ for all individuals in the domain. Similarly, I can know that she has proved $\exists x\phi(x)$ at i without knowing already of which individual a she proved that $\phi(a)$ was true, provided I know already that at no stage j at which she has proved at least as much as I know she has proved already it is the case that she hasn't proved $\phi(a)$ for every individual a in the domain of j.

The previous definitions allows for the following lemma:

Lemma 5.4.4. *Let $(\mathbf{X}, \{\alpha_i\}_{i\in X}, V^+, V^-)$ be an IP model. For any formula $\phi \in \mathfrak{L}$, we write $V_+(\phi)$ for the set $\{i \in X \ ; \ i \Vdash^+_{\alpha_i} \phi\}$ and $V_-(\phi)$ for the set $\{i \in X \ ; \ i \Vdash^-_{\alpha_i} \phi\}$. The following is true for any $\phi \in \mathfrak{L}$ and $i, j, k \in X$:*

1. *Disjointness: $V_+(\phi) \cap V_-(\phi) = \emptyset$.*
2. *Positive persistence: if $i \in V_+(\phi)$, then for any j such that $i \leq_1 j$, $j \in V_+(\phi)$.*
3. *Negative persistence: if $i \in V_-(\phi)$, then for any j such that $i \leq_2 j$, $j \in V_-(\phi)$.*
4. *Positive refinement: if $i \notin V_-(\phi)$, then there exists $j \in V_+(\phi)$ such that $i \leq_2 j$.*
5. *Negative refinement: if $i \notin V_+(\phi)$, then there exists $j \in V_-(\phi)$ such that $i \leq_1 j$.*

Proof. We prove items 1 to 5 by an induction on the complexity of ϕ.

- If ϕ is atomic, the claim follows from the definition of an IP model. If ϕ is $\top$ or $\bot$, it follows immediately from the definitions of Definition 5.4.3.
- $\phi := \psi \wedge \chi$.
  1. Assume $i \Vdash^-_{\alpha_i} \psi \wedge \chi$. Then in particular, since $i \leq_2 i$, $i \nVdash^+_{\alpha_i} \psi$ or $i \nVdash^+_{\alpha_i} \chi$. Hence $i \nVdash^+_{\alpha_i} \psi \wedge \chi$.
  2. Assume $i \Vdash^+_{\alpha_i} \psi \wedge \chi$ and $i \leq_1 j$. Then $i \Vdash^+_{\alpha_i} \psi$ and $i \Vdash^+_{\alpha_i} \chi$, hence, by IH, $j \Vdash^+_{\alpha_j} \psi$ and $j \Vdash^+_{\alpha_j} \chi$, which means that $j \Vdash^+_{\alpha_j} \psi \wedge \chi$.
  3. Assume $i \leq_2 j$. Then for any k such that $j \leq_2 k$, $i \leq_2 k$. Hence if $i \Vdash^-_{\alpha_i} \psi \wedge \chi$, it also follows that for every $k \leq_2$, $k \nVdash^+_{\alpha_k} \psi$ or $k \nVdash^-_{\alpha_k} \chi$, and hence $j \nVdash^-_{\alpha_j} \psi \wedge \chi$.
  4. If $i \nVdash^-_{\alpha_i} \psi \wedge \chi$, then there exists j such that $i \leq_2 j$ and $j \Vdash^+_{\alpha_j} \psi$ and $j \Vdash^+_{\alpha_j} \chi$. But then $j \Vdash^+_{\alpha_j} \psi \wedge \chi$.
  5. Assume $i \nVdash^+_{\alpha_i} \psi \wedge \chi$. Then either $i \nVdash^+_{\alpha_i} \psi$ or $i \nVdash^+_{\alpha_i} \chi$. By IH, this means that there exists j such that $i \leq_1 j$ and $j \Vdash^-_{\alpha_j} \psi$ or $j \Vdash^-_{\alpha_j} \chi$. But then, by IH, negative persistence and disjointness hold for ψ and χ, which means that either for any k such that $j \leq_2 k$, $k \nVdash^+_{\alpha_k} \psi$, or for any such k $k \nVdash^+_{\alpha_k} \chi$. Either way, it follows that $j \Vdash^-_{\alpha_j} \psi \wedge \chi$.

- $\phi := \psi \vee \chi$ The proof of items 1-5 in this case is completely similar to the proof of the previous case, by simply inverting $\Vdash^+$ and $\Vdash^-$ one one hand, and $\leq_1$ and $\leq_2$ on the other hand. Details are therefore left to the reader.

- $\phi := \psi \to \chi$

  1. Assume $i \Vdash^-_{\alpha_i} \psi \to \chi$. Then, in particular there exists k such that $i \leq_1 k$, $k \Vdash^+_{\alpha_k} \psi$ and $k \Vdash^-_{\alpha_k} \chi$. By IH, disjointness holds for χ, which means that $k \nVdash^+_{\alpha_k} \chi$. Hence $i \nVdash^+_{\alpha_i} \psi \to \chi$.
  2. Assume $i \leq_1 j$. Then for all k such that $j \leq_1 k$, $i \leq_1 k$. Hence if $i \Vdash^+_{\alpha_i} \psi \to \chi$, we also have $j \Vdash^+_{\alpha_j} \psi \to \chi$.
  3. Assume $i \leq_2 j$. Then for all k such that $j \leq_2 k$, $i \leq_2 k$. Hence if $i \Vdash^-_{\alpha_i} \psi \to \chi$, we also have $j \Vdash^-_{\alpha_j} \psi \to \chi$.
  4. Assume $i \nVdash^+_{\alpha_i} \psi \to \chi$. Then there exists j such that $i \leq_1 j$, $j \vDash^+_{\alpha_j} \psi$ and $j \nVdash^+_{\alpha_j} \chi$. By IH, there exists k such that $j \leq_1 k$ and $k \vDash^-_{\alpha_k} \chi$. But then $i \leq_1 k$, and $k \Vdash^-_{\alpha_k} \psi \to \chi$.
  5. Assume $i \nVdash^-_{\alpha_i} \psi \to \chi$. Then there exists j such that $i \leq_2 j$ and for all k such that $j \leq_1 k$, if $k \Vdash^+_{\alpha_k} \psi$, then $k \nVdash^-_{\alpha_k} \chi$. By IH, for any such k, if $k \nVdash^+_{\alpha_k} \chi$, then there exists k' such that $k \leq_1 k'$ and $k' \Vdash^-_{\alpha_{k'}} \chi$. But then $i \leq_1 k'$, and since $k \Vdash^+_{\alpha_k} \psi$, by IH we also have that $k' \Vdash^+_{\alpha_{k'}} \psi$, which contradicts our assumption. Hence $j \Vdash^+_{\alpha_j} \psi \to \chi$.

- $\phi := \forall x \psi(x)$

  1. Assume $i \Vdash^-_{\alpha_i} \forall x \psi(x)$. Then in particular $i \nVdash^+_{\alpha_i[a/x]} \psi(x)$ for some $a \in D_i$, and hence $i \nVdash^+_{\alpha_i} \forall x \psi(x)$.
  2. Assume $i \leq_1 j$ and $i \Vdash^+_{\alpha_i} \forall x \psi(x)$. Then for every $a \in D_j$, $i \Vdash^+_{\alpha_i[f^{-1}_{ij}(a)/x]} \psi(x)$. Hence, by IH, $j \Vdash^+_{\alpha_j[a/x]} \psi(x)$ for every $a \in D_j$, which means that $j \Vdash^+ \forall x \psi(x)$.
  3. Just like in the item 2 of the conjunction case, this follows directly from the definitions.
  4. Assume $i \nVdash^-_{\alpha_i} \forall x \psi(x)$. Then there exists some j such that $i \leq_2 j$ and $j \nVdash^+_{\alpha_j[a/x]} \psi(x)$ for all $a \in D_j$. But then $j \Vdash^+_{\alpha_j} \forall x \psi(x)$.
  5. Assume $i \nVdash^+_{\alpha_j} \forall x \psi(x)$. Then $i \nVdash^+_{\alpha_j[a/x]} \psi(x)$ for some $a \in D_i$. By IH, there exists j such that $i \leq_1 j$, and $j \Vdash^-_{\alpha_j[f_{ij}(a)/x]} \psi(x)$. But then since, by IH, disjointness and negative persistence hold for $\psi(x)$, it follows that $j \Vdash^-_{\alpha_j} \forall x \psi(x)$.

- $\phi := \exists x \psi(x)$ Once again, the proof of items 1-5 in this case is completely similar to the proof of the previous case, by simply inverting $\Vdash^+$ and $\Vdash^-$ one one hand, and $\leq_1$ and $\leq_2$ on the other hand. Details are therefore again left to the reader.

□

The previous lemma guarantees that the intuitive interpretation of the partial orders $\leq_1$ and $\leq_2$ given above are justified: the first order $\leq_1$ represents an increase in the amount of knowledge acquired by Eloise, while the second order represents an increase in information about what she has already proved and what she hasn't proved yet. It is also instrumental in the following proof, which corresponds to a proof of soundness of first-order intuitionistic logic with respect to IP-frames.

Definition 5.4.5. Let $\mathbf{X}$ be an IP-frame. A formula ϕ is *valid* on $\mathbf{X}$ if for every IP-model $(\mathbf{X}, \{\alpha_i\}_{i \in X}, V^+, V^-)$, $V_+(\phi) = X$.

Lemma 5.4.6. *Let* $(\mathbf{X}, \{\alpha_i\}_{i\in X}, V^+, V^-)$ *be an IP-model. Then* V_+ *is a HA-homomorphism from* $LT^{\mathfrak{L}}_{IPL}$ *into* $\mathsf{RO}_{12}(\mathbf{X})$.

Proof. It follows from Lemma 5.4.4 that for any $\phi \in \mathfrak{L}$, we have that $I_1C_2(V_+(\phi)) = V_+(\phi)$, where I_1 and C_2 are the interior and closure operator induced by the upset topology on $\leq_1$ and $\leq_2$ respectively. Moreover, it is straightforward to prove that Definition 5.4.3 entails that V_+ is a HA-homomorphism. □

Corollary 5.4.7. *IPL is sound with respect to IP-frames, i.e., for any theorem ϕ of IPL, ϕ is valid on any IP-frame* $\mathbf{X}$.

The converse of Corollary 5.4.7 is also true: IPL is complete with respect to IP-frames.

Theorem 5.4.8. *For any $\phi \in Form(\mathfrak{L}_{IPL})$, if ϕ is not derivable in IPL, then there is exists an IP- model $(\mathbf{X}, \{\alpha_i\}_{i\in X}, V^+, V^-)$ such that $V_+(\phi) \neq X$.*

Proof. Let $LT^{\mathfrak{L}}_{IPL}$ be the Lindenbaum-Tarski algebra of IPL for $\mathfrak{L}$ a first-order language, and let $X, \leq_1, \leq_2$ be the underlying frame of the canonical Q-space of $LT^{\mathfrak{L}}_{IPL}$. For every pseudo-complete Q-pair (F, I), let $D_{(F,I)} = Var\mathfrak{L}_{/\sim_F}$, where the equivalence relation $\sim_F$ is such that for any $x, y \in Var\mathfrak{L}$, $x \sim_F y$ iff $x = y \in F$. Moreover, for any pseudo-complete Q-pairs $(F, I), (F', I')$ such that $F \subseteq F'$, and any $x \in Var\mathfrak{L}$, let $f_{(F,I),(F',I')}(x^{\sim_F}) = x^{\sim_{F'}}$. Then it is straightforward to check that $\mathbf{X} = (X, \{D_i\}_{i\in X}, \leq_1, \leq_2, \{f_{ij}\}_{i\leq_1 j\in X})$ is an IP-frame. Moreover, for any $(F, I) \in \mathbf{X}$, let $\alpha_{(F,I)} : Var\mathfrak{L} \to D_{(F,I)}$ be such that $\alpha_{(F,I)}(x) = x^{\sim_F}$, and define V^+ and V^- such that for every atomic ϕ and every $(F, I) \in X$, $(F, I) \in V^+(\phi)$ iff $\phi \in F$, and $(F, I) \in V^-(\phi)$ iff $\phi \in I$. Then, by the definition of pseudo-complete Q-pairs, it follows that $(\mathbf{X}, \{\alpha_i\}_{i\in X}, V^+, V^-)$ is an IP-model. Moreover, a straightforward induction on the complexity of formulas yields that for any formula ϕ of $\mathfrak{L}_IPL$ and any $(F, I) \in \mathbf{X}$, $(F, I) \in V_+(\phi)$ iff $\phi \in F$. Hence for any formula ϕ, if $V_+(\phi) = X$, then $\{(F, I) \in X \ ; \ \phi \in F\} = X$. Now since $\mathbf{X}$ is the underlying frame of the canonical Q-space of $LT^{\mathfrak{L}}_{IPL}$, by Theorem 5.3.20, it follows that for any formula ϕ, $\{(F, I) \in X \ ; \ \phi \in F\} = X$ iff $\vdash_{IPL} \phi$. Hence if ϕ is not derivable in $LJ + CD$, then $V_+(\phi) \neq X$. This concludes the proof. □

5.5 Conclusion of This Chapter

In this chapter, we extended the framework of possibility semantics to intuitionistic logic, and provided a new semantics for first-order intuitionistic logic. The main interest of this semantics is that it relies on a completely choice-free representation theorem for Heyting algebras (Theorem 5.2.14), and on a generalization of Tarski's Lemma to the variety of Distributive lattices (Lemma 5.3.3) and Heyting algebras (Lemma 5.3.16) that uses only the Axiom of Dependent Choice.

Chapter 6

Generalizations of IP-Spaces and Related Work

In this chapter, we study in more detail the framework developed in chapter 5, and extend some of the results obtained there. In section 1, we generalize the methods and ideas developed in chapter 5 in order to apply them to the case of co-Heyting algebras. In Section 2, we generalize most of the results and ideas of Chapter 4, Section 3, and study the relationship between sets of compatible pairs over a lattice L and algebras of regular open sets that these sets of pairs induce. In particular, we obtain a choice-free representation theorem for bi-Heyting algebras. Finally, in Section 4, we compare IP-spaces with some related frameworks. In particular, we an equivalence between possibility spaces for Heyting algebras, Dragalin frames and Fairtlough-Mendler frames.

