What is a choice sequence? How a solution of Troelstra's paradox shows the way to an answer to this question

Joop Niekus

1 Introduction

In 1907 L.E.J.Brouwer, then an unknown Dutch mathematician, stepped into the running debate on the origin and certainty of mathematics with his thesis *On the foundations of mathematics* (Brouwer 1907). In his view mathematics consists of mental constructions only; there are no mathematical truths outside the human mind. The constructions are built up from the natural numbers N and an intuitively given continuum. With N Brouwer constructed the integers Z and the rational numbers Q, but he had no satisfactory way to introduce the real numbers. It would take more than ten years before Brouwer would present a solution of this problem.

In 1918 Brouwer, now a famous topologist, presented a complete new reconstruction of mathematics along lines set out in his thesis (Brouwer 1918). He called his reconstruction *intuitionism*, the existing mathematical practice *classical*. His foundational problem, reaching the power of the continuum from the discrete, he solved by the introduction of sequences not completely determined by a law: *choice sequences*.

In reaching the power of the continuum Brouwer used only global properties of a continuum with choice sequences. He did not use a particular choice sequence. The question what exactly is a choice sequence turned out to be an intriguing one. Brouwer added notes to his definitions, sometimes extending a former note, sometimes changing or withdrawing a former one, and may never have been fully satisfied.

The standard work on the subject, Troelstra's *Choice Sequences* concentrated on global properties. This corresponds to the fact that no instances of particular choice sequences had been found in the work of Brouwer, except one possible exception: his proof of the negative continuity theorem in Brouwer 1927. But that proof is very vague and allows many possible reconstructions. When Brouwer returned to the proof later he clearly uses a determined sequence. Our claim is that at this later time he was fully exploiting particular choice sequences in the method of the creating subject.

The method of the creating subject characterizes Brouwer's papers after 1945, when after a long delay he started to publish again. The method has always been supposed to be a radically new step in the work of Brouwer. The expression 'creating subject' was then interpreted as 'the idealized mathematician' and the generated sequences by the creating subject as completely determined. The notion of the idealized mathematician was formalized by Kreisel which resulted in the theory of the idealized mathematician. This theory does not reflect Brouwer's reasoning well and it was struck by a paradox, discovered by Troelstra, that could not be resolved satisfactorily.

We propose a solution of the paradox in which Kreisel's main assumptions are dropped. A consequence of our solution is that the generated sequences are no longer completely determined, they are choice sequences. We will conclude that the method of the creating subject is special, not because of the introduction of an idealized mathematician, but by the systematic application of particular choice sequences.

In search for evidence for the conception of choice sequence that arises from our solution, we made an investigation of Brouwer's earlier work. This investigation fully supports our position. Moreover, it shows that the method of the creating subject was not new at all.

We shall answer the question of the title for the particular choice sequences Brouwer actually used. This answer, which is not provided by the standard theories on the subject, opens up possibilities for research in a most characteristic part of Brouwer's work.

2 Logic

Brouwer's view that mathematics consists of mental constructions only, leads to an interpretation of the logical connectives different from the classical one.

Affirming a proposition A means stating to have a proof of A, i.e. a construction proving its affirmation.

$A \vee B$: I have a proof of A or I have a proof of B (or of both!).

$\neg A$: from any proof of A a contradiction can be derived.

The intuitionistic negation $\neg A$ is stronger than the negation of the natural language, where the negation of 'I have a proof of A' means: 'I do not have a proof of A'. This negation also occurs in intuitionistic reasonings, see the examples below, and is called the *weak negation*.

A consequence of the strong interpretation of $\vee$ and $\neg$ is that the law of the excluded middle, $A \vee \neg A$, does not hold as long as there are undecided mathematical questions. This non-validness would become the striking feature of intuitionism.

A peculiarity of intuitionism is also that the double negation of a proposition does not imply the proposition itself, i.e. $\neg\neg A$ does not imply A. Consequently $\neg\neg A \vee \neg A$ (called testedness), does not imply $A \vee \neg A$ (decidability). This distinction will play an important role is this paper.

We complete this section with the interpretation of $\to$ and $\wedge$.

$A \wedge B$: I have a proof of A and I have a proof of B;

$A \to B$: I have a construction that transforms any proof of A into a proof of B.

3 Choice sequences

The basic notion of intuitionism is that of *spread*. A spread is a law, a *building rule*, that regulates the construction of infinitely proceeding sequences, their terms chosen from a sequence A of mathematical objects already constructed, the *founding sequence*. The sequence A may for example consist of the natural numbers N, a (possibly finite) part of N, or the rational numbers Q. Brouwer had a preference to work with pairs of rational numbers, i.e. rational intervals.

The building rule decides whether a choice from A is admitted as the first term of a sequence under construction, and whether it is admitted as $n+1$-th term after a string of n already admitted choices. After n admitted choices there is always an admitted choice to continue the process.

A thus constructed sequence, that in general has no complete description, is called an element of the spread.

A simple example is the spread C, with founding sequence the natural numbers. The building rule is that each choice is admitted, for the first term as well as for every next. This spread is sometimes called the universal tree of natural numbers. Actually it is a description for the construction of such a tree.

In his German texts Brouwer used for spread the word Menge, which is the German word for set. Closer to the classical notion of set is Brouwer's *species*, which is a property that can be possessed by mathematical objects. An object possessing the property is an element of the species. A spread is a species, but a species need not be a spread. A simple example of a species which is not a spread are the rational numbers.

