Quantum Dialogues

MSc Thesis *(Afstudeerscriptie)*

written by

Laura Biziou-Van Pol
University of Amsterdam
(born 23.09.1986 in London, England)

under the supervision of **Prof Sonja Smets** and **Dr Soroush Rafiee Rad**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:	Members of the Thesis Committee:
03.07.2017	Prof Ronald De Wolf
	Dr Maria Aloni
	Dr Luca Incurvati
	Dr Dora Achourioti

Institute for Logic, Language and Computation

Abstract

This thesis explores Peter Mittelstaedt's dialogue-based system for quantum logic. It is argued that the approach is fundamentally similar to that of dialogical logic. Mittelstaedt's system is explored in detail and a comparison drawn between this approach and that of dialogical logic. The manner in which Mittelstaedt accommodates quantum features within this system is explored and some connections with the dynamic approach to quantum logic are suggested.

Introduction

Logics of scientific concepts have been of perennial interest to philosophers of science for a century or so. The logical positivists of the early twentieth century, for example, attempted to formalise the epistemology of science, the ways in which knowledge advances and scientific discoveries are made. Of course, quantum logic is a different endeavour, formalising the workings of physical systems themselves, or at least the results arising from measurements on those systems, depending on one's philosophical standpoint.

This paper aims to examine in detail a neglected dialogical approach to quantum logic developed by Peter Mittelstaedt in the 1970s[1]. As I argue in this thesis, this turns out to be very similar to the dialogical logic first developed by Lorenz and Lorenzen and now the subject of ongoing research, although it remains a niche field[2]. In our analysis we hope to clarify how we can view these quantum dialogical developments in a larger logical philosophical framework of other work on dialogics. In our conclusion section we will refer to further links to recent developments in the field of quantum logic, developments which may well be extended with interrogative elements and if they then they can benefit from the present study of quantum dialogues.

Quantum Mechanics and Quantum Logic

We begin with an extremely short summary of some key points about quantum phenomena and explain why they appear to give rise to a non-classical logic.

[1]See Mittelstaedt 1972, Mittelstaedt 1972, Mittelstaedt 1978, Mittelstaedt 1979.

[2]For an introduction to the dialogical approach, see eg. Keiff 2011, for early work on dialogical logic, see eg. Lorenzen 1955, 1965; Lorenz 1973

Quantum Phenomena

Quantum mechanics is the mathematical theory predicting the behaviour of microscopic particles. In mathematical terms, the theory is both successful and well-understood. However, in philosophical terms, it gives rise to a whole host of interpretational issues, as the behaviour of quantum systems is very much at odds with classical mechanics, and indeed with the way we usually conceive the functioning of the world at a macroscopic level (Jenann, 2015: §1). We now enumerate some of these distinctive features[3].

Physical quantities, in this context, refer to what we might think of as observables, the position, momentum and so-called 'spin' direction[4] of a particle are examples. To determine the values of such an observable, and hence to talk about physical quantity or a so-called experimental propositions, we have to perform measurements. However, in contrast to classical observations, we see that quantum measurements are non-deterministic, i.e. the outcome of a measurement is not fully determined by the input state (Baltag & Smets, 2011: 9). Initial state of a system ψ may assign to an experimental proposition a probability of 1, a probability of 0, or something in between. So, unlike in the classical setting, here there is no guarantee that a system in a given state will yield any one determinate value for all of the physical quantities when measured. The state of the system does not determine exact values for observables, but only probability distributions over these values[5].

Another distinctive feature is that there are physical quantities of quantum mechanical systems that cannot be measured simultaneously, we say these are incompatible. More precisely, as the accuracy of a measurement of one increases, the accuracy of a measurement of the other will decrease. The position and the momentum of a particle are the canonical examples of incompatible physical quantities (Hilgevoord & Uffink, 2016: §1). No quantum mechanical system ever has determinate simultaneous values for all observables pertaining to it (Ismael, 2015: §4). The mentioned case of position and momentum refers to the famous uncertainty principle, and has given rise to much philosophical discussion as to whether this is an inherent feature of the world or the result of some deficiency in either the theory or our ways of measuring. The notion of compatibility, or commensurability as

[3]There is, of course, an extensive body of literature on the subject, for a basic bibliography see for example Jenann 2015

[4]The angular momentum of a particle, which can be conceived as having a particular direction for our purposes

[5]This feature of quantum mechanics has, of course, given rise to debate about whether the theory can in fact be considered a complete description of reality, or whether there is some missing element that would facilitate a deterministic view.

it is referred to below, is a key feature for quantum logic to take into account, we will explore the logical workings of this notion in detail below.

A specific property of quantum states is that they can be in superposition. Mathematically, we can express this by taking a linear combination of given basis-states (Bacciagaluppi, 2007: 5).

In some sense, this means that a system can apparently be in several basis-states at the same time, until a measurement is made on the system causing it to 'collapse' into one of them. For instance, we can think of a particle in a superposition state of two properties P and Q as being in an indeterminate state in between exhibiting P and Q. The famous Schrödinger's cat thought experiment is intended to illustrate the peculiarity of this notion - a cat in a box in a superposition of live and dead states, which will be either alive or dead when the box is opened (i.e. when a measurement is made). Yet common sense dictates that the cat must be alive or dead regardless of whether the box has been opened or not[6]. Similarly, in classical mechanics a particle is assumed to have a well-defined determinate position, momentum etc. at all times, regardless of whether we know this information. And indeed, this conflicts with our common sense intuitions about the functioning of the macroscopic world.

This leads us to another particularity in the standard interpretation of quantum mechanics - measurements on a quantum system may change the state of the system. This is evident from the above stated fact that quantum theory is non-deterministic; and the possibility of a quantum system being in a superposition of different basis-states. Measurements, when made, yield determinate values for the physical quantity measured. And so it would appear that making a measurement somehow causes the system in question to assume some determinate value with regards to the observable measured. This is puzzling, and many different accounts of the reason for this *measurement problem* in the literature.

The above is by no means exhaustive of the manners in which quantum mechanics challenges our ordinary view of the world, but we need not delve any deeper into the practicalities of the physical theory in order to motivate the logical investigation thereof. Of course, the question now arises as to what logic has to do with the physical theory outlined.

[6] Of course, this thought experiment has precipitated reams of philosophising, which there is not space to go into here.

Non-Classical Features of Quantum Logic

From the above, it is clear that the failure of quantum systems to adhere to the laws of classical mechanics raises interesting questions regarding how to reason about them. Here we outline some of the non-classical features of quantum logic as traditionally done, in terms of both the behaviour of connectives and the failure of certain classical laws.

In mathematical terms the study of quantum logic uses the language of lattice theory, with specific reference to Hilbert spaces. In such a framework we can represent the different concepts and physical intuitions that we described above. To start, we view the experimental propositions as the closed linear subspaces (or equivalently as the projectors on these subspaces) of a given Hilbert space, where the Hilbert space provides the framework representing a quantum system.

Classical negation does not hold in the quantum setting. Given the form of the experimental propositions stated above, we can see that the classical negation of such a proposition is not necessarily an experimental proposition in its own right, in mathematical terms it does not necessarily correspond to a closed linear subspace of the Hilbert space[7]. In place of negation, quantum logic is generally set up in such a manner that the orthocomplement $\neg A$[8] of any proposition A fulfils the role of negation instead, satisfying the following conditions:

1. $A = \neg\neg A$
2. $A \subseteq B \Rightarrow \neg B \subseteq \neg A$
3. $A \vee \neg A = 1$

Standard quantum logic works with probability values of 0 and 1 only, although other probability values can be added in a different setting, different authors have their own approaches. State ψ assigns probability value 1 to event X if and only if it assigns probability value 0 to its orthocomplement, and vice versa. So in the case where we have the extreme ends of the probability spectrum the orthocomplement behaves as classical negation does (Dalla Chiara & Giuntini, 2008: 12).

[7]Indeed, as Baltag & Smets note, it is not even the case that all physically meaningful propositions about a quantum system deal with properties that are directly testable in the sense of corresponding to closed linear subspaces of a Hilbert space (Baltag & Smets, 2011: 5).

[8]The standard notation in quantum logic for orthocomplement and quantum join ('or') is different to that used below, I retain classical notation in the interest of consistency, as this is the notation used in Mittelstaedt's system, which forms the central component of this thesis.

Disjunction appears to be non-classical in the quantum setting, as mathematically we observe that the set-theoretic union of two closed linear subspaces is not necessarily itself a closed linear subspace, and therefore cannot automatically be taken to represent an experimental proposition about the system (Bacciagaluppi, 2013: 9). That is why quantum logicians work with the closure of the union, usually represented by $\sqcup$ to represent the quantum disjunction. Yet by doing so, the disjunction inherits a strange property, namely that in lattice-theoretic terms, state ψ may belong to a subspace corresponding to $X \sqcup Y$ whilst not belonging to either X or Y. Dalla Chiara & Giuntini give an illustration of this in physical terms. Quantum mechanics presents us with situations where we are dealing with alternatives that are 'semantically determined and true', while each of the alternatives in question is undetermined. Take for example a particle whose spin may assume one of two values only, up or down, where these, as mentioned, are incompatible physical quantities. It is undetermined what value the particle's spin will take upon testing, yet it must take one of the two. Hence it is untrue that the particle will have spin up, untrue that the particle will have spin down (as it will take either with a particular probability); yet true that it will take one of the two (Dalla Chiara & Giuntini, 2011: 8).

Because of the non-classical nature of quantum disjunction, distributivity ($A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$) fails here, and so, as mentioned, the algebra is not Boolean like that of classical propositional logic. The reason for the failure of distributivity hinges, of course, on the quantum disjunction just outlined. Just as we saw with the disjunction, a state ψ may belong to $A \wedge (B \vee C)$ without belonging to either $A \wedge B$ or $A \wedge C$. As Herman, Marsden & Piziack put it: 'The distinguishing feature of quantum mechanics, namely the existence of quantities which are not simultaneously measurable, led to an attack on the distributive law as the law of logic which is least tenable in quantum logic' (Herman, Marsden & Piziack, 1975: 305).

Non-classical disjunction and the failure of distributivity in the algebraic structure of quantum experimental propositions, in turn, have repercussions with regards to the implication, which is not classical in quantum logic. This will be discussed in some detail below, so I omit discussion of the matter here for the sake of brevity.

We now turn to a particularly crucial feature of quantum logic that will inform much of the following discussion. As mentioned, physical quantities of a quantum system may be compatible or incompatible (commensurable or incommensurable, in the terminology of the system to be discussed below). The intuitive meaning of this is that two observables are compatible if they can be measured simultaneously, as we saw above. The meaning of this in terms of the Hilbert space formulation outlined here is as follows. Two propositions A and B are said to be compatible if and only if:

$$A = (A \wedge B) \vee (A \wedge \neg B)$$

This relation is symmetric (Dalla Chiara, Giuntini & Greechie, 2011: 35). Mathematically speaking, compatibility equates to commutativity between the two projections corresponding to the observables at hand, i.e. $AB = BA$ (Dalla Chiara, Giuntini & Greechie, 2011: 202). In intuitive, physical terms, this means that making measurements for A and B will yield the same results regardless of the order in which the measurements are performed. More precisely, taking into account the probabilistic nature of the quantum theory at hand, the probabilities of a given result for each measurement remain the same, regardless of the order in which the measurements are performed. Given that we know that measurements can change the state of a system, this is not trivial - it may be that testing for A then affects the result of a subsequent measurement of B. Going back to the Hilbert space lattice, where the elements in question are compatible, we obtain a Boolean sublattice[9]. Conversely, propositions in the lattice are generally taken to be incompatible if neither $A \leq B$ nor $B \leq A$ holds. This will be of central importance later on when we explore Mittelstaedt's system and his way of conceptualising quantum logic.

Dialogue-Based Approaches to Logic

I will use dialogue-based approaches to develop the connections between Mittelstaedt's quantum version of this kind of dialogue-based logic and more standard variations thereon. In order to do this, I first examine a standard conception of dialogical logic (DL, henceforth) for first-order logic, disregarding the clauses for quantifiers, as Mittelstaedt's system is propositional.

It is important to note that the dialogical approach, although similar in technique to game-theoretic semantics developed by Hintikka, is fundamentally different, making no reference to the notion of truth in a model[10].

Dialogues in this context are two-player zero-sum games between idealised agents where what is at stake is a formula. One player (the Proponent) asserts a formula and must subsequently defend it from attacks by the Opponent, who attempts to refute it. If the Proponent wins the dialogue

[9]Indeed, the Foulis-Holland Theorem states that distributivity holds among any three elements of the lattice as long as one is compatible with the other two, see eg. Greechie, 1977.

[10]For an exposition of game theoretic semantics, see e.g. Hintikka, 2009.

the formula is valid, if the Opponent wins, it is not. The dialogical system comprises particle rules and structural rules. Particle rules are the means by which connectives are defined, seen in terms of the possibilities for players asserting, attacking and defending a proposition involving a connective. These essentially function as what we would usually think of as the semantics of the logic. The ways in which these can be combined in order to determine validities are then determined by means of structural rules specifying how dialogues are to proceed - this component therefore constitutes what we would usually think of as the syntax of the language.

Mittelstaedt's system has different kinds of dialogue, with a key distinction being between material and formal dialogue games. He has what he refers to as *argument rules* for both kinds of dialogue. In addition to these argument rules, he also has a set of what he terms *frame rules*, the most general meta-rules specifying how dialogues can proceed. As we will see, some of the frame rules in question correspond to DL's structural rules, and some of the argument rules for formal and material dialogues also correspond to structural rules of DL. Material dialogues will ultimately be seen to correspond to DL's particle rules; formal dialogues to its structural rules.

The means by which distinctively quantum features are incorporated into Mittelstaedt's system is via the rules for dialogues and the development of the means to accommodate the fact that there are statements about a quantum system that cannot be made at the same time; as well as the fact that making a measurement changes the state of the system and therefore what is true of it at one point is not necessarily the case later. This Mittelstaedt accounts for by means of commensurability and availability statements for the propositions involved in a given dialogue, as we shall see in detail later.

Dialogical Logic

The dialogical approach was formulated initially in the mid-twentieth century by Lorenz and Lorenzen, among others. It is a proof-theoretic approach making no reference to models but instead concerned with the notion of validity (Keiff, 2011: §5). Dialogues are used to specify whether or not a formula is valid in the system at hand - a successful strategy for the defence of a formula in a dialogue means that the formula is valid. This approach is constructive, making no reference to models, but only to finite dialogues, which are won or lost after a definite number of moves.

There are several philosophical motivations for espousing this kind of system as a means of developing first-order logic. If propositions are defined via the notion of proof, undecidable propositions are in fact not propositions at all. Therefore, says Keiff, the dialogical approach was developed to solve some of the issues with this (Keiff, 2011: §1). The dialogical method also has the neat side-effect that the differences between various logics can be explained with reference to different rules of the game.[11]

The notion of proof in this conception is defined in relation to the game - a proof of a proposition is just a dialogue in which the defender of the proposition has won the game. The games involve two kinds of rule - one specifying the general structure that the dialogue can take; and another set of rules specifying exactly which moves are acceptable on the part of the two players in any given set of circumstances. The latter specify the possible attacks and defences for any given position. The former specify more general structural rules[12]. As Keiff puts it, 'particle rules describe the local meaning of logical constants, structural rules determine their global semantics', so the notion of what constitutes a proof is determined by these structural rules. As Keiff notes, normative concepts may play a part in shaping the structural rules for a given dialogical system (Keiff, 2011 §2.1.2).

Of course, it is as well to note here the (fairly obvious, given the context) fact that the players in the dialogue games explored below are not taken to be humans, the games are not built to accommodate the inevitable strategic failings of human beings. The games in question are in this sense merely abstract, the players are generally conceived of as idealised agents. The notion of a game in this context is a device with which to define a process for checking the validity of a statement. And so the different permutations of dialogue games with which we deal below are in no way representative of actual reasoning processes, merely idealised ones. Yet this is instructive - how we ought to reason is at least as interesting as how we in fact do, if not more so.

Rules for Dialogical Logic

For the purposes of comparison with Mittelstaedt's system, we now state the basic particle rules and structural rules for a standard formulation of dialogical logic. Later in the paper, we examine the relations between the two. The structure of this section is as follows. First, particle rules are stated

[11] Of course, we shall examine this idea in much greater detail later with reference to quantum logic.