6.1 Representation Theorem for co-Heyting Algebras

Recall that the main result of the first section of the previous chapter was that for any refined bi-topological space (X, τ_1, τ_2), $\mathsf{RO}_{12}(X)$ is a Heyting algebra. In this section, we prove a dual statement, namely that for any refined bi-topological space (X, τ_1, τ_2), the algebra of regular closed sets RC_{12} is a co-Heyting algebra. We then define a new embedding from a distributive lattice into the regular closed sets of its canonical R-space, and show that if the original lattice is a co-Heyting algebra, then the embedding is a co-Heyting homomorphism. Finally, we prove an analogue of Theorem 5.3.20 for co-Heyting algebras, using the adequate version of the Q-Lemma in the case of co-Heyting algebras. Since most proofs in this section are simply dual versions of proofs that were given in the previous chapter, we have included these proofs in an Appendix.

6.1.1 Refined Regular Closed Sets

In our generalization of possibility semantics to co-Heyting algebras, it will prove easier to work with regular closed sets rather than regular open sets. To this end, we fix the following definition.

Definition 6.1.1. Let (X, τ_1, τ_2) be a refined bi-topological space. A subset $U \subseteq X$ is called *refined regular closed* if $C_1 I_2(U) = U$. We denote as $RC_{12}(X)$ the set of all refined regular closed sets of X.

Refined regular closed sets will play the same role as refined regular opens, as it appears from the following lemma.

Lemma 6.1.2. *Let (X, τ_1, τ_2) be a refined bi-topological space, and let C_1 be the co-Heyting algebra of closed sets under τ_1. Then $C_1 I_2$ is a co-nucleus on C_1.*

Proof. See Lemma A.1.1 in the Appendix. □

Corollary 6.1.3. *For any refined bi-topological space (X, τ_1, τ_2), the algebra of regular closed sets $\mathsf{RC}_{12}(X) = (RC_{12}(X), \wedge, \cup, -\!\!<, \bigwedge, \bigvee)$ is a co-Heyting algebra, where $A \wedge B = C_1 I_2(A \cap B)$, $A -\!\!< B = C_1(A \cap -B)$, $\bigwedge_{i \in I} A_i = C_1 I_2(\bigcap_{i \in I} A_i)$, and $\bigvee_{i \in I} A_i = C_1(\bigcup_{i \in I} A_i)$.*

Proof. This is a direct consequence of the general theory of co-nuclei. □

6.1.2 Representation for co-Heyting Algebras

Recall that for a distributive lattice L, the *canonical IP-space* of L is the refined bi-topological space (X, τ_1, τ_2), where X is the set of all pseudo-complete compatible pairs (F, I) over L, and τ_1 and τ_2 are the upset topologies induced by the filter inclusion ordering and the filter-ideal inclusion ordering respectively. Here, we work with the dual notion of a co-canonical space.

Definition 6.1.4. Let L be a distributive lattice. The *co-intuitionistic possibility space* of L (cIP-space) is the refined bi-topological space (X, τ_3, τ_2), where X is the set of all pseudo-complete compatible pairs over L, and τ_3 and τ_2 are the upset topologies induced by the ideal inclusion ordering $\leq_3$ and the filter-ideal inclusion ordering on X respectively.

As in the setting of Theorem 5.2.14 for Heyting algebras, our aim is to define a co-Heyting injective homomorphism from any co-Heyting algebra into the algebra of regular closed sets of its co-canonical R-space.

Lemma 6.1.5. *Let L be a distributive lattice, and let (X, τ_3, τ_2) be its co-canonical R-space, and let $|\cdot|$ and $|\cdot|^-$ be the positive and negative maps from L to $\mathscr{P}(X)$ respectively. For every $a \in L$, $C_3 I_2(-|a|^-) = -|a|^-$.*

Proof. See Lemma A.1.2 in the Appendix. □

A direct consequence of Lemma 6.1.5 is that the map: $-|\cdot|^- : L \to \mathsf{RC}_{32}(X)$ is well-defined. We now prove that it is a DL-homomorphism, and a co-Heyting homomorphism if L is a co-Heyting algebra.

Lemma 6.1.6. *For every DL L, the canonical map $-|\cdot|^-$ from L into the algebra of refined regular closed sets of its co-canonical R-frame has the following properties for any $a, b \in A$:*

1. $-|1|^- = X, -|0|^- = \emptyset$
2. $-|a \wedge b|^- = C_3 I_2(-|a|^- \cap -|b|^-)$
3. $-|a \vee b|^- = -|a|^- \cup -|b|^-$
4. $-|a -\!\!< b|^- = C_3(-|a|^- \cap |b|^-)$ *if L is a co-Heyting algebra.*

Proof.

1. For any $(F, I) \in X$, $1 \notin I$, and $0 \in I$.
2. Recall that for any $a, b \in L$, $|a| \cap |b| = |a \wedge b|$. Now since interior operators distribute over intersections, for any $a, b \in L$, we have that

$$C_3 I_2(-|a|^- \cap -|b|^-) = C_3(I_2 - |a|^- \cap I_2 - |b|^-) = C_3(|a| \cap |b|) = C_3(|a \wedge b|) = -|a \wedge b|^-.$$

3. By definition of an ideal, for any $(F, I) \in X$, and $a, b \in L$, $a \vee b \in I$ iff $a \in I$ and $b \in I$. Hence $|a \vee b|^- = |a|^- \cap |b|^-$, hence $-|a \vee b|^- = -|a|^- \cup -|b|^-$.

4. Note first that for any $(F, I) \in X$, $a, b \in L$, $a \leq a \prec b \vee b$, hence if $a \notin I$ and $b \in I$, this implies that $a \prec b \notin I$. Hence $-|a|^- \cap |b|- \subseteq -|a \prec b|^-$. Moreover, since $|a \prec b| \in \tau_3$, we only have to prove that

$$-|a \prec b|^- \subseteq C_3(-|a|^- \cap |b|^-).$$

Assume $a \prec b \notin I$ for some $(F, I) \in X$. Then $(\uparrow a, b \vee I)$ is a compatible pair, for otherwise $a \leq b \vee c$ for some $c \in I$, which means that $a \prec b \leq c$ and hence $a \prec b \in I$, contradicting our assumption. Hence $(\uparrow a, I)^* \in X$, which means that $(F, I) \in C_3(-|a|^- \cap |b|^-)$, which completes the proof.

□

Theorem 6.1.7 (Representation Theorem for Co-Heyting algebras)**.** *For any co-Heyting algebra L with co-canonical R-space (X, τ_3, τ_2), L is isomorphic to a subalgebra of the algebra $\mathsf{RC}_{32}(X)$ of regular closed sets of X.*

Proof. By Lemma 6.1.6, the map $-|\cdot|^- : L \to \mathsf{RC}_{32}(X)$ is a co-Heyting homomorphism, hence we only have to prove that it is injective. Assume $a, b \in L$ and $a \not\leq b$. Then $(\uparrow a, \downarrow b)$ is a compatible pair, and hence $(\uparrow a, \downarrow b)^* \in -|a|^- \cap |b|^-$, which means that $-|a|^- \not\subseteq -|b|^-$. hence $-|\cdot|^-$ is a co-Heyting embedding. □

6.1.3 Q-Completions of co-Heyting Algebras

Recall that in section 3 of the previous chapter, we proved that every Heyting algebra has a Q-completion by constructing for any Heyting algebra A an embedding of A into the regular open sets of its canonical Q-space (X_Q, τ_1, τ_2). Similarly, we prove that every co-Heyting algebra has a Q-completion by embedding any co-Heyting algebra L into the regular closed sets of its co-canonical Q-space (X_Q, τ_3, τ_2), where X_Q is the set of all Q-pairs over L, and τ_3 and τ_2 are the upset topologies over the ideal inclusion ordering and the filter-ideal inclusion ordering respectively. We first prove two technical lemmas analogous to Lemmas 5.3.13 and 5.3.15, which will be necessary to prove the Q-Lemma for co-Heyting algebras.

Lemma 6.1.8. *Let L be a co-Heyting algebra. Then for any $a, b, c \in L$, $a \prec b \leq ((a \wedge c) \prec b) \vee (a \prec (b \vee c))$*

Proof. The proof is the dual of the proof of Lemma 5.3.13, and is given in Lemma A.1.3 the Appendix. □

Definition 6.1.9. Let L be a co-Heyting algebra, and Q_M and Q_J be as in Definition 5.3.1. Q_M and Q_J are $\prec$-complete iff for any $a, b \in A$:

- if $X \in Q_J$, then $(a \wedge X) \prec b = \{(a \wedge x) \prec b \; ; \; x \in X\} \in Q_J$;
- if $Y \in Q_M$, then $a \prec (Y \vee b) = \{a \prec (y \vee b) \; ; \; y \in Y\} \in Q_J$;

Lemma 6.1.10. *Let L be a co-Heyting algebra, Q_M and Q_J as in Definition 6.1.9, and (F, I) a Q-pair such that $a \prec b \notin I$ for some $a, b \in L$. Then:*

1. *If $X \in Q_J$, then $(a \wedge x) \prec b \notin I$ for some $x \in X$, or $a \prec (\bigvee X \vee b) \notin I$*
2. *If $Y \in Q_M$, then $(a \wedge \bigwedge Y) \prec b) \notin I$ or $a \prec (y \vee b) \notin I$ for some $y \in Y$*

Proof. This proof is dual to the proof of Lemma 5.3.15, and is given in Lemma A.1.4 in the Appendix. □

We can now state and prove a version of the Q-Lemma for co-Heyting algebra:

Lemma 6.1.11. *(Q-Lemma for co-Heyting algebras) Let L be a co-Heyting algebra, Q_M and Q_J, as in Definition 6.1.9. Then for any Q-pair (F, I) and any $a, b \in L$, if $a \prec b \notin I$, then there exists a Q-pair (F', I') such that $I \cup \{b\} \in I'$ and $a \in F'$*

Proof. Similarly to Lemma 5.3.16, we first enumerate all $Z \in Q_J \cup Q_M$ and construct two sequences $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ as follows:

- $a_0 = a$, $b_0 = b$
- at stage $i+1$, we assume that $a_i \prec b_i \notin I$. Then there are two cases:
  - if $Z_{i+1} \in Q_J$, by Lemma 6.1.10, either $a_i \prec (b_i \vee \bigvee Z_{i+1}) \notin I$, or $(a_i \wedge z_{i+1}) \prec b_i \notin I$ for some $z_{i+1} \in Z$. In the first case, set $a_{i+1} = a_i$ and $b_{i+1} = b_i \vee \bigvee Z_{i+1}$. In the latter case, set $a_{i+1} = a_i \wedge z_{i+1}$ and $b_{i+1} = b_i$;
  - if $Z_{i+1} \in Q_M$, by Lemma 6.1.10, either $(a_i \wedge \bigwedge Z_{i+1}) \prec b_i \notin I$, or $a_i \prec (z_{i+1} \wedge b_i) \notin I$ for some $z_{i+1} \in Z_{i+1}$. In the first case, set $a_{i+1} = a_i \wedge \bigwedge Z_{i+1}$ and $b_{i+1} = b_i$. In the latter case, set $a_{i+1} = a_i$ and $b_{i+1} = b_i \vee z_{i+1}$.

Let $A' = \uparrow\{a_i\}_{i \in \mathbb{N}}$ and $B' = \downarrow\{b_i\}_{i \in \mathbb{N}}$. It is then routine to check that A' is a filter, B' is an ideal, and that $(A', B' \vee I)$ is the required compatible Q-pair. □

We conclude this section with the proof that every co-Heyting algebra has a Q-completion.

Definition 6.1.12 (Q-completions of co-Heyting algebras)**.** Let L be a co-Heyting algebra, Q_M and Q_J as in Definition 6.1.9. A Q-completion of L is a pair (C, α) such that C is a complete co-Heyting algebra, $\alpha : L \to C$ is an embedding, and for every $X \in Q_J$, $Y \in Q_M$, $\alpha(\bigvee_L X) = \bigvee_C \alpha[X]$ and $\alpha(\bigwedge_L Y) = \bigwedge_C \alpha[Y]$.

Theorem 6.1.13. *Let L be a co-Heyting algebra, Q_M and Q_J as in Definition 6.1.9, and let (X_Q, τ_3, τ_2) be the co-canonical Q-space of L. Then $(\mathsf{RC}_{32}(X_Q), -|\cdot|^-)$ is a Q-completion of L.*

Proof. Given Lemmas 5.3.9 and 6.1.6, and the Q-Lemma for distributive lattices, it is straightforward to see that $-|\cdot|^- : L \to \mathsf{RC}_{32}(X_Q)$ is an injective DL-homomorphism. Hence we only check the following for all $a, b \in L$, $X \in Q_J$ and $Y \in Q_M$:

1. $-|a \prec b|^- = C_3(-|a|^- \cap |b|^-)$
2. $-|\bigvee X|^- = C_3(\bigcup_{x \in X} -|x|^-)$
3. $-|\bigwedge Y|^- = C_3 I_2(\bigcap_{y \in Y} -|y|^-)$

1. By Lemma 6.1.6 -4., we only have to prove that $-|a \prec b|^- \subseteq C_3(-|a|^- \cap |b|^-)$. Let (F, I) be a pseudo-complete Q-pair such that $(a \prec b) \notin I$. Then by Lemma 6.1.11, there exists a Q-pair (F', I') such that $I \cup \{b\} \subseteq I'$ and $a \in F'$. Hence $(F', I')^* \subseteq -|a|^- \cap |b|^-$, which completes the proof.

2. Note first that since τ_3 is an upset topology, for any $X \in Q_J$, we have

$$C_3(\bigcup_{x\in X} -|x|^-) = \bigcup_{x\in X} -|x|^-.$$

Now since any ideal is downward closed, it is immediate that $-|x|^- \subseteq -|\bigvee X|^-$, and thus

$$\bigcup_{x\in X} -|x|^- \subseteq -|\bigvee X|^-.$$

For the converse, simply notice that for every pair $(F, I) \in X_Q$, if $x \in I$ for all $x \in X$, then $\bigvee X \in I$ since (F, I) is a Q-pair.