An element of a spread is constructed term by term. This process may be fixed from the first term onwards beforehand. Brouwer called these fixed elements sharp, or determined. We call them *lawlike*. What Brouwer's intuitionism gives its special character is that Brouwer accepts sequences not completely described as full-fledged elements. An element lacking a complete description we call *choice sequence*. In the most extreme case only an initial segment of an element is known. This is what Brouwer in 1918 uses to show that the spread C has a larger power than the natural numbers N, and thus solving his foundational problem from his thesis. He reasons as follows:

A function that assigns a natural number n to an element of C must fix that assignment on the basis of an initial segment of that element. But then every element of C sharing the same initial segment will be assigned the same natural number. So a $1-1$ function from N to C is impossible. Since a $1-1$ function from C to N is easily indicated, C has a larger power than N.[1]

In the first years of his intuitionistic reconstruction, as in his 1918 proof above, Brouwer only used global properties of a spread with choice sequences, he did not use a specific, a particular, choice sequence. That Brouwer thought about the use of such a sequence is proven by the notes he adds to the spread definition. The first is from 1925 where he noted at a similar phrase as 'A thus constructed sequence...' in our definition of spread:

> Including the feature of their freedom of continuation, which after each choice can be limited arbitrarily (possibly to being fully determined, but in any case according to a spread law) (Brouwer 1975, p. 302).

With the text of a lecture from 1927 an extension of this note was found:

> The freedom to proceed with a choice sequence can after every choice arbitrarily be restricted (possibly in dependence on events in the world of mathematical thought of the choosing person, imposed on the choosing person) (resulting e.g. in complete determination, or determination by a spread law).[2]

The note is much longer. The part we have deleted concerns the possible restrictions on choices and on restrictions. In Brouwer 1942 he changed his position, and in Brouwer 1952 even more so. He clearly struggled with this subject.

The standard text on choice sequences is Troelstra 1977, the result of research originated by Kreisel. It contains a host of technical results on formal systems of choice sequences. For these systems Troelstra proves elimination theorems, with as consequence that what can be proved with the choice sequences of the formal systems, can be stated with just lawlike sequences. Later, he would generalize this result to all choice sequences: they are dispensable (Troelstra 2001, p.227). This coincides with the fact that no instances of particular choice sequences had been found in the work of Brouwer, except one dubious example.

In his proof of the negative continuity theorem, preceding his famous and stronger continuity theorem in Brouwer 1927, Brouwer possibly uses a choice sequence. The proof is so vague that it allows many reconstructions. Troelstra discusses three possible ones in Troelstra 1982, and there are others, see Posy 1976, Martino 1985, Veldman 1982 and van Atten 2007, all different.

But what kind of sequence Brouwer used in this proof, it can not be characteristic for his use of particular choice sequences. For when he proved the theorem again, along the same lines but this time in a more explicit manner, he clearly uses a lawlike sequence, see Brouwer 1981, p. 81. And at that time, we claim, Brouwer was using particular choice sequences in an extensive way. But they were not recognized as such and have been misinterpreted.

The concept of choice sequence needed clarification. We found the key to this clarification in a solution of a paradox that arises from the misinterpretation. We shall show the misinterpretation, the paradox and our solution after the necessary definitions in the next section.

4 Real numbers

In intuitionism real numbers can be introduced by elements of the spread with founding sequence the rational numbers Q and as building rule

Each $q \in Q$ is admitted as first element; $q \in Q$ is admitted as successor a_{n+1} of a_n if $\mid a_n - q \mid < 2^{-n}$.

So elements of this spread are convergent sequences of rational numbers. Brouwer called them *points*. If for the points $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ the mutual difference between their terms converges to zero they are said to coincide. This is an equivalence relation, and its equivalence classes are Brouwer's real numbers. He called them *pointcores*. All handling of real numbers is done by means of points, and sometimes Brouwer treats the continuum as consisting of points.

An important distinction for the present article is the following: the lawlike elements generate the *reduced continuum*; all elements together generate the — *(full) continuum*.

In the examples below Brouwer will examine this distinction on basis of what he calls the naive order relation $<$. Let a be generated by $a_1, a_2, a_3, ...$ and b by $b_1, b_2, b_3, ...$ The relation $a < b$ holds if there exists an n_0 and an m_0 such that for all v $b_{n_0+v} - a_{n_0+v} > 2^{-m_0}$; so a positive distance between the terms from a certain index must be indicated.

This way of introducing real numbers is by no means unique. Another way to generate real numbers is by sequences of nesting rational intervals with their length converging to zero. Brouwer had a preference for $\lambda^{(n)}$-intervals, which are of the form $[\frac{a}{2^n}, \frac{(a+2)}{2^n}]$, a an integer, and n a natural number. Every $\lambda^{(1)}$ interval is admitted as first choice, a $\lambda^{(n)}$ is admitted as n-th term if it is contained in its predecessor. Two sequences are coincident if there is an overlap between any two of their terms, and the relation $<$ is defined in the natural way.

These two ways of introducing real numbers result in the same continuum; we will meet them both below.

5 The creating subject

After the Second World War Brouwer started to publish again allowing a delay of more then fifteen years. His papers of that period are characterized by a technique of drawing counterexamples against classical logic, with sequences constructed by a *creating subject*. The first example is from 1948, with the following definition:

> Let α be a mathematical assertion that cannot be tested, i.e. for which no method is known to prove either its absurdity or the absurdity of its absurdity. Then the creating subject can, in connection with this assertion α, create an infinitely proceeding sequence $a_1, a_2, a_3, \ldots$ according to the following direction: As long as, in the course of choosing the a_n, the

> creating subject has experienced neither the truth, nor the absurdity of α, a_n is chosen equal to 0. However, as soon as between the choice of a_{r-1} and a_r the creating subject has obtained a proof of the truth of α, a_r as well as a_{r+v} for every natural number v is chosen equal to 2^{-r}. And as soon as between the choice of a_{s-1} and a_s the creating subject has experienced the absurdity of α, a_s, as well as a_{s+v} for every natural number v is chosen equal to $-(2)^{-s}$. This infinitely proceeding sequence $a_1, a_2, a_3, \ldots$ is positively convergent, so it defines a real number ρ (Brouwer 1975, p. 478).