[12] A detailed explanation of these kinds of rules and the difference between them will be expounded in the context of Mittelstaedt's system below and so I keep the explanation here brief.

and explained. Structural rules for both classical and intuitionistic settings are then examined[13], and validity defined. Finally, the differences between classical and intuitionistic structural rules are discussed and examples of dialogues are given to demonstrate these differences.

First, we examine the argumentation forms for connectives, the *particle rules*. These state the possibilities for attack and defence of compound propositions. Atomic propositions cannot be attacked or defended, as the issue at hand is that of logical validity and so how one determines the truth or otherwise of atomic propositions is of no particular importance in this classical setting. As we will see later, Mittelstaedt does give a method for determining this, as he is working within an empirical quantum setting. The dialogues are conceived as an exchange between two player, the proponent stating the initial proposition, who must then attempt to defend it; and the opponent, who attempts to refute the proposition in question.

For the purposes of DL, an argumentation form is conceived as a tuple consisting of a player assignment (i.e. O or P); a set of possible attacks; a set of possible defences and a relation specifying which defences may be used against which attacks. In the following, we use $\forall$ and $\exists$ to denote a player assignment (O or P, where $\forall \neq \exists$ - the point here is that both participants may use these argumentation forms, subject to the structural rules to be enumerated below). A is used to denote the proposition being attacked or defended. The particle rules stating the possibilities for attack and defence of propositions containing the basic connectives are as follows[14].

	$\wedge$	$\vee$	$\neg$	$\rightarrow$
Assert	$\forall - ! - A \wedge B$	$\forall - ! - A \vee B$	$\forall - ! - \neg A$	$\forall - ! - A \rightarrow B$
Attack	$\exists - ? - 1$ or $\exists - ? - 2$	$\exists - ? - \wedge$	$\exists - ! - A$	$\exists - ! - A$
Defend	$\forall - ! - A$ and $\forall - ! - B$	$\forall - ! - A$ or $\forall - ! - B$	-	$\forall - ! - B$

In the above table, 1 and 2 refer to the different conjuncts. The 'force symbols' ! and ? refer to the assertion of a proposition; and to a request for the other player to prove a proposition, respectively. Note that attacks need not

[13]The difference will be important later in the paper, as Mittelstaedt's formal dialogues are intuitionistic.

[14]The formulation of the rules we use here is actually formulated for first-order logic, but because Mittelstaedt's system is propositional, I have disregarded the parts of the DL system concerning quantifiers. There are, of course, formulations of DL in purely propositional terms, see e.g. Rebuschi 2009. However, I have attempted to use what appears to me the most standard formulation.

always consist of a challenge to prove a proposition. Yet despite the attack on the implication consisting of an assertion of the antecedent, ultimately, as we can see from the corresponding defence, it amounts to a challenge to prove the consequent on the basis of the antecedent.

These essentially provide the equivalent of introduction and elimination rules, determining what it means for a proposition involving a connective to be valid in the system. So after $\forall$ has asserted a conjunction $A \wedge B$, $\exists$ must choose either the left or the right conjunct to attack, after which $\forall$ must defend the conjunct attacked. Subsequently $\exists$ attacks the other conjunct, which must also be defended by $\forall$. After a disjunction $A \vee B$ is asserted by $\forall$, $\exists$ attacks the disjunction as a whole and $\forall$ must choose either disjunct to defend (it is, of course, not necessary for $\forall$ to defend both disjuncts, as long as one of them can be defended the disjunction is true). If a negation $\neg A$ is asserted by $\forall$, $\exists$ may attack with the assertion of A, against which there is no defence. Lastly, if $\forall$ asserts an implication $A \rightarrow B$, $\exists$ may attack with an assertion of the antecedent A, after which $\forall$ must defend with an assertion of the consequent B. True to the aforementioned constructive origins of DL, to challenge an implication amounts to providing a proof of the antecedent and claiming that the other player will fail to build a proof of the consequent from this. Therefore the defence consists in providing a proof of the consequent[15] (Keiff, 2011: §2.1.1).

We now examine the structural rules of DL, which determine the overarching structure of dialogues and ultimately give us the tools with which to determine if a given proposition is valid. As mentioned, two variants of these rules are considered, for classical and intuitionistic logic. This will serve as an illustration of how different logics can be captured by different structural rules. First, we state several definitions and notational conventions used here, where A is any proposition:

> Dialogues have a (possibly empty) set of *premises* and a *thesis*, an initial assertion made by P. A *move* is any subsequent attack or defence.
>
> $D(A)$ denotes a dialogue for A, i.e. with a move of the form $P{-}!-A$ as the thesis from which the dialogue commences.
>
> $\Delta \in D(A)$ is a game within the dialogue $D(A)$. Games can be indexed to denote their position in the sequence of games constituting a dialogue - Δ_n denotes the n-th game in the sequence[16]. Moves within

[15]Note that this is also the reason that the attack against an implication takes the same form as that against a negation - the constructive negation is defined as $A \rightarrow \perp$, and $\perp$ cannot be defended (Keiff, 2011: §2.1.1).

[16]The reason for this distinction between dialogues and games will become clearer below,

a given Δ may also be indexed, for example $\Delta_1[n]$ denotes the n-th move in Δ_1.

Relevant structural rules of DL are as follows:

SR-0 (Starting Rule):

For any $\Delta \in D(A)$, the thesis has position 0[17], ie. $\Delta[0] = P{-}!-A$.

Every dialogue begins with the assertion of a thesis by a player who takes the defending role of proponent P. There may also, of course, be additional premises assumed for the purposes of the particular dialogue at hand. Every game in the dialogue begins with the assertion of a thesis by the proponent, which he must then defend (premises may be referred to at any point, of course).

For any $n > 0$ such that there is a $\Delta[n]$, this denotes a P-move where n is even, an O-move otherwise.

The game proceeds with P and O taking turns to make moves, in compliance with all particle and structural rules. Any move following the thesis is a reaction to a previous move by the other player. The next rule specifies different logics with reference to restrictions on which previous moves a player may react to at any given stage.

SR-1 (Round Closure Rule):

Classical: $\forall$ may attack any $\exists$-move as long as the rules allow, or defend against *any* attack of $\exists$ (even those she has already defended against).

Intuitionistic: $\forall$ may attack any move of $\exists$ as long as the rules allow, or defend against the *last* attack she has not already defended against.

The reason for the differences between these two variants of the rule will be explored in detail below, and so I leave the issue aside for the time being.

SR-2 (Branching Rule):

There are three cases in which $\Delta \in D(A)$ of even length (ie. where O is to make the next move) can be extended in such a way as to generate two distinct new games Δ_1 and Δ_2:

there are various points where a dialogue can 'split' into different possible subgames, for example the defence of a conjunction will generate distinct subgames as both conjuncts must be defended. Where propositions contain several, connectives distinct subgames dealing with the various components will be generated, so for example a proposition like $(A \wedge B) \rightarrow (A \rightarrow B)$ will require distinct games for $A \wedge B$, $A \rightarrow B$.

[17]Note that we begin with 0, not 1.

1. Assume for $0 < m < n$, that $\Delta[m] = O{-}! - B \vee C$; $\Delta[n] = P{-}? - \vee$. O may extend with the following moves:

$$\Delta_1[n+1] = O{-}! - B$$

$$\Delta_2[n+1] = O{-}! - C$$

In other words, when P attacks a disjunction asserted by O, which has then been attacked by P, the game can be extended in two ways, depending on which disjunct O chooses to defend with. As per the particle rules, O may choose either disjunct to defend, resulting in different games.

2. Assume $\Delta[n] = P{-}! - B \wedge C$. O may extend as follows:

$$\Delta_1[n+1] = O{-}? - L$$

$$\Delta_2[n+1] = O{-}? - R$$

Again, referring back to the particle rules, O may choose to attack a conjunction by challenging P to prove either conjunct, generating distinct games.

3. Assume for m such that $0 < m < n$, $\Delta[m] = O{-}! - B \rightarrow C$; $\Delta[n] = P{-}! - B$. O may extend with the following moves:

$$\Delta_1[n+1] = O{-}? - ...$$

$$\Delta_2[n+1] = O{-}! - C$$

Where ... denotes an attack concerning a relevant particle rule for a compound antecedent. This rule merely states the possibilities where the opponent must make a choice as to how to proceed. By SR-1, attacks are always possible, and so clearly it is also permissible for O to make the alternative attack after P has made the next move.

SR-3:

a) A dialogical game $\Delta \in D(A)$ is closed if and only if there is some atomic formula which has been played by both players. More precisely, let $Sub(A)$ denote all subformulas of A, then Δ is closed if and only if there exist two integers m and n and atomic formula $a \in Sub(A) \cup \{A\}$ such that the propositional content[18] of $\Delta[m]$ is a and the propositional content of $\Delta[n]$ is a and exactly one of $\{\Delta[m], \Delta[n]\}$ is a P-move. If Δ is closed, P wins, otherwise P loses.

[18] Let f denote ! or ?, e denote an expression. A move $\forall - f - e$ where has propositional content if e is a first-order formula, according to this formulation of DL (Keiff, 2011: §2.1.2), but one might just as well say a propositional formula in keeping with our restricting attention to the propositional aspects of this logic. This is another way of saying the expression concerned is a formula of the language.

A dialogical game is *finished* if and only if it is closed or the rules do not allow any move for the player whose turn it is to make the next move.

b) Let $\langle D(A), p \rangle$ be a play p of $D(A)$ of the form $p = \langle \Delta_0, ..., \Delta_n, \Delta \rangle$. $\langle D(A), p \rangle$ is finished iff all $\Delta_0, ..., \Delta_n$ are closed and Δ is finished. P wins the play iff $\langle D(A), p \rangle$ is closed.

These clauses specify the conditions for P winning or losing the game (i.e. the formula in question being valid or not, respectively). This formulation does not make explicit reference to winning strategies, however, it still essentially encapsulates this notion, given that if O has had to assent to a statement of P he has conceded that P is correct and has therefore lost the game.

SR-4:

Let $\Delta_1, \Delta_2 \in D(A)$ be distinct dialogical games. Assume O has chosen to play in Δ_1. O may only switch to Δ_2 if he has lost Δ_1, i.e. O cannot switch to another game before the one being played is finished.

Clearly, if O has won a part of the whole game, P has already lost the game. Yet there will be situations in which O loses part of the game but has yet to be defeated, as P still needs to defend another part of the formula to be proved.

SR-5:

- Atomic formula a is introduced in $\langle D(A), p \rangle$ by move $\Delta_i[n]$ if and only if:
  i) a is the propositional content of $\Delta_i[n]$
  ii) there is no previous move $\Delta_i[m]$ such that a is the propositional content of $\Delta_i[m]$ (i.e. if it has not been played in a previous move)
- P cannot introduce atomic formulas
- Atomic formulas cannot be attacked

P cannot introduce atomic formulas because the formula to be defended will be compound, and allowing P to introduce the atomic formulas needed to defend the proposition in question would mean that P could prove anything non-contradictory. O will introduce atomic formulas in the process of attacking P, after which P can then defend with the appropriate atomic formula (subject to the restrictions of SR-1). Atomic formulas cannot be attacked because we are not making reference to the truth of a statement in a model.

SR-6:

Def. Strict repetition:

- Let $\Delta[m]$ $\forall - ? - e$ be an attack against some previous move $\Delta[n]$. $\Delta[m]$ is a strict repetition if and only if there exists some move $\Delta[k]$ such that $n < k < m$ and $\Delta[k]$ is an attack against $\Delta[n]$ with an expression e identical to that in $\Delta[m]$
- Let $\Delta[m]$ $\forall - ! - e$ be a defence against some previous move $\Delta[n]$. $\Delta[m]$ is a strict repetition if and only if there exists some move $\Delta[k]$ such that $n < k < m$ and $\Delta[k]$ is a defence against $\Delta[n]$ with an expression e identical to that in $\Delta[m]$

Classical: No strict repetition is allowed.

Intuitionistic: If O has introduced an atomic formula that can now be used by P, P may perform a repetition of an attack. No other repetition is allowed.

This last rule excludes the possibility that a player repeats the same move indefinitely in a loop, effectively ensuring that the game terminates. Of course this is crucial to determining validities, it is impossible to prove that a proposition is valid according to these structural rules if the rules in question allow for indefinite repetition of arguments. As for SR-1, discussion of the reason for the difference between the two variants of the rule is postponed.

Validity in DL

Validity in DL is usually conceptualised in terms of there being a winning strategy for the statement in question. However, the formulation to which I adhere puts the notion slightly differently, although the definition below amounts to the same thing. Here we make reference to the definition of a closed game given in SR-3. This definition of validity will be explained further below in the context of connections between DL and Mittelstaedt's system.

Def. Validity: A sentence A is dialogically valid if all dialogue games belonging to the dialogue $D(A)$ are closed.

Essentially this definition captures the idea that one player has had to make a concession, playing a move previously played by another player because there was no other possible move to be made at that stage. Of course, the player who makes this concession (of necessity and not out of misjudgement or error, we are talking about an idealised notion of a game here, without fallible agents involved), loses the game, and the other player is therefore in possession of a winning strategy. Because P is always the player who starts the game by asserting a proposition to be defended, if P has a winning strategy the formula in question is valid, on the other hand, if O wins the dialogue, the initial proposition is not valid. Remember that the DL approach is a proof-theoretic framework, as such, no concept of truth is developed. That said, DL does not preclude the possibility of adding a notion of truth, indeed as we shall see, this is exactly what Mittelstaedt does in the quantum setting.

Classical and Intuitionistic Rules in DL

The difference between the classical and intuitionistic rules here merits some explanation. An in depth examination of intuitionistic logic is beyond the scope of this paper, but of course the two accept different rules, certain laws hold for classical logic that are intuitionistically unacceptable. One key difference between classical and intuitionistic logic is the acceptance or rejection, respectively, of the Law of Excluded Middle, $A \vee \neg A$ (Moschovakis, 2015:§1).

First, a note on the schematic representation of dialogues is in order[19]:

- The succession of arguments is enumerated by rows beginning at 0. From left to right, the first column contains this enumeration.
- The second column contains moves of O.
- The third column contains the number of the row O refers to with his current move.
- The fourth column contains moves of P.
- The fifth and final column contains the number(s) of the row(s) P refers to with her current move.
- Numbers in parentheses () denote attacks, i.e. (1) denotes an attack on a proposition of the other player at row 1.
- Numbers in angled brackets $\langle\rangle$ denote defences, i.e. $\langle 1 \rangle$ denotes a defence against an attack made by the other player at row 1.
- Where P takes over a proposition of O, we add an asterisk to number the row where the O-proposition in question occurs, i.e. 3^* denotes a

[19]Different presentations vary somewhat as to their representation of the dialogues, for the most part I follow Mittelstaedt's manner of representing them.

P-move taking over a proposition of O at row 3. This is important as P cannot introduce atomic propositions, but can only use them when they have been previously introduced by O.

- We refer to moves by player assignment and row number, $P3$ denotes P's move at 3.
- If a player does not defend at a given row, we place the empty argument [ ] in the relevant column and continue to the next row.

In order to provide a means of comparison, we begin with an example of a formula valid according to both classical and intuitionistic rules (so providing P with a winning strategy regardless of which set of rules is obeyed): $((A \to B) \wedge A) \to B$.

The dialogue here runs as follows:

	O		P	
0.			$((A \to B) \wedge A) \to B!$	
1.	$(A \to B) \wedge A!$	(0)	2?	(1)
2.	$A!$	$\langle 1 \rangle$	1?	(2)
3.	$A \to B!$	$\langle 2 \rangle$	$A!$	$(3), 2^*$
4.	B!	$\langle 3 \rangle$	$B!$	$\langle 0 \rangle, 4^*$

Of course, having stated the initial argument, P cannot defend against O's attack at 1 with $B!$ directly, as P may not introduce atomic formulas (by SR-5). Yet attacks can be made at any time according to both sets of rules, and so P has two avenues of attack against the conjunction asserted by O. O must eventually defend with B at 7, allowing P to take over the atomic proposition and win the dialogue - by SR-3 where the same atomic formula has been played twice and the game is therefore closed, P wins (Rebuschi, 2009: 232).

Now let us turn to a case in which the different sets of rules dictate different outcomes. We illustrate the difference between the two sets of rules with reference to the Law of Excluded Middle, which, of course, is classically, but not intuitionistically valid.