3. Notice first that since τ_2 is an upset topology, we have

$$I_2(\bigcap_{y\in Y} -|y|^-) = \bigcap_{y\in Y} I_2(-|y|^-) = \bigcap_{y\in Y} |y|$$

by Lemma 6.1.5. Now for any pair $(F, I) \in X_Q$, since (F, I) is a Q-pair, $\bigwedge Y \in F$ iff $y \in F$ for all $y \in F$. Hence

$$C_3 I_2(\bigcap_{y\in Y} -|y|^-) = C_3(\bigcap_{y\in Y} |y| = C_3|\bigwedge Y| = -|\bigwedge Y|^-.$$

□

We have therefore proved that every co-Heyting algebra embeds into the regular closed sets of its co-canonical R-space, and that every co-Heyting algebra also has a Q-completion, thus providing a counterpart of the results about Heyting algebras proved in the previous chapter. However, this work does not yield by itself a representation theorem for bi-Heyting algebras, since, although the canonical and co-canonical R-spaces of a bi-Heyting algebra have the same carrier, the embeddings into the canonical and co-canonical R-space are different in general. If we want to achieve such a result, we need to have better grasp of algebras of regular open and regular closed sets of bi-topological spaces, and of the kind of completion of lattices that they form. This is the goal of the next section, which can also be seen as a generalization of the results of chapter 3, section 3, to the setting of bi-topological possibility spaces.

6.2 Possibility Spaces and Completions of Lattices

In this section, we adopt a slightly more general view on canonical IP-spaces of Heyting algebras, in order to generalize to the setting of distributive lattices the study of the correspondence between properties of classes of filters over a Boolean algebra and properties of the regular opens of the induced possibility space that we started in chapter 3, section 3. Throughout this section, we state most of the results for algebras of regular open sets and for algebras of regular closed set. However, in most cases, we give a proof only for the regular open sets, since the corresponding dual statement for regular closed sets can be proved in a completely similar fashion. Moreover, some rather long and technical proofs have been moved to the Appendix for the convenience of the reader.

6.2.1 Generalized Possibility Spaces

We begin by slightly generalizing the framework we will be working with, and fix the following definitions.

Definition 6.2.1. Let L be a lattice, and $\mathscr{F}$, $\mathscr{I}$ the set of filters and ideals over L. Given a set $\mathscr{C} \subseteq \mathscr{F} \times \mathscr{I}$ of compatible pairs over L, we define the relations $\leq_1, \leq_2, \leq_3$ on $\mathscr{C}$ such that for any $(F, I), (F', I') \in \mathscr{C}$:

- $(F, I) \leq_1 (F', I')$ if $F \subseteq F'$
- $(F, I) \leq_2 (F', I')$ if $F \subseteq F'$ and $I \subseteq I'$
- $(F, I) \leq_3 (F', I')$ if $I \subseteq I'$.

Moreover, we define the following topologies on $(\mathscr{C}, \leq_1, \leq_2, \leq_3)$:

- τ_+ is the topology generated by the basis $\{|a|^+ \ ; \ a \in L\}$, where for any $a \in L$, $|a|^+ = \{(F, I) \in \mathscr{C} \ ; \ a \in F\}$.
- τ_- is the topology generated by the basis $\{|a|^- \ ; \ a \in L\}$, where for any $a \in L$, $|a|^- = \{(F, I) \in \mathscr{C} \ ; \ a \in I\}$.
- For $i \in \{1, 2, 3\}$, τ_i is the upset topology on $\mathscr{C}$ induced by $\leq_i$.

Definition 6.2.2. Let L be a lattice. A *generalized possibility space* for L is a bi-topological space $(\mathscr{C}, \sigma, \tau)$, where $\mathscr{C}$ is a set of compatible pairs over L.

Note that the main difference between generalized possibility spaces and IP-spaces is that in the former case we do not require that one topology is a finer than the other. This means that, given a generalized possibility space (X, σ, τ), it may not be the case in general that $I_\sigma C_\tau$ and $C_\sigma I_\tau$ are a nucleus on O_σ and a co-nucleus on C_τ respectively. However, we have the following result for any generalized possibility space.

Lemma 6.2.3. *Let (X, τ, σ) be a bi-topological space. Then $I_\tau C_\sigma$ is a closure operator on the complete Heyting algebra τ, and $C_\tau I_\sigma$ is a kernel operator on the closed sets in τ.*

Proof. We prove that $I_\tau C_\sigma$ is a closure operator on I_τ. The dual statement is proved similarly. Since both I_τ and C_σ are monotone operations on $\mathscr{P}(X)$, their composition is also monotonic. Moreover, for any $U \in \tau$, we have that $U = I_1(U) \subseteq I_\tau C_\sigma(U)$, hence $I_\tau C_\sigma$ is increasing on any set in τ. Finally, to see that it is also idempotent, if U is in τ, note that $U \subseteq I_\tau C_\sigma(U)$ implies that

$$C_\sigma(U) \subseteq C_\sigma I_\tau C_\sigma(U).$$

Conversely, for any V closed in σ, If $I_\tau C_\sigma(U) \subseteq V$, then we have that

$$C_\sigma I_\tau C_\sigma(U) \subseteq V.$$

But this immediately implies that

$$C_\sigma I_\tau C_\sigma(U) \subseteq C_\sigma(U).$$

Therefore

$$C_\sigma(U) = C_\sigma I_\tau C_\sigma(U),$$

which means that

$$I_\tau C_\sigma I_\tau C_\sigma(U) = I_\tau C_\sigma(U).$$

This completes the proof. ∎

The following is a generalization of the notion of separative sets of filters over a Boolean algebras.

Definition 6.2.4. Let L be a lattice. Then a set $\mathscr{C} \subseteq \mathscr{F} \times \mathscr{I}$ is a *separative* set of pairs if for any $a, b \in L$, if $a \nleq b$, then there is $(F, I) \in \mathscr{C}$ such that $a \in F$ and $b \in I$.

Lemma 6.2.5. *Let L be a lattice and $\mathscr{C}$ a separative set of pairs over L, and let $RO_{+-}(\mathscr{C})$ be the set of regular opens in $(\mathscr{C}, \tau_+, \tau_-)$, and $RC_{-+}(\mathscr{C})$ the set of regular closed sets in $(\mathscr{C}, \tau_-, \tau_+)$. Then:*

1. $\mathsf{RO}_{+-}(\mathscr{C}) = \{RO_{+-}(\mathscr{C}), \cap, \vee, \bigwedge, \bigvee, \emptyset, \mathscr{C}\}$ *is a complete lattice, where for any regular open sets A, B, we have $A \vee B = I_+C_-(A \cup B)$, and for any family $\{A_i\}_{i \in I}$ of regular open sets, $\bigwedge_{i \in I} A_i = I_+C_-(\bigcap_{i \in I} A_i)$ and $\bigvee_{i \in I} A_i = I_+C_-(\bigcup_{i \in I} A_i)$. Moreover, $(\mathsf{RO}_{+-}(\mathscr{C}), |\cdot|^+)$ is isomorphic to the MacNeille completion of L.*

2. *Dually, $\mathsf{RC}_{-+}(\mathscr{C}) = \{RC_{-+}(\mathscr{C}), \wedge, \cup, \bigwedge, \bigvee, \emptyset, \mathscr{C}\}$ is a complete lattice, where for any regular closed sets A, B, we have $A \wedge B = C_-I_+(A \cap B)$, and for any family $\{A_i\}_{i \in I}$ of regular open sets, $\bigwedge_{i \in I} A_i = C_-I_+(\bigcap_{i \in I} A_i)$ and $\bigvee_{i \in I} A_i = C_-I_+(\bigcup_{i \in I} A_i)$. Moreover, $(\mathsf{RC}_{-+}(\mathscr{C}), -|\cdot|^-)$ is isomorphic to the MacNeille completion of L.*

Proof. This proof has been moved to Lemma A.2.1 in the Appendix for the convenience of the reader. □

For any set of pairs $\mathscr{C}$, it is easy to see that the topologies τ_+ and τ_- on $\mathscr{C}$ are a generalization of the Stone topology on a set of filters over a Boolean algebra that we encountered in chapter 3. In light of this observation, the previous theorem can therefore be seen as a generalization of Lemma 4.3.3. Moreover, the following shows how Lemma 6.2.5 generalizes a result obtained in [6] and [34] under the assumption of (PFT):

Lemma 6.2.6. *Let L be a distributive lattice, and $(X_L, \tau, \leq)$ the dual Priestley space of L. Let σ_1, σ_2 be two topologies on X_L determined by the set of all open upsets and the set of all open downsets in $(X_L, \tau, \leq)$ respectively. Then $\mathsf{RO}_{12}(X_L)$ is isomorphic to the MacNeille completion of L.*

Proof. Recall that for any prime filter F over L, its complement F^c is a prime ideal. Let $\mathscr{C}$ be the set of all compatible pairs (F, F^c) for F a prime filter over L. Then it is straightforward to verify that the map $\alpha : (\mathscr{C}, \leq_1) \to (X_L, \leq)$ that sends every pair (F, F^c) in $\mathscr{C}$ to F is an order isomorphism such that for any $U \subseteq \mathscr{C}$, we have that $U \in \tau_+$ iff $\alpha[U] \in \sigma_1$. Moreover, for any $(F, F^c) \in \mathscr{C}$ and any $a \in L$, we have that $(F, F^c) \in |a|^-$ iff $a \notin F$, which implies that for any $U \in \mathscr{C}$, we have that $U \in \tau_-$ iff $U \in \sigma_2$. Hence $\mathsf{RO}_{12}(\mathscr{C})$ and $\mathsf{RO}_{12}(X_L)$ are isomorphic. Finally, by the Prime Filter Theorem, $\mathscr{C}$ is a separative set of compatible pairs over L. Therefore, by Lemma 6.2.5, $\mathsf{RO}_{12}(\mathscr{C})$ and $\mathsf{RO}_{12}(X_L)$ are both isomorphic to the MacNeille completion of L. □

As in the case of the Stone topology, the following definitions and lemmas will be useful to generalize the results about Alexandroff topologies on rich and separative sets of filters over Boolean algebras obtained in Lemma 4.3.5 and Lemma 4.3.7.

Definition 6.2.7. Let L be a lattice. A set $\mathscr{C}$ of pairs over L is *rich* if for any $(F, I) \in \mathscr{C}$ and any $a \in L$, if $a \notin F$, then there exists $(F', I') \in \mathscr{C}$ such that $F \subseteq F'$ and $a \notin I$, and dually, if $a \in I$, then there exists $(F', I') \in \mathscr{C}$ such that $a \in F$ and $I \subseteq I'$.

Definition 6.2.8. Let L be a lattice, and $\mathscr{F}$ and $\mathscr{I}$ be the set of filters and ideals over L. Then for any $F \in \mathscr{F}$, $I \in I$:

- the *positive cone* of F is defined as $\mathscr{O}^+(F) = \{(F', I') \in \mathscr{F} \times \mathscr{I} \ ; \ F \cap I = \emptyset \text{ and } F \subseteq F'\}$,
- the *negative cone* of I is defined as $\mathscr{O}^-(I) = \{(F', I') \in \mathscr{F} \times \mathscr{I} \ ; \ F \cap I = \emptyset \text{ and } I \subseteq I'\}$.

Moreover, for any set $\mathscr{C}$ of compatible pairs over L, and any $A \subseteq \mathscr{C}$, we fix the two sets:

- $G_A^{\mathscr{C}} = \{F \in \mathscr{F} \ ; \ \mathscr{O}^+(F) \cap \mathscr{C} \subseteq A\}$ and
- $J_A^{\mathscr{C}} = \{I \in \mathscr{I} \ ; \ \mathscr{O}^-(I) \cap A = \emptyset\}$.

The previous definition will be crucial for the proof of the following lemma:

Lemma 6.2.9. *Let L be a lattice, and $\mathscr{C}$ a separative and rich set of pairs over L. Then* $(\mathsf{RO}_{13}(\mathscr{C}), |\cdot|^+)$ *and* $\mathsf{RC}_{31}(\mathscr{C}), -|\cdot|^-$ *are a doubly-dense extensions of L.*

Proof. The full proof can be found in Lemma A.2.2 in the Appendix. However, the following fact from the proof will play an important role in some proofs below, and its statement is therefore reproduced. For any $A \in \mathsf{RO}_{13}(X)$ we have that:

1. $$A = I_1C_3(\bigcup_{F \in G_A} \bigcap_{a \in F} |a|^+);$$

2. $$A = \bigcap_{I \in J_A} I_1C_3(\bigcup_{b \in I} |b|^+).$$

□

Similarly to the case of Boolean algebras, we can now define conditions on a set $\mathscr{C}$ of compatible pairs over a lattice L that guarantee that the algebra of regular open sets based on $\mathscr{C}$ is not only a doubly-dense extension of L, but is moreover isomorphic to the canonical extension of L.

Definition 6.2.10. Let L be a lattice and $\mathscr{C}$ a set of pairs over L. Then $\mathscr{C}$ is *cofinal* if for any $F \in \mathscr{F}$, $I \in \mathscr{I}$ such that $F \cap I = \emptyset$, there exists $(F', I') \in \mathscr{C}$ such that $F \subseteq F'$ and $I \subseteq I'$.

