Dynamic Evidence Logics with Relational Evidence

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Dynamic evidence logics are logics for reasoning about the evidence and evidence-based beliefs of agents in a dynamic environment. This thesis develops a family of dynamic evidence logics which we call *relational evidence logics* (REL). Relational evidence logics aim to contribute to the existing work on evidence logics [1-5] in three main ways. First, while existing evidence logics model pieces of evidence as sets of possible states, REL models represent pieces of evidence as *evidence relations*. Evidence relations order states in terms of their relative plausibility, given a specific observation or instance of communication. Second, REL models include a representation of the *relative reliability* of the available pieces of evidence. This additional structure in the models is used to study reliability-sensitive forms of evidence aggregation. Third, various evidence aggregators are explored, to model alternative policies of the agent towards combining evidence.

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Outline of the thesis

Dynamic evidence logics [1–5] are logics for reasoning about the evidence and evidence-based beliefs of agents in a dynamic environment. The logics are typically presented in two parts. The *static* part enables reasoning about the evidence and beliefs held by an agent at a fixed point in time. The *dynamic* part allows us to reason about the way in which the agent changes her beliefs, and her stock of evidence, as she gathers new information about a situation of interest. Dynamic evidence logics belong to the framework of *dynamic epistemic logic* (DEL), which encompasses a large family of modal logics dealing with information flow in multi-agent systems (for an overview of DEL, see e.g. [6]).

Evidence logics are concerned with scenarios in which an agent collects several pieces of evidence about a situation of interest, from a number of sources, and uses this evidence to form and revise her beliefs about this situation. The agent is typically uncertain about the actual state of affairs, and as a result takes several alternative descriptions of this state as possible. In the logics introduced in [1–5], the agent is assumed to gather evidence of a specific type, which the authors call *binary* evidence. A piece of binary evidence is represented by a subset of the set of possible states. The members of the evidence set are taken to be compatible, or plausible, based on what the source reports. Hence the name ‘binary’; every state is either plausible (‘in’), or implausible (‘out’), according to the source’s report. Moreover, as observed in [3], in the logics of [1–5], the agent treats all evidence sets on a par. There is no explicit modeling of the relative *reliability* of pieces of evidence. Additionally, the evidence logics mentioned above study the evidence and beliefs held by an agent relying on a specific procedure for combining evidence.

This thesis develops a family of dynamic evidence logics which we call *relational evidence logics* (**REL**). Relational evidence logics aim to contribute to the existing work on evidence logics in three main ways.

- *Relax the assumption that all evidence is binary.* Instead of assuming that all evidence is binary, relational evidence logics deal with scenarios in which the evidence reported to the agent is modeled by *evidence relations*. Evidence relations are relations over the set of possible states that put an order over possible states. This ordering is meant to represent the relative plausibility of states, or the degree to which they fit a report given by some source. As discussed in Chapter II.1, a special type of evidence relation (dichotomous weak orders) can be used to model binary evidence in a relational way. Thus, in a way, evidence relations can be seen as a generalisation of evidence sets.

- *Model levels of evidence reliability.* In general, not all evidence is equally reliable, and a rational agent can (and should) calibrate her trust accordingly. To model evidence reliability, we equipped our models with *priority orders*, i.e., orderings of the family of evidence relations according to their relative reliability. Priority orders were introduced in [7], and have already been used in other DEL logics (see, e.g. [8, 9]). Here, we use them to define hierarchies of evidence.

- *Explore alternative evidence aggregation rules.* Our evidence models come equipped with an aggregator, which combines a family of evidence relations, with some priority order defined on it, into a single relation representing the combined plausibility of the possible states. The beliefs of the agent are then defined on the basis of this combined plausibility order. Various classes of relational evidence models, equipped with different evidence aggregators, are studied in some detail in different chapters of this thesis. This gives an insight into the different beliefs an agent may form, depending on the aggregator used.

This thesis is structured as follows. Chapter I.2 fixes some notational conventions that are used throughout the thesis. Chapter II.1 reviews existing models for belief and evidence. In this chapter, we pay special attention to the evidence models used in [1–5], which we call *neighborhood evidence models.* Chapter II.1 introduces relational evidence models, the class of models over which the various logics developed in this thesis are interpreted. We give examples of relational evidence, introduce notions of evidence-based belief for the **REL** setting and discuss ways to connect the neighborhood evidence logic (**NEL**) framework and the **REL** framework developed in this thesis. In Chapter II.2, we initiate our logical study of belief and evidence in the **REL** framework. We zoom into a specific class of **REL** models, the class of models in which all evidence is equally reliable and the evidence is aggregated in an unanimous way, taking the intersection of all the existing evidence relations. One main motivation for exploring this setting is the following; as we shall see, **NEL** models can be related to **REL** models of this type in a natural way. This relationship gives us one way to connect the **REL** framework back to the **NEL** framework which inspired it, before embarking on a more general study of **REL**. The chapter presents a number of dynamic logics for this class of models, which allow us to reason about relational variants of evidential actions first introduced in [3]: evidence addition, evidence upgrade and evidence updates. Chapter II.3 focuses on a different class of models, the ones featuring the *lexicographic rule* as the evidence aggregator. As discussed in Chapter II.1, this rule has appealing aggregative properties and uses the priority order in an intuitive way to revolve conflicts among pieces of evidence. In this chapter, we focus on a specific evidential action which we call *prioritized evidence addition.* Prioritized addition involves adding a piece of evidence to the stock of evidence, and placing it on top of the priority order, as the most reliable piece of evidence. This is reminiscent of the way information is treated in the AGM framework, in which new evidence is assigned a high level of priority (for details about the AGM framework to belief revision, see, e.g., [10]). Finally, in Chapter II.4 we study what we call General **REL**. This is the logic of the class of all **REL** models. In this chapter, we do not fix an aggregator. Instead, we are interested in reasoning about the beliefs that an agent would form, based on her evidence, *irrespective* of the aggregator used, as long as this aggregator satisfies the basic properties built into its definition.

Part I

PRELIMINARIES

Chapter 1

Notational conventions

This section provides some general definitions and notational conventions used throughout this thesis.

1.1 Relations and functions

Definition 1 (Preorder)**.** Let X be a set. A *preorder* on X is a binary relation $R \subseteq X^2$ that is reflexive (for all $x \in X$, Rxx) and transitive (for all $x, y, z \in X$: Rxy and Ryz implies Rxz). For a preorder R on X, we define the following associated relation:

- $R^{<} = \{(x, y) \in X^2 \mid Rxy \text{ and } \neg Ryx\}$
- $R^{\sim} = \{(x, y) \in X^2 \mid Rxy \text{ and } \neg Ryx\}$
- $R^{\bowtie} = \{(x, y) \in X^2 \mid \neg Rxy \text{ and } \neg Ryx\}$

The set of all preorders on X is denoted $Pre(X)$ and the set of all non-empty families of preorders on X is denoted

$$PRE(X) \coloneqq \{\mathscr{R} \mid \mathscr{R} \subseteq Pre(X), \mathscr{R} \neq \emptyset\}$$

◁

Definition 2 (Operations on relations)**.** Let R be a binary relation on X. The *reflexive transitive closure* of R is denoted R^*, and is defined as the smallest reflexive and transitive relation on X which contains R. For an element $x \in X$, $R[x] \coloneqq \{y \in X \mid Rxy\}$ denotes the set of R-successors of x in X. Let R_1 and R_2 be binary relations on X. We denote by $R_1 \circ R_2$ the *composition* of R_1 and R_2

$$R_2 \circ R_1 \coloneqq \left\{(x, z) \in X^2 \mid \exists y \in X : (x, y) \in R_1 \wedge (y, z) \in R_2\right\}$$

◁

1.2 Sequences

The set of all countable sequences of elements of a set X is denoted $S(X)$, and the set of all finite sequences is denoted $S_0(X)$. Elements of $S(X)$ are denoted $\vec{x} = \langle x_0, x_1, \ldots, \rangle$ or occasionally $\langle x_i \rangle_{i \in \alpha}$, where $\alpha \in \omega_1$ (i.e., α is a countable ordinal). The *concatenation* of two sequences $\vec{x_1}$ and $\vec{x_2}$ is denoted $\vec{x_1} \oplus \vec{x_2}$. We generalize sequence concatenation to several sequences (notation: $\bigoplus$) in the standard way:

- $\bigoplus(\vec{x}_1, \vec{x}_2) \coloneqq \vec{x}_1 \oplus \vec{x}_2$;
- $\bigoplus(\vec{x}_1, \ldots, \vec{x}_n) \coloneqq \bigoplus(\vec{x}_1, \ldots, \vec{x}_{n-1}) \oplus \vec{x}_n$

The set of sequences obtained by permuting the elements of $\langle x_i \rangle_{i\in\alpha}$ is denoted $Per(\langle x_i \rangle_{i\in\alpha}) \coloneqq \{\langle x_{\sigma(i)} \rangle_{i\in\alpha} \mid \sigma \text{ is a permutation of } \alpha\}$. The *length* of a sequence $\vec{s}$ (the number of elements it contains) is denoted $\mathsf{len}(\vec{s})$. The set of elements of a sequence $\vec{s}$ is denoted $\mathsf{set}(\vec{s}) \coloneqq \{s \mid s \text{ is an element of } s\}$. For a sequence $\vec{s}$, we denote by $N_{\vec{s}} \coloneqq \{i \mid 0 \leq i \leq \mathsf{len}(s)\}$ the set of non-negative integers up to $\mathsf{len}(s)$.

Chapter 2

Models for Belief and Evidence

This chapter is organized as follows. Section 2.1 reviews *plausibility models*, the most widely used models for epistemic-doxastic logic, and gives the standard plausibility-based notion of belief. Section 2.2 reviews *neighborhood evidence models* (**NEL** models, for short). NEL models are epistemic-doxastic models that include an explicit representation of the evidence underlying the agent's beliefs. Notions of evidence-based belief for the **NEL** setting, and other evidence-related notions are recalled in Sections 2.3-2.4. Section 2.5 reviews evidence dynamics for the **NEL** setting. Finally, Sections 2.6-2.7 review the evidence logic developed in [5], whose syntax will be used in the **REL** logics developed in this thesis.

2.1 Plausibility models

Doxastic logics are logics that allow one to reason about belief in some way. Many doxastic logics are based on modal languages, in which modal operators are used to describe belief, and which are typically interpreted over a certain type of Kripke model called *plausibility model*. In this section, we briefly recall plausibility models and their representation of belief. For more extensive readings on doxastic logic and plausibility models we refer to [11] and further literature in there. Throughout this chapter, we fix a set P of propositional variables.

Definition 3 (Plausibility Model). A (single-agent) *plausibility model* is a tuple $\mathcal{M} = (S, \preceq, V)$ where

- W is a non-empty set of *states* (or 'possible worlds');
- $\preceq \subseteq W^2$ is a *preorder*;
- $V : \mathsf{P} \to \mathscr{P}(W)$ is a *valuation function.*

◁

The idea behind plausibility models is the following. W is a set of *possible worlds*. Intuitively, these are all the ways a situation of interest could have been, from the agent's point of view. For example, when tossing a fair die, it is reasonable to consider six possible states, one for each of the ways the die could land. The formulas $p \in \mathsf{P}$ stand for basic facts about the world, such as 'the die landed 5'. The valuation function V tells us which of these basic facts hold at which states. The relation $\preceq$, called a *plausibility order*, represents the relative plausibility that the agent assigns to different states. For every two states $w, v \in W$, $w \preceq v$ reads as: 'the agent considers v at least as plausible as w'. The most plausible states are the agent's best candidates for the actual situation. Given a subset $U \subseteq W$, we denote by $Max_{\preceq}U$ the set of maximal $\preceq$-states of U:

$$Max_{\preceq}U := \{w \in U \mid \text{ for all } v \in U(w \preceq v \Rightarrow v \preceq w)\}$$

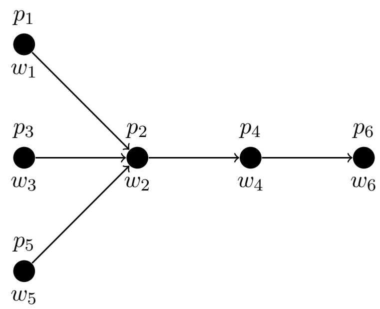
The objects that are believed by the agent are usually called *propositions*. Formally, a proposition P is just a set of possible worlds. We write BP to denote that the agent believes P.

Grove's notion of belief. The standard notion of belief in plausibility models, due to Grove, is the following

$$BP \text{ holds (at any state) iff } Max_{\preceq}W \subseteq P$$

That is, the agent believes P iff all the most plausible states are in P.

Example 1 (The biased die)**.** A die is tossed. The agent is interested in the outcome of this toss. The toss was hidden from the agent, so the agent considers possible all six outcomes. This is represented by a set W consisting of six possible worlds, $\{w_i \mid i = 1, \ldots, 6\}$. The world w_i is the one where the die landed i, for $i = 1, \ldots, 6$. We describe the basic facts about this situation with set of atomic formulas $\mathsf{P} = \{p_i \mid i = 1, \ldots, 6\}$; p_i stands for 'the die landed i'. Although the agent has not seen the outcome of the toss, she has spoken to the die-maker, who gave her the following information; "this die is biased. From previous rolls of the die, I can tell you that the most likely outcome is 6, followed by 4, followed by 2. The remaining outcomes are all less likely than an even numbered outcome, but their relative likelihood is unknown". Accepting this information as trust-worthy, the agent's initial hypothesis involves taking w_4 to be the most plausible state, followed by w_2, followed by w_6, followed by w_1-w_3 (which are all incomparable in terms of plausibility). The agent's point of view can be represented by the following plausibility model is as follows (reflexive and transitive edges are omitted):



◁

In plausibility models, the information held by the agent is not explicitly represented. It is typically understood that the agent arrived at this plausibility order by merging all her information, but what this information is remains unspecified. As a result, the model indicates whether the agent believes that P, but doesn't keep track of the evidence justifying this belief.

2.2 Neighborhood evidence models

Neighborhood evidence logics employ a different type of doxastic model in which the evidence underlying the agent's beliefs is encoded explicitly. In particular, they replace standard plausibility models with *neighborhood models*. In a neighborhood model, each state is assigned a collection of subsets of the set of states. These collections of subsets are viewed as the evidence that the agent has acquired.

Definition 4 (Neighborhood evidence model)**.** A *neighborhood evidence model* (**NEL** model, for short) is a tuple $M = \langle W, N, V \rangle$ where

- W is a non-empty set of states;
- $N : W \to \mathscr{P}(\mathscr{P}(W))$ is a neighborhood function (which is called an *evidence function* in this context) subject to the following constraints: for each $w \in W$, $\emptyset \notin N(w)$ and $W \in N(w)$;
- $V : \mathsf{P} \to \mathscr{P}(W)$ is a valuation function.

When N is a constant function, we get a *uniform evidence model* which can be conveniently written as $M = \langle W, E_0, V \rangle$, where $E_0 \subseteq \mathscr{P}(W)$ such that $\emptyset \notin E_0$ and $W \in E_0$. The elements $e \in E_0$ are called pieces of *basic evidence.* A uniform evidence model is called *feasible* if E_0 is finite. $\lhd$

In what follows, we will focus on *uniform* **NEL** models. To ease our presentation, we will drop the label 'uniform' and call uniform models simply **NEL** models. Whenever needed, we will always refer to the more general models as *non-uniform* models. The intuitive idea behind a **NEL** model is the following. As in plausibility models, the elements of W represent ways a situation of interest could have been, from the agent's point of view. The agent is assumed to have some basic evidence about this situation, which she has gathered from a variety of sources. These pieces of basic evidence are represented by the elements of E_0. If we have $e \in E_0$, this means that the agent has accepted e as a piece of evidence, which gives a justification for believing that the actual situation is described by one of the states included in e.

The following basic assumptions are implicit in the definition of a **NEL** model:

1. *Sources provide 'binary' evidence*: each piece of evidence $e \in E_0$ is modeled as a set of states $e \subseteq W$. The elements of e can be seen as the plausible states according to e (all equally so), while the remaining ones are deemed non-plausible. There are no 'degrees' of evidential support.
2. *Evidence may be erroneous*: a set recording a piece of evidence need not contain the actual world. Moreover, the agent may not know which evidence set is reliable.
3. *All evidence is equally reliable*: evidence sets are treated on a par, i.e., there is no ordering of evidence sets in terms of their weight or reliability.
4. *Evidence may be jointly inconsistent*: that is, the intersection of all the gathered evidence may be empty.
5. *Although pieces of evidence may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.*

Feasibility introduces an additional assumption; feasible models represent the evidential state of *bounded agents*; agents that can only collect, store and process finitely many pieces of evidence at any given moment.

2.3 Notions of evidence-based belief

Different notions of evidence-based belief have been proposed for **NEL** models. We will briefly review the notions presented by van Benthem and Pacuit [1, 3, 4, 12] and Baltag, Bezhanishvili, Özgün and Smets (BBOS) [5].

van Benthem-Pacuit's notion. There are two equivalent ways to define this notion of belief. The first approach is to introduce a plausibility order that is 'appropriately grounded' on the available evidence, and to define belief via this ordering. The second way is to define the agent's beliefs directly in terms of the available evidence. We will follow the first approach, introducing the notion of an *evidential plausibility order*, and referring the reader to e.g. [1] for details about the direct definition.

Definition 5 (Evidential plausibility order)**.** Given a **NEL** model $M = \langle W, E_0, V\rangle$ and a state $w \in W$, we write $E^w := \{e \in E_0 \mid w \in e\}$ to denote the largest family of evidence consistent with w. The *evidential plausibility order* over W is the preorder $\sqsubseteq_M$ given by

$$w \sqsubseteq_M v \textit{ iff } \forall e \in E_0(w \in e \Rightarrow v \in e) \textit{ iff } E^w \subseteq E^v$$

◁

That is, $w \sqsubseteq_M v$ holds iff every piece of evidence consistent with w is also consistent with v. The van Benthem-Pacuit's notion of evidence-based belief is the following

$$BP \text{ holds (at any state) iff } Max_{\sqsubseteq_M} W \subseteq P$$

As in plausibility models, the agent believes P iff P is true in all the most plausible states. But here the plausibility order is induced by the agent's evidence, rather than taken as a primitive. This notion of belief works well when the evidential plausibility relation is converse well-founded. However, like Grove's notion, it yields inconsistent beliefs in models in which there are no most plausible worlds.

BBOS's notion. In [5], the authors propose an alternative notion of evidence-based belief, which coincides with the one of van Benthem-Pacuit in feasible models, but it also ensures consistency of belief in non-feasible ones. The following notions are used in their definition.

Definition 6 (Body of evidence)**.** Given a **NEL** model $M = \langle W, E_0, V\rangle$, a *body of evidence* is a family $F \subseteq E_0$ such that every non-empty finite subfamily $F' \subseteq F$ is consistent, i.e., $\bigcap F' \neq \emptyset$. A body of evidence F *supports* a proposition P iff $\bigcap F \subseteq P$. We denote by $\mathcal{F}$ the family of all bodies of evidence in M, and by $\mathcal{F}^{\text{fin}}$ the family of all finite bodies of evidence. ◁

Definition 7 (Combined evidence)**.** Given a **NEL** model $M = \langle W, E_0, V\rangle$, a piece of *combined evidence* is any non-empty intersection of finitely many pieces of basic evidence. We denote by

$$E := \{\bigcap F \mid F \in \mathcal{F}^{\text{fin}}\}$$

the family of all combined evidence. A (combined) evidence $e \in E$ *supports* a proposition $P \subseteq W$ iff $e \subseteq P$. (In this case, we also say that e is evidence for P). ◁

Intuitively, basic pieces of evidence $e \in E_0$ are meant to represent information obtained directly by the agent, through observation, by the testimony of others, etc. On the other hand, a piece of combined evidence $e \in E$ represents derived evidence, obtained by the agent by collecting finite families of basic evidence and considering those states that are consistent with all of them.

The notion of evidence-based belief proposed in [5] is the following:

$$BP \text{ holds (at any state) iff } \forall F \in \mathcal{F}^{\text{fin}} \exists F' \in \mathcal{F}^{\text{fin}} (F \subseteq F' \text{ and } \bigcap F' \subseteq P)$$

That is, the agent believes P iff every finite body of evidence can be strengthened to a finite body supporting P.

2.4 Notions of evidence availability

The structure in **NEL** models can be used to introduce evidence-related notions. Note that the pieces of evidence in **NEL** models is not necessarily *factive*. A piece of evidence e is said to be factive at a state w if $w \in e$. That is, a piece of evidence is factive at a state w if, from the local perspective of w, the piece gives reason to think that w is a candidate for the actual state. In the framework of [1], an agent is said to have evidence for a proposition P if there is a piece of evidence e that supports P, i.e., $e \subseteq P$. In [5], notions based on factive evidence are are discussed. To define the various notions, we fix a model $M = \langle W, E_0, V \rangle$.

Basic evidence. The notion of having basic evidence in [5, 12] is as follows:

$$\text{the agent has basic evidence for } P \text{ (at any state) iff } \exists e \in E_0 (w \in e \subseteq P)$$

Basic factive evidence. The following notion concerning factive evidence was introduced in [5]:

$$\text{the agent has basic, factive evidence for } P \text{ at state } w \in W \text{ iff } \exists e \in E_0 (w \in e \subseteq P)$$

Combined evidence. This notion extends the one of having basic evidence to combined pieces of evidence:

$$\text{the agent has combined evidence for } P \text{ (at any state) iff } \exists e \in E (w \in e \subseteq P)$$

Combined, factive evidence. Similarly, we have a generalisation of the notion of having basic, factive evidence:

$$\text{the agent has basic, factive evidence for } P \text{ at state } w \in W \text{ iff } \exists e \in E (w \in e \subseteq P)$$

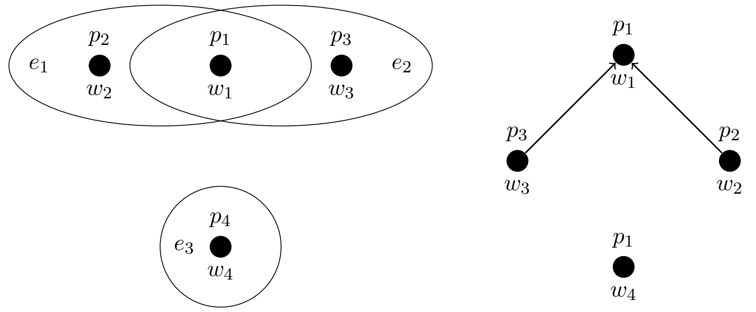


FIGURE 2.1: A **NEL** model (left) and its associated plausibility order $\sqsubseteq$ (right). e_1 is basic evidence for $p_2 \vee p_1$, $e_1 \cap e_2$ is combined evidence for p_1; e_2 is factive evidence for $p_1 \vee p_3$ at w_3

Having presented static notions of belief and evidence possession in **NEL** models, we turn next to evidence dynamics.

2.5 Evidence dynamics

In this section, we review some of the evidence dynamics for **NEL** models introduced in [3]: updates, evidence addition and evidence upgrade. Throughout this section, we fix a **NEL** model $M = \langle W, E_0, V \rangle$ and some proposition $P \subseteq W$, with $P \neq \emptyset$.

Updates (also known as public announcements) involve learning a new fact P with absolute certainty. Upon learning P, the agent rules out all possible states that are incompatible with it. The standard way of modeling this is via model restrictions. For **NEL** models, this means keeping only the worlds in P and only the pieces of evidence that are P-consistent.

Definition 8 (Update)**.** The model $M^{!P} = \langle W^{!P}, E_0^{!P}, V^{!P} \rangle$ has $\langle W^{!P} := P$, $E_0^{!P} := \{e \cap P \mid e \in E_0\}$ and for all $p \in \mathsf{P}$, $V^{!P}(p) := V(p) \cap P$ ◁

Next, we consider evidence addition $+P$, by which the agent accepts P as a new piece of evidence, without assuming P to be infallible information.

Definition 9 (Evidence addition)**.** The model $M^{+P} = \langle W^{+P}, E_0^{+P}, V^{+P} \rangle$ has $\langle W^{+P} := W$, $E_0^{+P} := E_0 \cup \{P\}$ and $V^{+P} := V$. ◁

This operation is weaker than than publicly announcing P, since the agent retains the ability to consistently condition on not P. Moreover, after adding it as a piece of evidence, the agent may not believe P.

Finally, we consider evidence upgrade $\Uparrow P$, which modifies each piece of existing evidence by integrating P into it.

Definition 10 (Evidence upgrade)**.** The model $M^{\Uparrow P} = \langle W^{\Uparrow P}, E_0^{\Uparrow P}, V^{\Uparrow P} \rangle$ has $\langle W^{\Uparrow P} := W$, $E_0^{\Uparrow P} := \{e \cup P \mid e \in E_0\} \cup \{P\}$ and $V^{\Uparrow P} := V$. ◁

This operation is stronger than simply adding P as evidence, since it also modifies each evidence set to make it consistent with P and is moreover sufficient to induce belief in P. But it is still weaker than update , since the agent retains the ability to consistently condition on not P.

In the following sections, we review the **NEL** logic presented in [5], which is based on the work of van Benthem, Pacuit and Fernández-Duque in [1, 3, 12]. We will often refer back to this logic in subsequent chapters, to contrast and relate it to the **REL** logics introduced therein.

2.6 Syntax and semantics for NEL

We first introduce a formal language, which we call $\mathscr{L}$. In [5], this language is called $\mathscr{L}_{\forall\Box\Box_0}$.

Definition 11 (Language $\mathscr{L}$)**.** Let P be a countably infinite set of propositional variables. The language $\mathscr{L}$ is defined by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi$$

where $p \in \mathsf{P}$. We define $\bot := p \wedge \neg p$ and $\top := \neg\bot$. The Boolean connectives $\vee$ and $\rightarrow$ are defined in terms of $\neg$ and $\wedge$ in the standard manner. The duals of the modal operators are defined in the following way: $\Diamond_0 := \neg\Box_0\neg$, $\Diamond := \neg\Box\neg$, $\exists := \neg\forall\neg$. $\lhd$

The intended interpretation of the modalities is as follows. $\Box_0\varphi$ reads as: 'the agent has basic, factive evidence for φ'; $\Box\varphi$ reads as: 'the agent has combined, factive evidence for φ'.

This language is interpreted over **NEL** models as follows.

Definition 12 (Satisfaction)**.** Let $M = \langle W, E_0, V\rangle$ be an **NEL** model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is defined as follows:

$$
\begin{array}{ll}
M, w \models p & \text{iff } w \in V(p) \\
M, w \models \neg\varphi & \text{iff } M, w \not\models \varphi \\
M, w \models \varphi \wedge \psi & \text{iff } M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \Box_0\varphi & \text{iff there is } e \in E_0 \text{ such that } w \in e \subseteq [\![\varphi]\!]_M \\
M, w \models \Box\varphi & \text{iff there is } e \in E \text{ such that } w \in e \subseteq [\![\varphi]\!]_M \\
M, w \models \forall\varphi & \text{iff } W = [\![\varphi]\!]_M
\end{array}
$$

where $[\![\cdot]\!]_M : \mathscr{L} \to 2^W$ is a *truth map* given by: $[\![\varphi]\!]_M = \{w \in W \mid M, w \models \varphi\}$. $\lhd$

Evidence. As expected, the notions of factive evidence availability introduced in Section 2.2.4 are matched by the semantics of the modalities $\Box_0$ and $\Box$. In particular, $\Box_0\varphi$ corresponds to the notion of 'having basic, factive evidence for φ' and $\Box\varphi$ corresponds to that of 'having combined, factive evidence for φ'. Moreover, the notions related to non-factive evidence are definable in the language. In particular, $\exists\Box_0\varphi$ corresponds to 'having basic evidence for φ', while $\exists\Box\varphi$ gives the notion of 'having combined evidence for φ'.

Belief. We recall the notion of evidence-based belief introduced in Section 2.2.3.

$$BP \text{ holds (at any state) iff } \forall F \in \mathcal{F}^{\text{fin}} \exists F' \in \mathcal{F}^{\text{fin}} (F \subseteq F' \text{ and } \bigcap F' \subseteq P)$$

As showed in [5], this notion of belief is definable in terms of $\forall$ and $\Box$. Specifically, we have to put

$$B\varphi := \forall\Diamond\Box\varphi$$

We now review the sound, complete and decidable proof system presented in [5]. We will refer back to this system in Chapter II.2.

2.7 The proof system L_0

Definition 13 (L_0)**.** The proof system L_0 includes the following *axiom schemas* for all formulas $\varphi, \psi \in \mathscr{L}$:

1. All tautologies of propositional logic
2. The $\mathsf{S5}$ axioms for $\forall$:

$$
\begin{array}{ll}
\mathsf{K}_\forall & : \forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi) \\
\mathsf{T}_\forall & : \forall\varphi \to \varphi \\
\mathsf{4}\forall & : \forall\varphi \to \forall\forall\varphi \\
\mathsf{5}_\forall & : \exists\varphi \to \forall\exists\varphi
\end{array}
$$

3. The $\mathsf{S4}$ axioms for $\Box$:

$\mathsf{K}_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\mathsf{T}_\Box$: $\Box\varphi \to \varphi$
$\mathsf{4}_\Box$: $\Box\varphi \to \Box\Box\varphi$

4. Axiom 4 for $\Box_0$:

   $\mathsf{4}_{\Box_0}$: $\Box_0\varphi \to \Box_0\Box_0\varphi$

5. The following interaction axioms:

   (a) $\forall\varphi \to \Box_0\varphi$ (Universality)
   (b) $(\Box_0\varphi \wedge \forall\psi) \leftrightarrow \Box_0(\varphi \wedge \forall\psi)$ (Pullout)
   (c) $\Box_0\varphi \to \Box\varphi$

The system L_0 includes the following *inference rules* for all formulas $\varphi, \psi \in \mathscr{L}$:

1. Modus ponens
2. Necessitation Rule for $\forall$: $\dfrac{\varphi}{\forall\varphi}$
3. Necessitation Rule for $\Box$: $\dfrac{\varphi}{\Box\varphi}$
4. Monotonicity Rule for $\Box_0$: $\dfrac{\varphi \to \psi}{\Box_0\varphi \to \Box_0\psi}$

$\triangleleft$

We denote by Λ_0 the logic generated by L_0. In [5], the authors prove the following result:

Theorem 1 (Theorem 6, [**1**]). *Λ_0 is sound, complete and has the finite model property with respect to the class of* ***NEL*** *models.*

In line with the work in [3], the authors of [5] also present several *dynamic extensions* of $\mathscr{L}$, obtained by adding to $\mathscr{L}$ dynamic modalities $[!\varphi]\psi$ for updates, $[+\varphi]\psi$ for evidence addition and $[\Uparrow \varphi]\psi$ for evidence upgrade. The truth conditions for dynamic formulas are given in terms of the corresponding model change, as standard in dynamic epistemic logic:

- $M, w \models [!\varphi]\psi$ iff $M, w \models \varphi$ implies $M^{!\varphi}, w \models \psi$
- $M, w \models [+\varphi]\psi$ iff $M, w \models \exists\varphi$ implies $M^{+\varphi}, w \models \psi$
- $M, w \models [\Uparrow \varphi]\psi$ iff $M, w \models \exists\varphi$ implies $M^{\Uparrow\varphi}, w \models \psi$

The precondition $M, w \models \varphi$ in the clause for update encodes the fact that updates are *factive*: the agent can only update with true sentences φ. The precondition $M, w \models \exists\varphi$ in the clauses for evidence addition and upgrade encodes the fact that, in order to qualify as (new) evidence, φ has to be *consistent*, that is, $[\![\varphi]\!]_M \neq \emptyset$. For each of the dynamic languages obtained by adding these modalities, the authors present sound and complete proof systems, which are obtained by adding so-called *reduction axioms* to L_0 . We refer the reader to [5] for details on these systems.

Part II

RELATIONAL EVIDENCE LOGIC

Chapter 1

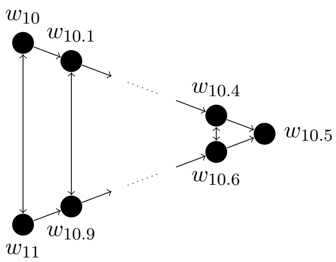
Relational Evidence Models

This chapter introduces *relational evidence models*, the class of models over which the various logics developed in this thesis are interpreted. The chapter is structured as follows. Section 1.1 introduces the notion of *relational evidence* and provides some examples of this type of evidence. Moreover, the formats that we use for modeling relational evidence (*evidence orders*) and its reliability (*priority orders*) are discussed. Section 1.2 introduces *evidence aggregators*, the rules used by the agent to combine relational evidence. Special attention is paid to the *lexicographic rule*, given its appealing aggregative properties and the role it plays in Chapter II.3. After introducing these notions, we present and exemplify relational evidence models in Section 1.3. Section 1.4 introduces notions of belief and evidence-possession for **REL** models. Section 1.5 explores ways to connect **NEL** models and **REL** models in a systematic way. Finally, Section 1.6 fixes the syntax and semantic of a basic static language for **REL**, which will be used throughout the thesis.

1.1 Relational evidence

An evidence item can be understood as an observation or a statement, possibly tainted with uncertainty, forwarded by some source, and describing what the current state of affairs is. The term 'source' has a very general scope here, encompassing anything that is capable of providing information to an agent; another agent, a sensor, memories, etc. The evidence item indicates which states in a set W are good candidates for the actual state, and which ones are not, according to the source. We call *relational evidence* any type of evidence that induces an ordering of states in terms of their relative plausibility. An example of a source that may generate relational evidence is an *imprecise sensor*. In general, real-world sensors have very different degrees of accuracy. For instance, a sonar sensor for measuring the distance to a wall, is relatively imprecise and can provide only a rough estimate of the actual distance. This means that different possible distances to the wall are compatible with the sonar reading, some of which may be taken to be more likely than others, given the particulars of the sensor (its bias, accuracy, etc.). The following is an example of relational evidence obtained from an imprecise sensor.

Example 2 (The thermometer)**.** Consider an agent interested in estimating the temperature in a room, which we assume to lie between 10°C and 11°C. To keep the example simple, let's suppose that the agent considers as possible only the values 10°C, 10.1°C, and so on, up to 11°C. This is represented by a set W consisting of eleven possible worlds, $\{w_{10}, w_{10.1}, \ldots, w_{10.9}, w_{11}\}$. The world w_i is the one where the temperature is i°C. To measure the temperature, the agent uses a well-calibrated thermometer. That a thermometer is well calibrated means here that its bias has been corrected for, so that the errors resulting from its use have a normal distribution, with zero mean and some unknown variance that is characteristic of the instrument. Plainly put, this means that if the thermometer reads 10.5°C, the agent takes 10.5°C to be the most likely value, 10.4°C and 10.6°C to be equiprobable and less likely than 10.5°C, and so on. As the error variance is unknown, assigning exact numerical probabilities to each possible temperature value is tricky. In this case, the agent has no need for exact probabilities; a plausibility ordering, asserting that some temperature values are more likely than others, will do. Suppose the thermometer does read 10.5°C. Given the particulars of the sensor indicated above, the measurement can be identified with the plausibility ordering depicted below (reflexive and transitive arrows are ommitted from the drawing):



◁

Throughout this chapter, we will further illustrate relational evidence discussing other potential sources for this type of evidence. For now, let us fix the general format that we will use to represent relational evidence.

Modeling relational evidence. An appropriate representation for relational evidence, which we adopt, is given by the class of *preorders*. We call preorders representing relational evidence, *evidence orders*. The reason for this specific representation choice is the generality provided by this kind of relation. As is well-known, preorders can represent most types of relational information, including comparisons with incomparable or tied alternatives.

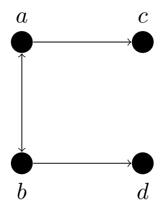


FIGURE 1.1: A preorder over four alternatives, illustrating the variety of pairwise comparisons that may be represented with preorders. c is *strictly preferred* to a. Moreover, the order expresses *indifference* between a and b (the two alternatives are tied), while c and d are *incomparable*.

The reflexivity and transitivity conditions met by any preorder can be seen as encoding basic rationality constraints on plausibility comparisons. Reflexivity simply requires that alternatives are equally plausible to themselves, an arguably rational assumption. Transitivity is also a very common assumption about rational preferences, which is standardly accepted when these preferences represent plausibility or comparative probability. Indeed, if a is less plausible than b, and b is less plausible that c, it seems natural to conclude that a is less plausible than c. More generally, given the famous 'money-pump argument' originally developed by Davidson, McKinsey, and Suppes in [13], it is clear that intransitive preferences can be problematic. Like Dutch book arguments regarding betting, in which the rationality of an agent is questioned because the agent is susceptible to having a book made against her (i.e., to accepting a series of bets which are such that she is bound to lose more than she can gain), the money-pump argument shows how agents with intransitive preferences are vulnerable to making a combination of choices that lead to a sure loss. To illustrate this, consider an agent with the intransitive preference order $\preceq$ over a set of goods. Suppose that $\preceq$ contains the following strict cycle $a \prec b \prec c \prec a$. Let us assume that the agent starts out with good a. Since the agent prefers b to a, he would be willing to pay some amount, let's say 1€, to attain b. So the agent buys b and her holdings are reduced by 1€. As the agent also prefers c to b, he is again willing to forego some money, let's say 1€, to attain c. The agent gives up 1€ and purchases c. As the agent also considers a to be strictly better than c, he is ready to relinquish c and re-purchase a (let's say again for 1€). After doing so, the agent arrives precisely back at the point she started, with a, only she now has lost 3€. Intransitive preferences (in particular, intransitive cyclic preferences) are thus shown to reflect problematic opinions about alternatives.

Ordering evidence in terms of reliability. In neighborhood evidence logics, evidence sets are treated on a par. In general, not all sources are equally trustworthy, so an agent combining evidence may be justified in giving priority to some evidence items over others. Thus, as suggested in [4], a next reasonable step in evidence logics would be to model levels of reliability of evidence. One general format for this is given by the *priority graphs* of [7], which have already been used extensively in dynamic epistemic logic (see, e.g., [8, 9]). In this thesis, we will use the related, yet simpler format of a 'priority order', as used in [14, 15], to represent hierarchy among pieces of evidence. Our definition of a priority order is as follows:

Definition 14 (Priority order)**.** Let $\mathscr{R}$ be a family of evidence orders over W. A *priority order for* $\mathscr{R}$ is a preorder $\preceq$ on $\mathscr{R}$. For $R, R' \in \mathscr{R}$, $R \preceq R'$ reads as: "the evidence order R' has at least the same priority as evidence order R". $\triangleleft$

Notation 1. We use the following abbreviations for priority orders: $R \prec R'$ denotes strict preference ($R \prec R'$ iff $R \preceq R'$ and $R \npreceq R'$), $R \sim R'$ denotes indifference ($R \sim R'$ iff $R \preceq R'$ and $R' \preceq R$) and $R \bowtie R'$ denotes incomparability ($R \bowtie R'$ iff $R \npreceq R'$ and $R' \npreceq R$). $\triangleleft$

Intuitively, priority orders tell us which pieces of evidence are more reliable according to the agent. They give the agent a natural way to break stalemates when faced with inconsistent evidence. Please note that in this thesis, I write $R \preceq R'$ to express that R' has at least the same priority as evidence order R, the opposite notation of that used in [2] (where the higher priority elements are the one lower in the priority order). Accordingly, I also draw pictures for priority orders by putting best evidence orders on top of the order, rather than below. The reader is asked to keep that in mind if reading this chapter and [7] in parallel.

Having discussed the notion of relational evidence, we next explore *evidence aggregators*. These are the rules followed by the agent to combine her available evidence.

1.2 Evidence aggregators

We are interested in modeling a situation in which an agent integrates evidence obtained from multiple sources to obtain and update a combined plausibility ordering, and to form beliefs based on this ordering. When we consider relational evidence with varying levels of priority, a natural way model the process of evidence combination is to define a function that takes as input a family of evidence orders $\mathscr{R}$ together with a priority order $\preceq$ defined on them, and combines them into a plausibility order. The agent's beliefs can then be defined in terms of this output. This is similar with what is done in preference aggregation theory, which studies how the preferences of a group of agents can be combined in a rational way. However, as noted in [2], in a setting such as ours we are working with a richer input that the one usually considered in preference aggregation theory. In the preference aggregation context, the input to the aggregator typically consists of binary relations only, without considerations of relative priority. In our setting, we let the relative reliability of evidence play a role in aggregation. Accordingly, we define our aggregators as functions taking a priority-ordered set of relations as input.

Definition 15 (Evidence aggregator)**.** Let W be a set of alternatives. An *evidence aggregator for* W is a function $Ag : (PRE(W) \times Pre(PRE(W))) \to Pre(W)$ mapping a preordered family $P = \langle \mathscr{R}, \preceq \rangle$, to a preorder $Ag(P)$ on W. $\mathscr{R}$ is seen here as a family of evidence orders over W, $\preceq$ as a priority order for $\mathscr{R}$, and $Ag(P)$ as an evidence-based plausibility order on W. ◁

We have built two properties into the definition of the aggregator. First, the aggregator admits as input any strict poset $P = \langle \mathscr{R}, \preceq \rangle$ based on any non-empty family of preorders. This condition is an analogue of the *Unrestricted Domain* axiom in preference aggregation theory, and it is arguably also desirable in the epistemic setting. It amounts to the demand that the agent should be capable of combining any family of ordered evidence. Secondly, the aggregator should always output a preorder. This condition is sometimes called *Collective Rationality*, and in our epistemic setting it means that the aggregator should output a relation qualifying as a plausibility order. As discussed above, reflexivity and transitivity correspond to natural rationality constraints on orderings, and arguably an aggregation system should be required to produce rational combined orderings.

At first glance, our definition of an aggregator may seem to impose mild constraints that are met by most natural aggergation functions. However, as it is well-known, the output of some common rules like the majority rule may not be transitive, and hence it doesn't count as an aggregator. A specific aggregator that *does* satisfy the constraints, and which will play a key role in this thesis, is the *lexicographic rule*. This aggregator was extensively studied in [7] and, as we will discuss below, it also satisfies several favorable aggregative properties. The definition of the aggregator is the following:

Definition 16. The (anti-)lexicographic rule is the aggregator lex given by

$$(w, v) \in \mathsf{lex}(\langle \mathscr{R}, \preceq \rangle) \text{ iff } \forall R' \in \mathscr{R} \ (R'wv \ \vee \ \exists R \in \mathscr{R}(R' \prec R \wedge R^{<}wv))$$

◁

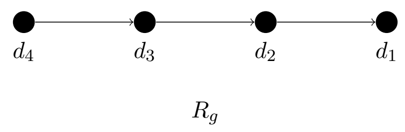
Please note that in this thesis, given that my definition of priority order puts 'more reliable' relations further up in the order, to define a rule that gives precedence to 'more reliable' relations I present the *anti*-lexicographic rule. In the setting of [7], the priority orders put 'better' relations *lower* in the order, so they present the lexicographic rule instead to define the same form of aggregation. To ease reading, I will hereafter leave the '*anti*' implicit in the expression '*anti*-lexicographic', always meaning by 'lexicographic rule' the one defined above. The reader is asked to keep this in mind if reading this chapter and [7] in parallel.

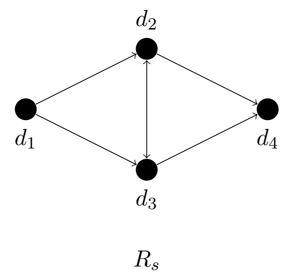
Intuitively, the lexicographic rule works as follows. Given a particular hierarchy $\preceq$ over a family of evidence $\mathscr{R}$, aggregation is done by giving priority to the evidence orders further up the hierarchy in a compensating way: the agent follows what all evidence orders agree on, if it can, or follows more influential pieces of evidence, in case of disagreement.

Observation. Note that whenever all evidence orders are taken to have the same priority, i.e., whenever $\preceq = \mathscr{R}^2$, or whenever all distinct evidence orders are taken to be incomparable, i.e., $\preceq = \{(R, R) \mid R \in \mathscr{R}\}$, the lexicographic rule reduces to the intersection rule on the input relations. That is, if $\preceq = \mathscr{R}^2$ or $\preceq = \{(R, R) \mid R \in \mathscr{R}\}$, then $\mathsf{lex}(\langle \mathscr{R}, \preceq \rangle) = \bigcap \mathscr{R}$. ◁

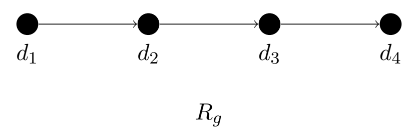
Before discussing the properties of the lexicographic rule, we consider an example of its application.