In the classical case, the dialogue will look as follows (Rahman & Tulenheimo, 2009: 168):

	O		P	
0.			$A \vee \neg A!$	
1.	$\vee?$	(0)	$\neg A!$	$\langle 1 \rangle$
2	$A!$	(1)	$A!$	$\langle 1 \rangle, 2^*$

Note that because of SR-5, P cannot assert atomic proposition A at 2, and so must choose the negation as defence $P1$. The classical permutation of SR-1 allows P to defend once more against the attack $O1$ (Rahman & Tulenheimo, 2009: 168), despite the fact that O has made another attack $O2$ in the interim. Because O has introduced atomic proposition A at $O2$, P may subsequently make use of it at $P2$, according to SR-5.

In the intuitionistic case, the last step of the classical dialogue is forbidden by the intuitionistic version of SR-1, which states that only the *last* attack made by the opposing player may be defended against.

The intuitionistic version of the same dialogue will therefore look as follows (Rahman & Tulenheimo, 2009: 168):

	O		P	
0.			$A \vee \neg A!$	
1.	$\vee?$	(0)	$\neg A!$	$\langle 1 \rangle$
2.	$A!$	(1)	-	

As we can see, in this game, P is forbidden from going back to defend against O's challenge once again at row 1., because O has attacked again at row 2., making $O2$ the most recent attack. As atomic formulas cannot be attacked, P therefore has no further possible moves to play and so loses the game.

The dialogue above reflects the fact that intuitionistically, we cannot infer the truth of A on the basis of the falsity of $\neg A$, as we would expect.

The effect of the intuitionistic rules is essentially dynamic, players must respond to what is happening at the current stage of the game, a new challenge effectively erases previous ones. The intuitionistic permutation of SR-1 ensures that at any stage of the game, P is able to provide grounds for his assertions without recourse to future concessions of O (Keiff, 2011: §2.2.3).

SR-6, dealing with strict repetition, the countering of a move with a proposition used previously against the very same move, works on the same principle. The intuitionistic variant forbids strict repetition unless an atomic formula has been introduced - this allows P to use the information available at that moment in order to counter the challenge posed at the current stage of the game. In the classical case (which forbids all strict repetition), this rule essentially ensures that dialogues are finite. Because defences against previous challenges are always allowed (by the classical permutation of SR-1) the dialogue about a formula could be endless, as the same challenge could be defended against again and again, yet the dialogue will always have the same end result.

It is worth mentioning here that the dialogical approach was developed in the context of philosophical concerns about logic and mathematics and its founders were of the opinion that the intuitionistic rules are the only correct ones[20]. This aside, as noted above, the differences between different logics are conceived as differences in the structural rules for the logics in question. There is, according to Keiff, no general theory of how structural rules ought to be formulated, but there are some of these rules that apply consistently (namely SR-0; SR-2; SR-3 a and b; SR-4), with changes being made to the other rules as necessary (Keiff, 2011: §2.2.2).

The discussion here will be relevant below when we explore the connections between DL and Mittelstaedt's system later in this paper, as Mittelstaedt's formal dialogues eventually result in an intuitionistic kind of logic.

We now move on to a reasonably thorough exposition of Mittelstaedt's dialogical system for quantum logic, which is somewhat more complex than the version of DL presented here, partly in virtue of the quantum setting. Subsequently the relations between the two systems will be explored. Mittelstaedt's system takes its key concepts from early permutations of DL and so the two approaches have a lot in common.

Mittelstaedt's Quantum Logic

As we shall see, Mittelstaedt's logic is in a very similar spirit to the dialogical approach discussed above. It provides an interesting example of a more practical, empirically focused use of this kind of approach, dealing as it does with scientific statements. Mittelstaedt aims to incorporate classical logic in his quantum system while allowing for the non-classical quantum aspect, how he does this will be explored below.

This section contains an exposition of the key elements of the logic in question. Later in the paper I will offer some discussion of the relation between dialogical logic and Mittelstaedt's system, as well as a more in depth analysis of how he deals quantum phenomena.

[20]For historical background, see e.g. Lorenz 2001.

Preliminaries

Mittelstaedt begins with what he terms *elementary propositions*, the results of measurements made on a quantum system. These are denoted by $a,\ b,\ c...$[21] For elementary propositions it is assumed that we have either an experimental procedure which proves a or an experimental procedure which disproves a (Mittelstaedt, 1978a: 50). $A,\ B,\ C...$ denote propositions more generally, A may be a compound proposition or an elementary proposition. Player assignments O and P work exactly as for DL, with P stating an initial proposition and O attempting to refute it.

Def. Proposition: The set $\mathscr{S}$ of propositions is defined recursively as:
i) Elementary propositions $a \in \mathscr{S}$ are propositions
ii) If $A \in \mathscr{S}$, $B \in \mathscr{S}$ are propositions, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\neg A$, $k(A,B)$, $\bar{k}(A,B)$ are propositions[22].

There are different varieties of dialogue in Mittelstaedt's system, fulfilling similar roles to the particle and structural rules for DL discussed above. *Material dialogues* fulfil the particle rule function, defining the connectives. *Formal dialogues* then give us the validities of the system, so the structure for determining logical truths.

The notion of truth is defined in a fairly standard manner for logical systems making use of dialogues, via the notion of a winning strategy. A participant has a *strategy of success* within dialogue-game D_x (material dialogues are denoted by D_m, formal ones by D_f) if there is a succession of arguments resulting in that participant's winning the dialogue regardless of the other player's moves.

Def. Truth: A is true (or $x - true$) iff P has a strategy of success within D_x about A.

Def. Falsity: A is false (or $x - false$) iff P has a strategy of success against A within D_x (i.e. if A is asserted by O, P wins D_x regardless of O's subsequent arguments. If A is true with respect to D_x we write $\vdash_{D_x} A$ (Mittelstaedt, 1978a: 53).

Of course, quantum logic is not classical, and Mittelstaedt must account for this in some way. He does this by introducing statements into the dialogues

[21] The same notation used for atomic propositions in DL is used for the elementary propositions here. This is because Mittelstaedt is effectively giving a notion of how the truth of atomic propositions is determined here, he is applying the dialogical technique to an empirical phenomenon.

[22] The meanings of $k(A,B)$ and $\bar{k}(A,B)$ will become clear shortly.

regarding the commensurability of the propositions at hand. Commensurability corresponds to the the idea of compatibility mentioned in the introduction and corresponds to the characteristic that there are properties of a quantum system that cannot be measured at the same time. We adhere to Mittelstaedt's terminology here. He introduces this notion into the material dialogues, capturing the fact that in the quantum setting a proposition proved earlier in a dialogue may not remain available to be used subsequently, if, for example, another measurement on the system has been made which has changed the state of the system in such a way that the previous proposition is no longer true. Mittelstaedt terms statements regarding whether two propositions are commensurable or incommensurable *availability propositions*. This notion will be made precise in the following sections. We now examine the language of this logic, starting with the general frame rules.

Frame Rules

Frame rules are meta-rules for the system, applying to both material and formal dialogues. They determine the permitted structure of dialogues in general, as follows:

F1: *P* asserts an initial argument, establishing the initial position of the dialogue.

F2: *O* attempts to refute *P*'s initial argument. Arguments are stated in turn by *P* and *O* according to proof rules.

F3: Arguments are either attacks or defences of previous arguments, not both.

The above are fairly self-explanatory, merely setting up how the dialogues proceed.

F4: a) An attack may be made at any point.

b) Having been attacked, participants must defend in reverse order of attacks (ie. most recent first).

F5: If a participant has no argument to continue they lose and the dialogue ends.

We will see later when the comparison is made with rules of DL that most of these rules also occur implicitly or explicitly in the structural rules for DL.

Material Dialogues

Dialogues containing proof-procedures that necessarily refer to empirical results, which must be proved outside the dialogue are called *material dialogues* (Mittelstaedt, 1978a: 60). As mentioned, they fulfil the same role as particle rules in DL, with additional specifications reflecting the empirical aspect of the enquiry. This will be discussed in detail below. Material dialogues deal with propositions where some element has to be proved outside the dialogue, i.e. that make reference to something empirical.

Elementary propositions and availability propositions must be proved outside the dialogues, and so they constitute what Mittelstaedt terms *material propositions*. Attacks and defences of material propositions are not arguments in the strict frame-rules sense, but questions and answers related to a quantum system, which ultimately determines the truth of material propositions (Mittelstaedt, 1978a: 62). Elementary propositions are proved by a measuring process on a quantum system, for availability propositions (to be discussed shortly), it is assumed that there is some empirical process for determining their truth or falsity. Compound propositions, on the other hand, are *dialogue-definite*, as their proofs must be performed within a dialogue.

The basic definitions of logical connectives concern elementary propositions related to one quantum system at a given time. The strategies available for the different connectives are summarised below, The possibilities for attack and defence for compound propositions in a material dialogue are as follows, with $A?$ representing a challenge to prove A and $A!$ a successful proof of A:

Proposition	Attack options	Defence options
A	$A?$	$A!$
$A \wedge B$	$1?, 2?$	A, B
$A \vee B$	$?$	A, B
$A \rightarrow B$	A	B
$\neg A$	A	

Examples of Mittelstaedt's dialogues follow the schematic conventions introduced above for the DL dialogues. An example of how the dialogue involving conjunction proceeds is as follows, dialogues involving the other binary connectives are analogous, subject to the attack and defence options given.

	O		P	
0.			$a \wedge b$	
1.	1?	(0)	a	$\langle 0 \rangle$
2.	a?	(1)	$a!$	$\langle 1 \rangle$
3.	2?	(0)	b	$\langle 0 \rangle$
4.	b?	(3)	$b!$	$\langle 3 \rangle$
5.	1?	(0)	a	$\langle 0 \rangle$
6.	a?	(5)		

P asserts $a \wedge b$ and is thereby committed to demonstrating elementary propositions a and b by means of experiments. O subsequently attacks the first conjunct, P retaliates by stating the conjunct a. Now O attacks the elementary proposition a, which P must prove with reference to a measurement on the system in question. The form of this kind of dialogue contrasts with the DL examples in that P stating a at 1. is not the same as P proving $a!$ with reference to a measurement at 2. (the same is true for b). This is not of great importance, merely serving to underline the empirical nature of the dialogues at hand. It is assumed in this example that P can prove a and b by measurements on the system. Note that at 5., O attacks the conjunction again. This is possible because of the aforementioned fact that, having yielded a positive result for a in one instance, the system is not guaranteed to do so again after a subsequent measurement[23].

The dialogue for negation runs as follows:

	O		P	
0.			$\neg a$	
1.	a	(0)	a?	(1)
2.	$a!$	$\langle 1 \rangle$		

O's successful proof (by measurement) of a at O2 definitively defeats P's initial proposition $\neg a$. If P were to repeat her attack the result would be the same, as no other measurement has been made on the system in the interim. The same would hold is the system did not yield a positive result for a. Therefore, this dialogue is automatically finite[24].

[23]To be a little more precise, the rationale for this is based on the assumption that if a is proved by measurement on a system at time t_1, then after a measurement for b at $t_2 > t_1$, given subsequent measurement for a at $t_3 > t_2$, the result a is not necessarily obtained, even for arbitrarily small time difference $\delta t = t_3 - t_1$ (Mittelstaedt, 1978: 60).

[24]This, of course, rests on the assumption that a measurement made at t_1 and repeated at t_2 will yield the same result, provided the time difference $\delta t = t_2 - t_1$ is sufficiently small (Mittelstaedt, 1978: 60).

The set of propositions in the material dialogues as seen so far comprises elementary propositions and compound propositions incorporating arbitrary iterations of connectives, as per the definition above (Mittelstaedt, 1978a: 59). To reiterate, material dialogues for two place connectives may, in principle, be infinite, in contrast to the DL dialogues we have seen. This holds for elementary propositions and material compound propositions. Because measurements affect quantum systems, a proposition A that was true of a system at some point in the game may no longer be true at a later stage of the game in question, and so we cannot restrict the material dialogues in the way that the DL dialogues above were restricted. Mittelstaedt develops a way to do this, to be discussed below. It is worth remembering here that material dialogues, despite the resemblance of some of their argument rules (detailed below) to those for the DL system, are actually providing a semantics for this kind of setup, as we shall see.

Commensurability - Introduction

Before stating the argument rules for material dialogues, an introductory look at commensurability and incommensurability is in order. This is a key notion in Mittelstaedt's system, allowing him to deal with quantum phenomena but also retain classical logic, in a manner to be discussed later in the paper. The lattice theoretic basis of Mittelstaedt's system comes into play here, I leave the details aside for the time being. We come back to this in the context of Mittelstaedt's notion of implication later on. The idea of commensurability in this system will be made formally precise below, we will also see how it fits in to the material dialogues. Commensurability fulfils two key functions aside from its role in retaining classical logic. Firstly, as we have seen, measurements performed on quantum systems may change the system, meaning that what is true of the system after one measurement does not necessarily remain so after subsequent measurements. It is also the case that certain measurements cannot both be made on the system with the same degree of accuracy, as mentioned in the introduction. Therefore, Mittelstaedt requires a way to factor this into his system, and commensurability provides both the conceptual and the formal means to do this. Relatedly, it provides the means to restrict the infinite material dialogues defining binary connectives to finite ones, eventually allowing a notion of validity captured in the formal dialogues (in exactly the manner of DL, where, as we saw, dialogues are restricted to finite length).

Mittelstaedt introduces procedures for proving availability propositions, which in turn can be used to restrict the length of the dialogues defining the connectives (as we will see at $A_m 2$ below). As stated above, availability propositions, like elementary propositions, are proved outside the dialogues - whether a given proposition is commensurable with another must be established empirically, ultimately. Availability propositions are of two varieties (Mittelstaedt, 1978a: 61):

Commensurability propositions $k(A, B)$ stating that A remains available after a material dialogue about B.

Def. Commensurability, $k(A, B)$: A and B are commensurable ($k(A, B)$ has been proved is for a given system), if A and B can be tested dialogically in an arbitrary sequence without thereby influencing the result of the dialogues.

Incommensurability propositions $\bar{k}(A, B)$ state that A is no longer available after a material dialogue about B.

Def. Incommensurability, $\bar{k}(A, B)$: A and B are incommensurable ($\bar{k}(A, B)$ has been proved) if, for a given system, the result of a dialogic test of one proposition can be changed by a dialogic test of the other.

We now move on to the argument rules for material dialogues, before showing how these propositions can be used within them. Specifically, we will see examples of how availability propositions are used to restrict dialogues about the binary connectives to finite length.

Argument Rules for Material Dialogues

$\underline{A_m\ 1}$: An elementary proposition a, once stated, may be attacked by a challenge $a?$ to prove it, for which the defence consists of a demonstration of a (by an empirical measurement).

Elementary propositions	Attack	Defence
a	$a?$	$a!$

$\underline{A_m\ 2}$:

Attacks and defences for availability propositions[25]:

[25]Examples of how these behave within the material dialogues will be gven in the next section.

Availability proposition	Attack	Defence
$k(A,B)$	$k(A,B)?$	$k(A,B)!$
$\bar{k}(A,B)$	$\bar{k}(A,B)?$	$\bar{k}(A,B)$

As for elementary propositions, commensurability and incommensurability propositions are demonstrated empirically. A challenge $k(A,B)?$ to prove the commensurability of A and B is countered with reference to some empirical procedure outside the dialogue.

$\underline{A_m\ 3}$: If a participant cannot defend against an attack of a material proposition (i.e. a, $k(A,B)$, $\bar{k}(A,B)$), he may assume a previous defence obligation.

$\underline{A_m\ 4}$:

Connective	Attacks	Defences
	$k(A,B)?$	$k(A,B)!$
$A \wedge B$	1?	A
	2?	B
	?	$\bar{k}(A,B)$
$A \vee B$		A
		B
$A \to B$	A	B
	$k(A,B)?$	$k(A,B)!$
$\neg A$	A	

Commensurability gives additional attack and defence possibilities for the binary connectives in the material dialogues. To attack $A \wedge B$, a player may question whether A and B can obtain together; likewise for $A \to B$. $A \vee B$, conversely, only acquires a new defence, the player defending this disjunction may now do so by demonstrating the incommensurability of A and B.

$\underline{A_m\ 5}$: If a participant attacks A, the other may check the availability of A using arguments $k(A,B)$. Here, B is any proposition in the dialogue after A, attack $k(A,B)?$ may be used once for every B.