Lemma 6.2.11. *Let L be a lattice and $\mathscr{C}$ a cofinal set of pairs over L. Then* $\mathsf{RO}_{12}(\mathscr{C})$ *and* $\mathsf{RC}_{31}(\mathscr{C})$ *are isomorphic to the canonical extension of L.*

Proof. Note first that it is easy to see that cofinality implies separativeness (since for any $a, b \in L$, if $a \nleq b$, then $\uparrow a \cap \downarrow b = \emptyset$) and richness (since for any $a \in L$, $F \in \mathscr{F}$, $I \in \mathscr{I}$, if $a \notin F$, then $F \cap \downarrow a = \emptyset$, and if $a \notin I$, then $\uparrow a \cap I = \emptyset$). Hence by Lemma 6.2.9, $\mathsf{RO}_{12}(\mathscr{C})$ is a doubly-dense extension of L. In order to prove that it is isomorphic to the canonical completion of L, it suffices to prove that it is compact, i.e. for any $X, Y \subseteq L$, if

$$\bigcap_{a \in X} |a|^+ \subseteq I_1C_3(\bigcup_{b \in Y} |b|^+),$$

then there exist finite sets $X' \subseteq X$ and $Y \subseteq Y'$ such that $\bigwedge X' \leq \bigvee Y'$. So let $X, Y \subseteq L$ such that

$$\bigwedge X' \nleq \bigvee Y'$$

for every finite X' and Y' such that $X' \subseteq X$ and $Y' \subseteq Y'$. Let $X^\wedge$ and $Y^\vee$ be the closure of X and Y under finite meets and finite joins respectively. Then we claim that

$$\uparrow X^\wedge \cap \downarrow Y^\vee = \emptyset$$

. For a contradiction, assume not. Then there must be $X' \subseteq X$ and $Y' \subseteq Y'$ such that $\bigwedge X' \leq \bigvee Y'$, contradicting our assumption. Hence since $\mathscr{C}$ is cofinal, this means that there exists $(F, I) \in \mathscr{C}$ such that $\uparrow X^\wedge \subseteq F$ and $\downarrow Y^\wedge \subseteq I$, which means that

$$\bigcap_{a \in X} |a|^+ \nsubseteq I_1 C_3(\bigcup_{b \in Y} |b|^+$$

. Hence $\mathsf{RO}_{12}(\mathscr{C})$ is isomorphic to the canonical extension of L.

The proof of the dual statement is left to the reader. □

We conclude this section with a generalization of Lemma 4.3.10.

Definition 6.2.12. Let L be a lattice. A set $\mathscr{C}$ of pairs over L is *normal* if for any $X, Y \subseteq L$ such that $\bigwedge X$ and $\bigwedge Y$ exist, and any $(F, I) \in \mathscr{C}$, if $X \subseteq F$, then $\bigwedge X \in F$, and if $Y \subseteq I$, then $\bigvee Y \in I$.

Lemma 6.2.13. *Let L be a lattice, and $\mathscr{C}$ a separative and rich set of pairs over L. Then $\mathsf{RO}_{+-}(\mathscr{C})$, $\mathsf{RO}_{13}(\mathscr{C})$, $\mathsf{RC}_{-+}(\mathscr{C})$ and $\mathsf{RC}_{31}(\mathscr{C})$ are all isomorphic to the MacNeille completion of L if on of the two following conditions is met:*

- *for any $(F, I) \in \mathscr{C}$, F is a principal filter and I is a principal ideal;*
- *$\mathscr{C}$ is normal and L is a complete lattice.*

Proof. We only prove the statement for $\mathsf{RO}_{+-}(\mathscr{C})$ and $\mathsf{RO}_{13}(\mathscr{C})$, since the proof is completely similar for $\mathsf{RC}_{-+}(\mathscr{C})$ and $\mathsf{RC}_{31}(\mathscr{C})$. Note first that, if $\mathscr{C}$ is separative, it follows from Lemma 6.2.5 that $\mathsf{RO}_{+-}(\mathscr{C})$ is isomorphic to the MacNeille completion of L. Moreover, if $\mathscr{C}$ is also rich, we have by Lemma 6.2.9 that for any $A \in RO_{12}(\mathscr{C})$,

$$A = I_1 C_3(\bigcup_{F \in G_A} \bigcap_{a \in F} |a|^+) = \bigcap_{I \in J_A} I_1 C_3(\bigcup_{b \in I} |b|^+).$$

Now if for any $(F, I) \in C$, F is a principal filter and I is a principal ideal, note that we can also assume without loss of generality that all filters in G_A and all ideals in J_A are principal. This means that for any $F \in G_A$, we have that

$$\bigcap_{a \in F} |a|^+ = |a_F|^+,$$

where $a_F \in L$ is the element that generates F, and for any $I \in J_A$, we have that

$$\bigcup_{b \in I} |b|^+ = |b_I|^+,$$

where $b_I \in L$ is the element that generates I. This implies that

$$A = I_1 C_3(\bigcup_{F \in G_A} |a_F|^+) = \bigcap_{I \in J_A} |b_I|^+,$$

which means that every $A \in \mathsf{RO}_{13}(\mathscr{C})$ is both a meet and a join of images of elements of L. Therefore if for every $(F, I) \in \mathscr{C}$, F is a principal and ideal and I is a principal ideal, $\mathsf{RO}_{12}(\mathscr{C})$ is isomorphic to the MacNeille completion of L.

Finally, it is immediate that this is also true if $\mathscr{C}$ is normal and L is a complete lattice, since if L is complete, then every filter that is closed under arbitrary meets is principal, and every ideal that is closed under arbitrary joins is principal. □

6.2.2 Refined Topologies

So far, the results we have proved hold for any lattice, and, contrary to the setting of the previous chapter, we did not require the bi-topological spaces to be refined. However, in the remainder of this section, we progressively restrict our attention to distributive lattice, Heyting algebras and co-Heyting algebras, and see how, in some sense, the results obtained in the previous chapter are a special case of the more general approach of the previous subsection. In particular, we provide characterizations of distributive lattices, Heyting algebras and co-Heyting algebras in terms of sets of compatible pairs, and prove a representation theorem for bi-Heyting algebras. We begin with the following definition.

Definition 6.2.14. Let L be lattice and $\mathscr{C}$ a set of pairs over L. Then $\mathscr{C}$ has the *left fusion property* if for any $(F, I), (F', I') \in \mathscr{C}$, if $I \subseteq I'$, then there is $(F^*, I^*) \in C$ such that $F \vee F' \subseteq F^*$ and $I \subseteq I^*$. Dually, $\mathscr{C}$ has the *right fusion property* if for any $(F, I), (F', I') \in \mathscr{C}$, if $F \subseteq F'$, then there is $(F^*, I^*) \in C$ such that $F \subseteq F^*$ and $I \vee I' \subseteq I^*$.

The following lemma connects the previous definition with the notion of pseudo-complete pairs defined in the previous chapter:

Lemma 6.2.15. *Let L be a distributive lattice, and (F, I), (F', I') two pseudo-complete compatible pairs. Then:*

1. *If $I \subseteq I'$, then $(F \vee F', I)$ is a compatible pair.*
2. *If $F \subseteq F'$, then $(F, I \vee I')$ is a compatible pair.*

Proof.

1. Assume $F \vee F' \cap I \neq \emptyset$. Then $a \wedge a' \leq b$ for some $a \in F$,$a' \in F'$ and $b \in I$. But then, since (F, I) has the RMP, $a' \in I$, and hence (F', I') is not a compatible pair, contradicting again our assumption.
2. Assume $F \cap I \vee I' \neq \emptyset$. Then there exists $a \in F$, $b \in I$ and $b' \in I'$ such that $a \leq b \vee b'$. But then, since (F, I) has the LJP, $b' \in F$, and hence $F' \cap I' \neq \emptyset$, contradicting our assumption. □

Corollary 6.2.16. *Let L be a distributive lattice. Then the set $\mathscr{C}_R$ of compatible pairs over L with the RMP has the left-fusion property, and the set $\mathscr{C}_L$ of compatible pairs over L with the LMP has the right-fusion property.*

Proof. Assume $(F, I), (F', I') \in \mathscr{C}_R$ and $I \subseteq I'$. Then, since (F, I) has the RMP, by Lemma 6.2.15 (1.) we have that $(F \vee F', I)$ is a compatible pair, hence $(F \vee F', I)^* \in \mathscr{C}_R$. Dually, if $(F, I), (F', I') \in \mathscr{C}_L$ and $F \subseteq F'$, then since (F, I) has the LJP, by Lemma 6.2.15 (2.) we have that $(F, I \vee I')$ is a compatible pair, which means that $(F, I \vee I')^* \in \mathscr{C}_L$. □

The following lemma draws a connection between refined bi-topological spaces and sets of pairs $\mathscr{C}$ with the left-fusion property or the right-fusion property.

Lemma 6.2.17. *Let L be a lattice and $\mathscr{C}$ a set of pairs over L. Then if $\mathscr{C}$ has the left fusion property, for any $U \in \tau_1$, $C_3(U) = C_2(U)$. Dually, if $\mathscr{C}$ has the right fusion property, for any V closed in τ_3, $I_1(V) = I_2(V)$.*

Proof. Assume $\mathscr{C}$ has the left-fusion property and let $U \in \tau_1$. Since $\leq_2 \subseteq \leq_3$, we have that $C_2(U) \subseteq C_3(U)$. To see the converse, assume $(F, I) \in C_3(U)$. This means that there exists $(F', I') \in U$ such that $I \subseteq I'$. Since $\mathscr{C}$ has the left-fusion property, this means that there is $(F^*, I^*) \in \mathscr{C}$ such that $F \vee F' \subseteq F^*$ and $I \subseteq I^*$. But since U is in τ_1, this implies that $(F^*, I^*) \in U$. Hence $(F, I) \in C_2(U)$.

Dually, assume $\mathscr{C}$ has the right-fusion property, and let V be closed in τ_3. Since $\leq_2 \subseteq \leq_1$, we have that $I_1(V) \subseteq I_2(V)$. To see the converse, assume $(F, I) \in -I_1(U)$. This means that there exists $(F', I') \notin U$ such that $F \subseteq F'$. Since $\mathscr{C}$ has the left-fusion property, this means that there is $(F^*, I^*) \in \mathscr{C}$ such that $F \subseteq F^*$ and $I \vee I' \subseteq I^*$. Since U is closed in τ_3, it follows that $(F^*, I^*) \notin U$. Hence $(F, I) \notin I_2(U)$. □

The previous lemma has the following consequences:

Lemma 6.2.18. *Let L be a lattice and $\mathscr{C}$ a set of pairs over L. Then:*

1. *If $\mathscr{C}$ is rich, separative, and has the right-fusion property, then $\mathsf{RO}_{12}(\mathscr{C})$ is a dense completion of L ; dually, if $\mathscr{C}$ is rich, separative, and has the left-fusion property, then $\mathsf{RO}_{32}(\mathscr{C})$ is a dense completion of L ;*

2. *If $\mathscr{C}$ is cofinal and has the right-fusion property, then $\mathsf{RO}_{12}(\mathscr{C})$ is isomorphic to the canonical extension of L; dually, if $\mathscr{C}$ is cofinal and has the left-fusion property, then $\mathsf{RC}_{32}(\mathscr{C})$ is isomorphic to the canonical extension of L;*

3. *If L is a complete lattice and $\mathscr{C}$ is normal, rich, separative, and has the right-fusion property, then $RO_{12}(\mathscr{C})$ is isomorphic to L. Dually, if $\mathscr{C}$ is normal, rich, separative and has the left-fusion property, then $\mathsf{RC}_{32}(\mathscr{C})$ is isomorphic to L.*

Proof. By Lemma 6.2.17, we have that if $\mathscr{C}$ has the left-fusion property, then for any $U \in \tau_1$, $I_1C_2(U) = I_1C_3(U)$, which means that $\mathsf{RO}_{12}(\mathscr{C})$ and $\mathsf{RO}_{13}(\mathscr{C})$ coincide. Dually, if $\mathscr{C}$ has the right-fusion property, then for any V closed in τ_3, $C_3I_2(V) = C_3I_1(V)$, which means that $\mathsf{RC}_{31}(\mathscr{C})$ and $\mathsf{RC}_{32}(\mathscr{C})$ coincide. But this implies at once the following:

1. if $\mathscr{C}$ is rich and separative, then by Lemma 6.2.9 both $\mathsf{RO}_{13}(\mathscr{C})$ and $\mathsf{RC}_{31}(\mathscr{C})$ are doubly-dense extensions of L. Hence if $\mathscr{C}$ has the left-fusion property, $\mathsf{RO}_{13}(\mathscr{C})$ is also a doubly-dense extension of L, and if $\mathscr{C}$ has the right-fusion property, $\mathsf{RC}_{31}(\mathscr{C})$ is also a doubly-dense extension of L.

2. By Lemma 6.2.11, if $\mathscr{C}$ is cofinal, then both both $\mathsf{RO}_{13}(\mathscr{C})$ and $\mathsf{RC}_{31}(\mathscr{C})$ are isomorphic to the canonical extension of L. Hence if $\mathscr{C}$ has the left-fusion property, $\mathsf{RO}_{13}(\mathscr{C})$ is also isomorphic to the canonical extension of L, and if $\mathscr{C}$ has the right-fusion property, $\mathsf{RC}_{31}(\mathscr{C})$ is also isomorphic to the canonical extension of L.

3. By Lemma 6.2.13, if L is a complete lattice and $\mathscr{C}$ is normal, then $\mathsf{RO}_{13}(\mathscr{C})$ and RC_{31} are isomorphic to the MacNeille completion of L. Hence if $\mathscr{C}$ has the left-fusion property, $\mathsf{RO}_{13}(\mathscr{C})$ is also isomorphic to the MacNeille completion of L, and if $\mathscr{C}$ has the right-fusion property, $\mathsf{RC}_{31}(\mathscr{C})$ is also isomorphic to the MacNeille completion of L.