For this real number ρ Brouwer proves that $\rho \neq 0$ holds, but that neither $\rho > 0$ nor $\rho < 0$ can be proved. Although Brouwer mentioned in his introductory words that he had used this example from 1927 upwards in his lectures, the method Brouwer applied has always been supposed to be radically new. The expression 'creating subject', later changed in 'creative subject', for short CS, has always been interpreted as 'the idealized mathematician', all his mathematical activity covered by an ω sequence of discrete stages. Kreisel formalized this notion, which resulted in the Theory of the Idealized Mathematician, the TIM (Kreisel 1967). The basic term of this theory is, for a mathematical assertion ϕ,

$\Box_n \phi$: the CS has a proof of ϕ at stage n.

Analysing the properties of the idealized mathematician Kreisel proposed the following axioms:

Axiom 1: $\Box_n \phi \rightarrow \Box_{n+m} \phi$ for all m and n;
Axiom 2: $\Box_n \phi \vee \neg \Box_n \phi$ for all n;
Axiom 3: $\exists n \Box_n \phi \rightarrow \phi$;
Axiom 4: $\phi \rightarrow \exists n \Box_n \phi$.

This set of axioms is sufficient to obtain Brouwer's result. Actually, it is more. Let us have a look at the proof of '$\rho > 0$ does not hold' Brouwer gave after defining ρ, with α the assertion that cannot be tested.

> If for this real number ρ the relation $\rho > 0$ were to hold, then $\rho < 0$ would be impossible, so it would be certain α could never be proved to be absurd, so the absurdity of the absurdity of α would be known, so α would be tested, which it is not. Thus the relation $\rho > 0$ does not hold.[3]

Translated in the language of the TIM, Brouwer derives $\neg\neg\alpha$ from $\rho > 0$ with the implications

$\rho > 0 \rightarrow \neg \rho < 0$
$\neg \rho < 0 \rightarrow \neg \exists n \Box_n \neg \alpha$
$\neg \exists n \Box_n \neg \alpha \rightarrow \neg\neg\alpha$

As we may observe, from the axioms of the TIM Brouwer only uses the contra-position of Axiom 4: $\phi \rightarrow \exists n \Box_n \phi$, with $\neg\alpha$ for ϕ. If Brouwer had wanted to use Axiom 3: $\exists n \Box_n \phi \rightarrow \phi$ he could have shortened his proof by using the implications

$\rho > 0 \rightarrow \exists n \Box_\alpha$ and
$\exists n \Box_n \alpha \rightarrow \alpha$.

He would not only shorten his proof by using Axiom 3, he also would have no need to resort to an untested proposition, i.e. a proposition for which $\neg\neg\phi \vee \neg\phi$ does not hold. Instead, an undecided proposition, i.e. an assertion ϕ for which $\phi \vee \neg\phi$ does not hold, would have been sufficient. We conclude that Brouwer does not want to use Axiom 3.

Further doubts about the TIM, as a theory for reconstructing Brouwer's arguments, comes from a paradox Troelstra discovered in elaborating Kreisel's theory.

6 The paradox

Before we present the paradox we remark that sequences created by the CS were taken to be lawlike in the TIM. The argument is that Axiom 2: $\Box_n \phi \vee \neg \Box_n \phi$ suggests a complete description of such a sequence. To distinguish these from sequences not involving the CS, the CS-sequences were called empirical, and the usual lawlike ones absolutely lawlike or mathematical (Troelstra 1969, p. 97). The paradox arises as follows:

Suppose the CS proves his results one by one. In that case we can narrow the stages down in such a way that at each stage there is only one new result. Let in this narrowing $A^{(1)}, A^{(2)}, A^{(3)}, \ldots$ be an enumeration of the new results corresponding to the stages. So $\Box_n A^m$ holds if $n \geq m$, else $\neg \Box_n A^m$ holds. Now let L be a predicate such that

$L(\alpha)$ holds iff α is a lawlike sequence

and define

$c(n) = \beta(n) + 1$ iff $A^n = L(\beta)$ for some sequence β, and $c(n) = 0$ otherwise.

Since sequences based on the activity of the CS are taken to be lawlike, $L(c)$ holds. Because of Axiom 4: $\phi \rightarrow \exists n \Box_n \phi$, there exists an n_0 such that $A^{n_0} = c$, hence $c(n_0) = c(n_0) + 1$: a contradiction. See Troelstra 1969, pp. 106-107.

Troelstra proposes two ways out. The first is to equip these stages with a type structure of levels of self-reflection of the CS, the second is to drop the assumption 'one new result at each stage'. He does not elaborate these suggestions in Troelstra 1969, but in van Dalen and Troelstra 1988(p. 846) he judges neither of them satisfactory and he concludes: 'Summing up, we can say that the attempts to formalize the theory of the IM as envisaged by Brouwer cannot be said to be satisfactory examples of 'informal rigour".

In formulating the paradox Troelstra is now more cautious than he was in 1969. He formulates the paradox with $L(\alpha)$ holds iff α 'fixed by a recipe' instead of 'lawlike'. But Troelstra does not abandon his main argument from Troelstra 1969 for calling a CS sequence, and thus the $c(n)$, lawlike. That is the decidability of $\Box_n\phi$ expressed by the TIM Axiom 2: $\Box_n\phi \vee \neg\Box_n\phi$. Neither does he question the conception underlying the axioms of the TIM: an idealized mathematician, all his mathematical activity covered by a sequence of stages. This questioning is the key of the solution of Niekus 1987. We present it in the next section.

