A simple propositional calculus for compact Hausdorff spaces

G. Bezhanishvili[*] N. Bezhanishvili[†] T. Santoli[‡] Y. Venema[§]

Abstract

We introduce a simple propositional calculus for compact Hausdorff spaces. Our approach is based on de Vries duality. The main new connective of our calculus is that of strict implication. We define the strict implication calculus SIC as our base calculus. We show that the corresponding variety SIA of strict implication algebras is a discriminator and locally finite variety. We prove that SIC is strongly sound and complete with respect to the universal subclass RSub of SIA, where the modality $\Box$ associated with the strict implication only takes on the values of 0 and 1. We develop Π_2-rules for strict implication algebras, and show that every Π_2-rule defines an inductive subclass of RSub. We prove that every derivation system axiomatized by Π_2-rules is strongly sound and complete with respect to the inductive subclass of RSub it defines. We introduce the de Vries calculus DVC and show that it is strongly sound and complete with respect to the class of compingent algebras, and then use MacNeille completions to prove that DVC is strongly sound and complete with respect to the class of de Vries algebras. We then utilize de Vries duality to introduce topological models of our calculus, and conclude that DVC is strongly sound and complete with respect to the class of compact Hausdorff spaces. We also develop strongly sound and complete calculi for zero-dimensional and connected compact Hausdorff spaces, and give a general criterion of admissibility for Π_2-rules. We finish the paper by comparing our approach to the existing approaches in the literature that are related to our work.

1 Introduction

Logic, algebra, and topology Extending Stone's seminal representation theorems for Boolean algebras [31] and distributive lattices [32], categorical dualities linking algebra and topology have been of fundamental importance in the development of the 20th century mathematics in general [23], and of logic and theoretical computer science in particular [20]. With algebras corresponding to the syntactic, deductive side of logical systems, and topological spaces to their semantics, Stone-type duality theory provides an elegant and useful mathematical framework for studying various properties of logical systems. In many particular cases, one sees natural specimens of logics, classes of algebras, and classes of topologies coming together. Out of a multitude of examples of such triples, we mention: (i) classical logic/Boolean

[*]NMSU, Las Cruces. Email: guram@nmsu.edu
[†]ILLC, Universiteit van Amsterdam. Email: N.Bezhanishvili@uva.nl
[‡]Mathematical Institute, University of Oxford. E-mail: thsantoli1@gmail.com.
[§]ILLC, Universiteit van Amsterdam. E-mail: y.venema@uva.nl.

algebras/Stone spaces [3]; (ii) intuitionistic logic/Heyting algebras/Esakia spaces [15]; (iii) geometric logic/spatial frames/sober spaces [36]; and (iv) modal logic/modal algebras/topological Kripke frames [8].

Our aim is to add to the study of these 'logic/algebra/topology' triples by providing a simple logical calculus for reasoning about compact Hausdorff spaces—a widely studied class of spaces, properly containing the class of Stone spaces. We do this by generalizing the classical setting of

classical logic/Boolean algebras/Stone spaces.

Namely, we extend the classical propositional language with a new logical connective of *strict implication*, which admits a natural topological interpretation; and we design a calculus consisting of finitely many axioms and rules that is sound and complete with respect to a category of algebras which is dual to the category KHaus of compact Hausdorff spaces and continuous maps. The resulting triple is reminiscent of the

modal logic/modal algebras/topological Kripke frames

triple, but has a different slant.

Over the years, a great number of dualities have been established for KHaus. Already the 1940s saw the development of Kakutani-Krein-Yosida duality [26, 27, 38], and of Gelfand-Naimark-Stone duality [19, 33], which have a distinctive ring-theoretic flavor. Later years saw the appearance of dualities with a more logical flavor, such as de Vries duality [13] and Isbell duality [21]. Unfortunately, however, none of the dually equivalent categories in the above-mentioned examples is finitary. In fact, there is no properly finitary dual algebraic category for KHaus [2].

De Vries algebras, finitarily Given the unavailability of a finitary duality for KHaus, and our goal of finding a finitary logical calculus for compact Hausdorff spaces, a natural approach is to see whether one of the dual algebraic categories can be somehow *approximated* by a finitary logic. In line with our earlier work [7], we will utilize de Vries duality, and work with the category DeV of de Vries algebras. Objects of this category are complete Boolean algebras B with a special binary relation $\prec$ (called by de Vries a *compingent relation*) satisfying certain conditions that resemble the definition of a proximity on a set [29] (precise definitions will be given in the next section).

In order to design a sound and complete finitary logical calculus for de Vries algebras, we need to overcome three obstacles. First of all, de Vries algebras are formally not algebras, due to the presence of the binary relation $\prec$. But any binary relation R on a Boolean algebra B can be equivalently described by means of its *characteristic function* $\chi_R : B \times B \to \{0, 1\}$. Since we may identify the values 0 and 1 of this map with the top and bottom element of B itself, we may actually represent the binary relation $\prec$ by means of the binary operation $\rightsquigarrow := \chi_\prec$ that we call a *strict implication* [7, Sec. 3]. Second, the fact that de Vries algebras are *complete* as Boolean algebras means that algebraically their signature involves infinite disjunctions. This problem can be solved rather easily by simply dropping the completeness condition from the definition; following de Vries himself, we call the resulting structures *compingent algebras*. As we will see, de Vries algebras can then be realized as MacNeille completions of compingent algebras. Consequently, our finitary calculus is in fact sound and complete with respect to both classes of algebras, compingent algebras and de Vries algebras. Third, when designing a sound and complete logical calculus for compingent algebras, we will see that two of the defining conditions of strict implication are not given by equations or even universal conditions, but rather, they are expressed by universal-existential statements or Π_2-statements. We overcome this final obstacle by first considering a wider class of algebras satisfying the universal part of the axiomatization of compingent algebras, and then adding particular *non-standard inference rules* which, as we will show, correspond to Π_2-statements. As a result, we obtain a sound and complete logical calculus for compingent algebras; we then use MacNeille completions to obtain completeness of the calculus with respect to de Vries algebras; and finally, we utilize de Vries duality to yield completeness for compact Hausdorff spaces.

Contribution of this paper The propositional calculus that we design is based on the extension of the language of classical propositional logic with a single binary connective $\rightsquigarrow$ of strict implication, which constitutes the backbone of our approach. For the topological interpretation of this language, we define a *compact Hausdorff model* as a pair (X, v) consisting of a compact Hausdorf space X and a valuation v assigning a regular open subset of X to each propositional letter, where we recall that $U \subseteq X$ is regular open if $\mathsf{Int}(\mathsf{Cl}(U)) = U$. The Boolean connectives of the language are then interpreted as the corresponding operations in the (complete) Boolean algebra $\mathcal{RO}(X)$ of regular open subsets of X, while the strict implication $\rightsquigarrow$ is interpreted as the binary operation $\rightsquigarrow: \mathcal{RO}(X) \times \mathcal{RO}(X) \to \mathcal{RO}(X)$ given by

$$U \rightsquigarrow V = \begin{cases} X, & \mathsf{Cl}(U) \subseteq V \\ \varnothing, & \text{otherwise.} \end{cases}$$

We introduce a finitary derivation system and prove that it is sound and complete with respect to the class of compact Hausdorff models. The proof is based on de Vries duality for compact Hausdorff spaces, and the fact that all but two axioms defining a strict implication can easily be rewritten as formulas, while the remaining two axioms can be rewritten as the so-called non-standard rules (we call them Π_2-rules due to the universal-existential statements they correspond to). The use of non-standard rules in modal logic is not new. One of the pioneers of this approach was Gabbay [17], who introduced a non-standard rule for irreflexivity. A precursor to this work was Burgess [9] who used such rules in the context of branching time logic. We also refer to [18] for the application of non-standard rules to axiomatize the logic of the real line in the language with the Since and Until modalities, and to [34] for a general completeness result for modal languages that are sufficiently expressive to define the so-called difference modality. Our approach is closest to that of Balbiani et al. [1] who use similar non-standard rules in the context of region-based theories of space (see below for a more detailed comparison to their work).

We introduce the variety SIA of strict implication algebras and show that it is a discriminator and locally finite variety. We mainly work with the class RSub of subdirectly irreducible strict implication algebras, which turns out to be a universal class. We show that Π_2-rules define subclasses of RSub axiomatized by universal-existential statements, and that every derivation system axiomatized by Π_2-rules is sound and complete with respect to the subclass of RSub it defines. We also give a criterion of when a Π_2-rule is admissible. Finally, we apply the developed theory of Π_2-rules to the derivation system for compingent algebras. We define the MacNeille completion of a compingent algebra, and show that it is a de Vries algebra. From this we deduce that the derivation system for compingent algebras is strongly sound and complete with respect to the class of de Vries algebras. By de Vries duality, this yields that the derivation system is strongly sound and complete with respect to the class of compact Hausdorff spaces. We also design finitary derivation systems that are strongly sound and complete for zero-dimensional and connected de Vries algebras, and hence for zero-dimensional and connected compact Hausdorff spaces. Finally, we show that the two non-standard rules used in the derivation system for compingent algebras are admissible.

Related work We next discuss several lines of research that are related to our work. The connection between compingent relations and strict implications on Boolean algebras was first discussed in [16]. Compingent relations on Boolean algebras generalize naturally to contact and pre-contact relations [14]. The dual concept of the latter is that of subordination [7]. In turn, pre-contact relations and subordinations are in 1-1 correspondence with quasi-modal operators of Celani [10]. In a different direction, proximity-like relations on Boolean algebras have an obvious generalization to lattices and posets [20]. One such relevant concept is that of proximity lattice (see, e.g., [37, 30, 25]). Jung et al. [24] and Moshier [28] developed the corresponding sequent calculi. One of the calculi in [28] is designed for compact Hausdorff spaces, but the approach of [28] and the duality it is based on is rather different from de Vries duality, and hence from our approach.

As we already pointed out, our approach is most closely related to that of Balbiani et al. [1], which in many ways inspired the current paper. Balbiani et al. develop two-sorted logical calculi for region-based theories of space. We instead use a simpler calculus, which extends the classical propositional calculus with one binary connective of strict implication. In Section 9 we establish a full and faithful translation from the two-sorted language of [1] to our language. The simplicity of our language allows the universal algebraic treatment of our framework, which we undertake in Section 3.

Our approach involving non-standard rules is similar to the one taken in [1]. In the body of the paper we will indicate which of our results have the same flavor as theirs. Going beyond [1], we give a general notion of a Π_2-rule, connect it to inductive classes of algebras, and prove a general soundness and completeness result for every derivational system axiomatized by these rules. We also give a general criterion of admissibility for Π_2-rules, show that the MacNeille completion of a compingent algebra is a de Vries algebra, and use this result to prove completeness of the corresponding derivational system with respect to de Vries algebras. Completeness with respect to compact Hausdorff spaces then follows from de Vries duality.

Overview The paper is organized as follows. In the next section we provide some basic information on compingent algebras, de Vries algebras, and de Vries duality. We also discuss subordinations and contact relations, and how to represent them by means of strict implications. In Section 3 we study our base variety SIA of strict implication algebras. In particular, we prove that it is a discriminator and locally finite variety. In Section 4 we introduce and study the strict implication calculus SIC, which corresponds to the variety of strict implication algebras. We prove that SIC is strongly sound and complete with respect to SIA, as well as with respect to the class RSub of subdirectly irreducible strict implication algebras. In Section 5 we introduce Π_2-rules and show that they correspond to inductive subclasses of RSub. In Section 6 we use two specific Π_2-rules to define the de Vries calculus DVC, which is strongly sound and complete with respect to the class of compingent algebras. Then, using MacNeille completions, we prove that DVC is also strongly sound and complete with respect to the class of de Vries algebras. In Section 7 we use de Vries duality to evaluate our language in compact Hausdorff spaces. This leads to the definition of topological semantics for our language; and the completeness results of the previous section yield strong completeness of our system with respect to the class of compact Hausdorff spaces. We also design strongly sound and complete derivation systems for zero-dimensional and connected compact Hausdorff spaces. In Section 8 we give a model-theoretic criterion for admissibility of Π_2-rules, using which we prove that the two Π_2-rules of our system are admissible. Finally, in Section 9, we give a detailed comparison of our work to that of Balbiani et al. [1].

2 Preliminaries

As mentioned in the introduction, the dual category of KHaus that we will be working with is the category DeV of de Vries algebras and de Vries morphisms. In this short preliminary section we provide the formal definition of the objects of this category, discuss their connection to compact Hausdorff spaces, and provide a presentation of de Vries proximity relations by means of strict implications.

Definition 2.1.

(1) A *compingent relation* on a Boolean algebra B is a binary relation $\prec$ satisfying the following conditions:

- (S1) $0 \prec 0$ and $1 \prec 1$;
- (S2) $a \prec b, c$ implies $a \prec b \wedge c$;
- (S3) $a, b \prec c$ implies $a \vee b \prec c$;
- (S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
- (S5) $a \prec b$ implies $a \leq b$;
- (S6) $a \prec b$ implies $\neg b \prec \neg a$;
- (S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;
- (S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

(2) A *compingent algebra* is a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a compingent relation on B.

(3) Compingent relations on complete Boolean algebras are called *de Vries proximities*, and *de Vries algebras* are compingent algebras whose underlying Boolean algebra is complete.