A proposition A is successfully defended if the dialogue-game starting from A is won by the player who initiates it (P). If P has a strategy of success for

A within D_m, A is materially true. If P has a strategy of success within D_m against A, A is materially false.

Availability Propositions in the Material Dialogues

We now illustrate how availability propositions allow us to replace the infinite dialogues defining connectives with finite ones.

Conjunction $a \wedge b$

This can be limited to finite length if a and b are known to be commensurable, as subsequent proof attempts will produce the same results. In the case that they are incommensurable, P is guaranteed to lose at some point, where either a or b cannot be defended (Mittelstaedt, 1978a: 63). Thinking back to the example of the conjunction dialogue above, and assuming as above that P can prove a and b with reference to measurements, we now have the following:

	O		P	
0.			$a \wedge b$	
1.	$k(a,b)?$	(0)	$k(a,b)!$	$\langle 0 \rangle$
2.	1?	(0)	a	$\langle 0 \rangle$
3.	$a?$	(2)	$a!$	$\langle 2 \rangle$
4.	2?	(0)	b	$\langle 0 \rangle$
5.	$b?$	(4)	$b!$	$\langle 4 \rangle$

With the additional attack and defence possibilities for commensurability $k(a,b)$ from A_m 2, O may attack immediately with a challenge to prove that a and b are commensurable. Assuming that P can prove this by the requisite empirical procedure, it is established that a measurement of a does not affect a measurement of b. Therefore, provided P can provide the empirical proofs of a and b, the dialogue does not continue indefinitely, as the same results will be repeated for all subsequent measurements. If a and b are incommensurable, a measurement of one would change the result of a subsequent measurement for the other. Therefore, at some point in the dialogue, P would be unable to defend one of the conjuncts, losing the dialogue, so $a \wedge b$ would not hold.

Disjunction $a \vee b$

Dialogues concerning a disjunction $A \vee B$ can be limited to finite length if it can be shown that a and b are incommensurable. Of course, if the player asserting a disjunction can prove one or other (or both) of the disjuncts, the problem does not arise. However, consider the situation where P has asserted $a \vee b$, yet has failed to prove both a and b by measurement. If a and b are commensurable, as above, the same results will continue to obtain regardless of how may times the measurements are performed. Yet if it can be shown that a and b are incommensurable, then P will at some point succeed in proving one of the two.

Material implication $a \rightarrow b$

This can be restricted to a finite dialogue if the availability of a remains after a b-proof - ie. a and b must be commensurable (Mittelstaedt, 1978a: 64). By way of illustration, the dialogue runs as follows:

	O		P	
0.			$a \rightarrow b$	
1.	a	(0)	[ ]	
2.	$a!$	$\langle 1 \rangle$	$a?$	(1)
3.	[ ]		b	$\langle 0 \rangle$
4.	$k(a,b)?$	(0)	$k(a,b)$	$\langle 0 \rangle$
5.	$b?$	(3)	$b!$	$\langle 3 \rangle$

Note that P cannot defend against an atomic proposition in row 1; and after O has proved a in row 2., it is then up to P to continue with his proof of b. Without P's proof that a and b are commensurable, even after a proof of b by P, O could attack anew with a, as there would be no guarantee of a successful b-proof after a new a-proof if a measurement of a may change the state of the system so as to preclude a proof of b (Mittelstaedt, 1978: 65).

Infinite dialogues can be restricted to finite ones in the same way for arbitrary compound propositions. Proofs by measurement are replaced by subdialogues relevant to proving the propositions at hand (Mittelstaedt, 1978a: 65).

Formal Dialogues - Introduction

Here we move on to formal dialogue games, the means by which validities are defined in Mittelstaedt's system. Formal dialogues give a set of rules allowing us to determine statements that are *always* true in a material dialogue. Of course some propositions can be defended in the material dialogue game regardless of the elementary and availability propositions that they contain, and we now turn to the details thereof. An example of such a material dialogue will serve to illustrate this point. As usual, *P* asserts a proposition which she then attempts to prove, and which *O* attempts to refute.

	O		P	
0.	$[\,]$		$(a \wedge b) \to (a \to b)$	
1.	$a \wedge b$	(0)	$[\,]$	
2.	a	$\langle 1 \rangle$	$1?$	(1)
3.	$a!$	$\langle 2 \rangle$	$a?$	(2)
4.	b	$\langle 1 \rangle$	$2?$	(1)
5.	$b!$	$\langle 4 \rangle$	$b?$	(4)
6.	$k(a,b)!$	$\langle 1 \rangle$	$k(a,b)?$	(1)
7.	$[\,]$		$a \to b$	$\langle 0 \rangle$
8.	a	(7)	b	$\langle 7 \rangle$
9.	$k(a,b)?$	(7)	$k(a,b)!$	$\langle 7 \rangle, 6^*$
10.	$b?$	(8)	$b!$	$\langle 8 \rangle, 5^*$

In row 1., *O* will attack with an assertion of the antecedent of the conditional proposition to be proved. As the latter takes the form of a conjunction, he must prove both conjuncts, with *P* attacking accordingly at rows 2.-5. *P* then attacks at 6. with a challenge to prove that a and b are commensurable, according to the rules for conjunction. As we saw above, if *O* fails to provide this, the dialogue about the conjunction could go on indefinitely. In this example, we deliberately take the situations most disadvantageous for *P*, where *O* is able to prove a, b and $k(a,b)$, at rows 3., 5., and 6., respectively. Yet *P* still wins the dialogue, as the crucial propositions $k(a,b)$ and b, which *P* requires to prove $a \to b$, the consequent of the implication statement she is trying to defend, are already proved by *O* in the course of his proof of $a \wedge b$. Therefore, this proposition can always be proved in the material dialogue game. The question then is how to systematise this notion. The argument rules Mittelstaedt develops for the formal dialogue games enable the construction of all these validities, eliminating the need to investigate every possibility for each material dialogue in order to determine validities (Mittelstaedt, 1978: 74).

As such, the formal dialogue games clearly fulfil a very different function to the material dialogues. While the latter determine what is true on the basis of the truth of elementary propositions and availability propositions referring to measurements; the former give us propositions true regardless of the truth of their constituent propositions.

Argument Rules for Formal Dialogues

Formal dialogues have their own argument rules analogous to those for material dialogues, despite the very different function of these rules. Mittelstaedt develops these in several steps, I adhere closely to these stages. The aim here, is to end up with a set of validities, which, clearly, should not depend on any empirical content such as measurement results.

The first step he takes is to eliminate reference to elementary propositions in the semi-formal dialogue game detailed in this section. Next, he eliminates availability propositions, by devising a system that captures propositions of the form $k(A, B)$ which always hold. This is the calculus of formal commensurabilities and will be explored in the next section. These formal commensurabilities can be proven to be equivalent to certain compound propositions, which replace them in the full formal dialogue game. The set of propositions that can be defended in the full formal dialogue game constitutes Q_{eff}, the calculus of intuitionistic quantum logic (Mittelstaedt, 1978: 72).

The rules establishing the semi-formal dialogue game are as follows:

$\underline{A_f 1}$:

(a) Elementary propositions cannot be attacked.

(b) O may always state elementary propositions; P may only do so where O has done so previously, and where the proposition in question is still available (Mittelstaedt, 1978a: 76).

Elementary propositions can no longer be attacked in the formal dialogues as we are no longer talking about the content thereof, but about propositions that are valid regardless of the content of elementary propositions. Because P must defend the initial proposition asserted (the validity of which is in question), he can no longer draw upon anything external to the dialogue, and he should prove that the proposition at hand is true independently of the content of elementary propositions contained within it.

$\underline{A_f 2}$: P may attack propositions asserted by O only if they are still available.

This rule is designed to avoid irrelevant attacks, clearly if a proposition is no longer available, any attack against it is redundant.

$\underline{A_f 3}$: Possibilities for attacks and defences of the connectives:

Connective	Attack options	Defence options
$A \wedge B$	$1?, 2?$	A, B
$A \vee B$	$?$	A, B
$A \rightarrow B$	A	B
$\neg A$	A	

A_f1 and A_f2 here do not eliminate the possibility that a given proposition may be infinitely available. Yet P must have a strategy of success. To limit the length of dialogues, we postulate that P sets a bound n on the number of times he may take over an elementary proposition to attack O (Mittelstadt, 1978a: 77). This is an artificial constraint, we then take the set of validities to be determined by the union of all D_f^n. A_f4 and A_f5 specify how this is done.

If A is true, P is assumed to have a strategy for success in the dialogue initiated by the assertion of A by P. So P can defend against all O's arguments in this kind of game. As such, for the purposes of the formal dialogue game we only need to consider the situation where P is in the worst possible position to defend A (recall the dialogue for $(a \wedge b) \rightarrow (a \rightarrow b)$ in the last section, where we deliberately assumed that O could prove everything he needed to prove). On this basis, we know that there is no need to go through every possibility for what O can prove in the dialogue. Hence, the attack and defence possibilities of both P and O are restricted to reflect this fact in the next rule. It is important to note here that this rule is just a technical device for expediency (Mittelstaedt, 1978: 78):

$\underline{A_f4}$:

(a) P may take over the same elementary proposition at most n times. P may attack the same proposition of O at most n times. This rule limits the length of dialogues in accordance with the bound n just mentioned.

(b) O may attack a proposition of P at most once. O must decide whether to defend against an attack from P or whether to attack P. If O defends, he may no longer attack, and the defence used is no longer possible.

A proposition A, once proved, only remains available if it is commensurable with every intervening proposition B stated between the initial proof of A and the subsequent citing of A (Mittelstaedt, 1978a: 78). As it happens, there are just three possible positions in the dialogue where the availability of a proposition may be called into question:

p_1: O attacks a material implication $A \to B$ of P with A.

p_2: O attacks a negation $\neg A$ of P with A.

p_3: O defends a disjunction $A \vee B$ against an attack from P.

These are the positions in which it will be necessary for O to introduce a new proposition in order to initiate the next step in the dialogue, hence the need to check the commensurability of the new proposition with the previous ones.

The next rule details how this may be done:

$\underline{A_f5}$:

(a) In positions p_1, p_2, p_3 of a dialogue, P may reduce the availability of all O's previous propositions to zero.

(b) (semi-formal) In such a position, where O can state a new proposition A, O may instead attack with $k(C, A)?$, where C is any previous O-proposition with non-zero availability. The defence consists in demonstrating $k(C, A)$[26].

c) In positions p_1, p_2, p_3 of a dialogue, P may challenge a proposition B. Then O may either assert B and continue the dialogue from this in the next row, or ask $((A_1 \wedge ... \wedge A_n) \to B)?$, where the antecedent is the conjunction of all O's previous propositions with non-zero availability. P must defend against the latter by asserting a new initial argument $(A_1 \wedge ... \wedge A_n) \to B$.

The rationale for the above rules will become clearer in the next section, which deals with commensurabilities in the formal dialogue-game - essentially the above rule plays a crucial role in eliminating commensurabilities in order to obtain the full formal dialogue-game.

Finally, to exclude infinite strategies again using the artificial bound n, we have:

$\underline{A_f6}$: P may make use of A_f5 c) at most n times.

The above rules establish the semi-formal dialogue game, the next step is to establish the formal commensurability calculus, so that availability propositions can be eliminated in order to give the full formal dialogue game.

[26] This part of A_f5 is modified in the full formal dialogue game in light of the formal commensurability calculus, as we will see below.

The Calculus of Formal Commensurabilities

In order to establish the full formal quantum dialogue-game it is necessary to eliminate material proofs regarding commensurabilities. There are some availability propositions that are formally true, holding regardless of the propositions contained within them (conversely, there are no formal incommensurabilities). These formal commensurabilities must be incorporated into the formal dialogues (Mittelstaedt, 1978a: 83), and to do this, Mittelstaedt develops a calculus of formal commensurabilities. The idea here, as we will see, is that if a proposition regarding commensurability can be proved within this calculus, it can actually be shown to be equivalent to a material proposition for which there exists a strategy for success, and so it can be replaced by such a proposition in the formal dialogue.

Recall the definition of commensurability from above - propositions are commensurable if they can be dialogically tested in arbitrary order without affecting the result of the dialogue. Now let us consider as an example the formal commensurability $k(A, A \rightarrow B)$.

Take a dialogue in which P needs to show both A and $A \rightarrow B$. In the event that P loses the dialogue about A, then, on attacking $A \rightarrow B$ (which is done with a proof of A), O will also lose the dialogue about A. Because only A has been tested, a repeat of the dialogue about A will give the same result. On the other hand, if we assume that P wins the dialogue about A, there are several possibilities as regards the subsequent dialogue about $A \rightarrow B$. P will lose the dialogue if he is unable the prove $k(A, B)$, but the proof attempt for $k(A, B)$ will yield the same result as the dialogue about A. If P is able to prove $A \rightarrow B$, then because A and B are commensurable the subsequent dialogue about B does not affect the previous result of the dialogue about A. Once $k(A, B)$ has been proved, the previous A-result will continue to hold. This means that $k(A, A \rightarrow B)$ holds across the board, it is a formal commensurability (Mittelstaedt, 1978: 83).

The full details of the formal commensurability calculus need not concern us overly here, it is enough for our purposes to see how they are incorporated into the formal dialogue game. As it turns out, some equivalences between formal commensurabilities and compound propositions can be proved. These allow formal commensurabilities to be incorporated into the formal dialogues, as required. I briefly state the relevant proofs here to illustrate why this is the case. We will show that if the formal commensurability $k(A, B)$ can be proved then it can be shown that there exists a strategy of success for $A \rightarrow (B \rightarrow A)$. Further, if a strategy of success for $A \rightarrow (B \rightarrow A)$ exists, then the obligation to prove $k(A, B)$ in a dialogue can be circumvented

(Mittelstaedt, 1978: 85).

Theorem: Formal commensurabilities

i) If the formal commensurability $k(A, B)$ can be proved then there exists a strategy of success for $A \to (B \to A)$.
Proof: Where $k(A, B)$ can be proved in K, we have a situation illustrated by the following dialogue:

	O		P	
0.	[ ]		$A \to (B \to A)$	
1.	A	(0)	$B \to A$	$\langle 0 \rangle$
2.	B	(1)	A	$\langle 1 \rangle$

Because $k(A, B)$ is presupposed, and the commensurability relation is symmetric, (i.e. $k(A, B) = k(B, A)$), O cannot attack P's assertion of $B \to A$ with $k(B, A)?$. Because of the assumed commensurability of A and B, the availability of A remains unchanged by a B-proof at $O2$ and therefore P can take A over from $O1$ to defend at $P2$ (Mittelstaedt, 1978a: 85).

ii) If there exists a strategy of success for the dialogue about $A \to (B \to A)$, the obligation to prove $k(A, B)$ can be disregarded.
Proof: An illustration will serve to clarify this proof:

	O		P	
.	.		.	
.	.		.	
.	.		.	
(m)	A		.	
.	.		.	
.	.		.	
(n)			$B \to A$	
$(n+1)$	$k(A, B)?$		[ ]	
$(n+2*)$	[ ]		$(B \to A)?$	
$(n+3*)$	$A \to (B \to A)?$		$A \to (B \to A)!$	
$(n+2)$	$(B \to A)!$		$(B \to A)?$	
$(n+3)$	B	(n)	[ ]	
$(n+4)$	A	$\langle n+2 \rangle$	B	$(n+2)$

We assume that P has a strategy of success for $A \rightarrow (B \rightarrow A)$. Here, $(n+2*)$ and $(n+3*)$ denote one 'branch', one way the dialogue could proceed; $(n+2)$... the other. Say O has asserted A at row m, and P has asserted $B \rightarrow A$ at row n. According to A_f5 b), O may attack with $k(A,B)$? at (n+1). By assumption, P has a strategy of success for $A \rightarrow (B \rightarrow A)$ and so can circumvent this attack by making use of A_f5 c), challenging $B \rightarrow A$ instead of defending with $k(A,B)$!. Thereupon, O has the two options illustrated at $(n+3*)$ or $(n+2)$. In the former case, P can defend successfully by assumption. In the latter, P may reduce the availability of A to zero. However, if O then asserts B as an attack against $P(n)$, P can attack with B. If O defends with A, P can then take this over and successfully defend, completing the dialogue.