7 A new solution

Undeniable in Brouwer's definition of the CS sequence above is that there is a creating subject that can choose values for a sequence, and that these choices are dependent on the mathematical results of the CS between these choices, the stages. But why call this CS an 'idealized mathematician'? Since in Brouwer's view mathematics is a creation of the human individual, the expression 'creating subject' may mean nothing but just 'the mathematician'. So it can be 'I', or 'we'. We shall use 'we' for CS.

According to the interpretation of the TIM the stages cover the whole mathematical activity of the CS. With 'we' for CS we let the stages cover our future only, and not our complete mathematical activity. The paradox is then avoided as follows.

Narrow the stages in such a way that there is only one new result at each stage, and let $A^{(1)}, A^{(2)}, A^{(3)}, ...$ be the list of new results. Remark that now this is a list of our future new results. Define as above the predicate $L(\alpha)$ holds iff 'α is fixed by a recipe' and define $c(n) = \beta(n) + 1$ iff $A^n = L(\beta)$ for some sequence β, $c(n) = 0$ otherwise. Then $L(c)$ holds *now*, so will not occur in the list of future new results, and there is no diagonalization as above in section 6.

Interpreting Brouwer's example this way, i.e. 'we' for 'creating subject' and the stages covering our future only, Brouwer's definition becomes a description of the construction of a sequence. In order to describe Brouwer's example interpreted this way in the language of the TIM, we change the interpretation of its basic term. We suppose the future existing of a sequence of ω discrete stages and we define

$\Box_n\phi$ holds if and only if at the n-th stage from now we shall have a proof of ϕ.

This new interpretation of $\Box_n\phi$ enables us to make a distinction in proofs. ϕ holds if we have a proof of ϕ which can be carried out *here and now* completely. $\Box_n\phi$ may hold if ϕ holds now, but a proof of $\Box_n\phi$ can also depend on information coming free before stage n. Under this new interpretation the axioms

Axiom 1: $\Box_n \phi \to \Box_{n+m} \phi$ for all m and n, and
Axiom 4: $\phi \to \exists n \Box_n \phi$

of the TIM are obvious, but

Axiom 2: $\Box_n \phi \vee \neg \Box_n \phi$ for all n, and
Axiom 3: $\exists n \Box_n \phi \to \phi$

are no longer valid.

For Axiom 3, take the sequence $a_1, a_2, a_3, \ldots$ defined by Brouwer above, with α the untested proposition. With 'we' for the CS, and the stages covering the future only, this sequence depends on our future results in connection with trying to prove α. For any n_0 we do not have $a_{n_0} = 0 \vee a_{n_0} \neq 0$ *now* , because we do not know whether we have made any result on working on α before stage n_0. But in stage $n_0 + 1$ we shall know whether we made a result at stage n_0, so $\Box_{n_0+1}(a_{n_0} = 0 \vee a_{n_0} \neq 0)$ does hold. Consequently, Axiom 3: $\exists n \Box_n \phi \to \phi$ is not valid, and explains Brouwer's avoidance of the principle.

On similar grounds Axiom 2: $\Box_n \phi \vee \neg \Box_n \phi$ is not valid: $\Box_{n_0} a_{n_0} = 0 \vee \neg \Box_{n_0} a_{n_0} = 0$ does not hold because we do not know now whether we will have made any result working on α before stage n_0. So the argument to call the CS sequences lawlike fails.

This is exactly as it should be. The description of the construction does not fix the sequence completely, it is made to depend on our future mathematical results. It is not a lawlike sequence, it is a choice sequence. What Brouwer's so called new method makes special is not the introduction of an idealized mathematician, but the introduction of particular choice sequences in his mathematical practice.

Summarizing our point of view we claim:

- Brouwer is using particular choice sequences in the method of the creating subject. They are given by a description of a construction, their terms be made to depend on the future mathematical results of the constructor of the sequence.
- In his postwar papers Brouwer denotes this constructor by 'the creating subject'. This is not an idealized mathematician, but just 'the mathematician', and it can be replaced by 'we' or 'I'.
- The reasoning over the choice sequence is done on the basis of the description only, before the construction has started.
- In this reasoning the principle 3: $\exists n \Box_n \phi \to \phi$ is not valid.

In the remainder of this paper we will expose evidence for this point view in the work of Brouwer. We start our investigation with the text of his Berlin lecture from 1927, because of Brouwer's reference to that year in his 1948 paper.

8 Brouwer1927 - Berlin

In his 1927 Berlin lecture Brouwer examines whether the naive order relation $<$ we defined in the section 'real numbers' is an order relation on the intuitionistic continuum in the usual sense of linear order.[4] We start with the following:

> Further, we denote with K_1 the smallest natural number n with the property that the n-th up to the $(n+9)$-th digit in the decimal expansion of π form the sequence 0123456789, and we define as follows a point r of the reduced continuum: the n-th λ-interval λ_n is a $\lambda^{(n-1)}$-interval centred around 0, as long as $n < K_1$; however, for $n \geq K_1$ λ_n is a $\lambda^{(n-1)}$-interval centred around $(-2)^{-K_1}$. The point core of the reduced continuum generated by r is neither =0, nor < 0 nor > 0, as long as the existence of K_1 neither has been proved nor has been proved to be absurd. So until one of these discoveries has taken place the reduced continuum is not completely ordered (Brouwer 1991, pp. 31–32 [5].)

Given an algorithm for the decimal expansion of π the sequence above is lawlike. And the result counts for the reduced continuum as Brouwer points out. Remark the role of time. The thus constructed real was neither > 0, nor < 0 for Brouwer, *then and there*. For us it is > 0, since it has been found out that $K_1 = 17.387.594.880$ (Borwein 1998). Of course another similar example replacing his is easily made.

