The Modal Logic of Generic Multiverses

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis, we investigate the modal logic of forcing and the modal logic of grounds of generic multiverses. Hamkins and Löwe showed that the ZFC-provable modal principles of forcing, as well as of grounds, are exactly the theorems of the modal logic S4.2 (see [16],[17]). We prove that the modal logic of forcing of any generic multiverse is also exactly S4.2 by showing that any model of ZFC has a ground whose modal logic of forcing is S4.2.

Moreover, we show that the modal logic of grounds of any generic multiverse is contained in S4.2Top. In particular, this implies that the modal logic of grounds of any generic multiverse with a bedrock is exactly S4.2Top. Furthermore, we show that the modal logic of any generic multiverse obtained by forcing with a progressively closed class product satisfying certain definability conditions – the only method known to us to produce multiverses without a bedrock – is contained in S5.

Contents

1 Introduction

In contemporary set theory, the technique of forcing is one of the main tools used to construct a vast variety of models. Introduced by Cohen [6] in 1962 to prove the independence of the axiom of choice and the continuum hypothesis, forcing subsequently was the key to many additional independence results. This technique enables us to extend models of set theory while ensuring a precise control over which sentences are true in the extension.

While forcing is a tool to show the independence of sentences from ZFC, the independence phenomenon is not random or without structure; instead it exhibits recognizable patterns. In the early 2000s, Hamkins and Löwe [16] started the study of the modal logic of forcing as the study of these general patterns. Since the relation "is a forcing extension of" can serve as an accessibility relation between models of ZFC, modal logic was the natural setting to do so.

Furthermore, Solovay [30] successfully studied provability in terms of modal logic – a somewhat similar endeavor. As Hamkins and Löwe put it:

> We aim to do for forceability what Solovay did for provability. [16, p. 1793]

An appropriate interpretation of modal logic in the context of forcing was already presented in Hamkins' 2003 paper [13] where he proposed a maximality principle stating that "anything forceable and not subsequently unforceable is true". The simple idea was to interpret the diamond $\Diamond$ as "there is a forcing extension in which" and the box $\Box$ as "in all forcing extensions". In this way, Hamkins' maximality principle can be expressed by the modal assertion $\Diamond\Box p \to p$.

Hamkins and Löwe then, in 2008, showed that the ZFC-provable modal principles of forcing amount exactly to the modal logic S4.2. All S4.2 formulas are valid under the forcing interpretation in every model of ZFC. As Hamkins had already proven the consistency of his maximality principle, it was furthermore clear that there are models in which more modal principles are valid: the formula $\Diamond\Box p \to p$ is not a theorem of S4.2, but it is valid under the forcing interpretation in models satisfying Hamkins's maximality principle. In particular, the maximality principle implies that all S5-formulas are valid under the forcing interpretation. Hamkins and Löwe provided those two logics, S4.2 and S5, as lower and upper bounds for the modal logic of forcing of models of ZFC.

A further step in understanding how forcing influences truth of set-theoretic sentences is to look in the downwards direction. We simply reverse the accessibility relation and look at models of which a given model is a forcing extension. Such inner models are called *grounds*. In 2012, Hamkins and Löwe [17] analyzed the modal logic of grounds with the result that it is S4.2 as well under the assumption that grounds are directed, i.e. that two grounds of a model always have a common ground. In 2016, Usuba [31] showed a strong version of this so called downwards directedness of grounds.

However, the situation is quite different to the upwards setting as there are models without proper ground. A prime example for such a model is the constructible universe L because it has no proper inner models and hence no proper grounds. If W is a model without proper grounds, the only model accessible via the relation "is a ground of" is W itself. Therefore, the modal logic of grounds of W is the modal logic of a Kripke frame consisting of one reflexive point. This modal logic exceeds S5.

Fuchs, Hamkins and Reitz called their study of grounds "set-theoretic geology" in their 2015 paper [9] using the geological metaphor providing the image of "brushing away the outermost layers of ... accumulated dust and sand to reveal the underlying ancient structure" [9, p. 496]. Minimal grounds, so called bedrocks, play a central role in this field. In his PhD thesis [26], Reitz constructed a model without a bedrock. The mentioned result by Usuba shows that any model with set-many grounds has a bedrock on the other hand. In addition, it implies that if a model has a bedrock then this bedrock is unique. So from the set-theoretic geology perspective, there are two fundamentally different types of models: models with a bedrock and models without a bedrock.

The two accessibility relations we use are "is a forcing extension of" and "is a ground of". So, starting with a model W of ZFC, the smallest family of models containing W and being closed under those accessibility relations is of great interest. Woodin [32] introduced this concept and called it the *generic multiverse of* W. These generic multiverses are the finest partition of the collection of all models of ZFC into parts which do not interact with each other through forcing.