The proofs for i) and ii) above can be generalised, as clearly they apply to each available O-proposition, so we have the following:

Theorem: Formal commensurabilities (generalised)

i) If formal commensurabilities $k(A_1, B)$, $k(A_2, B)$,... can be proved then there exists a strategy of success for the proposition $(A_1 \wedge ... \wedge A_n) \rightarrow (B \rightarrow (A_1 \wedge ... \wedge A_n))$.

i) If there exists a strategy for success for the proposition $(A_1 \wedge ... \wedge A_n) \rightarrow (B \rightarrow (A_1 \wedge ... \wedge A_n))$, the obligations to prove commensurabilities $k(A_1, B)$, $k(A_2, B)$,... can be circumvented (Mittelstaedt, 1978a: 86).

Rule A_f5 fulfils the following function, which can be seen at work in the dialogue above. In some situations in the formal dialogue game, it will be beneficial to allow P to challenge O to prove a formally true proposition B, in order for P to be able to take over propositions (usually atomic propositions) from B. After P has made such a move, there are two ways for O to proceed. Either he can assert B and continue the dialogue, or he can ask P to demonstrate the formal truth of B on the basis of all previous O-propositions in the dialogue with $(A_1 \wedge A_2 \wedge ... \wedge A_n) \rightarrow B$.

We will see below how this eliminates formal commensurabilities and completes the full formal dialogue-game.

Formal Dialogue Argument Rules Continued - The Full Formal Quantum Dialogue Game

To establish the full formal dialogue-game we must eliminate commensurabilities. The above generalised formal commensurability theorem gives us a new version of A_f5 b) taking into account the possibility of O attacking by questioning the availability of new proposition A in view of *the conjunction* of A with all previous propositions in the dialogue:

A_f5 b) (formal): In a position p_1, p_2, p_3 of a dialogue, where O can state a new proposition A, O may attack with $k(C, A)$? instead of asserting A (where C is the conjunction of all previous O-propositions with non-zero availabilities). The defence against $k(C, A)$? is the assertion of a new initial argument $C \rightarrow (A \rightarrow C)$.

A_f7: a) O may not attack the initial argument $A \rightarrow ((A \rightarrow B) \rightarrow A)$.

b) P may replace the initial argument $A \rightarrow (B \rightarrow A)$ with $B \rightarrow (A \rightarrow B)$.

c) P may replace the initial argument $(A_1 \star A_2) \rightarrow (B \rightarrow (A_1 \star A_2))$ with $A_1 \rightarrow (B \rightarrow A_1)$ or $A_2 \rightarrow (B \rightarrow A_2)$, with $\star \in \{\wedge, \vee, \rightarrow\}$ and where O must choose between the two possibilities.

d) P may replace initial argument $\neg A \rightarrow (B \rightarrow \neg A)$ with $A \rightarrow (B \rightarrow A)$.

e) P may state the same initial argument at most once.

A_f7 merely makes use of what was proved in the formal commensurability theorems above, with availability propositions of the formal variety giving way to compound propositions according to the equivalences proven. Initial arguments involving commensurabilities are replaced by P with the corresponding compound propositions; rules of the commensurability calculus K are replaced with rules concerning initial arguments (Mittelstaedt, 1978a: 87).

Dialogical Logic and Mittelstaedt's Dialogical Quantum Logic

We now move on to detailing the relationship between DL and Mittelstaedt's system. Mittelstaedt's work was based on early developments in DL[27], his aim was to develop a logic that could encapsulate the traits of both classical and quantum systems. He sees classical logic as a special case of his own system, where propositions are unrestrictedly available. Mittelstaedt's system is close to DL, but with added clauses and modifications to take into account: a) the empirical nature of the enquiry; b) (related) the temporal nature of dialogues; c) the quantum aspects of the logic. The two systems are rather different in their set up and they differ with respect to which rules are made explicit. Nevertheless, as briefly mentioned above, there is a correspondence between the two systems that is worthy of further elucidation.

[27] He refers particularly to the work of Lorenzen (Mittelstaedt,1978: 70).

Untangling the different terminology and sets of rules for DL and Mittelstaedt's system is somewhat intricate and it is not possible to compare the two in an entirely straightforward manner. Where DL has particle rules defining the connectives, which specify how to win the game in a specific situation; and structural rules, where the particle rules are abstracted out - this is the proof system, where validities are defined. Roughly, then, we can say that Mittelstaedt's material dialogues correspond to the particle rules of DL; whereas the formal dialogues (ie. those dealing with validity, what is always true regardless of the content of the propositions in question) correspond to the structural rules of DL.

More specifically, Mittelstaedt's formal dialogue-game gives proof rules for validity in the same way that DL's structural rules specify this for DL dialogues. His material dialogues differ from the particle rules for DL in that, although what they are fundamentally doing is giving the semantics for connectives, they are doing so within a particular empirical context. Mittelstaedt's system goes a layer deeper in the sense that he specifies the procedure for attacking and defending (ie. verifying) atomic formulas - with reference to a measurement process. DL is concerned solely with the logical side of things and so has no need to flesh this out. In DL, atomic propositions cannot be attacked or defended. Mittelstaedt's material dialogues specify criteria for truth of atomic (elementary) propositions, as well as setting up the rules for connectives. Thus, material dialogues fulfil the same role in Mittelstadt's system as particle rules do in DL, but go into more detail.

Rules for attacks and defences for the connectives appear in argument rules for both D_m and D_f (whereas these are just particle rules in DL). Attacks and defences for D_f turn out to be identical to the particle rules of DL, as might be expected. Attacks and defences for D_m include availability propositions, proved outside the main dialogue, DL does not have this construct, having no need to take into account the particulars of quantum theory. For Mittelstaedt, frame rules specify the general structure of both types of dialogue. For the most part, these meta-rules appear impicitly in the structural rules of DL, or are contained within more detailed rules, as will be illustrated below.

The two systems have rather different approaches, DL you might call a 'top-down' approach, where the starting point is to construct a proof-system for logical validities. Mittelstaedt, on the other hand, starts from a particular empirical phenomenon he is trying to model, and approches validities later through abstracting away from this. Nevertheless, they do end up doing the same thing, elaborating a system for determining validities.

To provide a more specific illustration, I now enumerate some of the correspondences between rules for DL and Mittelstaedt's system. The way DL is formulated in the presentation I have followed here is more formally explicit about the ingredients of the dialogue than some other accounts, as the formulation I refer to is more contemporary. This is, however, not of great conceptual importance, except in that it tends to make DL's rules somewhat more precise than Mittelstaedt's.

- SR-0 corresponds to F1 and F2. F1 corresponds to for any n greater than zero, and any $\Delta \in D(A)$ such that $\Delta[n]$ exists, if n is even, $\Delta[n]$ denotes a P-move, otherwise it denotes and O-move. F2 corresponds to the part of S-0 stating that O attempts to refute P's arguments, and arguments are stated in turn by P and O.

- F4 is the intuitionistic variant of SR-1. More specifically, F4a) is the first clause of the classical and intuitionistic variants of SR-1. F4b) states that participants are obliged to defend in reverse order of attacks, and that a player may postpone defence as long as there are possible attacks to put forth, which is the intuitionistic variant of SR-1.

- F3 is assumed in the preliminaries to DL's structural rules.

- SR-2's branching rules specify the courses a game can take where O has different options regarding the manner in which to proceed. While Mittelstaedt does not make this rule explicit for either variety of dialogue, it is implied by the definitions of the connectives specified for both the material dialogues (A_m4), and the formal dialogues (A_f3) together with the other rules. Different attack and defence possibilities are reflected in the argument rules for the material dialogue-game, A_m4 and A_m5, the difference being that the latter include stipulations regarding commensurability. The argument rules for A_f also include stipulations regarding attacks, defences and when in the semi-formal dialogue-game the availability of a proposition can be questioned, at A_f4.

- F5 states that if a participant has no argument to continue, he loses (and therefore the other participant wins). SR-3 puts this in far more precise terms, but the key idea is the same, both specify what counts as winning or losing in the context of the game.

- SR-5 and A_f1 correspond, both stipulate that atomic or elementary propositions cannot be attacked, and that P, having stated the initial argument, may not introduce such propositions. Of course, this rule does not apply to the material dialogues, where, according to A_m1 elementary propositions may indeed be attacked and defended, defences thereof consisting of an empirical demonstration of the proposition in question outside the dialogue.

- SR-6 (classical or intuitionistic) does not hold for material dialogues . For formal dialogues this restriction on the length of dialogues is fulfilled by $A_f(4^{(n)})$. There is a key difference at the level of rules for attacks and defences in DL and Mittelstaedt's corresponding rules in the material dialogues, which is that while the dialogues for particle rules in DL are finite, Mittelstaedt's are not, because of the possibility of a different outcome regarding the same proposition later in the dialogue in a quantum setting. Mittelstaedt therefore has to add commensurability and availability propositions to restrict the length of these dialogues.

As mentioned above, a core of DL structural rules are also present in Mittelstaedt's system, meaning that the two systems, though differently presented, have a lot in common. Mittelstaedt takes the dialogical methodology a step further in using it to essentially give a semantics for the proof-theoretic system of the formal dialogue games (on the basis of which he eventually specifies the effective quantum logic Q_{eff}).

At a general level, Mittelstaedt's system gives a semantics to DL's purely proof-theoretic approach, specifying the method by which atomic propositions (of the formal dialogue) are determined to be true - a material dialogue. Via the material dialogues, he gives a basis for interpreting the content of atomic propositions, namely in terms of statements about the state of a quantum system. The system therefore has an additional dimension, the meaning of compound propositions is explained, rather than left blank to fill in as one wishes (or a blank to be ignored entirely, depending on one's interests), as is the way with DL. This is interesting, of course there could be very many ways to do this, Mittelstaedt's quantum logical system is just one way to give this semantic meaning to the formal system developed in DL. Mittelstaedt's formal dialogue-game corresponds to the intuitionistic part of the full quantum calculus, this is reflected in the fact that some of Mittelstaedt's rules align, as we have seen, with the intuitionistic variants of the DL structural rules explained above.

Of course, Mittelstaedt also has to put in some work to deal with the quantum nature of the systems that are giving the logic its semantics, so to speak. The commensurability calculus he develops, which fits in to the formal dialogue-games is a requirement stemming from the semantic side, dealing as it does with whether propositions can, in principle, be asserted of one system at a given time. This important feature of the system is then incorporated into the syntactic rules by means of the equivalences that can be proved between formally true availability propositions from the commensurability calculus K (which always hold) and certain formally true material implication statements. Mittelstaedt places great importance on the possibility of eliminating material propositions, ie. those pertaining to the empirical, from his formal dialogues, as he wishes to formulate a logic that works for both classical and quantum systems, rather than one that only works for quantum systems. As such, in the end the formal quantum dialogue game actually bears great resemblance to the intuitionistic variant of the DL setup.

Truth and Validity in the Two Systems

The notions of validity given by DL and Mittelstaedt's formal dialogues are very similar, although the formulation of DL I have chosen to discuss does not make explicit reference to winning strategies. I now briefly revisit key notions regarding truth and validity for both systems in order to be able to compare them.

DL

- Def. Validity: A first-order sentence A is dialogically valid if all dialogue games belonging to the dialogue $D(A)$ are closed.
- Def. Closed: A dialogical game $\Delta \in D(A)$ is closed iff there is some atomic formula which has been played by both players. More precisely, Δ is closed iff there exist two integers m and n and atomic formula $\alpha \in Sub(A) \cup \{A\}$ such that the propositional content of $\Delta[m]$ is α and the propositional content of $\Delta[n]$ is α and exactly one of $\{\Delta[m], \Delta[n]\}$ is a P-move.
- DL makes no reference to truth in a model, focussing exclusively on validities in the proof system defined.
- This definition makes no explicit reference to strategy, but amounts to A being valid if P has a winning strategy for A, as P must be able to win against all possible moves by O, and the game is defined by the sequence of choices made by O. In other words, if a participant has been forced to play the same atomic formula as their counterpart, they have effectively been forced to concede that the other is correct.

As detailed previously, DL is a purely proof-theoretic approach, making no reference to models rendering a proposition true or false. It deals purely with a testing procedure for the validity of compound propositions, regardless of the atomic formulas contained within them, here we are interested in the mechanics of proving these validities without recourse to the meanings of their constituents.

Mittelstaedt's Quantum Logic
Mittelstaedt's characterisation of truth or validity is more standard for a dialogical approach, defining truth in terms of the proponent having a winning strategy for the statement at hand.

- A participant has a *strategy of success* within dialogue-game D_x if there is a succession of arguments resulting in that participant's winning the dialogue regardless of the arguments of the other.
- Def. Truth: A is true (or $x - true$) iff P has a strategy of success within D_x about A
- A is materially true if P has a strategy of success for A within D_m (which ultimately depends on the truth of the elementary propositions and availability propositions involved in what is being demonstrated).
- A is formally true if P has a strategy of success for A within D_f (so A is true in all material dialogues, regardless of the specifics of elementary and availability propositions contained within it - D_f deals with validities).

Notice that Mittelstaedt's terminology involves a notion of truth, in contrast to DL, which speaks only of validity. There are several factors at play here, one being that Mittelstaedt, of course, is dealing with empirical factors that do not arise in DL. Whereas in DL atomic propositions cannot be attacked and defended, they do not have any 'source' of truth or falsity, in some sense having meaning only insofar as they occur in compound propositions the validity or otherwise of which can then be ascertained; Mittlstaedt does have a substantive notion of truth for the material dialogues. In the empirical sense, this is just that an atomic proposition makes a statement about a physical system that can be true or false. Elementary proposition a being true is just to say that a measurement has been made on the system at hand verifying this. In mathematical terms, as his system is based on the standard lattice-theory for quantum logic, making use of Hilbert space models, the notion of material truth does amount to an idea of truth in a model. Conversely, the formal dialogues deal with those material propositions that are true regardless of the elementary propositions contained within them, i.e. validities in exactly the sense of DL.

The way the notion of truth and validity is presented for Mittelstaedt's different varieties of dialogue is similar, both being in terms of dialogues, yet the concepts of material and formal truth in his system are fundamentally different - essentially the former is semantic truth, the latter validity. DL has only the latter, being in principle unconcerned with the notion of truth in a model.

Rahman & Tulenheimo specify two approaches to adding a notion of truth to dialogical systems. Alethic dialogues relativise a dialogue to a model, adding (in the case of propositional logic) a valuation function; whereas what they term *material dialogues* avoid this by approximating a notion of truth by adding hypotheses to the structure in the form of initial concessions of the opponent. Effectively this amounts to specifying a model but in terms of the object language only, by means of specifying of every atomic proposition that either it or its negation holds (Rahman & Tulenheimo, 2001: 176).

Effectively, as the name of his material dialogues would suggest, this is what Mittelstaedt does, making no reference to the algebraic structures at hand while setting up the dialogues (although he links the two later on). Of course, he does make reference to how the truth of the elementary propositions is determined, but in the context of setting up the dialogues he does this without recourse to a mathematical model, it is simply taken as read that we have a physical method for determining the truth of the elementary propositions, and later the commensurability propositions.

Quantum Features of Mittelstaedt's System

I will now examine the distinctively quantum features of Mittelstaedt's system and how he squares these with the retention of crucial aspects of classicality to which he aspires.

As we have already seen in the exposition of Mittelstaedt's system above, a key contention is that classical logic corresponds to the part of quantum logic involving only commensurable propositions. The manner in which this is effected is explored in more detail here.

One of the key features of quantum logic is a non-classical implication, and there are many possible variants of quantum implication (see eg. Dalla Chiara & Giuntini, 2001; Pavičić & Megill, 2009; Herman, Marsden & Piziak, 1975). Mittelstaedt's notion of factual implication and its relation to the material implication that appears in the dialogues we have seen above bears further examination. In the following section we will see that Mittelstaedt's basic notion of implication in the quantum logical setting is in fact that of Sasaki Hook.

In the following section we summarise the issue of quantum implication, why it cannot be the classical implication. The particulars of implication in Mittelstaedt's system are the discussed, beginning with an exposition of the various notions of implication he has in lattice-theoretic terms, and the relations between these and the commensurability propositions examined above. We then revisit the formal dialogue games and elaborate on how the implication functions within these before showing how the logic of the formal dialogues is extended to the full quantum calculus.