Example 3 (The diagnosis)**.** Consider an agent seeking medical advice on an ongoing health issue. The agent has consulted two sources on this, a general practitioner and a doctor specialising in this type of health issue. To keep thing simple, we assume that there are four possible diseases that fit the agent's symptoms. We represent these distinct diseases by a set of states $W = \{d_1, \ldots, d_4\}$. Comparing the diseases in terms of how well they explain the observed symptoms, both the general practitioner and the specialist have arrived at a ranking of the possible diseases. Let us denote by R_g and R_s the plausibility orders representing the judgment of the general practitioner and the specialist, respectively, and assume they are as follows:





Given that the doctors have unequal expertise on the type of condition affecting the agent, it is reasonable for the agent to assign priority to the specialist's judgment. Accordingly, let $\preceq$ be the priority order over $\{R_g, R_s, triv\}$ (where $triv = W^2$ represents the trivial evidence) defined by putting $R_g \prec R_s \prec triv$ (and closing under reflexivity and transitivity). The trivial evidence can be seen as most reliable, as it just asserts full uncertainty (and thus full indifference) over the alternatives. Then $\mathsf{lex}(\langle\{R_g, R_s, triv\}, \preceq\rangle) = R_s^{<} \cup (R_s \cap R_g)$ looks as follows:



Where the doctors strictly disagree, priority is given to the specialist. For instance, although the general practitioner put d_4 below d_2, the specialist holds the opposite preference concerning these two options, and the latter's view is allowed to trump the former's. However, when the specialist is indifferent between two options, the strict preferences of the generalist doctor are adopted. Here, this is the case with respect to d_3 and d_2; the specialist's preferences are 'refined' by those of the general practitioner, leading the agent to strictly prefer d_3 over d_2. ◁

While some common rules such as the majority rule don't meet the constraints imposed to count as an aggregator, there is still room for choice. So how should the agent pick a suitable evidence aggregator? Perhaps the best known way to answer this question is to use the *axiomatic approach*, i.e., identify a set of desirable properties for an aggregator and then choose a rule that has these properties. This way to justify the choice of a particular aggregator was initiated by Arrow [16] and is still probably the best-known approach (for other 'rationalization' approaches, see e.g. [17], Chapter 8). We now review some attractive properties for an aggregator, interpreted in epistemic terms. These properties include some variants of Arrow's conditions, as presented in [18, 19]. Moreover, in [7], all these conditions are shown to be satisfied by the lexicographic rule. Fix a set of possible worlds W.

(I) *Independent of irrelevant alternatives*: the overall relative plausibility of any two states depends only on their relative plausibility according to the available evidence orders. That is,

$$\text{For all } W' \subseteq W,\ Ag(\langle\{R_i\}_{i\in I}, \preceq\rangle) \upharpoonright_{W'} = Ag(\langle\{R_i \upharpoonright_{W'}\}_{i\in I}, \prec\rangle)$$

(B) *Based on evidence only*: the combined plausibility order is a function of the ordered set of evidence only. Formally, let W and W' be two sets of states and let $\{R_i\}_{i\in I}$ and $\{R'_i\}_{i\in I}$ be families of evidence orders over W and W' respectively. Let $\preceq$ and $\preceq'$ be priority orders on $\{R_i\}_{i\in I}$ and $\{R'_i\}_{i\in I}$, respectively, with $R_i \preceq R_i$ iff $R'_i \preceq' R'_i$. If there is a bijection $f : W \to W'$ such that for all $i \in I$, $R_i wv$ iff $R'_i f(w)f(v)$, then

$$(w, v) \in\ Ag(\langle\{R_i\}_{i\in I}, \preceq\rangle) \text{ iff } (f(w), f(v)) \in Ag(\langle\{R'_i\}_{i\in I}, \preceq'\rangle)$$

(U) *Unanimous with abstentions*: if a nonempty subset of the evidence orders are unanimous (i.e., they have identical preferences) regarding w and v and the remaining evidence orders are neutral (i.e., they take w and v to be equally preferable), then the combined preferences coincide with those of the unanimous subset. That is, for all $\star \in \{<, \sim, \bowtie\}$

$$\text{if } \exists J \neq \emptyset \subseteq I : \forall j \in J, R^{\star}wv, \text{ and } \forall k \in I\backslash J, R^{\sim}wv, \text{ then } (w, v) \in Ag(\langle\{R_i\}_{i\in I}, \preceq\rangle)$$

(T) *Preserving transitivity*: the output of the aggregator is guaranteed to be transitive if the input relations are transitive. As noted in Section 1.1.2, this condition is built into the definition of an aggregator.

(N) *Non-dictatorial*: the aggregator does not return a fixed one of its arguments without regard to the others. Formally, If $|W| > 1$, then there is no $i \in I$ such that

$$Ag(\langle\{R_i\}_{i\in I}, \preceq\rangle) \text{ for all possible values of the } R_j, \text{ where } j \in I, j \neq i$$

It is also shown in [7] that the lexicographic rule preserves reflexivity, so it indeed counts as an aggregator. The rule also satisfies other well-known properties which can be derived from the ones above, such as *positive responsiveness* and the *Pareto criterion* (for more details about these properties, we refer the reader to [7]). Finally, the authors in [7] also prove a remarkable characterisation result: *any aggregation procedure that satisfies the $IBUTN$ conditions can be construed as a lexicographic rule with respect to some way of prioritizing the family of orders* (Theorem 3.2 in [7]). Again, for more information about the lexicographic rule and its axiomatic properties, we refer the reader to [7].

1.3 Relational evidence models

Having defined relational evidence and evidence aggregators, we are now ready to introduce relational evidence models. Their definition is as follows:

Definition 17 (Relational evidence model)**.** Let P be a set of propositional variables. A *relational evidence model* (**REL** model, for short) is a tuple $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, Ag\rangle$ where

- W is a non-empty set of *states*;

- $\langle \mathscr{R}, \preceq \rangle$ is a preordered set, where $\mathscr{R}$ is a family of preorders on W with $W^2 \in \mathscr{R}$ and $\preceq$ is a priority order for $\mathscr{R}$. The elements of $\mathscr{R}$ are called *basic evidence orders* and $\langle \mathscr{R}, \preceq \rangle$ is called an *ordered family of evidence*;
- $V : \mathsf{P} \to 2^W$ is a valuation function;
- $Ag : (PRE(W) \times Pre(PRE(W))) \to Pre(W)$ is an aggregator for W.

$W^2 \in \mathscr{R}$ is called the *trivial evidence order*. It represents the evidence stating that "the actual state is in W". This evidence represents full uncertainty and is taken to be always available to the agent as a starting point. ◁

We now fix some notation used to refer to specific classes of **REL** models. This notation will be uses throughout the thesis.

Notation 2. Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be a **REL** model.

- M is said to be an *f-model* iff $Ag = f$.
- Let P be a set of properties for an aggregator. M is said to be a *P-model* iff Ag satisfies all the properties in P.
- M is said to be *unordered* iff $\preceq = \emptyset$, i.e., if no piece of evidence is given priority over any other piece of evidence. Unordered evidence models represent scenarios in which all evidence is equally reliable. That is, all evidence is treated on a par by the agent, as done in neighborhood evidence models.

◁

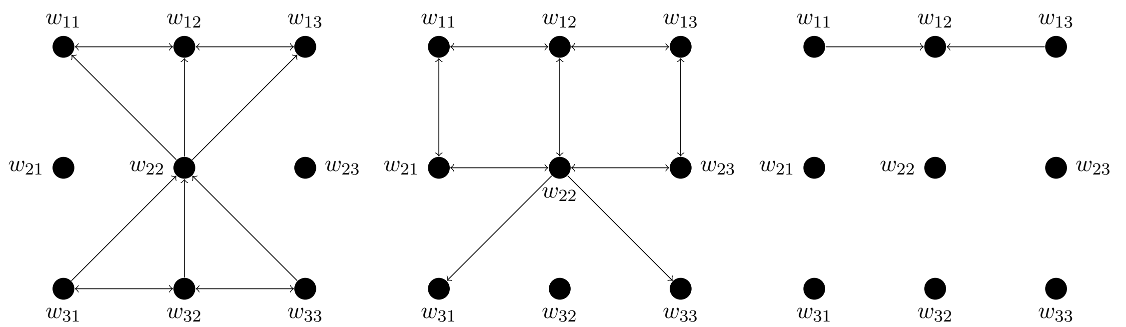
Evidential support and strength. Fix a **REL** model $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$. We say that a piece of evidence $R \in \mathscr{R}$ *supports* a proposition $P \subseteq W$ at w iff $R[w] \subseteq P$. That is, R supports P at w iff every state that is at least as likely as w, according to R, is a P-world. Let $R, R' \in \mathscr{R}$ be two pieces of evidence. We say that R is *at least as strong as* R' iff $R' \preceq R$.

To illustrate the kind of scenario described by **REL** models, let us consider an example.

Example 4 (Agent localization)**.** Consider an agent needing to determine its current location in an environment, given some initial information about the environment. We assume that the agent's environment is represented by a 3×3 grid. Each cell of the grid represents a possible current location for the agent. We represent this by a set W of possible locations $W = \{w_{ij} \mid i, j \in \{1, 2, 3\}\}$. To estimate its location within the environment, the agent senses the environment with three sensors (e.g., an accelerometer, a gyroscope and a magnetometer) which we label s_1, s_2 and s_3. Each sensor s_i provides a noisy reading r_i about the agent's location. As we did in Example 2, we assume that each reading induces an ordering over the alternatives; in this case, an ordering over cells. I.e., we have the following set of relational evidence $\mathscr{R} = \{r_1, r_2, r_3, triv\}$, where $triv = W^2$ is the trivial evidence order. We assume that the sensors have different levels of accuracy and as a result, the agent assigns different levels of reliability to the readings. In particular, suppose that sensor s_1 is more reliable and as a result r_1 is given priority over the other two sensor readings. The trivial evidence can be seen as most reliable, as it just asserts full uncertainty (and thus full indifference) over the alternatives. We represent this by the following priority order $\preceq$ over the available sensor readings in $\mathscr{R}$ (reflexive and transitive arrows are omitted):

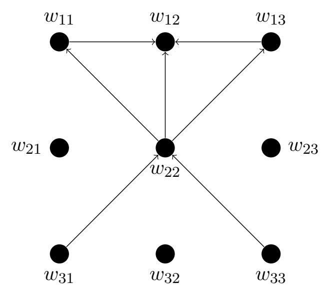
$$\begin{tikzcd} & triv & \\ & r_1 \arrow[u] & \\ r_2 \arrow[ur] & & r_3 \arrow[ul] \end{tikzcd}$$

Suppose the sensor readings are as follows (reflexive and transitive arrows are omitted):



s_1 s_2 s_3

Let's assume that the agent aggregates the ordered family of evidence $\langle \mathscr{R}, \preceq \rangle$ using the *lexicographic rule*. Then the aggregated evidence looks like this:



The most likely location is therefore w_{12}. r_1 is neutral among w_{11}, w_{12} and w_{13}, but the other two readings indicate that w_{11} is actually more likely than the other two, so the order provided by r_2 and r_3 is adopted. On the other hand, the orderings of, e.g., w_{22} and w_{31} are inconsistent among readings, and the inconsistency is resolved in favor of r_1. ◁

1.4 Notions of belief and evidence

We now introduce the notions of belief and evidence for **REL** models that we will be working with in subsequent chapters. In Section 4.4.1 of Chapter II.4, we will consider generalisations of these notions, together with the formulas encoding them in a basic language for **REL**.

Evidence-based belief in REL models. The notion of belief we will work with is based on the agent's plausibility order, which in **REL** models corresponds to the output of the aggregator. As we don't require the plausibility order to be converse-well founded, it may have no maximal elements, which means that Grove's definition of belief may yield inconsistent beliefs. For this reason, we adopt a usual generalization of Grove's definition, which defines beliefs in terms of truth in all 'plausible enough' worlds (see, e.g., [3, 20]). Given a **REL** model $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$, we put

$$BP \text{ holds (at any state) iff } \forall w (\exists v ((w, v) \in Ag(\langle \mathscr{R}, \preceq \rangle) \text{ and } Ag(\langle \mathscr{R}, \preceq \rangle)[v] \subseteq P))$$

That is, the agent believes P if for every state $w \in P$ state, we can always find a more plausible state $v \in P$, all whose successors are also in P. When the plausibility relation is indeed converse well-founded, this notion of belief coincides with Grove's one, while ensuring consistency of belief otherwise.

Evidence availability. As in the case of **NEL** models, different evidence-related notions can be introduced for **REL** models. Several notions make sense in the relational setting. We will focus on some natural ones, which, as we shall see in more detail in subsequent chapters, generalise the notions of evidence availability for **NEL** models that we reviewed in Section 2.2.4.

Basic factive evidence. We say that the agent has basic, factive evidence for P at w if there is a piece of evidence R that *supports* P at w. That is:

$$\text{the agent has basic evidence for } P \text{ at } w \in W \text{ iff } \exists R \in \mathscr{R}(R[w] \subseteq P)$$

Basic evidence. We also provide a non-factive version of the previous notion, which says that the agent has basic evidence for P if there is a piece of evidence R that *supports* P at *some* state. That is:

$$\text{the agent has basic evidence for } P \text{ (at any state) iff } \exists w(\exists R \in \mathscr{R}(R[w] \subseteq P))$$

Aggregated factive evidence. We propose a notion of aggregated evidence based on the output of the aggregator:

$$\text{the agent has aggregated, factive evidence for } P \text{ at } w \in W \text{ iff } Ag(\langle \mathscr{R}, \preceq \rangle)[w] \subseteq P$$

Aggregated evidence. The non-factive version of the previous notion is as follows:

$$\text{the agent has aggregated evidence for } P \text{ (at any state) iff } \exists w(Ag(\langle \mathscr{R}, \preceq \rangle)[w] \subseteq P)$$

1.5 Connecting NEL and REL models

In this section, we explore the relationship between neighborhood evidence models and relational evidence models. The models proposed here are not intended to replace neighborhood evidence models, but rather to complement them. So, what exactly is the relationship between these two frameworks for modeling beliefs? Here we will discuss a way to transform every **NEL** model into a **REL** model. To do that, we first relate binary evidence, the type of evidence considered in **NEL** models, with relational evidence.

Binary and relational evidence. A piece of binary evidence can be seen as dividing the set of alternatives into two subsets; the 'fully plausible' or 'good' ones and the 'least plausible' or 'bad' ones. The simplest encoding of this evidence item is as a set, containing the states that are considered fully plausible by the source. The idea is that, by default, if the information encoded in evidence set e is taken for granted, a first guess for the actual state should be an element of e. Another way to look at this kind of evidence is to view each source as reporting *dichotomous preferences* over the set of worlds. Specifically, each evidence set can be identified with a *dichotomous weak order* $\precsim$ over a set of alternatives W, i.e., a total preorder with at most two indifference classes. Formally, define the set of good alternatives associated with $\precsim$ as $G(\precsim) := \{w \in W \mid v \precsim w \text{ for all } v \in W\}$. Similarly, let $B(\precsim) := \{w \in W \mid w \precsim v \text{ for all } v \in W\}$ be the set of bad alternatives based on $\precsim$. An order $\precsim$ is said to be *dichotomous* if and only if every alternative belongs to at least one of these two sets: that is, if and only if $G(\precsim) \cup B(\precsim) = W$. Each evidence set $e \subseteq W$ can be turned into a dichotomous weak order $\preceq_e$, with $G(\precsim_e) = e$ and $B(\precsim_e) = W \setminus e$, by putting:

- $w \prec_e v$ iff $w \in B(\precsim_e)$ and $w \in G(\precsim_e)$;
- $w \sim_e v$ iff ($w \in B(\precsim_e)$ and $v \in B(\precsim_e)$) or ($w \in G(\precsim_e)$ and $v \in G(\precsim_e)$).

That is, an alternative is good if it is weakly preferred to all other alternatives and it is bad if every alternative is strictly preferred to it.

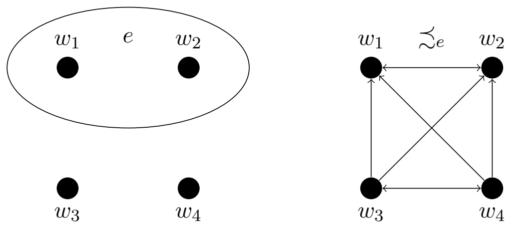


FIGURE 1.2: A piece of binary evidence, represented as an evidence set e (left) and as a dichotomous evidence order $\preceq_e$ (right).

Seen as a dichotomous order, binary evidence is a special case of relational evidence. We fix this relationship in the following definition:

Definition 18 (Evidence order associated with an evidence set)**.** Let W be a set. For each $e \subseteq W$, we denote by R_e the dichotomous weak order given by $G(R_e) = e$ and $B(R_e) = W \setminus e$. Equivalently, we define R_e by

$$(w, v) \in R_e \text{ iff } w \in e \Rightarrow v \in e$$

$\lhd$

Observation. We sometimes use the following facts, which follow immediately from the definition of R_e:

1. $w \in e \Leftrightarrow R_e[w] = e$
2. $w \notin e \Rightarrow R_e[w] = W$

$\lhd$

We discussed in Chapter II.1 how to induce an evidential plausibility order $\sqsubseteq_{E_0}$ from a given family of evidence sets E_0. As we also saw, this order can be used to define the notion of belief proposed in [1] in terms of truth in all $\sqsubseteq_{E_0}$-maximal elements, as well as the notion of belief introduced in [5] when we restrict our attention to feasible models (i.e., models with finitely many pieces of evidence). Probably the first question that comes to mind when we identify evidence sets with special evidence orders, is the following: given a family of evidence sets E_0, unordered as they come in a **NEL** model, what is an aggregator on their associated dichotomous orders $\{R_e \mid e \in E_0\}$ that outputs $\sqsubseteq_{E_0}$? In other words, what is an aggregator Ag such that $Ag(\langle\{R_e \mid e \in E_0\}, \preceq\rangle) = \sqsubseteq_{E_0}$, where $\preceq = \{R_e \mid e \in E_0\}^2$ so that the evidence is unordered? Given Definition 18, it is easy to see that such aggregator is the lexicographic rule, which given that $\preceq = \langle\{R_e \mid e \in E_0\}^2$, corresponds to the intersection of the R_e.

Proposition 1. *Let W be a set and let $E_0 \subseteq \mathscr{P}(W)$ be a family of evidence sets. Then $\sqsubseteq_{E_0} = \bigcap_{e \in E_0} R_e = \mathsf{lex}(\langle\{R_e \mid e \in E_0\}, \preceq\rangle)$, where $\preceq = \emptyset$.*

Proof. Let $w, v \in W$. We have

$$(w, v) \in \sqsubseteq_{E_0} \text{ iff } \forall e \in E_0(w \in e \Rightarrow v \in e) \text{ iff } \forall e \in E_0((w, v) \in R_e)$$

□

The perspective switch from evidence sets to evidence orders brings with it some insights. As the combination approach used on evidence sets to recover the plausibility order $\sqsubseteq_{E_0}$ turns out to be equivalent to doing lexicographic aggregation on the unordered set of the associated dichotomous orders, we can get a bit of extra information about the way the agent is handling evidence. As the lexicographic rule satisfies the $IBUTN$ axioms (and their implied axioms), the agent is can be seen 'as if' she was combining relational evidence in a specific way; being unanimous with abstentions, non-dictatorial, etc.

As noted above, in unordered **REL** models, i.e., models of the form $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ with $\preceq = \mathscr{R}^2$, the output of lexicographic rule always reduces to the intersection of the evidence relations. So given that **NEL** models are unordered (i.e., the families of evidence sets don't come with any ordering over them), it is perhaps most natural to think of the aggregator Ag such that $Ag(\langle \{R_e \mid e \in E_0\}, \preceq \rangle) = \sqsubseteq_{E_0}$ as simply being the *intersection rule*, i.e., the aggregator $Ag_\cap$ that generally disregards the priority order $\preceq$ and simply outputs the intersection of the available evidence $\bigcap_{e \in E_0} R_e$. We fix the definition for this basic *aggregator for unordered evidence* as follows:

Definition 19. The *intersection rule* is the aggregator $Ag_\cap$ given by

$$(w, v) \in Ag_\cap(\langle \mathscr{R}, \preceq \rangle) \text{ iff } (w, v) \in \bigcap \mathscr{R}$$

◁

Having fixed this connection evidence sets and evidence orders, and their associated aggregation procedures, we can now consider a natural way transform every **NEL** into an unordered, **REL** model in which each evidence order is dichotomous. To fix this connection, we define the following mapping between **NEL** and **REL** models.

Definition 20 (Relational model associated with a neighborhood model)**.** Let *Rel* be a map from neighborhood evidence models to **REL**-models given by

$$\langle W, E_0, V \rangle \mapsto \langle Rel(W), \langle Rel(E_0), \preceq \rangle, Rel(V), Ag_\cap \rangle$$

where $Rel(W) := W$, $Rel(V) := V$ $Rel(E_0) := \{R_e \mid e \in E_0\}$ and $\preceq = Rel(E_0)^2$. ◁

Observation. The choice of $\preceq = Rel(E_0)^2$ makes the associated model $Rel(M)$ be such that all evidence is equally reliable, as was the case in the original model M. Given the fact that the intersection rule effectively ignores the priority order, other choices of $\preceq$ would give us a **REL** model that is equivalent to $Rel(M)$, in the sense of including the same evidence orders and leading to the same evidence-based plausibility order. But these other choices of $\preceq$ make extra assumptions about the relative reliability of evidence that are alien to original model M. ◁

Proposition 2. *The map Rel is well defined.*

Proof. We need to show that, for each neighborhood model $M = \langle W, E_0, V \rangle$, the structure $Rel(M) = \langle Rel(W), \langle Rel(E_0), \preceq \rangle, Rel(V), Ag_\cap \rangle$ is indeed a relational model. Let $e \in E_0$. We need to show that R_e is a preorder.

- *Reflexivity.* Let $w \in W$. Either $w \in e$ or $w \notin e$, and either way we get $w \in e \Rightarrow w \in e$, so $R_e ww$.
- *Transitivity.* Let $R_e wv$ and $R_e vu$. If $w \notin e$ we have $w \in e \Rightarrow u \in e$ and thus $R_e wu$. Suppose now that $w \in e$. Then as $R_e wv$ we have $v \in e$, which given $R_e vu$ implies $u \in e$. Thus $R_e wu$.

$\square$

As we shall discuss in Chapter II.2, the mapping *Rel* turns every feasible **NEL** model into a feasible **REL** model in which the agent has evidence for, and believes, the same propositions as in the original **NEL** model. This intuitive connection will be proved in Chapter II.2; after interpreting the language of **NEL** over **REL** models with the semantics for the evidence and belief operators matching the definitions given for these notions in Section **??**, feasible **NEL** models and their images under *Rel* will be shown to be modally equivalent (in the sense of having point-wise equivalent modal theories). This connection between **NEL** models and unordered $Ag_{\cap}$-models motivates, in part, our interest in the logic of $Ag_{\cap}$-models, which we study in detail in Chapter II.2.

REL models as a generalisation of NEL models. Our previous remarks allow us to see the setting of **REL** models as generalising that of **NEL** models in three main ways. First, via the encoding of evidence sets as dichotomous orders, the type of binary evidence considered in **NEL** models can be represented in **REL** models, while the latter models also provide facilities for representing non-binary evidence. Second, given the presence of a priority order $\preceq$ in **REL** models, evidence pieces of varying reliability can be represented; putting $\preceq = \mathscr{R}^2$ we obtain the class of evidence models with unordered or equally reliable evidence as a special case. Finally, instead of fixing a specific procedure to aggregate evidence and defining an associated notion of evidence-based belief via this procedure, as done in [1, 5, 12], we propose a notion of evidence-based belief for **REL** models that relies on the output of an abstract aggregator Ag. This abstraction makes possible two related but distinct investigations: (i) logics of evidence and belief based on a specific aggregator. In this thesis, we do a first exploration in both directions; and (ii) logics of evidence and belief based on a class of aggregators characterised by certain properties. Chapters II.2 and II.3 follow the first path, presenting logics for reasoning about the evidence and beliefs of an agent that relies specifically on the intersection (Chapter II.3) and lexicographic rule (Chapter II.2). On the other hand, Chapter II.4 presents a 'minimal' logic for reasoning about the evidence and beliefs held by an agent irrespective of the aggregator applied, i.e., the evidence and beliefs obtained relying on any way of combining evidence that meets the constraints built into the definition of an aggregator: Unrestricted Domain and Collective Rationality.

1.6 Syntax and semantics for REL

To conclude this chapter, we recall the basic language $\mathscr{L}$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi$$

where $p \in \mathsf{P}$. The intended interpretation of the modalities is as follows. $\Box_0\varphi$ reads as: 'the agent has basic, factive evidence for φ'; $\Box\varphi$ reads as: 'the agent has aggregated evidence for φ'.

We introduced and interpreted this language over **NEL** models in Chapter I.2, Section 2.2.6. We will now interpret it over **REL** models. The language $\mathscr{L}$ is used as a basic static language for evidence and belief throughout this thesis. The language $\mathscr{L}$ is interpreted over **REL** models as follows.

Definition 21 (Satisfaction)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be an **REL** model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is defined as follows:

$M, w \models p$ iff $w \in V(p)$
$M, w \models \neg\varphi$ iff $M, w \not\models \varphi$
$M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \Box_0\varphi$ iff there is $R \in \mathscr{R}$ such that, for all $v \in W, Rwv$ implies $M, v \models \varphi$
$M, w \models \Box\varphi$ iff for all $v \in W, Ag(\langle \mathscr{R}, \preceq \rangle)wv$ implies $M, v \models \varphi$
$M, w \models \forall\varphi$ iff for all $v \in W, M, v \models \varphi$

◁

Definition 22 (Truth map)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be a **REL** model. We define a *truth map* $[\![\cdot]\!]_M : \mathscr{L} \to 2^W$ given by: $[\![\varphi]\!]_M = \{w \in W \mid M, w \models \varphi\}$ ◁

Evidence. As expected, the notions of factive evidence availability introduced in Section 1.1.4 is matched by the semantics of $\Box_0$ and $\Box$. In particular, $\Box_0\varphi$ corresponds to the notion of 'having basic, factive evidence for φ' and $\Box\varphi$ corresponds to that of 'having aggregated, factive evidence for φ'. Moreover, the notions related to non-factive evidence are definable in the language. As in the **NEL** setting, $\exists\Box_0\varphi$ corresponds to 'having basic evidence for φ', while $\exists\Box\varphi$ gives the notion of 'having aggregated evidence for φ'.

Belief. We recall the notion of evidence-based belief introduced in Section 1.1.4 of this Chapter.

$$B\varphi \text{ holds (at any state) iff } \forall w(\exists v((w, v) \in Ag(\langle \mathscr{R}, \preceq \rangle) \text{ and } Ag(\langle \mathscr{R}, \preceq \rangle)[v] \subseteq [\![\varphi]\!]_M))$$

That is, the agent believes φ if for every state $w \notin [\![\varphi]\!]_M$ state, we can always find a more plausible state $v \in [\![\varphi]\!]_M$, all whose successors are also in $[\![\varphi]\!]_M$. When the output of the aggregator is converse well-founded, this notion of belief coincides with Grove's one, but it ensures consistency of belief otherwise. It is easy to check that this notion of belief is definable in terms of $\forall$ and $\Box$. Specifically, we put

$$B\varphi \coloneqq \forall\Diamond\Box\varphi$$

We now generalise the notions of plain evidence and belief by introducing *conditional* versions of our evidence and belief operators: $B^\varphi\psi$, $\Box_0^\varphi\psi$ and $\Box^\varphi\psi$.

Conditional basic evidence. We can also give a conditional version of basic evidence. The intended interpretation of $\Box_0^\varphi\psi$ is "the agent has basic, factive evidence for ψ at w, conditional on φ being true". The definition of conditional basic evidence is as follows

$$\Box_0^\varphi\psi \text{ iff } \exists R \in \mathscr{R}(\forall v(Rwv \Rightarrow (v \in [\![\varphi]\!]_M \Rightarrow v \in [\![\psi]\!]_M)))$$

Conditional basic evidence is definable in the basic language by putting

$$\Box_0^\varphi\psi \coloneqq \Box_0(\varphi \to \psi)$$

The notion of conditional evidence reduces to that of plain evidence by setting $\varphi = \top$.

Conditional aggregated evidence. We also define a notion for aggregated evidence that mirrors the one given for basic evidence. The intended meaning of $\Box^{\varphi}\psi$ is "the agent has aggregated evidence for ψ at w, conditional on φ being true".

$$\Box^{\varphi}\psi \text{ iff } \forall v(Ag(\langle \mathscr{R}, \preceq\rangle)wv \Rightarrow (v \in [\![\varphi]\!]_M \Rightarrow v \in [\![\psi]\!]_M))$$

Conditional aggregated evidence is definable in the basic language by putting

$$\Box^{\varphi}\psi := \Box(\varphi \to \psi)$$

Again, the unconditional version is given by $\varphi = \top$.

Conditional belief. Conditional beliefs pre-encode the beliefs that we would have if we learnt that certain *propositions* are true. The intended interpretation of $B^{\varphi}\psi$ is "the agent believes ψ conditional on φ being true". In our setting, we define conditional belief as follows

$$B^{\varphi}\psi \text{ iff } \forall w(w \in [\![\varphi]\!]_M \Rightarrow \exists v(Ag(\langle \mathscr{R}, \preceq\rangle)wv \text{ and } v \in [\![\varphi]\!]_M \text{ and } Ag(\langle \mathscr{R}, \preceq\rangle)[v] \cap [\![\varphi]\!]_M \subseteq [\![\psi]\!]_M))$$

A simple inspection of the expression above should make it clear that this notion of belief is definable in terms of $\forall$ and $\Box$. Specifically, we put

$$B^{\varphi}\psi := \forall(\varphi \to \Diamond(\varphi \to (\Box\varphi \to \psi)))$$

As with the other conditional versions discussed above, this notion reduces to that of absolute belief when $\varphi = \top$.

1.7 Chapter review

In this chapter, we have introduce relational evidence models, the class of models over which the various logics developed in this thesis are interpreted. We have explored and exemplified the notion of *relational evidence*, as well as the formats that we use for modeling evidence (*evidence orders*) and its reliability (*priority orders*). We have also presented *evidence aggregators*, the rules used by the agent to combine relational evidence. After fixing the definition of a **REL** model, we have presented notions of belief and evidence-possession for this type of model. We have also explored ways to connect **NEL** models and **REL** models in a systematic way. Finally, we have fixed the syntax and semantic of a basic static language for **REL**, which will be used throughout the thesis.

Chapter 2

$\mathbf{REL}_\cap$: unanimous evidence merge

In this chapter, we initiate our logical study of belief and evidence in the **REL** setting. We zoom into a specific class of **REL** models, the class of $Ag_\cap$-models, and study logics for belief and evidence based on these models. Our motivation for exploring these logics is twofold. First, as we anticipated in Chapter II.1, Section 1.1.5, feasible **NEL** models can be turned into $Ag_\cap$-models model in which the agent has evidence for, and believes, the same propositions as in the original **NEL** model. This relationship gives us one way to connect the **REL** framework back to the **NEL** framework which inspired it, before embarking on a more general study of **REL**. Secondly, $Ag_\cap$-models come with an aggregator, the intersection rule, which recovers an evidence-based plausibility order only on the basis of the family of evidence; the priority order plays no role. This approach to aggregation fits well with some natural scenarios; those in which all the evidence is equally reliable, and those in which the agent has no information about the relative reliability of evidence. In this type of scenario, we say that the agent has *unordered* evidence. Putting together these two motivations, we can view the logics studied in this chapter as logics for reasoning about the evidence and beliefs of an agent that combines unordered evidence, using a procedure that agrees with, and generalises, the one proposed in [1, 2, 4, 5, 12] for the **NEL** setting.

2.1 Syntax and semantics

Here, we recall here the language $\mathscr{L}$, which is built recursively as follows:

$$\varphi ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi$$

In this chapter we focus on $Ag_\cap$-models, i.e., **REL** models of the form

$$M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, Ag_\cap\rangle$$

To simplify notation, hereafter we will refer to these models as $\cap$-models instead of $Ag_\cap$-models. As the intersection rule is insensitive to the priority order, when we consider $\cap$-models, it is convenient to treat the models as if they came with a family of evidence orders $\mathscr{R}$ only, instead of an ordered family $\langle \mathscr{R}, \preceq\rangle$. Accordingly, hereafter we will write $\cap$-models as follows:

$$M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$$

where $\mathscr{R}$ is a countable family of evidence orders over W. The semantics for formulas of $\mathscr{L}$ can then be stated as follows.

Definition 23 (Satisfaction)**.** Let $M = \langle W, \mathscr{R}, V, Ag_\cap \rangle$ be a $\cap$-model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is defined as follows:

$$\begin{array}{ll}
M, w \models p & \text{iff } w \in V(p) \\
M, w \models \neg\varphi & \text{iff } M, w \not\models \varphi \\
M, w \models \varphi \wedge \psi & \text{iff } M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \Box_0\varphi & \text{iff there is } R \in \mathscr{R} \text{ such that, for all } v \in W, Rwv \text{ implies } M, v \models \varphi \\
M, w \models \Box\varphi & \text{iff for all } v \in W, Ag_\cap(\mathscr{R})wv \text{ implies } M, v \models \varphi \\
M, w \models \forall\varphi & \text{iff for all } v \in W, M, v \models \varphi
\end{array}$$

◁

Before discussing axiomatizations of the logic of the class of $\cap$-models, we establish a fact anticipated in Chapter II.1, Section 1.1.5: the mapping *Rel* turns every feasible **NEL** model into a feasible **REL** model in which the agent has evidence for, and believes, the same propositions as in the original **NEL** model. More precisely, feasible **NEL** models and their images under *Rel* are modally equivalent, in the sense of having point-wise equivalent modal theories.

Proposition 3. *Let $M = \langle W, E_0, V \rangle$ be a feasible neighborhood model (i.e., a model with E_0 finite). For any $\varphi \in \mathscr{L}$ and any state $w \in W$*

$$M, w \models \varphi \text{ iff } Rel(M), w \models \varphi$$

Proof. By induction on the structure of φ. The base case for $\varphi = p$ ($p \in \mathsf{P}$) and the inductive step for $\varphi = \neg\psi, \varphi = \psi \wedge \chi$ and $\varphi = \forall\psi$ are shown by unfolding the definitions. We show now the cases involving $\Box_0$ and $\Box$ modalities.

- $\varphi = \Box_0\psi$. Note that:

$$\begin{aligned}
M, w \models \Box_0\psi &\text{ iff there is an } e \in E_0 \text{ such that } w \in e \subseteq [\![\psi]\!]_M \\
&\text{ iff there is an } R_e \in Rel(E_0) \text{ such that } R_e[w] = e \text{ and } e \subseteq [\![\psi]\!]_M \\
&\overset{i.h.}{\text{ iff}} \text{ there is an } R_e \in Rel(E_0) \text{ such that } R_e[w] \subseteq [\![\psi]\!]_{Rel(M)} \\
&\text{ iff } Rel(M), w \models \Box_0\psi
\end{aligned}$$

- $\varphi = \Box\psi$. Note that:

$$\begin{aligned}
M, w \models \Box\psi \text{ iff } & \text{ there is an } e \in E \text{ such that } w \in e \subseteq [\![\psi]\!]_M \\
\text{iff } & \text{ there are } e_1, \dots, e_n \in E_0 \text{ such that } \bigcap_{i=1}^{n} e_i = e \\
& \text{ and } w \in e \subseteq [\![\psi]\!]_M \\
\text{iff } & \text{ there are } R_{e_1}, \dots, R_{e_n} \in Rel(E_0) \text{ such that } R_{e_i}[w] = e_i \\
& \text{ and } w \in e \subseteq [\![\psi]\!]_M \\
\text{iff } & \text{ there are } R_{e_1}, \dots, R_{e_n} \in Rel(E_0) \text{ such that } (\bigcap_{i=1}^{n} R_{e_i})[w] \subseteq [\![\psi]\!]_M \\
\text{iff } & (\bigcap \mathscr{R})[w] \subseteq (\bigcap_{i=1}^{n} R_{e_i})[w] \subseteq [\![\psi]\!]_M \\
\overset{i.h.}{\text{iff}} & (\bigcap \mathscr{R})[w] \subseteq [\![\psi]\!]_{Rel(M)}
\end{aligned}$$

iff $Rel(M), w \models \Box\psi$

$\Box$

As the following proposition shows, this modal equivalence result does not extend to non-feasible **NEL** models. This is because, in models with infinitely many pieces of evidence, the notion of combined evidence presented in [5] differs from the one proposed for **REL** models. To clarify this, consider a **NEL** model $M = \langle W, E_0, V\rangle$. In the setting of [5], the agent has combined evidence for a proposition φ at w if there is a *finite* body of evidence containing w and supporting φ, i.e., if there is some finite $F \subseteq E_0$ such that $w \in \bigcap F$ and $\bigcap F \subseteq [\![\varphi]\!]_M$. Suppose M is a non-feasible models in which E_0 is such that $w \in E_0$ and $\bigcap E_0 \subseteq [\![\varphi]\!]_M$, while no finite subfamily $F \subseteq E_0$ is such that $w \in \bigcap F$ and $\bigcap F \subseteq [\![\varphi]\!]_M$. That is, the combination of *all* the evidence supports φ at w, but no combination of a finite subfamily of E_0 does. In a **NEL** model like this, the agent does not have combined evidence for φ. However, our proposed notion of aggregated evidence for **REL** models is based on combining all the available evidence (as opposed to finite subsets of it), and as a result in $Rel(M)$ the agent does have aggregated evidence for φ.

Proposition 4. *Non-feasible* ***NEL*** *models need not be modally equivalent to their images under* Rel*. In particular:*

1. *The left-to-right direction of Proposition 3 holds for non-feasible evidence models.*
2. *The right-to-left direction of Proposition 3 fails for non-feasible evidence models. In particular, there is an non-feasible neighborhood model M such that $Rel(M), w \models \Box\psi$ but $M, w \not\models \Box\psi$.*

Proof.

1. This is clear from the fact that the proofs for this direction don't depend on the cardinality of E_0.
2. The following is a counterexample. Let $M = \langle W, E_0, V\rangle$, with $W = \mathbb{N}$, $E_0 = \{\mathbb{N} \setminus \{2n+1\} \mid n \in \mathbb{N}\}$ and $V(p) = \{2n \mid n \in \mathbb{N}\}$. Let $w = 0$. Note that for all $e \in E$, $e \not\subseteq [\![p]\!]_M$ and thus $M, w \not\models \Box p$. Moreover, we have:

$$\Big(\bigcap_{R\in Rel(E_0)} R\Big)[w] = \bigcap_{e\in E_0} (R_e[w])$$

And as $w \in e$ for all $e \in E_0$, by **??** we have $R_e[w] = e$ for each $e \in E_0$, Hence

$$\bigcap_{e\in E_0} (R_e[w]) = \bigcap_{e\in E_0} (e) = \bigcap E_0$$

Note that $\bigcap E_0 = [\![p]\!]_M$, and thus, $(\bigcap_{R\in Rel(E_0)} R)[w] = [\![p]\!]_M$. By induction hypothesis, we have $[\![p]\!]_M = [\![p]\!]_{Rel(M)}$ and hence $(\bigcap_{R\in Rel(E_0)} R)[w] = [\![p]\!]_{Rel(M)}$, which implies $Rel(M), w \models \Box p$.

$\Box$

2.2 A proof system for $\mathbf{REL}_\cap$

In this section, we recall the proof system L_0. We first introduced this system in 2.2.7, where it was presented as a system to axiomatize the logic of the class of **NEL** models. In the following sections of this chapter, we revisit this system and show that it also axiomatizes the logic of the class of $\cap$-models.

The system includes the following *axiom schemas* for all formulas $\varphi, \psi \in \mathscr{L}$:

1. All tautologies of propositional logic
2. The $\mathsf{S5}$ axioms for $\forall$
3. The $\mathsf{S4}$ axioms for $\Box$
4. $\Box_0\varphi \to \Box_0\Box_0\varphi$
5. The following interaction axioms:
   (a) $\forall\varphi \to \Box_0\varphi$ (Universality)
   (b) $(\Box_0\varphi \wedge \forall\psi) \to \Box_0(\varphi \wedge \forall\psi)$ (Pullout)
   (c) $\Box_0\varphi \to \Box\varphi$

Moreover, the system of includes the following *inference rules* for all formulas $\varphi, \psi \in \mathscr{L}$:

1. Modus ponens
2. Necessitation Rule for $\forall$: $\dfrac{\varphi}{\forall\varphi}$
3. Necessitation Rule for $\Box$: $\dfrac{\varphi}{\Box\varphi}$
4. Monotonicity Rule for $\Box_0$: $\dfrac{\varphi \to \psi}{\Box_0\varphi \to \Box_0\psi}$

2.3 Soundness of L_0

In this section we prove that the logic generated by L_0, which we denote by Λ_0, is sound with respect to the class of $\cap$-models. Before that, we prove a more general result, which will be used also in other soundness proofs throughout the thesis.

Proposition 5. *The axioms listed under 1-4, 5(a), and 5(b) are valid in any **REL** model. Moreover, the inference rules listed 1-4 preserve truth.*

Proof. Let $M = \langle W, \langle \mathscr{R}, \prec\rangle, V, Ag\rangle$ be an **REL** model and w a world in M.

1. $\mathsf{S5}$ axioms for $\forall$:

   $\mathsf{K}_\forall$: $\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$. Let $M, w \models \forall(\varphi \to \psi)$ and suppose that $M, w \models \forall\varphi$. Take any $v \in W$. As $M, w \models \forall\varphi$, we have $M, v \models \varphi$. Thus given $M, w \models \forall(\varphi \to \psi)$, we have $M, v \models \psi$.

   $\mathsf{T}_\forall$: $\forall\varphi \to \varphi$. Let $M, w \models \forall\varphi$. Then every $v \in W$ is such that $M, v \models \varphi$. So in particular $M, w \models \varphi$.

   $\mathsf{4}\forall$: $\forall\varphi \to \forall\forall\varphi$. Let $M, w \models \forall\varphi$. Then every $v \in W$ is such that $M, v \models \varphi$. Hence every $v \in W$ is such that $M, v \models \forall\varphi$ and thus $M, w \models \forall\forall\varphi$.

   $\mathsf{5}_\forall$: $\exists\varphi \to \forall\exists\varphi$. Let $M, w \models \exists\varphi$. Then there is a $v \in W$ such that $M, v \models \varphi$. Take any $u \in W$. Then we have $M, u \models \exists\varphi$, and thus $M, w \models \forall\exists\varphi$.

2. $\mathsf{S4}$ axioms for $\Box$:

$\mathsf{K}_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. Let $M, w \models \Box(\varphi \to \psi)$ and suppose $M, w \models \Box\varphi$. Then $Ag(\langle \mathscr{R}, \prec \rangle)[w] \subseteq [\![\varphi \to \psi]\!]_M$ and $Ag(\langle \mathscr{R}, \prec \rangle)[w] \subseteq [\![\varphi]\!]_M$. Take any $v \in Ag(\langle \mathscr{R}, \prec \rangle)[w]$. Then $M, v \models \varphi \to \psi$ and $M, v \models \varphi$, so $M, v \models \psi$.

$\mathsf{T}_\Box$: $\Box\varphi \to \varphi$. Let $M, w \models \Box\varphi$. Then $Ag(\langle \mathscr{R}, \prec \rangle)[w] \subseteq [\![\varphi]\!]_M$. Since $codom(Ag) = Pre(W)$, $Ag(\langle \mathscr{R}, \prec \rangle)$ is reflexive and thus $w \in Ag(\langle \mathscr{R}, \prec \rangle)[w]$. Hence $M, w \models \varphi$.

$\mathsf{4}_\Box$: $\Box\varphi \to \Box\Box\varphi$. Let $M, w \models \Box\varphi$. Then $Ag(\langle \mathscr{R}, \prec \rangle)[w] \subseteq [\![\varphi]\!]_M$. Take any $v \in Ag(\langle \mathscr{R}, \prec \rangle)[w]$, i.e., any v such that $(w, v) \in Ag(\langle \mathscr{R}, \prec \rangle)$. We need to show that for all $u \in Ag(\langle \mathscr{R}, \prec \rangle)[v]$, $M, u \models \varphi$. Take any $u \in Ag(\langle \mathscr{R}, \prec \rangle)[v]$, i.e., $(v, u) \in Ag(\langle \mathscr{R}, \prec \rangle)$. Since $codom(Ag) = Pre(W)$, $Ag(\langle \mathscr{R}, \prec \rangle)$ is transitive and thus given $(w, v), (v, u) \in Ag(\langle \mathscr{R}, \prec \rangle)$ we have $(w, u) \in Ag(\langle \mathscr{R}, \prec \rangle)$. Hence $M, u \models \varphi$.

3. Axiom $\mathsf{4}$ for $\Box_0$:

   Let $M, w \models \Box_0\varphi$. Thus, there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi]\!]_M$. We need to show that there is a $R' \in \mathscr{R}$ such that $R'[w] \subseteq [\![\Box_0\varphi]\!]_M$. Take $R = R'$. Consider any $v \in R[w]$, i.e, Rwv. We need to show that $R[v] \subseteq [\![\varphi]\!]_M$. Take any u such that Rvu. Since $dom(Ag) = (PRE(W) \times Pre(PRE(W)))$, R is transitive and thus from Rwv and Rvu we get Rwu. Hence $M, u \models \varphi$.