Remark 2.2. In presence of (S6), (S2) and (S3) are interdefinable. If $(B, \prec)$ is a de Vries algebra, then (S8) is equivalent to $a = \bigvee\{b \mid b \prec a\}$.

For a compact Hausdorff space X, let $\mathcal{RO}(X)$ be the complete Boolean algebra of regular open subsets of X. Define $\prec$ on $\mathcal{RO}(X)$ by

$$U \prec V \text{ iff } \mathsf{Cl}(U) \subseteq V.$$

Then $(\mathcal{RO}(X), \prec)$ is a de Vries algebra. Conversely, suppose $(B, \prec)$ is a compingent algebra. A *round filter* of $(B, \prec)$ is a filter F of B satisfying $a \in F$ implies $\exists b \in F$ with $b \prec a$. An *end* of $(B, \prec)$ is a maximal proper round filter. Let X be the set of ends of $(B, \prec)$. For $a \in B$, let

$$\beta(a) = \{x \in X \mid a \in x\}.$$

Then $\{\beta(a) \mid a \in B\}$ generates a compact Hausdorff topology on X. Moreover, if X is compact Hausdorff, then X is homeomorphic to the dual of $(\mathcal{RO}(X), \prec)$; if $(B, \prec)$ is a compingent algebra and X is its dual, then $(B, \prec)$ embeds in $(\mathcal{RO}(X), \prec)$; and $(B, \prec)$ is isomorphic to $(\mathcal{RO}(X), \prec)$ iff $(B, \prec)$ is a de Vries algebra. These correspondences extend to contravariant functors, which yield a dual equivalence of the categories KHaus and DeV. We refer to [13] for missing details and proofs.

Axioms (S1)–(S6) of Definition 2.1 are universal statements, while axioms (S7), (S8) are universal-existential statements. By deleting them we arrive at the concepts of subordination and contact relation.

Definition 2.3.

(1) A *subordination* on a Boolean algebra B is a binary relation $\prec$ satisfying (S1)–(S4).

(2) A subordination is *reflexive* if it satisfies (S5), and it is a *contact relation* if in addition it satisfies (S6).

(3) A *contact algebra* is a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a contact relation on B.

(4) Let Sub be the class of all pairs $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a subordination on B; let RSub be the subclass of Sub consisting of the pairs $(B, \prec)$, where $\prec$ is a reflexive subordination on B; and let Con be the subclass of RSub consisting of contact algebras.

Remark 2.4. Subordinations correspond to the quasi-modal operators of [10] and to the precontact relations of [14]. Morphisms between objects of Sub were studied in [7].

As was pointed out in [7, Sec. 3], subordinations on B can be described by means of strict implications.

Definition 2.5. A *strict implication* on a Boolean algebra B is a binary operation $\rightsquigarrow : B \times B \to B$ with values in $\{0, 1\}$ satisfying

(I1) $0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$;
(I2) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$;
(I3) $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$.

If $\prec$ is a subordination on B, then define $\rightsquigarrow: B \times B \to B$ by

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\rightsquigarrow$ is a strict implication on B. Conversely, if $\rightsquigarrow$ is a strict implication on B, then define $\prec$ by setting

$$a \prec b \text{ iff } a \rightsquigarrow b = 1.$$

It is easy to see that $\prec$ is a subordination on B, and that this correspondence is 1-1. Moreover, the axioms (S5)–(S8) correspond, respectively, to the axioms:

(I4) $a \rightsquigarrow b \leq a \to b$;
(I5) $a \rightsquigarrow b = \neg b \rightsquigarrow \neg a$;
(I6) $a \rightsquigarrow b = 1$ implies $\exists c : a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$;
(I7) $a \neq 0$ implies $\exists b \neq 0 : b \rightsquigarrow a = 1$.

As we will see in the next section, adding (I4) to (I1)–(I3) is very useful in algebraic as well as logical calculations. Therefore, as our base variety, we will consider the variety generated by the algebras $(B, \rightsquigarrow)$, where B is a Boolean algebra and $\rightsquigarrow$ is a strict implication on B satisfying (I4). The corresponding subordinations are reflexive.

3 The variety of strict implication algebras

From now on, when we write $(B, \rightsquigarrow) \in \mathsf{RSub}$, we mean that $\rightsquigarrow$ is a strict implication on B satisfying (I4), and hence the corresponding subordination is reflexive. In this section we will study the variety $\mathcal{V}$ generated by RSub. We will prove that $\mathcal{V}$ is axiomatized by adding to (I1)–(I4) the following axioms:

(I8) $\Box(a \to b) \wedge (b \rightsquigarrow c) \leq a \rightsquigarrow c$;
(I9) $(a \rightsquigarrow b) \wedge \Box(b \to c) \leq a \rightsquigarrow c$;
(I10) $(a \rightsquigarrow b) \leq c \rightsquigarrow (a \rightsquigarrow b)$;
(I11) $\neg(a \rightsquigarrow b) \leq c \rightsquigarrow \neg(a \rightsquigarrow b)$.

But first we observe that $\mathcal{V}$ is a discriminator variety. Let $(B, \rightsquigarrow) \in \mathsf{RSub}$. For $a \in B$, define

$$\Box a = 1 \rightsquigarrow a.$$

If $a = 1$, then $1 \rightsquigarrow a = 1 \rightsquigarrow 1 = 1$ by (I1). On the other hand, if $a \neq 1$, then by (I4), $1 \rightsquigarrow a \leq 1 \to a = a \neq 1$, so $1 \rightsquigarrow a \neq 1$. But $1 \rightsquigarrow a \in \{0, 1\}$, so $1 \rightsquigarrow a = 0$. Thus,

$$\Box a = \begin{cases} 1 \text{ if } a = 1 \\ 0 \text{ if } a \neq 1. \end{cases}$$

In other words, $\Box$ is the Boolean dual of the so-called *unary discriminator term* [22]. From this, and the fact that the class RSub is axiomatized by universal first-order formulas, the following observations are immediate [35, Sec. 8.2.].

Proposition 3.1.

(1) *The variety $\mathcal{V}$ is a discriminator variety, and hence a semisimple variety.*

(2) *The simple algebras in $\mathcal{V}$ are exactly the members of* RSub.

We already saw that (I1)–(I4) hold in every member of RSub, and hence in every member of $\mathcal{V}$.

Lemma 3.2. (I8)–(I11) *hold in every member of* RSub.

Proof. Let $(B, \rightsquigarrow) \in \mathsf{RSub}$ and let $a, b, c \in B$. First we show that (I8) holds. Since $\Box(a \to b) \wedge (b \rightsquigarrow c) \in \{0, 1\}$, the only interesting case is when $\Box(a \to b) \wedge (b \rightsquigarrow c) = 1$. But then $\Box(a \to b) = 1$ and $b \rightsquigarrow c = 1$, so $a \leq b$ and $b \prec c$, yielding $a \prec c$ by (S4). Therefore, $a \rightsquigarrow c = 1$, and so (I8) holds.

Second we show that (I9) holds. Again, the only interesting case is when $(a \rightsquigarrow b) \wedge \Box(b \to c) = 1$. But then $a \prec b$ and $b \leq c$, yielding $a \prec c$ by (S4). Thus, $a \rightsquigarrow c = 1$, and hence (I9) holds.

Third we show that (I10) holds. The only interesting case is when $a \rightsquigarrow b = 1$. But then $c \rightsquigarrow (a \rightsquigarrow b) = c \rightsquigarrow 1 = 1$ by (I1). Therefore, (I10) holds.

Finally we show that (I11) holds. Again, the only interesting case is when $\neg(a \rightsquigarrow b) = 1$. But then $c \rightsquigarrow \neg(a \rightsquigarrow b) = 1$ by (I1). Thus, (I11) holds. $\Box$

Definition 3.3. We call a pair $(B, \rightsquigarrow)$ a *strict implication algebra* if B is a Boolean algebra and $\rightsquigarrow$ is a binary operation on B satisfying (I1)–(I4) and (I8)–(I11). Let SIA be the variety of all strict implication algebras.

As we observed, $\mathsf{RSub} \subseteq \mathsf{SIA}$, and hence $\mathcal{V} \subseteq \mathsf{SIA}$. To prove the converse, we require some preparation.

Lemma 3.4. *Let $(B, \rightsquigarrow) \in \mathsf{SIA}$. Then the following hold for every $a, b, c \in B$:*

(1) $\Box(a \wedge b) = \Box a \wedge \Box b$.
(2) $a \leq b$ *implies* $\Box a \leq \Box b$.
(3) $\Box 1 = 1$.
(4) $\Box a \leq a$.
(5) $a \rightsquigarrow b = \Box(a \rightsquigarrow b)$.
(6) $\Box\Box a = \Box a$.
(7) $\neg(a \rightsquigarrow b) = \Box\neg(a \rightsquigarrow b)$.
(8) $\Box\neg\Box a = \neg\Box a$.
(9) $\Box a \leq c \rightsquigarrow a$.

Proof. (1) By (I3), $\Box(a \wedge b) = 1 \rightsquigarrow (a \wedge b) = (1 \rightsquigarrow a) \wedge (1 \rightsquigarrow b) = \Box a \wedge \Box b$.

(2) If $a \leq b$, then $a = a \wedge b$. Therefore, by (1), $\Box a = \Box(a \wedge b) = \Box a \wedge \Box b$. Thus, $\Box a \leq \Box b$.

(3) By (I1), $\Box 1 = 1 \rightsquigarrow 1 = 1$.

(4) By (I4), $\Box a = 1 \rightsquigarrow a \leq 1 \to a = a$.

(5) It is immediate from (I10) that $a \rightsquigarrow b \leq \Box(a \rightsquigarrow b)$. That $\Box(a \rightsquigarrow b) \leq a \rightsquigarrow b$ follows from (4).

(6) By (5), $\Box a = 1 \rightsquigarrow a = \Box(1 \rightsquigarrow a) = \Box\Box a$.

(7) It is immediate from (I11) that $\neg(a \rightsquigarrow b) \leq \Box\neg(a \rightsquigarrow b)$. That $\Box\neg(a \rightsquigarrow b) \leq \neg(a \rightsquigarrow b)$ follows from (4).

(8) By (7), $\neg\Box a = \neg(1 \rightsquigarrow a) = \Box\neg(1 \rightsquigarrow a) = \Box\neg\Box a$.

(9) By (I1), $c \rightsquigarrow 1 = 1$, so $\Box(c \rightsquigarrow 1) = 1$ by (3). Therefore, by (I8),

$$\Box a = 1 \wedge \Box a = \Box(c \rightsquigarrow 1) \wedge (1 \rightsquigarrow a) \leq c \rightsquigarrow a.$$

□

Remark 3.5. By (5) and (9) of Lemma 3.4, $a \rightsquigarrow b = \Box(a \rightsquigarrow b) \leq c \rightsquigarrow (a \rightsquigarrow b)$. Since the proof of (9) does not require (I10), we see that $a \rightsquigarrow b = \Box(a \rightsquigarrow b)$ implies (I10). Similarly, $\neg(a \rightsquigarrow b) = \Box\neg(a \rightsquigarrow b)$ implies (I11).

It is well known that congruences of Boolean algebras correspond to filters, and this correspondence is obtained as follows. If θ is a congruence on a Boolean algebra B, then $F_\theta = \{a \in B \mid a\theta 1\}$ is a filter of B. If F is a filter of B, then θ_F defined by $a\theta_F b$ iff $a \leftrightarrow b \in F$ is a congruence of B. Moreover, $\theta_{F_\theta} = \theta$ and $F_{\theta_F} = F$. We next characterize congruences of strict implication algebras.

Proposition 3.6. *For $(B, \rightsquigarrow) \in \mathsf{SIA}$, there is a 1-1 correspondence between*

1. *congruences of $(B, \rightsquigarrow)$;*
2. *congruences θ of B such that $a\theta b$ implies $(a \rightsquigarrow c)\theta(b \rightsquigarrow c)$ and $(c \rightsquigarrow a)\theta(c \rightsquigarrow b)$;*
3. *filters F of B such that $a \in F$ implies $\Box a \in F$;*
4. *filters F of B such that $a \to b \in F$ implies $(b \rightsquigarrow c) \to (a \rightsquigarrow c), (c \rightsquigarrow a) \to (c \rightsquigarrow b) \in F$;*
5. *filters F of B such that $a \to b, b \rightsquigarrow c, c \to d \in F$ imply $a \rightsquigarrow d \in F$.*

Proof. (1)⇒(2): This is obvious.

(2)⇒(1): Suppose $a\theta b$ and $c\theta d$. By (2), $(a \rightsquigarrow c)\theta(b \rightsquigarrow c)$ and $(b \rightsquigarrow c)\theta(b \rightsquigarrow d)$. Therefore, $(a \rightsquigarrow c)\theta(b \rightsquigarrow d)$. Thus, θ is a congruence of $(B, \rightsquigarrow)$.

(3)⇒(4): Suppose F satisfies (3) and $a \to b \in F$. Then $\Box(a \to b) \in F$. By (I8) and (I9), for any $c \in B$, we have $\Box(a \to b) \leq (b \rightsquigarrow c) \to (a \rightsquigarrow c)$ and $\Box(a \to b) \leq (c \rightsquigarrow a) \to (c \rightsquigarrow b)$. Therefore, $(b \rightsquigarrow c) \to (a \rightsquigarrow c), (c \rightsquigarrow a) \to (c \rightsquigarrow b) \in F$, and so F satisfies (4).