This means that the modal logics of forcing or of grounds in different generic multiverses are not directly related to each other. Hence, one could imagine two generic multiverses which look very different from a modal logic point of view. This raises the main question we try to answer in this thesis: What can the modal logic of forcing or of grounds of a generic multiverse be?

For the downwards direction, we have to deal with the fundamental distinction of models from the set-theoretic geology perspective. If one model in a generic multiverse has a bedrock, then this bedrock is a bedrock of all models in the generic multiverse. So for the downwards direction, the distinction between models with a bedrock and models without a bedrock is mirrored by the distinction between generic multiverses with a bedrock and generic multiverses without a bedrock. For the upwards direction, there are no similar complications as we can conduct any forcing over any model as we wish.

To the best of our knowledge, the study of the modal logic of generic multiverses is new to the literature.

Outline of this Thesis

This thesis contains the following four main results:

Result 1: The modal logic of forcing of any generic multiverse is S4.2.

Result 2: The modal logic of grounds of any generic multiverse is contained in S4.2Top.

Result 3: The modal logic of grounds of any generic multiverse with a bedrock is exactly S4.2Top.

Result 4: The modal logic of grounds of any generic multiverse obtained by the only method known to us to obtain multiverses without a bedrock is contained in S5. In particular, it is not S4.2Top.

In the following, we present the structure of the thesis, explain the four main results, and make them precise.

In *chapter 2*, we expose two forcing notions in detail. The first one is forcing with partial functions. These forcings allow to add a function with well-controllable properties. The second forcing notion is a generalized version of Sacks's perfect set forcing. A remarkable property of these forcings is that they generate minimal extensions, i.e. extensions of a model W such that there are no proper intermediate models between W and the extension. These two forcing notions will be used in our forcing arguments throughout the thesis.

Chapter 3 discusses the global multiverse, the collection of all countable transitive models of ZFC. The two relations on this multiverse we want to study are the relations "is a forcing extension of" and its inverse "is a ground of". Hence, we begin the chapter by introducing the notion of grounds together with a definability result.

Then, we turn our attention to the structure of the multiverse with these two relations. Besides straightforward observations about reflexivity and transitivity the directedness of grounds is a main characteristic. In appendix B, we give an exposition of Usuba's unpublished proof of the strong downwards directedness of grounds.

At the end of this chapter, we focus on the connected components of the global multiverse with the two relations. We call these connected components *generic multiverses* and they are the main object of study in chapters 6 and 7. For our later analysis of generic multiverses, we introduce the concept of the mantle of a model, the intersection of all its grounds.

In *chapter 4*, we first give a short summary of results from basic modal logic, in particular characterization results for some modal logics in terms of classes of finite frames. While most modal logics under discussion are well-known, we also display results for the modal logic S4.2Top. In his Master's thesis [18], Inamdar showed that S4.2Top is the modal logic of inner models, and this modal logic will also play a central role in our inspection of the modal logic of grounds.

Following this, we present a general set-up for the investigation of model constructions or other notions relating models of some theory in terms of modal logic. The main tools for such an investigation are control statements and labelings.

In *chapter 5*, we transfer the set-up of the previous chapter to forcing extensions and grounds. We also show how the expressibility of truth in forcing extensions or grounds within the language of set theory allows for a different perspective, and how it leads to a connection of the existence of a certain family of control statements to the modal logic S4.2Top. This result, theorem 5.3, states that the existence of a certain family of control statements in a model is sufficient to establish S4.2Top as an upper bound for the corresponding modal logic.

Subsequently, we present the main results about the modal logic of forcing and the modal logic of grounds. Here, class forcing comes into play and we generalize a step in the determination of the modal logic of grounds which will become important in section 7.2.2.

To illustrate the differences between the upwards and the downwards direction, we put examples of models obtained by different forcing constructions in the modal logic of grounds context. Furthermore, we provide two examples which illustrate that the modal logic view talks about the behavior of set-theoretic truth rather than the structure of the multiverses: We present two models with the same modal logic of grounds but with a very different ground structure.

In *chapter 6,* we start the investigation of the modal logic of forcing of generic multiverses and settle the question about its possible outcomes. Main theorem 6.8 states:

Result 1: Each model W of ZFC has a ground W' with modal logic of forcing S4.2. Therefore, the modal logic of forcing of any generic multiverse is S4.2 (p. 45).

Section 6.1 generalizes a family of control statements provided by Friedman, Fuchino and Sakai [8]. In *section 6.2*, we extend our new family of control statements with a variation of another family of control statements used by Hamkins and Löwe [16] and show that the two families can be handled without affecting each other.

In *section 6.3*, we show for any model how to find a ground of this model with modal logic S4.2 using these control statements and the strong downwards directedness of grounds.