4. Interaction axioms:

   (a) $\forall\varphi \to \Box_0\varphi$ (Universality). Let $M, w \models \forall\varphi$. Then $W = [\![\varphi]\!]_M$. For any $R \in \mathscr{R}$, $R[w] \subseteq W = [\![\varphi]\!]_M$ and hence $M, w \models \Box_0\varphi$.

   (b) $(\Box_0\varphi \wedge \forall\psi) \leftrightarrow \Box_0(\varphi \wedge \forall\psi)$ (Pullout). $(\Rightarrow)$. Suppose $M, w \models \Box_0\varphi \wedge \forall\psi$. Then there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi]\!]_M$ and $[\![\psi]\!]_M = W$. Hence $R[w] \subseteq ([\![\varphi]\!]_M \cap [\![\forall\psi]\!]_M)$, i.e., $R[w] \subseteq ([\![\varphi \wedge \forall\psi]\!]_M)$ and thus $M, w \models \Box_0(\varphi \wedge \forall\psi)$. $(\Leftarrow)$. Suppose $M, w \models \Box_0(\varphi \wedge \forall\psi)$. Then there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi \wedge \forall\psi]\!]_M$, i.e., $R[w] \subseteq ([\![\varphi]\!]_M \cap [\![\forall\psi]\!]_M)$. Either $[\![\forall\psi]\!]_M = W$ or $[\![\forall\psi]\!]_M = \emptyset$. But if $[\![\forall\psi]\!]_M = \emptyset$, we would have $R[w] \subseteq ([\![\varphi]\!]_M \cap \emptyset$ and thus $R[w] = \emptyset$. However, as $dom(Ag) = (PRE(W) \times Pre(PRE(W)))$, R is reflexive and thus $(w, w) \in R[w]$. Hence $[\![\forall\psi]\!]_M \neq \emptyset$ and thus we must have $[\![\forall\psi]\!]_M = W$. So $R[w] \subseteq ([\![\varphi]\!]_M \cap [\![\forall\psi]\!]_M) = [\![\varphi]\!]_M$, which together with $[\![\forall\psi]\!]_M = W$ implies $M, w \models \Box_0\varphi \wedge \forall\psi$.

5. Inference rules:

   (a) Necessitation Rule for $\forall$: Let $M \models \varphi$. Then $W = [\![\varphi]\!]_M$ and thus $M \models \forall\varphi$.

   (b) Necessitation Rule for $\Box$: Let $M \models \varphi$. Then $W = [\![\varphi]\!]_M$. Take any world $w \in W$. As $Ag(\langle \mathscr{R}, \prec \rangle)[w] \subseteq W$, we have $M, w \models \Box\varphi$ and thus $M \models \Box\varphi$.

   (c) Monotonicity Rule for $\Box_0$: Let $M \models \varphi$. Then $W = [\![\varphi]\!]_M$. Take any world $w \in W$ and any $R \in \mathscr{R}$. As $R[w] \subseteq W$, we have $M, w \models \Box_0\varphi$ and thus $M \models \Box_0\varphi$.

$\Box$

We now show the main soundness result for Λ_0.

Theorem 2. Λ_0 *is sound with respect to the class of* $\cap$*-models. That is, for all* $\varphi \in \mathscr{L}$ *and any* $\cap$*-model* M*:* $\vdash_{\mathsf{L}_0} \varphi$ *implies* $M \models \varphi$.

Proof. It suffices to show that each axiom is valid and that the inference rules preserve truth. Given Proposition 5, we know that the axioms listed under 1-4, 5(a), and 5(b) are valid in any **REL** model, and that the inference rules listed 1-4 preserve truth. Hence as the class of $\cap$-models is contained in the class of **REL** models, these axioms are still valid in the former class. Thus, it remains to be shown that the axiom 5(c), $\Box_0\varphi \to \Box\varphi$, is valid in all $\cap$-models. Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model and let w a world in M. Suppose that $M, w \models \Box_0\varphi$. Then there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi]\!]_M$. Note that $\bigcap \mathscr{R}[w] = \bigcap_{R' \in \mathscr{R}}(R'[w]) \subseteq R[w]$. Hence $M, w \models \Box\varphi$. $\square$

2.4 Completeness of L_0

This section proves strong completeness of Λ_0 with respect to the class of $\cap$-models. The completeness proof follows directly from the fact that the logic generated by Λ_0 is complete with respect to finite (and hence feasible) **NEL** models. This, together with the fact that feasible **NEL** models are modally equivalent to their images under *Rel*, as established in Proposition 3, gives us completeness.

As shown in [21, pp. 194-195], a logic is complete if every consistent set of formulas is satisfiable on some model:

Proposition 6. *A logic Λ is strongly complete with respect to a class of models $\mathcal{C}$ iff every Λ-consistent set of formulas is satisfiable on some model $M \in \mathcal{C}$.*

Theorem 3. *Λ_0 is strongly complete with respect to the class of $\cap$-models.*

Proof. By Proposition 6, it suffices to show that every L_0-consistent set of formulas is satisfiable on some $\cap$-model. Let Γ be an L_0-consistent set of formulas. As Λ_0 is complete and has the finite model property with respect to **NEL** models, there is a finite (and hence feasible) neighborhood evidence model M and a state w in M such that $M, w \models \varphi$ for all $\varphi \in \Gamma$. By Proposition 3, we have $Rel(M), w \models \varphi$ for all $\varphi \in \Gamma$. Thus, Γ is satisfiable on a $\cap$-model. $\square$

2.5 Evidence dynamics

Having established the soundness and completeness of the static logic Λ_0, we now turn to evidence dynamics. In line with the work on **NEL** logics, we consider update, evidence addition and evidence upgrade actions for $\cap$-models. Throughout this section, we fix a $\cap$-model $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$, some proposition $P \subseteq W$ and some evidence order $R \in Pre(W)$.

Updates. 'Hard information' is naturally represented as a proposition. Thus, as done in the **NEL** setting and more generally in dynamic epistemic logic, we will consider here updates that involve learning a new fact P with absolute certainty. Upon learning P, the agent rules out all possible states that are incompatible with it. Following standard practice, we model this is via model restrictions. For **REL** models, this means keeping only the worlds in $[\![\varphi]\!]_M$ and restricting each evidence order accordingly.

Definition 24 (Update)**.** The model $M^{!P} = \langle W^{!P}, \mathscr{R}^{!P}, V^{!P}, Ag_\cap^{!P}\rangle$ has $W^{!P} := P$, $\mathscr{R}^{!P} := \{R \cap P^2 \mid R \in \mathscr{R}\}$, $Ag_\cap^{!P} := Ag_\cap$ restricted to P, and for all $p \in \mathsf{P}$, $V^{!P}(p) := V(p) \cap P$. $\triangleleft$

Relational evidence addition. Unlike our notion of update, which is standardly defined in terms of an incoming proposition $P \subseteq W$, our proposed notion of evidence addition for

$\cap$-models involves accepting a new piece of *relational evidence* R from a trusted source. That is, relational evidence addition consists of adding a new piece of relational evidence $R \subseteq Pre(W)$ to the family $\mathscr{R}$.

Definition 25 (Relational evidence addition)**.** The model $M^{+P} = \langle W^{+R}, \mathscr{R}^{+R}, V^{+R}, Ag_{\cap}^{+R} \rangle$ has $W^{+R} := W$, $\mathscr{R}^{+R} := \mathscr{R} \cup \{R\}$, $V^{+R} := V$ and $Ag_{\cap}^{+R} := Ag_{\cap}$. $\lhd$

As expected, after adding R_P as a piece of evidence, the agent may not believe P.

Observation. In the general **REL** setting, evidence addition can be seen as a complex action involving two transformations on the initial model: (i) adding a piece of relational evidence to $\mathscr{R}$; and (ii) updating the priority order $\preceq$ to 'place' the new evidence item where it fits, according to its reliability. We will discuss this more general notion in subsequent chapters. As we indicated at the beginning of this chapter, we take $\cap$-models to be appropriate to model situations involving *unordered evidence*, i.e., scenarios in which all evidence is equally reliable, or in which the agent has no information about relative reliability. Accordingly, the notion of relational addition for $\cap$-models corresponds to adding a new piece of evidence that is on a par with the previous ones; either as equally reliable, or as fully incomparable. $\lhd$

Relational evidence upgrade. Finally, we consider an action of upgrade with a piece of relational evidence R. This upgrade action is based on the notion of *binary lexicographic merge* from Andréka et. al. [7]. The action is similar in spirit to the evidence upgrade introduced in the **NEL** setting (and the general notion of lexicographic upgrade in epistemic logic), as it modifies the existing evidence giving priority to the new evidence relation. Moreover, an upgrade with R_P induces belief in P, and thus upgrade is stronger than addition, as usual.

Definition 26 (Evidence upgrade)**.** The model $M^{\Uparrow R} = \langle W^{\Uparrow R}, \mathscr{R}^{\Uparrow R}, V^{\Uparrow R}, Ag_{\cap}^{\Uparrow R} \rangle$ has $W^{\Uparrow R} := W$, $\mathscr{R}^{\Uparrow R} := \{R^{<} \cup (R \cap R') \mid R' \in \mathscr{R}\}$, $V^{\Uparrow R} := V$ and $Ag_{\cap}^{\Uparrow R} := Ag_{\cap}$. $\lhd$

Intuitively, this operation modifies each existing piece of evidence R' with R following the rule: "keep whatever R and R' agree on, and where they conflict, give priority to R'".

2.6 A PDL language for relational evidence

To encode the evidential actions described above, we will present dynamic extensions of $\mathscr{L}$, obtained by adding to $\mathscr{L}$ dynamic modalities for update, evidence addition and evidence upgrade. The modalities for update will be standard, i.e., modalities of the form $[!\varphi]\psi$. However, to encode syntactically the relational evidence featured in addition and upgrade, we need to add formulas to the language standing for evidence orders. A natural way to introduce order-defining expressions, in a modal setting such as ours, is to employ suitable program expressions from Propositional Dynamic Logic (PDL). We will follow this approach, augmenting $\mathscr{L}$ with dynamic modalities of the form $[+\pi]\psi$ for addition and $[\Uparrow \pi]\psi$ for upgrade, where the symbol π occurring inside the modality is a PDL program that stands for a piece of relational evidence.

As evidence orders are preorders, we will employ a set of program expressions whose terms are guaranteed to always define preorders. An natural fragment of PDL meeting this condition is the one provided by programs of the form π^*, which always define reflexive transitive closure of some relation.

Definition 27 (Π)**.** The set of program symbols Π is defined recursively as follows:

$$\pi ::= \ A \,|?\varphi \mid \pi \cup \pi \mid \pi ; \pi \mid \pi^*$$

where $\varphi \in \mathscr{L}$. Here A denotes the universal program, while the rest of the programs have their usual PDL meanings (for details about the PDL language, we refer the reader to [22]). We denote by $\Pi_* := \{\pi^* \mid \pi \in \Pi\}$ the set of $*$-programs. We call Π_* the set of *evidence programs.* ◁

Π_* will be our program set of choice; every program symbol inside a dynamic modality will come from this set. To assign meaning to the programs in **NEL** models, we extend the truth map $[\![\cdot]\!]_M$ as follows:

Definition 28 (Truth map)**.** Let $M = \langle W, \langle \mathscr{R}, \prec \rangle, V, Ag \rangle$ be an **REL** model. We define an extended *truth map* $[\![\cdot]\!]_M : \mathscr{L} \cup \Pi \to 2^W \cup 2^{W^2}$ given by:

$$\begin{aligned}
[\![\varphi]\!]_M &= \{w \in W \mid M, w \models \varphi\} \\
[\![A]\!]_M &= W^2 \\
[\![?\varphi]\!]_M &= \{(w, w) \in W^2 \mid w \in [\![\varphi]\!]^M\} \\
[\![\pi \cup \pi']\!]_M &= [\![\pi]\!]_M \cup [\![\pi']\!]_M \\
[\![\pi ; \pi']\!]_M &= [\![\pi]\!]_M \circ [\![\pi']\!]_M \\
[\![\pi^*]\!]_M &= [\![\pi]\!]_M^*
\end{aligned}$$

◁

Before introducing dynamic languages, in the remainder of this section, we consider some examples of types of relational evidence that can be defined with programs from Π_*. After that, we show some syntactic facts about Π_* that we will often draw upon in completeness proofs for dynamic extensions of $\mathscr{L}$.

Some examples of evidence programs. Here are some natural types of relational evidence that can be defined with expressions from Π_*.

Dichotomous evidence orders. For a formula φ, define $\pi_\varphi := (A; ?\varphi) \cup (?\neg\varphi; A; ?\neg\varphi)$. π_φ puts the φ worlds strictly above the $\neg\varphi$ worlds, and makes every world equally plausible within each of these two regions. That is, π_φ defines the type of dichotomous evidence order that we considered in Chapter II.1.

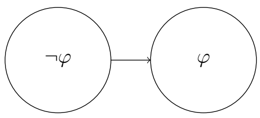


FIGURE 2.1: The dichotomous order defined by π_φ.

Total evidence orders. Several programs can be used to define total orders. For example, for formulas $\varphi_1, \ldots, \varphi_n$, we can define

$$\begin{aligned}
\pi^t(\varphi_1, \ldots, \varphi_n) := (A; ?\varphi_1) &\cup (?\neg\varphi_1; A; ?\neg\varphi_1; ?\varphi_2) \\
&\cup (?\neg\varphi_1; \neg\varphi_2; A; ?\neg\varphi_1; ?\varphi_2; ?\varphi_3) \\
&\cup \ldots \\
&\cup (?\neg\varphi_1; \ldots; ?\neg\varphi_n; A; ?\neg\varphi_1; \ldots; ?\neg\varphi_{n-1}; ?\varphi_n)
\end{aligned}$$

This type of program is described in [23]. $\pi^t(\varphi_1, \ldots, \varphi_n)$ puts the φ_1 worlds above everything else, the $\neg\varphi_1 \wedge \varphi_2$ worlds above the $\neg\varphi_1 \wedge \neg\varphi_2$ worlds, and so on, and the $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \cdots \wedge \neg\varphi_{n-1} \wedge \varphi_n$ above the $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \cdots \wedge \neg\varphi_n$ worlds. Such relations are indeed connected well-preorders.

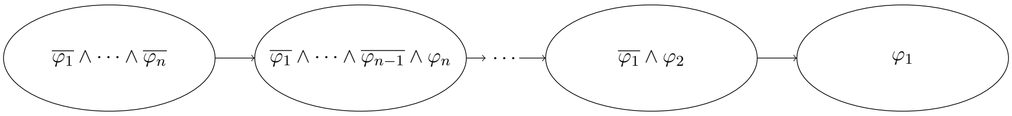


FIGURE 2.2: The total evidence order defined by $\pi^t(\varphi_1, \ldots, \varphi_n)$.

Ceteris paribus evidence orders. Relational comparisons are often incomplete, i.e., they involve a ranking of states with incomparable elements. This is often the case, for instance, when a source reports so-called *ceteris paribus* (CP) preferences. As an example, suppose that a source provides the following piece of evidence: "other things being equal, φ_1 is more plausible than $\neg\varphi_1$, and φ_2 is more plausible than $\neg\varphi_2$. However, the plausibility of φ_3 depends on φ_1 and φ_2; if both or none of them are true, I consider φ_3 to be more plausible than $\neg\varphi_3$, but I deem $\neg\varphi_3$ more plausible than φ_3 otherwise". Following the standard jargon from the CP-nets literature, φ_1 and φ_2 are here conditionally preferentially independent, while φ_3 is dependent on both φ_1 and φ_2. We can succinctly represent this ceteris paribus evidence statement with a CP-net (for details about CP-nets, we refer the reader to [24]):

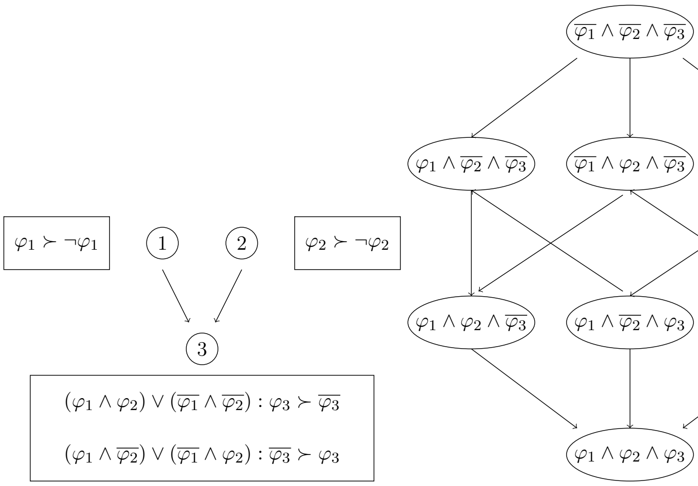


FIGURE 2.3: A CP-net for the statement (left) and its induced evidence order (right).

The evidence order induced by the statement from the source can be defined in our language as follows:

$$
\begin{aligned}
&(?\neg\varphi_1; A; ?\varphi_1) \cup (?\neg\varphi_2; A; ?\neg\varphi_2) \\
&\qquad \cup (?(((\varphi_1 \wedge \varphi_2) \vee (\neg\varphi_1 \wedge \varphi_2)) \wedge \neg\varphi_3); A; ?(((\varphi_1 \wedge \varphi_2) \vee (\neg\varphi_1 \wedge \varphi_2)) \wedge \varphi_3)) \\
&\qquad \cup (?(((\varphi_1 \wedge \neg\varphi_2) \vee (\neg\varphi_1 \wedge \varphi_2)) \wedge \varphi_3); A; ?(((\varphi_1 \wedge \neg\varphi_2) \vee (\neg\varphi_1 \wedge \varphi_2)) \wedge \neg\varphi_3))
\end{aligned}
$$

Next, we will prove some syntactic facts about the programs Π and Π_*. The main results in a *normal form lemma* for the programs in Π. This lemma shows that, for any evidence program $\pi \in \Pi$, we can find another program $\pi' \in \Pi$, which is a union of certain programs, and which is equivalent to π. Of special interest for us is the normal form established for programs of the shape π^*. The fact that every evidence program π^* is equivalent to a program with a specific syntactic shape will be put to use extensively in the completeness proofs for dynamic extensions of $\mathscr{L}$. Thus, the small detour that we take now to study the program set Π will pay off later on when we tackle the main goal of encoding evidence dynamics.

Notation 3. For programs $\pi_1, \ldots, \pi_n \in \Pi$ we write $\bigcup_{i=1}^{n} \pi_i$ to denote the program $\pi_1 \cup \cdots \cup \pi_n$. $\lhd$

We first introduce the notion of program equivalence that we will use:

Definition 29 (Program equivalence)**.** Two programs $\pi, \pi' \in \Pi$ are *equivalent* (notation $\pi \equiv \pi'$) iff for every **REL** model M, $[\![\pi]\!]_M = [\![\pi']\!]_M$. $\lhd$

Next, we give the definition of a union form:

Definition 30 (Union form)**.** A program π is in *union form* if it has the form:

$$\bigcup_{i \in I} (?\varphi_i; A; ?\psi_i) \cup (?\theta)$$

where $\varphi_i, \psi_i, \theta \in \mathscr{L}$ and I is a suitable finite index set. That is, a program is in union form if it is a *union of clauses*, where a clause is a program of the form $(?\varphi; A; ?\psi)$ or a test $(?\theta)$. $\lhd$

We now define a normal form as follows:

Definition 31 (Normal form)**.** A *normal form* for a program $\pi \in \Pi$ is a program $\pi' \in \Pi$ such that:

1. π' is in union form;
2. π and π' are equivalent.

$\lhd$

The following well-known results about relational composition will be used in the normal form lemma.

Proposition 7. *Relational composition distributes over arbitrary unions. That is, for any binary relation R and any indexed family of binary relations Q_i,*

1. $R \circ (\bigcup_i Q_i) = \bigcup_i (R \cup Q_i)$
2. $(\bigcup_i Q_i) \circ R = \bigcup_i (Q_i \circ R)$

Proof. See, e.g., [22, p. 8]. $\square$

The following facts will also be used frequently.

Proposition 8. *Let M be a **REL** model. Then:*

1. $(x, y) \in [\![?\varphi; A; ?\psi]\!]_M$ *iff* $x \in [\![\varphi]\!]_M$ *and* $x \in [\![\psi]\!]_M$.
2. $(x, y) \in [\![?\varphi; ?\psi]\!]_M$ *iff* $(x, y) \in [\![?(\varphi \wedge \psi)]\!]_M$.
3. $(x, y) \in [\![?\varphi_1; A; ?(\psi_1 \wedge \varphi_2); A; ?\psi_2]\!]_M$ *iff* $(x, y) \in [\![?(\varphi_1 \wedge \exists(\psi_1 \wedge \varphi_2); A; ?\psi_2]\!]_M$

Proof.

1. $(x,y) \in [\![?\varphi; A; ?\psi]\!]_M$
   iff $(x,y) \in [\![?\varphi]\!]_M \circ [\![A; ?\psi]\!]_M$
   iff $\exists z : (x,z) \in [\![?\varphi]\!]_M$ and $(z,y) \in [\![A; ?\psi]\!]_M$
   iff $\exists z : x = z$ and $z \in [\![\varphi]\!]_M$ and $(z,y) \in [\![A; ?\psi]\!]_M$
   iff $\exists z : x = z$ and $z \in [\![\varphi]\!]_M$ and $(z,y) \in [\![A]\!]_M \circ [\![?\psi]\!]_M$
   iff $\exists z : (x = z$ and $z \in [\![\varphi]\!]_M$ and $\exists u : ((z,u) \in [\![A]\!]_M$ and $(u,y) \in [\![?\psi]\!]_M))$
   iff $\exists z : (x = z$ and $z \in [\![\varphi]\!]_M$ and $\exists u : (u \in W$ and $u = y$ and $y \in [\![\psi]\!]_M))$
   iff $x \in [\![\varphi]\!]_M$ and $y \in [\![\psi]\!]_M$

2. $(x,y) \in [\![?\varphi; ?\psi]\!]_M$
   iff $(x,y) \in [\![?\varphi]\!]_M \circ [\![?\psi]\!]_M$
   iff $\exists z : (x,z) \in [\![?\varphi]\!]_M$ and $(z,y) \in [\![?\psi]\!]_M$
   iff $\exists z : x = z$ and $z \in [\![\varphi]\!]_M$ and $z = y$ and $y \in [\![\varphi]\!]_M$
   iff $x = y$ and $x \in [\![\varphi \wedge \psi]\!]_M$
   iff $(x,y) \in [\![?(\varphi \wedge \psi)]\!]_M$

3. $(x,y) \in [\![?\varphi_1; A; ?(\psi_1 \wedge \varphi_2); A; ?\psi_2]\!]_M$
   iff $(x,y) \in [\![?\varphi_1; A; ?(\psi_1 \wedge \varphi_2)]\!]_M \circ [\![A; ?\psi_2]\!]_M$
   iff $\exists z : (x,z) \in [\![?\varphi_1; A; ?(\psi_1 \wedge \varphi_2)]\!]_M$ and $(z,y) \in [\![A; ?\psi_2]\!]_M$
   iff $\exists z : x \in [\![\varphi_1]\!]_M$ and $z \in [\![\psi_1 \wedge \varphi_2]\!]_M$ and $(z,y) \in [\![A; ?\psi_2]\!]_M$ (by Item 2 of this Prop.)
   iff $x \in [\![\varphi_1 \wedge \exists(\psi_1 \wedge \varphi_2)]\!]_M$ and $y \in [\![\psi_2]\!]_M$
   iff $(x,y) \in [\![?(\varphi_1 \wedge \exists(\psi_1 \wedge \varphi_2)); A; ?\psi_2]\!]_M$ and $y \in [\![\psi_2]\!]_M$(by Item 2 of this Prop.)

□

In the step of the normal form lemma concerning $*$-programs, we will make use of the following definitions and results.

Definition 32 (Walks and paths)**.** Let $R \subseteq W \times W$. An *walk* along R is a sequence of (not necessarily distinct) vertices $w_1, w_2, \ldots, w_k$, where $w_i \in W$ for $i = 1, 2, \ldots, k$, such that $(v_i, v_{i+1}) \in R$ for $i = 1, 2, \ldots, k-1$. A *path* is a walk in which all vertices are distinct (except possibly the first and last). A wv-walk is a walk with first vertex w and last vertex v. A wv-path is defined similarly. The *length* of a walk (path) is its number of edges. ◁

Proposition 9. *Let $R \subseteq W \times W$. Every wv-walk along R contains a wv-path along R.*

Proof. This is a standard result. For a proof, see, e.g., [25, p. 19]. □

Proposition 10. *Let M be a **REL** model. Every wv-path along $[\![\bigcup_{i=1}^n (?\varphi_i; A; ?\psi_i)]\!]_M$ of length $\ell > n$ contains a wv-path along $[\![\bigcup_{i=1}^n (?\varphi_i; A; ?\psi_i)]\!]_M$ of length at most n.*

Proof. We prove the claim by induction on the length ℓ of a wv-path $\mathcal{P} = wu_1u_2 \ldots u_\ell v$.

- Base step: $\ell = n + 1$. For each edge $(u_j, u_{j+1}) \in [\![\bigcup_{i=1}^n (?\varphi_i; A; ?\psi_i)]\!]_M$, where $j \in \{1, \ldots, n\}$, there is an $i \in \{1, \ldots, n\}$ such that $M, u_j \models \varphi_i$ and $M, u_{j+1} \models \psi_i$. There are only n indices, but $\mathcal{P}$ has $n + 1$ edges, so some index $d \in \{1, \ldots, n\}$ has to be used twice. That is, there are $(u_{j_1}, u_{j_2}) \in [\![(?\varphi_d; A; ?\psi_d)]\!]_M$ and $(u_{j_3}, u_{j_4}) \in$

$[\![(?\varphi_d; A; ?\psi_d)]\!]_M$. Then, (u_{j_1}, u_{j_4}) and (u_{j_3}, u_{j_1}) are both in $[\![(?\varphi_d; A; ?\psi_d)]\!]_M$. Without loss of generality, suppose that $j_1 < j_3$. Then as $\mathcal{P}$ is a path, all vertices (except possibly the endpoints) are distinct, and thus $u_{j_1} \neq u_{j_3}$. Removing the segment $u_{j_1} u_{j_1+1} \dots u_{j_3}$ yields a strictly shorter wv-path $\mathcal{P}'$ contained in $\mathcal{P}$. Hence, as the length of $\mathcal{P}$ is $n+1$, that of $\mathcal{P}'$ is at most n.

- Inductive step: $\ell > n+1$. We suppose that the claim holds for paths of length less than ℓ. Let $\mathcal{P}$ be a wv-path of length ℓ. There are only n indices, but $\mathcal{P}$ has $\ell > n+1$ edges, so some index $d \in \{1, \dots, n\}$ has to be used twice. Hence, there are $(u_{j_1}, u_{j_2}) \in [\![(?\varphi_d; A; ?\psi_d)]\!]_M$ and $(u_{j_3}, u_{j_4}) \in [\![(?\varphi_d; A; ?\psi_d)]\!]_M$. Then, (u_{j_1}, u_{j_4}) and (u_{j_3}, u_{j_1}) are both in $[\![(?\varphi_d; A; ?\psi_d)]\!]_M$. Without loss of generality, suppose that $j_1 < j_3$. Then as $\mathcal{P}$ is a path, all vertices (except possibly the endpoints) are distinct, and thus $u_{j_1} \neq u_{j_3}$. Removing the segment $u_{j_1} u_{j_1+1} \dots u_{j_3}$ yields a strictly shorter wv-path $\mathcal{P}'$ contained in $\mathcal{P}$. By the induction hypothesis, $\mathcal{P}'$ contains a wv-path $\mathcal{P}''$ along $[\![\bigcup_{i \in I}(?\varphi_i; A; ?\psi_i)]\!]_M$ of length at most n. As $\mathcal{P}''$ is contained in $\mathcal{P}'$ it is also contained in $\mathcal{P}$.

□

Proposition 11. *Let M be an **REL** model. Then $[\![\pi \cup ?\varphi]\!]^*_M = [\![\pi]\!]^*_M$.*

Proof. ($\subseteq$) Let $(x, y) \in [\![\pi \cup ?\varphi]\!]^*_M$. Then $x = y$, or there is a finite xy-walk along $[\![\pi \cup ?\varphi]\!]_M$. Thus, given Proposition 10, $x = y$, or there is a finite xy-path along $[\![\pi \cup ?\varphi]\!]_M$ of length ℓ, i.e., there are $z_1, z_2, \dots, z_\ell, z_{\ell+1}$ such that $z_1 = x$ and $z_{\ell+1} = y$, all states are different, except possibly the first and last, and for each $k \in \{1, \dots, \ell\}$, $(z_k, z_{k+1}) \in [\![\pi \cup ?\varphi]\!]_M$. Suppose $x = y$. Then as $[\![\pi]\!]^*_M$ is reflexively closed, we have $(x, y) \in [\![\pi]\!]^*_M$. Suppose now that $x \neq y$. Then the xy-path cannot be $x[\![?\varphi]\!]_M y$. Moreover, since all the z_i are different, except possibly the first and last, the xy-path contains no reflexive edges, so we must have $(z_k, z_{k+1}) \in [\![\pi]\!]_M$ for each $k \in \{1, \dots, \ell\}$. Thus the xy-path is a path along $[\![\pi]\!]_M$ and thus $(x, y) \in [\![\pi]\!]^*_M$. ($\supseteq$) Let $(x, y) \in [\![\pi]\!]^*_M$. Then $x = y$, or there is a finite xy-walk along $[\![\pi \cup ?\varphi]\!]_M$. This xy-walk is also an xy-walk along $[\![\pi \cup ?\varphi]\!]_M$ and thus $(x, y) \in [\![\pi \cup ?\varphi]\!]^*_M$. □

After proving some auxiliary results, we consider next the normal form lemma.

Lemma 1 (Normal Form Lemma)**.** *Given any program $\pi \in \Pi$ we can find a normal form π' for it.*

Proof. The proof is by induction on the structure of π. Let M be any **REL** model.

- $\pi := A$. Let π' be the union form $\pi' := (?\top; A; ?\top) \cup (?\bot)$. We now show that π' is equivalent to π.

  $(x, y) \in [\![\pi']\!]_M$
  iff $(x, y) \in [\![?\top; A; ?\top]\!]_M$ or $(x, y) \in [\![?\bot]\!]_M$
  iff $x \in [\![?\top]\!]_M$ and $y \in [\![?\top]\!]_M$, or $x = y$ and $y \in [\![\bot]\!]$ (by Prop. 8)
  iff $x \in W$ and $y \in W$
  iff $(x, y) \in [\![A]\!]_M$

- $\pi :=?\varphi$. Let π' be the union form $\pi' := (?\bot; A; ?\bot) \cup (?\varphi)$.

$$\begin{aligned}
&(x, y) \in [\![\pi']\!]_M \\
&\text{iff } (x, y) \in [\![?\bot; A; ?\bot]\!]_M \text{ or } (x, y) \in [\![?\varphi]\!]_M \\
&\text{iff } x \in [\![\bot]\!]_M \text{ and } y \in [\![\bot]\!]_M, \text{ or } (x, y) \in [\![?\varphi]\!]_M \quad \text{(by Proposition 8)} \\
&\text{iff } (x, y) \in [\![?\varphi]\!]_M
\end{aligned}$$

- $\pi := \pi_1 \cup \pi_2$. By induction hypothesis, we can find normal forms for π_1 and π_2. Let the forms be $\pi_1' := \bigcup_{i\in I}(?\varphi_i; A; ?\psi_i) \cup ?\theta$ and $\pi_2' := \bigcup_{j\in J}(?\varphi_j; A; ?\psi_j) \cup ?\theta'$ respectively. Let π' be the union form $\pi' := (\bigcup_{k\in I\cup J} ?\varphi_k; A; ?\psi_k) \cup (?\theta \vee \theta')$. We will show that π' is a normal form for π.

$$\begin{aligned}
&(x, y) \in [\![\pi]\!]_M \\
&\text{iff } (x, y) \in [\![\pi_1 \cup \pi_2]\!]_M \\
&\text{iff } (x, y) \in [\![\pi_1]\!]_M \cup [\![\pi_2]\!]_M \\
&\text{iff } (x, y) \in [\![\pi_1]\!]_M \text{ or } (x, y) \in [\![\pi_2]\!]_M \\
&\text{iff } (x, y) \in [\![\pi_1']\!]_M \text{ or } (x, y) \in [\![\pi_2']\!]_M \quad \text{(by induction hypothesis)} \\
&\text{iff } (x, y) \in [\![\bigcup_{i\in I}(?\varphi_i; A; ?\psi_i) \cup ?\theta]\!]_M \text{ or } (x, y) \in [\![\bigcup_{j\in J}(?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M \\
&\text{iff } (x, y) \in \bigcup_{i\in I}[\![?\varphi_i; A; ?\psi_i]\!]_M \text{ or } (x, y) \in \bigcup_{j\in J}[\![?\varphi_j; A; ?\psi_j]\!]_M \text{ or } (x, y) \in [\![?\theta]\!]_M \\
&\quad \text{or } (x, y) \in [\![?\theta']\!]_M) \\
&\text{iff } (x, y) \in \bigcup_{k\in I\cup J}[\![?\varphi_k; A; ?\psi_k]\!]_M \text{ or } (x, y) \in [\![?\theta]\!]_M \text{ or } (x, y) \in [\![?\theta']\!]_M) \\
&\text{iff } (x, y) \in \bigcup_{k\in I\cup J}[\![?\varphi_k; A; ?\psi_k]\!]_M \text{ or } (x = y \text{ and } y \in [\![\theta]\!]_M) \text{ or } (x = y \text{ and } y \in [\![\theta']\!]_M) \\
&\text{iff } (x, y) \in \bigcup_{k\in I\cup J}[\![?\varphi_k; A; ?\psi_k]\!]_M \text{ or } x = y \text{ and } (y \in [\![\theta]\!]_M \text{ or } y \in [\![\theta']\!]_M) \\
&\text{iff } (x, y) \in \bigcup_{k\in I\cup J}[\![?\varphi_k; A; ?\psi_k]\!]_M \text{ or } x = y \text{ and } y \in [\![\theta \vee \theta']\!]_M \\
&\text{iff } (x, y) \in \bigcup_{k\in I\cup J}[\![?\varphi_k; A; ?\psi_k]\!]_M \text{ or } (x, y) \in [\![?(\theta \vee \theta')]\!]_M \\
&\text{iff } (x, y) \in [\![\bigcup_{k\in I\cup J}(?\varphi_k; A; ?\psi_k)]\!]_M \text{ or } (x, y) \in [\![?(\theta \vee \theta')]\!]_M \\
&\text{iff } (x, y) \in [\![\bigcup_{k\in I\cup J}(?\varphi_k; A; ?\psi_k) \cup \big(?(\theta \vee \theta')\big)]\!]_M
\end{aligned}$$

- $\pi = \pi_1; \pi_2$. By induction hypothesis, we can find normal forms for π_1 and π_2. Let the forms be $\pi_1' := \bigcup_{i\in I}?(\varphi_i; A; ?\psi_i) \cup ?\theta$ and $\pi_2' := \bigcup_{j\in J}(?\varphi_j; A; ?\psi_j) \cup ?\theta'$ respectively. We will find a normal form for π in two steps. First, we transform π into a more convenient shape, essentially using (several times) the fact that composition distributes over union (Proposition 7). The first step is as follows:

$$\begin{aligned}
&(x, y) \in [\![\pi]\!]_M \\
&\text{iff } (x, y) \in [\![\pi_1; \pi_2]\!]_M \\
&\text{iff } (x, y) \in [\![\pi_1]\!]_M \circ [\![\pi_2]\!]_M
\end{aligned}$$

iff $(x,y) \in [\![\pi_1']\!]_M \circ [\![\pi_2']\!]_M$ (by induction hypothesis)

iff $(x,y) \in [\![(\bigcup_{i\in I} ?\varphi_i; A; ?\psi_i) \cup ?\theta]\!]_M \circ [\![(\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M$

iff $(x,y) \in ([\![\bigcup_{i\in I} ?\varphi_i; A; ?\psi_i]\!]_M \cup [\![?\theta]\!]_M) \circ [\![(\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M$

iff $(x,y) \in ([\![\bigcup_{i\in I} ?\varphi_i; A; ?\psi_i]\!]_M \circ [\![(\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M) \cup ([\![?\theta]\!]_M \circ [\![(\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$

(by Prop. 7)

iff $(x,y) \in ([\![\bigcup_{i\in I} ?\varphi_i; A; ?\psi_i]\!]_M \circ ([\![\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j]\!]_M \cup [\![?\theta']\!]_M)) \cup ([\![?\theta]\!]_M \circ [\![(\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$

iff $(x,y) \in ([\![\bigcup_{i\in I} ?\varphi_i; A; ?\psi_i]\!]_M \circ [\![\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j]\!]_M) \cup ([\![\bigcup_{i\in I} ?\varphi_i; A; ?\psi_i]\!]_M \circ [\![?\theta']\!]_M))$

$\cup ([\![?\theta]\!]_M \circ [\![(\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$ (by Prop. 7)

iff $(x,y) \in ((\bigcup_{i\in I} [\![?\varphi_i; A; ?\psi_i]\!]_M) \circ [\![\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j]\!]_M) \cup (\bigcup_{i\in I} ([\![?\varphi_i; A; ?\psi_i]\!]_M) \circ [\![?\theta']\!]_M))$

$\cup ([\![?\theta]\!]_M \circ \bigcup_{j\in J} [\![?\varphi_j; A; ?\psi_j \cup ?\theta']\!]_M)$

iff $(x,y) \in (\bigcup_{i\in I} ([\![?\varphi_i; A; ?\psi_i]\!]_M \circ [\![\bigcup_{j\in J} ?\varphi_j; A; ?\psi_j]\!]_M)) \cup (\bigcup_{i\in I} ([\![?\varphi_i; A; ?\psi_i]\!]_M \circ [\![?\theta']\!]_M))$

$\cup ([\![?\theta]\!]_M \circ \bigcup_{j\in J} [\![(?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$

iff $(x,y) \in (\bigcup_{i\in I} \bigcup_{j\in J} ([\![?\varphi_i; A; ?\psi_i]\!]_M \circ [\![?\varphi_j; A; ?\psi_j]\!]_M)) \cup (\bigcup_{i\in I} ([\![?\varphi_i; A; ?\psi_i]\!]_M \circ [\![?\theta']\!]_M))$

$\cup ([\![?\theta]\!]_M \circ \bigcup_{j\in J} [\![(?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$ (by Prop. 7 repeatedly)

iff $(x,y) \in (\bigcup_{i\in I} \bigcup_{j\in J} [\![?\varphi_i; A; ?\psi_i; ?\varphi_j; A; ?\psi_j]\!]_M) \cup (\bigcup_{i\in I} [\![?\varphi_i; A; ?\psi_i; ?\theta']\!]_M)$

$\cup ([\![?\theta]\!]_M \circ \bigcup_{j\in J} [\![(?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$

iff $(x,y) \in (\bigcup_{i\in I} \bigcup_{j\in J} [\![?\varphi_i; A; ?\psi_i; ?\varphi_j; A; ?\psi_j]\!]_M \cup [\![?\varphi_i; A; ?\psi_i; ?\theta']\!]_M)$

$\cup \bigcup_{j\in J} ([\![?\theta]\!]_M \circ [\![(?\varphi_j; A; ?\psi_j) \cup ?\theta']\!]_M)$ (by Prop. 7 and properties of arbitrary unions)

iff $(x,y) \in (\bigcup_{i\in I} \bigcup_{j\in J} [\![?\varphi_i; A; ?\psi_i; ?\varphi_j; A; ?\psi_j]\!]_M \cup [\![?\varphi_i; A; ?\psi_i; ?\theta']\!]_M)$

$\cup \bigcup_{j\in J} ([\![?\theta]\!]_M \circ ([\![?\varphi_j; A; ?\psi_j]\!]_M \cup [\![?\theta']\!]_M))$

iff $(x,y) \in (\bigcup_{i\in I} \bigcup_{j\in J} [\![?\varphi_i; A; ?\psi_i; ?\varphi_j; A; ?\psi_j]\!]_M \cup [\![?\varphi_i; A; ?\psi_i; ?\theta']\!]_M)$

$\cup \bigcup_{j\in J} (([\![?\theta]\!]_M \circ [\![?\varphi_j; A; ?\psi_j]\!]_M) \cup ([\![?\theta]\!]_M \circ [\![?\theta']\!]_M))$ (by Prop. 7)

iff $(x,y) \in (\bigcup_{i\in I} \bigcup_{j\in J} [\![?\varphi_i; A; ?\psi_i; ?\varphi_j; A; ?\psi_j]\!]_M \cup [\![?\varphi_i; A; ?\psi_i; ?\theta']\!]_M)$

$$
\begin{aligned}
& \cup \bigcup_{j\in J}(([\![?\theta;?\varphi_j;A;?\psi_j]\!]_M)\cup([\![?\theta;?\theta']\!]_M)) \\
\text{iff } & (x,y)\in\bigcup_{i\in I}\bigcup_{j\in J}([\![?\varphi_i;A;?\psi_i;?\varphi_j;A;?\psi_j]\!]_M\cup[\![?\varphi_i;A;?\psi_i;?\theta']\!]_M \\
& \cup[\![?\theta;?\varphi_j;A;?\psi_j]\!]_M)\cup[\![?\theta;?\theta']\!]_M \\
\text{iff } & (x,y)\in\bigcup_{i\in I}\bigcup_{j\in J}([\![?(\varphi_i\wedge\exists(\psi_i\wedge\varphi_j));A;?\psi_j]\!]_M\cup[\![?\varphi_i;A;?\psi_i\wedge\theta']\!]_M \\
& \cup[\![?\theta\wedge\varphi_j;A;?\psi_j]\!]_M)\cup[\![?(\theta\wedge\theta')]\!]_M \quad \text{(by Prop. 8)}
\end{aligned}
$$

We now construct a different enumeration of the formulas to simplify the double union in the program above, converting it to a simpler form. First, we define the following sets of formulas: $\varphi_1 := \{\varphi_i \mid i \in I\}$; $\varphi_2 := \{\varphi_i \wedge \exists(\psi_i \wedge \varphi_j) \mid i \in I, j \in J\}$; $\varphi_3 := \{\theta \wedge \varphi_j \mid j \in J\}$; $\Psi_1 := \{\psi_i \wedge \theta \mid i \in I\}$; $\Psi_2 := \{\psi_j \mid j \in J\}$. Let $\Gamma := \varphi_1 \cup \varphi_2 \cup \varphi_3$ and $\Delta := \Psi_1 \cup \Psi_2$. Now we define a new index set $K := (I \times \{0\}) \cup (J \times \{0\}) \cup (I \times J)$. We enumerate the formulas $\gamma \in \Gamma$ with indices from K as follows: $\gamma_{(i,0)} = \varphi_i$ for each $i \in I$; $\gamma_{(j,0)} = \theta \wedge \varphi_j$ for each $j \in J$; $\gamma_{(i,j)} = \varphi_i \wedge \exists(\psi_i \wedge \varphi_j)$ for each $i \in I, j \in J$. We enumerate the formulas $\delta \in \Delta$ as follows: $\delta_{(i,0)} = \psi_i \wedge \theta$, for each $i \in I$; $\delta_{(j,0)} = \psi_j$, for each $j \in J$; $\delta_{(i,j)} = \psi_j$ for each $i \in I, j \in J$. Substituting the indices in the program above with the new enumeration, we can now write the program in union form:

$$
\begin{aligned}
& (x,y)\in\bigcup_{i\in I}\bigcup_{j\in J}([\![?\varphi_i\wedge\exists(\psi_i\wedge\varphi_j);A;?\psi_j]\!]_M\cup[\![?\varphi_i;A;?\psi_i\wedge\theta']\!]_M \\
& \cup[\![?\theta\wedge\varphi_j;A;?\psi_j]\!]_M)\cup[\![?\theta\wedge\theta']\!]_M \\
\text{iff } & (x,y)\in(\bigcup_{k\in K}[\![\gamma_k;A;\delta_k]\!]_M)\cup[\![?\theta\wedge\theta']\!]_M \\
\text{iff } & (x,y)\in[\![\bigcup_{k\in K}(\gamma_k;A;\delta_k]\!]_M\cup[\![?\theta\wedge\theta']\!]_M \\
\text{iff } & (x,y)\in[\![\bigcup_{k\in K}(\gamma_k;A;\delta_k)\cup\theta\wedge\theta']\!]_M
\end{aligned}
$$

Hence, the program $\pi' := \bigcup_{k\in K}(\gamma_k; A; \delta_k) \cup \theta \wedge \theta'$ is a normal form for π.