(4)⇒(5): Suppose F satisfies (4) and $a \to b, b \rightsquigarrow c, c \to d \in F$. From $a \to b \in F$ it follows that $(b \rightsquigarrow c) \to (a \rightsquigarrow c) \in F$. Therefore, since $b \rightsquigarrow c \in F$, we have $a \rightsquigarrow c \in F$. Also, from $c \to d \in F$ it follows that $(a \rightsquigarrow c) \to (a \rightsquigarrow d) \in F$. This together with $a \rightsquigarrow c \in F$ yields $a \rightsquigarrow d \in F$. Thus, F satisfies (5).

(5)⇒(3): Suppose F satisfies (5) and $a \in F$. Since $1 \to 1 = 1 \rightsquigarrow 1 = 1$ and $1 \to a = a$, we have $1 \to 1, 1 \rightsquigarrow 1, 1 \to a \in F$. Therefore, by (4), $\Box a = 1 \rightsquigarrow a \in F$. Thus, F satisfies (3).

(2)⇒(3): Suppose θ is a congruence of $(B, \rightsquigarrow)$ and $a \in F_\theta$. Then $a\theta 1$. Therefore, $(1 \rightsquigarrow a)\theta(1 \rightsquigarrow 1)$. Thus, $\Box a\theta 1$, and so $\Box a \in F_\theta$.

(4)⇒(2): Suppose F satisfies (4), $a\theta_F b$, and $c \in B$. Then $a \to b \in F$ and $b \to a \in F$. Therefore, by (4), $(b \rightsquigarrow c) \to (a \rightsquigarrow c), (c \rightsquigarrow a) \to (c \rightsquigarrow b) \in F$ and $(a \rightsquigarrow c) \to (b \rightsquigarrow c), (c \rightsquigarrow b) \to (c \rightsquigarrow a) \in F$. Thus, $(a \rightsquigarrow c) \leftrightarrow (b \rightsquigarrow c), (c \rightsquigarrow a) \leftrightarrow (c \rightsquigarrow b) \in F$. Consequently, $(a \rightsquigarrow c)\theta_F(b \rightsquigarrow c)$ and $(c \rightsquigarrow a)\theta_F(c \rightsquigarrow b)$, and hence θ_F satisfies (2). □

Definition 3.7. Let $(B, \rightsquigarrow)$ be a strict implication algebra. We call a filter F of B a $\Box$-*filter* provided F satisfies Proposition 3.6(3); that is, $a \in F$ implies $\Box a \in F$.

By Proposition 3.6, congruences of strict implication algebras correspond to their $\Box$-filters.

Lemma 3.8. *Let $(B, \rightsquigarrow)$ be a strict implication algebra, $a \in B$, and F a $\Box$-filter. Then the filter generated by $F \cup \{\Box a\}$ is a $\Box$-filter. In particular, we have that $\uparrow\Box a$ and $\uparrow\neg\Box a$ are $\Box$-filters.*

Proof. Let F' be the filter generated by $F \cup \{\Box a\}$, and let $b \in F'$. Then there is $c \in F$ such that $c \wedge \Box a \leq b$. Lemma 3.4(2) yields $\Box(c \wedge \Box a) \leq \Box b$. Since F is a $\Box$-filter, $\Box c \in F$. Also, by Lemma 3.4(6), $\Box a = \Box\Box a$. Therefore, by Lemma 3.4(1), $\Box(c \wedge \Box a) = \Box c \wedge \Box\Box a = \Box c \wedge \Box a \in F'$. Thus, $\Box b \in F'$, which shows that F' is a $\Box$-filter.

In particular, as $\{1\}$ is a $\Box$-filter, it follows that $\uparrow\Box a$ is a $\Box$-filter, and by Lemma 3.4(8), the same holds for $\uparrow\neg\Box a$. $\Box$

Lemma 3.9. *If $(B, \rightsquigarrow) \in \mathsf{SIA}$ is subdirectly irreducible, then $(B, \rightsquigarrow) \in \mathsf{RSub}$.*

Proof. Let $(B, \rightsquigarrow) \in \mathsf{SIA}$ be subdirectly irreducible. Since members of SIA satisfy (I1)–(I4), it is sufficient to show that for every $a, b \in B$, we have $a \rightsquigarrow b \in \{0, 1\}$. By Lemma 3.4(5), $a \rightsquigarrow b = \Box(a \rightsquigarrow b)$. Therefore, it suffices to show that $\Box a \in \{0, 1\}$ for each $a \in B$.

Let $a \in B$. If $\Box a \notin \{0, 1\}$, then $\neg\Box a \notin \{0, 1\}$. By Lemma 3.8, $F := \uparrow\Box a$ and $G := \uparrow\neg\Box a$ are $\Box$-filters. Moreover, $F, G \neq \{1\}$, but if $b \in F \cap G$, then $b \geq \Box a, \neg\Box a$, so $b = 1$, and hence $F \cap G = \{1\}$. Therefore, among the $\Box$-filters different from $\{1\}$, there is no least one. This contradicts to $(B, \rightsquigarrow)$ being subdirectly irreducible. Thus, $\Box a \in \{0, 1\}$, and hence $(B, \rightsquigarrow) \in \mathsf{RSub}$. $\Box$

Theorem 3.10. $\mathcal{V} = \mathsf{SIA}$.

Proof. We already observed that $\mathcal{V} \subseteq \mathsf{SIA}$. Conversely, by Lemma 3.9, every subdirectly irreducible member of SIA belongs to $\mathsf{RSub} \subseteq \mathcal{V}$. Therefore, $\mathsf{SIA} \subseteq \mathcal{V}$. Thus, the equality. $\Box$

Applying Proposition 3.1 yields:

Corollary 3.11. *The variety SIA is semisimple, and $(B, \rightsquigarrow) \in \mathsf{SIA}$ is simple iff $(B, \rightsquigarrow) \in \mathsf{RSub}$.*

We next show that our base variety SIA is locally finite, and study subvarieties and inductive subclasses of SIA.

Proposition 3.12. *The variety SIA is locally finite.*

Proof. Let $(B, \rightsquigarrow) \in \mathsf{RSub}$ be n-generated, with generators $a_1, \dots, a_n \in B$. For each $a \in B$, there is a term $t(x_1, \dots, x_n)$ such that $a = t(a_1, \dots, a_n)$. Since $(B, \rightsquigarrow) \in \mathsf{RSub}$, for each $b, c \in B$, we have $b \rightsquigarrow c \in \{0, 1\}$. Therefore, by replacing each subterm of $t(x_1, \dots, x_n)$ of the form $x \rightsquigarrow y$ with either 0 or 1, we obtain a Boolean term $t'(x_1, \dots, x_n)$ such that $a = t'(a_1, \dots, a_n)$. Thus, B is n-generated as a Boolean algebra, and hence has at most 2^{2^n} elements. By Corollary 3.11, there is a uniform bound $m(n) = 2^{2^n}$ on all n-generated subdirectly irreducible members of SIA. Consequently, by [4, Thm. 3.7(4)], SIA is locally finite. $\Box$

As an immediate consequence we obtain:

Corollary 3.13. *Every subvariety of* SIA *is generated by its finite members.*

While SIA has many subvarieties, we will be interested in the subvariety $\mathcal{V}_{\mathsf{Con}}$ obtained by postulating the identity (I5). Our interest is motivated by the fact that $\mathcal{V}_{\mathsf{Con}}$ is exactly the subvariety of SIA generated by the class Con of contact algebras. We further restrict Con to the class Com of compingent algebras, by postulating (I6) and (I7). But unlike (I5), neither (I6) nor (I7) is an identity. However, both (I6) and (I7) are Π_2-statements (i.e., statements of the form $\forall \overline{x} \exists \overline{y} \Phi(\overline{x}, \overline{y})$, where $\overline{x}, \overline{y}$ are tuples of variables and $\Phi(\overline{x}, \overline{y})$ is a quantifier-free formula). By the Chang-Łoś-Suszko Theorem (see, e.g., [11, Thm. 3.2.3]), the classes corresponding to Π_2-statements are inductive classes, where we recall that a class is *inductive* provided it is closed under unions of chains (equivalently, closed under directed limits). While we will be mainly interested in the inductive class Com, in Section 5 we will show that all inductive subclasses of RSub can be axiomatized by non-standard rules.

We conclude this section by observing that, unlike subvarieties of SIA, not every inductive (already universal) subclass of SIA is determined by its finite algebras. For example, consider the universal subclass $\mathcal{C}$ of SIA obtained by postulating

(I12) $a \rightsquigarrow b = 0$ or $a \rightsquigarrow b = 1$.

Clearly $\mathcal{C} \subseteq \mathsf{RSub}$. Let $(B, \rightsquigarrow)$ correspond to the de Vries algebra of regular opens of $[0, 1]$. Since $[0, 1]$ is a connected space, $x \rightsquigarrow x = 1$ is refuted on $(B, \rightsquigarrow)$. However, a finite $(B, \rightsquigarrow) \in \mathcal{C}$ corresponds to the de Vries algebra of some finite discrete space, and all such algebras validate $x \rightsquigarrow x = 1$. Thus, $\mathcal{C}$ is not determined by its finite algebras.

4 The strict implication calculus

We next present a sound and complete deductive system for SIA. We will work with the language of classical propositional logic, which we will enrich with one binary connective $\rightsquigarrow$ of strict implication. For a formula φ, we will abbreviate $\top \rightsquigarrow \varphi$ as $\Box\varphi$.

A *valuation* on $(B, \rightsquigarrow)$ is an assignment of elements of B to propositional letters of our language $\mathcal{L}$, which extends to all formulas of $\mathcal{L}$ in the usual way. We say that a valuation v on $(B, \rightsquigarrow)$ *satisfies* a formula φ if $v(\varphi) = 1$. If all valuations on $(B, \rightsquigarrow)$ satisfy φ, then we say that $(B, \rightsquigarrow)$ *validates* φ, and write $(B, \rightsquigarrow) \models \varphi$. For a set of formulas Γ, we write $(B, \rightsquigarrow) \models \Gamma$ if $(B, \rightsquigarrow) \models \varphi$ for every $\varphi \in \Gamma$.

Suppose $\mathcal{U}$ is a class of algebras, φ is a formula, and Γ is a set of formulas. We say that φ is a *semantic consequence* of Γ over $\mathcal{U}$, and write $\Gamma \models_{\mathcal{U}} \varphi$, provided for all $(B, \rightsquigarrow) \in \mathcal{U}$, from $(B, \rightsquigarrow) \models \Gamma$ it follows that $(B, \rightsquigarrow) \models \varphi$.

Definition 4.1. The *strict implication calculus* SIC is the derivation system containing all theorems of classical propositional calculus CPC, the axiom schemes:

(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$
(A2) $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \leftrightarrow (\varphi \rightsquigarrow \psi \wedge \chi)$
(A3) $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \leftrightarrow (\varphi \vee \psi \rightsquigarrow \chi)$
(A4) $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$

(A8) $\Box(\varphi \to \psi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$
(A9) $(\varphi \rightsquigarrow \psi) \wedge \Box(\psi \to \chi) \to (\varphi \rightsquigarrow \chi)$
(A10) $(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow (\varphi \rightsquigarrow \psi))$
(A11) $\neg(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \psi))$,

and is closed under the inference rules:

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(R) $\dfrac{\varphi}{\Box\varphi}$

Remark 4.2. The numbering of the axioms in Definition 4.1 matches the numbering of the axioms for strict implication algebras.

For a set of formulas Γ and a formula φ, we write $\Gamma \vdash_{\mathsf{SIC}} \varphi$ if φ is derivable in SIC from Γ. Since each axiom of SIC has an equational counterpart in the axiomatization of SIA, the standard Lindenbaum construction yields that SIC is strongly sound and complete with respect to SIA; that is, for a set of formulas Γ and a formula φ,

$$\Gamma \vdash_{\mathsf{SIC}} \varphi \text{ iff } \Gamma \models_{\mathsf{SIA}} \varphi.$$

We next show that SIC is in fact strongly sound and complete with respect to RSub. For this we require the following lemma.

Lemma 4.3. *Let* $(B, \rightsquigarrow) \in \mathsf{SIA}$.

(1) *For a proper* $\Box$*-filter* F *in* $(B, \rightsquigarrow)$*, the following are equivalent:*
  (a) F *is a maximal* $\Box$*-filter.*
  (b) *For each* $a \in B$*, we have* $\Box a \in F$ *or* $\neg\Box a \in F$*.*
  (c) $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$*.*

(2) *If* F *is a* $\Box$*-filter and* $a \notin F$*, then there is a maximal* $\Box$*-filter* M *such that* $F \subseteq M$ *and* $a \notin M$*.*

Proof. (1) (a)$\Rightarrow$(b): Suppose $\Box a \notin F$. Let G be the filter generated by F and $\Box a$. By Lemma 3.8, G is a $\Box$-filter. Since F is a maximal $\Box$-filter, G is improper. Therefore, $0 = \Box a \wedge b$ for some $b \in F$. Thus, $b \le \neg\Box a$, and so $\neg\Box a \in F$.