This example was not new, Brouwer already used it in a lecture in 1923 (Brouwer 1975, p. 270). But the next was new:

> Therefore we consider a mathematical entity or species S, a property E, and we define as follows the point s of the continuum: the n-th λ-interval λ_n is a $\lambda^{(n-1)}$-interval centred around 0, as long as neither the validity nor the absurdity of E for S is known, but it is a $\lambda^{(n)}$-interval centred around 2^{-m} (-2^{-m}), if $n \geq m$ and between the choice of the $(m-1)$-th and the m-th interval a proof of the validity (absurdity) of E for S has been found. The point core belonging to s is $\neq 0$, but as long as neither the absurdity, nor the absurdity of the absurdity of E for S is known, neither > 0 nor < 0. Until one of these discoveries has taken place, the continuum cannot be ordered (Brouwer 1991, p. 31–32).

Because of the result and the formulation (the double negation!) it can not be otherwise than that this is the example Brouwer is referring to in his 1948 article. And because of the distinction reduced continuum versus continuum the sequence s cannot be otherwise than a choice sequence. Note that Brouwer uses 'we' where he would later use the expression 'creating subject'.

Brouwer had found a way to apply individual choice sequences in his mathematical practice. It is this kind of choice sequence, depending on the mathematical experience of the constructor, mentioned in a note with his spread definition we showed in the section 'choice sequences', that Brouwer would explore to full extent in his post-war papers.

This example has not played a role in the research on choice sequences or on creating subject arguments, for it was not published until 1991. That is different for the next example.

9 Brouwer1928 - Vienna

A year after Berlin Brouwer lectured in Vienna. In the meantime he had generalized the technique based on the expansion of π used in the first Berlin example, by introducing the notion of a *fleeing property f for natural numbers.* It satisfies the following conditions: for each natural number it is decidable whether it possesses f or not, no natural number possessing f is known, and the assumption of the existence of a number possessing f is not known to be contradictory. The critical number λ_f of a fleeing property f is the smallest natural number possessing f.

Brouwer's standard example of a fleeing property is being the smallest k, the k-th up to the $k+9$-th digit in π's expansion of which form the sequence $0, 1, 2, \ldots, 9$, used in the first cited Berlin example above. The real number defined over there is an example of *a dual pendulum number.* As we mentioned in the previous section, for Brouwer's standard example the critical number has become known, so for us this property is not fleeing any more.

In Brouwer 1930 Brouwer examines the continuum on seven properties, all valid classically for the reals, but not intuitionistically. Whenever it is possible, he uses a lawlike sequence, as in the following example. With 0 replacing 1/2 it is the same as our first cited Berlin example.

> That the continuum (and also the reduced continuum) is *not discrete* follows from e.g. the fact that the number $1/2 + p_f$, where p_f is the dual pendulum number of the fleeing property f, is neither equal to 1/2, nor apart from 1/2.[6]

But if necessary he uses a choice sequence

> That $<$ is not *an order* on the continuum is demonstrated by the real number p, generated by the sequence (c_n), its terms chosen such that $c_1 = 0$ and $c_v = c_{v+1}$ with only the following exception. Whenever I find the critical number of some particular fleeing property f, I choose the next c_v equal to -2^{-v}, and when I find a proof this critical number does not exist, I choose c_v equal to 2^{-v}. This number p is unequal to 0, but nevertheless it is not apart from zero.[7]

The definition of fleeing property does not exclude that it is tested. Since testedness excludes one of the disjunctive parts of the definition, we may fairly assume that Brouwer presupposed it, and so this example is the same as the 1948 one. Note that Brouwer uses 'I' this time.

In two other cases Brouwer used a choice sequence to demonstrate the difference between the classical and the intuitionistic continuum. It is remarkable that Brouwer after his first full use of individual choice sequences, stopped publishing for more than fifteen years.

10 Brouwer1934 - Geneva

Brouwer stopped publishing after 1930 but he did not stop lecturing. Brouwer 1934 is the text of his Geneva lectures of 1934. This manuscript is not prepared for publication at all, it is full of crossings out and improvements, and Brouwer's tone is loose. Most important is that there is no other place in the work of Brouwer were he is so explicit about the difference between lawlike sequences and choice sequence.

After the introduction of the continuum existing of points, not pointcores, Brouwer examines the continuum on order. He starts with the choice sequence:[8]

> The n-th interval $\lambda^{(n-1)}$ is of length $2/(n-1)$ and centred around 0. This is how one starts. But at the same time one works on a difficult problem, to know whether the property E for a species S is true, for example Fermat's problem. If for that problem a solution has been found between the $(n-1)$-th and the n-th choice, the choice of the intervals will be different.[9]

As we pointed out in the example of Brouwer (1930), if E could be tested, one of the possibilities in the definition would be excluded.

> If the property is true for the species S, then the v-th interval will be for $v \geq n$ the interval $\lambda^{(v)}$ centered around 2^{-n}. The next interval will be placed according to this law, within its predecessor with the same centre. If, on the other side, one finds that the property E is absurd for the species S, than the intervals will be centred around -2^{-n}.[10]

For the thus defined point s Brouwer proves that $s = 0$ is impossible, but neither $s > 0$ nor $s < 0$ can be proved. If their is any doubt that Brouwer uses a choice sequence here, the following remark seems to be decisive:

> We show the same for the reduced continuum. The point above is not a sharp point, because the construction is not completely determined, but depends on the intelligence of the constructor relative to the posed problem.[11]

That $>$ is not an order on the reduced continuum (actually Brouwer only shows that it not a complete order) is shown by the point r defined with the decimal expansion of π, just as in Berlin. About the difference between this point r and s, the choice sequence above, Brouwer remarks:

> When one hundred different persons are constructing the number r, one is always certain that any interval chosen by one of these persons is always covered, at least partly, by every interval chosen by one of the others. That is different for s. If I would give the definition of s to one hundred different persons, who are all going to work in a different room, it is possible that one of these one hundred persons at one time will choose an interval not covered by an interval chosen by one of the others.[12]

We think it is an important quote. Above we saw some uncertainty when to call a sequence lawlike ('fixed by a recipe'). Brouwer gives a clear condition here: different persons should come up with the same result. But above all, this fragment supports our conception of a choice sequence. It is given by a definition that does not determine the terms completely: different persons can reach different results. The reasoning is done on basis of the incomplete description only. It is exactly this incompleteness that makes the strong results against classical mathematics possible.