□

Moreover, from previous work on refined bi-topological spaces we already have the following results:

Lemma 6.2.19. *Let L be a lattice and $\mathscr{C}$ set of pairs over L. $RO_{12}(\mathscr{C})$ is a complete Heyting algebra and $RC_{32}(\mathscr{C})$ is a complete co-Heyting algebra.*

Proof. This a direct consequence of the fact that I_1C_2 is a nucleus on τ_1, and C_3I_2 is a co-nucleus on the co-Heyting algebra of closed sets in τ_2. □

Combining the last two facts with previous work, we obtain the following characterization of distributive lattices:

Theorem 6.2.20. *Let L be a lattice. Then the following are equivalent:*

1. *L is a distributive lattice;*
2. *There exists a rich enough and separative set of pairs over L that has the left-fusion property;*
3. *There exists a rich enough and separative set of pairs over L that has the right-fusion property.*

Proof. To see that $1 \Rightarrow 2$ and $1 \Rightarrow 3$, note that by Lemma 5.2.6, for any distributive lattice L, the set of pseudo-complete pairs over L is cofinal, hence also a separative and rich class. Moreover, by Lemma 6.2.15, it also has the right-fusion property since, every pair has the LJP, and the left-fusion property, since every pair has the RMP. To see that $2 \Rightarrow 1$, note that if $\mathscr{C}$ is rich enough and separative, then L embeds into $\mathsf{RO}_{13}(\mathscr{C})$ by Lemma 6.2.9. Moreover, if $\mathscr{C}$ has the left-fusion property, then by Lemma 6.2.18 $\mathsf{RO}_{13}(\mathscr{C})$ is isomorphic to $\mathsf{RO}_{12}(\mathscr{C})$. But $\mathsf{RO}_{12}(\mathscr{C})$ is always a distributive lattice by Lemma 6.2.19. Hence L is isomorphic to a subalgebra of a distributive lattice, which means that it is a distributive lattice. The proof that $3 \Rightarrow 1$ is completely similar and is therefore left to the reader. □

The previous characterization theorem can moreover be extended to complete Heyting algebra and complete co-Heyting algebra. For this however, we first need the following lemma.

Lemma 6.2.21. *Let L be a complete distributive lattice, and (F, I) a compatible pair over L such that F preserves all meets in L and I preserves all joins in L. Then:*

1. *If L is a Heyting algebra, then there exists $I' \in \mathscr{I}$ such that $I \subseteq I'$, I' preserves all joins in L, and (F, I') is a compatible pair with the RMP.*
2. *If L is a co-Heyting algebra, then there exists $F' \in \mathscr{F}$ such that $F \subseteq F'$, F' preserves all meets in L, and (F', I) is a compatible pair with the RMP.*

Proof.

1. Note first that since L is a complete lattice, if F preserves all meets in L, then F is a principal filter, and if I preserves all joins in L, then I is a principal ideal. Now let $I' = {\downarrow}\{c \in L \,;\, a \wedge c \leq b \text{ for some } a \in F, b \in I\}$. By Lemma 5.2.4, (F, I') is a compatible pair with the RMP and $I \subseteq I'$. In order to show that I' preserves all joins, we show that $I' = {\downarrow}(a_F \to b_I)$, where a_F and b_I are the elements of L that generate F and I respectively. Let $c \in I'$. Then there are $a \in F$, $b \in I$ such that $a \wedge c \leq b$. Now $a_F \leq a$ and $b \leq b_I$, which means that $a_F \wedge c \leq b_I$. But this implies, by residuation, that $c \leq a_F \to b_I$.
2. Let $F' = {\uparrow}\{c \in L \,;\, a \leq c \vee b \text{ for some } a \in F, b \in I\}$. The proof that F' is the required filter is completely similar to the previous one and is therefore left to the reader.

□

Theorem 6.2.22. *Let L be a complete lattice. Then the following are equivalent:*

1. *L is a Heyting algebra*
2. *There is a normal, separative, rich set $\mathscr{C}$ of compatible pairs over L such that $\mathscr{C}$ has the left-fusion property.*

Dually, the following are equivalent:

3. *L is a co-Heyting algebra*
4. *There is a normal, separative, rich set $\mathscr{C}$ of compatible pairs over L such that $\mathscr{C}$ has the right-fusion property.*

Proof. We prove only the first equivalence, since the second one is proved in a completely similar way. For $1 \Rightarrow 2$, assume that L is a Heyting algebra, and let $\mathscr{C}$ be the set of all compatible pairs (F, I) with the RMP and such that F and I are a principal filter and a principal ideal respectively. Then it is easy to see that, by Lemma 5.2.4 and Lemma 6.2.15, $\mathscr{C}$ is separative and rich, and has the left-fusion property.

For $2 \Rightarrow 1$, assume $\mathscr{C}$ is a normal, separative and rich set of pairs over (F, I). Then by Lemma 6.2.18, $RO_{12}(\mathscr{C})$ is isomorphic to the MacNeille completion of L. But since L is complete, it is isomorphic to its own MacNeille completion. Hence L is isomorphic to $\mathsf{RO}_{12}(\mathscr{C})$, and since $\mathsf{RO}_{12}(\mathscr{C})$ is a complete Heyting algebra, so is L. $\square$

We conclude this section by solving the problem that we raised at the end of section 1. In particular, we show how the algebraic investigations of this section yield a representation theorem for bi-Heyting algebras.

Theorem 6.2.23. *Let L be a distributive lattice, and $\mathscr{C}$ a cofinal set of pairs over L. Then:*

1. *If $\mathscr{C}$ has the left-fusion property, every pair in $\mathscr{C}$ has the LJP, and L is a co-Heyting algebra, then $\mathsf{RO}_{12}(\mathscr{C})$ is co-Heyting algebra.*
2. *If $\mathscr{C}$ has the right-fusion property, every pair in $\mathscr{C}$ has the RMP, and L is a Heyting algebra, then $\mathsf{RC}_{32}(\mathscr{C})$ is a Heyting algebra.*

Proof.

1. By Lemma 6.2.18, we have that, if $\mathscr{C}$ has the left-fusion property, then $\mathsf{RO}_{12}(\mathscr{C})$ is a doubly-dense extension of L and a distributive lattice. Now for any $A, B \in \mathsf{RO}_{12}(\mathscr{C})$, let

$$A \multimap^* B = I_1C_2(\bigcup_{F_1 \in G_A, I_1 \in J_B} \bigcap \{|a \multimap b|^+ \ ; \ a \in F_1, b \in I_1\})$$

We claim that for any $C \in \mathsf{RO}_{12}(\mathscr{C})$, we have

$$A \multimap^* B \subseteq C \Leftrightarrow A \subseteq I_1C_2(B \cup C).$$

For the right-to-left direction, assume $C \in \mathsf{RO}_{12}(X)$ is such that $A \subseteq I_1C_2(B \cup C)$. We claim that this means that for any $F_1 \in G_A$, $I_1 \in J_B$, $I_2 \in J_C$, there exists $a_1 \in F_1$ $b_1 \in I_1$ and $b_2 \in I_2$ such that $a_1 \leq b_1 \vee b_2$. Otherwise, it is straightforward to see that $F_1 \cap I_1 \vee I_2 = \emptyset$, and, since $\mathscr{C}$ is cofinal, this means there exists $(F^*, I^*) \in \mathscr{C}$ such that

$F_1 \subseteq F^*$ and $I_1 \vee I_2 \subseteq I^*$. But this means that for any (F^{**}, I^{**}) such that $F^* \subseteq F^{**}$ and $I^* \subseteq I^{**}$, we have that

$$(F^{**}, I^{**}) \notin \bigcup_{b \in I_1} |b|^+$$

and

$$(F^{**}, I^{**}) \notin \bigcup_{c \in I_2} |c|^+,$$

and hence

$$(F^*, I^*) \notin I_1C_3(B \cup C).$$

On the other hand, since $F_1 \in G_A$, we have that $(F^*, I^*) \in A$, which contradicts the assumption that $A \subseteq I_1C_2(B \cup C)$. Hence there must be $a_1 \in F_1$, $b_1 \in I_1$, $b_2 \in I_2$ such that $a_1 \leq b_1 \vee b_2$. Now assume that

$$(F, I) \in \bigcup_{F_1 \in G_A, I_1 \in J_B} \bigcap \{|a \prec\!\!- b|^+ \ ; \ a \in F_1, b \in I_1\}.$$

Then there exists $F_1 \in \mathbb{G}_A$, and $I_1 \in \mathbb{J}_B$ such that $a_1 \prec\!\!- b_1 \in F$ for all $a_1 \in F_1$, $b_1 \in I_1$. Moreover, for any $I_2 \in \mathbb{J}_C$, there exists $a_1 \in F_1$, $b_1 \in I_1$ and $b_2 \in I_2$ such that $a_1 \leq b_1 \vee b_2$, and hence by residuation $a_1 - \!\!< b_1 \leq b_2$. But then this means that for any $I_2 \in \mathbb{J}_C$, $F \cap I_2 \neq \emptyset$, which means that $(F, I) \in C$. Hence

$$A \prec\!\!-^* B \subseteq I_1C_2(C) = C.$$

For the converse, we prove that

$$A \subseteq I_1C_2(A \prec\!\!-^* B \cup B).$$

To see this, assume $(F, I) \in A$, and notice that this implies that $F \in G_A$. Now let (F', I') be such that $(F, I) \leq_1 (F', I')$, and assume that $F' \cap I_1 = \emptyset$ for some $I_1 \in J_B$. Then, since (F', I') has the LJP, by Lemma 6.2.15 (2.) this means that $(F', I_1 \vee I')$ is a compatible pair, and by upward-completeness of $\mathscr{C}$ this means that there is $(F^*, I^*) \in \mathscr{C}$ such that $F' \subseteq F^*$ and $I_1 \vee I' \subseteq I^*$. Now for any $a_1 \in F$, $b_1 \in I_1$, $a_1 \leq a_1 \prec\!\!- b_1 \vee b_1$. Hence, since $F \subseteq F^*$, $I_1 \subseteq I^*$, and (F^*, I^*) has the LJP, it follows that $a_1 \prec\!\!- b_1 \in F^*$ for any $a_1 \in F$, $b_1 \in I_1$. Hence $(F^*, I^*) \in A \prec\!\!-^* B$. Hence for any (F', I') such that $(F, I) \leq_1 (F', I')$, either $(F', I') \in B$, or there exists $(F^*, I^*) \in A \prec\!\!-^* B$ such that $(F', I') \leq_2 (F^*, I^*)$, and therefore

$$A \subseteq I_1C_2(A \prec\!\!-^* B \cup B).$$

2. For any $A, B \in RC_{32}(\mathscr{C})$, let:

$$A \to^* B = C_3I_2(\bigcap_{F_1 \in H_A, I_1 \in K_B} \bigcup \{-|a_1 \to |b_1|^- \ ; \ a_1 \in F_1, b_1 \in I_1\})$$

The proof that for any $C \in \mathsf{RC}_{32}(\mathscr{C})$, $C_3I_2(A \cap C) \subseteq B$ iff $C \subseteq A \to^* B$ is completely similar to that of the dual statement above, and details are therefore left to the reader.

□

Theorem 6.2.24. *Let L be a bi-Heyting algebra, and $\mathscr{C}$ the set of all pseudo-complete compatible pairs over L. Then L is isomorphic to a subalgebra of $RO_{12}(\mathscr{C})$ and to a subalgebra of $RC_{32}(\mathscr{C})$.*

Proof. Since the set of all pseudo-compatible pairs over L is cofinal and has the left-fusion property, it follows from the previous theorem that $\mathsf{RO}_{12}(\mathscr{C})$ is a bi-Heyting algebra. Moreover, by Theorem 5.2.14, we also have that $|\cdot|^+$ is a Heyting-homomorphism. Hence in order to prove that it is a bi-Heyting homomorphism, we only have to prove that for any $a, b \in L$, $|a -\!\!< b| = |a| -\!\!<^* |b|$. First of all, notice that

$$G_{|a|} = \{F \; ; \; a \in F\},$$

and that

$$J_{|b|} = \{I \; ; \; b \in I\}.$$

Now assume that $a -\!\!< b \in F$ for some $(F, I) \in X$. Since $-\!\!<$ is monotone in the first component and antitone in the second one, it follows that $a -\!\!< b \leq c -\!\!< d$ for c, d such that $a \leq c$ and $d \leq b$. Hence

$$\{c -\!\!< d \; ; \; c \in \uparrow a, \; d \in \downarrow b\} \subseteq F,$$

which means that $(F, I) \in A -\!\!<^* B$. For the converse, simply notice that if $\{c -\!\!< d \; ; \; c \in F_1, d \in I_1\} \subseteq F$ for some $(F, I) \in X$, $F_1 \in G_{|a|}$ and $I_1 \in J_{|b|}$, then in particular $a -\!\!< b \in F$, since $a \in F_1$ and $b \in J_1$. Hence

$$A -\!\!<^* B \subseteq I_1 C_2(|a -\!\!< b|) = |a -\!\!< b|,$$

which completes the proof. The proof that $(\mathsf{RC}_{32}(\mathscr{C}), -|\cdot|^-)$ is also a completion of L is completely similar, and therefore left to the reader. $\square$

It is worth noting however that the previous theorem does not immediately yield the existence of Q-completions for bi-Heyting algebras. The reason is that one of the hypotheses of the theorem is that $\mathscr{C}$ is a cofinal set. But this implies that the completion $\mathsf{RO}_{12}(\mathscr{C})$ obtained is isomorphic to the canonical extension of L, and not its Q-completion. We left this question as an open problem.

6.3 Comparison of IP-spaces with Related Frameworks

In the last section of this chapter, we compare IP-spaces with some related work. We first show how IP-spaces generalize both classical possibility semantics and Priestley spaces. Moreover, we show that there exists a tight connection between IP-spaces and some frameworks that have been proposed as generalizations of possibility semantics to intuitionistic semantics.

6.3.1 Canonical IP-Spaces and Classical Possibility Frames

In this section, we prove that the canonical R-space of a Heyting algebra is a generalization of the canonical possibility frame of a Boolean algebra in the following sense: whenever B is a Boolean algebra, then the algebra of regular opens of its canonical possibility frame is isomorphic to the algebra of regular opens of its canonical R-space. In particular, the following lemma shows why the use of pairs (F, I) rather than filters is superfluous whenever B is a Boolean algebra:

Lemma 6.3.1. *Let B be a Boolean algebra. For any filter F and ideal I over B, (F, I) is a pseudo-complete pair iff $I = F^\delta$, where $F^\delta = \{\neg a \; ; \; a \in F\}$*

Proof. We prove that the following holds for any Boolean algebra B and F and I a filter and an ideal over B respectively:

- If (F, I) has the RMP, then $a \in F \Rightarrow \neg a \in I$, and if (F, I) has the LJP, then $\neg a \in I \Rightarrow a \in F$. The first implication follows from the fact that $0 \in I$ for any I and $a \wedge \neg a \leq 0$ for any $a \in B$. The second implication follows from the fact that $1 \in F$ for any F and $1 \leq a \vee \neg a$ for any $a \in B$.