(b)$\Rightarrow$(c): Let $a \in B$. Then $\Box a \in F$ or $\neg\Box a \in F$. If $\Box a \in F$, then $\Box_F[a] = [\Box a] = 1_F$. On the other hand, if $\Box a \notin F$, then $\neg\Box a \in F$, so $\neg\Box_F[a] = [\neg\Box a] = 1_F$, and hence $\Box_F[a] = 0_F$. Thus, $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$.

(c)$\Rightarrow$(a): Suppose G is a $\Box$-filter properly containing F. Then there is $a \in G \setminus F$. Since G is a $\Box$-filter and $\Box a \le a$, we see that $\Box a \in G \setminus F$. Therefore, $[\Box a] \ne 1_F$. Since $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$, we conclude that $[\Box a] = 0_F$. Thus, $[\neg\Box a] = 1_F$, yielding that $\neg\Box a \in F \subseteq G$. Consequently, G is an improper $\Box$-filter, and hence F is a maximal $\Box$-filter.

(2) Since $a \notin F$, by Zorn's lemma there is a $\Box$-filter M such that $F \subseteq M$, $a \notin M$, and M is maximal with this property. If M is not a maximal $\Box$-filter, then by (1), there is $b \in B$ such that $\Box b, \neg\Box b \notin M$. Let G be the filter generated by M and $\Box b$ and H be the filter generated by M and $\neg\Box b$. By Lemma 3.8, both G and H are $\Box$-filters that properly extend F. Therefore, $a \in G, H$, so there exist $c, d \in M$ such that $a \ge \Box b \wedge c$ and $a \ge \neg\Box b \wedge d$. Thus,

$a \geq (\Box b \vee \neg\Box b) \wedge (\Box b \vee d) \wedge (c \vee \neg\Box b) \wedge (c \vee d) \in M$. The obtained contradiction proves that M is a maximal $\Box$-filter. $\Box$

Theorem 4.4. *The system* SIC *is strongly sound and complete with respect to* RSub*; that is, for a set of formulas* Γ *and a formula* φ,

$$\Gamma \vdash_{\mathsf{SIC}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathsf{RSub}} \varphi.$$

Proof. Since $\mathsf{RSub} \subseteq \mathsf{SIA}$ and SIC is strongly sound and complete with respect to SIA, from $\Gamma \vdash_{\mathsf{SIC}} \varphi$ it follows that $\Gamma \models_{\mathsf{RSub}} \varphi$. Conversely, if $\Gamma \nvdash_{\mathsf{SIC}} \varphi$, then in the Lindenbaum algebra $(B, \rightsquigarrow)$ of SIC, the $\Box$-filter generated by $\{[\psi] \mid \psi \in \Gamma\}$ does not contain $[\varphi]$. By Lemma 4.3(2), there is a maximal $\Box$-filter F such that $\{[\psi] \mid \psi \in \Gamma\} \subseteq F$ and $[\varphi] \notin F$. But then $(B/F, \rightsquigarrow_F) \models \Gamma$ and $(B/F, \rightsquigarrow_F) \not\models \varphi$. By Lemma 4.3(1), $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$. Thus, $\Gamma \not\models_{\mathsf{RSub}} \varphi$. $\Box$

5 Π_2-rules

As follows from the previous section, SIC is sound and complete with respect to SIA. Therefore, normal extensions of SIC correspond to subvarieties of SIA. To work with inductive subclasses of SIA, we require non-standard rules. Since SIA is generated by RSub (which is a universal class), it is sufficient to work with inductive subclasses of RSub.

Definition 5.1 (Π_2-rule)**.** A Π_2-*rule* is a rule of the form

$$(\rho) \qquad \frac{F(\overline{\varphi}, \overline{p}) \to \chi}{G(\overline{\varphi}) \to \chi}$$

where F, G are formulas, $\overline{\varphi}$ is a tuple of formulas, χ is a formula, and $\overline{p}$ is a tuple of propositional letters.

The non-standard feature of Π_2-rules is that their application is subject to the *side condition* that the proposition letters $\overline{p}$ do not occur in any assumptions.

With the rule ρ, we associate the first-order formula

$$\Phi_\rho := \forall \overline{x}, z \Big(G(\overline{x}) \not\leq z \to \exists \overline{y} : F(\overline{x}, \overline{y}) \not\leq z \Big).$$

Our purpose will be to show that the system $\mathcal{S}$ obtained by adding the Π_2-rules $\{\rho_i \mid i \in I\}$ to SIC is strongly sound and complete with respect to the inductive subclass $\mathcal{U}$ of RSub defined by the statements $\{\Phi_{\rho_i} \mid i \in I\}$. The proof of the next theorem is similar to that of [1, Lem. 7.10].

Theorem 5.2. *Let* $\mathcal{S} = \mathsf{SIC} + \{\rho_i \mid i \in I\}$*,* $\mathcal{U} = \mathsf{RSub} + \{\Phi_{\rho_i} \mid i \in I\}$*, and* $\mathcal{V}$ *be the variety generated by* $\mathcal{U}$*. For a set of formulas* Γ *and a formula* φ*, we have:*

(1) $\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \Gamma \models_{\mathcal{U}} \varphi$.
(2) $\vdash_{\mathcal{S}} \varphi \Leftrightarrow \models_{\mathcal{V}} \varphi$.

Proof. (1) That $\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \Gamma \models_{\mathcal{U}} \varphi$ can be shown by a fairly straightforward inductive proof on the length of derivations. We only show that Π_2-rules preserve validity. Suppose ρ is a Π_2-rule and v is a valuation into $(B, \rightsquigarrow) \in \mathsf{SIA}$ satisfying Φ_ρ. If $G(v(\varphi)) \nleq v(\chi)$, then since $(B, \rightsquigarrow)$ satisfies Φ_ρ, there is a tuple $\bar{c}$ in B such that $F(v(\varphi), \bar{c}) \nleq v(\chi)$. Consider the valuation v' which coincides with v everywhere, except maps $\bar{p}$ to $\bar{c}$. Then $v'(F(\varphi, \bar{p})) = F(v(\varphi), \bar{c}) \nleq v(\chi) = v'(\chi)$, so $v'(F(\varphi, \bar{p}) \to \chi) \neq 1$. Therefore, if the conclusion of ρ is refuted on $(B, \rightsquigarrow)$, then so is the premise of ρ.

To complete the proof of (1), it remains to show that $\Gamma \nvdash_{\mathcal{S}} \varphi \Rightarrow \Gamma \not\models_{\mathcal{U}} \varphi$. We do this by slightly modifying the construction of the Lindenbaum algebra.

Suppose $\Gamma \nvdash_{\mathcal{S}} \varphi$. For each rule ρ_i, we add a countably infinite set of fresh propositional letters to the set of existing propositional letters, build the Lindenbaum algebra $(B, \rightsquigarrow)$ over the expanded set of propositional letters, and construct a maximal $\Box$-filter M of $(B, \rightsquigarrow)$ such that $\{[\psi] \mid \psi \in \Gamma\} \cup \{\neg\Box[\varphi]\} \subseteq M$ and for every rule ρ_i and formulas $\bar{\varphi}, \chi$:

(†) if $[G_i(\bar{\varphi}) \to \chi] \notin M$, then there is a tuple $\bar{p}$ of variables such that $[F_i(\bar{\varphi}, \bar{p}) \to \chi] \notin M$.

To construct M, since $\Gamma \nvdash_{\mathcal{S}} \varphi$, the $\Box$-filter M_0 generated by $\{[\psi] \mid \psi \in \Gamma\} \cup \{\neg\Box[\varphi]\}$ is proper. Enumerate all formulas φ and all tuples $(\bar{\varphi}, \chi)$ of formulas, and build the filters $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \dots$ as follows:

- For $n = ki$, if $\Box[\varphi_n] \notin M_n$, let M_{n+1} be the filter generated by M_n and $\neg\Box[\varphi_n]$.
- For $n = ki + l$ ($l \leq i$), if $[G_l(\bar{\varphi}) \to \chi] \notin M_n$, let M_{n+1} be the filter generated by M_n and $\neg\Box[F_l(\bar{\varphi}, \bar{p}) \to \chi]$, where $\bar{p}$ is a tuple of variables for ρ_l not occurring in $\bar{\varphi}, \chi$, and any of ψ with $[\psi] \in M_n$.

It is easy to see by induction that each M_n is proper. Indeed, if the filter generated by M_n and $\neg\Box[\varphi_n]$ is improper, then there is $[\psi] \in M_n$ such that $[\psi] \wedge \neg\Box[\varphi_n] = 0$. Therefore, $[\psi] \leq \Box[\varphi_n]$, and so $\Box[\varphi_n] \in M_n$, a contradiction. On the other hand, if the filter generated by M_n and $\neg\Box[F_l(\bar{\varphi}, \bar{p}) \to \chi]$ is improper, then there is $[\psi] \in M_n$ such that $[\psi] \wedge \neg\Box[F_l(\bar{\varphi}, \bar{p}) \to \chi] = 0$. Thus, $[\psi] \leq \Box[F_l(\bar{\varphi}, \bar{p}) \to \chi] \leq [F_l(\bar{\varphi}, \bar{p})] \to [\chi]$. This yields $[F_l(\bar{\varphi}, \bar{p})] \leq [\psi \to \chi]$, so $[F_l(\bar{\varphi}, \bar{p})] \to [\psi \to \chi] = 1$. Applying ρ_l gives $[G_l(\bar{\varphi})] \to [\psi \to \chi] = 1$, so $[\psi] \to [G_l(\bar{\varphi}) \to \chi] = 1$, and hence $[\psi] \leq [G_l(\bar{\varphi}) \to \chi]$. Consequently, $[G_l(\bar{\varphi}) \to \chi] \in M_n$, a contradiction.

It also follows from Lemma 3.8 that each M_n is a $\Box$-filter. Let $M = \bigcup_{n \in \omega} M_n$. Then it is clear that M is a proper $\Box$-filter. It is a maximal $\Box$-filter by Lemma 4.3(1), and it follows from the construction that M satisfies (†).

By (†), the quotient of $(B, \rightsquigarrow)$ by M satisfies each Φ_{ρ_i}. By Lemma 4.3(1), the quotient belongs to RSub. Therefore, the quotient belongs to $\mathcal{U}$. Moreover, since $\neg\Box[\varphi] \in M$, we have that $\neg\Box[\varphi]$ maps to 1, so $\Box[\varphi]$ maps to 0 in the quotient. Thus, $[\varphi]$ does not map to 1 in the quotient, and hence $\Gamma \not\models_{\mathcal{U}} \varphi$.

(2) Observe that $\mathcal{U}$ consists of the subdirectly irreducible members of $\mathcal{V}$, and apply (1). □

It follows that the class of subdirectly irreducible algebras in SIA validating a set of Π_2-rules is an inductive subclass of RSub. We next show that the converse is also true. Namely, for every inductive subclass K of RSub, there is a set of Π_2-rules $\{\rho_i \mid i \in I\}$ such that

$\mathcal{S} = \mathsf{SIC} + \{\rho_i \mid i \in I\}$ is sound and complete with respect to K. To obtain such a set of Π_2-rules, it is sufficient to show that every Π_2-statement is equivalent to a statement of the form Φ_ρ for some Π_2-rule ρ. Without loss of generality we may assume that all atomic formulas $\Phi(\bar{x}, \bar{y})$ are of the form $t(\bar{x}, \bar{y}) = 1$ for some term t.

Definition 5.3. Given a quantifier-free first-order formula $\Phi(\overline{x}, \overline{y})$, we associate with the tuples of variables $\overline{x}, \overline{y}$ the tuples of propositional letters $\overline{p}, \overline{q}$, and define the formula $\Phi^*(\overline{p}, \overline{q})$ in the language of SIC as follows:

$$\begin{aligned}(t(\overline{x}, \overline{y}) = 1)^* &:= 1 \rightsquigarrow t(\overline{p}, \overline{q})\\ (\neg\Psi)^*(\overline{x}, \overline{y}) &:= \neg\Psi^*(\overline{p}, \overline{q})\\ (\Psi_1(\overline{x}, \overline{y}) \wedge \Psi_2(\overline{x}, \overline{y}))^* &:= \Psi_1^*(\overline{p}, \overline{q}) \wedge \Psi_2^*(\overline{p}, \overline{q})\end{aligned}$$

Let $(B, \rightsquigarrow) \in \mathsf{RSub}$. For each term $t(\overline{x}, \overline{y})$, if we evaluate $\overline{x}, \overline{p}$ as $\overline{a}$ and $\overline{y}, \overline{q}$ as $\overline{b}$, then it is obvious that $(B, \rightsquigarrow)$ satisfies $t(\overline{x}, \overline{y}) = 1$ iff $(B, \rightsquigarrow)$ satisfies $1 \rightsquigarrow t(\overline{p}, \overline{q})$. Therefore, an easy induction shows that $(B, \rightsquigarrow)$ satisfies $\Phi(\overline{x}, \overline{y})$ iff $(B, \rightsquigarrow)$ satisfies $\Phi^*(\overline{p}, \overline{q})$.