Quantum Implication - Background

We start with the briefest of comments on the problem of implication in quantum logic. It is not my intention to discuss the lattice theory involved in any great depth here, but a short look at it will serve to clarify the issue at hand. In this section, we look at the lattice theoretic aspects of Mittelstaedt's logic with a view to explaining his notion of implication. The necessity of modifying the classical notion of implication is one of the defining features of quantum logic, and so is a useful way to explore what makes Mittelstaedt's system distinctively quantum. We will see that the notion of commensurability plays a crucial role here, and we will look in slightly more depth at some lattice theoretic aspects of this mentioned earlier.

The material implication is among the classical connectives that must be altered to accommodate quantum phenomena. This problem hinges on the fact that the lattice underlying the logic is that of the closed subspaces of a Hilbert space, which differs in certain key respects from the Boolean lattice of classical logic.

The lattice of closed subspaces of a Hilbert space, unlike a Boolean lattice, is not distributive, i.e. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ does not, in general, hold. Any logical system with material implication as main operator will be distributive, and so the Hilbert space lattice cannot have a classical material implication. We will see why in more detail below.

Experimental propositions of the form $A \rightarrow B$ do not seem to encode a static experimental property of a system, but rather a relation between properties (Baltag & Smets, 2011: 4).

A propositional calculus, as Mittelstaedt puts it, is a lattice L, with elements A, B, C... which form a partially ordered system with respect to $\leq$, the implication relation, as he would have it. Of course, to have a logical system the implication $A \leq B$ is also used in iterated form (Mittelstaedt, 1972: 1359).

According to Dalla Chiara & Giuntini, an implication connective $\overset{*}{\to}$ ought to satisfy the following conditions:

Identity: $A \overset{*}{\to} A$

Modus Ponens: If A is true and $A \overset{*}{\to} B$ is true, then B is true (Dalla Chiara & Giuntini, 2008: 22).

As a side note, it is instructive to say a word or two about why some form of implication is so important for a logical system. Modus Ponens is essential for deduction - it is difficult to capture the notion that a statement follows from another without it, and so we must be able to express it in lattice theoretic terms if the lattice in question is to be considered a logic. As Zeman puts it, if a formalism is to have a reasonable interpretation as a theory of deduction, it must have some relation that, when it holds between two elements of the formalism, might reasonably be interpreted as expressing that one follows from the other (Zeman, 1979: 723). This is the implication relation, crucial for a formalism to be considered a logic[28].

Various different quantum implications have been proposed, and it is not my intention to discuss them all here, but implication connectives that can be defined for quantum logic are all to some degree anomalous (for further details, see Dalla Chiara & Giuntini, 2008).

We focus here on the variety of quantum implication known as Sasaki hook, which, as we shall see, is the implication of the full quantum logic of Mittelstaedt's system. This is defined as follows:

$A \overset{S}{\to} B := A' \vee (A \wedge B)$

Sasaki hook does exhibit certain non-classical features such as the failure of contraposition, ie. $A \overset{S}{\to} B \Rightarrow B' \overset{S}{\to} A'$ does not hold (Herman et al. 1975: 312). Nevertheless, it is arguably the best behaved of the quantum conditionals that can be defined, being equivalent to the material implication in Boolean sublattices of the quantum lattice, as we shall see below[29].

[28] See for example Greechie & Gudder, 1971 for an argument that quantum logic is not in fact a logic.

[29] Philosophical discussion as to the meaning of Sasaki hook abounds. Hardegree 1979, for example, argues that it encodes the Stalnaker conditional; whereas according to Smets, it should not be conceived of as a static implication, but a dynamic one (Smets, 2001: 308).

Implication in Mittelstaedt's System

A Digression on Lattices

In what follows, a brief sketch of the lattice theoretic aspects of Mittelstaedt's system is given, with a view to clarifying the manner in which he deals with the quantum implication discussed above. The lattice theory in question is relevant because it is the means by which he defines *factual implication*, the partial order relation, which is to be distinguished from the implication connectives that we saw above in the dialogues.

I begin by introducing some notation and defining several key properties lattices may have, which will differentiate the lattices discussed below.

We define a zero element 0 and a unit element 1[30] satisfying the following for any lattice element A (Mittelstaedt, 1978: 28):

$0 \leq A$, $A \leq 1$[31]

Orthocomplement: For an orthocomplemented lattice, there exists an automorphism $\Theta_{\neg} : \mathscr{S} \to \mathscr{S}$, denoted by $\Theta_{\neg}(A) = \neg A$ satisfying the following:

$A \wedge \neg A \leq \bigwedge$
$0 \leq A \wedge \neg A$
$A = \neg\neg A$
$A \leq B \Rightarrow \neg B \leq \neg A$

Pseudocomplement: The pseudocomplement of A, $\neg A$ in a pseudocomplemented lattice satisfies the following:

$0 \leq A$
$A \wedge \neg A \leq 0$
$A \wedge C \leq 0 \Rightarrow C \leq \neg A$

Quasi-pseudocomplement:

$0 \leq A$
$A \wedge \neg A \leq 0$
$A \wedge C \leq 0 \Rightarrow A \to C \leq \neg A$
$A \leq B \to A \Rightarrow \neg A \leq B \to \neg A$

Quasimodularity: $B \leq A,\ C \leq \neg A \Rightarrow A \wedge (B \vee C) \leq B$

Distributivity: $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge \neg C)$

[30] I have modified Mittelstaedt's original notation here, he uses $\bigwedge$ for 0 and $\bigvee$ for 1.

[31] When we come to interpret this lattice as a logic, 0 and 1 take on the meaning of canonical falsity and canonical truth, respectively.

Mittelstaedt introduces several lattices in the course of his explanation of the relation between dialogues and lattices. A key distinction is that between L_q, the quasimodular lattice; and L_{qi}, the *quasi-implicative* lattice of the logic Q_{eff} arising from the formal dialogue games. The latter corresponds to the intuitionistic part of L_q, as we shall see in greater detail below. As mentioned above, Mittelstaedt bases his novel methodology for quantum logic on the traditional conception of the lattice of closed linear subspaces of a Hilbert space. The lattice he eventually arrives at in his exposition is the orthocomplemented quasimodular lattice L_q, which is actually less specific than the algebra of closed subspaces of a Hilbert space (the latter having some properties not shared by L_q) (Mittelstaedt, 1978: 29). Yet he does this by way of the lattice L_{qi} corresponding to the propositional calculus Q_{eff} arising from the formal dialogue games detailed above.

The specifics of Mittelstaedt's exposition are rather detailed, so only the briefest outline is given here. He makes reference to the following lattice structures:

L_H: The lattice of closed linear subspaces of a Hilbert space.

L_q: The orthocomplemented quasimodular lattice. This has as a model the lattice of closed linear subspaces of a Hilbert space (Mittelstaedt, 1978: 27), but is a weakening of L_H, lacking certain conditions that hold there[32].

L_o: The orthocomplemented lattice.

L_{qi}: The quasi-implicative lattice. This is the lattice of the logic Q_{eff}, the propositional calculus arising from the formal dialogue-games. It is quasi-pseudocomplemented (rather than orthocomplemented or pseudocomplemented), and not quasimodular or distributive. It corresponds to the intuitionistic part of L_q.

L_i: The implicative lattice. This is the lattice of ordinary intuitionistic logic. It is distributive and pseudocomplemented. Mutually commensurable elements of L_{qi} form implicative sublattices of L_{qi}.

L_B: The Boolean lattice. This is the lattice of classical logic, which, of course, is orthocomplemented and distributive. Mutually commensurable elements of L_q form a Boolean lattice.

Relations between the different lattices Mittelstaedt discusses can be summarised as follows (Mittelstaedt, 1978: 118):

[32] Among these are atomicity and the covering law, which L_q does not satisfy (Mittelstaedt, 1978: 29).

L_{qi} extended by $A \leq B \to A$ results in L_i

L_{qi} extended by $1 \leq A \vee \neg A$ results in L_q

L_q extended by $A \leq B \to A$ results in L_B

L_i extended by $1 \leq A \vee \neg A$ results in L_B

Mittelstaedt develops logic Q_{eff} on the basis of the formal dialogue-game, as we shall see in more detail below. The axioms of this logic are intuitionistically defendable propositions from the games. These in turn correspond to the axioms of the lattice L_{qi}. Eventually, he extends this logic with the law of excluded middle (making certain assumptions about the value-definiteness of material propositions) to the full quantum logic, corresponding to the lattice L_q.

Propositions $A \in L_{qi}$ which can be shown to be true (ie. for which the relation $1 \leq A$ can be derived in L_{qi}) are the compound propositions for which there exists a strategy for success within a dialogue game. Therefore, a particular definition of a dialogue game counts as a semantics for L_i. This cannot be directly applied to L_q, which validates $A \vee \neg A$. However, a generalisation of the dialogic method can be used to give an interpretation of L_{qi}, the intuitionistic part of L_q[33]. The generalised quantum dialogue-game leads to an interpretation of L_{qi}, which can then be extended to an interpretation of L_q (Mittelstaedt, 1978: 46).

The reasons for his starting from the intuitionistic variant of the logic here appear to be at least to some degree of a historical bent. As has been noted above in the exposition of DL, the founders of the dialogical tradition were convinced that intuitionistic logic was the correct logic and were looking to establish a way of doing intuitionistic logic, a proof system for it. As such, this is the way Mittelstaedt sets up his dialogues, so as to result in an intuitionistic permutation of quantum logic. Nevertheless, he also later gives reasons based in the physical nature of the theory, when extending L_{qi} to L_q and hence building the calculus of full quantum logic, stating that the reason for the missing principle of excluded middle is the lack of value-definiteness for commensurability propositions. We will revisit this below.

[33] To be more precise, if excluded middle is added to the axioms of L_{qi}, the resulting lattice is isomorphic to L_q

Partial Orders as Implications

In a Boolean lattice L_B, the partial order $\leq$, as well as being considered a relation, can also be used to introduce a two-place operation *material implication*, which is the classical implication. For L_B, material implication is connected with $\leq$ by the condition $A \leq B \Leftrightarrow 1 \leq A \rightarrow B$. Here, $\rightarrow$ satisfies the following (Mittelstaedt, 1978: 37):

MI1: $A \wedge (A \rightarrow B) \leq B$

MI2: $A \wedge C \leq B \Rightarrow C \leq A \rightarrow B$

Of course, it is well known that the material implication can be defined in terms of the other classical connectives as follows: $A \rightarrow B := \neg A \vee B$.

In the lattices L_q and L_{qi}, we cannot have the material implication in general, as only distributive lattices have this form of implication, and L_q is not distributive[34]. Note that MI1 can be seen as a version of Modus Ponens and MI2 as one direction of the deduction theorem. These ingredients are crucial and any implication that satisfies them will necessarily be distributive, see, eg. Hardegree 1981.

Nevertheless, Mittelstaedt contends that postulates can be formulated for L_q guaranteeing the existence of a lattice element satisfying some intuitive properties of material implication such as Modus Ponens and additionally it has to relate to the order relation as follows: $1 \leq A \rightarrow B \Leftrightarrow A \leq B$ (Mittelstaedt, 1978: 44). According to Mittelstaedt, this new implication is the analogue of the implication that holds for Boolean and implicative lattices. Implications can be formulated differently for different lattices, but must satisfy Modus Ponens.

Therefore, we can have an implication that retains the most important features of the classical material implication for L_q, the analogue for the quasi-modular lattice of what we find in the Boolean lattice, which fulfils the conditions specified above. (Mittelstaedt, 1978: 38). I will denote this different variety of implication here by $\overset{S}{\rightarrow}$, as it refers to the above-mentioned Sasaki Hook. Mittelstaedt calls this implication variant *material quasi-implication* (the more commonplace contemporary terminology is Sasaki hook), and, to recall, it is defined as follows:

$A \overset{S}{\rightarrow} B = \neg A \vee (A \wedge B)$

[34] We will see later in the context of commensurability how Mittelstaedt obtains more standard varieties of implication for the dialogues.

Material quasi-implication (as defined for L_q) satifies the following conditions:

$A \wedge (A \xrightarrow{S} B) \leq B$
$A \wedge C \leq B \Rightarrow \neg A \vee (A \wedge C) \leq A \xrightarrow{S} B$

We now turn to the quasi-implication of L_{qi}, the lattice of the logic of the formal dialogues.

The quasi-implication of L_{qi} satisfies the following:

$L_{qi}(4.1)$ $A \wedge (A \rightarrow B) \leq B$
$L_{qi}(4.2)$ $A \wedge C \leq B \Rightarrow A \rightarrow C \leq A \rightarrow B$
$L_{qi}(4.3)$ $A \leq B \rightarrow A \Rightarrow B \leq A \rightarrow B$
$L_{qi}(4.4)$ $B \leq A \rightarrow B,\ C \leq A \rightarrow C \Rightarrow B * C \leq A \rightarrow (B * C)$ for $* \in \{\wedge, \vee, \rightarrow\}$

Under certain circumstances, the order on L_{qi} gives rise to the intuitionistic implication of Q_{eff}, on others it will give rise to the Sasaki hook, depending on the additional conditions present.

If L_{qi} is extended by the following axioms then the quasi-implication here goes over into the element $\neg A \vee (A \wedge B)$, the material quasi-implication of L_q. A sketch of the proof is given here by way of illustration[35] (Mittelstaedt, 1978: 114):

5.27 $1 \leq A \vee \neg A$
5.28 $A = \neg\neg A$
5.29 $A \leq B \Rightarrow \neg B \leq \neg A$

To prove that with the addition of these axioms, the quasi-implication of L_{qi} is equivalent to the material quasi-implication of L_q (ie, the quasi-implication $A \rightarrow B = A \xrightarrow{S} B = \neg A \vee (A \wedge B)$), we proceed as follows, referring specifically to $L_{qi}(4.2)$ ($A \wedge C \leq B \Rightarrow A \rightarrow C \leq A \rightarrow B$):

From $A \wedge (A \wedge B) \leq B$ we obtain $A \rightarrow (A \wedge B) \leq A \rightarrow B$. Because of the formal commensurability $A \wedge B \leq A \rightarrow (A \wedge B)$ (Mittelstaedt, 1978: 105), we have that $A \wedge B \leq A \rightarrow B$, from $L_{qi}(4.2)$.

From $L_{qi}(4.2)$, again, we have that $\neg A \leq A \rightarrow B$, as follows. From $A \wedge \neg A \leq B$, it follows from $L_{qi}(4.2)$ that $A \rightarrow \neg A \leq A \rightarrow B$. From the formal commensurability $A \rightarrow B \leq A \rightarrow (A \rightarrow B)$ (Mittelstadt, 1978: 105), we have that $\neg A \leq A \rightarrow B$.

It follows that $\neg A \vee (A \wedge C) \leq A \rightarrow C$.

[35]Indeed, L_{qi} can be extended to L_q with the addition of some axioms, see Chapter 5 of Mittelstaedt, 1978.

It is worth noting here that not everyone believes that taking the partial order as one's implication relation is legitimate. For example, Herman, Marsden & Piziack raise an objection, namely that implication, classically, is a binary connective; whereas $\leq$ is a relation between lattice elements. They contend that this amounts to a confusion of object-language and meta-language, and that $\leq$ says something about deducibility, rather than implication (Herman, Marsden & Piziack., 1975: 307). Zeman, for example, contends that $\leq$ is in fact the deduction ($\vdash$), rather than the implication relation (Zeman, 1979: 724).

Mittelstaedt essentially mitigates against this challenge to viewing $\leq$ as the implication by saying that while it is a relation, where it holds between A and B, a particular kind of proposition concerning A and B is also true. Here, it is worth briefly clarifying the difference between the partial ordering relation on the lattices denoted by $\leq$ and the closely linked, but different operator $\rightarrow$ (in its various forms, as we have seen, both these concepts differ depending on the lattice under discussion). The partial order on a lattice is a relation that holds between elements of the lattice in question. The implication operator $\rightarrow$, on the other hand, gives rise to a proposition in its own right, and hence another element of the lattice. The proposition $A \rightarrow B$ holds precisely when $A \leq B$, this is when the additional lattice element $A \rightarrow B$ is generated.

So in the case of L_q, for example, the Sasaki hook, $A \overset{S}{\rightarrow} B$ is a proposition about the relation between A and B, and so is, in fact, another, separate element of the lattice. This proposition is true when $A \leq B$, and is equivalent to the proposition $\neg A \vee (A \wedge B)$, as we have seen. This relationship between the partial order and the implication operator holds regardless of the lattice in question (at least as far as the lattices Mittelstaedt discusses are concerned), but for different lattice structures, the element generated by, so to speak, talking about the relation $\leq$ is different.