That in this reasoning there is no place for Axiom 3: $\exists n \Box_n \phi \rightarrow \phi$ is shown in the next section by our final quotes.

11 Brouwer1949 - Handwritten note

As we have showed in section 4, Brouwer avoids the use of Axiom 3: $\exists n \Box_n \phi \rightarrow \phi$ in his proof, and we have rejected it in section 6 on the basis of our interpretation of $\Box_n \phi$. The following text contains an even more decisive argument against Axiom 3. It is from a handwritten note found in Brouwer's papers referring to his proof of Brouwer 1949, translated from the Dutch by A.Heyting.

> Further distinctions in connection with the excluded middle.
>
> $\overline{a}$ will mean: a is non-contradictory.
>
> $\underline{a}$ will mean: a is contradictory
>
> [...]
>
> The principle of testability can assert: either: from now on either $\overline{a}$ or $\underline{a}$ holds, notation: $|a$. Or: from a certain moment in future on either $\overline{a}$ or $\underline{a}$ will hold, notation $a|$.
>
> Then $a|$ is non-contradictory, but $|a$ need not to be non-contradictory (Brouwer 1975, pp. 603-604).

Of course $\overline{a}$ is $\neg\neg a$, $\underline{a}$ is $\neg a$, $|a$ is $\neg\neg a \vee \neg a$, and $a|$ is nothing but our $\exists n \Box_n (\neg\neg a \vee \neg a)$. So in his attempt to handle choice sequences, Brouwer introduces a distinction between what holds here and now, and what holds from a certain moment in he future, which is the same distinction we make. Brouwer continues his note with:

> For instance, let p be a point of the continuum in course of development, whose continuation is free at this moment, but may be restricted at any moment in the future; then (p is rational)| is non-contradictory, but |(p is rational) is contradictory, for the complete freedom which exists at this moment makes it impossible to be sure that the rationality of p is contradictory, but also to be sure that it is contradictory that the rationality of p is contradictory (Brouwer 1975, pp. 603-604).

We shall discuss this example at another time. At the end of this note Brouwer expresses doubts about introducing $a|$ as a mathematical notion, without further argument. But in his creating arguments from around that time and later, he adds to 'α can not be tested' also 'and α is not recognized as testable' (e.g. Brouwer 1975, p. 491 and p. 525). For this last expression we see no other candidate than Brouwer's $a|$, which is our $\exists n \Box_n (\neg\neg a \vee \neg a)$. Finally, that Brouwer, given the distinction, would accept Axiom 3: $\exists n \Box_n \phi \rightarrow \phi$ is of course out of question. For stating that $\neg\neg a \vee \neg a$ is contradictory and that $\exists n \Box_n (\neg\neg a \vee \neg a)$ is not, refutes Axiom 3 in a very strong way.

12 Conclusions

The introduction of choice sequences, called incomplete objects in Niekus 2010, has made Brouwer's intuitionism unique, also within constructivism. Initially Brouwer only used global properties of choice sequences, as in his reaching the continuum from the discrete. But later he found a way to apply particular choice sequences in his mathematical practice.

The particular choice sequences Brouwer used are given by the description of a construction of a sequence. The description does not determine the sequence completely. The terms of the choice sequences are made to depend on the mathematical discoveries of the constructor of the sequence. The reasoning over a choice sequence is done on the basis of the incomplete description only, before the construction has started, and is independent of the constructor.

Brouwer started the use of these sequences in the nineteen twenties and he came to full exploitation of the concept in his papers after the Second World War. In these postwar papers he denoted the constructor of a choice sequence by 'creating subject'. We have concentrated on one example, but there are others. In Brouwer 1930 (these are also in Niekus 2002), in Brouwer 1948, in Brouwer 1949, and in Brouwer 1954. As far as we know these other examples have never been studied seriously. It seems to us that an understanding of these examples is blocked by the intuitionistic standard theories.

In the intuitionistic textbooks, for example Heyting 1956, van Dalen and Troelstra 1988 and Dummett 2000 Brouwer's choice sequences are treated in the last few pages, often as a controversial subject, misinterpreted in the theory of the idealized mathematician. While considerable room is dedicated to Troelstra's choice sequences. In van Atten 2004 creating subject sequences are recognized as being non-lawlike, but they are treated after the choice sequence chapter, without abandoning the standard theory TIM, clearly as a controversial subject. In van Atten 2007, a study on the phenomenology of choice sequences, Brouwer's particular choice sequences, except the dubious example of section 3, are not even mentioned.

We claim that our conception of a choice sequence, arising from our solution of Troelstra's paradox, opens up a most characteristic part of Brouwer's mathematics for further research.

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Joop Niekus, Amsterdam 29-12-2016, j.m.niekus@uva.nl

Notes

[1]The original text, with A denoting the natural numbers is: 'Die Menge C ist größer als die Menge A. Ein Gesetz, das jedem Elemente g von C ein Element h von A zuordnet, muss nämlich das Element h vollständig bestimmt haben nach dem Bekanntwerden eines gewissen Anfangssegmentes α der Folge von Ziffernkomplexen von g. Dann aber wird jedem Elemente von C, welches α als Anfangssegment besitzt, dasselbe Element h von A zugeordnet. Es ist mithin unmöglich, jedem Elemente von C ein verschiedenes Element von A zuzuordnen. Weil man anderseits in mannigfacher Weise jedem Elemente von A ein verschiedenes Element von C zuordnen kann, so ist hiermit der aufgestellte Satz bewiesen.' (Brouwer 1918, p. 18.)