- $\pi := \pi_1^*$. By induction hypothesis, we can find a normal form for π_1. Let this normal form be $\pi_1' := \bigcup_{i\in I}(?\varphi_i; A; ?\psi_i) \cup ?\theta$. We recall here that $S_0(I)$ denotes the set of all finite sequences of elements from I. Let π' be the union form:

$$\pi' := \bigcup_{s\in S_0(I)}\Big(?(\varphi_{s_1}\wedge\bigwedge_{k=2}^{\mathsf{len}(s)}(\exists(\psi_{s_{k-1}}\wedge\varphi_{s_k})));A;?\psi_{s_{\mathsf{len}(s)}})\Big)\cup(?\top)$$

We will show that π' is a normal form for π. Observe that:

$$
\begin{aligned}
& (x,y)\in[\![\pi_1^*]\!]_M \\
\text{iff } & (x,y)\in[\![\pi_1]\!]_M^* \\
\text{iff } & (x,y)\in[\![\bigcup_{i\in I}(?\varphi_i;A;?\psi_i)\cup?\theta]\!]_M \\
\text{iff } & (x,y)\in[\![\bigcup_{i\in I}(?\varphi_i;A;?\psi_i)]\!]_M \quad \text{(by Prop. 11)}
\end{aligned}
$$

iff there is a finite xy-walk along $[\![\bigcup_{i\in I}(?\varphi_i; A; ?\psi_i)]\!]_M$ of length ℓ, or $x = y$

iff there is a finite xy-path along $[\![\bigcup_{i\in I}(?\varphi_i; A; ?\psi_i)]\!]_M$ of length ℓ', or $x = y$ (by Prop. 9)

iff there is a finite xy-path along $[\![\bigcup_{i\in I}(?\varphi_i; A; ?\psi_i)]\!]_M$ of length at most $|I|$, or $x = y$ (by Prop. 10)

iff for some $s \in S_0(I)$, there are $z_1, z_2, \ldots, z_{\mathsf{len}(s)}, z_{\mathsf{len}(s)+1}$ such that $z_1 = x$ and $z_{\mathsf{len}(s)+1} = y$ and for each $k \in \{1, \ldots, \mathsf{len}(s)\}$, $(z_k, z_{k+1}) \in [\![?\varphi_{s_k}; A; ?\psi_{s_k}]\!]_M$, or $x = y$

iff for some $s \in S_0(I)$, there are $z_1, z_2, \ldots, z_{\mathsf{len}(s)}, z_{\mathsf{len}(s)+1}$ such that $z_1 = x$ and $z_{\mathsf{len}(s)+1} = y$ and for each $k \in \{1, \ldots, \mathsf{len}(s)\}$, $z_k \in [\![\varphi_{s_k}]\!]_M$ and $z_{k+1} \in [\![\psi_{s_k}]\!]_M$ and $y \in [\![\psi_{s_{\mathsf{len}(s)}}]\!]_M$ or $x = y$ (by Prop. 8)

iff for some $s \in S_0(I)$, $x \in [\![\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))]\!]_M$ and $y \in [\![?\psi_{s_{\mathsf{len}(s)}}]\!]_M$, or $x = y$

iff for some $s \in S_0(I)$, $(x, y) \in [\![?(\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))); A; ?\psi_{s_{\mathsf{len}(s)}})]\!]_M$ or $x = y$ (by Prop. 8)

iff $(x, y) \in \bigcup_{s\in S_0(I)} [\![?(\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))); A; ?\psi_{s_{\mathsf{len}(s)}})]\!]_M$, or $x = y$

iff $(x, y) \in [\![\bigcup_{s\in S_0(I)} ?\Big((\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))); A; ?\psi_{s_{\mathsf{len}(s)}}\Big)]\!]_M$, or $x = y$

iff $(x, y) \in [\![\bigcup_{s\in S_0(I)} ?\Big((\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))); A; ?\psi_{s_{\mathsf{len}(s)}}\Big)]\!]_M$, or $(x, y) \in [\![?\top]\!]_M$

iff $(x, y) \in [\![\bigcup_{s\in S_0(I)} \Big(?(\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))); A; ?\psi_{s_{\mathsf{len}(s)}})\Big) \cup (?\top)]\!]_M$

iff $(x, y) \in [\![\pi']\!]_M$

□

Notation 4. The normal form for $*$-programs, i.e.,

$$\bigcup_{s\in S_0(I)} \Big(?(\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k}))); A; ?\psi_{s_{\mathsf{len}(s)}})\Big) \cup (?\top)$$

will appear often in the completeness proof for dynamic logics. As it is a rather long program, we will generally use the following abbreviation to ease reading. Let I be an index set. For a sequence $s \in S_0(I)$, let $\boldsymbol{\varphi} := \langle \varphi_{s_1}, \ldots, \varphi_{s_{\mathsf{len}(s)}} \rangle$ and $\boldsymbol{\psi} := \langle \psi_{s_1}, \ldots, \psi_{s_{\mathsf{len}(s)}} \rangle$. We will use the following abbreviation:

$$s(\boldsymbol{\varphi}, \boldsymbol{\psi}) := (\varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k})))$$

With this abbreviation, the union normal form for $*$-programs will usually be written as:

$$\bigcup_{s \in S_0(I)} \Big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \Big) \cup (?\top)$$

$\triangleleft$

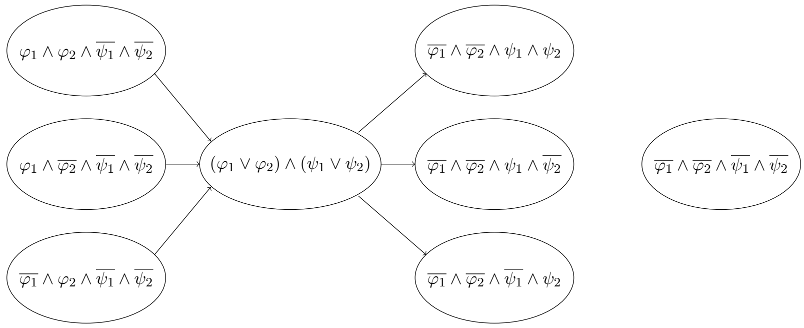


FIGURE 2.4: The relation defined by the program $\pi := ((?\varphi_1; A; ?\psi_1) \cup (?\varphi_2; A; ?\psi_2))^*$. By the Normal Form Lemma, π is equivalent to the normal form program $(?\varphi_1; A; ?\psi_1) \cup (?\varphi_2; A; ?\psi_2) \cup (?(\varphi_1 \wedge \exists(\psi_1 \wedge \varphi_2); A; ?\psi_2) \cup (?(\varphi_2 \wedge \exists(\psi_2 \wedge \varphi_1); A; ?\psi_1)$.

Notation 5 (Normal form programs)**.** For a program $\pi \in \Pi$, we let $\mathsf{nf}(\pi)$ denote some normal form for π. $\triangleleft$

2.7 REL$_\cap^+$: dynamics of evidence addition

Having established how the normal form for an arbitrary evidence program π^* looks like, we start now our study of evidence dynamics. In this section, we focus on the action of evidence addition that we introduced in Section 2.2.5. As anticipated in 2.2.6, we encode the dynamics of evidence addition by extending $\mathscr{L}$ with modal operators of the form $[+\pi]$ that describe evidence-addition actions. The new formulas of the form $[+\pi]\varphi$ are used to express the statement: "φ is true after the evidence order defined by π is added as a piece of evidence".

2.7.1 Syntax and semantics of REL$_\cap^+$

Definition 33 (Language $\mathscr{L}^+$)**.** Let P be a countably infinite set of propositional variables. The language $\mathscr{L}^+$ is defined by mutual recursion:

$$\begin{aligned} \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi \mid [+\pi^*]\varphi \\ \pi &::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^* \end{aligned}$$

where $p \in \mathsf{P}$. $\triangleleft$

We recall here the model transformation induced by evidence addition.

Definition 34. Let $M = \langle W, \mathscr{R}, V, Ag_\cap \rangle$ be a $\cap$-model and $\pi \in \Pi_*$. The model $M^{+\pi} = \langle W^{+\pi}, \mathscr{R}^{+\pi}, V^{+\pi}, Ag_\cap^{+\pi} \rangle$ has $W^{+\pi} := W$, $V^{+\pi} := V$, $Ag_\cap^{+\pi} := Ag_\cap$ and $\mathscr{R}^{+\pi} := \mathscr{R} \cup \{[\![\pi]\!]_M\}$ $\triangleleft$

We extend the satisfaction relation $\models$ to cover formulas of the form $[+\pi]\varphi$ as follows:

Definition 35 (Satisfaction for $[+\pi]\varphi$)**.** Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model, $w \in W$ and $\pi \in \Pi_*$. The satisfaction relation $\models$ between pairs (M, w) and formulas $[+\pi]\varphi \in \mathscr{L}^+$ is defined as follows:

$$M, w \models [+\pi]\varphi \text{ iff } M^{+\pi}, w \models \varphi$$

$\lhd$

2.7.2 A proof system for $\mathbf{REL}_\cap^+$: $\mathsf{L}_\cap^+$

This section introduces a proof system for $\mathbf{REL}_\cap^+$. In the next section, the logic generated by this proof system will be shown to be sound and complete with respect to $\cap$-models. The soundness and completeness proofs work via a standard reductive analysis, appealing to *reduction axioms*. Reduction axioms are valid formulas of $\mathscr{L}^+$ that indicate how to translate a formula with evidence addition modalities into a provably equivalent one without them. Adding these reduction axioms to our base system L_0 we obtain a system $\mathsf{L}_\cap^+$. The completeness of the logic generated by L^+ will then follow from the known completeness of the logic generated by L_0. We refer to [26] for an extensive explanation of this technique.

As we anticipated in Section 2.2.6, *normal forms* play a key role in the reduction axioms that we present. This lemma shows that for every evidence program $\pi \in \Pi_*$, we can find a normal form $\mathsf{nf}(\pi)$, i.e., a program which is equivalent to π and has the specific form

$$\mathsf{nf}(\pi) := \bigcup_{s \in S_0(I)} \left(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\right) \cup (?\top)$$

Here is how normal forms enter the picture. A reduction axiom works by describing the effects of an action in terms of what is true *before* the execution of the action. In dynamic epistemic logic, this is sometimes called 'pre-encoding' the effects of the action. In our setting, the goal is to 'pre-encode' the effects of adding the relations defined by arbitrary evidence programs $\pi \in \Pi_*$. We don't know how the evidence order defined by an arbitrary program π looks like, and this makes the 'pre-encoding' unfeasible. However, we *do* know that π is equivalent to a program in normal form, i.e., $\mathsf{nf}(\pi)$. Hence, if we pre-encode the effects of adding $\mathsf{nf}(\pi)$, we will have pre-encoded those of π as well. This is exactly what we will do; we use what we know about the syntactic structure of $\mathsf{nf}(\pi)$ to pre-encode the effects of adding $\mathsf{nf}(\pi)$. And given its equivalence to π, by doing so we effectively pre-encode the effects of adding π.

Definition 36 ($\mathsf{L}_\cap^+$)**.** Let $\chi, \chi' \in \mathscr{L}^+$ and let $\pi \in \Pi_*$ be an evidence program with normal form

$$\mathsf{nf}(\pi) := \bigcup_{s \in S_0(I)} \left(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\right) \cup (?\top)$$

The proof system of $\mathsf{L}_\cap^+$ includes all *axioms schemas and inference rules* of L_0. Moreover, it includes the following *reduction axioms*:

EA1$_\cap$: $[+\pi]p \leftrightarrow p$ for all $p \in \mathsf{P}$

EA2$_\cap$: $[+\pi]\neg\chi \leftrightarrow \neg[+\pi]\chi$

EA3$_\cap$: $[+\pi]\chi \wedge \chi' \leftrightarrow [+\pi]\chi \wedge [+\pi]\chi'$

EA4$_\cap$: $[+\pi]\Box_0\chi \leftrightarrow \Box_0[+\pi]\varphi \vee \left([+\pi]\chi \wedge \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))\right)$

EA5$_\cap$: $[+\pi]\Box\chi \leftrightarrow \left([+\pi]\chi \wedge \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))\right)$

EA6$_\cap$: $[+\pi]\forall\chi \leftrightarrow \forall[+\pi]\chi$

$\triangleleft$

2.7.3 Soundness and completeness of $\mathsf{L}^+_\cap$

We denote by $\Lambda^+_\cap$ the logic generated by $\mathsf{L}^+_\cap$. This section proves soundness and completeness of $\Lambda^+_\cap$ with respect to the class of $\cap$-models. As indicated above, the proofs works via a standard reductive analysis. The key part of the proofs is to show that the reduction axioms are valid.

Theorem 4. *$\Lambda^+_\cap$ is sound with respect to the class of $\cap$-models.*

Proof. It suffices to show that the axioms EA1$_\cap$ – EA6$_\cap$ are valid in all $\cap$-models. Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model, w a world in M, $\pi \in \Pi_*$ be a program with $\mathsf{nf}(\pi) := \bigcup_{s\in S_0(I)}(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$.

1. The validity of EA1$_\cap$ follows from the fact that the evidence addition transformer does not change the valuation function. The validity of the Boolean reduction axioms EA2$_\cap$ and EA3$_\cap$ can be proven by unfolding the definitions.

2. Axiom EA4$_\cap$: We first prove the following:

   Claim. $[\![\pi]\!]_M[w] \subseteq [\![[+\pi]\chi]\!]_M$ iff $M, w \models [+\pi]\chi \wedge \bigwedge_{s\in S_0(I)}(s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$.

   Proof. ($\Rightarrow$) Suppose $[\![\pi]\!]_M[w] \subseteq [\![[+\pi]\chi]\!]_M$. As π is a $*$-program, $[\![\pi]\!]_M$ is reflexive and thus $M, w \models [+\pi]\chi$. It remains to be shown that

   $$M, w \models \bigwedge_{s\in S_0(I)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$$

   Take any $s \in S_0(I)$ and suppose that $M, w \models s(\boldsymbol{\varphi},\boldsymbol{\psi})$. We need to show that $M, w \models \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi)$. Take any $v \in W$ and suppose $M, v \models \psi_{s_{\mathsf{len}(s)}}$. If we show that $M, v \models [+\pi]\chi$, we are done. Given $M, w \models s(\boldsymbol{\varphi},\boldsymbol{\psi})$ and $M, v \models \psi_{s_{\mathsf{len}(s)}}$, by Proposition 8, we have $(w, v) \in [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$. Thus

   $$(w, v) \in \bigcup_{s\in S_0(I_k)} [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$$

   Hence as

   $$\begin{aligned}[\![\pi]\!]_M = [\![\mathsf{nf}(\pi)]\!]_M &= [\![\bigcup_{s\in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\ &= [\![\bigcup_{s\in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \cup [\![?\top]\!]_M \\ &= \bigcup_{s\in S_0(I)} [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \cup [\![?\top]\!]_M\end{aligned}$$

   we have $(w, v) \in [\![\pi]\!]_M$. Hence, given $[\![\pi]\!]_M[w] \subseteq [\![[+\pi]\chi]\!]_M$ we have $M, v \models [+\pi]\chi$, as required.

   ($\Leftarrow$) Suppose that $M, w \models [+\pi]\chi \wedge \bigwedge_{s\in S_0(I)}(s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$. We will show that $[\![\pi]\!]_M[w] \subseteq [\![[+\pi]\chi]\!]_M$. Take any v and suppose $(w, v) \in [\![\pi]\!]_M$. We need to show that $v \in [\![[+\pi]\chi]\!]_M$. If $v = w$, given $M, w \models [+\pi]\chi$ we are done. So suppose $v \neq w$. Note that

$$\begin{aligned}
&(w,v) \in [\![\pi]\!]_M \\
\text{iff } &(w,v) \in [\![\mathsf{nf}(\pi)]\!]_M \\
\text{iff } &(w,v) \in [\![\bigcup_{s \in S_0(I)} ?\big(s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
\text{iff } &(w,v) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } (w,v) \in [\![?\top]\!]_M \\
\text{iff } &(w,v) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } w = v \\
\text{iff } &(w,v) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \quad \text{(as } w \neq v \text{ by assumption)} \\
\text{iff } &(w,v) \in \bigcup_{s \in S_0(I)} [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \\
\text{iff } &\text{ for some } s' \in S_0(I), (w,v) \in [\![?s'(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s'_{\mathsf{len}(s')}}]\!]_M \\
\text{iff } &\text{ for some } s' \in S_0(I), w \in [\![s'(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M \text{ and } v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M \quad \text{(by Proposition 8)}
\end{aligned}$$

Since we have $M, w \models \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$, we get in particular

$$M, w \models s'(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to [+\pi]\chi)$$

Thus from $w \in [\![s'(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M$ we get $M, w \models \forall(\psi_{s'_{\mathsf{len}(s')}} \to [+\pi]\chi)$. And given $v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M$ we get $M, v \models [+\pi]\chi$, as required. □

Given the Claim, we have

$$\begin{aligned}
&M, w \models [+\pi]\Box_0\chi \\
\text{iff } &M^{+\pi}, w \models \Box_0\chi \\
\text{iff } &\text{there is an } R \in \mathscr{R} \cup \{[\![\pi]\!]_M\} \text{ such that } R[w] \subseteq [\![\chi]\!]_{M^{+\langle\pi_i\rangle_{i<n}}} \\
\text{iff } &\text{there is an } R \in \mathscr{R} \cup \{[\![\pi]\!]_M\} \text{ such that } R[w] \subseteq [\![[+\langle\pi_i\rangle_{i<n}]\chi]\!]_M \\
\text{iff } &\text{there is an } R \in \mathscr{R} \text{ such that } R[w] \subseteq [\![[+\pi]\chi]\!]_M \\
&\quad \text{or } [\![\pi]\!]_M[w] \subseteq [\![[+\pi]\chi]\!]_M \\
\text{iff } &M, w \models \Box_0[+\pi]\chi \\
&\quad \text{or } M, w \models [+\pi]\chi \wedge \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi)) \quad \text{(by the Claim above))} \\
\text{iff } &M, w \models \Box_0[+\pi]\chi \vee \Big([+\pi]\chi \wedge \bigwedge_{s \in S_0(I_i)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+\langle\pi_i\rangle_{i<n}]\chi))\Big)
\end{aligned}$$

3. Axiom EA5$_\cap$: We first prove the following:

Claim. $\bigcap(\mathscr{R} \cup \{[\![\pi]\!]_M\})[w] \subseteq [\![[+\pi]\chi]\!]_M$ iff $M, w \models [+\pi]\chi \wedge \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$.

Proof. ($\Rightarrow$) Suppose $\bigcap(\mathscr{R}\cup\{[\![\pi]\!]_M\})[w] \subseteq [\![[+\pi]\chi]\!]_M$. As $\bigcap(\mathscr{R}\cup\{[\![\pi]\!]_M\})$ is reflexive, we have $M, w \models [+\pi]\chi$. It remains to be shown that

$$M, w \models \bigwedge_{s\in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$$

Take any $s \in S_0(I)$ and suppose that $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$. We need to show that $M, w \models \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi)$. Take any $v \in \bigcap(\mathscr{R})[w]$ and suppose $M, v \models \psi_{s_{\mathsf{len}(s)}}$. If we show that $M, v \models [+\pi]\chi$, we are done. Given $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$ and $M, v \models \psi_{s_{\mathsf{len}(s)}}$, by Proposition 8, we have $(w, v) \in [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$. Thus

$$(w, v) \in \bigcup_{s\in S_0(I_k)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$$

Hence as

$$\begin{aligned}
[\![\pi]\!]_M = [\![\mathsf{nf}(\pi)]\!]_M &= [\![\bigcup_{s\in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
&= [\![\bigcup_{s\in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \cup [\![?\top]\!]_M \\
&= \bigcup_{s\in S_0(I)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \cup [\![?\top]\!]_M
\end{aligned}$$

we have $(w, v) \in [\![\pi]\!]_M$. Hence, given $\bigcap(\mathscr{R} \cup \{[\![\pi]\!]_M\})[w] \subseteq [\![\pi]\!]_M[w] \subseteq [\![[+\pi]\chi]\!]_M$ we have $M, v \models [+\pi]\chi$, as required.

($\Leftarrow$) Suppose that $M, w \models [+\pi]\chi \wedge \bigwedge_{s\in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$. We will show that $\bigcap(\mathscr{R} \cup \{[\![\pi]\!]_M\})[w] \subseteq [\![[+\pi]\chi]\!]_M$. Take any v and suppose $(w, v) \in \bigcap(\mathscr{R}\cup\{[\![\pi]\!]_M\})$. We need to show that $v \in [\![[+\pi]\chi]\!]_M$. If $v = w$, given $M, w \models [+\pi]\chi$ we are done. So suppose $v \neq w$. Since $(w, v) \in \bigcap(\mathscr{R} \cup \{[\![\pi]\!]_M\}) = \bigcap(\mathscr{R}) \cap [\![\pi]\!]_M$, we have $(w, v) \in [\![\pi]\!]_M$. Reasoning as we did in the proof of EA4$_\cap$, we get

$$(w, v) \in [\![\pi]\!]_M \text{ iff for some } s' \in S_0(I), w \in [\![s'(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M \text{ and } v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M$$

Given that $M, w \models \bigwedge_{s\in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi))$, we have in particular

$$M, w \models s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s'_{\mathsf{len}(s)}} \to [+\pi]\chi)$$

Thus from $w \in [\![s'(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M$ we get $M, w \models \Box(\psi_{s'_{\mathsf{len}(s)}} \to [+\pi]\chi)$. And given $(w, v) \in \bigcap\mathscr{R}$ and $v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M$, we get $M, v \models [+\pi]\chi$, as required. □

Given the Claim, we have

$$\begin{aligned}
&M, w \models [+\pi]\Box\chi \\
\text{iff } &M^{+\pi}, w \models \Box\chi \\
\text{iff } &\bigcap(\mathscr{R} \cup \{[\![\pi]\!]_M\})[w] \subseteq [\![\chi]\!]_{M^{+\pi}} \\
\text{iff } &\bigcap(\mathscr{R} \cup \{[\![\pi]\!]_M\})[w] \subseteq [\![[+\pi]\chi]\!]_M \\
\text{iff } &M, w \models [+\pi]\chi \wedge \bigwedge_{s\in S_0(I_i)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+\pi]\chi)) \quad (\text{ by the Claim above})
\end{aligned}$$

4. Axiom EA6$_\cap$:

$$M, w \models [+\pi]\forall\chi \text{ iff } M^{+\pi}, w \models \forall\chi \text{ iff } [\![\chi]\!]_{M^{+\pi}} = W^{+\pi} \text{ iff } [\![[+\pi]\chi]\!]_M = W \text{ iff } M, w \models \forall[+\pi]\chi$$

□

Theorem 5. $\Lambda_\cap^+$ *is complete with respect to the class of* $\cap$*-models.*

Proof. The proof is standard, following the approach presented, e.g., in [27], Chapter 7. Here we indicate the key steps. The soundness of the reduction axioms implies that for any formula $\varphi \in \mathscr{L}^+$, there exists a semantically equivalent formula ψ in the static language $\mathscr{L}$. Moreover, the recursion axioms give us an inductive algorithm to reduce a formula in the dynamic language $\mathscr{L}^+$ to a formula in the static language $\mathscr{L}$. In other words, using the recursion axioms, we can also show that any formula $\varphi \in \mathscr{L}^+$ is provably equivalent to a formula $\psi \in \mathscr{L}$ (the details can be found in [27], Section 7.4). The completeness of $\Lambda_\cap^+$ follows then from the completeness of Λ_0 and the soundness of the recursion axioms as follows. Let $\varphi \in \mathscr{L}^+$ be such that $\not\vdash_{\mathsf{L}_\cap^+} \varphi$. Then, by the recursion axioms, there is a $\psi \in \mathscr{L}$ with $\vdash_{\mathsf{L}_\cap^+} \varphi \leftrightarrow \psi$. As $\Lambda_0 \subset \Lambda_\cap^+$ and $\psi \in \mathscr{L}$ we have $\not\vdash_{\mathsf{L}_0} \psi$. By the completeness of Λ_0 (Theorem 3), there is a $\cap$-model M such that $[\![\psi]\!]_M \neq W$. Thus, by the validity of the reduction axioms of $\mathsf{L}_\cap^+$ we conclude that $[\![\varphi]\!]_M \neq W$. □

2.8 REL$_\cap^\Uparrow$: evidence upgrade

In this section, we study the upgrade action introduced in Section 2.2.5. As we did with evidence addition, we encode the dynamics of evidence upgrade by extending $\mathscr{L}$ with modal operators of the form $[\Uparrow \pi]$ that describe evidence-upgrade actions. The new formulas of the form $[\Uparrow \pi]\varphi$ are used to express the statement: "φ is true after the existing evidence is upgraded with the relation defined by π".

2.8.1 Syntax and semantics of REL$_\cap^\Uparrow$

Definition 37 (Language $\mathscr{L}^\Uparrow$)**.** Let P be a countably infinite set of propositional variables. The language $\mathscr{L}^\Uparrow$ is defined by mutual recursion:

$$\begin{aligned} \varphi &::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi \mid [\Uparrow \pi^*]\varphi \\ \pi &::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^* \end{aligned}$$

where $p \in \mathsf{P}$. ◁

We recall here the model transformation induced by evidence upgrade.

Definition 38. Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model and $\pi \in \Pi_*$. The model $M^{\Uparrow\pi} = \langle W^{\Uparrow\pi}, \mathscr{R}^{\Uparrow\pi}, V^{\Uparrow\pi}, Ag_\cap^{\Uparrow\pi}\rangle$ has $W^{\Uparrow\pi} := W$, $V^{\Uparrow\pi} := V$, $Ag_\cap^{\Uparrow\pi} := Ag_\cap$ and

$$\mathscr{R}^{\Uparrow\pi} := \{[\![\pi]\!]_M^< \cup ([\![\pi]\!]_M \cap R) \mid R \in \mathscr{R}\}$$

◁

The truth conditions of $[\Uparrow \pi]\varphi$ are given by extending the satisfaction relation $\models$ as follows:

Definition 39 (Satisfaction for $[\Uparrow \pi]\varphi$)**.** Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model, $w \in W$ and $\pi \in \Pi_*$. The satisfaction relation $\models$ between pairs (M, w) and formulas $[\Uparrow \pi]\varphi \in \mathscr{L}^\Uparrow$ is defined as follows:

$$M, w \models [\Uparrow \pi]\varphi \text{ iff } M^{\Uparrow\pi}, w \models \varphi$$

◁

2.8.2 A proof system for $\mathbf{REL}_{\cap}^{\Uparrow}$: $\mathsf{L}_{\cap}^{\Uparrow}$

This section introduces the proof system $\mathsf{L}_{\cap}^{\Uparrow}$. In the next section, the logic generated by this proof system will be shown to be sound and complete with respect to $\cap$-models. The proofs works via a reductive analysis, like the one presented for $\Lambda_{\cap}^{+}$.

Before presenting the proof system $\mathsf{L}_{\cap}^{\Uparrow}$, we introduce some abbreviations that will be used in the definition of the reduction axioms.

Notation 6. Let π be a normal form $\pi := \bigcup_{s\in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)$. For each $J \subseteq I$, we define the abbreviations:

$$J(\boldsymbol{\varphi}) := \bigwedge_{j\in J} \varphi_j \wedge \bigwedge_{j'\in I\setminus J} \neg\varphi_{j'}$$

$$J(\boldsymbol{\psi}) := \bigwedge_{j\in J} \psi_j \wedge \bigwedge_{j'\in I\setminus J} \neg\psi_{j'}$$

Moreover, for a formula $[\Uparrow \pi]\chi$, we define the following abbreviations:

$$\pi^{\cap}(\chi) := [\Uparrow \pi]\chi \wedge \bigvee_{J\subseteq I} \Big(J(\boldsymbol{\varphi}) \wedge \Box_0\big(\big(\bigvee_{\substack{s\in S_0(I):\\ s_1\in J}} (\exists(s(\boldsymbol{\varphi},\boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})\big) \to [\Uparrow \pi]\chi\big)\Big)$$

$$\pi^{<}(\chi) := \bigvee_{J\subseteq I} \big(J(\boldsymbol{\psi}) \wedge \mathsf{suc}^{<}(\chi)\big)$$

$$\mathsf{suc}^{<}(\chi) := \bigwedge_{s\in S_0(I)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall\big((\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s'\in S_0(I)} (s'(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j\in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi\big)\big)$$

$\lhd$

As we now show, $\pi^{\cap}(\chi)$ is true at a state w in a $\cap$-model $M = \langle W, \mathscr{R}, V, Ag_{\cap}\rangle$ iff there is a piece of evidence $R \in \mathscr{R}$ such that $(R \cap [\![\pi]\!]_M)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. That is, after restricting R with the upgrading input $[\![\pi]\!]_M$, every successor of w satisfies $[\![[\Uparrow \pi]\chi]\!]_M$. Moreover, $[\Uparrow \pi]\chi \wedge \pi^{<}(\chi)$ is true at a state w in a model $M = \langle W, \mathscr{R}, V, Ag_{\cap}\rangle$ if $[\Uparrow \pi]\chi$ is true at w and $[\![\pi]\!]_M^{<}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. That is, every state v that is strictly more plausible than w according to $[\![\pi]\!]_M$ satisfies $[\Uparrow \pi]\chi$. These formulas will do most of the work in the the reduction axioms of $\mathsf{L}_{\cap}^{\Uparrow}$.

Lemma 2. *Let $M = \langle W, \mathscr{R}, V, Ag_{\cap}\rangle$ be a $\cap$-model, w a world in M, $\pi \in \Pi_*$ be a program with $\mathsf{nf}(\pi) := \bigcup_{s\in S_0(I)}(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$. Then*

1. *$M, w \models \pi^{\cap}(\chi)$ iff there is an $R \in \mathscr{R} : (R \cap [\![\pi]\!]_M)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$*
2. *$M, w \models [\Uparrow \pi]\chi \wedge \pi^{<}(\chi)$ iff $w \in [\![[\Uparrow \pi]\chi]\!]_M$ and $[\![\pi]\!]_M^{<}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$*

Proof.
Item 1:

($\Rightarrow$) Let $M, w \models \pi^{\cap}(\chi)$, i.e.,

$$M, w \models [\Uparrow \pi]\chi \wedge \bigvee_{J\subseteq I} \Big(J(\boldsymbol{\varphi}) \wedge \Box_0\big(\big(\bigvee_{\substack{s\in S_0(I):\\ s_1\in J}} (\exists(s(\boldsymbol{\varphi},\boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})\big) \to [\Uparrow \pi]\chi\big)\Big)$$

We need to show that there is an $R \in \mathscr{R} : (R \cap [\![\pi]\!]_M)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. Note first that we have

$$M, w \models \bigvee_{J \subseteq I} \big(J(\boldsymbol{\varphi}) \wedge \Box_0\big((\bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})) \rightarrow [\Uparrow \pi]\chi\big)\big)$$

Then, there is a $J \subseteq I$ such that $M, w \models J(\boldsymbol{\varphi})$ and

$$M, w \models \Box_0\big((\bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})) \rightarrow [\Uparrow \pi]\chi\big)$$

Hence, there is some $R \in \mathscr{R}$ such that, for all v with Rwv

$$M, v \models (\bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})) \rightarrow [\Uparrow \pi]\chi \tag{2.1}$$

Now take any u such that $(w, u) \in R \cap [\![\pi]\!]_M$. If we show that $u \in [\![[\Uparrow \pi]\chi]\!]_M$, we are done. Note first that given $M, w \models \pi^{\cap}(\chi)$, we have $M, w \models [\Uparrow \pi]\chi$, so if $w = u$ we are done. Suppose $w \neq u$. As $(w, u) \in R \cap [\![\pi]\!]_M$, we have $(w, u) \in [\![\pi]\!]_M$. Note that

$$\begin{aligned}
& (w, u) \in [\![\pi]\!]_M \\
\text{iff } & (w, u) \in [\![\mathsf{nf}(\pi)]\!]_M \\
\text{iff } & (w, u) \in [\![\bigcup_{s \in S_0(I)} ?\big(s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
\text{iff } & (w, u) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } (w, u) \in [\![?\top]\!]_M \\
\text{iff } & (w, u) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } w = u \\
\text{iff } & (w, u) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ (since by assumption } w \neq u) \\
\text{iff } & (w, u) \in \bigcup_{s \in S_0(I)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \\
\text{iff } & \exists s^\star \in S_0(I)((w, u) \in [\![?s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s^\star_{\mathsf{len}(s^\star)}}]\!]_M) \\
\text{iff } & \exists s^\star \in S_0(I)(w \in [\![s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M \text{ and } u \in [\![\psi_{s^\star_{\mathsf{len}(s^\star)}}]\!]_M) \quad \text{(by Prop. 8)}
\end{aligned}$$

Thus, we have $M, w \models s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi})$ and hence $M, u \models \exists(s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi}))$. Recall that

$$s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \varphi_{s^\star_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s^\star)} (\exists(\psi_{s^\star_{k-1}} \wedge \varphi_{s^\star_k})))$$

Hence $M, w \models \varphi_{s^\star_1}$ and as $M, w \models J(\boldsymbol{\varphi})$, we must have $\varphi_{s^\star_1} = \varphi_j$ for some $j \in J$. This, together with $M, u \models \psi_{s^\star_{\mathsf{len}(s^\star)}}$ gives us

$$M, u \models \bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})$$

which, given 2.1, implies $M, u \models [\Uparrow \pi]\chi$, as required.

($\Leftarrow$) Suppose that there is an $R \in \mathscr{R} : (R \cap [\![\pi]\!]_M)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. First, note that $R \cap [\![\pi]\!]_M$ is reflexive, and hence $M, w \models [\Uparrow \pi]\chi$. Hence it remains to be shown that

$$M, w \models \bigvee_{J \subseteq I} \big(J(\boldsymbol{\varphi}) \wedge \Box_0\big((\bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})) \to [\Uparrow \pi]\chi)\big)$$

It is clear that there is some $J \subseteq I$ such that $M, w \models J(\boldsymbol{\varphi})$, so we must show for this J that

$$M, w \models \Box_0\big((\bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})) \to [\Uparrow \pi]\chi\big)$$

Consider R and take any v such that Rwv. We need to show that

$$M, v \models (\bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})) \to [\Uparrow \pi]\chi$$

Suppose that

$$M, v \models \bigvee_{\substack{s \in S_0(I): \\ s_1 \in J}} (\exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}})$$

Then there is some $s \in S_0(I)$ with $s_1 \in J$ such that $M, v \models \exists(s(\boldsymbol{\varphi}, \boldsymbol{\psi})) \wedge \psi_{s_{\mathsf{len}(s)}}$. Hence there is some u such that $M, u \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$. Recall that

$$s(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \varphi_{s_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s)} (\exists(\psi_{s_{k-1}} \wedge \varphi_{s_k})))$$

Given $s_1 \in J$ and $M, w \models J(\boldsymbol{\varphi})$, we have $M, w \models \varphi_{s_1}$ and thus $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$. This, together with $M, v\psi_{s_{\mathsf{len}(s)}}$, implies $(w, v) \in [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$ and hence $(w, v) \in [\![\mathsf{nf}(\pi)]\!]_M$, which means $(w, v) \in [\![\pi]\!]_M$. As $(w, v) \in R$ and $(w, v) \in [\![\pi]\!]_M$ we have $(w, v) \in R \cap [\![\pi]\!]_M$. Thus given $(R \cap [\![\pi]\!]_M)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$ we have $M, v \models [\Uparrow \pi]\chi$, as required.

Item 2:

($\Rightarrow$) Let $M, w \models [\Uparrow \pi]\chi \wedge \pi^<(\chi)$. Then $w \in [\![[\Uparrow \pi]\chi]\!]_M$, so it remains to be shown that $[\![\pi]\!]_M^<[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. We have $M, w \models \pi^<(\chi)$, i.e.,

$$M, w \models \bigvee_{J \subseteq I} \big(J(\boldsymbol{\psi}) \wedge \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall\big(\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s' \in S_0(I)} \big(s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j)\big) \to [\Uparrow \pi]\chi)))$$

Then, there is a $J \subseteq I$ such that $M, w \models J(\boldsymbol{\psi})$ and

$$M, w \models \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall\big((\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi)\big) \tag{2.2}$$

We need to show that $[\![\pi]\!]_M^<[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. Take any v such that $(w, v) \in [\![\pi]\!]_M^<$, i.e., $(w, v) \in [\![\pi]\!]_M$ and $(v, w) \notin [\![\pi]\!]_M$. We will show that $v \in [\![[\Uparrow \pi]\chi]\!]_M$. First, observe that

$$\begin{aligned} & (w, v) \in [\![\pi]\!]_M \\ \text{iff } & (w, v) \in [\![\mathsf{nf}(\pi)]\!]_M \end{aligned}$$

$$\text{iff } (w,v) \in [\![\bigcup_{s \in S_0(I)} ?\big(s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M$$

$$\text{iff } (w,v) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } (w,v) \in [\![?\top]\!]_M$$

$$\text{iff } (w,v) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } w = v$$

$$\text{iff } (w,v) \in [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } w = v$$

$$\text{iff } (w,v) \in \bigcup_{s \in S_0(I)} [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \text{ or } w = v$$

$$\text{iff } \exists s^\star \in S_0(I)((w,v) \in [\![?s^\star(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s^\star_{\mathsf{len}(s')}}]\!]_M) \text{ or } w = v$$

$$\text{iff } \exists s^\star \in S_0(I)(w \in [\![s^\star(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M \text{ and } v \in [\![\psi_{s^\star_{\mathsf{len}(s^\star)}}]\!]_M) \text{ or } w = v \quad \text{(by Prop. 8)}$$

Moreover, note that

$$\begin{aligned}
&(v,w) \notin [\![\pi]\!]_M \\
\text{iff } &(w,v) \notin [\![\mathsf{nf}(\pi)]\!]_M \\
\text{iff } &(v,w) \notin [\![\bigcup_{s \in S_0(I)} ?\big(s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
\text{iff } &(v,w) \notin [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ and } (w,v) \notin [\![?\top]\!]_M \\
\text{iff } &(v,w) \notin [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ and } w \neq v \\
\text{iff } &(v,w) \notin \bigcup_{s \in S_0(I)} [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \text{ and } w \neq v \\
\text{iff } &\forall s \in S_0(I)((v,w) \notin [\![?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M) \text{ and } w \neq v \\
\text{iff } &\forall s \in S_0(I)(v \notin [\![s(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M \text{ or } w \notin [\![\psi_{s_{\mathsf{len}(s)}}]\!]_M) \text{ and } w \neq v \\
\text{iff } &\forall s \in S_0(I)(v \in [\![s(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M \text{ implies } w \notin [\![\psi_{s_{\mathsf{len}(s)}}]\!]_M) \text{ and } w \neq v
\end{aligned}$$

Hence we have $\exists s^\star \in S_0(I)(w \in [\![s^\star(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M$ and $v \in [\![\psi_{s^\star_{\mathsf{len}(s^\star)}}]\!]_M)$ and $\forall s \in S_0(I)(v \in [\![s(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M$ implies $w \notin [\![\psi_{s_{\mathsf{len}(s)}}]\!]_M)$. From 2.2, we have in particular

$$M, w \models s^\star(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall\big((\psi_{s^\star_{\mathsf{len}(s^\star)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi)$$

We have $w \in [\![s^\star(\boldsymbol{\varphi},\boldsymbol{\psi})]\!]_M$, from which we get

$$M, w \models \forall\big((\psi_{s^\star_{\mathsf{len}(s^\star)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi)$$

Thus, in particular

$$M, v \models (\psi_{s^\star_{\mathsf{len}(s^\star)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi$$

We already have $v \in [\![\psi_{s^\star_{\mathsf{len}(s^\star)}}]\!]_M$, so if we show that

$$M, v \models \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j)) \tag{2.3}$$

we will get $M, v \models [\Uparrow \pi]\chi$, as required. Take any $s' \in S_0(I)$ and suppose that $M, v \models s'(\boldsymbol{\varphi}, \boldsymbol{\psi})$. We need to show that $M, v \models \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))$. Consider any $u \in W$ and suppose $M, u \models \psi_{s'_{\mathsf{len}(s')}}$. Towards a contradiction, suppose that $M, u \models \varphi_j$, for some $j \in J$. As $M, w \models J(\boldsymbol{\psi})$, we have $M, w \models \psi_j$. Consider the sequence $s'' := s' \oplus \langle j \rangle$. From $\forall s \in S_0(I)(v \in [\![s(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M$ implies $w \notin [\![\psi_{s_{\mathsf{len}(s)}}]\!]_M)$ we have: $v \in [\![s''(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M$ implies $w \notin [\![\psi_{s''_{\mathsf{len}(s'')}}]\!]_M$. Note that

$$s''(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \varphi_{s''_1} \wedge \bigwedge_{k=2}^{\mathsf{len}(s'')} (\exists(\psi_{s''_{k-1}} \wedge \varphi_{s''_k}))) = s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \wedge \exists(\psi_j \wedge \varphi_{s_j})$$

Given $M, u \models \varphi_j$ and $M, w \models \psi_j$, we have $M, v \models \exists(\psi_j \wedge \varphi_{s_j})$, which together with $M, v \models s'(\boldsymbol{\varphi}, \boldsymbol{\psi})$ gives us $M, v \models s''(\boldsymbol{\varphi}, \boldsymbol{\psi})$. This implies that $w \notin [\![\psi_{s''_{\mathsf{len}(s'')}}]\!]_M$, i.e., $M, w \not\models \psi_j$ (contradiction). Thus we have $M, v \models \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))$. As s' was arbitrarily picked we get 2.3, which together with $M, v \models \psi_{s^\star_{\mathsf{len}(s^\star)}}$ implies $M, v \models [\Uparrow \pi]\chi$. Since v was picked arbitrarily, we get $[\![\pi]\!]^<_M[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$, as required.

($\Leftarrow$) Suppose that $w \in [\![[\Uparrow \pi]\chi]\!]_M$ and $[\![\pi]\!]^<_M[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$. We need to show $M, w \models [\Uparrow \pi]\chi \wedge \pi^<(\chi)$. $M, w \models [\Uparrow \pi]\chi$ is immediate, so it remains to be shown that $M, w \models \pi^<(\chi)$, i.e.,

$$M, w \models \bigvee_{J \subseteq I} \big(J(\boldsymbol{\psi}) \wedge \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall\big((\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi)\big)\big)$$

Clearly, there is a $J \subseteq I$ such that $M, w \models J(\boldsymbol{\psi})$, so it remains to be shown that;

$$M, w \models \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall\big((\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi)\big)$$

Take any $s \in S_0(I)$ and suppose that $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$. We need to show that

$$M, w \models \forall\big((\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))) \to [\Uparrow \pi]\chi\big)$$

Take any $v \in W$ and suppose that

$$M, v \models \psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))$$

We need to show that $M, v \models [\Uparrow \pi]\chi$. Note that, if $v = w$ we are done, so suppose that $v \neq w$. Since $[\![\pi]\!]^<_M[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$, if we show that $(w, v) \in [\![\pi]\!]^<_M$ we are done. We show this next, i.e., $(w, v) \in [\![\pi]\!]_M$ and $(v, w) \notin [\![\pi]\!]_M$. Note that $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$ and $M, v \models \psi_{s_{\mathsf{len}(s)}}$. Hence $(w, v) \in [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$ and thus $(w, v) \in [\![\mathsf{nf}(\pi)]\!]_M$, which gives $(w, v) \in [\![\pi]\!]_M$. Towards a contradiction, suppose that $(v, w) \in [\![\pi]\!]_M$. Then

$(v, w) \in [\![\mathsf{nf}(\pi)]\!]_M$, i.e.,

$$(v, w) \in [\![\bigcup_{s \in S_0(I)} (?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)]\!]_M$$

Since $v \neq w$ by assumption, $(v, w) \notin [\![?\top]\!]_M$, so we must have

$$(v, w) \in [\![\bigcup_{s \in S_0(I)} (?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}})]\!]_M = \bigcup_{s \in S_0(I)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$$

By Lemma 1, this means that there is a finite vw-path along $[\![\bigcup_{i \in I}(?\varphi_i; A; ?\psi_i)]\!]_M$. That is, for some $s'' \in S_0(I)$, $v \in [\![\varphi_{s''_1}]\!]_M$ and there are $z_1, z_2, \ldots, z_{\mathsf{len}(s'')}, z_{\mathsf{len}(s'')+1}$ such that $z_1 = v$ and $z_{\mathsf{len}(s'')+1} = w$ and for each $k \in \{1, \ldots, \mathsf{len}(s'')\}$, $(z_k, z_{k+1}) \in [\![?\varphi_{s''_k}; A; ?\psi_{s''_k}]\!]_M$. Given $M, w \models J(\boldsymbol{\psi})$, we must have $\psi_{s''_{\mathsf{len}(s'')}} = \psi_j$ for some $j \in J$. Let $s^\star = \langle s''_1, \ldots, s''_{\mathsf{len}(s'')-1} \rangle$. As we have

$$M, v \models \bigwedge_{s' \in S_0(I)} (s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j \in J} \neg\varphi_j))$$

In particular, we also have

$$M, v \models s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s^\star_{\mathsf{len}(s^\star)}} \to \bigwedge_{j \in J} \neg\varphi_j)$$

As $M, v \models s''(\boldsymbol{\varphi}, \boldsymbol{\psi})$, we also have $M, v \models s^\star(\boldsymbol{\varphi}, \boldsymbol{\psi})$, which gives us

$$M, v \models \forall(\psi_{s^\star_{\mathsf{len}(s^\star)}} \to \bigwedge_{j \in J} \neg\varphi_j) \tag{2.4}$$

Given that $\psi_{s^\star_{\mathsf{len}(s^\star)}} = \psi_{s''_{\mathsf{len}(s'')-1}}$, we get

$$M, z_{\mathsf{len}(s'')} \models \psi_{s^\star_{\mathsf{len}(s^\star)}}$$

and thus from 2.4 we get

$$M, z_{\mathsf{len}(s'')} \models \bigwedge_{j \in J} \neg\varphi_j$$

But then $(z_{\mathsf{len}(s'')}, w) \notin [\![?\varphi_{s''_{\mathsf{len}(s'')}}; A; ?\psi_{s''_{\mathsf{len}(s'')}}]\!]_M$ (contradiction).