Commensurability in Lattice Theoretic Terms

We now turn our attention to commensurability from a lattice theoretic point of view, in order to better understand the relationships between the lattice-theoretic implications $\leq$ in L_q and L_{qi} and the implication connectives in their associated logics. The key point here is that commensurable propositions generate distributive sublattices, thereby allowing for non-quantum implication connectives to hold between propositions in the logics at hand.

We now define the commensurability relation K, which we saw above in the context of dialogues, in lattice theoretic terms, for L_q, and L_{qi}[36]

$K \subseteq L_q \times L_q := (A, B) \in K \Leftrightarrow (A \wedge B) \vee (A \wedge \neg B)$

Implication relations are contained in K for both L_q and L_{qi}. This means that commensurability is a prerequisite for the implication relation $\leq$ to hold between two elements.

In L_q, the conditions satisfied by $K \subseteq L_q \times L_q$ can be summarised as follows (Mittelstaedt, 1978: 34).

(KL_q1) K is symmetric
(KL_q2) $R \subseteq K$
(KL_q3) If $\mathscr{S} \subseteq L_q$ is a subset of elements with $\mathscr{S} \times \mathscr{S} \subseteq K$, then $\mathscr{S}$ generates a Boolean sublattice $L_B(\mathscr{S}) \subseteq L_q$
(KL_q4) If $L_B \subset L_q$ is a Boolean sublattice, the elements of any subset $\mathscr{S} \subseteq L_B$ are commensurable, ie. $\mathscr{S} \times \mathscr{S} \subseteq K$.

The proof that mutually commensurable elements $A, B, C \in L_q$ generate a Boolean sublattice runs as follows:

From $B = (B \wedge A) \vee (B \wedge \neg A)$ and $C = (C \wedge A) \vee (C \wedge \neg A)$, we get $A \wedge (B \vee C) = A \wedge ((B \wedge A) \vee (B \wedge \neg A) \vee (C \wedge A) \vee (C \wedge \neg A))$. Applying quasimodularity we get the desired result (Mittelstaedt, 1978: 33).

In fact, this result holds for any orthomodular lattice; the result can be stated in what is called a Foulis-Holland set in the literature. A Foulis-Holland set is a set of elements A, B, C of an orthomodular lattice where at least one element commutes with the other two (ie. one element is commensurable with the other two, casting our minds back to the definition of compatibility in the introduction). This implies distributivity where the set in question is a subset of the orthomodular lattice[37].

We can easily see how where we have commensurable propositions A and B (and hence, a distributive sublattice of L_q), the Sasaki hook defined above becomes equivalent to the material conditional. Recall the definition of commensurability, $A = (A \wedge B) \vee (A \wedge \neg B)$. Sasaki hook $A \rightarrow B$, as we have seen, is defined as $\neg A \vee (A \wedge B)$. By distributivity, from the latter we have $(\neg A \vee A) \wedge (\neg A \vee B)$, in other words $\neg A \vee B$, the classical material implication, as long as excluded middle is a validity. This does not automatically hold for L_{qi}, where, in the first instance, the law of excluded middle does not hold[38], and in the second, the quasi-implication of L_{qi} does not give

[36]The relevant definitions are found in Mittelstaedt 1978, pages 35-35, and 112, respectively.
[37]For the proof of this, see eg. Greechie, 1977.
[38]Recall the failure of a dialogical proof for the law of excluded middle working with intuitionistic rules in DL above

us the Sasaki hook unless some axioms are added. This will become clearer when we look at the relation between L_{qi} and the corresponding logic Q_{eff}, which is intuitionistic and so does not admit of an implication connective definable in terms of the other connectives.

Analogous to the conditions for L_q above, in L_{qi}, the conditions satisfied by $K \subseteq L_{qi} \times L_{qi}$ can be summarised as follows (Mittelstaedt, 1978: 34).

(KL_{qi}1) K is symmetric
(KL_{qi}2) $R \subseteq K$
(KL_{qi}3) If $\mathscr{S} \subseteq L_{qi}$ is a subset of elements with $\mathscr{S} \times \mathscr{S} \subseteq K$, then $\mathscr{S}$ generates an implicative sublattice $L_i(\mathscr{S}) \subseteq L_{qi}$
(KL_{qi}4) If $L_i \subset L_{qi}$ is an implicative sublattice, the elements of any subset $\mathscr{S} \subseteq L_i$ are commensurable, ie. $\mathscr{S} \times \mathscr{S} \subseteq K$.

Neither of the lattices above is distributive, yet commensurable subsets of elements generate distributive sublattices for both, albeit different varieties thereof - the implicative lattices of ordinary intuitionistic logic for L_{qi}; the Boolean lattices of classical logic for L_q.

Quasi-implication, the Formal Dialogue Game and Effective Quantum Logic

For the sake of convenience in formulating the calculus Q_{eff} of effective quantum logic, the calculus we get from the formal dialogue game, Mittelstaedt introduces some of the notions from the lattice theoretic setting we have just seen, to which we now turn our attention. He adds to the set S of propositions two new propositions, canonical truth denoted by 1 and canonical falsity denoted by 0. In the dialogical context, 1 cannot be questioned by either player, and the player attempting to argue 0 loses the dialogue. It follows that propositions $A \rightarrow 1$ and $0 \rightarrow A$ can be defended in a dialogue regardless of the content of A (Mittelstaedt, 1978: 89).

He also introduces the now familiar relation $\leq$ in the context of the dialogues, which holds if and only if the intuitionistic implication can be defended in a dialogue, defined as:

$A \leq B \Leftrightarrow \vdash_{D_f} A \rightarrow B$

From the dialogues and the relation above, we also have that $A \leq 1$ and $0 \leq A$ for all A. From these and $L_{qi}(4.3)$ detailed in the section on commensurability above, it also follows that 1 and 0 are commensurable with all other propositions. Aribitrary A can be defended in a formal dialogue if and only if the relation $1 \leq A$ holds; similarly $\neg A$ holds if and only if $A \leq 0$ does (Mittelstaedt, 1978: 90). And so we have $1 \leq \neg A \Leftrightarrow A \leq 0$.

It can be shown in the dialogues that conditions 1.1 and 1.2 from above hold, the first two conditions satisfied by the quasi-implication, as follows:

1.1: $A \leq A$

	O	P		
0	[ ]		$A \to A$	
1.	A	(0)	A	$\langle 0 \rangle$

1.2: $A \leq B, B \leq C \Rightarrow A \leq C$

	O		P	
$-2.$	$A \to B$			
$-1.$	$B \to C$			
0.			$A \to C$	
1.	A	(0)	[ ]	
2.	B	$\langle -2 \rangle$	A	(-2)
3.	C	$\langle -1 \rangle$	B	(-1)
4	[ ]		C	$\langle 0 \rangle$

It is worth noting that whether propositions A, B and C in this second dialogue are mutually commensurable is irrelevant for the strategy of success in question (Mittelstaedt, 1978: 92). For the sake of ilustration, I also include the dialogical proof of Modus Ponens, given its central importance discussed above.

Modus Ponens: $A \wedge (A \to B) \leq B$

	O		P	
0.			$(A \wedge (A \to B)) \to B$	
1.	$A \wedge (A \to B)$	(0)	[ ]	
2.	A	$\langle 1 \rangle$	1?	(1)
3.	$A \to B$	$\langle 1 \rangle$	2?	(1)
4.	B	$\langle 3 \rangle$	A	(3)
5.	[ ]		B	$\langle 0 \rangle$

Note that this dialogue makes use of the formal commensurability $k(A, A \rightarrow B)$ at $P(4)$, where P takes over A asserted by O at (2) (Mittelstaedt, 1978: 93).

As we have seen in the discussion of the various lattices involved, where propositions are commensurable, they form implicative sublattices of the quasi-implicative lattice L_{qi} (which, again, corresponds to the calculus Q_{eff} of effective quantum logic). This is the manner in which Mittelstaedt manages to maintain more standard connectives in his logic whilst also accounting for quantum features.

However, there are validities provable in the formal dialogues that hold regardless of whether the propositions concerned are commensurable, see the dialogical proof of 1.2 above, for example.

The quasi-implicative lattice is a weakening of the implicative lattice, the lattice of ordinary intuitionistic logic. This can be seen from the following dialogically provable propositions, which can be strengthened to intuitionistic validities in the case that the propositions involved are commensurable:

Q_{eff} theorem: $A \wedge C \leq B \Rightarrow A \rightarrow C \leq A \rightarrow B$. The dialogue in question runs as follows:

	O		P	
$-1.$	$A \wedge C \rightarrow B$			
0.	[ ]		$(A \rightarrow C) \rightarrow (A \rightarrow B)$	
1.	$A \rightarrow C$	(0)	$A \rightarrow B$	$\langle 0 \rangle$
2.	A	(1)	[ ]	
3.	[ ]		$A \wedge C$	(-1)
4.	2?	(3)	[ ]	
5.	C	$\langle 1 \rangle$	A	(1)
6.	B	$\langle -1 \rangle$	C	$\langle 3 \rangle$
7.	[ ]		B	$\langle 1 \rangle$

This can, if the propositions in question are commensurable, be strengthened to $A \wedge C \leq B \Rightarrow C \leq A \rightarrow B$, which cannot be proved in the formal dialogue game. Mittelstaedt states that this is the decisive difference between ordinary logic and quantum logic (Mittelstaedt, 1978: 94). As we saw above, any implicative lattice satisfies this condition.

Likewise, we have Q_{eff} theorem $A \wedge C \leq 0 \Rightarrow A \rightarrow C \leq \neg A$, provable in the formal dialogue game as follows:

	O		P	
$-1.$	$\neg(A \wedge C)$			
0.	[ ]		$(A \rightarrow C) \rightarrow \neg A$	
1.	$A \rightarrow C$	(0)	$\neg A$	$\langle 0 \rangle$
2.	A	(1)	[ ]	
3.	[ ]		$A \wedge C$	
4.	2?	(3)	$A(1)$	
5.	C	$\langle 1 \rangle$	C	$\langle 3 \rangle$

Where A and C are commensurable, we can strengthen this to $A \wedge C \leq B \Rightarrow C \leq A \rightarrow B$, as in ordinary intuitionistic logic, however, this is not generally provable in the formal dialogue game.

In the context of Q_{eff}, the implication $A \leq B$ holds if and only if the intuitionistic implication[39] $A \rightarrow B$ can be defended in a formal dialogue game (Mittelstaedt, 1978: 90). Every implication derivable from Q_{eff} can be proved in a formal dialogue. Q_{eff} is consistent with regards to the class of implications provable in the formal dialogue, also complete, ie. $\vdash_{Q_{eff}} A \leq B \Rightarrow \vdash_{Q_{eff}} A \rightarrow B$, and conversely, $\vdash_{Q_{eff}} A \rightarrow B \Rightarrow \vdash_{Q_{eff}} A \leq B$. This amounts to being able to prove the same things in the logical and lattice-theoretic settings (Mittelstaedt, 1978: 99).

If all elements of a lattice L_{qi} are 'pairwise commensurable', ie. each element is commensurable with every other, in other words, we can add the relation $A \leq B \rightarrow A$ to its axioms, then we have an implicative lattice, as we saw above (Mittelstaedt, 1978: 112). Quasi-implication (the partial order) is contained in the commensurability relation (and so any propositions between which the quasi-implication holds will be commensurable), but the connectives give no indication as to whether the propositions in question are commensurable. So, for example, there is a strategy of success in the formal dialogue game for $A \wedge B \leq A$ regardless of whether A and B are commensurable or not, yet, of course, A is commensurable with itself (Mittelstaedt, 1978: 92).The propositions involved in a successful formal dialogue proof, at the lattice theoretic level, form implicative sublattices of L_{qi} (ie. lattices of ordinary intuitionistic logic). However, this does not necessarily mean that the propositions involved in each different formal dialogue game are mutually commensurable, hence the fact that L_{qi} is not

[39]Mittelstaedt uses the term *material implication* to refer to two separate concepts - the classical material implication connective and the intuitionistic implication connective, I have differentiated between the two here to avoid confusion.

itself an implicative lattice.

Q_{eff} is intuitionistic, so the implication we have here is not exactly that of classical logic, definable in terms of $A \rightarrow B := \neg A \vee B$. Intuitionistic logic does not have this kind of definition in terms of other operators for the implication, as Mittelstaedt says, L_i has an implication connective that cannot be expressed in terms of the other connectives[40] (Mittelstaedt, 1978: 43). Rather, what $A \rightarrow B$ in this context tells us is that there is some method by which a proof of B can be deduced from a proof of A, in this case, clearly via the means of a winning strategy for B in the material dialogue derived from a prior proof of A.

Therefore, we end up with an implication connective that expresses something more along the lines of 'if A is provable in a material dialogue, then from that proof, B is provable in a material dialogue'. This stands to reason, given that the connectives in the dialogue games are defined by the possible attacks and defences they admit, and in the formal dialogue game the whole idea is to construct validities, which hold regardless of the content of material propositions.

Concerning validity, we use the dialogues to capture what is valid, eventually ending up with a propositional calculus based on a series of formally true implications from which other formally true implications can be derived (Mittelstaedt, 1978: 97). Of course, Mittelstaedt does in fact provide us with a semantics for this logic as well in the form of the material dialogues, he does specify what it is for a material proposition to be true, but he also abstracts away from this in the formal dialogue games, eliminating reference to material propositions in favour of propositions that are provable in the material dialgues no matter what the content of the elementary propositions contained therein. Yet the material dialogues then become relevant again when extending the logic to the full quantum logic, as we want to have some reference to the quantum systems this is intended to capture.

I now give a very brief sketch of how Mittelstaedt extends Q_{eff} to obtain the calculus of full quantum logic (which he calls Q).

Q_{eff} (with corresponding lattice L_{qi} differs from the propositional structure in Hilbert space (given by lattice L_q) in that the law of excluded middle cannot be derived. For elementary propositions the law of excluded middle 'is valid for physical reasons', elementary propositions have definite truth values, whereas for compound propositions the law of excluded middle cannot be proved. This is due to the fact that commensurability propo-

[40] For a detailed exposition of intuitionistic logic, see for example Bezhanishvili & de Jongh, 2006

sitions can never be shown conclusively by a finite process to be value definite[41], and so it cannot be guaranteed that the elementary propositions in a compound proposition containing a two-place connective are always commensurable (Mittelstaedt, 1978: 119). Mittelstaedt instead postulates a 'strategy of confirmation' on the philosophical basis that science proceeds without such stringent proofs, meaning that the law of excluded middle can be introduced in the calculus (Mittelstaedt, 1978: 138).

Concluding Thoughts

Mittelstaedt's Dialogues in the Quantum Logical Setting

Mittelstaedt's setup is, as he explicitly states, the traditional quantum logic setup, conceptualised via the notion of dialogues we have looked at in detail above, describing a process or algorithm for checking the validity of a proposition. The lattice theoretic background he describes is the traditional variety, however, as we have seen, he looks to develop the dialogical system independently of its lattice-theoretic basis, before unifying the dialogues with the the lattices generated by the logics in question.

Mittelstaedt speaks of the dialogic conceptualisation as a pragmatic device (Mittelstaedt, 1978a: 48), and emphasises that the construction of his language does not hinge on any empirical results, contending instead that 'the syntactic rules of this language are rather the linguistic preconditions of the possibility of experience' (Mittelstaedt, 1978a: 48). The procedures described for proving propositions about the world determine the syntax of this scientific language (Mittelstaedt, 1978a: 48), they also determine the semantics, the meaning of a proposition is conceptualised in terms of its possibilities in a dialogue.

Ultimately, his system is both a means to determine truth in a particular kind of mathematical model and a means to determine validites. Mittelstaedt is keen on the fact that the dialogues can be divorced from the lattice theory, standing on their own as purely syntactic devices without reference to empirical results, although he does, as we have seen, need to make some concession to the fact that he is dealing with empirical phenomena.

[41] This point has a distinct Popperian feel to it, essentially what Mittelstaedt is running up against here is just the problem of induction - no finite sequence of successful demonstrations of a commensurability proposition guarantees that the next attempt will not have a different answer.