[2] The text is from Troelstra 1982 p. 473. The author remarks that the note has been found in the Brouwer Archive together with the text of Brouwer's Berlin lecture from 1927, and that it has certainly been written before the Second World War. It is not included in Brouwer 1991, which contains the text of that lecture.

[3]The remainder of Brouwer's proof, i.e. that $\rho < 0$ does not hold and that $\rho = 0$ is contradictory, is as follows: 'Further, if for the real number ρ the relation $\rho < 0$ were to hold, then $\rho > 0$ would be impossible, so it would be certain that α could never be proved to be true, so the absurdity of α would be known, so again α would be tested, which it is not. Thus the relation $\rho < 0$ does not hold. Finally let us suppose that the relation $\rho = 0$ holds. In this case neither $\rho < 0$ nor $\rho > 0$ could ever be proved, so neither the absurdity nor the truth of α could ever be proved, so the absurdity as well as the absurdity of the absurdity of α would be known. This is a contradiction, so the relation $\rho = 0$ is absurd, in other words the real numbers ρ and 0 are different.' (Brouwer 1975 pp. 478-479).

[4] A species is ordered by a relation R if R fulfils the following conditions: 1. If $a = u$, $b = v$ and aRb, than also uRv; 2. If aRb and bRc, than also aRc. If, moreover for any elements a and b of the species aRb, or $a = b$ or bRa holds, R is called a complete order.

[5]The two citations in this section were interwoven in the text of the lecture, which is:

'Dazu betrachten wir eine mathematische Entität oder Species S, eine Eigenschaft E, und definieren wie folgt den Punkt s des Kontinuums: Das n-te λ-Intervall λ_n ist eine symmetrisch um den Nullpunkte gelegenes $\lambda^{(n-1)}$-Intervall, so lange man die Gültigheit noch die Absurdität von E für S kennt, dagegen ist es ein symmetrisch um den Punkt 2^{-m}, bzw. um den Punkt 2^{-m} gelegenes $\lambda^{(n)}$-Intervall , wenn $n \geq m$ und zwischen der Wahl des $(m-1)$-ten und der Wahl des m-ten Intervalles ein Beweis der Gültigkeit bzw. der Absurdität von E für S gefunden worden ist.

Weiter bezeichnen wir mit k_1 die kleinste natürliche Zahl n mit der Eigenschaft, daß die n-te bis $(n+9)$-te Ziffer der Dezimalbruchenentwicklung von π eine Sequenz 0123456789 bilden und dazu definieren wir wie folgt den Punkt r des reduzierten Kontinuums: Das n-te λ-Intervall λ_n ist ein symmetrisch um den Nullpunkt gelegenes λ^{n-1}-Intervall, solange $n < k_1$; für $n \geq k_1$ aber ist λ_n das symmetrisch um den Punkt $(-2)^{k_1}$ gelegene λ^n-Intervall.

Alsdann ist der zu s gehörende Punktkern des Kontinuums $\neq 0$, aber solange man weder die Absurdität noch die Absurdität der Absurdität von E für S kennt, weder > 0 noch < 0. Bis zum stattfinden einer dieser beide Entdeckungen kann also das Kontinuum nicht geordnet sein.

Weiter ist der zu r gehörende Punktkern des reduziertes Kontinuums, solange die Existenz von k_1 weder bewiesen noch noch ad absurdum geführt ist, weder $= 0$, noch > 0, noch < 0. Bis zum stattfinden einer dieser beiden Entdeckungen ist also das reduzierte Kontinuum nicht vollständig geordnet' (Brouwer 1991, pp.31-32

[6] 'Daß das Kontinuum (und ebenso das reduzierte Kontinuum) nicht diskret ist, folgt z. B. daraus, daß die Zahl $1/2 + p_f$, wo p_f die duale pendelzahl der fliehenden Eigenschaft f vorstellt, weder gleich 1/2 noch von 1/2 verschieden ist' (Brouwer 1975, p. 435).

[7]'Daß das Kontinuum durch die der Anschauung entnommene Reihenfolge ihrer Elemente nicht geordnet ist, erweist sich am Elemente p, für dessen bestimmte konvergente Folge $c_1, c_2, ...c1$ im Nullpunkt und jedes $c_{v+1} = c_v$ gewählt wird, mit der einzige Ausnahme, daß ich, sobald von einer bestimmten fliehenden Eigenschaft f mir eine L"ösungszahl λ_f bekannt wird, das nächste c_v gleich -2^{-v-1} wähle, und daß ich, sobald mir eine Beweis der Absurdität dieser Lösungszahl bekannt wird, das nächste c_v gleich 2^{-v-1} wähle. Dieses Element p ist von Null verschieden, ist aber trotzdem weder kleiner als Null noch größer als Null' (Brouwer 1975, p. 435-436).

[8]Brouwer 1934 consists of six parts, probably corresponding to six lectures. All citations of this section are from the second part, pp. 22-26, translated by the author from the French original.

[9]'Le n-ieme intervalle $\lambda^{(n-1)}$ est de longueur $2/(n-1)$ et il est centré a l'origine. C'est ainsi qu'on commense. Mais en même temps on se pose un problème difficile, a savoir si la propriete E est vraie pour une espace S, par example le problème de Fermat. L'on trouvait une solution de ce problême entre le $(n-1)$-ieme choix et le n-ieme, alors a commencer du n-ieme intervalle, on choisirait cet intervalle d'une autre maniere.'

[10]'Si la propriété est vraie pour l'espce S, alors le v-ieme intervalle sera pour $v \leq n$ l'intervalle $\lambda^{(v)}$ centre au point 2^{-n}. L'intervalle suivant sera situé en vertu de cette loi, dans le précèdent et aura le même centre. Si, au contraire, on trouvait que la propriété E est absurde pour l'espece S, alors centrerait les intervalles autour du point -2^{-n}'.