□

Having shown the lemmma above, we now present the proof system $\mathsf{L}^{\Uparrow}_{\cap}$.

Definition 40 ($\mathsf{L}^{\Uparrow}_{\cap}$)**.** Let $\chi, \chi' \in \mathscr{L}^{\Uparrow}$ and let $\pi \in \Pi_*$ be an evidence program with normal form

$$\mathsf{nf}(\pi) := \bigcup_{s \in S_0(I)} \left(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\right) \cup (?\top)$$

The proof system of $\mathsf{L}^{\Uparrow}_{\cap}$ includes all *axioms schemas and inference rules* of L_0. Moreover, it includes the following *reduction axioms*:

EU1$_\cap$: $[\Uparrow \pi]p \leftrightarrow p$ for all $p \in \mathsf{P}$

EU2$_\cap$: $[\Uparrow \pi]\neg\chi \leftrightarrow \neg[\Uparrow \pi]\chi$

EU3$_\cap$: $[\Uparrow \pi]\chi \wedge \chi' \leftrightarrow [\Uparrow \pi]\chi \wedge [\Uparrow \pi]\chi'$

EU4$_\cap$: $[\Uparrow \pi]\Box_0\chi \leftrightarrow [\Uparrow \pi]\chi \wedge \pi^{<}(\chi) \wedge \pi^{\cap}(\chi)$

EU5$_\cap$: $[\Uparrow \pi]\Box\chi \leftrightarrow [\Uparrow \pi]\chi \wedge \pi^{<}(\chi) \wedge \bigwedge_{s\in S_0(I)}(s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [\Uparrow \pi]\chi))$

EU6$_\cap$: $[\Uparrow \pi]\forall\chi \leftrightarrow \forall[\Uparrow \pi]\chi$

$\triangleleft$

2.8.3 Soundness and completeness of $\mathsf{L}_\cap^\Uparrow$

We denote by $\Lambda_\cap^\Uparrow$ the logic generated by $\mathsf{L}_\cap^\Uparrow$. This section proves soundness and completeness of $\Lambda_\cap^\Uparrow$ with respect to the class of $\cap$-models. The proof works via a standard reductive analysis, like the one presented for $\Lambda_\cap^+$.

Theorem 6. *$\Lambda_\cap^\Uparrow$ is sound with respect to the class of $\cap$-models.*

Proof. It suffices to show that the axioms EU1$_\cap$ – EU6$_\cap$ are valid in all $\cap$-models. Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model, w a world in M, $\pi \in \Pi_*$ be a program with $\mathsf{nf}(\pi) := \bigcup_{s\in S_0(I)}(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$.

1. The validity of EU1$_\cap$ follows from the fact that the evidence addition transformer does not change the valuation function. The validity of the Boolean reduction axioms EU2$_\cap$ and EU3$_\cap$ can be proven by unfolding the definitions.

2. Axiom EU4$_\cap$:

   $M, w \models [\Uparrow \pi]\Box_0\chi$
   iff $M^{\Uparrow\pi}, w \models \Box_0\chi$
   iff there is an $R \in \mathscr{R}^{+\pi}$ such that $R[w] \subseteq [\![\chi]\!]_{M^{\Uparrow\pi}}$
   iff there is an $R \in \mathscr{R}$ such that $\big([\![\pi]\!]_M^{<} \cup ([\![\pi]\!]_M \cap R)\big)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$
   iff there is an $R \in \mathscr{R}$ such that $[\![\pi]\!]_M^{<}[w] \cup ([\![\pi]\!]_M \cap R)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$
   iff $[\![\pi]\!]_M^{<}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$ and there is an $R \in \mathscr{R}$ such that $([\![\pi]\!]_M \cap R)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$
   iff $[\![\pi]\!]_M^{<}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$ and there is an $R \in \mathscr{R}$ such that $([\![\pi]\!]_M \cap R)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$
   and $w \in [\![[\Uparrow \pi]\chi]\!]_M$ (as $w \in ([\![\pi]\!]_M \cap R)[w]$)
   iff $M, w \models [\Uparrow \pi]\chi \wedge \pi^{<}(\chi) \wedge \pi^{\cap}(\chi)$ (by Lemma 2)

3. Axiom EU5$_\cap$: We first prove the following:

   Claim. $([\![\pi]\!]_M \cap \bigcap\mathscr{R})[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M$ iff $M, w \models [\Uparrow \pi]\chi \wedge \bigwedge_{s\in S_0(I)}(s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [\Uparrow \pi]\chi))$.

   Proof. The proof is identical to the one used for the Claim inside Proposition 19, under item 'Axiom EA5$_\cap$'. $\Box$

   Note next that

$$\bigcap\mathscr{R}^{\Uparrow\pi} = \bigcap_{R\in\mathscr{R}}\Big([\![\pi]\!]_M^{<} \cup ([\![\pi]\!]_M \cap R)\Big)$$
$$= [\![\pi]\!]_M^{<} \cup \bigcap_{R\in\mathscr{R}}([\![\pi]\!]_M \cap R) = [\![\pi]\!]_M^{<} \cup ([\![\pi]\!]_M \cap \bigcap\mathscr{R})$$

Thus,

$$\begin{aligned}
&M, w \models [\Uparrow \pi]\Box\chi\\
&\text{iff } M^{\Uparrow\pi}, w \models \Box\chi\\
&\text{iff } \bigcap \mathscr{R}^{\Uparrow\pi}[w] \subseteq [\![\chi]\!]_{M^{\Uparrow\pi}}\\
&\text{iff } \bigcap \mathscr{R}^{\Uparrow\pi}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M\\
&\text{iff } \big([\![\pi]\!]_M^{<} \cup ([\![\pi]\!]_M \cap \bigcap \mathscr{R})\big)[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M\\
&\text{iff } [\![\pi]\!]_M^{<}[w] \cup ([\![\pi]\!]_M \cap \bigcap \mathscr{R})[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M\\
&\text{iff } [\![\pi]\!]_M^{<}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M \text{ and } ([\![\pi]\!]_M \cap \bigcap \mathscr{R})[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M\\
&\text{iff } w \in [\![[\Uparrow \pi]\chi]\!]_M \text{ and } [\![\pi]\!]_M^{<}[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M \text{ and } ([\![\pi]\!]_M \cap \bigcap \mathscr{R})[w] \subseteq [\![[\Uparrow \pi]\chi]\!]_M\\
&(\text{ as } w \in \big([\![\pi]\!]_M \cap R\big)[w])\\
&M, w \models [\Uparrow \pi]\chi \wedge \pi^{<}(\chi) \wedge \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [\Uparrow \pi]\chi))\\
&(\text{by Lemma 2 and the Claim above})
\end{aligned}$$

4. Axiom EU6$_\cap$:

$$M, w \models [\Uparrow \pi]\forall\chi \text{ iff } M^{\Uparrow\pi}, w \models \forall\chi \text{ iff } [\![\chi]\!]_{M^{\Uparrow\pi}} = W^{\Uparrow\pi} \text{ iff } [\![[\Uparrow \pi]\chi]\!]_M = W \text{ iff } M, w \models \forall[\Uparrow \pi]\chi$$

□

Theorem 7. $\Lambda_\cap^{\Uparrow}$ *is complete with respect to the class of* $\cap$*-models.*

Proof. Once we have established the validity of the reduction axioms, the proof is standard and follows the same steps used to prove completeness of $\Lambda^\cap$ (see Theorem 5). □

2.9 REL$_\cap^!$: evidence update

2.9.1 Syntax and semantics of REL$_\cap^!$

In this section, we study the evidence update action introduced in Section 2.2.5. As we did with the previous actions discussed in this Chapter, we encode the dynamics of evidence update by extending $\mathscr{L}$ with modal operators of the form $[!\varphi]$ that describe specific announcements. The new formulas of the form $[!\varphi]\psi$ are used to express the statement: "ψ is true after φ is publicly announced".

2.9.2 Syntax and semantics of REL$_\cap^!$

Definition 41 (Language $\mathscr{L}^!$)**.** Let P be a countably infinite set of propositional variables. The language $\mathscr{L}^!$ is defined recursively by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi \mid [!\varphi]\varphi$$

where $p \in \mathsf{P}$. ◁

We recall here the model transformation induced by public announcements.

Definition 42. Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model and $\varphi \in \mathscr{L}^!$. The model $M^{!\varphi} = \langle W^{!\varphi}, \mathscr{R}^{!\varphi}, V^{!\varphi}, Ag^{!\varphi}_\cap\rangle$ has $W^{!\varphi} := [\![\varphi]\!]_M$, for each $p \in \mathsf{P}$, $V^{!\varphi}(p) := V(p) \cap W^{!\varphi}$, $Ag^{!\varphi}_\cap := Ag_\cap$ and

$$\mathscr{R}^{!\varphi} := \{R \cap [\![\varphi]\!]^2_M \mid R \in \mathscr{R}\}$$

$\lhd$

The truth conditions of $[!\varphi]\psi$ are given by extending the satisfaction relation $\models$ as follows:

Definition 43 (Satisfaction for $[!\varphi]\psi$)**.** Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $[!\varphi]\psi \in \mathscr{L}^!$ is defined as follows:

$$M, w \models [!\varphi]\psi \text{ iff } M, w \models \varphi \text{ implies } M^{!\varphi}, w \models \psi$$

$\lhd$

2.9.3 A proof system for $\mathbf{REL}^!_\cap$: $\mathsf{L}^!_\cap$

Definition 44 ($\mathsf{L}^!_\cap$)**.** The proof system $\mathsf{L}^!_\cap$ includes all *axioms schemas and inference rules* of L_0. Moreover, it includes the following *reduction axioms* for all formulas $\theta, \chi, \chi' \in \mathscr{L}_{\mathbf{REL}^!_\cap}$.

PA1$_\cap$: $[!\varphi]p \leftrightarrow (\varphi \to p)$ for all $p \in \mathsf{P}$

PA2$_\cap$: $[!\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[!\varphi]\psi)$

PA3$_\cap$: $[!\varphi]\psi \wedge \psi' \leftrightarrow [!\varphi]\psi \wedge [!\varphi]\psi'$

PA4$_\cap$: $[!\varphi]\Box_0\psi \leftrightarrow (\varphi \to \Box_0(\varphi \to [!\varphi]\psi))$

PA5$_\cap$: $[!\varphi]\Box\psi \leftrightarrow (\varphi \to \Box(\varphi \to [!\varphi]\psi))$

PA6$_\cap$: $[!\varphi]\forall\psi \leftrightarrow (\varphi \to \forall[!\varphi]\psi)$

$\lhd$

2.9.4 Soundness and completeness of $\mathsf{L}^!_\cap$

We denote by $\Lambda^!_\cap$ the logic generated by $\mathsf{L}^!_\cap$. This section proves soundness and completeness of $\Lambda^!_\cap$ with respect to the class of $\cap$-models. The proof works via a standard reductive analysis, like the ones presented for $\Lambda^+_\cap$ and $\Lambda^\Uparrow_\cap$.

Theorem 8. *$\Lambda^!_\cap$ is sound with respect to the class of $\cap$-models.*

Proof. It suffices to show that the axioms PA1$_\cap$ – PA6$_\cap$ are valid in all $\cap$-models. Let $M = \langle W, \mathscr{R}, V, Ag_\cap\rangle$ be a $\cap$-model and w a world in M.

1. The validity of PA1$_\cap$ follows from the fact that the public announcement transformer does not change the valuation function. The validity of the Boolean reduction axioms PA2$_\cap$ and PA3$_\cap$ can be proven by unfolding the definitions.

2. Axiom PA4$_\cap$: ($\Rightarrow$). Suppose that $M, w \models [!\varphi]\Box_0\psi$. We need to show that $M, w \models \varphi \to \Box_0(\varphi \to [!\varphi]\psi)$. Suppose that $M, w \models \varphi$. Given $M, w \models [!\varphi]\Box_0\psi$, we get $M^{!\varphi}, w \models \Box_0\psi$. Hence there is an $R \in \mathscr{R}^{!\varphi}$ such that $R[w] \subseteq [\![\psi]\!]_{M^{!\varphi}}$. Note that $R = R' \cap [\![\varphi]\!]^2_M$ for some $R' \in \mathscr{R}$. We have to show that $M, w \models \Box_0(\varphi \to [!\varphi]\psi)$, so if we show that for all v such that $R'wv$, $M, v \models \varphi \to [!\varphi]\psi$, we are done. So take any v such that $R'wv$ and suppose that $M, v \models \varphi$. Given $M, w \models \varphi$ we have

$(w, v) \in [\![\varphi]\!]^2_M$, so Rwv. Hence $M^{!\varphi}, v \models \psi$ and thus $M, v \models [!\varphi]\psi$, as required.

($\Leftarrow$). Suppose that $M, w \models \varphi \to \Box_0(\varphi \to [!\varphi]\psi)$. We need to show that $M, w \models [!\varphi]\Box_0\psi$, i.e., $M, w \models \varphi$ implies $M^{!\varphi}, w \models \Box_0\psi$. Suppose that $M, w \models \varphi$. Then we have $M, w \models \Box_0(\varphi \to [!\varphi]\psi)$. Thus there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi \to [!\varphi]\psi]\!]_M$. Consider $R' = R \cap [\![\varphi]\!]^2_M \in \mathscr{R}^{!\varphi}$. If we show that $R'[w] \subseteq [\![\psi]\!]_{M^{!\varphi}}$ we are done. Take any v such that $R'wv$. Then we have $(w, v) \in [\![\varphi]\!]^2_M$ and thus $M, v \models \varphi$. Hence, given $R[w] \subseteq [\![\varphi \to [!\varphi]\psi]\!]_M$ we get $M, v \models [!\varphi]\psi$. And given $M, v \models \varphi$ this gives us $M^{!\varphi}, v \models \psi$, as required.

3. Axiom PA5$_\cap$:

($\Rightarrow$). Suppose that $M, w \models [!\varphi]\Box\psi$. We need to show that $M, w \models \varphi \to \Box(\varphi \to [!\varphi]\psi)$. Suppose that $M, w \models \varphi$. Given $M, w \models [!\varphi]\Box\psi$, we get $M^{!\varphi}, w \models \Box\psi$. Hence $\bigcap(\{R \cap [\![\varphi]\!]^2_M \mid R \in \mathscr{R}\})[w] \subseteq [\![\psi]\!]_{M^{!\varphi}}$. That is, $((\bigcap\mathscr{R}) \cap [\![\varphi]\!]^2_M)[w] \subseteq [\![\psi]\!]_{M^{!\varphi}}$. We have to show that $M, w \models \Box(\varphi \to [!\varphi]\psi)$. Take any v such that $(\bigcap\mathscr{R})wv$ and suppose that $M, v \models \varphi$. Given $M, w \models \varphi$ we have $(w, v) \in [\![\varphi]\!]^2_M$, so $((\bigcap\mathscr{R}) \cap [\![\varphi]\!]^2_M)wv$. Hence $M^{!\varphi}, v \models \psi$ and thus $M, v \models [!\varphi]\psi$, as required.

($\Leftarrow$). Suppose that $M, w \models \varphi \to \Box(\varphi \to [!\varphi]\psi)$. We need to show that $M, w \models [!\varphi]\Box\psi$, i.e., $M, w \models \varphi$ implies $M^{!\varphi}, w \models \Box\psi$. Suppose that $M, w \models \varphi$. Then we have $M, w \models \Box(\varphi \to [!\varphi]\psi)$. Thus $(\bigcap\mathscr{R})[w] \subseteq [\![\varphi \to [!\varphi]\psi]\!]_M$. We have to show that $M^{!\varphi}, w \models \Box\psi$, i.e., $((\bigcap\mathscr{R}) \cap [\![\varphi]\!]^2_M)[w] \subseteq [\![\psi]\!]_{M^{!\varphi}}$. Take any v such that $((\bigcap\mathscr{R}) \cap [\![\varphi]\!]^2_M)wv$. Then we have $(w, v) \in [\![\varphi]\!]^2_M$ and thus $M, v \models \varphi$. Moreover, we have $(\bigcap\mathscr{R})wv$ which together with $(\bigcap\mathscr{R})[w] \subseteq [\![\varphi \to [!\varphi]\psi]\!]_M$ implies $M, v \models [!\varphi]\psi$. And given $M, v \models \varphi$ this gives us $M^{!\varphi}, v \models \psi$, as required.

4. Axiom PA6$_\cap$:

($\Rightarrow$). Suppose that $M, w \models [!\varphi]\forall\psi$. We need to show that $M, w \models \varphi \to \forall[!\varphi]\psi$. Suppose that $M, w \models \varphi$. Given $M, w \models [!\varphi]\forall\psi$, we get $M^{!\varphi}, w \models \forall\psi$. Hence $[\![\varphi]\!]_M = [\![\psi]\!]_{M^{!\varphi}}$. That is, for all $v \in W^{!\varphi}$, $M^{!\varphi}, v \models \psi$ and thus $M^{!\varphi}, w \models \forall[!\varphi]\psi$, as required.

($\Leftarrow$). Suppose that $M, w \models \varphi \to \forall[!\varphi]\psi$. We need to show that $M, w \models [!\varphi]\forall\psi$, i.e., $M, w \models \varphi$ implies $M^{!\varphi}, w \models \forall\psi$. Suppose that $M, w \models \varphi$. Then we have $M, w \models \forall[!\varphi]\psi$. Hence, for all $v \in W$, $M, v \models [!\varphi]\psi$. I.e., for all $v \in W$, $M, v \models \varphi$ implies $M^{!\varphi}, v \models \psi$. This means that for all $v \in W^{!\varphi}$, $M^{!\varphi}, v \models \psi$, i.e., $M^{!\varphi}, w \models \forall\psi$, which gives us $M, w \models [!\varphi]\forall\psi$.

$\Box$

2.10 Chapter review

In this chapter, we have started our logical study of belief and evidence in the **REL** setting. Focusing on the specific class of $Ag_\cap$-models, we have first shown a natural way to turn feasible **NEL** models into $Ag_\cap$-models model in which the agent has evidence for, and believes, the same propositions as in the original **NEL** model. In this way, we have connected the **REL** framework back to the **NEL** framework which inspired it. After that, we have studied dynamic logics for $Ag_\cap$-models. We have considered a number of variants of the evidential actions introduced in [3], and presented sound and complete logics for reasoning about the effects of these evidential actions.

Chapter 3

$\mathbf{REL}_{\mathsf{lex}}$: lexicographic evidence merge

In this chapter, we continue our logical study of belief and evidence in the **REL** setting. We now zoom into the class of lex-models, and study logics for belief and evidence based on these models. lex-models are interesting from the **REL** perspective, since these models use the priority structure of evidence models in a natural way to resolve conflicts between pieces of evidence. This chapter is structured as follows. Section 3.1 reviews the syntax and semantics of the basic static language of **REL** models, fixing the specific truth-conditions for formulas in lex models. We then begin the study of the static logic of lex-models. Section 3.2 presents a proof system for this logic. Section 3.3 proves soundness of the logic generated by this system, and Section 3.4 provides a completeness proof. The approach to the proof is similar to the one used by Fagin et. al. [28] to prove completeness for the logic of distributed knowledge. Section 3.5 provides a first look at the dynamics of evidence addition over lex models. In a setting with *ordered evidence*, as the one modeled by lex models, evidence addition can be seen as a complex action involving two simultaneous transformations on an initial lex model: (i) adding a piece of relational evidence to the body of evidence $\mathscr{R}$; and (ii) updating the priority order $\preceq$ to 'place' the new evidence item where it fits, according to its reliability. Consequently, multiple types of evidence addition can be considered. As a starting point, here we will study an action of *prioritized addition*. This action involves adding a piece of evidence to the stock of evidence, and placing it on top of the priority order, as the most reliable piece of evidence. This is reminiscent of the way information is treated in the AGM framework, in which new evidence is assigned a high level of priority. This type of action is also interesting because it relates lex-models and $\cap$-models. In particular, prioritized addition in lex-models coincides with the action of evidence upgrade $\mathsf{up}\pi$ introduced for $\cap$-models in the previous chapter.

3.1 Syntax and semantics

Here, we recall here the language $\mathscr{L}$, which is built recursively as follows:

$$\varphi ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi$$

In this chapter we focus on lex-models, i.e., **REL** models of the form

$$M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, \mathsf{lex} \rangle$$

The semantics for formulas of $\mathscr{L}$ in lex-models is as follows.

Definition 45 (Satisfaction)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, \mathsf{lex} \rangle$ be a lex-model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is defined as follows:

$M, w \models p$ iff $w \in V(p)$
$M, w \models \neg\varphi$ iff $M, w \not\models \varphi$
$M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \Box_0\varphi$ iff there is $R \in \mathscr{R}$ such that, for all $v \in W$, Rwv implies $M, v \models \varphi$
$M, w \models \Box\varphi$ iff for all $v \in W$, $\mathsf{lex}(\langle \mathscr{R}, \preceq \rangle)wv$ implies $M, v \models \varphi$
$M, w \models \forall\varphi$ iff for all $v \in W, M, v \models \varphi$

$\lhd$

3.2 A proof system for $\mathbf{REL}_{\mathsf{lex}}$: $\mathsf{L}_{\mathsf{lex}}$

This section introduces the proof system $\mathsf{L}_{\mathsf{lex}}$. In the coming sections, the logic generated by $\mathsf{L}_{\mathsf{lex}}$ will be shown to be sound and complete with respect to the class of lex-models.

Definition 46 ($\mathsf{L}_{\mathsf{lex}}$)**.** The proof system of $\mathsf{L}_{\mathsf{lex}}$ includes the following *axiom schemas* for all formulas $\varphi, \psi \in \mathscr{L}$:

1. All tautologies of propositional logic
2. The $\mathsf{S5}$ axioms for $\forall$:

   $\mathsf{K}_\forall$: $\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$
   $\mathsf{T}_\forall$: $\forall\varphi \to \varphi$
   $4\forall$: $\forall\varphi \to \forall\forall\varphi$
   $5_\forall$: $\exists\varphi \to \forall\exists\varphi$

3. The $\mathsf{S4}$ axioms for $\Box$:

   $\mathsf{K}_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
   $\mathsf{T}_\Box$: $\Box\varphi \to \varphi$
   $4_\Box$: $\Box\varphi \to \Box\Box\varphi$

4. The T, 4 and N axioms for $\Box_0$:

   $\mathsf{T}_{\Box_0}$: $\Box_0\varphi \to \varphi$
   $4_{\Box_0}$: $\Box_0\varphi \to \Box_0\Box_0\varphi$
   $\mathsf{N}_{\Box_0}$: $\Box_0\top$

5. The following interaction axioms:
   (a) $\forall\varphi \to \Box_0\varphi$ (Universality for $\Box_0$)
   (b) $\forall\varphi \to \Box\varphi$ (Universality for $\Box$)
   (c) $(\Box_0\varphi \wedge \forall\psi) \leftrightarrow \Box_0(\varphi \wedge \forall\psi)$ (Pullout$^{\leftrightarrow}$)

The proof system $\mathsf{L}_{\mathsf{lex}}$ includes the following *inference rules* for all formulas $\varphi, \psi \in \mathscr{L}$:

1. Modus ponens
2. Necessitation Rule for $\forall$: $\dfrac{\varphi}{\forall\varphi}$

3. Necessitation Rule for $\Box$: $\dfrac{\varphi}{\Box\varphi}$

4. Monotonicity Rule for $\Box_0$: $\dfrac{\varphi \to \psi}{\Box_0\varphi \to \Box_0\psi}$

$\lhd$

3.3 Soundness of $\mathsf{L_{lex}}$

In this section we prove that the logic generated by $\mathsf{L_{lex}}$, which we denote by Λ_{lex}, is sound with respect to the class of lex-models.

Theorem 9. Λ_{lex} *is sound with respect to the class of* lex *models.*

Proof. It suffices to show that each axiom is valid and that the inference rules preserve truth. Note that in Theorem 12 of Chapter II.4 we show that the $\forall$ and $\Box_0$ axioms, the interaction axioms and the inference rules of $\mathsf{L_{lex}}$ are all valid in **REL** models. Hence, they are still valid in any lex model. Thus, it remains to be shown that the $\mathsf{S4}$ axioms for $\Box$ axioms are valid. Let $M = \langle W, \langle \mathscr{R}, \prec\rangle, V, \mathsf{lex}\rangle$ be a lex model and w a world in M.

- $\mathsf{K}_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. Let $M, w \models \Box(\varphi \to \psi)$ and suppose $M, w \models \Box\varphi$. Then $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w] \subseteq [\![\varphi \to \psi]\!]_M$ and $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w] \subseteq [\![\varphi]\!]_M$. Take any $v \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w]$. Then $M, v \models \varphi \to \psi$ and $M, v \models \varphi$, so $M, v \models \psi$.

- $\mathsf{T}_\Box$: $\Box\varphi \to \varphi$. Let $M, w \models \Box\varphi$. Then $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w] \subseteq [\![\varphi]\!]_M$. Since $codom(\mathsf{lex}) = Pre(W)$, $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)$ is reflexive and thus $w \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w]$. Hence $M, w \models \varphi$.

- $\mathsf{4}_\Box$: $\Box\varphi \to \Box\Box\varphi$. Let $M, w \models \Box\varphi$. Then $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w] \subseteq [\![\varphi]\!]_M$. Take any $v \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w]$. Take any $u \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[v]$. Since $codom(\mathsf{lex}) = Pre(W)$, $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)$ is transitive, given $(w, v), (v, u) \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)$ we have $(w, u) \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)$. Hence $M, u \models \varphi$. Thus, $M, w \models \Box\Box\varphi$.

$\Box$

3.4 Completeness of $\mathsf{L_{lex}}$

This section proves strong completeness of Λ_{lex} with respect to the class of lex-models. Before going into the details of the proof, we give an outline of the main steps in it.

1. Step 1: *Completeness of* Λ_{lex} *with respect to pre-models.* First, we define a specific type of canonical **REL** model for each Λ_{lex}-consistent theory T_0, which we call a *pre-model* for T_0. Then we prove completeness of Λ_{lex} via canonical pre-models.

2. Step 2: *Unraveling.* In the second step, we unravel the canonical pre-model for T_0 (see Chapter 4.5 in [21] for details about this technique). This involves creating all possible histories in the pre-model rooted at T_0. The histories are the paths of the canonical pre-model that start at T_0. These histories are related in such a way that they form a tree.

3. Step 3: *Completeness of* Λ_{lex} *with respect to* lex *models.* In the third step, we take the tree we just constructed, and from we define a lex model for T_0. Then we define a variant of a *bounded morphism* between the canonical pre-model and the lex model generated from the tree, which makes completeness with respect to those models immediate.

3.4.1 Step 1: Completeness with respect to pre-models

We start Step 1 with the construction of pre-models. We will build a canonical pre-model for each Λ_{lex}-consistent set of formulas T_0. Before defining the model, we fix some basic definitions:

Definition 47. The *theorems* of the logic Λ_{lex} are just the formulas in this logic, i.e., φ is a theorem of Λ_{lex} (notation: $\vdash_{\mathsf{L_{lex}}} \varphi$) iff $\varphi \in \Lambda_{\mathsf{lex}}$. ◁

Definition 48 (Deducibility)**.** We write $T_0 \vdash_{\mathsf{L_{lex}}} \psi$ iff there are formulas $\varphi_1, \ldots, \varphi_n$, where $\varphi_i \in T_0$ for $0 \leq i \leq n$ such that $\vdash_{\mathsf{L_{lex}}} (\varphi_1 \wedge \ldots \varphi_n) \to \psi$. ◁

Definition 49 (Consistency)**.** A formula φ is Λ_{lex}-*consistent* iff $\nvdash_{\mathsf{L_{lex}}} \neg\varphi$. Otherwise it is inconsistent. ◁

Definition 50 (Maximal consistency)**.** A set of formulas T_0 is *maximally consistent* iff T_0 is Λ_{lex}-consistent and for all $\varphi \notin T_0$, $T_0 \cup \{\varphi\}$ is inconsistent. ◁

As the following lemma shows, any consistent set of formulas can be extended to a maximally consistent one:

Lemma 3 (Lindenbaum's Lemma)**.** *Every consistent set of formulas of $\mathscr{L}$ can be extended to a maximally consistent one.*

Proof. The proof is a special case of [21, p. 197]. The language $\mathscr{L}$ is countable. Let $\varphi_0, \varphi_1, \ldots$ be an enumeration of the formulas in the language. Let Γ be a consistent set of formulas. Γ^+ is defined as the union of a chain of consistent sets:

$$\Gamma_0 := \Gamma$$
$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if this is consistent} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$
$$\Gamma^+ := \bigcup_{n\in\mathbb{N}} \Gamma_n$$

By construction, $\Gamma \subseteq \Gamma^+$. One can easily check that Γ^+ is maximally consistent. □

Maximally consistent sets have some handy properties:

Proposition 12. *Let T_0 be a maximally consistent set. The following hold:*

1. *For any formula φ: $\varphi \in T_0$ or $\neg\varphi \in T_0$.*
2. *$\varphi \in T_0$ iff $T_0 \vdash_{\mathsf{L_{lex}}} \varphi$.*
3. *T_0 is closed under modus ponens: $\varphi, \varphi \to \psi \in T_0$ implies $\psi \in T_0$.*
4. *$\neg\varphi \in T_0$ iff $\varphi \notin T_0$.*
5. *$\varphi \wedge \psi \in T_0$ iff $\varphi \in T_0$ and $\psi \in T_0$.*

Proof. The proofs are all standard. See, e.g., [29, p. 53]. □

We are now ready to define the notion of a canonical pre-model that we will use in the completeness proof of Step 1.

Definition 51 (Canonical pre-model for T_0)**.** Let T_0 be a Λ_{lex}-*consistent* set of formulas. A canonical pre-model for T_0 is a structure $M^c = \langle W^c, \langle \mathscr{R}^c, \preceq^c \rangle, V^c, Ag^c \rangle$ with:

- $W^c := \{T \mid T \text{ is a maximally consistent theory and } R^{\forall} T_0 T\}$.

- $\mathscr{R}^c := \{R^{\Box_0\varphi} \mid \varphi \in \mathscr{L} \text{ and } (\exists\Box_0\varphi) \in T_0\} \cup \{R'\}$.
- $\preceq^c$ is a preorder on $\mathscr{R}^c$ with

$$R \prec^c R^{\Box_0\top} \text{ for all } R \in \mathscr{R} \setminus \{R^{\Box_0\top}\}$$

  and

$$R \prec^c R^{\Box} \text{ for all } R \in \mathscr{R} \setminus \{R^{\Box}, R^{\Box_0\top}\}$$

- V^c is a valuation function given by $V^c(p) := \|p\|$.
- Ag^c is an aggregator for W^c given by

$$Ag^c(\langle\mathscr{R}, \preceq\rangle) = \begin{cases} R^{\Box} & \text{if } \langle\mathscr{R}, \preceq\rangle = \langle\mathscr{R}^c, \preceq^c\rangle \\ W^c \times W^c & \text{otherwise} \end{cases}$$

Where:

- $R^{\forall}$ is the relation on W^c given by: $R^{\forall}TS$ iff for all $\varphi \in \mathscr{L}$: $(\forall\varphi) \in T \Rightarrow (\forall\varphi) \in S$.
- for each $\varphi \in \mathscr{L}$, $R^{\Box_0\varphi}$ is the relation on W^c given by: $R^{\Box_0\varphi}TS$ iff $\Box_0\varphi \in T \Rightarrow \Box_0\varphi \in S$.
- $R^{\Box}$ is the relation on W^c given by: $R^{\Box}TS$ iff for all $\varphi \in \mathscr{L}$: $\Box\varphi \in T \Rightarrow \varphi \in S$.
- $F : W^c \times \mathscr{L} \to W^c$ is a function given by cases:
  (a) for every pair $(T, \varphi) \in W^c \times \mathscr{L}$ such that $(\Box_0\varphi) \notin T$, choose some theory $S \in W^c$ such that $\varphi \notin S$, and put $F(T, \varphi) := S$;
  (b) for every pair $(T, \varphi) \in W^c \times \mathscr{L}$ not satisfying the condition of case (a), put $F(T, \varphi) := T$.
- $R' := (R^{\Box} \cup \{(T, F(T, \varphi)) \mid T \in W^c \text{ and } \varphi \in \mathscr{L}\})^*$
- for each $\varphi \in \mathscr{L}$, $\|\varphi\|_c := \{T \in W^c \mid \varphi \in T\}$

$\triangleleft$

We first show that this canonical pre-model is indeed a **REL** model.

Proposition 13. *M^c is a **REL** model.*

Proof. In order to show that M^c is an **REL** model, we have to show that:

1. $\mathscr{R}^c$ is a family of evidence, i.e., every $R \in \mathscr{R}$ is a preorder.
2. $W^c \times W^c \in \mathscr{R}^c$.
3. $R^{\Box}$ is a preorder, and thus Ag^c is well-defined.

The rest of the model meets the conditions of a **REL** model, so let's turn to the three points just indicated.

For item 1, let $\varphi \in \mathscr{L}$ be arbitrary. Let $R \in \mathscr{R}$ be arbitrary. Then either $R = R'$ or $R = R^{\Box_0\varphi}$ for some φ. As R' is the reflexive transitive closure of the relation $R^{\Box} \cup \{(T, F(T, \varphi)) \mid T \in W^c \text{ and } \varphi \in \mathscr{L}\}$, it is a preorder, as required. Now consider $R = R^{\Box_0\varphi}$ for some φ. The reflexivity of R is immediate from the definition of $R^{\Box_0\varphi}$. For the transitivity, let $T, S, U \in M^c$ and suppose that $R^{\Box_0\varphi}TS$ and $R^{\Box_0\varphi}SU$. Either $\Box_0\varphi \notin T$ or $\Box_0\varphi \in T$. Note that, by definition of $R^{\Box_0\varphi}$, if $\Box_0\varphi \notin T$, then $R^{\Box_0\varphi}[T] = W^c$ and thus $R^{\Box_0\varphi}TU$. Suppose now that $\Box_0\varphi \in T$. Then by definition of $R^{\Box_0\varphi}$, given $R^{\Box_0\varphi}TS$ we have $\Box_0\varphi \in S$, and thus as $R^{\Box_0\varphi}SU$ we get $\Box_0\varphi \in U$, which implies $R^{\Box_0\varphi}TU$.

For item 2, observe that $\mathsf{N}_\Box$, i.e., $\Box_0\top$, is an axiom of our system. Thus it is a member of any maximal consistent set, which implies that $R^{\Box_0\top} = W^c \times W^c$.

For item 3, take any $R^\Box$. For reflexivity, suppose that $(\Box\varphi) \in T$ for some $T \in M^c$. As $\mathsf{T}_\Box$ is an axiom and T is maximal consistent, $(\Box\varphi \to \varphi) \in T$. As $(\Box\varphi) \in T$ and T is closed under modus ponens, we have $\varphi \in T$. Thus $R^\Box TT$. For transitivity, let $T, S, U \in M^c$ and suppose that $R^\Box TS$ and $R^\Box SU$. Suppose $(\Box\varphi) \in T$. As $\mathsf{4}_\Box$ is an axiom and T is maximally consistent, $(\Box\varphi \to \Box\Box\varphi) \in T$. As $(\Box\varphi) \in T$ and T is closed under modus ponens, we have $\Box\Box\varphi \in T$. As $R^\Box TS$, we then have $\Box\varphi \in S$. Hence, as $R^\Box SU$, we have $\varphi \in U$. As φ was arbitrary, this holds for each φ and hence we have $R^\Box TU$.

□

Having established that M^c is a **REL** model, we prove now the standard lemmas to show that the canonical pre-model works as expected.

Lemma 4 (Existence Lemma for $\forall$)**.** $\|\exists\varphi\| \neq \emptyset$ *iff* $\|\varphi\| \neq \emptyset$.

Proof.
($\Rightarrow$). Assume $T \in \|\exists\varphi\|$, i.e., $(\exists\varphi) \in T \in W^c$. We first prove the following:

Claim. The set $\Gamma := \{\forall\psi \mid (\forall\psi) \in T\} \cup \{\varphi\}$ is consistent.

Proof. Suppose that Γ is inconsistent, i.e., $\Gamma \vdash_{\mathsf{L}_{\mathsf{lex}}} \bot$. Then there are finitely many sentences $\forall\psi_1, \ldots, \forall\psi_n \in T$ such that $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall\psi_1 \wedge \cdots \wedge \forall\psi_n \to \neg\varphi$. By Necessitation for $\forall$ we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall(\forall\psi \wedge \cdots \wedge \forall\psi_n \to \neg\varphi)$ and from this, by $\mathsf{K}_\forall$ and modus ponens we get $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall(\forall\psi_1 \wedge \cdots \wedge \forall\psi_n) \to \forall\neg\varphi$. The system $\mathsf{S5}$ has the theorem $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\forall\forall\psi_1 \wedge \cdots \wedge \forall\forall\psi_n) \to \forall(\forall\psi_1 \wedge \cdots \wedge \forall\psi_n)$ (see, e.g., [29, p. 20]). Hence by propositional logic we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\forall\forall\psi_1 \wedge \cdots \wedge \forall\forall\psi_n) \to \forall\neg\varphi$. Given $\mathsf{4}_\forall$ we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall\psi_1 \to \forall\forall\psi_1, \ldots, \vdash_{\mathsf{L}_{\mathsf{lex}}} \forall\psi_n \to \forall\forall\psi_n$, which by propositional logic implies $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n) \to \forall\forall\psi_1 \wedge \cdots \wedge \forall\forall\psi_n$. Thus we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n) \to \forall\neg\varphi$. Hence as T is maximal consistent and closed under modus ponens, we get $(\forall\neg\varphi) \in T$. But we also have $(\exists\varphi) \in T$, i.e., $(\neg\forall\neg\varphi) \in T$, and since T is maximal consistent, this means that $(\forall\neg\varphi) \notin T$. Contradiction. □

Given the Claim, by Lindenbaum's Lemma, there is some maximally consistent theory S such that $\Gamma \subseteq S$. As $\varphi \in \Gamma$ we have $\varphi \in S$. Moreover, as $\{\forall\psi \mid (\forall\psi) \in T\} \subseteq \{\forall\chi \mid (\forall\chi) \in S\}$ we have $R^\forall TS$. As $T \in W^c$, we also have $R^\forall T_0 T$. That is, $\{\forall\theta \mid (\forall\theta) \in T_0\} \subseteq \{\forall\psi \mid (\forall\psi) \in T\}$. Thus $\{\forall\theta \mid (\forall\theta) \in T_0\} \subseteq \{\forall\chi \mid (\forall\chi) \in S\}$ and thus $R^\forall T_0 S$. Hence $S \in W^c$, which together with $\varphi \in S$ gives us $S \in \|\varphi\|$.

($\Leftarrow$) Assume $T \in \|\varphi\|$, i.e., $\varphi \in T$. Given $\mathsf{T}_\forall$ we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall\neg\varphi \to \neg\varphi$, and by contraposition we get $\vdash_{\mathsf{L}_{\mathsf{lex}}} \neg\neg\varphi \to \neg\forall\neg\varphi$, i.e., $\vdash_{\mathsf{L}_{\mathsf{lex}}} \varphi \to \exists\varphi$. Hence $(\varphi \to \exists\varphi) \in T$ and as T is closed under modus ponens, given also $\varphi \in T$ we get $(\exists\varphi) \in T$, i.e., $T \in \|\exists\varphi\|$. □

Lemma 5 (Existence Lemma for $\Box$)**.** $T \in \|\Diamond\varphi\|$ *iff there is an* $S \in \|\varphi\|$ *such that* $R^\Box TS$.

Proof.
($\Rightarrow$). Assume $T \in \|\Diamond\varphi\|$, i.e., $\Diamond\varphi \in T \in W^c$. We first prove the following:

Claim. The set $\Gamma := \{\psi \mid (\Box\psi) \in T\} \cup \{\forall\theta \mid \forall\theta \in T_0\} \cup \{\varphi\}$ is consistent.

Proof. Suppose that Γ is inconsistent. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{L}_{\mathsf{lex}}} \bot$. By the theorems $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box(\psi_{i_1} \wedge \cdots \wedge \psi_{i_n}) \leftrightarrow (\Box\psi_{i_1} \wedge \cdots \wedge \Box\psi_{i_n})$ and $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall(\theta_{j_1} \wedge \cdots \wedge \theta_{j_n}) \leftrightarrow (\forall\theta_{j_1} \wedge \cdots \wedge \forall\theta_{j_n})$ we can assume that $\Gamma_0 = \{\Box\psi, \forall\theta, \neg\varphi\}$ for some $\Box\psi, \forall\theta \in T$. That is, we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box\psi \wedge \forall\theta \to \neg\varphi$. By Necessitation for $\Box$ we obtain $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box(\Box\psi \wedge \forall\theta \to \neg\varphi)$. From this, by $\mathsf{K}_\Box$ we get $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box(\Box\psi \wedge \forall\theta) \to \Box\neg\varphi$. By the theorem $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box(\Box\psi \wedge \forall\theta) \leftrightarrow (\Box\Box\psi \wedge \Box\forall\theta)$, from propositional logic we get $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\Box\Box\psi \wedge \Box\forall\theta) \to \Box\neg\varphi$. Given the axioms in our system we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box\psi \to \Box\Box\psi$ and $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall(\forall\theta) \to \Box(\forall\theta)$. Using these, by propositional logic we obtain $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\Box\psi \wedge \forall\forall\theta) \to \Box\neg\varphi$. Given our axioms, we also have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall\theta \to \forall\forall\theta$. Hence by propositional logic we get $\vdash_{\mathsf{L}_{\mathsf{lex}}} (\Box\psi \wedge \forall\theta) \to \Box\neg\varphi$. As $\Box\psi, \forall\theta \in T$ and T is closed under modus ponens, we get $(\Box\neg\varphi) \in T$. But we also have $(\Diamond\varphi) \in T$, i.e., $(\neg\Box\neg\varphi) \in T$, and since T is maximal consistent, this means that $(\Box\neg\varphi) \notin T$. Contradiction. $\Box$

Given the Claim, by Lindenbaum's Lemma, there is some maximally consistent theory S such that $\Gamma \subseteq S$. As $\varphi \in \Gamma$ we have $\varphi \in S$. Moreover, as $\{\psi \mid (\Box\psi) \in T\} \subseteq S$, we have $R^\Box TS$. Additionally, we have $\{\forall\theta \mid (\forall\theta) \in T_0\} \subseteq S$ and thus $R^\forall T_0 S$. Hence $S \in W^c$, which together with $\varphi \in S$ gives us $S \in \|\varphi\|$.

($\Leftarrow$) Assume $T \in \|\varphi\|$, i.e., $\varphi \in T$. Given $\mathsf{T}_\Box$ we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box\neg\varphi \to \neg\varphi$, and by contraposition we get $\vdash_{\mathsf{L}_{\mathsf{lex}}} \neg\neg\varphi \to \neg\Box\neg\varphi$, i.e., $\vdash_{\mathsf{L}_{\mathsf{lex}}} \varphi \to \Diamond\varphi$. Hence, $(\varphi \to \Diamond\varphi) \in T$ and as T is closed under modus ponens, given also $\varphi \in T$ we get $(\Diamond\varphi) \in T$, i.e., $T \in \|\Diamond\varphi\|$. $\Box$

Lemma 6 (Existence Lemma for $\Box_0$)**.** *$T \in \|\Box_0\varphi\|$ iff there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq \|\varphi\|$.*

Proof. ($\Rightarrow$). Assume $T \in \|\Box_0\varphi\|$, i.e., $(\Box_0\varphi) \in T \in W^c$. We first prove the following:

Claim. $\exists\Box_0\varphi \in T_0$.

Proof. Suppose not. As T_0 is maximal consistent, we have $\neg\exists\Box_0\varphi \in T_0$, i.e., $\forall\neg\Box_0\varphi \in T_0$. As $T \in W^c$, we have $R^\forall T_0 T$. So given $\forall\neg\Box_0\varphi \in T_0$ we have $\forall\neg\Box_0\varphi \in T$. By $\mathsf{T}_\forall$ we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \forall\neg\Box_0\varphi \to \neg\Box_0\varphi$, i.e., $(\forall\neg\Box_0\varphi \to \neg\Box_0\varphi) \in T$. As T is closed under modus ponens, given $(\forall\neg\Box_0\varphi) \in T$ we get $(\neg\Box_0\varphi) \in T$. But we also have $(\Box_0\varphi) \in T \in W^c$ and thus T is inconsistent. Contradiction. $\Box$

Hence $R^{\Box_0\varphi} \in \mathscr{R}^c$. We will show that $R^{\Box_0\varphi}[T] \subseteq \|\varphi\|$. Let $S \in W^c$ be arbitrary and suppose that $R^{\Box_0\varphi}TS$. By definition of $R^{\Box_0\varphi}$, we have $(\Box_0\varphi) \in T$ implies $(\Box_0\varphi) \in S$. As $(\Box_0\varphi) \in T$ we get $(\Box_0\varphi) \in S$. Given $\mathsf{T}_{\Box_0}$ we have $\vdash_{\mathsf{L}_{\mathsf{lex}}} \Box_0\varphi \to \varphi$ and thus $(\Box_0\varphi \to \varphi) \in S$. Since S is closed under modus ponens we thus get $\varphi \in S$, i.e., $S \in \|\varphi\|$. As S was picked arbitrarily, we have $R^{\Box_0\varphi}[T] \subseteq \|\varphi\|$.