- (F, F^δ) has the RMP and the LJP. To see that (F, F_δ) has the RMP, assume $a \wedge b \leq c$ for some $a \in F, c \in F^\delta$. Then $\neg c \leq \neg(a \wedge b) = \neg a \vee \neg b$. But since $c \in F^\delta$, $\neg c \in F$, and moreover $(\neg a \vee \neg b) \wedge a \leq \neg b$, hence $\neg b \in F^\delta$, which means that $b \in F^\delta$. To see that (F, F^δ) has the LJP, assume $a \leq b \vee c$ for some $a \in F, c \in F^\delta$. Then $\neg b \wedge \neg c \leq \neg(b \vee c) \leq \neg a$. Moreover, $\neg a \in F^\delta$ since $a \in F$, and $\neg b \leq (\neg b \wedge \neg c) \vee c$, and hence $\neg b \in F^\delta$, which meant that $b \in F$.

□

We can therefore prove that IP-spaces are a generalization of classical possibility spaces in the following sense:

Lemma 6.3.2. *Let B be a Boolean algebra, $(X_1, \leq)$ its canonical possibility frame, and $(X, \leq_1, \leq_2)$ its canonical R-frame. Then $\mathsf{RO}(X_1)$ is isomorphic to $\mathsf{RO}_{12}(X)$.*

Proof. By Lemma 6.3.1, for any $(F, I) \in X$, $I = F^\delta$. Hence for any filter F over B, there is a unique $(F', I') \in X$ such that $F = F'$, and hence $(X_1, \leq)$ is order isomorphic to $(X, \leq_1, \leq_2)$. Moreover, for any filters F and F' over B, $F \subseteq F'$ iff $F^\delta \subseteq F'^\delta$. Hence the filter inclusion ordering $\leq_1$ and the filter-ideal inclusion ordering $\leq_2$ coincide on X. From this it follows that for any $U \subseteq X$, $I_1C_2(U) = I_1C_1(X)$. Hence $\mathsf{RO}_{12}(X) \cong \mathsf{RO}_{11}(X) \cong \mathsf{RO}(X_1)$. □

Moreover, Lemma 6.3.1 also implies the following important fact regarding the relationship between Lemma 5.3.3 (the Q-Lemma for distributive lattices) and Tarski's Lemma:

Lemma 6.3.3. *Lemma 5.3.3 and Tarski's Lemma are equivalent over ZF.*

Proof. Let B be a Boolean algebra, Q a countable set of meets, and let $a \in B$ such that $a \neq 0$. Since all meets are distributive in a Boolean algebra, and any Boolean algebra is a distributive lattice, we can apply Lemma 5.3.3 with $Q_M = Q$ and $Q_J = \emptyset$. This means that we obtain a compatible Q-pair (F, I) such that $a \in F$. Now consider $(F^*, I^*) = (F, I)^*$, and note that by Lemma 6.3.1, we have that if $a \in I^*$, then $\neq a \in F^*$. Moreover, since (F, I) is a Q-pair and we have that $F \subseteq F^*$ and $I \subseteq I^*$, it follows that $a \in F^*$, and that (F^*, I^*) is also a Q-pair. Now consider $\bigwedge X \in Q$. Then either $\bigwedge X \in F^*$, or $a \in I^*$, which, by Lemma 6.3.1, implies that $\neg a \in F^*$. Therefore F^* is the required filter.
For the converse direction, recall that the only non-constructive part of the proof of Lemma 5.3.3 was the use of the Axiom of Dependent Choices. Since Tarski's Lemma (TL) is equivalent to (DC) over ZF[1], it follows that $ZF + TL$ implies $ZF + QDL$. □

We can conclude with the following corollary:

Corollary 6.3.4. *The Rasiowa-Sikorski Lemma for Boolean algebras (RS(BA)), the conjunction of BPI and TL, the conjunction of PFT and QDL, and the Rasiowa-Sikorski Lemma for distributive lattices (RS(DL)) are all equivalent over ZF.*

Proof. Immediate from the previous lemma and Lemma 5.3.17 □

[1]By Goldblatt's theorem [29]

6.3.2 Canonical IP-spaces and Dual Priestley Spaces

In this section, we show how, for any distributive lattice L, the algebra of regular opens of its canonical IP-space is isomorphic to the upsets of its dual Priestley space. In order to achieve this result, however, we will make use of the Prime Filter Theorem.

Lemma 6.3.5. *Let L be a distributive lattice, and $(\mathscr{C}, \tau_1, \tau_2)$ and $(X_L, \tau, \leq)$ be the canonical IP-space of L and its dual Priestley space respectively. Then $\mathsf{RO}_{12}(\mathscr{C})$ is isomorphic to $Up(X_L)$.*

Proof. Let us define two maps: $\Gamma : \mathscr{P}(\mathscr{C}) \to \mathscr{P}(X_L)$ and $\Delta : \mathscr{P}(X_L) \to \mathscr{P}(\mathscr{C})$ such that for any $A \in \mathscr{C}$ and $B \in X_L$,

$$\Gamma(A) = \{F \in X_L \ ; \ \mathscr{O}^+(F) \subseteq A\}$$

and

$$\Delta(B) = \{(F, I) \in \mathscr{C} \ ; \ \mathscr{O}^+(F) \cap X_L \subseteq B\}.$$

- We first prove that Γ and Δ reduce to maps with codomains $Up(X_L)$ and $\mathsf{RO}_{12}(\mathscr{C})$ respectively. First, it is straightforward to see that $\Gamma(A) \in Up(X_L)$ for any $A \in \mathscr{P}(\mathscr{C})$. Moreover, note that for any prime filter F, (F, F^c) is a pseudo-complete pair that is maximal with respect to $\leq_2$. Now for any $B \in X_L$, and any $(F, I) \in \mathscr{C}$, if $(F, I) \notin \Delta(\mathscr{C})$, this means that there exists a prime filter $F' \notin B$ such that $F \subseteq F'$. But then $(F, I) \leq_1 (F', F'^c)$ and $(F', F'^c) \in I_2(-\Delta(B))$. Hence $I_1C_2(\Delta(B)) \subseteq \Delta(B)$. Moreover, it is easy to see that $\Delta(B)$ is an upset with respect to $\leq_1$, and therefore $\Delta(B) = I_1C_2(\Delta(B))$.

- Second, we prove that both $\Delta : \mathsf{RO}_{12}(\mathscr{C}) \to Up(X_L)$ and $\Gamma : Up(X_L) \to \mathsf{RO}_{12}(\mathscr{C})$ are injective. Let $A \neq A' \in \mathsf{RO}_{12}(\mathscr{C})$, and assume w.l.o.g. that $(F, I) \in A$ and $(F, I) \notin A'$. This means that there is $(F', I') \in \mathscr{C}$ such that $(F, I) \leq_1 (F', I')$ and for all $(F^*, I^*) \in \mathscr{C}$, if $(F', I') \leq_2 (F^*, I^*)$, then $(F^*, I^*) \notin A'$. By the Prime Filter Theorem, (F', I') extends to a pair (F^*, F^{*c}) such that $F' \leq F^*$ and $I' \leq F^{*c}$. Hence $F^* \notin \Gamma(A')$. On the other hand, since $(F, I) \in A$ and $F \subseteq F^*$, it follows at once that $F^* \in \Gamma(A)$.
  For the other direction, assume $B \neq B' \in Up(X_L)$. Then, w.l.o.g., there is $(F \in X_L)$ such that $F \in B$ but $F \in B'$. But then $(F, F^c) \in \Delta(B)$ and $(F, F^c) \notin \Delta(B')$.

- Third, we check that $\Delta\Gamma(A) = A$ for any $A \in RO_{12}(\mathscr{C})$, and $\Gamma\Delta(B) = B$ for any $B \in Up(X_L)$. Notice first that it is straightforward to see that both $\Delta\Gamma$ and $\Gamma\Delta$ are increasing on $\mathsf{RO}_{12}(\mathscr{C})$ and $Up(X_L)$ respectively. Moreover, as noted above, for any $(F, I) \in \mathscr{C}$ and $A \in \mathsf{RO}_{12}(\mathscr{C})$, if $(F, I) \notin A$, then by the PFT there exists a prime filter F^* such that $F \leq F^*$ and $F^* \notin A$. Hence $\Delta\Gamma(A) \leq A$ for any $A \in RO_{12}(\mathscr{C})$. Moreover, for any $F \in X_L$ and any $B \in Up(X_L)$, if $F \in \Gamma\Delta(B)$, then in particular $(F, F^c) \in \Delta(B)$. But this in turn implies that $F \in B$. Hence $\Gamma\Delta(B) \subseteq B$ for any $B \ Up(X_L)$.

- Finally, it is straightforward to check that both Γ and Δ are DL homomorphisms, and Heyting homomorphisms if L is a Heyting algebra. Details for this part are therefore left to the reader.

$\square$

Note that a shorter, more abstract proof of this result also follows from the previous section. Recall first that the algebra of upsets of the dual Priestley space of a distributive lattice L is always isomorphic to the canonical extension of L. Moreover, since the set $\mathscr{C}$ of all pseudo-complete pairs over L forms a cofinal class with the left-fusion property, it follows from by Lemmas 6.2.11 and 6.2.19 that $\mathsf{RO}_{12}(\mathscr{C})$ is also isomorphic to the canonical extension of L. However, the maps Γ and Δ between $RO_{12}(\mathscr{C})$ and $Up(X_L)$ that we constructed in the proof of

Lemma 6.3.5 show how the refined regular open sets of compatible pairs are precisely the sets that are completely determined by their prime pairs. This completely mirrors the situation in classical possibility semantics, since the regular open sets of the canonical possibility space of a Boolean algebra B are precisely the sets of filters over B that are completely determined by the ultrafilters they contain.

6.3.3 FM-frames and Dragalin frames

Finally, we can now precisely determine the level of generality of intuitionistic possibility spaces for Heyting algebras. In fact, building on the work of Bezhanishvili and Holliday [7], we prove an equivalence between IP-spaces, Fairtlough-Mendler frames, and Dragalin frames.[2] We first recall some definitions for both frameworks.

Definition 6.3.6. [19] A *Fairtlough-Mendler frame* (FM-frame) $\mathfrak{F}$ is a structure $(X, \leq_1, \leq_2, V)$, where $\leq_1$ and $\leq_2$ are two preorders such that $\leq_2 \subseteq \leq_1$, and $V \subseteq X$ is an upset with respect to $\leq_1$. We say that $\mathfrak{F}$ is *normal* if $V = \emptyset$.

Definition 6.3.7. [15], [7] A Dragalin frame is a triple $\mathfrak{F} = (S, \leq, D)$ such that $(S, \leq)$ is a poset and $D : S \to \mathscr{P}(\mathscr{P}(S)$ is a Dragalin funciton, i.e. satisfies the following requirements for any $x, y \in S$ and $X, Y \subseteq S$:

1. $\emptyset \notin D(x)$;
2. if $y \in X$ and $X \in D(x)$, then there is $z \in X$ such that $x \leq z$ and $y \leq z$;
3. if $x \leq y$, then for all $Y \in D(y)$, there is $X \in D(x)$ such that $X \subseteq \downarrow Y$;
4. if $y \in X$ and $X \in D(x)$, then there is $Y \in D(y)$ such that $Y \subseteq \downarrow X$.

$\mathfrak{F}$ is *normal* if for any $x \in S$, $D(x) \neq \emptyset$.

As the next lemma shows, FM-frames and Dragalin frames are tightly connected to intuitionistic possibility spaces, since every FM-frame and every Dragalin frame gives rise to a nuclear algebra (L, j), i.e. a complete Heyting algebra L with a nucleus j on L.

Lemma 6.3.8. *(Fairtlough-Mendler, Dragalin)*

1. *Let $\mathfrak{F} = (X, \leq_1, \leq_2, V)$ be a FM-frame, and let $Up(\mathfrak{F})_F = \{U \subseteq Y \ ; \ V \subseteq U\} \cap Up(\mathfrak{F})$, where $Up(\mathfrak{F})$ is the set of upsets with respect to $\leq_1$ in $\mathfrak{F}$. Moreover, define an operation $\Box_{\leq_1}\Diamond_{\leq_2}$ on $Up(\mathfrak{F})_F$ such that for any $U \in Up(\mathfrak{F})_F$, we have*
$$\Box_{\leq_1}\Diamond_{\leq_2}U = \{x \in X \ ; \ \forall y \in X : x \leq_1 y \Rightarrow \exists z \in U \ ; \ y \leq_2 z\}.$$
Then $Up(\mathfrak{F})_F$ is a complete Heyting algebra, and $\Box_{\leq_1}\Diamond_{\leq_2}$ is a nucleus on $Up(\mathfrak{F})_F$.
2. *Let $\mathfrak{G} = (S, \leq, D)$ be a Dragalin frame, and let $Up(\mathfrak{G})$ be the set of upsets of $\mathfrak{F}$. Then the operation $j : Up(\mathfrak{G} \to Up(\mathfrak{G})$ defined by $j(U) = \{x \in S \ ; \ \forall X \in D(x) X \cap U \neq \emptyset\}$ is a nucleus on the complete Heyting algebra $Up(\mathfrak{G})$.*

Note that this means that we can associate to any Dragalin frame and any FM-frame a nuclear algebra (L, j), and more importantly another complete Heyting algebra L' by taking the fixpoints of j in L. Dragalin [15] showed that every spatial locale, i.e. every complete Heyting algebra isomorphic to the lattice of open sets of some topological space, is isomorphic to the algebra

[2] We thank Wes Holliday for raising this question in a private conversation.

of fixpoints of some Dragalin frame. Moreover, Bezhanishvili and Holliday [7] strengthened Dragalin's result and proved that *any* complete Heyting algebra can be represented in this way. Here, we provide another proof of this result based on our previous work on possibility spaces. The key fact is the following theorem, which shows that possibility spaces, FM-frames and Dragalin frames are equivalent in the following sense:

Theorem 6.3.9.