Lemma 5.4. *Let $(B, \rightsquigarrow) \in \mathsf{RSub}$. For any quantifier-free formula $\Phi(\overline{x}, \overline{y})$, we have $(B, \rightsquigarrow) \models \forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$ iff $(B, \rightsquigarrow) \models \forall\overline{x}, z\Big(1 \not\leq z \to \exists\overline{y} : \Phi^*(\overline{x}, \overline{y}) \not\leq z\Big)$.*

Proof. ($\Rightarrow$) Suppose $(B, \rightsquigarrow) \models \forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$. Let $\overline{a}$ be a tuple of elements of B and $c \in B$. By assumption, there exists a tuple $\overline{b}$ in B such that $(B, \rightsquigarrow) \models \Phi(\overline{x}, \overline{y})[\overline{a}, \overline{b}]$. Therefore, if $1 \not\leq c$, then $\Phi^*(\overline{a}, \overline{b}) = 1 \not\leq c$. Thus, $(B, \rightsquigarrow) \models \forall\overline{x}, z\Big(1 \not\leq z \to \exists\overline{y} : \Phi^*(\overline{x}, \overline{y}) \not\leq z\Big)$.

($\Leftarrow$) Suppose $(B, \rightsquigarrow) \models \forall\overline{x}, z\Big(1 \not\leq z \to \exists\overline{y} : \Phi^*(\overline{x}, \overline{y}) \not\leq z\Big)$. Let $\overline{a}$ be a tuple of elements of B. Since $1 \not\leq 0$, there exists a tuple $\overline{b}$ in B such that $\Phi^*(\overline{a}, \overline{b}) \not\leq 0$. Therefore, since $\Phi^*(\overline{a}, \overline{b})$ evaluates only to 0 or 1, we obtain $\Phi^*(\overline{a}, \overline{b}) = 1$. Thus, $(B, \rightsquigarrow) \models \Phi(\overline{x}, \overline{y})[\overline{a}, \overline{b}]$. This shows that $(B, \rightsquigarrow) \models \forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$. ☐

Consequently, an arbitrary Π_2-statement $\forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$ is equivalent to the Π_2-statement associated to the Π_2-rule

$$(\rho_\Phi) \qquad \frac{\Phi^*(\bar{\varphi}, \bar{p}) \to \chi}{\chi}$$

Thus, by Theorem 5.2, we obtain:

Theorem 5.5. *If T is a Π_2-theory of first-order logic axiomatizing an inductive subclass K of RSub, then the system $\mathsf{SIC} + \{\rho_\Phi \mid \Phi \in T\}$ is sound and complete with respect to K.*

6 Π_2-rules for compingent relations

We recall from Section 2 that a binary relation $\prec$ on a Boolean algebra B is a compingent relation if it satisfies (S1)–(S8), and that a compingent algebra is a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a compingent relation on B. In this section we will use our results of the previous two sections to construct a sound and complete system for the class Com of compingent algebras.

Recall that the axioms (A1)–(A4) and (A8)–(A9) defining the system SIC correspond to the equations (I1)–(I4) and (I8)–(I11) defining the variety SIA. We will now add an axiom and two derivation rules that correspond, respectively, to the equation (I5) and the ∀∃-statements (I6) and (I7).

Definition 6.1. The *de Vries calculus* DVC is the extension of SIC with the axiom

(A5) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$

and the Π_2-rules ($\rho 6$) and ($\rho 7$):

$$(\rho 6)\quad \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

$$(\rho 7)\quad \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

It is obvious that (A5) corresponds to the equation (I5). The Π_2-statements corresponding to the rules ($\rho 6$) and ($\rho 7$) are:

$$(\Phi_{\rho 6})\ \forall x_1, x_2, y\Big(x_1 \rightsquigarrow x_2 \not\leq y \to \exists z : (x_1 \rightsquigarrow z) \wedge (z \rightsquigarrow x_2) \not\leq y\Big)$$

$$(\Phi_{\rho 7})\ \forall x, z\Big(x \not\leq z \to \exists y : y \wedge (y \rightsquigarrow x) \not\leq z\Big)$$

These directly correspond to the respective conditions (I6) and (I7), as we will see now.

Lemma 6.2. *Let* $(B, \rightsquigarrow) \in$ RSub.

(1) $(B, \rightsquigarrow) \models$ (I6) *iff* $(B, \rightsquigarrow) \models \Phi_{\rho 6}$.

(2) $(B, \rightsquigarrow) \models$ (I7) *iff* $(B, \rightsquigarrow) \models \Phi_{\rho 7}$.

Proof. (1) (⇒) Suppose $(B, \rightsquigarrow) \models$ (I6). Let $a, b, d \in B$ be such that $a \rightsquigarrow b \not\leq d$. Then $d \neq 1$ and $a \rightsquigarrow b \neq 0$, so $a \rightsquigarrow b = 1$. By (I6), there is $c \in B$ such that $a \rightsquigarrow c = c \rightsquigarrow b = 1$. Therefore, $1 = (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq d$. Thus, $(B, \rightsquigarrow) \models \Phi_{\rho 6}$.

(⇐) Suppose $(B, \rightsquigarrow) \models \Phi_{\rho 6}$. Let $a, b \in B$ be such that $a \rightsquigarrow b = 1$. Then $a \rightsquigarrow b \not\leq 0$. By $\Phi_{\rho 6}$, there is $c \in B$ such that $(a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq 0$. Therefore, since $(B, \rightsquigarrow) \in$ RSub, we have $a \rightsquigarrow c = c \rightsquigarrow b = 1$. Thus, $(B, \rightsquigarrow) \models$ (I6).

(2) (⇒) Suppose $(B, \rightsquigarrow) \models$ (I7). Let $a, c \in B$ be such that $a \not\leq c$. Then $a \wedge \neg c \neq 0$. By (I7), there is $b \neq 0$ such that $b \rightsquigarrow (a \wedge \neg c) = 1$. By (I3), $b \rightsquigarrow (a \wedge \neg c) = (b \rightsquigarrow a) \wedge (b \rightsquigarrow \neg c)$. Therefore, $b \rightsquigarrow a = 1$ and $b \rightsquigarrow \neg c = 1$. The latter equality, by (I4), yields $b \leq \neg c$. Since $b \neq 0$, we must have $b \not\leq c$. Thus, we have found $b \in B$ such that $b \wedge (b \rightsquigarrow a) = b \not\leq c$. This shows that $(B, \rightsquigarrow) \models \Phi_{\rho 7}$.

(⇐) Suppose $(B, \rightsquigarrow) \models \Phi_{\rho 7}$. Let $a \neq 0$ be an element of B. By $\Phi_{\rho 7}$, there is $b \in B$ such that $b \wedge (b \rightsquigarrow a) \not\leq 0$. Therefore, $b \neq 0$ and $b \rightsquigarrow a = 1$. Thus, $(B, \rightsquigarrow) \models$ (I7). □

Theorem 6.3. DVC *is strongly sound and complete with respect to* Com.

Proof. By Theorem 5.2(2), DVC is strongly sound and complete with respect to the inductive subclass of RSub satisfying (A5), $\Phi_{\rho 6}$, and $\Phi_{\rho 7}$. Since (A5) obviously corresponds to (I5), by Lemma 6.2, this inductive subclass coincides with Com. □

We recall that a de Vries algebra is a compingent algebra whose underlying Boolean algebra is complete. To show that DVC is strongly sound and complete with respect to the class of de Vries algebras, we will work with MacNeille completions of compingent algebras.

We will view a Boolean algebra B as a subalgebra of its MacNeille completion $\overline{B}$, so every element of $\overline{B}$ is a join and a meet of elements from B. We denote elements of B by $a, b, c, \dots$ and elements of $\overline{B}$ by $x, y, z, \dots$

Definition 6.4. Let B be a Boolean algebra and $\prec$ be a binary relation on B. Define $\lhd$ on the MacNeille completion $\overline{B}$ of B by setting

$$x \lhd y \text{ iff there exist } a, b \in B \text{ such that } x \leq a \prec b \leq y.$$

We call $(\overline{B}, \lhd)$ the *MacNeille completion* of $(B, \prec)$.

Lemma 6.5. *If $\prec$ is a subordination on a Boolean algebra B, then $\lhd$ is a subordination on $\overline{B}$. Moreover, if $\prec$ is a compingent relation on B, then $\lhd$ is a compingent relation on $\overline{B}$.*

Proof. Suppose that $\prec$ is a subordination on B. We show that $\lhd$ satisfies (S1)–(S4).

(S1) We have $0 \leq 0 \prec 0 \leq 0$ and $1 \leq 1 \prec 1 \leq 1$, so $0 \lhd 0$ and $1 \lhd 1$.

(S2) Suppose $x \lhd y, z$. Then there are $a, a', b, b' \in B$ such that $x \leq a \prec b \leq y$ and $x \leq a' \prec b' \leq z$. By (S2) and (S4) applied to $(B, \prec)$, we have $x \leq a \wedge a' \prec b \wedge b' \leq y \wedge z$. Therefore, $x \lhd y \wedge z$.

(S3) Suppose $x, y \lhd z$. Then there are $a, a', b, b' \in B$ such that $x \leq a \prec b \leq z$ and $y \leq a' \prec b' \leq z$. By (S3) and (S4) applied to $(B, \prec)$, we have $x \vee y \leq a \vee a' \prec b \vee b' \leq z$. Therefore, $x \vee y \lhd z$.

(S4) Suppose $x \leq y \lhd z \leq u$. From $y \lhd z$ it follows that there are $a, b \in B$ such that $y \leq a \prec b \leq z$. Therefore, $x \leq a \prec b \leq u$, and so $x \lhd u$.

Consequently, $\lhd$ is a subordination on $\overline{B}$. Next suppose $\prec$ is a compingent relation on B. We show that $\lhd$ satisfies (S5)–(S8).

(S5) Suppose $x \lhd y$. Then there are $a, b \in B$ such that $x \leq a \prec b \leq y$. By (S5) applied to $(B, \prec)$, we have $a \leq b$. Therefore, $x \leq y$.

(S6) Suppose $x \lhd y$. Then there are $a, b \in B$ such that $x \leq a \prec b \leq y$. By (S6) applied to $(B, \prec)$, we have $\neg y \leq \neg b \prec \neg a \leq \neg x$. Therefore, $\neg y \lhd \neg x$.

(S7) Suppose $x \lhd y$. Then there are $a, b \in B$ such that $x \leq a \prec b \leq y$. Since $a \prec b$, by (S7) applied to $(B, \prec)$, there is $c \in B$ such that $a \prec c \prec b$. Therefore, $x \leq a \prec c \leq c$ and $c \leq c \prec b \leq y$. Thus, $x \lhd c \lhd y$.

(S8) Suppose $x \neq 0$. Then there is $a \neq 0$ such that $a \leq x$. By (S8) applied to $(B, \prec)$, there is $b \neq 0$ such that $b \prec a$. Therefore, $b \leq b \prec a \leq x$. Thus, we have found $b \neq 0$ such that $b \lhd x$.

This shows that, indeed, $\lhd$ is a compingent relation on $\overline{B}$. □

Let DeV be the class of all de Vries algebras. We view $(B, \prec) \in$ DeV as the strong implication algebra $(B, \rightsquigarrow)$, where B is a complete Boolean algebra and the subordination corresponding to $\rightsquigarrow$ is a compingent relation.

Theorem 6.6. *The system* DVC *is strongly sound and complete with respect to* DeV*; that is, for a set of formulas* Γ *and a formula* φ,

$$\Gamma \vdash_{\mathsf{DVC}} \varphi \textit{ iff } \Gamma \models_{\mathsf{DeV}} \varphi.$$

Proof. Since $\mathsf{DeV} \subseteq \mathsf{Com}$, by Theorem 6.3, $\Gamma \vdash_{\mathsf{DVC}} \varphi$ implies $\Gamma \models_{\mathsf{DeV}} \varphi$. Conversely, suppose $\Gamma \not\vdash_{\mathsf{DVC}} \varphi$. By Theorem 6.3, there is $(B, \rightsquigarrow) \in \mathsf{Com}$ satisfying Γ and refuting φ. Let $\prec$ be the corresponding compingent relation. By Lemma 6.5, $\lhd$ is a compingent relation on $\overline{B}$. Let $\rightsquigarrow'$ be the strict implication corresponding to $\lhd$. Since B is a subalgebra of $\overline{B}$ and $\lhd$ extends $\prec$, we see that $(B, \rightsquigarrow)$ is a subalgebra of $(\overline{B}, \rightsquigarrow')$. Therefore, $(\overline{B}, \rightsquigarrow')$ satisfies Γ and refutes φ. Thus, $\Gamma \models_{\mathsf{DeV}} \varphi$. □

7 Topological completeness

In this section we show how our results from the previous sections directly imply completeness results for the topological interpretation of our language.

We recall that the de Vries algebra of a compact Hausdorff space X is the pair $(\mathcal{RO}(X), \prec)$, where $\mathcal{RO}(X)$ is the complete Boolean algebra of regular open subsets of X and $U \prec V$ iff $\mathsf{Cl}(U) \subseteq V$. By de Vries duality [13], every de Vries algebra is isomorphic to the de Vries algebra of some compact Hausdorff space. This allows us to define topological semantics for our language.

Definition 7.1. A *compact Hausdorff model* is a pair (X, v), where X is a compact Hausdorff space and v is a valuation assigning a regular open set to each propositional letter.