($\Leftarrow$) Let $T \in W^c$ and suppose there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq \|\varphi\|$. By definition of R^c, (i) $R = R'$ or (ii) $R = R^{\Box_0\theta}$ for some $\theta \in \mathscr{L}$ such that $(\exists\Box_0\theta) \in T_0$.

We first consider the case (i), i.e., $R = R'$. We need to show that $T \in \|\Box_0\varphi\|$. Suppose not, i.e., $\Box_0\varphi \notin T$. Recall that, given the definition of $F : W^c \times \mathscr{L} \to W^c$, given that $(\Box_0\varphi) \notin T$, we have $F(T, \varphi) = S$ for some theory $S \in W^c$ such that $\varphi \notin S$. Moreover, R' is the reflexive transitive closure of the relation $R^\Box \cup \{(T, F(T, \varphi)) \mid T \in W^c \text{ and } \varphi \in \mathscr{L}\}$. Hence $(T, F(T, \varphi)) = (T, S) \in \{(T, F(T, \varphi)) \mid T \in W^c \text{ and } \varphi \in \mathscr{L}\}$ and thus $(T, S) \in R'$. But then, as $S \in W^c$ and $\varphi \notin S$, we have $S \notin \|\varphi\|$. This implies $R'[T] \not\subseteq \|\varphi\|$, contradicting our assumption to the contrary.

We now consider case (ii), i.e., $R = R^{\Box_0\theta}$ for some $\theta \in \mathscr{L}$ such that $(\exists\Box_0\theta) \in T_0$. Either $\Box_0\theta \in T$ or $\Box_0\theta \notin T$. We consider both cases.

Case 1: Suppose that $\Box_0\theta \in T \in W^c$. We first prove the following:

Claim. The set $\Gamma := \{\Box_0\theta\} \cup \{\forall\psi \mid (\forall\psi) \in T\} \cup \{\neg\varphi\}$ is inconsistent.

Proof. Suppose that Γ is consistent. By Lindenbaum's Lemma there is some maximal consistent theory S such that $\Gamma \subseteq S$. Moreover, as $\{\forall\psi \mid (\forall\psi) \in T_0\} \subseteq \{\forall\psi \mid (\forall\psi) \in T\} \subseteq S$, we have $R^{\forall}T_0S$ and thus $S \in W^c$. As $\neg\varphi \in \Gamma$ we have $\neg\varphi \in S$. Since S is consistent we have $\varphi \notin S$, i.e., $S \notin \|\varphi\|$. From $\Box_0\theta \in \Gamma$ we have $\Box_0\theta \in S$. By definition of $R^{\Box_0\theta}$, we get $R^{\Box_0\theta}TS$. But then, given $S \notin \|\varphi\|$, we have $R^{\Box_0\theta}[T] \not\subseteq \|\varphi\|$. Contradiction. $\Box$

Given the Claim, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{L_{lex}}} \bot$. By the theorem $\vdash_{\mathsf{L_{lex}}} \forall(\psi_1 \wedge \cdots \wedge \psi_n) \leftrightarrow (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n)$ we can assume that $\Gamma_0 = \{\Box_0\theta, \forall\psi, \neg\varphi\}$ for some $\psi \in T$. Since $\Gamma_0 \vdash_{\mathsf{L_{lex}}} \bot$ we have $\vdash_{\mathsf{L_{lex}}} (\Box_0\theta \wedge \forall\psi \wedge \neg\varphi) \to \bot$, so by propositional logic $\vdash_{\mathsf{L_{lex}}} (\Box_0\theta \wedge \forall\psi) \to (\neg\varphi \to \bot)$, i.e., $\vdash_{\mathsf{L_{lex}}} (\Box_0\theta \wedge \forall\psi) \to (\neg\neg\varphi)$, i.e., $\vdash_{\mathsf{L_{lex}}} (\Box_0\theta \wedge \forall\psi) \to \varphi$. Given the Pullout axiom, we have $\vdash_{\mathsf{L_{lex}}} \Box_0(\theta \wedge \forall\psi) \to (\Box_0\theta \wedge \forall\psi)$ and thus $\vdash_{\mathsf{L_{lex}}} \Box_0(\theta \wedge \forall\psi) \to \varphi$. By the Monotonicity Rule for $\Box_0$, we get $\vdash_{\mathsf{L_{lex}}} \Box_0\Box_0(\theta \wedge \forall\psi) \to \Box_0\varphi$. By $\mathsf{4}_{\Box_0}$, we have $\vdash_{\mathsf{L_{lex}}} \Box_0(\theta \wedge \forall\psi) \to \Box_0\Box_0(\theta \wedge \forall\psi)$ and thus $\vdash_{\mathsf{L_{lex}}} \Box_0(\theta \wedge \forall\psi) \to \Box_0\varphi$. By the Pullout axiom, we have $\vdash_{\mathsf{L_{lex}}} (\Box_0\theta \wedge \forall\psi) \to \Box_0(\theta \wedge \forall\psi)$. Hence $\vdash_{\mathsf{L_{lex}}} (\Box_0\theta \wedge \forall\psi) \to \Box_0\varphi$. Therefore $((\Box_0\theta \wedge \forall\psi) \to \Box_0\varphi) \in T$. As $(\Box_0\theta) \in T$ and $(\forall\psi) \in T$, by closure under modus ponens, we have $\Box_0\varphi \in T$. That is, $T \in \|\Box_0\varphi\|$.

Case 2: Suppose that $\Box_0\theta \notin T$. Note that $\Box_0\theta \notin T$ implies that $R^{\Box_0\theta}[T] = W^c$, and since we have $R = R^{\Box_0\theta}$ and $R[T] \subseteq \|\varphi\|$, all this gives us that $W^c \subseteq \|\varphi\|_c$, i.e. all theories in the canonical model contain φ. We now prove the following:

Claim. The set $\Gamma := \{\forall\psi \mid (\forall\psi) \in T\} \cup \{\neg\varphi\}$ is inconsistent.

Proof. Suppose that Γ is consistent. By Lindenbaum's Lemma there is some maximal consistent theory S such that $\Gamma \subseteq S$. Moreover, as $\{\forall\psi \mid (\forall\psi) \in T_0\} \subseteq \{\forall\psi \mid (\forall\psi) \in T\} \subseteq S$, we have $R^{\forall}T_0S$ and thus $S \in W^c$. As $\neg\varphi \in \Gamma$ we have $\neg\varphi \in S$ and thus $S \in \|\neg\varphi\|$. Therefore $W^c \not\subseteq \|\varphi\|$ (contradiction). $\Box$

Given the Claim, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{L_{lex}}} \bot$. By the theorem $\vdash_{\mathsf{L_{lex}}} \forall(\psi_1 \wedge \cdots \wedge \psi_n) \leftrightarrow (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n)$ we can assume that $\Gamma_0 = \{\forall\psi, \neg\varphi\}$ for some $\psi \in T$. Since $\Gamma_0 \vdash_{\mathsf{L_{lex}}} \bot$ we have $\vdash_{\mathsf{L_{lex}}} (\forall\psi \wedge \neg\varphi) \to \bot$, so by propositional logic $\vdash_{\mathsf{L_{lex}}} (\forall\psi) \to (\neg\varphi \to \bot)$, i.e., $\vdash_{\mathsf{L_{lex}}} (\forall\psi) \to (\neg\neg\varphi)$, i.e., $\vdash_{\mathsf{L_{lex}}} (\forall\psi) \to \varphi$. By propositional logic, given $\vdash_{\mathsf{L_{lex}}} (\forall\psi) \to \varphi$ we can strengthen the antecedent getting $\vdash_{\mathsf{L_{lex}}} (\Box_0\top \wedge \forall\psi) \to \varphi$. Given the Pullout axiom$^{\leftrightarrow}$, we have $\vdash_{\mathsf{L_{lex}}} \Box_0(\top \wedge \forall\psi) \leftrightarrow (\Box_0\top \wedge \forall\psi)$ and thus $\vdash_{\mathsf{L_{lex}}} \Box_0(\top \wedge \forall\psi) \to \varphi$. By the Monotonicity Rule for $\Box_0$, we get $\vdash_{\mathsf{L_{lex}}} \Box_0\Box_0(\top \wedge \forall\psi) \to \Box_0\varphi$. By $\mathsf{4}_{\Box_0}$, we have $\vdash_{\mathsf{L_{lex}}} \Box_0(\top \wedge \forall\psi) \to \Box_0\Box_0(\top \wedge \forall\psi)$ and thus $\vdash_{\mathsf{L_{lex}}} \Box_0(\top \wedge \forall\psi) \to \Box_0\varphi$. By the Pullout axiom$^{\leftrightarrow}$, we have $\vdash_{\mathsf{L_{lex}}} (\Box_0\top \wedge \forall\psi) \leftrightarrow \Box_0(\top \wedge \forall\psi)$. Hence $\vdash_{\mathsf{L_{lex}}} (\Box_0\top \wedge \forall\psi) \to \Box_0\varphi$. Therefore $((\Box_0\top \wedge \forall\psi) \to \Box_0\varphi) \in T$. As $\Box_0\top$ is an axiom of our system, we have $(\Box_0\top) \in T$ and $(\forall\psi) \in T$. Hence by closure under modus ponens, we have $\Box_0\varphi \in T$. That is, $T \in \|\Box_0\varphi\|$. $\Box$

Lemma 7 (Truth Lemma)**.** *For every formula $\varphi \in \mathscr{L}$, we have: $[\![\varphi]\!]_{M^c} = \|\varphi\|$.*

Proof. The proof is by induction on the complexity of φ. The base case follows from the definition of V^c. For the inductive case, suppose that for all $T \in W^c$ and all formulas ψ of lower complexity than φ, we have $[\![\psi]\!]_{M^c} = \|\psi\|$. The Boolean cases where $\varphi = \neg\psi$ and $\varphi = \psi_1 \wedge \psi_2$ follow from the induction hypothesis together with the standard facts about maximal consistent theories included in Proposition 12. Only the modalities remain. Let $\varphi = \exists\psi$ and consider any $T \in M^c$. We have $T \in \|\exists\psi\|$ iff (Proposition 4) $\|\varphi\| \neq \emptyset$ iff (induction hypothesis) $[\![\psi]\!]_{M^c}$ iff $[\![\exists\psi]\!]_{M^c} = W^c$ iff $T \in [\![\exists\psi]\!]_{M^c}$. Now let $\varphi = \Box_0\psi$ and consider any $T \in M^c$. We have $T \in \|\Box_0\psi\|$ iff (Proposition 6) there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq \|\psi\|$ iff (induction hypothesis) there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq [\![\psi]\!]_{M^c}$ iff $T \in [\![\Box_0\psi]\!]_{M^c}$. Let $\varphi = \Diamond\psi$. We have $T \in \|\Diamond\psi\|$ iff (Proposition 5) there is an $S \in \|\psi\|$ such that $R^{\Box}TS$ iff (induction hypothesis) there is an $S \in [\![\psi]\!]_{M^c}$ such that $R^{\Box}TS$ iff there is an $S \in [\![\psi]\!]_{M^c}$ such that $(T,S) \in Ag^c(\langle \mathscr{R}^c, \preceq^c\rangle)$ iff $T \in [\![\Diamond\psi]\!]_{M^c}$. ☐

Lemma 8. Λ_{lex} *is strongly complete with respect to the class of pre-models (and hence it is also complete with respect to **REL** models).*

Proof. By Proposition 6, it suffices to show that every Λ_{lex}-consistent set of formulas is satisfiable on some lex model. Let Γ be an Λ_{lex}-consistent set of formulas. By Lindenbaum's Lemma, there is a maximally consistent set T_0 such that $\Gamma \subseteq T_0$. Choose any canonical pre-model M^c for T_0. By Lemma 16, $M^c, T_0 \models \varphi$ for all $\varphi \in T_0$. ☐

3.4.2 Step 2: Unravelling the canonical pre-model

Next, we will unravel the canonical pre-model. We first fix some preliminary notions.

We first define a set of "evidential indices"

$$I := \{\Box_0\varphi \mid (\exists\Box_0\varphi) \in T_0\} \cup \{\Box\} \cup \{(\varphi, j) \mid \varphi \in \mathscr{L}, j \in \{l, r\}\},$$

where l is a symbol for "left" copy and r is a symbol for "right" copy. We use ϵ, ϵ' as meta-variables ranging over evidential indices in I. To each $\epsilon \in I$, we associate a corresponding relation R^ϵ on the canonical model W^c, as follows: $R^{\Box_0\varphi}$ and $R^{\Box}$ are as before (the relations in the canonical pre-model), and $R^{(\varphi,l)} = R^{(\varphi,r)} := \{(T,S) \mid S = F(T,\varphi)\}$.

Definition 52 (Histories)**.** Let $M^c = \langle W^c, \langle \mathscr{R}^c, \preceq^c\rangle, V^c, Ag^c\rangle$ be a canonical pre-model for T_0. The set of histories rooted at T_0 is the following set of finite sequences:

$$\tilde{W} := \{(T_0, \epsilon_1, T_1, \epsilon_2, \ldots, \epsilon_n, T_n) \mid n \geq 0, \epsilon_i \in I \text{ and } T_{i-1}R^{\epsilon_i}T_i \text{ for all } i \leq n\}$$

The set $\tilde{W}$ forms the set of worlds of the unravelled tree. ◁

Basically, histories record all finite sequences of worlds in M^c starting with T_0 and passing to R^ϵ-successors at each step, where $\epsilon \in I$.

Definition 53 (β)**.** We denote by $\beta : \tilde{W} \to W^c$ the map returning the last theory in each history, i.e.

$$\beta(T_0, \epsilon_1, T_1, \epsilon_2, \ldots, \epsilon_n, T_n) := T_n$$

for all histories in $\tilde{W}$. ◁

We now define the relations that will feature in the unravelling of M^c around T_0.

Definition 54 ($\to^\epsilon$ relations)**.** For a history $w = (T_0, \epsilon_1, T_1, \epsilon_2, \ldots, \epsilon_n, T_n) \in \tilde{W}$, we denote by

$$(w, \epsilon, T) := (T_0, \epsilon_1, T_1, \epsilon_2, \ldots, \epsilon_n, T_n, \epsilon, T)$$

the history obtained by extending the history w with the sequence (ϵ, T) (where $T \in W^c$). Using this notation, we define the following relations $\to^\epsilon$ over $\tilde{W}$, labelled by indices in I:

$$w \to^\epsilon w' \text{ iff } w' = (w, \epsilon, T) \text{ for some } T \in W^c$$

◁

We now define the unravelled tree for T_0.

Definition 55 (Unravelled tree)**.** Let $M^c = \langle W^c, \langle \mathscr{R}^c, \preceq^c \rangle, V^c, Ag^c \rangle$ be a canonical pre-model for T_0. The unravelling of M^c around T_0 is the structure $\tilde{K} = \langle \tilde{W}, \{\rightarrow^\epsilon \mid \epsilon \in I\}, \tilde{V} \rangle$ with

$$\tilde{V}(p) := \{w \in \tilde{W} \mid \beta(w) \in V^c(p)\}$$

◁

In the tree unravelling, one history has another history accessible if the second is one step longer than the first. The valuation on histories is copied from that on their last nodes. We now define paths on this tree of histories.

Definition 56 ($\mathcal{R}$-path)**.** Let $w, w' \in \tilde{W}$ and let $\mathcal{R} \subseteq \{\rightarrow^\epsilon \mid \epsilon \in I\}$. An $\mathcal{R}$-path from w to w' is a finite sequence

$$p = (w_0, \epsilon_1, w_1, \epsilon_2, \ldots, \epsilon_{n-1}, w_n)$$

where $w_0 = w$, $w_n = w'$, $w_k \in \tilde{W}$ for $k = 1, 2, \ldots, n$, $\epsilon_k \in I$ for $k = 1, 2, \ldots, n-1$ and $w_k \rightarrow^{\epsilon_k} w_{k+1}$ for $k = 1, 2, \ldots, n-1$. For an $\mathcal{R}$-path $p = (w_0, \epsilon_1, w_1, \epsilon_2, \ldots, \epsilon_{n-1}, w_n)$ from w to w', we denote by

$$(p, \epsilon, w'') := (w_0, \epsilon_1, w_1, \epsilon_2, \ldots, \epsilon_n, w_n, \epsilon, w'')$$

the path obtained by extending the path p with (ϵ, w''). If $\mathcal{R}$ is not specified, we speak of a *path*. For any path $p = (w_0, \epsilon_1, w_1, \epsilon_2, \ldots, \epsilon_{n-1}, w_n)$ we define $first(p) = w_0$ and $last(p) = w_n$. ◁

The following is a standard results about (unravelled) trees, which we will refer to later on.

Lemma 9 (Uniqueness of paths)**.** *Let $\tilde{K}$ be the unravelling of M^c around T_0. Let $w, w' \in \tilde{W}$ and $\mathcal{R} \subseteq \{\rightarrow^\epsilon \mid \epsilon \in I\}$. Then, there is at most one $\mathcal{R}$-path p from w to w'.*

3.4.3 Step 3: Completeness with respect to lex-models

Step 2 unravelled the canonical pre-model from step 1. Using the structure from the unravelled tree, we now define a **REL** model $\tilde{M}$ from it. We then show that this model is in fact a lex-model. Finally, we define a variant of a bounded morphism defined for **REL** models, which we call *bounded aggregation-morphism.* Bounded aggregation-morphisms work on **REL** models in the same way as standard bounded-morphisms do on Kripke models: for **REL** models, modal satisfaction is invariant under bounded aggregation-morphisms. We then show that M^c is a bounded-morphic image of $\tilde{M}$, which gives us completeness.

We first define the model $\tilde{M}$.

Definition 57 ($\tilde{M}$)**.** Let $\tilde{K}$ be the unravelling of M^c around T_0. The structure $\tilde{M} = \langle \tilde{W}, \langle \tilde{\mathscr{R}}, \tilde{\preceq} \rangle, \tilde{V}, \tilde{Ag} \rangle$ has:

$$\tilde{\mathscr{R}} := \{\tilde{R}^{\Box_0 \varphi} \mid (\exists \Box_0 \varphi) \in T_0\} \cup \{R'_l, R'_r\} \cup \{\tilde{W} \times \tilde{W}\}$$

where:

$$\tilde{R}^{\Box_0 \varphi} := (\rightarrow^{\Box_0 \varphi})^*$$

$$R'_l = (\rightarrow^{\Box} \cup \bigcup \{\rightarrow^{(\varphi, l)} \mid \varphi \in \mathscr{L}\})^*$$

$$R'_r = (\to^{\Box} \cup \bigcup\{\to^{(\varphi,r)} \mid \varphi \in \mathscr{L}\})^*$$

Moreover, the priority order $\tilde{\preceq}$ is the preorder on $\tilde{\mathscr{R}}$ with:

$$R\tilde{\prec}R'_l, R'_r \text{ for all } R \in \tilde{\mathscr{R}} \setminus \{R'_l, R'_r, \tilde{W} \times \tilde{W}\}$$

and

$$R'_l, R'_r\tilde{\prec}\tilde{W} \times \tilde{W}$$

Finally, the aggregator $\tilde{Ag}$ is given by:

$$\tilde{Ag}(\langle \mathscr{R}, \preceq \rangle) = \begin{cases} \tilde{R}^{\Box} := (\to^{\Box})^* & \text{if } \langle \mathscr{R}, \preceq \rangle = \langle \tilde{\mathscr{R}}, \tilde{\preceq} \rangle \\ \mathsf{lex}(\langle \mathscr{R}, \preceq \rangle) & \text{otherwise} \end{cases}$$

◁

Proposition 14. *All the evidence relations in $\tilde{\mathscr{R}} \setminus \{\tilde{W} \times \tilde{W}\}$ are reflexive, transitive and anti-symmetric.*

Proof. Reflexivity and transitivity follow immediately from the fact that each $R \in \tilde{\mathscr{R}} \setminus \{\tilde{W} \times \tilde{W}\}$ is the reflexive transitive closure of some other relation, and $\tilde{W} \times \tilde{W}$ is reflexive and transitive. Hence we just need to show the anti-symmetry of the relations. Let $R \in \tilde{\mathscr{R}} \setminus \{\tilde{W} \times \tilde{W}\}$ and suppose Rwv and Rvw. First, we consider the case $R = \tilde{R}^{\Box_0\varphi}$ for some φ such that $\exists\Box_0\varphi \in T_0$. I.e. $R = (\to^{\Box_0\varphi})^*$. Given Rwv there is some $n \geq 0$ such that:

$$w = w_0 \to^{\Box_0\varphi} w_1 \to^{\Box_0\varphi} \cdots \to^{\Box_0\varphi} w_n = v$$

Similarly, given Rvw there is some $m \geq 0$ such that:

$$v = w'_0 \to^{\Box_0\varphi} w'_1 \to^{\Box_0\varphi} \cdots \to^{\Box_0\varphi} w'_m = w$$

By definition of $\to^{\Box_0\varphi}$, we have $w_1 = (w, \Box_0\varphi, T_1)$ for some $T_1 \in W^c$, $w_2 = (w, \Box_0\varphi, T_1, \Box_0\varphi, T_2)$ for some $T_2 \in W^c$, and proceeding in this way we get

$$w_n = v = (w, \Box_0\varphi, T_1, \Box_0\varphi, T_2, \ldots, \Box_0\varphi, T_n) \text{ where } T_i \in W^c, \text{for } i \leq n \tag{3.1}$$

Similarly, we have $w'_1 = (v, \Box_0\varphi, T'_1)$ for some $T'_1 \in W^c$, $w'_2 = (v, \Box_0\varphi, T'_1, \Box_0\varphi, T'_2)$ for some $T'_2 \in W^c$, and proceeding in this way we get

$$w'_m = w = (v, \Box_0\varphi, T'_1, \Box_0\varphi, T'_2, \ldots, \Box_0\varphi, T'_n) \text{ where } T'_i \in W^c, \text{for } i \leq m \tag{3.2}$$

Hence we must have $n = m = 0$. For otherwise, substituting v in 3.2 with the expression in 3.1 we get

$$w = (w, \Box_0\varphi, T_1, \Box_0\varphi, T_2, \ldots, \Box_0\varphi, T_n, v, \Box_0\varphi, T'_1, \Box_0\varphi, T'_2, \ldots, \Box_0\varphi, T'_m) \text{ for } n > 0 \text{ or } m > 0$$

which is impossible. Therefore $w = w_0 = w_n = v$, as required.

Now we consider the case $R = R'_l$, i.e. $R = (\to^{\Box} \cup \bigcup\{\to^{(\varphi,l)} \mid \varphi \in \mathscr{L}\})^*$. Given Rwv there is some $n \geq 0$ such that:

$$w = w_0 \to^{i_0} w_1 \to^{i_2} \cdots \to^{i_{n-1}} w_n = v \text{ where } i_k \in \{\Box\} \cup \{(\varphi, l) \mid \varphi \in \mathscr{L}\}, \text{ for } k = 1, \ldots, n-1$$

Similarly, given Rvw there is some $m \geq 0$ such that:

$$v = w'_0 \to^{j_0} w'_1 \to^{j_2} \cdots \to^{j_{m-1}} w'_n = v \text{ where } j_k \in \{\Box\} \cup \{(\varphi, l) \mid \varphi \in \mathscr{L}\}, \text{ for } k = 1, \ldots, m-1$$

Reasoning as we did in the case of $R = \tilde{R}^{\Box_0\varphi}$, we conclude that $m = n = 0$ and hence $w = v$. The case of $R = R'_r$ is analogous to the one just discussed, and we are done.

$\Box$

Proposition 15. *In $\tilde{M}$ we have:*

$$R'_l \cap R'_d = \tilde{R}^{\Box}$$

Proof. $\Box$

($\subseteq$) Let $(w, v) \in R'_l \cap R'_d$. Then we have $(w, v) \in R'_l$, i.e.,

$$(w, v) \in (\to^{\Box} \cup \bigcup\{\to^{(\varphi,l)} | \ \varphi \in \mathscr{L}\})^*$$

Hence, there is some $n \geq 0$ such that:

$w = w_0 \to^{i_0} w_1 \to^{i_2} \cdots \to^{i_{n-1}} w_n = v$ *where* $i_k \in \{\Box\} \cup \{(\varphi, l) \mid \varphi \in \mathscr{L}\},$ *for* $k = 1, \ldots, n{-}1$

Similarly, we have $(w, v) \in R'_r$, i.e.,

$$(w, v) \in (\to^{\Box} \cup \bigcup\{\to^{(\varphi,r)} | \ \varphi \in \mathscr{L}\})^*$$

Hence, there is some $m \geq 0$ such that:

$w = w'_0 \to^{j_0} w'_1 \to^{j_2} \cdots \to^{j_{m-1}} w'_m = v$ *where* $j_k \in \{\Box\} \cup \{(\varphi, r) \mid \varphi \in \mathscr{L}\},$ *for* $k = 1, \ldots, m{-}1$

By definition of $\to^{i_k}$, we have $w_1 = (w, i_0, T_1)$ for some $T_1 \in W^c$, $w_2 = (w, i_0, T_1, i_1, T_2)$ for some $T_2 \in W^c$, and proceeding in this way we get

$$w_n = v = (w, i_0, T_1, i_1, T_2, \ldots, i_{n-1}, T_n) \tag{3.3}$$

where $T_i \in W^c$ and $i_k \in \{\Box\} \cup \{(\varphi, l) \mid \varphi \in \mathscr{L}\}$, for $k = 1, \ldots, n-1$. Reasoning in a similar way, we get

$$w'_m = v = (w, j_0, T'_1, j_1, T'_2, \ldots, j_{m-1}, T'_m) \tag{3.4}$$

where $T'_i \in W^c$ and $j_k \in \{\Box\} \cup \{(\varphi, r) \mid \varphi \in \mathscr{L}\}$, for $k = 1, \ldots, m-1$.

Given the expressions 3.3 and 3.4, we have $w_n = v = w'_m$. Hence $n = m$ and for all $k < n$, $i_k = j_k$. Hence we must have $i_k = \Box = j_k$ for all $k < n$. This means that the path

$$w = w_0 \to^{i_0} w_1 \to^{i_2} \cdots \to^{i_{n-1}} w_n = v$$

can be rewritten as

$$w = w_0 \to^{\Box} w_1 \to^{\Box} \cdots \to^{\Box} w_n = v$$

which is an $\{\Box\}$-path from w to v. Hence $(w, v) \in \tilde{R}^{\Box} = (\to^{\Box})^*$.

($\supseteq$) Let $(w, v) \in \tilde{R}^{\Box} = (\to^{\Box})^*$. Then there is some $n \geq 0$ such that:

$$w = w_0 \to^{\Box} w_1 \to^{\Box} \cdots \to^{\Box} w_n = v$$

The $\{\Box\}$-path described above is also an $\{\Box\} \cup \bigcup\{(\varphi, l) \mid \varphi \in \mathscr{L}\}$-path and an $\{\Box\} \cup \bigcup\{(\varphi, r) \mid \varphi \in \mathscr{L}\}$-path. Hence we have $(w, v) \in R'_l$ and $(w, v) \in R'_r$. Thus $(w, v) \in R'_l \cap R'_r$.

Proposition 16. *$\tilde{M}$ is a* lex *model.*

Proof. To establish that $\tilde{M}$ is a lex model, we need to show that it meets the condition of a **REL** model and that $\tilde{Ag} = \mathsf{lex}$. That is, we have to show:

1. $\tilde{\mathscr{R}}$ is a family of evidence, i.e., every $R \in \tilde{\mathscr{R}}$ is a preorder.
2. $\tilde{W} \times \tilde{W} \in \tilde{\mathscr{R}}$, i.e., the trivial evidence order is a piece of available evidence.
3. $\tilde{R}^{\Box} = \mathsf{lex}(\langle \tilde{\mathscr{R}}, \tilde{\preceq} \rangle)$ (which given the definition of $\tilde{Ag}$, gives $\tilde{Ag} = \mathsf{lex}$ as required)

Item 1 follows from 14, and Item 2 follows from the definition of $\tilde{M}$. Hence Item 3 remains to be shown. Note first that by 15, we have $R'_l \cap R'_d = \tilde{R}^{\Box}$.

($\subseteq$) Suppose that $(w, v) \in \mathsf{lex}(\langle \tilde{\mathscr{R}}, \tilde{\preceq})$. Note that lex is given here by

$$(w, v) \in \mathsf{lex}(\langle \tilde{\mathscr{R}}, \tilde{\preceq} \rangle) \text{ iff } \forall R' \in \mathscr{R}\ (R'wv\ \vee\ \exists R \in \tilde{\mathscr{R}}(R' \tilde{\prec} R \wedge R^{<}wv)) \tag{3.5}$$

Suppose for reductio that $(w, v) \notin R'_l \cap R'_d$. Then $(w, v) \notin R'_l$ or $(w, v) \notin R'_r$. Without loss of generality, suppose $(w, v) \notin R'_l$. Given 3.5, we have in particular:

$$(R'_l wv\ \vee\ \exists R \in \tilde{\mathscr{R}}(R'_l \tilde{\prec} R \wedge R^{<}wv)) \tag{3.6}$$

Note that the definition of $\tilde{\preceq}$ is such that R'_l has no relation strictly above it other than $\tilde{W} \times \tilde{W}$. And $\tilde{W} \times \tilde{W}$ is symmetric and thus it is not the case that $(\tilde{W} \times \tilde{W})^{<}wv$. Hence the right disjunct in 3.6 is false. Therefore we must have $R'_l wv$, contradicting our assumption to the contrary.

Suppose that $(w, v) \in R'_l \cap R'_d$. Then $R'_l wv$ and $R'_r wv$. Suppose first that $w = v$. As $\mathsf{lex}(\langle \tilde{\mathscr{R}}, \tilde{\preceq} \rangle)$ is a preorder, we have $(w, v) \in \mathsf{lex}(\langle \tilde{\mathscr{R}}, \tilde{\preceq} \rangle)$ and we are done. Suppose now that $w \neq v$. By Proposition 14, R'_l and R'_r are antisymmetric. Thus from $w \neq v$, $R'_l wv$ and $R'_r wv$, we get $(R'_l)^{<}wv$ and $(R'_r)^{<}wv$. Hence, as we have

$$R \tilde{\prec} R'_l, R'_r \text{ for all } R \in \tilde{\mathscr{R}} \setminus \{R'_l, R'_r\}$$

from the definition of lex we get $(w, v) \in \mathsf{lex}(\langle \tilde{\mathscr{R}}, \tilde{\preceq})$ as required. □

We now introduce the notion of a bounded aggregation-morphism. This is, as we will show, a truth-preserving map between **REL** models, which works similarly to standard bounded morphisms for Kripke models.

Definition 58 (Bounded aggregation-morphism)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ and $M' = \langle W', \langle \mathscr{R}', \preceq' \rangle, V', Ag' \rangle$ be two **REL** models. A mapping $f : W \to W'$ is a bounded aggregation-morphism if the following hold:

1. Valuation condition: for all $w \in W$, $w \in V(p)$ iff $f(w) \in V'(p)$
2. Forth conditions:
   (a) for all $R \in \mathscr{R}$, for all $w \in W$, there exists some $R' \in \mathscr{R}'$ such that $R'[f(w)] \subseteq \{f(v) \mid Rwv\}$
   (b) for all $w, v \in W$, if $Ag(\langle \mathscr{R}, \preceq \rangle)wv$ then $Ag'(\langle \mathscr{R}', \preceq' \rangle)f(w)f(v)$
3. Back conditions:
   (a) for all $R' \in \mathscr{R}'$ and all $w \in W$ there exists some $R \in \mathscr{R}$ such that $\{f(v) \mid Rwv\} \subseteq R'[f(w)]$.

(b) for all $w \in W$, $v' \in W'$, if $Ag'(\langle \mathscr{R}', \preceq' \rangle)f(w)v'$ then there exists some world $v \in W$ such that $Ag(\langle \mathscr{R}, \preceq \rangle)wv$ and $f(v) = v'$.

$\triangleleft$

Proposition 17. *Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ and $M' = \langle W', \langle \mathscr{R}', \preceq' \rangle, V', Ag' \rangle$ be two **REL** models. Let $f : W \to W'$ be a surjective bounded aggregation-morphism. Then for all $w \in W$ and $\varphi \in \mathscr{L}$: $M, w \models \varphi$ iff $M', f(w) \models \varphi$. That is: modal satisfaction is invariant under surjective bounded aggregation-morphisms.*

Proof. By induction on the structure of φ. The base case holds by the valuation condition. The boolean cases are shown by unfolding the definitions, so we consider the cases involving modalities.

Suppose $M, w \models \Diamond_0 \psi$. Then for all $R \in \mathscr{R}$ there is some $v \in W$ such that Rwv and $M, v \models \psi$. Now we want to show: $M, f(w) \models \Diamond_0 \psi$. That is, for all $R' \in \mathscr{R}'$ there is some $v' \in W'$ such that $R'f(w)v'$ and $M', v' \models \psi$. Let $R' \in \mathscr{R}'$ be arbitrary. By the back condition 3(a), there is some $R \in \mathscr{R}$ such that $\{f(v) \mid Rwv\} \subseteq R'[f(w)]$. Hence given Rwv, we have $f(v) \in R'[f(w)]$. That is, $R'f(w)f(v)$. By induction hypothesis, given $M, v \models \psi$ we have $M', f(v) \models \psi$. As R' was arbitrarily picked, this holds for all relations in $\mathscr{R}'$. Hence $M', f(w) \models \Diamond_0 \psi$.

Suppose now that $M', f(w) \models \Diamond_0 \psi$. Then for all $R' \in \mathscr{R}'$ there is some $v' \in W'$ such that $R'f(w)v'$ and $M', v' \models \psi$. Now we want to show: $M, w \models \Diamond_0 \psi$. That is, for all $R \in \mathscr{R}$ there is some $v \in W$ such that Rwv and $M, v \models \psi$. Let $R \in \mathscr{R}$ be arbitrary. By the forth condition $2(a)$, there exists some $R' \in \mathscr{R}'$ such that $R'[f(w)] \subseteq \{f(v) \mid Rwv\}$. We have $R'f(w)v'$ and $M', v' \models \psi$ for some $v' \in W'$. As f is surjective, we have $v' = f(u)$ for some $u \in W$. Hence given $R'[f(w)] \subseteq \{f(v) \mid Rwv\}$ and $f(u) \in R'[f(w)]$, we get $f(u) \in \{f(v) \mid Rwv\}$. Hence Rwu. By induction hypothesis, given $M', f(u) \models \psi$ we get $M, u \models \psi$. As R was arbitrarily picked, this holds for all relations in $\mathscr{R}$. Hence we have $M, w \models \diamond_0 \psi$.

Now suppose $M, w \models \Diamond \psi$. Then there is some $v \in W$ such that $Ag(\rangle \mathscr{R}, \preceq \langle)wv$ and $M, v \models \psi$. By the forth condition $2(b)$, we have $Ag'(\rangle \mathscr{R}', \preceq' \langle)f(w)f(v)$. By induction hypothesis, $M', f(v) \models \psi$. Hence $M', f(w) \models \psi$.

Lastly, suppose $M, f(w) \models \Diamond \psi$. Then there is some $v' \in W'$ such that $Ag'(\rangle \mathscr{R}', \preceq' \langle)wv$ and $M', v' \models \psi$. Hence by the back condition $3(b)$, there is some world $v \in W$ such that $Ag(\rangle \mathscr{R}, \preceq \langle)wv$ and $f(v) = v'$. By induction hypothesis, we get $M, v \models \psi$. Hence $M, w \models \Diamond \psi$.

$\square$

Proposition 18. *The map $\beta : \tilde{W} \to W^c$ is a surjective bounded aggregation-morphism.*

Proof. We need to check that β satisfies the conditions of a surjective bounded aggregation-morphism.

1. Surjectivity: Let $T \in W^c$ be arbitrary. We need to show that there is some $h \in \tilde{W}$ such that $\beta(h) = T$. Recall that we showed in 13.2. that $W^c \times W^c = R^{\Box_0 \top} \in \mathscr{R}$. Hence $R^{\Box_0 \top} T_0 T$. Thus the history $h = (T_0, \Box_0 \top, T)$ is an element of $\tilde{W}$ with $\beta(h) = T$, as required.

2. Valuation condition. This follows from the definition of $\tilde{V}$, i.e.,

$$\tilde{V}(p) \coloneqq \{h \in \tilde{W} \mid \beta(h) \in V^c(p)\}$$

3. Forth conditions:

   (a) We need to show that for all $R \in \tilde{\mathscr{R}}$, for all $w \in \tilde{W}$, there exists some $R'' \in \mathscr{R}^c$ such that $R''[\beta(w)] \subseteq \{\beta(v) \mid Rwv\}$. Let $R \in \tilde{\mathscr{R}}$ and $w \in \tilde{W}$ be arbitrary. Suppose first that $R = \tilde{R}^{\Box_0\varphi}$ for some φ with $\exists\Box_0\varphi \in T_0$. Consider $R^{\Box_0\varphi} \in \mathscr{R}^c$. Take any $T \in R^{\Box_0\varphi}[\beta(w)]$, i.e., $R^{\Box_0\varphi}\beta(w)T$. We will show that $T \in \{\beta(v) \mid \tilde{R}^{\Box_0\varphi}wv\}$. Note that, given $R^{\Box_0\varphi}\beta(w)T$, the history $w' = (w, \Box_0\varphi, T)$ is in $\tilde{W}$. This means that $w \rightarrow^{\Box_0\varphi} w'$. Hence $(\rightarrow^{\Box_0\varphi})^*ww'$, i.e., $\tilde{R}^{\Box_0\varphi}ww'$. Given $\beta(w') = T$, we get $T \in \{\beta(v) \mid \tilde{R}^{\Box_0\varphi}wv\}$, as required.

   Suppose now that $R = R'_l = (\rightarrow^{\Box} \cup \bigcup\{\rightarrow^{(\varphi,l)} \mid \varphi \in \mathscr{L}\})^*$. Consider $R' = (R^{\Box} \cup \bigcup\{R^{(\varphi,l)}\})^* \in \mathscr{R}^c$. Take any $T \in R'[\beta(w)]$, i.e., $R'\beta(w)T$. We will show that $T \in \{\beta(v) \mid R'_l wv\}$. Given $R'\beta(w)T$, for some $n \geq 0$, there is a path:

   $$\beta(w) = S_0 R^{\epsilon_0} S_1 R^{\epsilon_2}, \ldots, R^{\epsilon_{n-1}} S_n = T$$

   where $S_i \in W^c$, $\epsilon_k \in \{\Box\} \cup \{(\varphi, l) \mid \varphi \in \mathscr{L}\}$, for $k < n$. Hence there are histories $w_1 = (w, \epsilon_0, S_1)$, $w_2 = (w, \epsilon_0, S_1, \epsilon_1, S_2)$, up to $w_n = (w, \epsilon_0, S_1, \epsilon_1, S_2, \ldots, \epsilon_{n-1}, T)$. Hence, by definition of $\rightarrow^{\epsilon_k}$, for each $k < n$, we have $w_k \rightarrow^{\epsilon_k} w_{k+1}$. Hence there is a path

   $$w = w_0 \rightarrow^{\epsilon_0} w_1 \rightarrow^{\epsilon_2}, \ldots, \rightarrow^{\epsilon_{n-1}} w_n$$

   And hence $R'_l ww_n$. Given $\beta(w_n) = T$, we get $T \in \{\beta(v) \mid R'_l wv\}$, as required.

   The case of $R = R'_r$ is analogous to the one above. Hence we have left the case $R = \tilde{W} \times \tilde{W}$. Consider $R^{\Box_0\top} = W^c \times W^c \in \mathscr{R}^c$. Take any $T \in R^{\Box_0\top}[\beta(w)]$, i.e., $R^{\Box_0\top}\beta(w)T$. This just means that $T \in W^c$. We will show that $T \in \{\beta(v) \mid (w, v) \in \tilde{W} \times \tilde{W}\} = \{\beta(v) \mid v \in \tilde{W}\}$. As β is surjective, we know that there is some $u \in \tilde{W}$ such that $\beta(u) = T$, and we are done.

   (b) We need to show that for all $w, v \in \tilde{W}$, if $\tilde{Ag}(\langle\tilde{\mathscr{R}}, \tilde{\preceq}\rangle)wv$ then $Ag^c(\langle\mathscr{R}^c, \preceq^c\rangle)\beta(w)\beta(v)$. Let $w, v \in \tilde{W}$ be arbitrary and suppose that $\tilde{Ag}(\langle\tilde{\mathscr{R}}, \tilde{\preceq}\rangle)wv$. By Proposition 16.3, given $\tilde{Ag}(\langle\tilde{\mathscr{R}}, \tilde{\preceq}\rangle)wv$ we have $\tilde{R}^{\Box}wv$, i.e., $(\rightarrow^{\Box})^*wv$. Hence, for some $n \geq 0$, there is a path:

   $$w = w_0 \rightarrow^{\Box} w_1 \rightarrow^{\Box} \cdots \rightarrow^{\Box} w_n = v$$

   Hence there are histories $w_1 = (w, \Box, S_1)$, $w_2 = (w, \Box, S_1, \Box, S_2)$, up to $w_n = v = (w, \Box, S_1, \Box, S_2, \ldots, \Box, S_n)$. Hence by definition of w_n we have

   $$\beta(w)R^{\Box}S_1R^{\Box}S_2, \ldots, R^{\Box}S_n$$

   And since $R^{\Box}$ is transitive, we get $R^{\Box}\beta(w)S_n$, i.e., $R^{\Box}\beta(w)\beta(v)$, as required.

4. Back conditions:

   (a) We need to show that for all $R'' \in \mathscr{R}^c$ and all $w \in \tilde{W}$ there exists some $R \in \tilde{\mathscr{R}}$ such that $\{\beta(v) \mid Rwv\} \subseteq R''[\beta(w)]$. Let $R'' \in \mathscr{R}^c$ and $w \in \tilde{W}$ be arbitrary. We reason by cases. First, suppose that $R'' = R^{\Box_0\varphi}$. Consider $\tilde{R}^{\Box_0\varphi} \in \tilde{\mathscr{R}}$. We will show that $\{\beta(v) \mid \tilde{R}^{\Box_0\varphi}wv\} \subseteq R^{\Box_0\varphi}[\beta(w)]$. Take any $\beta(u) \in \{\beta(v) \mid \tilde{R}^{\Box_0\varphi}wv\}$.

We have $\tilde{R}^{\Box_0\varphi}wu$, i.e., $(\to^{\Box_0\varphi})^*wu$. Hence for some $n \geq 0$, there is a path:

$$w = w_0 \to^{\Box_0\varphi} w_1 \to^{\Box_0\varphi}, \ldots, \to^{\Box_0\varphi} w_n = u$$

where $w_i \in \tilde{W}$, for $k \leq n$. Hence there are histories $w_1 = (w, \Box_0\varphi, S_1)$, $w_2 = (w, \Box_0\varphi, S_1, \Box_0\varphi, S_2)$, up to $w_n = u = (w, \Box_0\varphi, S_1, \Box_0\varphi, S_2, \ldots, \Box_0\varphi, S_n)$. Hence, by definition of w_n, we have

$$\beta(w)R^{\Box_0}S_1R^{\Box_0}, \ldots, R^{\Box_0}\beta(u)$$

And since $R^{\Box_0}$ is transitive, we get $R^{\Box_0}\beta(w)\beta(u)$, as required.

Suppose now that $R'' = R' \in W^c$. Consider $R'_l \in \tilde{\mathscr{R}}$. We will show that $\{\beta(v) \mid R'_l wv\} \subseteq R'[\beta(w)]$. Take any $\beta(u) \in \{\beta(v) \mid R'_l wv\}$. We have $R'_l wu$, i.e., $(\to^{\Box} \cup \bigcup\{\to^{(\varphi,l)} \mid \varphi \in \mathscr{L}\})^*wu$. Hence for some $n \geq 0$, there is a path:

$$w = w_0 \to^{\epsilon_0} w_1 \to^{\epsilon_1}, \ldots, \to^{\epsilon_{n-1}} w_n = u$$

where $w_i \in \tilde{W}$, $\epsilon_k \in \{\Box\} \cup \bigcup\{(\varphi, l) \mid \varphi \in \mathscr{L}\}$ for $k < n$. By definition of $\to^{\epsilon_k}$, $w_1 = (w, \epsilon_0, S_1)$ for some $S_1 \in W^c$, $w_2 = (w, \epsilon_0, S_1, \epsilon_1\varphi, S_2)$ for some $S_2 \in W^c$, up to $w_n = u = (w, \epsilon_0, S_1, \epsilon_1, S_2, \ldots, \epsilon_{n-1}, S_n)$ where $S_i \in W^c$ for $i \leq n$. Hence, by definition of w_n, we have a path

$$\beta(w)R^{\epsilon_0}S_1R^{\epsilon_1}S_2, \ldots, R^{\epsilon_{n-1}}\beta(u)$$

Since $R' = (R^{\Box} \cup \bigcup\{R^{(\varphi,l)}\})^*$, the path above is a path from $\beta(w)$ to $\beta(u)$ along R', i.e., we have $R'\beta(w)\beta(u)$, as required.