1. *For any refined Alexandroff possibility space (X, τ_1, τ_2), there exists a normal FM-frame $\mathfrak{G}$ such that the algebra of regular opens of (X, τ_1, τ_2) is isomorphic to the algebra of fixpoints of $\mathfrak{G}$.*

2. *For any normal FM-frame $\mathfrak{G}$, there exists a normal Dragalin frame $\mathfrak{F}$ such that the algebra of fixpoints of $\mathfrak{G}$ is isomorphic to the algebra of fixpoints $\mathfrak{F}$.*

3. *For any Dragalin frame $\mathfrak{F}$, there exists a refined Alexandroff possibility space (X, τ_1, τ_2) such that the algebra of fixpoints $\mathfrak{F}$ is isomorphic to the algebra of regular open sets of (X, τ_1, τ_2).*

Proof.

1. Let (X, τ_1, τ_2) be a refined Alexandroff possibility space, and let $\leq_1$, $\leq_2$ be the specialization preorder of τ_1 and τ_2 respectively. Then, since $\tau_1 \subseteq \tau_2$, it follows that $(X, \leq_1, \leq_2, \emptyset)$ is a normal FM-frame. Moreover it is immediate to see that for any $U \subseteq X$, $I_1 C_2(U) = U$ iff $U = \{x \in X ; \forall y \in X : x \leq_1 y \Rightarrow \exists z \in U ; y \leq_2 z\}$. Hence $\mathsf{RO}_{12}(X)$ is isomorphic to the algebra of fixpoints of $\Box_{\leq_1} \Diamond_{\leq_2}$.

2. This result was proved by Bezhanishvili and Holliday [7].

3. Let $\mathfrak{F}$ be a Dragalin frame. Then the algebra L of fixpoints of $\mathfrak{F}$ is a complete Heyting algebra, and therefore, by Theorem 6.2.22, there is a set $\mathscr{C}$ of pairs over L such that L is isomorphic to the algebra of regular opens of $(\mathscr{C}, \tau_1, \tau_2)$. But $(\mathscr{C}, \tau_1, \tau_2)$ is a refined Alexandroff possibility space, which completes the proof.

□

To conclude the proof of our claim, it now suffices to prove the following consequence of Theorem 6.3.9 :

Corollary 6.3.10. *Any complete Heyting algebra L is isomorphic to the algebra of fixpoints of some FM-frame and of some Dragalin frame.*

Proof. Let L be a complete Heyting algebra. Then, by Theorem 6.2.22, L is isomorphic to the algebra of regular opens $\mathsf{RO}_{12}(X)$ of some refined Alexandroff possibility space (X, τ_1, τ_2). But then, by Corollary 5.4.7, there exists a FM-frame $\mathfrak{F}$ and a Dragalin frame $\mathfrak{G}$ such that $\mathsf{RO}_{12}(X)$ is isomorphic to the algebra of fixpoints of $\mathfrak{F}$ and the algebra of fixpoints of $\mathfrak{G}$. □

An important consequence of Corollary 6.3.10 is that FM-frames, Dragalin frames and IP-space all provide the basis for a semantics for propositional intuitionistic logic that is strictly more general than Kripke semantics, and at least as general as topological semantics. In particular, this means that IP-spaces may be suitable framework for the study of superintuitionistic logics, although this is a topic that we have not touched in this thesis.

6.4 Conclusion of this chapter

In this chapter, we have studied in more detail several aspects of the framework of IP-spaces defined in the previous chapter. In particular, we have shown that the ideas and methods developed for Heyting algebras could also be applied in a straightforward way in the setting of co-Heyting algebras, and to some extent in the case of bi-Heyting algebras. Moreover, we have used a slight generalization of IP-spaces to provide topological representations for completions of lattices, thus generalizing some of the results of Chapter 4, section 3. Finally, we have shown that IP-spaces generalize in some sense both classical possibility spaces and Priestley spaces, and we have related them to other frameworks that have been proposed as a generalization of possibility semantics for intuitionistic logic.

Chapter 7

Conclusion and Future Work

We conclude with a brief recap of the main elements of the thesis, and list some ideas for future research.

7.1 Summary of the Thesis

The starting point of this thesis was the combination of two results: the first is Goldblatt's topological proofs of the Rasiowa-Sikorski Lemma for distributive lattices and Heyting algebras [30]. The second is the observation he makes in [29] that the Rasiowa-Sikorski Lemma is equivalent to the conjunction of the Boolean Prime Ideal Theorem and Tarski's Lemma. In this thesis, we proved some straightforward generalizations of the Rasiowa-Sikorski Lemma using Goldblatt's method, and used those versions of the Rasiowa-Sikorski Lemma to give algebraic completeness proofs for several first-order calculi. Moreover, we showed how Tarski's Lemma and the framework of possibility semantics enable us to provide an alternative semantics for CPL. In order to achieve similar results for intuitionistic logic, we proposed a generalization of possibility semantics to intuitionistic possibility spaces, and defined a new semantics for IPL. In particular, the proof relies on an adaptation of Tarski's Lemma to the setting of distributive lattices, called the Q-Lemma, and on a similar statement for Heyting algebras. Finally, we showed how the ideas and methods developed for Heyting algebras can easily be adapted in the case of co-Heyting algebras, and compared our framework with some related work.

7.2 Future Work

- Further generalizations of possibility semantics and Tarski's Lemma:
  It seems natural to wonder about possible generalizations of the ideas involved in intuitionistic possibility spaces to some other logics that we have introduced. In particular, we have not investigated possible analogues of Tarski's Lemma for modal Heyting and co-Heyting algebras. However, it seems very likely that a straightforward combination of ideas from Lemmas 4.4.6, 5.3.16 and 6.1.11 could yield a proof of the following conjecture:

  Conjecture 7.2.1.

  – *Let $(L, \Box)$ be a modal Heyting algebra, and Q_M, Q_J, two countable $(\Box, \to)$-complete sets of Barcan meets and joins over L. Then for any compatible Q-pair (F, I) over L, and any $a \in L$, if $\Box a \notin F$, then there exists a compatible Q-pair (F', I') such that $F^{\Box} \subseteq F'$ and $a \in I'$.*

– *Dually, let $(M, \Diamond)$ be a modal co-Heyting algebra, and Q_M, Q_J, two countable $(\Diamond, \prec)$-complete sets of Barcan meets and joins over L. Then for any compatible Q-pair (F, I) over M, and any $a \in M$, if $\Diamond a \notin I$, then there exists a compatible Q-pair (F', I') such that $a \in F'$ and $I_\Diamond \subseteq I'$.*

Similarly, one can expect that the method applied in Lemma 4.4.2, which yields a choice-free representation theorem for BAO's via canonical possibility spaces, could generalize in a straightforward way to the setting of modal Heyting and co-Heyting algebras. However, it is not clear whether this would be enough to obtain an immediate proof that every modal Heyting algebra and every modal co-Heyting has a Q-completion, or if we would encounter the same problem as in the case of bi-Heyting algebras.

- Topological representations of non-distributive lattices:
  In the last chapter, we prove that every lattice can be embedded into the regular open sets of some bi-topological space, and show how this yields a characterization of distributive lattices in terms of properties of sets of compatible pairs. These preliminary results could motivate new investigations in the representation theory of non-distributive lattices via bi-topological spaces. Here, a thorough comparison of generalized possibility spaces with the literature on canonical extensions could be of great interest. In particular, the ongoing effort to classify completions of lattices and study them via polarities between sets of filters and ideals (in particular in [16] and [25]) seems tightly connected to the approach in terms of compatible pairs that we have undertaken in this work. If this connection were to be made explicit, one could hope that this would open promising leads for the study of non-distributive lattices, and in particular for the rich and complex family of residuated lattices (see [23], [24]).

- Further applications of the Q-Lemma:
  As we repeatedly emphasized, one of the main interests of the Q-Lemma for distributive lattices is that, unlike the Rasiowa-Sikorski Lemma, it does not presuppose the Prime Filter Theorem, nor any topological representation of distributive lattices. This means in particular that it could be of some interest to some areas of set theory where the full axiom of choice is not available. [1] Moreover, since the proof of the Q-Lemma is purely algebraic, one could think that the method used there can be generalized more easily to non-distributive lattices. A simple argument shows however that the technical lemma (Lemma 5.2.5) that is used in the inductive step of the proof of the Q-Lemma is true in a lattice L if and only if L is distributive.[2] This seems to suggest that, as in the case of Tarski's Lemma, a generalization of the Q-Lemma to non-distributive lattices might require a substantial modification of its statement.

[1] As an example, it is an important result in contemporary set theory that the Axiom of Determinacy (AD) is inconsistent with the Axiom of Choice (AC), but not with the Axiom of Dependent Choices (DC) (see [50]). In fact, (AD) implies that every ultrafilter on ω is principal, and therefore is not even consistent with (BPI). This means that any model of $ZF + AD + DC$ provides an example of a set-theoretic universe where QDL holds, while RS(DL) does not.

[2] For the left-to-right direction: it is a well-known fact that a lattice L is distributive if and only if neither the pentagon N5 nor the diamond M5 are sublattices of L, and it is straightforward to see that Lemma 5.2.5 is false for both lattices.

Appendix A

Appendix

A.1 Proofs for Section 6.1

Lemma A.1.1 (Lemma 6.1.2). *Let (X, τ_1, τ_2) be a refined bi-topological space, and let C_1 be the co-Heyting algebra of closed sets under τ_1. Then $C_1 I_2$ is a co-nucleus on C_1.*

Proof. We first prove that $C_1 I_2$ is a kernel operator.

i) Monotonicity is obvious, since both C_1 and I_2 are monotone operators.

ii) To prove that $C1I_2(U) \subseteq U$ for all $U \in \mathsf{C}_1$, notice that $I_2(U) \subseteq U$ and that $C_1(U) \subseteq U$ for any such U. Hence $C_1 I_2(U) \subseteq C_1(U) \subseteq U$.

iii) For idempotence, recall that for any $V \in \tau_1$, $I_1 C_2 I_1 C_2(V) = I_1 C_2(V)$ since $I_1 C_2$ is a nucleus on O_1. Now for every $U \in \mathsf{C}_1$, $-U \in \mathsf{O}_1$, Hence $C_1 I_2(U) = -I_1 C_2(-U) = -I_1 C_2 I_1 C_2(-U) = C_1 I_2 C_1 I_2(U)$.

It remains to be proved that for any $U, V \in \mathsf{C}_1$, $C_1 I_2(U \cup V) = C_1 I_2(U) \cup C_1 I_2(V)$. Again, recall that since $I_1 C_2$ is a nucleus on O_1, $I_1 C_2(A \cap B) = I_1 C_2(A) \cap I_1 C_2(B)$ for any $A, B \in \mathsf{O}_1$. Hence for any $U, V \in \mathsf{C}_1$,

$$C_1 I_2(U \cup V) = -I_1 C_2 - (U \cup V) = -I_1 C_2(-U \cap -V) =$$

$$= -(I_1 C_2(-U) \cap I_1 C_2(-V)) = -I_1 C_2(-U) \cup -I_1 C_2(-V) = C_1 I_2(U) \cup C_1 I_2(V).$$

□

Lemma A.1.2 (Lemma 6.1.5). *Let L be a distributive lattice, and let (X, τ_3, τ_2) be its co-canonical R-space, and let $|\cdot|$ and $|\cdot|^-$ be the positive and negative maps from L to $\mathscr{P}(X)$ respectively. For every $a \in L$, $C_3 I_2(-|a|^-) = -|a|^-$.*

Proof. Let $a \in L$. We first prove that $C_2(|a|^-) = -|a|$. Note first that $|a|^- \subseteq -|a|$ since $F \cap I = \emptyset$ for every $(F, I) \in X$, and since $|a| \in \tau_2$, it follows that $C_2(|a|^-) \subseteq -|a|$. For the converse, consider $(F, I) \in X$ such that $a \notin F$. Then since (F, I) is pseudo-complete, by Lemma 5.2.2 it follows that $(F, I \vee a)$ is a compatible pair, hence $(F, I \vee a)^* \in X$, which means that $(F, I) \in C_2(|a|-)$. Hence $C_2(|a|^-) = -|a|$, which means that $I_2(-|a|^-) = -C_2(|a|^-) = |a|$. Hence if we prove that $C_3(|a|) = -|a|^-$, we are done. Notice first that since all pairs in X are compatible, $|a| \subseteq -|a|^-$, and since $|a|^- \in \tau_3$, it follows that $C_3(|a|) \subseteq -|a|^-$. For the converse, assume that $a \notin I$ for some pair $(F, I) \in X$. Then $(\uparrow a, I)$ is a compatible pair, and hence $(\uparrow a, I)^* \in X$, which means that $(F, I) \in C_3(|a|)$. Hence $C_3(|a|) = -|a|^-$, which completes the proof. □

Lemma A.1.3 (Lemma 6.1.8)**.** *Let L be a co-Heyting algebra. Then for any $a, b, c \in L$, $a \prec b \leq ((a \wedge c) \prec b) \vee (a \prec (b \vee c))$*

Proof. Note first that for any $a, b, c, d \in L$,

$$a \wedge (c \vee d) \leq (a \vee d) \wedge (c \vee d) \leq (a \wedge c) \vee d \leq (((a \wedge c) \prec b) \vee b) \vee d \quad (1)$$

Moreover, since $a \leq (a \prec (b \vee c)) \vee (b \vee c)$, by (1) we have that

$$a \leq a \wedge a \leq a \wedge (((a \prec (b \vee c)) \vee b) \vee c) \leq (((a \wedge c) \prec b) \vee b) \vee ((a \prec (b \vee c)) \vee b) \quad (2)$$

But then (2) entails that

$$a \leq (((a \wedge c) \prec b) \vee (a \prec (b \vee c))) \vee b \quad (3)$$

Hence, by residuation,

$$a \prec b \leq ((a \wedge c) \prec b) \vee (a \prec (b \vee c))$$

□

Lemma A.1.4 (Lemma 6.1.10)**.** *Let L be a co-Heyting algebra, Q_M and Q_J as in Definition 6.1.9, and (F, I) a Q-pair such that $a \prec b \notin I$ for some $a, b \in L$. Then:*

1. *If $X \in Q_J$, then $(a \wedge x) \prec b \notin I$ for some $x \in X$, or $a \prec (\bigvee X \vee b) \notin I$*
2. *If $Y \in Q_M$, then $(a \wedge \bigwedge Y) \prec b) \notin I$ or $a \prec (y \vee b) \notin I$ for some $y \in Y$*

Proof. 1. Assume $X \in Q_J$, and $(a \wedge x) \prec b \in I$ for all $x \in X$. Then, since (F, I) is a Q-pair, it follows that $\bigvee((a \wedge x) \prec b) = (a \wedge \bigvee X) \prec b) \in I$. But then, if $a \prec (\bigvee X \vee b) \in I$, then, since by Lemma 6.1.8 $a \prec b \leq ((a \wedge \bigvee X) \prec b) \vee (a \prec (\bigvee X \vee b))$, it follows that $a \prec b \in I$, contradicting our assumption.