Mittelstaedt's language takes as primitive the results of measurements - 'elementary propositions concerning measuring results' (Mittelstaedt, 1978a: 48). Thus his conceptual foundation fits with the operational tradiditon in quantum logic. Rather than seeing a projection as encoding a physical property of a system, and assuming that it is physical properties that ultimately count as meaningful or fundamental, the operational approach, originating with the work of Piron, views quantum logic in terms of the results of measurements made on a system (Coecke, Moore & Wilce, 2001: 6). Mittelstaedt's system follows this general idea, the elementary propositions of material dialogues are taken to be the results of measurements made on a quantum system.

This idea chimes with the notion of a dialogue in that the making of measurements on a system can be conceptualised as a process with a variety of question-answer structure, as mentioned in the introduction. 'The operationist [sic]...will see the experimenter as the asker of questions, he will view the theory as supplying the answers and he will view quantum logic as describing the logical structure of those questions and answers', as Gibbins words it (Gibbins, 1982: 210). Of course, the measurements made on a system, which determine the truth of elementary propositions do not form part of Mittelstaedt's dialogical structure, the means of determining the truth of elementary and availability propositions is taken to be a process that happens outside the dialogues. Mittelstaedt seeks to relate the empirical and the logical in such a way as to delineate the two clearly, he is at pains not to allow the empirical to directly affect his logic, which ultimately abstracts away from the material dialogues, creating a system that does not hinge on quantum mechanics, although it contains devices designed to deal with the non-classical features that must be accomodated by any logic thereof. The dialogical approach appears convenient for this, allowing as it does a kind of heirarchy of dialogues, material dialogues concerning empirical notions; then formal dialogues that eliminate the empirical aspects. The empirical aspect reappears in the full quantum logic, as Mittelstaedt has to justify the law of excluded middle with reference to quantum measurements resulting in value-definite propositions.

As is the way with philosophers and logicians, Mittelstaedt wishes to have his cake and eat it, locating the non-classicality of quantum logic outside of the main dialogues, via availability propositions that must be proved outside these dialogues. An ingenious feature of his setup is that he can incorporate classical logic into his system, as a special case where propositions are unrestrictedly available. So it seems that (like Putnam 1968, among others), he believes his quantum logic to be the 'actual' logic of the world. As he puts it, 'the 'true' object-logic of propositions about physical systems is given by the calculus of full quantum logic' (Mittelstaedt, 1978: 143). He contends that the calculus of quantum logic is the most general propositional system applicable to propositions about classical and quantum systems, and that classical propositional logic is a special case of this (Mittelstaedt, 1978: 140).

One clear motivation, as I see it, for Mittelstaedt to espouse a dialogical approach is that it captures the ways in which he introduces and explains the quantum features of his system very intuitively. Commensurabilities are easily explained in the dialogical structure, in a way that has perhaps more intuitive appeal than an explanation in terms of commuting projection operators, although of course they amount to the same thing as Mittelstaedt's theory is also based on the Hilbert space foundation.

To say that two statements are commensurable or otherwise and check for availability in view of what has previously been proved in a particular dialogue is a neat way to keep track of what can and cannot be said of a given quantum system at the level of material dialogues. It is a useful way to think about the information available about a particular system that again, gives a procedure for checking what information is available at a given point in a dialogue.

Gibbins makes a suggestion not made by Mittelstaedt himself regarding the motivations for the dialogical approach, along the lines that a dialogical construction is more natural than an axiomatic or natural deduction account in the quantum setting. Gibbins' reasons for this centre on logic representing a reasoning process, and quantum logic representing a way of reasoning about quantum systems. He suggests that axiomatic approaches distort this fact, presenting logic as a mathematical system, rather than a representation of a reasoning process. It minimises the need for rules of inference, but at the cost of requiring axioms. Natural deduction approaches, conversely, eliminate some (or even, at the limit, all) axioms in favour of rules of inference, thereby arguably mirroring reasoning processes more accurately (DL is a prime example of this manner of thinking, although the focus is not necessarily on the accurate reflection of reasoning but on more technical advantages). Gibbins thinks that Mittelstaedt's approach takes this further, and that Mittelstaedt might argue that dialogic in this context was yet more natural, that 'logic arises because of, and perhaps in the course of *arguing*' (Gibbins, 1982: 211). As we saw above, in DL, a proposition is true when it can be defended by its proponent. It is formally true if it can be defended regardless of other propositions in the dialogue. A proof in this context, rather than being sequence of well-formed formulas or a tree of valid sequents, is a dialogue where the proponent of the proposition proved has successfully defeated the opponent attempting to disprove the proposition. Gibbins contends that this structure fits well with Mittelstaedt's operationalist bent regarding the philosophy of quantum mechanics (Gibbins, 1982: 212).

It appears to me that this analysis is somewhat flawed, for two reasons. Firstly, there is the fact that, although quantum logic formalises an empirical phenomenon, it still does so on the basis of the mathematical models used to formulate the theory of that phenomenon. With regards to quantum mechanics we have experimental testing procedures, but it isn't as if we are able to experience the phenomena tested in the sense of our everyday experience of the macroscopic world. Measurements are made by an intricate set of equipment and procedures, and the mathematical formalisms are essentially what gives meaning to the very idea of performing such tests - there would be precious little point in doing so if not to compare the results to the predictions of the theory in question, which is a mathematical model of the workings of the microscopic world.

Secondly as far as the reasoning process is concerned, the dialogical approach makes no attempt to model any real world, human reasoning process, quite the contrary, it has nothing to do with actual reasoning, everything to do with an idealised reasoning process, where the agents involved in the dialogues are idealised. As mentioned, it essentially provides an algorithm for a normative kind of reasoning. This point is potentially somewhat uncharitable towards Gibbins, admittedly, clearly he is not suggesting that there is no normatively correct way to reason about quantum systems. Yet he does speak of the players of the game as if they were real word agents, contrary to the spirit of the dialogical approach, and this seems to me to be confusing the issue.

The fact that Mittelstaedt's modus operandi for expounding his system is entirely dialogical seems worthy of some thought. Partly this is presumably a desire for consistency and a unified and systematic exposition. However, other dialogue-based approaches such as Hintikka's game-theoretic semantics take a different stance, incorporating a traditional semantics making reference to truth in a model, whilst using the dialogues as the means of showing a proposition to be true or false. While the game-theoretic semantics is also dialogue-based, they start from a different perspective compared to the proof-theoretic accounts explained in this thesis. Part of the interest of Mittelstaedt's approach, as I see it, is that he defines both the syntax and (part of) the semantics (ie. the definitions of connectives) in dialogical terms, referring to model only for the urposes of determining the truth of elementary propositions; whereas GTS defines having a winning strategy in a dialogue with reference to truth in a model without the particle rules characteristic of the dialogical approach.

Dialogical Quantum Logic and Dynamic Quantum Logic - Further Research Possibilities

I conclude this thesis with some thoughts on avenues for further research, in particular with regards to relations between dialogical logics and dynamic logics, a more recent development that has proven highly successful in modelling many real world phenomena including knowledge and belief. Epistemic and quantum variants thereof differ. The latter aim to capture processes of information exchange, identifying states of a system (of whatever kind) with the *actions* that can be performed upon the system, the ways it can be changed. Thus, this approach posits actions, as opposed to propositions, as the subject matter of the logics in question (Baltag & Smets, 2008: 2). Modal operators are used to formalise these actions. In particular, a dynamic quantum logic is developed by Baltag & Smets[42], and some comparisons between this and Mittelstaedt's dialogical quantum logic will be sketched in this section.

It is clear that there are fruitful connections to be explored between dynamic and dialogical logics generally. Dialogical variants of modal logic have been devised[43]. The challenge for dialogical approaches in this context is to dispense with model theoretic notions such as possible worlds and define modalities in purely dialogical terms; also to capture the notion of an accessibility relation between the 'worlds' in question. One way of doing this is by introducing 'worlds' as hypotheses; and relations between them as specification questions narrowing down the possibilities, so we get to more specific statements from the original hypothesis by means of the dialogues[44]. Another is to add an extra feature to the structural rules, a series of labels for different contexts (roughly speaking) (Magnier, 2012: 15).

True to its modal nature, dynamic quantum logic is built up from a set of states and a family of binary relations between those states, the latter representing different operations that can be performed on a state, we can then detail whether these operations result in a new state, and what the state resulting from such an operation on an initial state will be. The language, we have set of propositional formulas and a set of programs, unitary transfor-

[42]See, eg. Baltag & Smets 2004; Baltag & Smets 2005

[43]See, eg. Rebuschi 2009 for a dialogical reconstruction of (modal) epistemic logic; Magnier 2012 for a dialogical reconstruction of (dynamic) Public Announcement Logic; Blackburn 2001 for an argument that modal logic is fundamentally dialogical.

[44]Rahman 2015 gives such a formulation of modal logic in dialogical terms.

mations capturing the actions that can be performed on a quantum system. For the set of propositional formulas the classical connectives are defined in the usual manner, the quantum programs, however, behave non-classically, reflecting quantum features, which are accounted for as a non-classical dynamics of information flow. As such, we can express what may occur to a system, as opposed to just what is true of it. The dynamic approach is more expressive than Mittelstaedt's, which has no way to differentiate between different types of measurements. Dynamic quantum logic also encodes unitary evolutions of a system over time, which Mittelstaedt has no way to capture.

Of course, dynamic quantum logic has the more usual logical setup, being generally conceptualised in terms of a quantum frame with a valuation function on this frame (Baltag & Smets, 2004: 49) - as such it refers to truth in a given model, which the dialogical approach looks to avoid in general, being proof-theoretic in nature. Yet, as we have seen, Mittelstaedt in fact does have a semantic aspect to his logic, eventually referring back to the algebraic Hilbert space realisation as a model of the quantum lattice.

The two approaches have certain interesting features in common, to which we now turn. Contrary to traditional quantum logic, where laws of classical logic are generally revised to account for quantum features, both can be united with a classical logic within their respective systems. Of course, the ways in which they conceptualise this are markedly different. Dynamic quantum logic can capture classical logic as the static part of its system, Mittelstaedt conceptualises it as the logic of commensurable propositions.

For dynamic quantum logic, it is the actions that can be performed on a system that are non-classical, yet a static, classical component remains. In Mittelstaedt's system it is the incommensurability of propositions that make the logic quantum and he gives us a method for determining validities within this system, some of which require the propositions in question to be commensurable and some of which do not. As we have seen, commensurable elements of the quantum lattice generate Boolean sublattices, the classical parts exist within the wider, quantum whole. As we would expect, there is a parallel here, the dynamic part of the dynamic quantum logic encapsulates the tests that can be performed on a system; availability propositions for Mittelstaedt tell us whether two propositions are such that each can be tested for without affecting the result of a test for the other. Yet the dynamic approach differs in that the quantum features are expressed in terms of actions on the static parts, whereas Mittelstaedt's leaves the notion of testing outside the dialogues, outside the formalisation itself.

The dynamic approach conceives of the non-classicality of quantum logic in terms of non-classical information flow. Mittelstaedt does not put it in these terms, but of course commensurability is his way of expressing restrictions on the information that can be gleaned about a system.

One interesting feature of the dynamic approach in relation to Mittelstaedt's system concerns the implication. As we saw, in Mittelstaedt's full quantum logic, the implication connective is the Sasaki hook. This is also the case for dynamic quantum logic. One of the actions that can be performed on a system is a test for a given property, and the Sasaki hook $R \stackrel{S}{\rightarrow} T$ is captured in the dynamic part of the language, as the dynamic modality for tests. It expresses the idea of the antecedent being the *weakest precondition* for the consequent. It holds if after successfully performing $R?$, the test for property R on a system, property T will surely hold[45] (Baltag & Smets, 2011: 10). The terminology is slightly different within the two systems, with the dynamic approach conceived in terms of states and properties thereof, modified by measurement processes, Mittelstaedt's in terms of the results of measurements.

To introduce the details captured by dynamic quantum logic would entail the introduction of a modal aspect to Mittelstaedt's system. This would require clauses for modalities in the material dialogues (analogous, as we saw, to the particle rules of DL); as well as additional argument rules for the formal dialogues detailing how the modal operators may be used there (Rebuschi, 2009: 233). The formal dialogues are intuitionistic, Rebuschi 2009 details how an S4 modal 'simulation' or translation of intuitionistic logic can be implemented in a dialogical structure (in this case in the context of an S4 epistemic logic) (Rebuschi, 2009: 239). This suggests that, with appropriate modifications, something similar might be implemented with quantum modalities in the formal dialogue game, although the details would need to be worked out.

The dialogical system deals primarily with rules, with a process, rather than with the notion of information central to the dynamic approach. Yet it seems to me that Mittelstaedt is still making use of a notion of information. In the case of availability propositions in the material dialogues, we might say that here we deal with the specifics of information we can have about the system we are talking about. Ultimately in the formal dialogues the notion of what information we can have about a system at any one time

[45] We also have a weaker notion in the dynamic setup, $\langle R?\rangle T$, meaning that one *can* perform a test R on the current state and end up with property T, as well as the ordinary $\Box$ and $\Diamond$ modalities for tests, meaning that the property in question will hold after any test, or that there is some test one can perform resulting in that property, respectively (Baltag & Smets, 2004: 42).

is amalgamated into the system of validities by determining certain commensurability propositions (about simultaneously available information) that always hold. These are then translated into a series of propositions that can always be used within the proof system. Commensurability is set up to capture the fact that there are physical quantities of a quantum system that cannot be simultaneously known, where measuring one will affect a subsequent measurement of the other, therefore it appears that they encode part of the meaning of the dynamic test operators[46]. Nevertheless, it is important to bear in mind that the formal dialogue games do not yet give us the full quantum logic where the Sasaki hook implication holds, so it is not the case that the intuitionistic implication of the formal dialogues coincides with the weakest precondition of dynamic quantum logic.

In this respect the two systems differ, dynamic quantum logic has classical implication and Sasaki hook, the two varieties of implication that hold in Mittelstaedt's full quantum logic. Mittelstaedt has the calculus of the formal dialogues as a stepping stone to the full quantum calculus, but no analogue of this exists for dynamic quantum logic. Nevertheless, the formal dialogues give us the means to define calculus Q_{eff} which is extended to calculus Q. Sasaki hook is the definition of implication for Q and therefore expresses the same thing as its dynamic equivalent. The necessity element is not couched in modal terms but in terms of validity. Rather than a notion of weakest precondition, in Mittelstaedt's system a proposition of the form $A \rightarrow B$ in Q means that a material dialogue proving A guarantees a successful material dialogue about B. This is tantamount to saying that this holds for all material dialogues. The latter is essentially a proof-theoretic account; whereas dyanmic quantum logic is a semantic one.

As outlined above, modal structural rules must place restrictions on the accessibility relation between states. This is somewhat akin to Mittelstaedt's development of the formal commensurability calculus in the formal dialogues, which fulfils the role of establishing propositions that can be simultaneously true. The dynamic approach allows us to express (among other things) the idea that a particular proposition is true after a given measurement has been made. It would therefore make sense that commensurability relations would need to be expressed in modal terms, if modal operators were to be added to the system, and the ways in which this could be done would need to be worked out in detail to unify the two approaches.

[46] Albeit less detailed, given that, for example, we have no explicit notion of a property being potentially satisfied. It seems to me that this is connected to the fact that the formal dialogues are set up to capture validity, truths that alays hold, without regard for the expression of possibility.

Conclusion

Here, I have aimed to give a thorough exposition of Mittelstaedt's dialogical quantum logic, which appears to have been somewhat neglected in more contemporary studies of quantum logic. A basic account of the workings of the dialogical approach has been given, with a view to facilitating a comparison with the quantum dialogical system. We have seen in some detail that Mittelstaedt's approach is a modification of the intuitionistic permutation of standard dialogical logic, incorporating a semantic element (in the form of material dialogues) that pertains directly to quantum systems, but which ultimately abstracts away from empirical content to give us a dialogical system for proving validities (formal dialogues). This system gives a method for the elimination of direct references to the quantum phenomena it must accommodate in the formal commensurability calculus, the validities of which are then shown to be equivalent propositions provable in the formal dialogues. An intuitionistic propositional calculus is then developed on the basis of these validities, which can be extended to the full quantum logic. One of the decisive differences between classical and quantum logics, the implication connective, was then examined, first with reference to the lattice theoretic basis of the system and then with reference to the dialogues, with the aim of demonstrating how the quantum implication known as Sasaki hook arises in the system. Subsequently some general thoughts on the use of the dialogical approach in the quantum setting have been given, as well some indications of possible avenues for futher research in this field on the basis of connections between Mittelstaedt's dialogical system and recent dynamic approaches to quantum logic.

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