(b) We need to show that for all $w \in \tilde{W}$, $T \in W^c$, if $Ag^c(\langle \mathscr{R}^c, \preceq^c\rangle)\beta(w)T$ then there exists some history $v \in \tilde{W}$ such that $\tilde{Ag}(\langle\tilde{\mathscr{R}}, \tilde{\preceq}\rangle)wv$ and $\beta(v) = T$. Let $w \in \tilde{W}$ and $T \in W^c$ be arbitrary, and suppose that $Ag^c(\langle \mathscr{R}^c, \preceq^c\rangle)\beta(w)T$, i.e., $R^{\Box}\beta(w)T$. Then the history $w' = (w, \Box, T)$ is in $\tilde{W}$ and $\beta(w') = T$, as required.

$\Box$

Theorem 10. Λ_{lex} *is complete with respect to the class of* lex *models.*

Proof. Let Γ be a Λ_{lex}-consistent set of formulas. By Lindenbaum's Lemma, Γ can be extended to a maximal consistent set T_0. Choose any canonical pre-model M^c for T_0. By Lemma 14, $M^c, T_0 \models \varphi$ for all $\varphi \in T_0$. Let $\tilde{K}$ be the unraveling of M^c around T_0 and let $\tilde{M}$ be the lex model generated from $\tilde{K}$. Note that the history $(T_0) \in \tilde{W}$. Let $\beta : \tilde{W} \to W^c$ be the map defined above. By Proposition 18, β is a surjective bounded aggregation-morphism. By Proposition 17, we have $M^c, T_0 \models \psi$ iff $\tilde{M}, \beta(T_0) \models \psi$. Hence, in particular, $\tilde{M}, \beta(T_0) \models \varphi$ for all $\varphi \in T_0$. $\Box$

3.5 REL$^{+}_{\mathsf{lex}}$: prioritized evidence addition

This Section provides a first look at the dynamics of evidence addition over lex models. Here we will study an action of *prioritized addition.* For generality, we describe this action over **REL** models.

Prioritized addition. Let$M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, Ag\rangle$ be a **REL** model and $R \in Pre(W)$ a piece of relational evidence. The prioritized addition of R consists of adding R to the set of available evidence $\mathscr{R}$, giving the highest priority to the new evidence. To clarify this action, let us first fix the model transformation describing prioritized addition.

Definition 59 (Prioritized addition)**.** The model $M^{\mathsf{up}R} = \langle W^{\mathsf{up}R}, \langle \mathscr{R}^{\mathsf{up}R}, \preceq^{\mathsf{up}R}\rangle, V^{\mathsf{up}R}, Ag^{\mathsf{up}R}\rangle$ has $W^{\mathsf{up}R} := W$, $\mathscr{R}^{\mathsf{up}R} := \mathscr{R} \cup \{R\}$, $V^{\mathsf{up}R} := V$, $Ag^{\mathsf{up}R} := Ag$ and

$$\preceq^{\mathsf{up}R} := \preceq \cup \{(R', R) \mid R' \in \mathscr{R}\}$$

◁

Observation. The letters up are meant to help remind the reader that the relation R is placed 'up' and above every other evidence relation. ◁

Addition actions with ordered evidence. On lexicographic models, there are other possible choices of evidence addition besides prioritize addition. For instance, one could add R by simply adding it to the stock of evidence, without essentially changing the priority relation at all (i.e. changing the priority order only by adding the loop for R, to make it reflexive on this extended domain). In other words, the new evidence is not comparable with the old one. This form of addition resembles a bit more the way addition works on $\cap$-models models, and might be called *non-prioritized addition.* It is interesting to highlight the relationship between prioritized addition in lex-models and evidence upgrade in $\cap$-models. In particular, prioritized addition in lex-models coincides with the action of evidence upgrade $\Uparrow \pi$ introduced for $\cap$-models in the previous chapter.

3.5.1 Syntax and semantics of $\mathbf{REL}^+_{\mathsf{lex}}$

In this section we introduce the logic $\mathbf{REL}^+_{\mathsf{lex}}$ of prioritized addition. The language of $\mathbf{REL}^+_{\mathsf{lex}}$ is obtained from $\mathscr{L}$ by adding modal operators of the form $[+^{\mathsf{up}}\pi]$ that describe prioritized evidence-addition actions. If $[+^{\mathsf{up}}\pi]$ is such a modality, then new formulas of the form $[+^{\mathsf{up}}\pi]\varphi$ are used to express the statement that φ is true after the prioritized addition of the evidence order defined by π. Here, as we did with $\mathbf{REL}^+_{\cap}$, the programs π occurring inside the dynamic modalities are expressions from the program set Π_*.

Definition 60 (Language $\mathscr{L}^+_{\mathsf{lex}}$)**.** Let P be a countably infinite set of propositional variables. The language $\mathscr{L}^+_{\mathsf{lex}}$ is given by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid [+^{\mathsf{up}}\pi]\varphi$$

where $\pi \in \Pi_*$. ◁

The truth clause for the dynamic modalities is given by extending the satisfaction relation $\models$ for $\mathscr{L}_{\mathbf{REL}}$ as follows:

Definition 61 (Satisfaction for $[+^{\mathsf{up}}\pi]\varphi$)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, Ag\rangle$ be an **REL** model, $w \in W$ and $\pi \in \Pi_*$. The satisfaction relation $\models$ between pairs (M, w) and formulas $[+^{\mathsf{up}}]\varphi \in \mathscr{L}^+_{\mathsf{lex}}$ is defined as follows:

$$M, w \models [+^{\mathsf{up}}\pi]\varphi \text{ iff } M^{\mathsf{up}[\![\pi]\!]_M}, w \models \varphi$$

◁

We are now ready to introduce the proof system for $\mathbf{REL}^+_{\mathsf{lex}}$.

3.5.2 A proof system for $\mathbf{REL}^+_{\mathsf{lex}}$: $\mathsf{L}^+_{\mathsf{lex}}$

Before presenting the proof system $\mathbf{REL}^+_{\mathsf{lex}}$, we introduce some abbreviations that will be used in the definition of the reduction axioms. The reader may recall that these abbreviations were also used in the reduction axioms for evidence upgrade in $\cap$-models.

Notation 7. Let π be a normal form $\pi \coloneqq \bigcup_{s\in S_0(I)} \big(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)$. For each $J \subseteq I$, we define the abbreviations:

$$J(\boldsymbol{\varphi}) \coloneqq \bigwedge_{j\in J} \varphi_j \wedge \bigwedge_{j'\in I\setminus J} \neg\varphi_{j'}$$

$$J(\boldsymbol{\psi}) \coloneqq \bigwedge_{j\in J} \psi_j \wedge \bigwedge_{j'\in I\setminus J} \neg\psi_{j'}$$

Moreover, for a formula $[\mathsf{up}\pi]\chi$, we define the following abbreviations:

$$\pi^{<}(\chi) \coloneqq \bigvee_{J\subseteq I} \big(J(\boldsymbol{\psi}) \wedge \mathsf{suc}^{<}(\chi)\big)$$

$$\mathsf{suc}^{<}(\chi) \coloneqq \bigwedge_{s\in S_0(I)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall\big((\psi_{s_{\mathsf{len}(s)}} \wedge \bigwedge_{s'\in S_0(I)} (s'(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \forall(\psi_{s'_{\mathsf{len}(s')}} \to \bigwedge_{j\in J} \neg\varphi_j))) \to [+^{\mathsf{up}}\pi]\chi)\big)$$

$\triangleleft$

During our discussion of evidence upgrade in $\cap$-models, we showed in Lemma 2 from Chapter II.2 that $[\mathsf{up}\pi]\chi \wedge \pi^{<}(\chi)$ is true at a state w in a $\cap$-model M if $[\mathsf{up}\pi]\chi$ is true at w and $[\![\pi]\!]^{<}_{M}[w] \subseteq [\![[\mathsf{up}\pi]\chi]\!]_M$. That is, every state v that is strictly more plausible than w according to $[\![\pi]\!]_M$ satisfies $[\mathsf{up}\pi]\chi$. It is easy to see that, since the formula $\pi^{<}(\chi)$ contains no occurrences of $\Box$ (only $\forall$), the result transfers to lex models, as the semantics of $\forall$ is the same throughout **REL** models. Hence we have:

Lemma 10. *Let $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, \mathsf{lex}\rangle$, w a world in M, $\pi \in \Pi_*$ be a program with $\mathsf{nf}(\pi) \coloneqq \bigcup_{s\in S_0(I)}(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$. Then*

$$M, w \models [+^{\mathsf{up}}\pi]\chi \wedge \pi^{<}(\chi) \text{ iff } w \in [\![[+^{\mathsf{up}}\pi]\chi]\!]_M \textit{ and } [\![\pi]\!]^{<}_{M}[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M$$

Proof. Follows from Lemma 2. $\square$

We now introduce another lemma which will be useful in the proof of the reduction axioms.

Lemma 11. *Let $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, \mathsf{lex}\rangle$, w a world in M, $\pi \in \Pi_*$ be a program with $\mathsf{nf}(\pi) \coloneqq \bigcup_{s\in S_0(I)}(?s(\boldsymbol{\varphi},\boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$. Then*

$$(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M$$

iff

$$M, w \models [+\pi]\chi \wedge \bigwedge_{s\in S_0(I)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi))$$

.

Proof. ($\Rightarrow$) Suppose $(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M$. As $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M$ is reflexive, we have $M, w \models [+^{\mathsf{up}}\pi]\chi$. It remains to be shown that

$$M, w \models \bigwedge_{s\in S_0(I)} (s(\boldsymbol{\varphi},\boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi))$$

Take any $s \in S_0(I)$ and suppose that $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$. We need to show that $M, w \models \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi)$. Take any $v \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w]$ and suppose $M, v \models \psi_{s_{\mathsf{len}(s)}}$. If we show that $M, v \models [+^{\mathsf{up}}\pi]\chi$, we are done. Given $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$ and $M, v \models \psi_{s_{\mathsf{len}(s)}}$, by Proposition 8, we have $(w, v) \in [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$. Thus

$$(w, v) \in \bigcup_{s \in S_0(I_k)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$$

Hence as

$$\begin{aligned}
[\![\pi]\!]_M = [\![\mathsf{nf}(\pi)]\!]_M &= [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
&= [\![\bigcup_{s \in S_0(I)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \cup [\![?\top]\!]_M \\
&= \bigcup_{s \in S_0(I)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \cup [\![?\top]\!]_M
\end{aligned}$$

we have $(w, v) \in [\![\pi]\!]_M$. Hence we have $v \in (\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w]$. Thus, given $(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M$ we have $M, v \models [+^{\mathsf{up}}\pi]\chi$, as required.

($\Leftarrow$) Suppose that $M, w \models [+^{\mathsf{up}}\pi]\chi \wedge \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi))$. We will show that $(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M$. Take any v and suppose $(w, v) \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M$. We need to show that $v \in [\![[+^{\mathsf{up}}\pi]\chi]\!]_M$. If $v = w$, given $M, w \models [+^{\mathsf{up}}\pi]\chi$ we are done. So suppose $v \neq w$. Since $(w, v) \in (\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)$, we have $(w, v) \in [\![\pi]\!]_M$. Reasoning as we did in the proof of $\mathrm{EA4}_\cap$ (see Theorem 19), we get

$$(w, v) \in [\![\pi]\!]_M \text{ iff } \text{ for some } s' \in S_0(I), w \in [\![s'(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M \text{ and } v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M$$

Given that $M, w \models \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi))$, we have in particular

$$M, w \models s'(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s'_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi)$$

Thus from $w \in [\![s'(\boldsymbol{\varphi}, \boldsymbol{\psi})]\!]_M$ we get $M, w \models \Box(\psi_{s'_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi)$. And given that $(w, v) \in \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)$ and $v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M$, we get $M, v \models [+^{\mathsf{up}}\pi]\chi$, as required. $\Box$

We now show one more lemma, before presenting the proof system $\mathsf{L}^{+}_{\mathsf{lex}}$. With these lemmas in place, the proofs of the validity of the reduction axioms in this system will be almost immediate.

Lemma 12. *Let $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, \mathsf{lex}\rangle$, w a world in M, $\pi \in \Pi_*$ be a program and φ a formula. Then*

$$(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![\varphi]\!]_M \textit{ and } [\![\pi]\!]^{<}_M[w] \subseteq [\![\varphi]\!]_M$$

iff

$$\mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)[w] \subseteq [\![\varphi]\!]_M$$

Proof. ($\Rightarrow$) Suppose that

$$(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![\varphi]\!]_M \text{ and } [\![\pi]\!]^{<}_M[w] \subseteq [\![\varphi]\!]_M$$

Towards a contradiction, suppose that there is a $v \in \mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)[w]$ such that $v \notin [\![\varphi]\!]_M$. Then

$$v \notin (\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \textit{ and } v \notin [\![\pi]\!]^{<}_M[w]$$

Given $v \notin [\![\pi]\!]^{<}_M[w]$ we have $\neg[\![\pi]\!]_M wv$ or $[\![\pi]\!]_M vw$. But note that $v \in \mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)[w]$ and $[\![\pi]\!]_M$ has no other relation strictly above it in the priority order $\preceq^{\mathsf{up}\pi}$, and thus by definition of lex we must have $[\![\pi]\!]_M wv$, Hence we have $[\![\pi]\!]_M vw$. Moreover, given $v \notin (\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w]$, we must have $v \notin \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w]$ or $\neg[\![\pi]\!]_M wv$. We know that $\neg[\![\pi]\!]_M wv$ cannot be, so we have $v \notin \mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)[w]$. This means that

$$\exists R \in \mathscr{R}(\neg Rwv \text{ and } \forall R' \in \mathscr{R}(R \not\prec R' \text{ or } \neg R'^{<}wv)$$

Given $v \in \mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)[w]$ we have

$$\forall Q \in \mathscr{R}^{\mathsf{up}\pi}(Qwv \text{ or } \exists Q' \in \mathscr{R}^{\mathsf{up}\pi}(Q \prec^{\mathsf{up}\pi} Q' \text{ and } Q'^{<}wv))$$

Note that $R \in \mathscr{R}^{\mathsf{up}\pi}$. Hence in particular we have

$$Rwv \text{ or } \exists Q' \in \mathscr{R}^{\mathsf{up}\pi}(R \prec^{\mathsf{up}\pi} Q' \text{ and } Q'^{<}wv)$$

As $\neg Rwv$, we must have

$$\exists Q' \in \mathscr{R}^{\mathsf{up}\pi}(R \prec^{\mathsf{up}\pi} Q' \text{ and } Q'^{<}wv)$$

From the statements above, we know that $Q' \notin \mathscr{R}$, and hence $Q' = [\![\pi]\!]_M$. But then we have $[\![\pi]\!]^{<}_M wv$, contradicting our assumption to the contrary.

($\Leftarrow$) Suppose that

$$\mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)[w] \subseteq [\![\varphi]\!]_M$$

We then have

$$\big(\forall Q \in \mathscr{R}^{\mathsf{up}\pi}(Qwv \text{ or } \exists Q' \in \mathscr{R}^{\mathsf{up}\pi}(Q \prec^{\mathsf{up}\pi} Q' \text{ and } Q'^{<}wv))\big) \Rightarrow v \in [\![\varphi]\!]_M$$

Towards a contradiction, suppose that

$$(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \not\subseteq [\![\varphi]\!]_M \text{ or } [\![\pi]\!]^{<}_M[w] \not\subseteq [\![\varphi]\!]_M$$

Suppose first that

$$(\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w] \not\subseteq [\![\varphi]\!]_M$$

Then there is some $v \in (\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle) \cap [\![\pi]\!]_M)[w]$ such that $v \notin [\![\varphi]\!]_M$. Then $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)wv$ and $[\![\pi]\!]_M wv$ From $\mathsf{lex}(\langle \mathscr{R}, \preceq\rangle)wv$ we have

$$\forall R \in \mathscr{R}(Rwv \text{ or } \exists R' \in \mathscr{R}(R \prec^{\mathsf{up}\pi} R' \text{ and } R'^{<}wv))$$

Since $\mathscr{R}^{\mathsf{up}\pi} = \mathscr{R} \cup \{[\![\pi]\!]_M\}$, together with $[\![\pi]\!]_M wv$, this gives us

$$\forall Q \in \mathscr{R}^{\mathsf{up}\pi}(Qwv \text{ or } \exists Q' \in \mathscr{R}^{\mathsf{up}\pi}(Q \prec^{\mathsf{up}\pi} Q' \text{ and } Q'^{<}wv))$$

Hence $\mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)wv$ and thus $v \in [\![\varphi]\!]_M$ (contradiction). Suppose now that

$$[\![\pi]\!]^{<}_M[w] \not\subseteq [\![\varphi]\!]_M$$

Then there is some v such that $[\![\pi]\!]^{<}_M wv$ and $v \notin [\![\varphi]\!]_M$. But, as $[\![\pi]\!]$ is the top element of $\preceq^{\mathsf{up}\pi}$, given $\mathsf{lex}(\langle \mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi}\rangle)[w] \subseteq [\![\varphi]\!]_M$ and $[\![\pi]\!]^{<}_M wv$ we have $v \in [\![\varphi]\!]_M$ (contradiction).

□

Having established these lemmas, we now present the proof system $\mathsf{L}^+_{\mathsf{lex}}$. In the next section, the logic generated by this proof system will be shown to be sound and complete with respect to lex models.

Definition 62 ($\mathsf{L}^+_{\mathsf{lex}}$)**.** Let $\chi, \chi' \in \mathscr{L}^+_{\mathsf{lex}}$ and let $\pi \in \Pi_*$ be an evidence program with normal form

$$\mathsf{nf}(\pi) := \bigcup_{s \in S_0(I)} (?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$$

The proof system $\mathsf{L}^+_{\mathsf{lex}}$ includes all *axioms schemas and inference rules* of L_0. Moreover, it includes the following *reduction axioms*:

upEA1 : $[+^{\mathsf{up}}\pi]p \leftrightarrow p$ for all $p \in \mathsf{P}$

upEA2 : $[+^{\mathsf{up}}\pi]\neg\chi \leftrightarrow \neg[+\vec{\pi}]\chi$

upEA3 : $[+^{\mathsf{up}}\pi]\chi \wedge \chi' \leftrightarrow [+^{\mathsf{up}}\pi]\chi \wedge [+\vec{\pi}]\chi'$

upEA4 : $[+^{\mathsf{up}}\pi]\Box_0\chi \leftrightarrow \Box_0[+^{\mathsf{up}}\pi]\varphi \vee \big([+\pi]\chi \wedge \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi))\big)$

upEA5 : $[+^{\mathsf{up}}\pi]\Box\chi \leftrightarrow [\mathsf{up}\pi]\chi \wedge \pi^{<}(\chi) \wedge \bigwedge_{s \in S_0(I)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi))$

upEA6 : $[+^{\mathsf{up}}\pi]\forall\chi \leftrightarrow \forall[+^{\mathsf{up}}\vec{\pi}]\chi$

◁

3.5.3 Soundness and completeness of $\mathsf{L}^+_{\mathsf{lex}}$

Let be the logic generated by $\mathsf{L}^+_{\mathsf{lex}}$. This section proves soundness and completeness of Λ^+_{lex} with respect to the class of lex-models. As in previous chapters, the proof works via a standard reductive analysis. The idea of the proof is to show the reduction axioms are valid. Their validity is sufficient for turning every formula of our dynamic language $\mathscr{L}^+_{\mathsf{lex}}$ into one of our static language $\mathscr{L}$.

Proposition 19. *The axioms* up *EA1-*up *EA6 are valid in all* lex*-models.*

Proof. Let $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, \mathsf{lex}\rangle$, w a world in M, $\pi \in \Pi_*$ be a program with $\mathsf{nf}(\pi) := \bigcup_{s \in S_0(I)}(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$.

1. The validity of upEA1 follows from the fact that the prioritized evidence addition transformer does not change the valuation function. The validity of the Boolean reduction axioms upEA2 and upEA3 can be proven by unfolding the definitions.

2. Axiom upEA4: Note that this same axiom was called EA4$_\cap$ in the system $\mathsf{L}^+_\cap$ for $\cap$-models (Chapter II.2, Definition 36). The effects on evidence possession (as expressed by $\Box_0$-formulas) of evidence addition in $\cap$-models are the same as the effects of prioritized evidence addition in lex models; in both cases, the piece of evidence $[\![\pi]\!]_M$ is added to the initial body of evidence $\mathscr{R}$. Thus, it is easy to see that the proof of the validity of EA4$_\cap$ in $\cap$-models can be straightforwardly adapted to show the validity of upEA4 in lex models.

3. Axiom upEA5:

$$\begin{aligned}
& M, w \models [+^{\mathsf{up}}\pi]\Box\chi \\
\text{iff } & M^{+\pi}, w \models \Box\chi \\
\text{iff } & \mathsf{lex}(\mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi})[w] \subseteq [\![\chi]\!]_{M^{+\mathsf{up}\pi}} \\
\text{iff } & \mathsf{lex}(\mathscr{R}^{\mathsf{up}\pi}, \preceq^{\mathsf{up}\pi})[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M \\
\text{iff } & (\mathsf{lex}(\langle \mathscr{R}, \preceq \rangle) \cap [\![\pi]\!]_M)[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M \text{ and } [\![\pi]\!]^{<}_M[w] \subseteq [\![[+^{\mathsf{up}}\pi]\chi]\!]_M \quad \text{(by Lemma 12)} \\
\text{iff } & M, w \models [+^{\mathsf{up}}\pi]\chi \wedge \pi^{<}(\chi) \wedge \bigwedge_{s \in S_0(I)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \Box(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\pi]\chi)) \\
& \text{(by Lemmas 10, 11)}
\end{aligned}$$

4. Axiom EA6$_\cap$:

$$M, w \models [+^{\mathsf{up}}\pi]\forall\chi \text{ iff } M^{+^{\mathsf{up}}\pi}, w \models \forall\chi \text{ iff } [\![\chi]\!]_{M^{+\mathsf{up}\pi}} = W^{+^{\mathsf{up}}\pi}$$
$$\text{iff } [\![[+^{\mathsf{up}}\pi]\chi]\!]_M = W \text{ iff } M, w \models \forall[+^{\mathsf{up}}\pi]\chi$$

□

Theorem 11. $\Lambda^{+}_{\mathsf{lex}}$ *is complete with respect to the class of* lex *models.*

Proof. Once we have established the validity of the reduction axioms, the proof is standard and follows the same steps used to prove completeness of $\Lambda_\cap$ (see Theorem 5). □

3.6 Chapter review

In this chapter, we have studied a logic for belief and evidence based over lex-models. First, we explored the static logic of lex-models, presenting a sound and complete proof system for this logic. In Section 3.5 we gave a first look at the dynamics of evidence addition over lex models. In a setting with *ordered evidence*, several forms of evidence addition are natural. We focused on one of those, which we called *prioritized addition*, and presented a matching dynamic logic.

Chapter 4

General Relational Evidence Logic

This chapter introduces *General Relational Evidence Logic*, **REL** for short. **REL** is a logic of belief based on aggregated evidence, in which the aggregated evidence is the output of an aggregator characterised by certain intuitive properties. This aggregator is not fixed. Instead, we are interested in reasoning about the beliefs that an agent would form, based on her evidence, *irrespective* of the aggregator used, as long as this aggregator satisfies the basic properties built into its definition. The chapter is organized as follows. Section 4.4.1 recalls the basic syntax and semantics of *static* **REL**. This static logic allows us to reason about the beliefs and evidence of an agent at a specific point in time. After discussing the static logic, we take a first look at dynamics over **REL** models with an abstract aggregator. In particular, we will focus on the action of *prioritized addition* introduced in the previous chapter. As a first step towards pre-encoding the effects of prioritized addition, we present a language with *conditional aggregated evidence* modalities. These modalities allow us to reason about the propositions that the agent's aggregated evidence would support, if the agent performed the prioritized addition of some piece of evidence. Section 4.2.2 presents a proof system for this conditional language, which is then shown to be sound and complete for the class of **REL** models in Sections 4.2.3 and 4.2.4. After that, we present a full dynamic language with dynamic modalities pre-encoding the effects of prioritized addition on basic and aggregated evidence, and belief. We also present a proof system for this dynamic language based on reduction axioms, which are then used to show that the dynamic logic of prioritized addition is sound and complete with respect to **REL** models.

4.1 Syntax and semantics

Here, we recall here the static language of **REL**, $\mathscr{L}$, which is built recursively as follows:

$$\varphi ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi$$

In this chapter we focus on general **REL** models,i.e., models of the form

$$M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$$

where Ag is some aggregator for W. We recall the semantics for formulas of $\mathscr{L}$, which are as follows.

Definition 63 (Satisfaction)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be a **REL** model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\varphi \in \mathscr{L}$ is defined as follows:

$M, w \models p$ iff $w \in V(p)$
$M, w \models \neg\varphi$ iff $M, w \not\models \varphi$
$M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \Box_0\varphi$ iff there is $R \in \mathscr{R}$ such that, for all $v \in W, Rwv$ implies $M, v \models \varphi$
$M, w \models \Box\varphi$ iff for all $v \in W, Ag(\langle \mathscr{R}, \preceq \rangle)wv$ implies $M, v \models \varphi$
$M, w \models \forall\varphi$ iff for all $v \in W, M, v \models \varphi$

$\lhd$

4.2 REL$^+$: dynamics of prioritized addition

This Section provides a first look at the dynamics of evidence addition over **REL** models with an abstract aggregator. As a starting point in the logical analysis of addition in **REL** models, here we focus on the action of *prioritized addition* introduced for lex-models in Section 3.3.5. As a reminder, let us fix here the model transformation describing prioritized addition.

Prioritized addition. Let$M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be a **REL** model and $R \in Pre(W)$ a piece of relational evidence. The prioritized addition of R consists of adding R to the set of available evidence giving the highest priority to the new evidence.

Definition 64 (Prioritized addition)**.** The model $M^{\mathsf{up}R} = \langle W^{\mathsf{up}R}, \langle \mathscr{R}^{\mathsf{up}R}, \preceq^{\mathsf{up}R} \rangle, V^{\mathsf{up}R}, Ag^{\mathsf{up}R} \rangle$ has $W^{\mathsf{up}R} := W$, $\mathscr{R}^{\mathsf{up}R} := \mathscr{R} \cup \{R\}$, $V^{\mathsf{up}R} := V$, $Ag^{\mathsf{up}R} := Ag$ and

$$\preceq^{\mathsf{up}R} := \preceq \cup \{(R', R) \mid R' \in \mathscr{R}\}$$

$\lhd$

In the remaining part of the chapter, we will consider an *iterated* version of prioritized addition, defined with a (possibly empty) sequence of evidence orders $\vec{R} = \langle R_1, \ldots, R_n \rangle$ as input.

Definition 65 (Iterated prioritized addition)**.** Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be a **REL** model and $\vec{R} = \langle R_1, \ldots, R_n \rangle$ be a sequence of evidence orders (i.e., $R_i \in Pre(W)$ for $i \in \{1, \ldots n\}$).The model $M^{\mathsf{up}R} = \langle W^{\mathsf{up}\vec{R}}, \langle \mathscr{R}^{\mathsf{up}\vec{R}}, \preceq^{\mathsf{up}\vec{R}} \rangle, V^{\mathsf{up}\vec{R}}, Ag^{\mathsf{up}\vec{R}} \rangle$ has $W^{\mathsf{up}\vec{R}} := W$, $\mathscr{R}^{\mathsf{up}\vec{R}} := \mathscr{R} \cup \{R_i \mid i \in \{1, \ldots n\}\}$, $V^{\mathsf{up}\vec{R}} := V$, $Ag^{\mathsf{up}\vec{R}} := Ag$ and

$$\begin{aligned}\preceq^{\mathsf{up}\vec{R}} := \preceq & \cup \{(R, R_1) \mid R \in \mathscr{R}\} \\ & \cup \{(R, R_2) \mid R \in \mathscr{R} \cup \{R_1\}\} \\ & \cup \ldots \\ & \cup \{(R, R_n) \mid R \in \mathscr{R} \cup \{R_j \mid j \in \{1, \ldots, n-1\}\}\}\end{aligned}$$

$\lhd$

That is, first R_1 is added as the highest priority evidence, then R_2 is added as the highest priority evidence, on top of every other evidence (including R_1), and so on, up to R_n. Naturally, when the sequence $\vec{R}$ has one element, we are back to the basic notion of prioritized addition.

4.2.1 Syntax and semantics of conditional REL

To pre-encode part of the dynamics of iterated prioritized addition, we will modify our basic language $\mathscr{L}$ with *conditional aggregated evidence modalities* of the form $\Box^{\vec{\pi}}$, where $\vec{\pi}$ is a finite, possibly empty sequence of evidence programs $\pi_1, \ldots, \pi_n$ (i.e., $\pi_i \in \Pi_*$, for $i \in \{1, \ldots n\}$). The intended interpretation of $\Box^{\vec{\pi}}\varphi$ is "the agent would have aggregated evidence for φ, if she performed the iterated prioritized addition of the evidence orders defined by $\vec{\pi}$". The language with conditional addition modalities, denoted $\mathscr{L}_{\mathbf{REL}}$, is as follows:

Definition 66 (Language $\mathscr{L}_{\mathbf{REL}}$)**.** Let P be a set of propositional variables. The language $\mathscr{L}_{\mathbf{REL}}$ is defined by mutual recursion:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box^{\vec{\pi}}\varphi \mid \forall\varphi$$
$$\pi ::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*$$

where $p \in \mathsf{P}$, A is the symbol for the universal program and $\vec{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$ is a (possibly empty) finite sequence of programs, with $\pi_i \in \Pi_*$, for $i \in \{1, \ldots n\}$. The dual of the modal operator for conditional aggregated evidence is defined in the usual way: $\Diamond^{\vec{\pi}} := \neg\Box^{\vec{\pi}}\neg$. $\triangleleft$

Through the conditional formulas $\Box^{\vec{\pi}}\varphi$, the language $\mathscr{L}_{\mathbf{REL}}$ pre-encodes the changes in *aggregated evidence* due to iterated prioritized addition. Our main goal, however, is to pre-encode the induced changes also at the level of belief and basic evidence. To do this, in subsequent sections we present a dynamic language extending $\mathscr{L}_{\mathbf{REL}}$ with dynamic modalities of the form $[+^{\mathsf{up}}\vec{\pi}]\varphi$ with the intended meaning: "φ is true after the iterated prioritized addition of the evidence sequence defined by $\vec{\pi}$". But first, we will axiomatize the logic for the language without dynamic modalities. Once this is in place, a sound and complete system for the logic with dynamic modalities will be obtained by adding reduction axioms to the system presented for the language without them.

Notation 8. We often abuse the notation for the truth map $[\![\cdot]\!]_M$ and write $[\![\vec{\pi}]\!]_M$ to denote $\langle [\![\pi_1]\!]_M, \ldots, [\![\pi_n]\!]_M \rangle$, where $\vec{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$. $\triangleleft$

The truth clause for the new conditional modalities is as follows:

Definition 67. Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be an **REL** model and $w \in W$. The satisfaction relation $\models$ between pairs (M, w) and formulas $\Box^{\vec{\pi}}\varphi \in \mathscr{L}_{\mathbf{REL}}$ is defined as follows:

$$M, w \models \Box^{\vec{\pi}}\varphi \text{ iff } Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M} \rangle)[w] \subseteq [\![\varphi]\!]_M$$

where $\mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}$ and $\preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}$ are the family of evidence and the priority order given by the iterated prioritized addition of $[\![\vec{\pi}]\!]_M$ (as indicated in Definition 65 above). $\triangleleft$

That is, $\Box^{\vec{\pi}}\varphi$ is true at a state w if the agent would have aggregated evidence for φ, assuming that the current ordered body of evidence is transformed by the iterated prioritized addition of $[\![\vec{\pi}]\!]_M$. Note that, as we allow $\vec{\pi}$ to be empty, $\Box^{\vec{\pi}}$ reduces the standard $\Box\varphi$ from $\mathscr{L}$ when $\vec{\pi}$ is the empty sequence.

4.2.2 The proof system $\mathsf{L}_{\mathbf{REL}}$

This section introduces the proof system for the language with conditional modalities $\Box^{\vec{\pi}}$. In the next section, the logic generated by this proof system will be shown to be sound and complete with respect to **REL** models. The completeness proof works via a canonical model construction.

Definition 68 ($\mathsf{L}_{\mathbf{REL}}$)**.** The proof system of $\mathsf{L}_{\mathbf{REL}}$ includes the following *axiom schemas* for all formulas $\varphi, \psi \in \mathscr{L}_{\mathbf{REL}}$ and program sequences $\vec{\pi} \in S_0(\Pi_*)$ (we remind the reader that $S_0(\Pi_*)$ is the set of all finite sequences of elements of Π_*):

1. All tautologies of propositional logic
2. The $\mathsf{S5}$ axioms for $\forall$:

$$\begin{aligned}
\mathsf{K}_\forall &: \forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)\\
\mathsf{T}_\forall &: \forall\varphi \to \varphi\\
\mathsf{4}\forall &: \forall\varphi \to \forall\forall\varphi\\
\mathsf{5}_\forall &: \exists\varphi \to \forall\exists\varphi
\end{aligned}$$

3. The $\mathsf{S4}$ axioms for $\Box^{\vec{\pi}}$:

$$\begin{aligned}
\mathsf{K}_\Box^{\vec{\pi}} &: \Box^{\vec{\pi}}(\varphi \to \psi) \to (\Box^{\vec{\pi}}\varphi \to \Box^{\vec{\pi}}\psi)\\
\mathsf{T}_\Box^{\vec{\pi}} &: \Box^{\vec{\pi}}\varphi \to \varphi\\
\mathsf{4}_\Box^{\vec{\pi}} &: \Box^{\vec{\pi}}\varphi \to \Box^{\vec{\pi}}\Box^{\vec{\pi}}\varphi
\end{aligned}$$

4. The T, $\mathsf{4}$ and N axioms for $\Box_0$:

$$\begin{aligned}
\mathsf{T}_{\Box_0} &: \Box_0\varphi \to \varphi\\
\mathsf{4}_{\Box_0} &: \Box_0\varphi \to \Box_0\Box_0\varphi\\
\mathsf{N}_{\Box_0} &: \Box_0\top
\end{aligned}$$

5. The following interaction axioms:
   (a) $\forall\varphi \to \Box_0\varphi$ (Universality for $\Box_0$)
   (b) $\forall\varphi \to \Box^{\vec{\pi}}\varphi$ (Universality for $\Box^{\vec{\pi}}$)
   (c) $(\Box_0\varphi \wedge \forall\psi) \leftrightarrow \Box_0(\varphi \wedge \forall\psi)$ (Pullout$^{\leftrightarrow}$)

The proof system of **REL** includes the following *inference rules* for all formulas $\varphi, \psi \in \mathscr{L}_{\mathbf{REL}}$ and program sequences $\vec{\pi} \in S_0(\Pi_*)$:

1. Modus ponens
2. Necessitation Rule for $\forall$: $\dfrac{\varphi}{\forall\varphi}$
3. Necessitation Rule for $\Box^{\vec{\pi}}$: $\dfrac{\varphi}{\Box^{\vec{\pi}}\varphi}$
4. Monotonicity Rule for $\Box_0$: $\dfrac{\varphi \to \psi}{\Box_0\varphi \to \Box_0\psi}$

$\triangleleft$

4.2.3 Soundness of $\mathsf{L}_{\mathbf{REL}}$

We denote by Λ_{REL} the logic generated by $\mathsf{L}_{\mathbf{REL}}$. In this section we prove the soundness of Λ_{REL} with respect to the class of **REL** models.

Theorem 12. Λ_{REL} *is sound with respect to the class of* ***REL*** *models.*

Proof. Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be an **REL** model and w a world in M.

1. S5 axioms for $\forall$:

   $\mathsf{K}_\forall$: $\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$. Let $M, w \models \forall(\varphi \to \psi)$ and suppose that $M, w \models \forall\varphi$. Take any $v \in W$. As $M, w \models \forall\varphi$, we have $M, v \models \varphi$. Thus given $M, w \models \forall(\varphi \to \psi)$, we have $M, v \models \psi$.

   $\mathsf{T}_\forall$: $\forall\varphi \to \varphi$. Let $M, w \models \forall\varphi$. Then every $v \in W$ is such that $M, v \models \varphi$. So in particular $M, w \models \varphi$.

   $4\forall$: $\forall\varphi \to \forall\forall\varphi$. Let $M, w \models \forall\varphi$. Then every $v \in W$ is such that $M, v \models \varphi$. Hence every $v \in W$ is such that $M, v \models \forall\varphi$ and thus $M, w \models \forall\forall\varphi$.

   $5_\forall$: $\exists\varphi \to \forall\exists\varphi$. Let $M, w \models \exists\varphi$. Then there is a $v \in W$ such that $M, v \models \varphi$. Take any $u \in W$. Then we have $M, u \models \exists\varphi$, and thus $M, w \models \forall\exists\varphi$.

2. S4 axioms for $\Box^{\vec{\pi}}$:

   $\mathsf{K}^{\vec{\pi}}_\Box$: $\Box^{\vec{\pi}}(\varphi \to \psi) \to (\Box^{\vec{\pi}}\varphi \to \Box^{\vec{\pi}}\psi)$. Let $M, w \models \Box^{\vec{\pi}}(\varphi \to \psi)$ and suppose $M, w \models \Box^{\vec{\pi}}\varphi$. Then $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w] \subseteq [\![\varphi \to \psi]\!]_M$ and $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w] \subseteq [\![\varphi]\!]_M$. Take any $v \in Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w]$. Then $M, v \models \varphi \to \psi$ and $M, v \models \varphi$, so $M, v \models \psi$.

   $\mathsf{T}^{\vec{\pi}}_\Box$: $\Box^{\vec{\pi}}\varphi \to \varphi$. Let $M, w \models \Box^{\vec{\pi}}\varphi$. Then $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w] \subseteq [\![\varphi]\!]_M$. Since $codom(Ag) = Pre(W)$, $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)$ is reflexive and thus $w \in Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w]$. Hence $M, w \models \varphi$.

   $4^{\vec{\pi}}_\Box$: $\Box^{\vec{\pi}}\varphi \to \Box^{\vec{\pi}}\Box^{\vec{\pi}}\varphi$. Let $M, w \models \Box^{\vec{\pi}}\varphi$. Then $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w] \subseteq [\![\varphi]\!]_M$. Take any $v \in Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w]$. Take any $u \in Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[v]$. Since $codom(Ag) = Pre(W)$, $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)$ is transitive, given $(w, v), (v, u) \in Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)$ we have $(w, u) \in Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)$. Hence $M, u \models \varphi$. Thus, $M, w \models \Box^{\vec{\pi}}\Box^{\vec{\pi}}\varphi$.

3. Axiom T, 4 and N for $\Box_0$:

   $\mathsf{T}_{\Box_0}$: $\Box_0\varphi \to \varphi$. Let $M, w \models \Box_0\varphi$. Thus, there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi]\!]_M$. Since $dom(Ag) = (PRE(W) \times Pre(PRE(W)))$, R is reflexive and thus Rww. Hence $M, w \models \varphi$.

   $4_{\Box_0}$: $\Box_0\varphi \to \Box_0\Box_0\varphi$. Let $M, w \models \Box_0\varphi$. Thus, there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi]\!]_M$. We need to show that there is an $R' \in \mathscr{R}$ such that $R'[w] \subseteq [\![\Box_0\varphi]\!]_M$. Take $R = R'$. Consider any $v \in R[w]$, i.e, Rwv. We need to show that $R[v] \subseteq [\![\varphi]\!]_M$. Take any u such that Rvu. Since $dom(Ag) = (PRE(W) \times Pre(PRE(W)))$, R is transitive and thus from Rwv and Rvu we get Rwu. Hence $M, u \models \varphi$.

   $\mathsf{N}_{\Box_0}$: $\Box_0\top$. Take any $R \in \mathscr{R}$. Then $R[w] \subseteq W = [\![\top]\!]_M$ and hence $M, w \models \Box_0\top$.

4. Interaction axioms:

   (a) $\forall\varphi \to \Box_0\varphi$ (Universality for $\Box_0$). Let $M, w \models \forall\varphi$. Then $W = [\![\varphi]\!]_M$. For any $R \in \mathscr{R}$, $R[w] \subseteq W = [\![\varphi]\!]_M$ and hence $M, w \models \Box_0\varphi$.

   (b) $\forall\varphi \to \Box^{\vec{\pi}}\varphi$ (Universality for $\Box^{\vec{\pi}}\varphi$). Let $M, w \models \forall\varphi$. Then $W = [\![\varphi]\!]_M$. As $Ag(\langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle)[w] \subseteq W = [\![\varphi]\!]_M$ we have $M, w \models \Box^{\vec{\pi}}\varphi$.

   (c) $(\Box_0\varphi \wedge \forall\psi) \leftrightarrow \Box_0(\varphi \wedge \forall\psi)$ (Pullout$^\leftrightarrow$). ($\Rightarrow$). Suppose $M, w \models \Box_0\varphi \wedge \forall\psi$. Then there is an $R \in \mathscr{R}$ such that $R[w] \subseteq [\![\varphi]\!]_M$ and $[\![\psi]\!]_M = W$. Hence $R[w] \subseteq ([\![\varphi]\!]_M \cap [\![\forall\psi]\!]_M)$, i.e., $R[w] \subseteq ([\![\varphi \wedge \forall\psi]\!]_M)$ and thus $M, w \models \Box_0(\varphi \wedge \forall\psi)$. ($\Leftarrow$). Suppose $M, w \models \Box_0(\varphi \wedge \forall\psi)$. Then there is an $R \in \mathscr{R}$ such that $R[w] \subseteq$

$[\![\varphi \wedge \forall\psi]\!]_M$, i.e., $R[w] \subseteq ([\![\varphi]\!]_M \cap [\![\forall\psi]\!]_M)$. Either $[\![\forall\psi]\!]_M = W$ or $[\![\forall\psi]\!]_M = \emptyset$. But if $[\![\forall\psi]\!]_M = \emptyset$, we would have $R[w] \subseteq ([\![\varphi]\!]_M \cap \emptyset$ and thus $R[w] = \emptyset$. However, as $dom(Ag) = (PRE(W) \times Pre(PRE(W)))$, R is reflexive and thus $(w, w) \in R[w]$. Hence $[\![\forall\psi]\!]_M \neq \emptyset$ and thus we must have $[\![\forall\psi]\!]_M = W$. So $R[w] \subseteq ([\![\varphi]\!]_M \cap [\![\forall\psi]\!]_M) = [\![\varphi]\!]_M$, which together with $[\![\forall\psi]\!]_M = W$ implies $M, w \models \Box_0\varphi \wedge \forall\psi$.

5. Inference rules:

   (a) Necessitation Rule for $\forall$: Let $M \models \varphi$. Then $W = [\![\varphi]\!]_M$ and thus $M \models \forall\varphi$.

   (b) Necessitation Rule for $\Box^{\vec{\pi}}$: Let $M \models \varphi$. Then $W = [\![\varphi]\!]_M$. Take any world $w \in W$. As $Ag(\langle \mathscr{R}, \prec\rangle)[w] \subseteq W$, we have $M, w \models \Box^{\vec{\pi}}\varphi$ and thus $M \models \Box^{\vec{\pi}}\varphi$.

   (c) Monotonicity Rule for $\Box_0$: Let $M \models \varphi$. Then $W = [\![\varphi]\!]_M$. Take any world $w \in W$ and any $R \in \mathscr{R}$. As $R[w] \subseteq W$, we have $M, w \models \Box_0\varphi$ and thus $M \models \Box_0\varphi$.

$\Box$

4.2.4 Completeness of $\mathsf{L}_{\mathbf{REL}}$

This section proves the strong completeness of $\Lambda_{\mathbf{REL}}$ with respect to the class of **REL** models. The proof is based on the the completeness-via-canonicity approach. In particular, we construct of a canonical **REL** model for each $\mathsf{L}_{\mathbf{REL}}$-consistent theory T_0.