If $\rightsquigarrow$ is the strict implication corresponding to $\prec$, then the formulas of our language are interpreted in $(\mathcal{RO}(X), \rightsquigarrow) \in \mathsf{DeV}$. Since each $(B, \rightsquigarrow) \in \mathsf{DeV}$ is isomorphic to $(\mathcal{RO}(X), \rightsquigarrow)$ for some compact Hausdorff space X, as an immediate consequence of Theorem 6.6, we obtain the following result.

Theorem 7.2. *The system* DVC *is strongly sound and complete with respect to compact Hausdorff models.*

In the completeness proof of DVC with respect to de Vries algebras, we have used the fact that being a compingent relation is preserved by the MacNeille completion. Since every de Vries algebra is isomorphic to the de Vries algebra of a compact Hausdorff space, a first-order statement in the language of Boolean algebras with a binary relation, when satisfied by a de Vries algebra, can be regarded as expressing a property which is satisfied by the regular open subsets of a compact Hausdorff space. This fact can be used to obtain deductive systems that are strongly sound and complete with respect to some interesting subclasses of the class KHaus of compact Hausdorff spaces.

More precisely, suppose $\rho_1, \dots, \rho_k$ are Π_2-rules such that their corresponding first-order statements $\Phi_{\rho_1}, \dots, \Phi_{\rho_k}$ are preserved by MacNeille completions. Then the system $\mathsf{DVC} + \rho_1 + \cdots + \rho_k$ is strongly sound and complete with respect to the compact Hausdorff spaces satisfying the property $\Phi_{\rho_1} \wedge \cdots \wedge \Phi_{\rho_k}$. By the results of the previous sections, this system is strongly sound and complete with respect to the algebras in RSub which satisfy $\Phi_{\rho_1} \wedge \cdots \wedge \Phi_{\rho_k}$.

Therefore, $\mathsf{DVC}+\rho_1+\cdots+\rho_k$ is strongly sound and complete with respect to de Vries algebras satisfying $\Phi_{\rho_1} \wedge \cdots \wedge \Phi_{\rho_k}$. By de Vries duality, this can be seen as a completeness result with respect to compact Hausdorff spaces which satisfy the topological property expressed by $\Phi_{\rho_1} \wedge \cdots \wedge \Phi_{\rho_k}$. The same applies not only to Π_2-rules, but also to formulas. Therefore, we have:

Proposition 7.3. *Suppose the formulas $\varphi_1, \dots, \varphi_n$ and the Π_2-rules $\rho_1, \dots, \rho_k$ are preserved by MacNeille completions. Then the system*

$$\mathsf{DVC} + \varphi_1 + \cdots + \varphi_n + \rho_1 + \cdots + \rho_k$$

is strongly sound and complete with respect to the subclass of KHaus *satisfying the conditions $\varphi_1, \dots, \varphi_n$ and $\Phi_{\rho_1}, \dots, \Phi_{\rho_k}$.*

In the remainder of this section, we will construct deductive systems for, respectively, zero-dimensional and connected compact Hausdorff spaces.

Starting with zero-dimensionality, we consider the following property, studied in [5, 6]:

(S9) $a \prec b$ implies $\exists c : c \prec c$ and $a \prec c \prec b$.

Also consider the Π_2-rule

$$(\rho 9) \quad \frac{(p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

and the corresponding $\forall\exists$-statement

$(\Phi_{\rho 9})$ $\forall x, y, z\Big(x \rightsquigarrow y \nleq z \to \exists u : (u \rightsquigarrow u) \wedge (x \rightsquigarrow u) \wedge (u \rightsquigarrow y) \nleq z\Big)$.

Lemma 7.4. *Let $(B, \rightsquigarrow) \in \mathsf{Com}$. Then $(B, \rightsquigarrow) \models$ (S9) iff $(B, \rightsquigarrow) \models \Phi_{\rho 9}$.*

Proof. ($\Rightarrow$) Suppose $a \rightsquigarrow b \nleq d$. Then $d \neq 1$ and $a \rightsquigarrow b \neq 0$, so $a \rightsquigarrow b = 1$. Therefore, $a \prec b$, and so by (S9), there is c such that $c \prec c$ and $a \prec c \prec b$. Thus, $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) = 1 \nleq d$. Consequently, $(B, \rightsquigarrow) \models \Phi_{\rho 9}$.

($\Leftarrow$) Suppose $a \prec b$. Then $a \rightsquigarrow b = 1 \nleq 0$. Therefore, by $\Phi_{\rho 9}$, there is c such that $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \nleq 0$, which implies $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) = 1$. Thus, $c \prec c$ and $a \prec c \prec b$. Consequently, $(B, \rightsquigarrow) \models$ (S9). □

It follows that $\mathsf{DVC} + (\rho 9)$ is strongly sound and complete with respect to the class of compingent algebras satisfying (S9).

Lemma 7.5. *The property* (S9) *is preserved by MacNeille completions.*

Proof. Let $(B, \prec)$ be a compingent algebra satisfying (S9), and let $(\overline{B}, \lhd)$ be its MacNeille completion. Suppose $x \lhd y$ in $\overline{B}$. Then there are $a, b \in B$ such that $x \leq a \prec b \leq y$. By (S9) for $\prec$, there is $c \in B$ such that $c \prec c$ and $a \prec c \prec b$. By (S4), $x \lhd c \lhd y$. Thus, $(\overline{B}, \lhd)$ satisfies (S9). □

Consequently, $\mathsf{DVC}+(\rho 9)$ is strongly sound and complete with respect to de Vries algebras satisfying (S9). It is proved in [5, Sec. 4] that a de Vries algebra satisfies (S9) iff its dual compact Hausdorff space is zero-dimensional. Thus, we arrive at the following.

Theorem 7.6. *The system* DVC + (ρ9) *is strongly sound and complete with respect to the class of zero-dimensional compact Hausdorff spaces.*

Turning to the connectedness, we now consider the following property:

(S10) $a \prec a$ implies $a = 0$ or $a = 1$.

Clearly $(B, \rightsquigarrow) \in$ Com satisfies (S10) iff $(a \rightsquigarrow a) \to (1 \rightsquigarrow a) \vee (1 \rightsquigarrow \neg a)$ is satisfied in $(B, \rightsquigarrow)$. Therefore, $(B, \rightsquigarrow)$ satisfies (S10) iff $(B, \rightsquigarrow) \models$ (C), where (C) is the formula

(C) $(\varphi \rightsquigarrow \varphi) \to (\top \rightsquigarrow \varphi) \vee (\top \rightsquigarrow \neg\varphi)$.

Consequently, DVC + (C) is strongly sound and complete with respect to the class of compingent algebras satisfying (S10).

Lemma 7.7. *The property* (S10) *is preserved by MacNeille completions.*

Proof. Let $(B, \prec)$ be a compingent algebra satisfying (S10), and let $(\overline{B}, \lhd)$ be its MacNeille completion. Suppose $x \lhd x$ in $\overline{B}$. Then it follows from the definition of $\lhd$ that $x \in B$. Therefore, (S10) for $\prec$ yields that $x = 0$ or $x = 1$. □

It follows that DVC + (C) is strongly sound and complete with respect to the class of de Vries algebras satisfying (S10).

Lemma 7.8. *A de Vries algebra* $(B, \prec)$ *satisfies* (S10) *iff its dual space* X *is connected.*

Proof. Suppose U is a clopen subset of X. Then $U \prec U$ in $\mathcal{RO}(X)$. Since $(\mathcal{RO}(X), \prec)$ is isomorphic to $(B, \prec)$ and $(B, \prec)$ satisfies (S10), we see that $U = \varnothing$ or $U = X$. Thus, X is connected. Conversely, if X is connected, then $\varnothing, X$ are the only clopen subsets of X. Therefore, for $U \in \mathcal{RO}(X)$, we have $U \prec U$ implies $U = \varnothing$ or $U = X$. Thus, $(\mathcal{RO}(X), \prec)$ satisfies (S10). Since $(\mathcal{RO}(X), \prec)$ is isomorphic to $(B, \prec)$, we conclude that $(B, \prec)$ satisfies (S10). □

As a result, we arrive at the following:

Theorem 7.9. *The system* DVC + (C) *is strongly sound and complete with respect to the class of connected compact Hausdorff spaces.*

8 Admissibility of Π_2-rules

In this section we show that (ρ6) and (ρ7) are admissible in SIC + (A5). To prove this, we develop a semantic criterion for establishing when a given Π_2-rule is admissible in SIC.

Definition 8.1. A rule ρ is *admissible* in a system $\mathcal{S}$ if for each formula φ, from $\vdash_{\mathcal{S}+\rho} \varphi$ it follows that $\vdash_{\mathcal{S}} \varphi$.

Lemma 8.2. *A* Π_2*-rule*

$$(\rho) \qquad \frac{F(\overline{\varphi}, \overline{p}) \to \chi}{G(\overline{\varphi}) \to \chi}$$

is admissible in SIC *iff for any set of formulas* Γ *and any tuple* $\overline{\varphi}, \chi$ *of formulas, if* $\Gamma \vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$ *and* $\overline{p}$ *does not appear in* $\Gamma, \overline{\varphi}, \chi$*, then* $\Gamma \vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$.

Proof. (⇒) Suppose $\Gamma \vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p}) \to \chi$ and $\overline{p}$ does not appear in $\Gamma, \overline{\varphi}, \chi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p}) \to \chi$. Let $\psi = \bigwedge \Gamma_0$, so $\{\psi\} \vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p}) \to \chi$. By Theorem 4.4, $\{\psi\} \models_{\mathsf{RSub}} F(\overline{\varphi},\overline{p}) \to \chi$. We show that $\models_{\mathsf{RSub}} \Box\psi \to (F(\overline{\varphi},\overline{p}) \to \chi)$. Suppose $(B, \rightsquigarrow) \in \mathsf{RSub}$ and v is a valuation on $(B, \rightsquigarrow)$ such that $v(\Box\psi) = 1$. Then $v(\psi) = 1$, so $v(F(\overline{\varphi},\overline{p}) \to \chi) = 1$. Therefore, applying Theorem 4.4 again yields $\vdash_{\mathsf{SIC}} \Box\psi \to (F(\overline{\varphi},\overline{p}) \to \chi)$, so $\vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p}) \to (\Box\psi \to \chi)$. Since $\overline{p}$ does not appear in $\overline{\varphi}, \Box\psi \to \chi$, by admissibility of ρ, we have $\vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to (\Box\psi \to \chi)$. Therefore, $\models_{\mathsf{RSub}} G(\overline{\varphi}) \to (\Box\psi \to \chi)$, so $\models_{\mathsf{RSub}} \Box\psi \to (G(\overline{\varphi}) \to \chi)$, and hence the same argument as above yields $\{\psi\} \models_{\mathsf{RSub}} G(\overline{\varphi}) \to \chi$. Thus, $\{\psi\} \vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$, and so $\Gamma \vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$.

(⇐) Suppose $\vdash_{\mathsf{SIC}+\rho} \psi$. Let $\psi_1, \dots, \psi_n$ be a proof of ψ in $\mathsf{SIC} + \rho$. We show by induction on $i = 1, \dots, n$ that $\vdash_{\mathsf{SIC}} \psi_i$, and hence $\vdash_{\mathsf{SIC}} \psi$. If $i = 1$, then ψ_1 is an instance of an axiom of SIC, so $\vdash_{\mathsf{SIC}} \psi_1$. Suppose $i > 1$. If ψ_i is an instance of an axiom of SIC or is obtained by (MP) or (R) from ψ_j, ψ_k with $j, k < i$, then it is obvious that $\vdash_{\mathsf{SIC}} \psi_i$. Let $\psi_i = G(\overline{\varphi}) \to \chi$ be obtained by ρ from $\psi_j = F(\overline{\varphi},\overline{p}) \to \chi$ with $j < i$ and $\overline{p}$ not appearing in $\overline{\varphi}, \chi$. By inductive hypothesis, $\vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p}) \to \chi$. By assumption, $\vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$. Thus, $\vdash_{\mathsf{SIC}} \psi$. □

Theorem 8.3 (Admissibility Criterion). *A Π_2-rule ρ is admissible in SIC iff for each $(B, \rightsquigarrow) \in \mathsf{RSub}$ there is $(C, \rightsquigarrow) \in \mathsf{RSub}$ such that $(B, \rightsquigarrow)$ is a substructure of $(C, \rightsquigarrow)$ and $(C, \rightsquigarrow) \models \Phi_\rho$.*

Proof. (⇒) Let $(B, \rightsquigarrow) \in \mathsf{RSub}$ and let $(B_0, \rightsquigarrow)$ be a countable elementary substructure of $(B, \rightsquigarrow)$. Consider the set $\{p_a \mid a \in B_0\}$ of propositional letters and let

$$U = \{\Phi^*(\overline{p_a}) \mid (B_0, \rightsquigarrow) \models \Phi[\overline{a}/\overline{x}]\},$$

where $\Phi(\overline{x})$ is quantifier-free and Φ^* is defined as in Definition 5.3.

Since $(B_0, \rightsquigarrow) \models U$, by Theorem 4.4, U is consistent in SIC. Therefore, as ρ is admissible in SIC, U is also consistent in $\mathsf{SIC} + \rho$. Thus, as in the proof of Theorem 5.2, we can construct $(D, \rightsquigarrow) \in \mathsf{RSub}$ and a valuation v on $(D, \rightsquigarrow)$ such that $(D, \rightsquigarrow) \models \Phi_\rho$ and v satisfies all formulas in U. Sending a to $v(p_a)$ is then an embedding of $(B_0, \rightsquigarrow)$ into $(D, \rightsquigarrow)$.