[11]'Nous demomtrerons la même chose pour le continu reduit. Le point de tout a l'heure n'est pas un point predestiné, puisqu'il reste dans sa construction quelque chose qui n'est pas entierement determine mais dependre de l'intelligence du constructeur relativement au probleme pose'.

[12]'Si cent personnes differentes s'occupent de la construction du nombre r, on est toujours sur que chaque intervalle choisi par une de ces personnes, sera couvert, du moins a partie, par chaque intervalle choisi par une autre de ces personnes. Il n'est pas de même pour s. Si je donne la definition de s a cent personne differentes, qui travaillent dans un local different, il se peut que l'une de ces cent personnes choisisse une fois un intervalle ne couvrait pas l'intervalle choisi par une autre de ces personnes.lintervalle choisipar une autre de ces personnes.'

References

van Atten, M. (2004). *On Brouwer*. Wadsworth Philosophers Series. Thomson Learning, London.

van Atten, M. (2007). *Brouwer meets Husserl*. Synthese Library 335. Springer, Dordrecht, The Netherlands.

Borwein, J. M. (1998). Brouwer-Heyting sequences converge. *The Mathematical Intelligencer*, 20:14–15.

Brouwer, L. E. J. (1907). *Over de Grondslagen van de Wiskunde*. Mathematisch Centrum.

Brouwer, L. E. J. (1918). Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossen Dritten. Erster Teil: Allgemeine Mengenlehre. *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam*, 5:1–43. Also Brouwer (1975), 150–191.

Brouwer, L. E. J. (1927). Über Definitionsbereiche von Funktionen. *Mathematische annalen*, 97:60–75. Also Brouwer (1975), 390–405.

Brouwer, L. E. J. (1930). Die Struktur des Kontinuums. Vienna: Gottlieb Gistel. Also *Brouwer (1975)*, 429–440.

Brouwer, L. E. J. (1934). Geneva lectures 1934. Brouwer Archive BMS 44 - unpublished.

Brouwer, L. E. J. (1942). Zum freien werden von mengen und functionen. *Proceedings Koninklijke Nederlandse Akademie van Wetenschappen*, 45:107–108. Also in Brouwer (1975) 459–460.

Brouwer, L. E. J. (1948). Conciousness, philosophy and mathematics. In *Proc. 10th Intern. Congress Philosophy*, pages 1235–1249, Amsterdam. Also in Brouwer (1975), 480–495.

Brouwer, L. E. J. (1949). The non-equivalence of the constructive and the negative order relation on the continuum. 11:37–39. Also in Brouwer (1975),495-497.

Brouwer, L. E. J. (1952). Historical background, principles and methods of intuitionism. *South African J. Sci.*, 49:139–146. Also in Brouwer (1975), 508–516.

Brouwer, L. E. J. (1954). Points and spaces. *Canadian Journal of Mathematics*, 6:1–17. Also in Brouwer (1975), 522–538.

Brouwer, L. E. J. (1975). *Collected Works*, volume I. North Holland Publishing Company. Editor A. Heyting.

Brouwer, L. E. J. (1981). *Brouwer's Cambridge lectures on intuitionism.* Cambridge University Press. Editor D. van Dalen.

Brouwer, L. E. J. (1991). *Intuitionismus.* B.I. Wissenschaftverlag. Editor D. van Dalen.

van Dalen, D. and Troelstra, A. S., editors (1982). *The L. E. J. Brouwer centenary symposium.* North Holland Publishing Company.

van Dalen, D. and Troelstra, A. S. (1988). *Constructivity in mathematics*, volume I and II. North Holland Publishing Company, Amsterdam.

Dummett, M. (2000). *Elements of intuitionism, second edition.* Oxford Logic Guides 39. Clarendon Press, Oxford, second edition.

Heyting, A. (1956). *Intuitionism, an introduction.* North-Holland Publishing Company, Amsterdam.

Kreisel, G. (1967). *Informal rigour and completeness proofs.* in *Lakatos (1967)*, 138–186.

Lakatos, I., editor (1967). *Problems in the philosophy of mathematics*, volume I. North-Holland Publishing Company. Proceedings of the International Colloquium in the Philosophy of Science, London, 1965.

Martino, E. (1985). On the Brouwerian concept of negative continuity. *Journal of Philosophical Logic*, 14:379–398.

Niekus, J. M. (1987). The method of the creative subject. *Proceedings of the Koninklijke Akademie van Wetenschappen te Amsterdam*, Series A(4):431–443.

Niekus, J. M. (2002). Individual choice sequences in the work of L. E. J. Brouwer. PP 2002-05, ILLC.

Niekus, J. M. (2010). Brouwer's incomplete objects. *History and Philosophy of Logic*, 31(1):31–46.

Posy, C. (1976). Varieties of indeterminacy in the theory of general choice sequences. *Journal of Philosophical Logic*, 5:431–443.

Troelstra, A. S. (1969). *Principles of intuitionism*, volume 95 of *Springer Lecture Notes*. Springer Verlag.

Troelstra, A. S. (1977). *Choice sequences*. Oxford Logic Guides. Clarendon Press.

Troelstra, A. S. (1982). On the origin and development of Brouwer's concept of choice sequences. in van Dalen and Troelstra (1982), pp.465–486.

Troelstra, A. S. (2001). Honderd jaar keuzerijen. Verslag van de gewone vergadering van de afdeling natuurkunde, 24-09-2001, volume 110, nr.6, 223–227. Lecture for the ordinary meeting of the section physics of the Royal Dutch Academy of Science, 24-09-2001.

Veldman, W. (1982). On the continuity of functions in intuitionistic real analysis. Technical Report 8210, KU Nijmegen.