Definition 69 (Canonical model for T_0)**.** A canonical model for T_0 is a structure $M^c = \langle W^c, \langle \mathscr{R}^c, \preceq^c\rangle, V^c, Ag^c\rangle$ with:

- $W^c := \{T \mid T \text{ is a maximally consistent theory and } R^{\forall}T_0T\}$
- $\mathscr{R}^c := \{R^{\Box_0\varphi} \mid \varphi \in \mathscr{L}_{\mathbf{REL}} \text{ and } (\exists\Box_0\varphi) \in T_0\}$
- $\preceq^c$ is a priority order on $\mathscr{R}^c$ with $R' \prec R^{\Box_0\top}$ for all $R' \in \mathscr{R}^c \setminus \{R^{\Box_0\top}\}$
- V^c is a valuation function given by $V^c(p) := \|p\|$
- Ag^c is an aggregator for W^c with

$$Ag^c(\langle \mathscr{R}, \preceq\rangle) = \begin{cases} R^{\vec{\pi}} & \text{if } \langle \mathscr{R}, \preceq\rangle = \langle \mathscr{R}^{\mathsf{cup}[\![\vec{\pi}]\!]_{M^c}}, \preceq^{\mathsf{cup}[\![\vec{\pi}]\!]_{M^c}}\rangle \\ W^c \times W^c & \text{otherwise} \end{cases}$$

where:

- $R^{\forall}$ is the relation on W^c given by: $R^{\forall}TS$ iff for all $\varphi \in \mathscr{L}_{\mathbf{REL}}$: $(\forall\varphi) \in T \Rightarrow (\forall\varphi) \in S$.
- for each $\varphi \in \mathscr{L}_{\mathbf{REL}}$, $R^{\Box_0\varphi}$ is the relation on W^c given by: $R^{\Box_0\varphi}TS$ iff $\Box_0\varphi \in T \Rightarrow \Box_0\varphi \in S$.
- for each $\varphi \in \mathscr{L}_{\mathbf{REL}}$, $\|\varphi\| := \{T \in W^c \mid \varphi \in T\}$
- for each $\vec{\pi} \in S_0(\Pi_*)$, $R^{\vec{\pi}}$ is the relation on W^c given by: $R^{\vec{\pi}}TS$ iff for all $\varphi \in \mathscr{L}_{\mathbf{REL}}$: $\Box^{\vec{\pi}}\varphi \in T \Rightarrow \varphi \in S$.

$\triangleleft$

We first show that this canonical model is indeed a **REL** model.

Proposition 20. *M^c is a **REL** model.*

Proof. In order to show that M^c is an **REL** model, we have to show that:

1. $\mathscr{R}^c$ is a family of evidence, i.e., every $R \in \mathscr{R}$ is a preorder.
2. $R^{\Box_0 \top} = W^c \times W^c$, i.e., the trivial evidence order is an element of $\mathscr{R}^c$, as required.
3. $R^{\vec{\pi}}$ is a preorder for each $\vec{\pi}$, and thus Ag^c is well-defined.

The rest of the model meets the conditions of a **REL** model, so let's turn to the three points just indicated.

For item 1, let $\varphi \in \mathscr{L}$ be arbitrary. Let $R \in \mathscr{R}$ be arbitrary. Then $R = R^{\Box_0\varphi}$ for some φ. The reflexivity of R is immediate from the definition of $R^{\Box_0\varphi}$. For the transitivity, let $T, S, U \in M^c$ and suppose that $R^{\Box_0\varphi}TS$ and $R^{\Box_0\varphi}SU$. Either $\Box_0\varphi \notin T$ or $\Box_0\varphi \in T$. Note that, by definition of $R^{\Box_0\varphi}$, if $\Box_0\varphi \notin T$, then $R^{\Box_0\varphi}[T] = W^c$ and thus $R^{\Box_0\varphi}TU$. Suppose now that $\Box_0\varphi \in T$. Then by definition of $R^{\Box_0\varphi}$, given $R^{\Box_0\varphi}TS$ we have $\Box_0\varphi \in S$, and thus as $R^{\Box_0\varphi}SU$ we get $\Box_0\varphi \in U$, which implies $R^{\Box_0\varphi}TU$.

For item 2, observe that $\mathsf{N}_\Box$, i.e., $\Box_0\top$, is an axiom of our system. Thus it is a member of any maximal consistent set, which implies that $R^{\Box_0\top} = W^c \times W^c$.

For item 3, take any $R^{\vec{\pi}}$. For reflexivity, suppose that $(\Box^{\vec{\pi}}\varphi) \in T$ for some $T \in M^c$. As $\mathsf{T}^{\vec{\pi}}_\Box$ is an axiom and T is maximal consistent, $(\Box^{\vec{\pi}}\varphi \to \varphi) \in T$. As $(\Box^{\vec{\pi}}\varphi) \in T$ and T is closed under modus ponens, we have $\varphi \in T$. Thus $R^{\vec{\pi}}TT$. For transitivity, let $T, S, U \in M^c$ and suppose that $R^{\vec{\pi}}TS$ and $R^{\vec{\pi}}SU$. Suppose $(\Box^{\vec{\pi}}\varphi) \in T$. As $\mathsf{4}^{\vec{\pi}}_\Box$ is an axiom and T is maximally consistent, $(\Box^{\vec{\pi}}\varphi \to \Box^{\vec{\pi}}\Box^{\vec{\pi}}\varphi) \in T$. As $(\Box^{\vec{\pi}}\varphi) \in T$ and T is closed under modus ponens, we have $\Box^{\vec{\pi}}\Box^{\vec{\pi}}\varphi \in T$. As $R^{\vec{\pi}}TS$, we then have $\Box^{\vec{\pi}}\varphi \in S$. Hence, as $R^{\vec{\pi}}SU$, we have $\varphi \in U$. As φ was arbitrary, this holds for each φ and hence we have $R^{\vec{\pi}}TU$.

□

Having established that M^c is a **REL** model, we prove now the standard lemmas to show that the canonical model works as expected.

Lemma 13 (Existence Lemma for $\forall$)**.** *$\|\exists\varphi\| \neq \emptyset$ iff $\|\varphi\| \neq \emptyset$.*

Proof.
$(\Rightarrow)$. Assume $T \in \|\exists\varphi\|$, i.e., $(\exists\varphi) \in T \in W^c$. We first prove the following:

Claim. The set $\Gamma := \{\forall\psi \mid (\forall\psi) \in T\} \cup \{\varphi\}$ is consistent.

Proof. Suppose that Γ is inconsistent, i.e., $\Gamma \vdash_{\mathsf{L}_{\mathbf{REL}}} \bot$. Then there are finitely many sentences $\forall\psi_1, \ldots, \forall\psi_n \in T$ such that $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall\psi_1 \wedge \cdots \wedge \forall\psi_n \to \neg\varphi$. By Necessitation for $\forall$ we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall(\forall\psi \wedge \cdots \wedge \forall\psi_n \to \neg\varphi)$ and from this, by $\mathsf{K}_\forall$ and modus ponens we get $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall(\forall\psi_1 \wedge \cdots \wedge \forall\psi_n) \to \forall\neg\varphi$. The system $\mathsf{S5}$ has the theorem $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\forall\forall\psi_1 \wedge \cdots \wedge \forall\forall\psi_n) \to \forall(\forall\psi_1 \wedge \cdots \wedge \forall\psi_n)$ (see, e.g., [29, p. 20]). Hence by propositional logic we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\forall\forall\psi_1 \wedge \cdots \wedge \forall\forall\psi_n) \to \forall\neg\varphi$. Given $\mathsf{4}_\forall$ we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall\psi_1 \to \forall\forall\psi_1, \ldots, \vdash_{\mathsf{L}_{\mathbf{REL}}} \forall\psi_n \to \forall\forall\psi_n$, which by propositional logic implies $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n) \to \forall\forall\psi_1 \wedge \cdots \wedge \forall\forall\psi_n$. Thus we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n) \to \forall\neg\varphi$. Hence as T is maximal consistent and closed under modus ponens, we get $(\forall\neg\varphi) \in T$. But we also have $(\exists\varphi) \in T$, i.e., $(\neg\forall\neg\varphi) \in T$, and since T is maximal consistent, this means that $(\forall\neg\varphi) \notin T$. Contradiction. □

Given the Claim, by Lindenbaum's Lemma, there is some maximally consistent theory S such that $\Gamma \subseteq S$. As $\varphi \in \Gamma$ we have $\varphi \in S$. Moreover, as $\{\forall\psi \mid (\forall\psi) \in T\} \subseteq \{\forall\chi \mid (\forall\chi) \in S\}$ we have $R^{\forall}TS$. As $T \in W^c$, we also have $R^{\forall}T_0T$. That is, $\{\forall\theta \mid (\forall\theta) \in T_0\} \subseteq \{\forall\psi \mid (\forall\psi) \in T\}$. Thus $\{\forall\theta \mid (\forall\theta) \in T_0\} \subseteq \{\forall\chi \mid (\forall\chi) \in S\}$ and thus $R^{\forall}T_0S$. Hence $S \in W^c$, which together with $\varphi \in S$ gives us $S \in \|\varphi\|$.

($\Leftarrow$) Assume $T \in \|\varphi\|$, i.e., $\varphi \in T$. Given $\mathsf{T}_{\forall}$ we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall\neg\varphi \to \neg\varphi$, and by contraposition we get $\vdash_{\mathsf{L}_{\mathbf{REL}}} \neg\neg\varphi \to \neg\forall\neg\varphi$, i.e., $\vdash_{\mathsf{L}_{\mathbf{REL}}} \varphi \to \exists\varphi$. Hence $(\varphi \to \exists\varphi) \in T$ and as T is closed under modus ponens, given also $\varphi \in T$ we get $(\exists\varphi) \in T$, i.e., $T \in \|\exists\varphi\|$. $\square$

Lemma 14 (Existence Lemma for $\Box^{\vec{\pi}}$)**.** *Let $\vec{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$ be arbitrary. $T \in \|\Diamond^{\vec{\pi}}\varphi\|$ iff there is an $S \in \|\varphi\|$ such that $R^{\vec{\pi}}TS$.*

Proof.
($\Rightarrow$). Assume $T \in \|\Diamond^{\vec{\pi}}\varphi\|$, i.e., $\Diamond^{\vec{\pi}}\varphi \in T \in W^c$. We first prove the following:

Claim. The set $\Gamma := \{\psi \mid (\Box^{\vec{\pi}}\psi) \in T\} \cup \{\forall\theta \mid \forall\theta \in T_0\} \cup \{\varphi\}$ is consistent.

Proof. Suppose that Γ is inconsistent. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{L}_{\mathbf{REL}}} \bot$. By the theorems $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}(\psi_{i_1} \wedge \cdots \wedge \psi_{i_n}) \leftrightarrow (\Box^{\vec{\pi}}\psi_{i_1} \wedge \cdots \wedge \Box^{\vec{\pi}}\psi_{i_n})$ and $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall(\theta_{j_1} \wedge \cdots \wedge \theta_{j_n}) \leftrightarrow (\forall\theta_{j_1} \wedge \cdots \wedge \forall\theta_{j_n})$ we can assume that $\Gamma_0 = \{\Box^{\vec{\pi}}\psi, \forall\theta, \neg\varphi\}$ for some $\Box^{\vec{\pi}}\psi, \forall\theta \in T$. That is, we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}\psi \wedge \forall\theta \to \neg\varphi$. By Necessitation for $\Box^{\vec{\pi}}$ we obtain $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}(\Box^{\vec{\pi}}\psi \wedge \forall\theta \to \neg\varphi)$. From this, by $\mathsf{K}_{\Box^{\vec{\pi}}}$ we get $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}(\Box^{\vec{\pi}}\psi \wedge \forall\theta) \to \Box^{\vec{\pi}}\neg\varphi$. By the theorem $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}(\Box^{\vec{\pi}}\psi \wedge \forall\theta) \leftrightarrow (\Box^{\vec{\pi}}\Box^{\vec{\pi}}\psi \wedge \Box^{\vec{\pi}}\forall\theta)$, from propositional logic we get $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\Box^{\vec{\pi}}\Box^{\vec{\pi}}\psi \wedge \Box^{\vec{\pi}}\forall\theta) \to \Box^{\vec{\pi}}\neg\varphi$. Given the axioms in our system we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}\psi \to \Box^{\vec{\pi}}\Box^{\vec{\pi}}\psi$ and $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall(\forall\theta) \to \Box^{\vec{\pi}}(\forall\theta)$. Using these, by propositional logic we obtain $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\Box^{\vec{\pi}}\psi \wedge \forall\forall\theta) \to \Box^{\vec{\pi}}\neg\varphi$. Given our axioms, we also have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall\theta \to \forall\forall\theta$. Hence by propositional logic we get $\vdash_{\mathsf{L}_{\mathbf{REL}}} (\Box^{\vec{\pi}}\psi \wedge \forall\theta) \to \Box^{\vec{\pi}}\neg\varphi$. As $\Box^{\vec{\pi}}\psi, \forall\theta \in T$ and T is closed under modus ponens, we get $(\Box^{\vec{\pi}}\neg\varphi) \in T$. But we also have $(\Diamond^{\vec{\pi}}\varphi) \in T$, i.e., $(\neg\Box^{\vec{\pi}}\neg\varphi) \in T$, and since T is maximal consistent, this means that $(\Box^{\vec{\pi}}\neg\varphi) \notin T$. Contradiction. $\square$

Given the Claim, by Lindenbaum's Lemma, there is some maximally consistent theory S such that $\Gamma \subseteq S$. As $\varphi \in \Gamma$ we have $\varphi \in S$. Moreover, as $\{\psi \mid (\Box^{\vec{\pi}}\psi) \in T\} \subseteq S$, we have $R^{\vec{\pi}}TS$. Additionally, we have $\{\forall\theta \mid (\forall\theta) \in T_0\} \subseteq S$ and thus $R^{\forall}T_0S$. Hence $S \in W^c$, which together with $\varphi \in S$ gives us $S \in \|\varphi\|$.

($\Leftarrow$) Assume $T \in \|\varphi\|$, i.e., $\varphi \in T$. Given $\mathsf{T}_{\Box^{\vec{\pi}}}$ we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \Box^{\vec{\pi}}\neg\varphi \to \neg\varphi$, and by contraposition we get $\vdash_{\mathsf{L}_{\mathbf{REL}}} \neg\neg\varphi \to \neg\Box^{\vec{\pi}}\neg\varphi$, i.e., $\vdash_{\mathsf{L}_{\mathbf{REL}}} \varphi \to \Diamond^{\vec{\pi}}\varphi$. Hence, $(\varphi \to \Diamond^{\vec{\pi}}\varphi) \in T$ and as T is closed under modus ponens, given also $\varphi \in T$ we get $(\Diamond^{\vec{\pi}}\varphi) \in T$, i.e., $T \in \|\Diamond^{\vec{\pi}}\varphi\|$. $\square$

Lemma 15 (Existence Lemma for $\Box_0$)**.** *$T \in \|\Box_0\varphi\|$ iff there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq \|\varphi\|$.*

Proof. ($\Rightarrow$). Assume $T \in \|\Box_0\varphi\|$, i.e., $(\Box_0\varphi) \in T \in W^c$. We first prove the following:

Claim. $\exists\Box_0\varphi \in T_0$.

Proof. Suppose not. As T_0 is maximal consistent, we have $\neg\exists\Box_0\varphi \in T_0$, i.e., $\forall\neg\Box_0\varphi \in T_0$. As $T \in W^c$, we have $R^{\forall}T_0T$. So given $\forall\neg\Box_0\varphi \in T_0$ we have $\forall\neg\Box_0\varphi \in T$. By $\mathsf{T}_{\forall}$ we have $\vdash_{\mathsf{L}_{\mathbf{REL}}} \forall\neg\Box_0\varphi \to \neg\Box_0\varphi$, i.e., $(\forall\neg\Box_0\varphi \to \neg\Box_0\varphi) \in T$. As T is closed under modus ponens, given $(\forall\neg\Box_0\varphi) \in T$ we get $(\neg\Box_0\varphi) \in T$. But we also have $(\Box_0\varphi) \in T \in W^c$ and thus T is inconsistent. Contradiction. $\square$

Hence $R^{\Box_0\varphi} \in \mathscr{R}^c$. We will show that $R^{\Box_0\varphi}[T] \subseteq \|\varphi\|$. Let $S \in W^c$ be arbitrary and suppose that $R^{\Box_0\varphi}TS$. By definition of $R^{\Box_0\varphi}$, we have $(\Box_0\varphi) \in T$ implies $(\Box_0\varphi) \in S$. As $(\Box_0\varphi) \in T$ we get $(\Box_0\varphi) \in S$. Given $\mathsf{T}_{\Box_0}$ we have $\vdash_{\mathbf{L_{REL}}} \Box_0\varphi \to \varphi$ and thus $(\Box_0\varphi \to \varphi) \in S$. Since S is closed under modus ponens we thus get $\varphi \in S$, i.e., $S \in \|\varphi\|$. As S was picked arbitrarily, we have $R^{\Box_0\varphi}[T] \subseteq \|\varphi\|$.

($\Leftarrow$) Let $T \in W^c$ and suppose there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq \|\varphi\|$. By definition of R^c, $R = R^{\Box_0\theta}$ for some $\theta \in \mathscr{L}_{\mathbf{REL}}$ such that $(\exists\Box_0\theta) \in T_0$. Either $\Box_0\theta \in T$ or $\Box_0\theta \notin T$. We consider both cases.

Case 1: Suppose that $\Box_0\theta \in T \in W^c$. We first prove the following:

Claim. The set $\Gamma := \{\Box_0\theta\} \cup \{\forall\psi \mid (\forall\psi) \in T\} \cup \{\neg\varphi\}$ is inconsistent.

Proof. Suppose that Γ is consistent. By Lindenbaum's Lemma there is some maximal consistent theory S such that $\Gamma \subseteq S$. Moreover, as $\{\forall\psi \mid (\forall\psi) \in T_0\} \subseteq \{\forall\psi \mid (\forall\psi) \in T\} \subseteq S$, we have $R^\forall T_0 S$ and thus $S \in W^c$. As $\neg\varphi \in \Gamma$ we have $\neg\varphi \in S$. Since S is consistent we have $\varphi \notin S$, i.e., $S \notin \|\varphi\|$. From $\Box_0\theta \in \Gamma$ we have $\Box_0\theta \in S$. By definition of $R^{\Box_0\theta}$, we get $R^{\Box_0\theta}TS$. But then, given $S \notin \|\varphi\|$, we have $R^{\Box_0\theta}[T] \not\subseteq \|\varphi\|$. Contradiction. $\Box$

Given the Claim, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathbf{L_{REL}}} \bot$. By the theorem $\vdash_{\mathbf{L_{REL}}} \forall(\psi_1 \wedge \cdots \wedge \psi_n) \leftrightarrow (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n)$ we can assume that $\Gamma_0 = \{\Box_0\theta, \forall\psi, \neg\varphi\}$ for some $\psi \in T$. Since $\Gamma_0 \vdash_{\mathbf{L_{REL}}} \bot$ we have $\vdash_{\mathbf{L_{REL}}} (\Box_0\theta \wedge \forall\psi \wedge \neg\varphi) \to \bot$, so by propositional logic $\vdash_{\mathbf{L_{REL}}} (\Box_0\theta \wedge \forall\psi) \to (\neg\varphi \to \bot)$, i.e., $\vdash_{\mathbf{L_{REL}}} (\Box_0\theta \wedge \forall\psi) \to (\neg\neg\varphi)$, i.e., $\vdash_{\mathbf{L_{REL}}} (\Box_0\theta \wedge \forall\psi) \to \varphi$. Given the Pullout axiom, we have $\vdash_{\mathbf{L_{REL}}} \Box_0(\theta \wedge \forall\psi) \to (\Box_0\theta \wedge \forall\psi)$ and thus $\vdash_{\mathbf{L_{REL}}} \Box_0(\theta \wedge \forall\psi) \to \varphi$. By the Monotonicity Rule for $\Box_0$, we get $\vdash_{\mathbf{L_{REL}}} \Box_0\Box_0(\theta\wedge\forall\psi) \to \Box_0\varphi$. By $\mathsf{4}_{\Box_0}$, we have $\vdash_{\mathbf{L_{REL}}} \Box_0(\theta\wedge\forall\psi) \to \Box_0\Box_0(\theta\wedge\forall\psi)$ and thus $\vdash_{\mathbf{L_{REL}}} \Box_0(\theta\wedge\forall\psi) \to \Box_0\varphi$. By the Pullout axiom, we have $\vdash_{\mathbf{L_{REL}}} (\Box_0\theta\wedge\forall\psi) \to \Box_0(\theta\wedge\forall\psi)$. Hence $\vdash_{\mathbf{L_{REL}}} (\Box_0\theta \wedge \forall\psi) \to \Box_0\varphi$. Therefore $((\Box_0\theta \wedge \forall\psi) \to \Box_0\varphi) \in T$. As $(\Box_0\theta) \in T$ and $(\forall\psi) \in T$, by closure under modus ponens, we have $\Box_0\varphi \in T$. That is, $T \in \|\Box_0\varphi\|$.

Case 2: Suppose that $\Box_0\theta \notin T$. Note that $\Box_0\theta \notin T$ implies that $R^{\Box_0\theta}[T] = W^c$, and since we have $R = R^{\Box_0\theta}$ and $R[T] \subseteq \|\varphi\|$, all this gives us that $W^c \subseteq \|\varphi\|_c$, i.e. all theories in the canonical model contain φ. We now prove the following:

Claim. The set $\Gamma := \{\forall\psi \mid (\forall\psi) \in T\} \cup \{\neg\varphi\}$ is inconsistent.

Proof. Suppose that Γ is consistent. By Lindenbaum's Lemma there is some maximal consistent theory S such that $\Gamma \subseteq S$. Moreover, as $\{\forall\psi \mid (\forall\psi) \in T_0\} \subseteq \{\forall\psi \mid (\forall\psi) \in T\} \subseteq S$, we have $R^\forall T_0 S$ and thus $S \in W^c$. As $\neg\varphi \in \Gamma$ we have $\neg\varphi \in S$ and thus $S \in \|\neg\varphi\|$. Therefore $W^c \not\subseteq \|\varphi\|$ (contradiction). $\Box$

Given the Claim, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathbf{L_{REL}}} \bot$. By the theorem $\vdash_{\mathbf{L_{REL}}} \forall(\psi_1 \wedge \cdots \wedge \psi_n) \leftrightarrow (\forall\psi_1 \wedge \cdots \wedge \forall\psi_n)$ we can assume that $\Gamma_0 = \{\forall\psi, \neg\varphi\}$ for some $\psi \in T$. Since $\Gamma_0 \vdash_{\mathbf{L_{REL}}} \bot$ we have $\vdash_{\mathbf{L_{REL}}} (\forall\psi \wedge \neg\varphi) \to \bot$, so by propositional logic $\vdash_{\mathbf{L_{REL}}} (\forall\psi) \to (\neg\varphi \to \bot)$, i.e., $\vdash_{\mathbf{L_{REL}}} (\forall\psi) \to (\neg\neg\varphi)$, i.e., $\vdash_{\mathbf{L_{REL}}} (\forall\psi) \to \varphi$. By propositional logic, given $\vdash_{\mathbf{L_{REL}}} (\forall\psi) \to \varphi$ we can strengthen the antecedent getting $\vdash_{\mathbf{L_{REL}}} (\Box_0\top \wedge \forall\psi) \to \varphi$. Given the Pullout axiom$^\leftrightarrow$, we have $\vdash_{\mathbf{L_{REL}}} \Box_0(\top \wedge \forall\psi) \leftrightarrow (\Box_0\top \wedge \forall\psi)$ and thus $\vdash_{\mathbf{L_{REL}}} \Box_0(\top \wedge \forall\psi) \to \varphi$. By the Monotonicity Rule for $\Box_0$, we get $\vdash_{\mathbf{L_{REL}}} \Box_0\Box_0(\top \wedge \forall\psi) \to \Box_0\varphi$. By $\mathsf{4}_{\Box_0}$, we have $\vdash_{\mathbf{L_{REL}}} \Box_0(\top \wedge \forall\psi) \to \Box_0\Box_0(\top \wedge \forall\psi)$ and thus $\vdash_{\mathbf{L_{REL}}} \Box_0(\top \wedge \forall\psi) \to \Box_0\varphi$. By the Pullout axiom$^\leftrightarrow$, we have $\vdash_{\mathbf{L_{REL}}} (\Box_0\top \wedge \forall\psi) \leftrightarrow \Box_0(\top \wedge \forall\psi)$. Hence $\vdash_{\mathbf{L_{REL}}} (\Box_0\top \wedge \forall\psi) \to \Box_0\varphi$. Therefore $((\Box_0\top \wedge \forall\psi) \to \Box_0\varphi) \in T$. As $\Box_0\top$ is an axiom of our system, we have

$(\Box_0\top) \in T$ and $(\forall\psi) \in T$. Hence by closure under modus ponens, we have $\Box_0\varphi \in T$. That is, $T \in \|\Box_0\varphi\|$. □

Lemma 16 (Truth Lemma). *For every formula $\varphi \in \mathscr{L}_{\boldsymbol{REL}}$, we have: $[\![\varphi]\!]_{M^c} = \|\varphi\|$.*

Proof. The proof is by induction on the complexity of φ. The base case follows from the definition of V^c. For the inductive case, suppose that for all $T \in W^c$ and all formulas ψ of lower complexity than φ, we have $[\![\psi]\!]_{M^c} = \|\psi\|$. The Boolean cases where $\varphi = \neg\psi$ and $\varphi = \psi_1 \wedge \psi_2$ follow from the induction hypothesis together with the standard facts about maximal consistent theories included in Proposition 12. Only the modalities remain. Let $\varphi = \exists\psi$ and consider any $T \in M^c$. We have $T \in \|\exists\psi\|$ iff (Proposition 13) $\|\varphi\| \neq \emptyset$ iff (induction hypothesis) $[\![\psi]\!]_{M^c}$ iff $[\![\exists\psi]\!]_{M^c} = W^c$ iff $T \in [\![\exists\psi]\!]_{M^c}$. Now let $\varphi = \Box_0\psi$ and consider any $T \in M^c$. We have $T \in \|\Box_0\psi\|$ iff (Proposition 15) there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq \|\psi\|$ iff (induction hypothesis) there is an $R \in \mathscr{R}^c$ such that $R[T] \subseteq [\![\psi]\!]_{M^c}$ iff $T \in [\![\Box_0\psi]\!]_{M^c}$. Finally, let $\vec{\pi}$ be arbitrary and let $\varphi = \Diamond^{\vec{\pi}}\psi$. We have $T \in \|\Diamond^{\vec{\pi}}\psi\|$ iff (Proposition 14) there is an $S \in \|\psi\|$ such that $R^{\vec{\pi}}TS$ iff (induction hypothesis) there is an $S \in [\![\psi]\!]_{M^c}$ such that $R^{\vec{\pi}}TS$ iff there is an $S \in [\![\psi]\!]_{M^c}$ such that $(T, S) \in Ag^c(\langle \mathscr{R}^{\mathsf{cup}[\![\vec{\pi}]\!]_{M^c}}, \preceq^{\mathsf{cup}[\![\vec{\pi}]\!]_{M^c}}\rangle)$ iff $T \in [\![\Diamond^{\vec{\pi}}\psi]\!]_{M^c}$. □

Lemma 17. *$\Lambda_{\boldsymbol{REL}}$ is strongly complete with respect to the class of **REL** models.*

Proof. By Proposition 6, it suffices to show that every $\Lambda_{\mathbf{REL}}$-consistent set of formulas is satisfiable on some **REL** model. Let Γ be an $\Lambda_{\mathbf{REL}}$-consistent set of formulas. By Lindenbaum's Lemma, there is a maximally consistent set T_0 such that $\Gamma \subseteq T_0$. Choose any canonical model M^c for T_0. By Lemma 16, $M^c, T_0 \models \varphi$ for all $\varphi \in T_0$. □

Now that we have shown soundness and completeness for the system $\mathsf{L}_{\mathbf{REL}}$, we study next the dynamic logic of iterated prioritized evidence addition, **REL**$^+$.

4.2.5 Syntax and semantics of REL$^+$

As anticipated in Section 4.2.1, we will encode the dynamics of prioritized evidence addition by extending $\mathscr{L}_{\mathbf{REL}}$ with modal operators of the form $[+^{\mathsf{up}}\vec{\pi}]$. The new formulas of the form $[+^{\mathsf{up}}\vec{\pi}]\varphi$ are used to express the statement: "φ is true after the iterated evidence addition of the evidence sequence defined by $\vec{\pi}$".

Definition 70 (Language $\mathscr{L}^+_{\mathbf{REL}}$). Let P be a countably infinite set of propositional variables. The language $\mathscr{L}^+_{\mathbf{REL}}$ is defined by mutual recursion:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box^{\vec{\pi}}\varphi \mid \forall\varphi \mid [+^{\mathsf{up}}\vec{\pi}]\varphi$$
$$\pi ::= A \mid ?\varphi \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*$$

where, as in the case of $\mathscr{L}_{\mathbf{REL}}$, $\vec{\pi}$ is a (possibly empty) finite sequence of evidence programs. ◁

The truth clause for the dynamic modalities is given by extending the satisfaction relation $\models$ for $\mathscr{L}_{\mathbf{REL}}$ as follows:

Definition 71 (Satisfaction for $[+^{\mathsf{up}}\vec{\pi}]\varphi$). Let $M = \langle W, \langle \mathscr{R}, \preceq\rangle, V, Ag\rangle$ be an **REL** model, $w \in W$ and $\vec{\pi} \in S_0(\Pi_*)$. The satisfaction relation $\models$ between pairs (M, w) and formulas $[+^{\mathsf{up}}\vec{\pi}]\varphi \in \mathscr{L}^+_{\mathbf{REL}}$ is defined as follows:

$$M, w \models [+^{\mathsf{up}}\vec{\pi}]\varphi \text{ iff } M^{\mathsf{up}[\![\vec{\pi}]\!]_M}, w \models \varphi$$

◁

4.2.6 A proof system for $\mathbf{REL}^+$: $\mathsf{L}^+_{\mathbf{REL}}$

This section introduces the proof system $\mathsf{L}^+_{\mathbf{REL}}$. In the next section, the logic generated by this proof system will be shown to be sound and complete with respect to **REL** models. The soundness and completeness proofs work via a standard reductive analysis, appealing to *reduction axioms.*

Definition 72 (Proof system of $\mathbf{REL}^+$)**.** Let $\chi, \chi' \in \mathscr{L}^+_{\mathbf{REL}}$ and let $\vec{\pi} = \langle \pi_1, \ldots, \pi_n \rangle \in S_0(\Pi_*)$ be a sequence of evidence programs with each π_i, for $i \in \{1, \ldots, n\}$, has a normal form

$$\mathsf{nf}(\pi_i) := \bigcup_{s \in S_0(I_i)} (?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$$

The proof system of $\mathsf{L}^+_{\mathbf{REL}}$ includes all *axioms schemas and inference rules* of $\mathsf{L}_{\mathbf{REL}}$. Moreover, it includes the following *reduction axioms*:

PEA1 : $[+^{\mathsf{up}}\vec{\pi}]p \leftrightarrow p$ for all $p \in \mathsf{P}$

PEA2 : $[+^{\mathsf{up}}\vec{\pi}]\neg\chi \leftrightarrow \neg[+\vec{\pi}]\chi$

PEA3 : $[+^{\mathsf{up}}\vec{\pi}]\chi \wedge \chi' \leftrightarrow [+^{\mathsf{up}}\vec{\pi}]\chi \wedge [+\vec{\pi}]\chi'$

PEA4 : $[+^{\mathsf{up}}\vec{\pi}]\Box_0\chi \leftrightarrow \Box_0[+^{\mathsf{up}}\vec{\pi}]\varphi \vee \big([+\vec{\pi}]\chi \wedge \bigvee_{i<n}(\bigwedge_{s \in S_0(I_i)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \rightarrow \forall(\psi_{s_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi)))\big)$

PEA5 : $[+^{\mathsf{up}}\vec{\pi}]\Box^{\vec{\rho}}\chi \leftrightarrow \Box^{\vec{\pi}\oplus\vec{\rho}}\chi$, for $\vec{\rho} \in S_0(\Pi_*)$

PEA6 : $[+^{\mathsf{up}}\vec{\pi}]\forall\chi \leftrightarrow \forall[+^{\mathsf{up}}\vec{\pi}]\chi$

We remind the reader that $\vec{\pi} \oplus \vec{\rho}$ denotes the concatenation of the sequences $\vec{\pi}$ and $\vec{\rho}$. $\triangleleft$

4.2.7 Soundness and completeness of $\mathsf{L}^+_{\mathbf{REL}}$

Let $\Lambda^+_{\mathbf{REL}}$ denote the logic generated by $\mathsf{L}^+_{\mathbf{REL}}$. This section proves soundness and completeness of the logic $\Lambda^+_{\mathbf{REL}}$. As indicated above, the proofs works via a standard reductive analysis. The key part of the proofs is to show that the reduction axioms are valid.

Proposition 21. *The axioms* PEA1-PEA6 *are valid.*

Proof. Let $M = \langle W, \langle \mathscr{R}, \preceq \rangle, V, Ag \rangle$ be an **REL** model, w a world in M and $\vec{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$ be a sequence of evidence programs with $\mathsf{nf}(\pi_i) := \bigcup_{s \in S_0(I_i)}(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}) \cup (?\top)$ for each $i \in \{1, \ldots, n\}$.

1. The validity of PEA1 follows from the fact that the evidence addition transformer does not change the valuation function. The validity of the Boolean reduction axioms PEA2 and PEA3 can be proven by unfolding the definitions.

2. Axiom PEA4: We first prove the following:

   Claim. There is an $k \in \{1, \ldots, n\}$ such that $[\![\pi_k]\!]_M[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M$ iff $M, w \models [+^{\mathsf{up}}\vec{\pi}]\chi \wedge \bigvee_{i<n}(\bigwedge_{s \in S_0(I_i)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \rightarrow \forall(\psi_{s_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi)))$.

   Proof. ($\Rightarrow$) Suppose there is an $k \in \{1, \ldots, n\}$ such that $[\![\pi_k]\!]_M[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M$. As π_k is an evidence program, $[\![\pi_k]\!]_M$ is reflexive and thus $M, w \models [+^{\mathsf{up}}\vec{\pi}]\chi$. It remains to be shown that

$$M, w \models \bigvee_{i<n}(\bigwedge_{s \in S_0(I_i)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \rightarrow \forall(\psi_{s_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi))) \tag{4.1}$$

To show (4.1), it suffices to find one $i \in \{1, \ldots, n\}$ such that

$$M, w \models \bigwedge_{s \in S_0(I_i)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\vec{\pi}]\chi))$$

Consider $i = k$, take any $s \in S_0(I_k)$ and suppose that $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$. We need to show that $M, w \models \forall(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\vec{\pi}]\chi)$. Take any $v \in W$ and suppose $M, v \models \psi_{s_{\mathsf{len}(s)}}$. If we show that $M, v \models [+^{\mathsf{up}}\vec{\pi}]\chi$, we are done. Given $M, w \models s(\boldsymbol{\varphi}, \boldsymbol{\psi})$ and $M, v \models \psi_{s_{\mathsf{len}(s)}}$, by Proposition 8, we have $(w, v) \in [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$. Thus

$$(w, v) \in \bigcup_{s \in S_0(I_k)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M$$

Hence as

$$\begin{aligned}
[\![\pi_k]\!]_M &= [\![\mathsf{nf}(\pi_k)]\!]_M \\
&= [\![\bigcup_{s \in S_0(I_k)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
&= [\![\bigcup_{s \in S_0(I_k)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \cup [\![?\top]\!]_M \\
&= \bigcup_{s \in S_0(I_k)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \cup [\![?\top]\!]_M
\end{aligned}$$

we have $(w, v) \in [\![\pi_k]\!]_M$. Hence, given $[\![\pi_k]\!]_M[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M$ we have $M, v \models [+^{\mathsf{up}}\vec{\pi}]\chi$, as required.

($\Leftarrow$) Suppose that $M, w \models [+^{\mathsf{up}}\vec{\pi}]\chi \wedge \bigvee_{i<n}(\bigwedge_{s \in S_0(I_i)}(s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\vec{\pi}]\chi)))$. Then there is some $k \in \{1, \ldots, n\}$ such that

$$M, w \models \bigwedge_{s \in S_0(I_k)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \to \forall(\psi_{s_{\mathsf{len}(s)}} \to [+^{\mathsf{up}}\vec{\pi}]\chi))) \tag{4.2}$$

We will show that $[\![\pi_k]\!]_M[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M$. Take any v and suppose $(w, v) \in [\![\pi_k]\!]_M$. We need to show that $v \in [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M$. If $v = w$, given $M, w \models [+^{\mathsf{up}}\vec{\pi}]\chi$ we are done. So suppose $v \neq w$. Note that

$$\begin{aligned}
&(w, v) \in [\![\pi_k]\!]_M \\
\text{iff } &(w, v) \in [\![\mathsf{nf}(\pi_k)]\!]_M \\
\text{iff } &(w, v) \in [\![\bigcup_{s \in S_0(I_k)} ?\big(s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big) \cup (?\top)]\!]_M \\
\text{iff } &(w, v) \in [\![\bigcup_{s \in S_0(I_k)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } (w, v) \in [\![?\top]\!]_M \\
\text{iff } &(w, v) \in [\![\bigcup_{s \in S_0(I_k)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \text{ or } w = v \\
\text{iff } &(w, v) \in [\![\bigcup_{s \in S_0(I_k)} \big(?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}\big)]\!]_M \quad (\text{as } w \neq v \text{ by assumption }) \\
\text{iff } &(w, v) \in \bigcup_{s \in S_0(I_k)} [\![?s(\boldsymbol{\varphi}, \boldsymbol{\psi}); A; ?\psi_{s_{\mathsf{len}(s)}}]\!]_M \\
\text{iff } &\text{ for some } s' \in S_0(I_k), (w, v) \in [\![?s'(\varphi); A; ?\psi_{s'_{\mathsf{len}(s')}}]\!]_M
\end{aligned}$$

$$\text{iff for some } s' \in S_0(I_k), w \in [\![s'(\varphi)]\!]_M \text{ and } v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M \quad \text{(by Prop. 8)}$$

Given (4.2), we have in particular

$$M, w \models s'(\varphi) \rightarrow \forall(\psi_{s'_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi)$$

Thus from $w \in [\![s'(\varphi)]\!]_M$ we get $M, w \models \forall(\psi_{s'_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi)$. And given $v \in [\![\psi_{s'_{\mathsf{len}(s')}}]\!]_M$ we get $M, v \models [+^{\mathsf{up}}\vec{\pi}]\chi$, as required. □

Given the Claim, we have

$$\begin{aligned}
& M, w \models [+^{\mathsf{up}}\vec{\pi}]\Box_0\chi \\
\text{iff } & M^{\mathsf{up}+\vec{\pi}}, w \models \Box_0\chi \\
\text{iff } & \text{there is an } R \in \mathscr{R} \cup \{[\![\pi_i]\!]_M \mid i = 1, \ldots, n\} \text{ such that } R[w] \subseteq [\![\chi]\!]_{M^{+\vec{\pi}}} \\
\text{iff } & \text{there is an } R \in \mathscr{R} \cup \{[\![\pi_i]\!]_M \mid i = 1, \ldots, n\} \text{ such that } R[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M \\
\text{iff } & \text{there is an } R \in \mathscr{R} \text{ such that } R[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M \\
& \quad \text{or there is an } i \in \{1, \ldots, n\} \text{ such that } [\![\pi_i]\!]_M[w] \subseteq [\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M \\
\text{iff } & M, w \models \Box_0[+^{\mathsf{up}}\vec{\pi}]\chi \\
& \quad \text{or } M, w \models [+^{\mathsf{up}}\vec{\pi}]\chi \wedge \bigvee_{i<n}\Big(\bigwedge_{s \in S_0(I_i)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \rightarrow \forall(\psi_{s_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi))\Big) \\
& \quad (\text{ by the Claim above}) \\
\text{iff } & M, w \models \Box_0[+^{\mathsf{up}}\vec{\pi}]\chi \vee \Big([+^{\mathsf{up}}\vec{\pi}]\chi \wedge \bigvee_{i<n}\Big(\bigwedge_{s \in S_0(I_i)} (s(\boldsymbol{\varphi}, \boldsymbol{\psi}) \rightarrow \forall(\psi_{s_{\mathsf{len}(s)}} \rightarrow [+^{\mathsf{up}}\vec{\pi}]\chi))\Big)\Big)
\end{aligned}$$

3. Axiom EA5: $M, w \models [+^{\mathsf{up}}\vec{\pi}]\Box^{\vec{\rho}}\chi$ iff $M^{\mathsf{up}\vec{\pi}}, w \models \Box^{\vec{\rho}}\chi$ iff $\langle W^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M} \rangle, V^{\mathsf{up}[\![\vec{\pi}]\!]_M}, Ag^{\mathsf{up}[\![\vec{\pi}]\!]_M}\rangle, w \models \Box^{\vec{\rho}}\chi$ iff

   $$\langle W^{\mathsf{up}[\![\vec{\pi}]\!]_M \oplus [\![\vec{\rho}]\!]_M}, \langle \mathscr{R}^{\mathsf{up}[\![\vec{\pi}]\!]_M \oplus [\![\vec{\rho}]\!]_M}, \preceq^{\mathsf{up}[\![\vec{\pi}]\!]_M \oplus [\![\vec{\pi}]\!]_M}\rangle, V^{\mathsf{up}[\![\vec{\pi}]\!]_M \oplus [\![\vec{\rho}]\!]_M}, Ag^{\mathsf{up}[\![\vec{\pi}]\!]_M \oplus [\![\vec{\pi}]\!]_M}\rangle, w \models \chi$$

   iff $M, w \models \Box^{\vec{\pi} \oplus \vec{\rho}}\chi$

4. Axiom EA6: $M, w \models [+^{\mathsf{up}}\vec{\pi}]\forall\chi$ iff $M^{+^{\mathsf{up}}\vec{\pi}}, w \models \forall\chi$ iff $[\![\chi]\!]_{M^{+\mathsf{up}\vec{\pi}}} = W^{+\mathsf{up}\vec{\pi}}$ iff $[\![[+^{\mathsf{up}}\vec{\pi}]\chi]\!]_M = W$ iff $M, w \models \forall[+^{\mathsf{up}}\vec{\pi}]\chi$.

□

Theorem 13. $\Lambda^{+}_{\boldsymbol{REL}}$ *is complete with respect to the class of* $\boldsymbol{REL}$ *models.*

Proof. Once we have established the validity of the reduction axioms, the proof is standard and follows the same steps used to prove completeness of $\Lambda_{\cap}$ (see Theorem 5). □

4.3 Chapter review

In this chapter, we studied General Relational Evidence Logic. This is a logic of belief based on aggregated evidence, in which the aggregator is not fixed. Hence, this logic gives a means to reason about the beliefs that an agent would form, based on her evidence, *irrespective* of the aggregator used, as long as this aggregator satisfies the basic properties built into its definition. We then considered the dynamics of prioritized addition over general models. We presented a proof system with conditional modalities that pre-encode the effects of prioritized addition on **REL** models, and provided a matching proof system. After that, we presented a full dynamic language with dynamic modalities, as well as a proof system for this dynamic language, and showed that this system is sound and complete with respect to **REL** models.

Conclusion

In this thesis, we have studied a family of *dynamic relational evidence logics*, i.e., logics for reasoning about the relational evidence and evidence-based beliefs of agents in a dynamic environment. Our goal was to contribute to existing work on evidence logics [1–5] in three main ways.

- *Relax the assumption that all evidence is binary.* Instead of assuming that all evidence is binary, we modeled evidence with evidence relations, ordering states in terms of plausibility. As discussed in Chapter II.1, a special type of evidence relation (dichotomous weak orders) can be used to model binary evidence in a relational way. Thus, in a way, evidence relations can be seen as a generalisation of evidence sets.

- *Model levels of evidence reliability.* We equipped our models with *priority orders*, i.e., orderings of the family of evidence relations according to their relative reliability to model the relative reliability of pieces of evidence. This enabled the study of evidence aggregation based on reliability-sensitive rules, such as the lexicographic rule.

- *Explore alternative evidence aggregation rules.* We studied logics involving unanimous evidence aggregation of equally reliable evidence ($\mathbf{REL}_{\cap}$), as well as logics based on reliability-sensitive rules ($\mathbf{REL}_{\mathsf{lex}}$), such as the lexicographic rule. Moreover, we explored the general logic of the class of **REL** models.

Clearly, many open problems remain. Here are a few more specific avenues for future research:

- *Additional aggregators*: We studied two natural aggregators. As we know from the social choice literature, many other aggregators have nice properties. An interesting extension to this work could involve developing logics based on other well-known aggregators.

- *Additional evidential actions*: As we saw, in a setting with ordered evidence, evidence actions are complex transformations, both of the stock of evidence and the priority order. For the lexicographic case, we studied a form of prioritized addition. It could be interesting to consider more general forms of addition, or actions that transform the priority order (re-evaluation of reliability) without affecting the stock of evidence.

- *Probabilistic evidence*: We moved from the binary evidence case to the relational evidence case. Another important form of evidence is probabilistic evidence, i.e., evidence that comes in the form of a probability distribution over the set of states. The aggregation of probabilistic information is studied in the area of *probabilistic opinion pooling* [30] and *pure inductive logic* [31]. The logical study of these aggregation settings is also an interesting open avenue of research.

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