Claim. There is $(C, \rightsquigarrow) \in \mathsf{RSub}$ such that $(D, \rightsquigarrow)$ is an elementary substructure of $(C, \rightsquigarrow)$ and $(B, \rightsquigarrow)$ is a substructure of $(C, \rightsquigarrow)$.

Proof of claim. Let $\mathrm{Th}(\mathsf{RSub})$ be the first-order theory of RSub, and consider the first-order theory

$$T = \mathrm{Th}(\mathsf{RSub}) \cup \{\Phi(\overline{x}) \mid (D, \rightsquigarrow) \models \Phi(\overline{x})\} \cup \{\Psi(\overline{x}) \mid (B, \rightsquigarrow) \models \Psi(\overline{x})\},$$

where $\Phi(\overline{x})$ is a first-order formula and $\Psi(\overline{x})$ is quantifier-free. Suppose for contradiction that T is inconsistent. Then by compactness there exist $\overline{a} \in B_0, \overline{b} \in B \setminus B_0, \overline{c} \in D$, a first-order formula $\Phi(\overline{x}, \overline{z})$, and a quantifier-free formula $\Psi(\overline{x}, \overline{y})$ such that

$$(D, \rightsquigarrow) \models \Phi(\overline{a}, \overline{c}), \quad (1)$$

$$(B, \rightsquigarrow) \models \Psi(\overline{a}, \overline{b}), \quad (2)$$

$$\mathrm{Th}(\mathsf{RSub}) \models \Phi(\overline{a}, \overline{c}) \to \neg\Psi(\overline{a}, \overline{b}). \quad (3)$$

Since the constants $\overline{a}, \overline{b}, \overline{c}$ do not occur among the formulas of $\mathrm{Th}(\mathsf{RSub})$, by (3), $\mathrm{Th}(\mathsf{RSub}) \models \exists \overline{z}\Phi(\overline{a}, \overline{z}) \to \forall \overline{y}\neg\Psi(\overline{a}, \overline{y})$. Because $(D, \rightsquigarrow)$ is a model of $\mathrm{Th}(\mathsf{RSub})$, (1) yields $(D, \rightsquigarrow) \models$

$\forall\overline{y}\neg\Psi(\overline{a},\overline{y})$. Therefore, as $\forall\overline{y}\neg\Psi(\overline{a},\overline{y})$ is a universal statement and $(B_0,\rightsquigarrow)$ is a substructure of $(D,\rightsquigarrow)$, we obtain $(B_0,\rightsquigarrow)\models\forall\overline{y}\neg\Psi(\overline{a},\overline{y})$. On the other hand, by (2), $(B,\rightsquigarrow)\models\exists\overline{y}\Psi(\overline{a},\overline{y})$. Since $(B_0,\rightsquigarrow)$ is an elementary substructure of $(B,\rightsquigarrow)$, we have $(B_0,\rightsquigarrow)\models\exists\overline{y}\Psi(\overline{a},\overline{y})$, and so we have arrived at the desired contradiction. In other words, we have proved that T is consistent. But then T must have a model $(C,\rightsquigarrow)$, which is easily seen to satisfy the conditions mentioned in the Claim. □

From the Claim it is immediate that $(B,\rightsquigarrow)$ is a substructure of $(C,\rightsquigarrow)$ and $(C,\rightsquigarrow)\models\Phi_\rho$.

($\Leftarrow$) Suppose $\vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p})\to\chi$ with $\overline{p}$ not occurring in $\overline{\varphi},\chi$. Let $(B,\rightsquigarrow)\in\mathsf{RSub}$ and let v be a valuation on $(B,\rightsquigarrow)$. By assumption, there is $(C,\rightsquigarrow)\in\mathsf{RSub}$ such that $(B,\rightsquigarrow)$ is a substructure of $(C,\rightsquigarrow)$ and $(C,\rightsquigarrow)\models\Phi_\rho$. Let $i:B\hookrightarrow C$ be the inclusion. Then $v' := i\circ v$ is a valuation on $(C,\rightsquigarrow)$. For any $\overline{c}\in C$, let v'' be the valuation $(v')^{\overline{c}}_{\overline{p}}$. Since $\vdash_{\mathsf{SIC}} F(\overline{\varphi},\overline{p})\to\chi$, we have $v''(F(\overline{\varphi},\overline{p})\to\chi)=1_C$. This means that for all $\overline{c}\in C$, we have $F(v'(\overline{\varphi}),\overline{c})\le v'(\chi)$. Therefore, $(C,\rightsquigarrow)\models\forall\overline{y}\Big(F(v'(\overline{\varphi}),\overline{y})\le v'(\chi)\Big)$. Since $(C,\rightsquigarrow)\models\Phi_\rho$, we have $(C,\rightsquigarrow)\models G(v'(\overline{\varphi}))\le v'(\chi)$. Thus, as $G(v'(\overline{\varphi}))\le v'(\chi)$ in C, we have $G(v(\overline{\varphi}))\le v(\chi)$ in B. Consequently, $v(G(\overline{\varphi})\to\chi)=1_B$. This shows that $\vdash_{\mathsf{SIC}} G(\overline{\varphi})\to\chi$, and hence, by Lemma 8.2, ρ is admissible in SIC. □

Remark 8.4. Lemma 8.2 and Theorem 8.3 hold also if we replace SIC with SIC+(A5) and RSub with Con.

Next we will utilize Theorem 8.3 to prove that both rules (ρ6) and (ρ7) are admissible in SIC+(A5). For the proof of the following simple lemma consult [14, 7].

Lemma 8.5. *Let R be a binary relation on a set X. Define $\prec_R$ on $\mathcal{P}(X)$ by $U\prec_R V$ iff $R[U]\subseteq V$.*

(1) *$\prec_R$ is a subordination on $\mathcal{P}(X)$.*
(2) *R is reflexive iff $(\mathcal{P}(X),\prec_R)$ satisfies* (S5).
(3) *R is symmetric iff $(\mathcal{P}(X),\prec_R)$ satisfies* (S6).
(4) *R is transitive iff $(\mathcal{P}(X),\prec_R)$ satisfies* (S7).

We use Lemma 8.5 to show an analogue of [1, Lem. 2.5] in our setting. We recall [7, Sec. 2.1] that if $(B,\prec)\in\mathsf{Sub}$, then the dual of $(B,\prec)$ is the pair (X,R), where X is the Stone space of B and R is given by xRy iff $\Uparrow x\subseteq y$; here $\Uparrow S=\{a\in B\mid \exists b\in S \text{ with } b\prec a\}$ for $S\subseteq B$. Moreover, R is reflexive iff $(B,\prec)$ satisfies (S5), R is symmetric iff $(B,\prec)$ satisfies (S6), and R is transitive iff $(B,\prec)$ satisfies (S7). Furthermore, if $\mathsf{Clop}(X)$ is the Boolean algebra of clopens of X, then $(B,\prec)$ is isomorphic to $(\mathsf{Clop}(X),\prec_R)$, where $U\prec_R V$ iff $R[U]\subseteq V$.

Lemma 8.6. *Every $(B,\prec)\in\mathsf{Con}$ can be embedded into $(C,\prec)\in\mathsf{Con}$ satisfying* (S7).

Proof. Suppose (X,R) is the dual of $(B,\prec)$. Then R is reflexive and symmetric. Let $Y=\{\{x,y\}\subseteq X\mid xRy\}$ and let

$$X'=\{(x,\alpha)\in X\times Y\mid x\in\alpha\}.$$

Define R' on X' by

$$(x,\alpha)R'(y,\beta)\Leftrightarrow\alpha=\beta.$$

Clearly R' is an equivalence relation on X' and $f : X' \to X$ given by $f(x, \alpha) = x$ is onto. Therefore, $f^{-1} : \mathsf{Clop}(X) \to \mathcal{P}(X')$ is a Boolean embedding. Since R is reflexive and symmetric, it follows from the definition of R' that $(x, \alpha)R'(y, \beta)$ implies xRy.

Claim. For $U, V \in \mathsf{Clop}(X)$, we have $U \prec_R V$ iff $f^{-1}(U) \prec_{R'} f^{-1}(V)$.

Proof of claim. Since $(x, \alpha)R'(y, \beta)$ implies $f(x, \alpha)Rf(y, \beta)$, we see that $U \prec_R V$ implies $f^{-1}(U) \prec_{R'} f^{-1}(V)$. For the converse, suppose $U \not\prec_R V$. Then $R[U] \nsubseteq V$. Therefore, there are $x \in U$ and $y \notin V$ such that xRy. Let $\alpha = \{x, y\}$. Then $(x, \alpha)R'(y, \alpha)$, $(x, \alpha) \in f^{-1}(U)$, and $(y, \alpha) \notin f^{-1}(V)$. Thus, $R'[f^{-1}(U)] \nsubseteq f^{-1}(V)$, and hence $f^{-1}(U) \not\prec_{R'} f^{-1}(V)$. □

Let $(C, \prec) = (\mathcal{P}(X'), \prec_{R'})$. By Lemma 8.5, $(C, \prec)$ satisfies (S1)–(S7), and by the Claim, f^{-1} is an embedding of $(B, \prec)$ into $(C, \prec)$. □

Theorem 8.7. *(ρ6) is admissible in* $\mathsf{SIC} + (A5)$.

Proof. Apply Theorem 8.3 and Lemmas 6.2(1) and 8.6. □

Lemma 8.8. *Suppose* $(B, \prec) \in \mathsf{RSub}$. *Let* $B' = B \times B$ *and define* $\prec'$ *on* B' *by*

$$(a, b) \prec' (c, d) \Leftrightarrow a \prec c \text{ and } b \le d.$$

Then $(B', \prec') \in \mathsf{RSub}$. *Moreover, if* $(B, \prec) \in \mathsf{Con}$, *then* $(B', \prec') \in \mathsf{Con}$.

Proof. It is sufficient to show that $(B', \prec')$ satisfies (S1)–(S6).

(S1) Since $0 \prec 0$ and $1 \prec 1$, it is obvious that $(0, 0) \prec' (0, 0)$ and $(1, 1) \prec' (1, 1)$.

(S2) Suppose $(a, b) \prec (c, d), (c', d')$. Then $a \prec c, c'$ and $b \le d, d'$. Therefore, $a \prec c \wedge c'$ and $b \le d \wedge d'$. Thus, $(a, b) \prec' (c, c') \wedge (d, d')$.

(S3) Suppose $(a, b), (a', b') \prec' (c, d)$. Then $a, a' \prec c$ and $b, b' \le d$. Therefore, $a \vee a' \prec c$ and $b \vee b' \le d$. Thus, $(a \vee a', b \vee b') \prec' (c, d)$.

(S4) Suppose $(a, b) \le (a', b') \prec (c', d') \le (c, d)$. Then $a \le a' \prec c' \le c$ and $b \le b' \le d' \le d$. Thus, $a \prec c$ and $b \le d$, and so $(a, b) \prec' (c, d)$.

(S5) Suppose $(a, b) \prec' (c, d)$. Then $a \prec c$ and $b \le d$. From $a \prec c$ it follows that $a \le c$. Thus, $(a, b) \le (c, d)$.

(S6) Suppose $(a, b) \prec' (c, d)$. Then $a \prec c$ and $b \le d$. Therefore, $\neg c \prec \neg a$ and $\neg d \le \neg b$. Thus, $\neg(c, d) \prec' \neg(a, b)$. □

Lemma 8.9. *Every* $(B, \prec) \in \mathsf{RSub}$ *can be embedded into* $(C, \prec) \in \mathsf{RSub}$ *satisfying* (S8).

Proof. Starting from $(B, \prec)$, we inductively build a chain

$$(B, \prec) \hookrightarrow (B_1, \prec) \hookrightarrow (B_2, \prec) \hookrightarrow (B_3, \prec) \hookrightarrow \cdots$$

in RSub such that the union $(C, \prec) := \bigcup_{n \in \omega} (B_n, \prec)$ satisfies (S8).

If $(B_n, \prec)$ is already defined, define $(B_{n+1}, \prec) := (B_n, \prec) \times (B_n, \le)$. By Lemma 8.8, $(B_{n+1}, \prec) \in \mathsf{RSub}$. Moreover, $a \mapsto (a, a)$ is an embedding of $(B_n, \prec)$ into $(B_{n+1}, \prec)$. We prove that $(C, \prec)$ satisfies (S8).

Let $0 \ne a \in C$. Then there is n such that $a \in B_n$. Therefore, $(a, a) \in B_{n+1}$. Let $b := (0, a) \in B_{n+1}$. We have $b \ne 0$ and $b \prec (a, a)$. Thus, $(B, \prec)$ satisfies (S8). □

Theorem 8.10. *($\rho 7$) is admissible in* SIC.