2. Assume $Y \in Q_M$, and $a \prec (y \vee b) \in I$ for all $y \in Y$. Then, since (F, I) is a Q-pair, it follows that $\bigvee(a \prec (Y \vee b)) = a \prec (\bigwedge Y \vee b) \in I$. Hence, if $(a \wedge \bigwedge Y) \prec b) \in I$, then, since by Lemma 6.1.8, $a \prec b \leq ((a \wedge \bigwedge Y) \prec b) \vee (a \prec (\bigwedge Y \vee b))$, it follows that $a \prec b \in I$, contradicting again our assumption.

□

A.2 Proofs for Section 6.2

Lemma A.2.1 (Lemma 6.2.5)**.** *Let L be a lattice and $\mathscr{C}$ a separative set of pairs over L, and let $RO_{+-}(\mathscr{C})$ be the set of regular opens in $(\mathscr{C}, \tau_+, \tau_-)$, and $RC_{-+}(\mathscr{C})$ the set of regular closed sets in $(\mathscr{C}, \tau_-, \tau_+)$. Then:*

1. *$\mathsf{RO}_{+-}(\mathscr{C}) = \{RO_{+-}(\mathscr{C}), \cap, \vee, \bigwedge, \bigvee, \emptyset, \mathscr{C}\}$ is a complete lattice, where for any regular open sets A, B, we have $A \vee B = I_+C_-(A \cup B)$, and for any family $\{A_i\}_{i \in I}$ of regular open sets, $\bigwedge_{i \in I} A_i = I_+C_-(\bigcap_{i \in I} A_i)$ and $\bigvee_{i \in I} A_i = I_+C_-(\bigcup_{i \in I} A_i)$. Moreover, $(\mathsf{RO}_{+-}(\mathscr{C}), |\cdot|^+)$ is isomorphic to the MacNeille completion of L.*
2. *Dually, $\mathsf{RC}_{-+}(\mathscr{C}) = \{RC_{-+}(\mathscr{C}), \wedge, \cup, \bigwedge, \bigvee, \emptyset, \mathscr{C}\}$ is a complete lattice, where for any regular closed sets A, B, we have $A \wedge B = C_-I_+(A \cap B)$, and for any family $\{A_i\}_{i \in I}$ of regular open sets, $\bigwedge_{i \in I} A_i = C_-I_+(\bigcap_{i \in I} A_i)$ and $\bigvee_{i \in I} A_i = C_-I_+(\bigcup_{i \in I} A_i)$. Moreover, $(\mathsf{RC}_{-+}(\mathscr{C}), -|\cdot|^-)$ is isomorphic to the MacNeille completion of L.*

Proof. Once again, we only give the proof for $\mathsf{RO}_{+-}(\mathscr{C})$. The dual statement is proved similarly. Note first that, by the previous lemma, I_+C_- is a closure operator on a complete Heyting algebra, and this implies at once that $\mathsf{RO}_{+-}(\mathscr{C})$ is a complete lattice.

In order to show that $(\mathsf{RO}_{+-}(\mathscr{C}), |\cdot|^+)$ is isomorphic to the MacNeille completion of L, we first prove that $|\cdot|^+ : L \to \mathsf{RO}_{+-}(\mathscr{C})$ is an injective homomorphism, and then that the image of L is dense in $(\mathsf{RO}_{+-}(\mathscr{C})$. Note first that, since $\mathscr{C}$ is separative, for any $a, b \in L$, we have

$$|a|^+ \subseteq -|b|^- \Leftrightarrow a \leq b.$$

Moreover, by definition of τ_+ and τ_-, for any $a \in L$, we have

$$C_-|a|^+ = \bigcap\{-|b|^- \ ; \ b \in L, \ |a|^+ \subseteq -|b|^-\},$$

and also

$$C_+|a|^- = \bigcap\{-|c|^+ \ ; \ c \in L, \ |a|^- \subseteq -|c|^+\}.$$

Hence for any $a \in L$, we have

$$C_-|a|^+ = \bigcap\{-|b|^- \ ; \ a \leq b\}$$

and

$$C_+|a|^- = \bigcap\{-|c|^+ \ ; \ c \leq a\}.$$

From this it follows at once that for any $a \in L$, we have $C_-|a|^+ \subseteq -|a|^-$ and $C_+|a|^- \subseteq -|a|^+$.

For the converse of these two statements, note that for any $b, c \in L$, if $b \leq c$, then $|b|^+ \subseteq |c|^+$ and $|c|^- \subseteq |b|^-$, which means that $-|c|^+ \subseteq -|b|^+$ and that $-|b|^- \subseteq -|c|^-$. But then for any $b \in L$ such that $a \leq b$, $-|a|^- \subseteq -|b|^-$, which means that

$$-|a|^- \subseteq C|a|^+,$$

and for any $c \in L$ such that $c \leq a$, $-|a|^+ \subseteq -|c|^+$ which implies that

$$-|a|^+ \subseteq C|a|^-.$$

Therefore for any $a \in L$, we have $C_-|a|^+ = |a|^-$ and $C_+|a|^- = -|a|^+$, which means that

$$I_+C_-|a|^+ = -C_+ - C_-|a|^+ = -C_+|a|^- = |a|^+.$$

Moreover, we have the following for any $a, b \in L$:

- $I_+C_-|0|^+ = \emptyset$, $I_+C_-|1|^+ = X$;
- $I_+C_-|a \wedge b|^+ = I_+C_-|a|^+ \cap I_+C_-|b|^+$, since I_+C_- is a closure operator on τ_+, and $|a \wedge b|^+ = |a|^+ \cap |b|^+$;
- $I_+C_-|a \vee b|^+ = I_+(-|a \vee b|^-) = I_+(-|a|^- \cup -|b|^-) = I_+(C_-|a|^+ \cup C_-|b|^+) = = I_+C_-(|a|^+ \cup |b|^+)$.
- if $a \not\leq b$, then $|a|^+ \not\subseteq |b|^+$, since $\mathscr{C}$ is separative.

Therefore $(\mathsf{RO}_{+-}(\mathscr{C}), |\cdot|^+)$ is a completion of L.

Finally, we show that the image of L is dense in $\mathsf{RO}_{+-}(\mathscr{C})$. Notice first that for any $U \in RO_{+-}(\mathscr{C})$, U is in τ_+, which means that

$$U = \bigcup_{a \in A_U} |a|^+,$$

where

$$A_U = \{a \in L \ ; \ |a|^+ \subseteq U\}.$$

Therefore U is join of images of elements of L, since

$$U = I_+C_-(U) = I_+C_-(\bigcup_{a \in A_U} |a|^+).$$

To see that U is also a meet of images of elements of L, let $B_U = \{b \in L \ ; \ U \subseteq |b|^+\}$. Note first that it is straightforward to see that

$$U \subseteq \bigcap_{b \in B_U} |b|^+,$$

which means that

$$U \subseteq I_+C_-(U) \subseteq I_+C_-(\bigcap_{b \in B_U} |b|^+).$$

For the converse, note that for any $b \in B_U$, we have $|b|^+ \subseteq -|b|^-$. Moreover, for any $c, d \in L$ such that $U \subseteq -|c|^-$, and $|d|^+ \subseteq U$, by separativeness of $\mathscr{C}$ we have that $d \leq c$. But since $U = \bigcup_{a \in A_U} |a|^+$, this means that for any $c \in L$, we have $U \subseteq |c|^+$ iff $U \subseteq -|c|^-$. Hence

$$\bigcap_{b \in B_U} -|b|^- = C_-(U),$$

from which it follows that

$$\bigcap_{b \in B_U} |b|^+ \subseteq \bigcap_{b \in B_U} -|b|^- = C_-(U),$$

and therefore

$$I_+C_-(\bigcap_{b \in B_U} |b|^+) \subseteq I_+C_-C_-(U) = I_+C_-(U) = U.$$

This means that

$$U = I_+C_-(\bigcap_{b \in B_U} |b|^+),$$

which completes the proof that $\mathsf{RO}_{+-}(\mathscr{C})$ is the MacNeille completion of L. □

Lemma A.2.2 (Lemma 6.2.9). *Let L be a lattice, and $\mathscr{C}$ a separative and rich set of pairs over L. Then $(\mathsf{RO}_{13}(\mathscr{C}), |\cdot|^+)$ and $\mathsf{RC}_{31}(\mathscr{C}), -|\cdot|^-$ are a doubly-dense extensions of L.*

Proof. In light of Lemmas 5.2.9 to 5.2.11, in order to prove that $|\cdot|^+ : L \to \mathsf{RO}_{13}(\mathscr{C})$ is a lattice homomorphism, it is enough to prove that for any $a \in L$, we have $C_3(|a|^+) = -|a|^-$ and $C_1(|a|^-) = |a|^+$. Note first that for any $a \in L$, $-|a|^+$ is closed in τ_1 and $-|a|^-$ is closed in τ_2. Moreover, since all pairs in $\mathscr{C}$ are compatible, we have that $C_3(|a|^+) \subseteq -|a|^-$ and $C_1(|a|^-) \subseteq -|a|^+$.

To see the converse in both cases, note that for any $(F, I) \in \mathscr{C}$, by richness of $\mathscr{C}$, if $a \notin F$, then there exists $(F_1, I_1) \in \mathscr{C}$ such that $F \subseteq F'$ and $a \in I_1$, which means that $(F, I) \in C_1(|a|^-)$. Dually, if $a \notin I$, then there exists $(F_2, I_2) \in \mathscr{C}$ such that $I \subseteq I_2$ and $a \in F$, and hence $(F, I) \in C_3(|a|^+)$.

For any $a \in L$, we thus have that $C_3(|a|^+) = -|a|^-$ and $C_1(|a|^-) = -|a|^+$. From this it follows at once that for $a, b \in L$:

- $I_1C_3(|a|^+) = -C_1(|a|^-) = |a|^+$, which means that $|\cdot|^+$ is well-defined
- $I_1C_3(|0|^+) = \emptyset$, $I_1C_3(|1|^+) = \mathscr{C}$
- $I_1C_3(|a \wedge b|^+) = |a \wedge b|^+ = |a|^+ \wedge |b|^+$
- $I_1C_3|a \vee b|^+ = I_1(-|a \vee b|^-) = I_1(-|a|^- \cup -|b|^-) = I_1(C_3|a|^+ \cup C_3|b|^+) = = I_1C_3(|a|^+ \cup |b|^+)$.
- if $a \nleq b$, then $|a|^+ \nsubseteq |b|^+$, since $\mathscr{C}$ is a separative.

Hence $(\mathsf{RO}_{13}(\mathscr{C}), |\cdot|^+)$ is a completion of L. To see that the image of L is doubly dense in $\mathsf{RO}_{13}(\mathscr{C})$, we claim that for any $A \in \mathsf{RO}_{13}(X)$,

1. $$A = I_1C_3(\bigcup_{F \in G_A} \bigcap_{a \in F} |a|^+);$$
2. $$A = \bigcap_{I \in J_A} I_1C_3(\bigcup_{b \in I} |b|^+).$$

For the first equality, note that since A is a upset with respect to $\leq_1$, for any $(F, I) \in \mathscr{C}$, $(F, I) \in A$ iff $F \in G_A$. Hence
$$A = \bigcup_{F \in G_A} \bigcap_{a \in F} |a|^+,$$
which means that
$$A = I_1C_3(A) = I_1C_3(\bigcup_{F \in G_A} \bigcap_{a \in F} |a|^+).$$
For the second equality, assume that $(F, I) \in A$ and that $J \in J_A$. Let (F', I') be a pair in $\mathscr{C}$ such that $F \subseteq F'$. Note that this means that $(F', I') \in A$. Now for any $b \in J$, if there is not $(F^*, I^*) \in \mathscr{C}$ such that $I' \subseteq I^*$ and $b \in F^*$, this means that $(F', I') \in -C_3(|b|^+) = |b|^-$. Hence if
$$(F', I') \in -C_3(\bigcup_{b \in J} |b|^+),$$
it follows that
$$(F', I') \in \bigcap_{b \in J} |b|^-,$$
i.e. that $J \subseteq I'$. But then since $(F', I') \in A$, this means that $J \notin J_A$, contradicting our assumption. Therefore we have
$$A \subseteq \bigcap_{I \in J_A} I_1C_3(\bigcup_{b \in I} |b|^+).$$

For the converse, assume that $(F, I) \notin A$. Since $A = I_1C_3(A)$, this means that there exists $(F', I') \in \mathscr{C}$ such that $F \subseteq F'$ and $\mathscr{O}^-(F', I') \cap A = \emptyset$. But this means that $I' \in J_A$. Moreover, since

$$(F', I') \notin C_3(\bigcup_{b \in I'} |b|^+,$$

this implies that

$$(F, I) \notin \bigcap_{I' \in J_A} I_1C_3(\bigcup_{b \in I'} |b|^+$$

. Therefore, by taking complements, we arrive at

$$\bigcap_{I \in J_A} I_1C_3(\bigcup_{b \in I} |b|^+) \subseteq A.$$

This concludes the proof that the image of L is doubly-dense in $\mathsf{RO}_{12}(\mathscr{C})$, and therefore also that $(\mathsf{RO}_{12}(\mathscr{C}), |\cdot|)$ is a doubly-dense extension of L.

The proof of the dual statement for $\mathsf{RC}_{31}(\mathscr{C})$ is left to the reader. $\square$

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