Proof. Apply Theorem 8.3 and Lemmas 6.2(2) and 8.9. □

Remark 8.11. By Theorems 8.7 and 8.10, the logic of DVC (the set of formulas derivable in DVC) is equal to $\mathsf{SIC} + \text{(A5)}$. This means that the logic of compingent algebras is the same as the logic of contact algebras. On the other hand, the Π_2-theories (the sets of derivable Π_2-rules) of DVC and $\mathsf{SIC} + \text{(A5)}$ are obviously different—the Π_2-rules ($\rho 6$) and ($\rho 7$) are derivable in the former but not in the latter. These two rules capture the very essence of the theory of compact Haudorff spaces in our language. This generates an interesting methodological question of what the right logical formalism should be to reason about compact Haudroff spaces. Should we be concerned only with the logics or should we also consider the theories of Π_2-rules? Although in this paper we are only concerned with logics, the results in this section suggest that a theory of Π_2-rules may be a more appropriate framework to reason about compact Hausdorff spaces. We leave it as a future work to develop the Π_2-theory for compact Hausdorff spaces together with the general theory of such calculi. The connection of Π_2-rules and inductive classes developed in Section 5 should play a key role in these investigations.

9 Comparison with relevant work

In this final section we compare our approach to that of Balbiani et al. [1]. Namely, we show how to translate fully and faithfully the language $\mathsf{L}(\mathsf{C}, \leq)$ of [1] into our language.

We recall that the formulas of the language $\mathsf{L}(\mathsf{C}, \leq)$ are built from atomic formulas using Boolean connectives $\neg, \wedge, \vee, \to, \bot, \top$; atomic formulas are of the form $t\mathsf{C}s$ and $t \leq s$, where t, s are Boolean terms (C stands for the contact relation and $\leq$ for the inclusion relation). In turn, Boolean terms are built from Boolean variables using Boolean operations $\sqcap, \sqcup, (-)^*, 0, 1$.

As usual, a *Kripke frame* is a pair (W, R), where W is a nonempty set and R is a binary relation on W, and a *valuation* is a map v from the set of Boolean variables to the powerset $\mathcal{P}(W)$. It extends to the set of all Boolean terms as follows:

$$\begin{aligned} v(t \sqcap s) &= v(t) \cap v(s), \\ v(t \sqcup s) &= v(t) \cup v(s), \\ v(t^*) &= W \setminus v(t), \\ v(0) &= \varnothing, \\ v(1) &= W. \end{aligned}$$

A *Kripke model* is a triple (W, R, v) consisting of a Kripke frame (W, R) and a valuation v. Atomic formulas are interpreted in (W, R, v) as follows:

$$\begin{aligned} (W, R, v) \models (t \leq s) &\iff v(t) \subseteq v(s), \\ (W, R, v) \models (t\mathsf{C}s) &\iff R[v(t)] \cap v(s) \neq \varnothing. \end{aligned}$$

Complex formulas are then interpreted by the induction clauses for propositional connectives.

In [1, Sec. 6] the authors define the propositional calculus PWRCC in the language $\mathsf{L}(\mathsf{C}, \leq)$ and prove that PWRCC is sound and complete with respect to the class of Kripke frames where the binary relation R is reflexive and symmetric. Such Kripke frames are closely related to contact algebras. Namely, as we already pointed out in Section 8, the following lemma holds.

Lemma 9.1.

1. *Suppose* (W, R) *is a reflexive and symmetric Kripke frame. Define* $\prec_R$ *on* $\mathcal{P}(W)$ *by* $U \prec_R V$ *iff* $R[U] \subseteq V$*. Then* $(\mathcal{P}(W), \prec_R)$ *is a contact algebra.*

2. *Suppose* $(B, \prec)$ *is a contact algebra and* (X, R) *is the dual of* $(B, \prec)$*. Then* (X, R) *is a reflexive and symmetric Kripke frame, and the Stone map* $\beta : B \to \mathcal{P}(X)$*, given by* $\beta(a) = \{x \in X \mid a \in x\}$*, is an embedding of* $(B, \rightsquigarrow)$ *into* $(\mathcal{P}(X), \rightsquigarrow_R)$*.*

We next translate $\mathsf{L}(\mathsf{C}, \leq)$ into our language $\mathcal{L}$. We identify the set of Boolean variables of $\mathsf{L}(\mathsf{C}, \leq)$ with the set of proposition letters of $\mathcal{L}$. Then Boolean terms can be translated into formulas of $\mathcal{L}$ as follows:

$$
\begin{aligned}
a^T &= a, \text{ for a Boolean variable } a,\\
(t \sqcap s)^T &= t^T \wedge s^T,\\
(t \sqcup s)^T &= t^T \vee s^T,\\
(t^*)^T &= \neg(t^T),\\
0^T &= \bot,\\
1^T &= \top.
\end{aligned}
$$

For atomic formulas, we define:

$$
\begin{aligned}
(t \leq s)^T &= \Box(t^T \to s^T),\\
(t\mathsf{C}s)^T &= \neg(t^T \rightsquigarrow \neg s^T).
\end{aligned}
$$

Finally, complex formulas are translated inductively as follows:

$$
\begin{aligned}
(\neg\varphi)^T &= \neg\varphi^T,\\
(\varphi \wedge \psi)^T &= \varphi^T \wedge \psi^T,\\
(\varphi \vee \psi)^T &= \varphi^T \vee \psi^T,\\
(\varphi \to \psi)^T &= \varphi^T \to \psi^T,\\
\bot^T &= \bot,\\
\top^T &= \top.
\end{aligned}
$$

Theorem 9.2. *For any formula* φ *of* $\mathsf{L}(\mathsf{C}, \leq)$*, we have*

$$\mathsf{PWRCC} \vdash \varphi \textit{ iff } \mathsf{DVC} \vdash \varphi^T.$$

Proof. By [1, Cor. 6.1], PWRCC is sound and complete with respect to the class of reflexive and symmetric Kripke frames (W, R); and by Theorem 6.3 and Remark 8.11, DVC is sound and complete with respect to the class of contact algebras. Given a Kripke model (W, R, v), the valuation v of Boolean variables of $\mathsf{L}(\mathsf{C}, \leq)$ into $\mathcal{P}(W)$ can be seen as a valuation of propositional letters of $\mathcal{L}$ into the algebra $(\mathcal{P}(W), \rightsquigarrow_R)$.

Claim 9.3. $(W, R, v) \models \varphi$ *iff* $(\mathcal{P}(W), \rightsquigarrow_R, v) \models \varphi^T$.

Proof of Claim. For a Boolean term t, we have $v(t) = v(t^T) \subseteq W$. If φ is an atomic formula of the form $t \leq s$, then

$$\begin{aligned}(W, R, v) \models \varphi \text{ iff } & v(t) \subseteq v(s)\\ \text{iff } & v(t^T) \leq v(s^T) \text{ in } \mathcal{P}(W)\\ \text{iff } & (\mathcal{P}(W), \rightsquigarrow_R, v) \models t^T \to s^T\\ \text{iff } & (\mathcal{P}(W), \rightsquigarrow_R, v) \models \Box(t^T \to s^T)\\ \text{iff } & (\mathcal{P}(W), \rightsquigarrow_R, v) \models \varphi^T.\end{aligned}$$

If φ is an atomic formula of the form $t\mathbf{C}s$, then

$$\begin{aligned}(W, R, v) \models \varphi \text{ iff } & R[v(t)] \cap v(s) \neq \emptyset\\ \text{iff } & R[v(t)] \nsubseteq W \setminus v(s)\\ \text{iff } & R[v(t^T)] \nsubseteq v(\neg s^T)\\ \text{iff } & (\mathcal{P}(W), \rightsquigarrow_R, v) \models \neg(t^T \rightsquigarrow \neg s^T)\\ \text{iff } & (\mathcal{P}(W), \rightsquigarrow_R, v) \models \varphi^T.\end{aligned}$$

Finally, if φ is a complex formula, then a straightforward induction completes the proof. □

Now, if $\mathsf{PWRCC} \nvdash \varphi$, then there is a reflexive and symmetric Kripke model (W, R, v) refuting φ. By Claim 9.3, φ^T is refuted in $(\mathcal{P}(W), \rightsquigarrow_R, v)$. Therefore, $\mathsf{DVC} \nvdash \varphi^T$. Conversely, if $\mathsf{DVC} \nvdash \varphi^T$, then there is a contact algebra $(B, \prec)$ and a valuation v on $(B, \prec)$ refuting φ^T. By Lemma 9.1(2), φ^T is refuted in $(\mathcal{P}(X), \rightsquigarrow_R, v)$. By Claim 9.3, φ is refuted in (X, R, v). Therefore, $\mathsf{PWRCC} \nvdash \varphi$. □

References

[1] Ph. Balbiani, T. Tinchev, and D. Vakarelov. Modal logics for region-based theories of space. *Fund. Inform.*, 81(1-3):29–82, 2007.

[2] B. Banaschewski. More on compact Hausdorff spaces and finitary duality. *Canad. J. Math.*, 36(6):1113–1118, 1984.

[3] J. L. Bell and A. B. Slomson. *Models and ultraproducts: An introduction.* North-Holland Publishing Co., Amsterdam-London, 1969.

[4] G. Bezhanishvili. Locally finite varieties. *Algebra Universalis*, 46(4):531–548, 2001.

[5] G. Bezhanishvili. Stone duality and Gleason covers through de Vries duality. *Topology Appl.*, 157(6):1064–1080, 2010.

[6] G. Bezhanishvili. Lattice subordinations and Priestley duality. *Algebra Universalis*, 70(4):359–377, 2013.

[7] G. Bezhanishvili, N. Bezhanishvili, S. Sourabh, and Y. Venema. Irreducible equivalence relations, Gleason spaces, and de Vries duality. *Applied Categorical Structures*, 2016. DOI 10.1007/s10485-016-9434-2.

[8] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*. Cambridge University Press, Cambridge, 2001.

[9] J. P. Burgess. Decidability for branching time. *Studia Logica*, 39(2-3):203–218, 1980.

[10] S. Celani. Quasi-modal algebras. *Math. Bohem.*, 126(4):721–736, 2001.

[11] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[12] M. de Rijke. The modal logic of inequality. *J. Symbolic Logic*, 57(2):566–584, 1992.

[13] H. de Vries. *Compact spaces and compactifications. An algebraic approach*. PhD thesis, University of Amsterdam, 1962.

[14] I. Düntsch and D. Vakarelov. Region-based theory of discrete spaces: a proximity approach. *Ann. Math. Artif. Intell.*, 49(1-4):5–14, 2007.

[15] L. Esakia. *Heyting algebras I. Duality theory (Russian)*. "Metsniereba", Tbilisi, 1985.

[16] L. Esakia. Boolean algebras with subordination. In *Methods for research in logic (Russian)*, pages 75–82. "Metsniereba", Tbilisi, 1987.

[17] D. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In *Aspects of philosophical logic (Tübingen, 1977)*, volume 147 of *Synthese Library*, pages 67–89. Reidel, Dordrecht-Boston, Mass., 1981.

[18] D. Gabbay and I. Hodkinson. An axiomatization of the temporal logic with Until and Since over the real numbers. *J. Logic Comput.*, 1(2):229–259, 1990.

[19] I. Gelfand and M. Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.

[20] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[21] J. R. Isbell. Atomless parts of spaces. *Math. Scand.*, 31:5–32, 1972.

[22] P. Jipsen. Discriminator varieties of Boolean algebras with residuated operators. In C. Rauszer, editor, *Algebraic Methods in Logic and Computer Science*, volume 28 of *Banach Center Publications*, pages 239–252. Polish Academy of Sciences, 1993.

[23] P. T. Johnstone. *Stone spaces*. Cambridge University Press, Cambridge, 1982.

[24] A. Jung, M. Kegelmann, and M. Moshier. Multi lingual sequent calculus and coherent spaces. *Fundam. Inform.*, 37(4):369–412, 1999.

[25] A. Jung and Ph. Sünderhauf. On the duality of compact vs. open. In *Papers on general topology and applications (Gorham, ME, 1995)*, volume 806 of *Ann. New York Acad. Sci.*, pages 214–230. New York Acad. Sci., New York, 1996.

[26] S. Kakutani. Weak topology, bicompact set and the principle of duality. *Proc. Imp. Acad. Tokyo*, 16:63–67, 1940.

[27] M. Krein and S. Krein. On an inner characteristic of the set of all continuous functions defined on a bicompact Hausdorff space. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 27:427–430, 1940.

[28] M. A. Moshier. On the relationship between compact regularity and Gentzen's cut rule. *Theoret. Comput. Sci.*, 316(1-3):113–136, 2004.

[29] S. A. Naimpally and B. D. Warrack. *Proximity spaces*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 59. Cambridge University Press, London, 1970.

[30] M. B. Smyth. Stable compactification. I. *J. London Math. Soc. (2)*, 45(2):321–340, 1992.

[31] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.

[32] M. H. Stone. Topological representation of distributive lattices and Brouwerian logics. *Časopis Pešt. Mat. Fys.*, 67:1–25, 1937.

[33] M. H. Stone. A general theory of spectra. I. *Proc. Nat. Acad. Sci. U. S. A.*, 26:280–283, 1940.

[34] Y. Venema. Derivation rules as anti-axioms in modal logic. *J. Symbolic Logic*, 58(3):1003–1034, 1993.

[35] Y. Venema. Algebras and coalgebras. In *Handbook of Modal Logic*, pages 331–426. 2006.

[36] S. Vickers. *Topology via logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 1989.

[37] A. J. Ward. Representations of proximity lattices. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 17:41–57 (1975), 1974.

[38] K. Yosida. On vector lattice with a unit. *Proc. Imp. Acad. Tokyo*, 17:121–124, 1941.