Chapter 7 gives partial answers to the question about the modal logic of grounds of generic multiverses. The first main result of this chapter is theorem 7.4:

Result 2: For any model W of ZFC there is a model U in the generic multiverse of W whose modal logic of grounds is contained in S4.2Top. Hence, the modal logic of grounds of the generic multiverse of W is contained in S4.2Top (p. 48).

In *section 7.1*, we again provide a new family of control statements and prove that in any generic multiverse there is a model in which these control statements satisfy the condition of our earlier mentioned result. This provides S4.2Top as an upper bound.

For generic multiverses with a bedrock, we see that all S4.2Top-formulas are valid under the ground interpretation. So, we can establish the following result, theorem 7.5:

Result 3: The modal logic of grounds in any generic multiverse with bedrock is exactly S4.2Top (p. 7.5).

This also shows that S4.2Top is the best upper bound for the modal logic of grounds of a generic multiverse we can formulate without imposing additional constraints on the generic multiverse.

In *section 7.2.1*, we discuss one possibility how the modal logic of grounds of a generic multiverse without a bedrock could be S4.2Top. Namely, we state observations about models with only elementarily equivalent grounds, but without a bedrock.

Finally, we show that all models without a bedrock we know have a modal logic of grounds contained in S5. Theorem 7.8 states:

Result 4: Let W be a countable transitive model of $\mathrm{ZFC} + \mathrm{GA}$ and $\mathbb{P} = \prod_{\alpha \in \mathrm{Ord}} P_\alpha$ be a *reasonable* progressively closed product. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over W, then the modal logic of grounds of $W[G]$ is contained in S5 (p. 50).

The ground axiom, GA, states that there are no non-trivial grounds. The technical conditions which we call reasonable here are made precise in definition 7.7. All models without a bedrock we know have been constructed by a progressively closed product meeting these conditions. In particular, that means that the modal logic of the generic multiverse of such a model is not S4.2Top. The proof of this result relies on lemma 5.8 where we generalized a step of Reitz's proof for the existence of a model without a bedrock in [26].

2 Forcing

We assume that the reader is familiar with the basics of forcing, including preservation theorems, the forcing equivalence of any partial order to some complete Boolean algebra, products and two-step iterations. Our notation and terminology follows [22]. In appendix A, an overview over the notation and the basic results we use can be found.

Whenever we say forcing without specification, we mean set forcing. Nevertheless, one special case of class forcing, progressively closed products, is important in this thesis as well. We follow the terminology of Reitz's PhD thesis [26] for these class forcings, and a short summary thereof can be found in section A.5 in appendix A.

In this chapter, we first settle our background theory before we introduce two forcing notions which will be used throughout the thesis.

2.1 Background Theory

We work in a meta-universe V which is a model of ZFC + "there is a transitive countable model of ZFC". This is the case, e.g. if there is an inaccessible cardinal κ in V. The existence of an inaccessible cardinal κ ensures that V_κ is a transitive set model of ZFC, and hence by the Skolem-Löwenheim theorem and the Mostowski collapse lemma, there are countable transitive models of ZFC.

Note that this meta-theory is stronger than it would have to be as we could carry out all the forcing arguments in a purely syntactic way. For more details on the different approaches to forcing, we refer the reader to [22, chapter VII, §9]. Nevertheless, the semantic approach where we can actually generate forcing extensions is able to express the ideas behind our endeavor much clearer. As we want to look at a collection of models as a Kripke structure with an accessibility relation between different models, our approach via countable transitive models of ZFC seems to be appropriate. The collection of all countable transitive models is a set in our meta-universe V and also the generic multiverses we define in section 3.3 are just sets and can serve as the underlying set of a Kripke model.

2.2 Two Forcing Notions

In this section, we present the two forcing notions which are the basis for our forcing arguments in the sequel.

2.2.1 Forcing with Partial Functions

We first give an overview over a class of forcing notions which use partial functions. All results of this section can be found in [22, chapter VII, §6]. Throughout this section, let W be a model of ZFC.

Definition 2.1. Given two sets I and J as well as an infinite cardinal λ we define the following partial order:

$$\mathrm{Fn}(I, J, \lambda) := \{p \subseteq I \times J : p \text{ is a function and } |p| < \lambda\},$$

ordered by $\supseteq$, i.e. for $p, q \in \mathrm{Fn}(I, J, \lambda)$ we say $p \leq q$ iff $p \supseteq q$.

If $|I| \geq \lambda$ forcing with this partial order leads to a non-trivial forcing extension as in this case every $p \in \mathrm{Fn}(I, J, \lambda)$ has two incompatible extensions because $|p| < |I|$. If G is an $\mathrm{Fn}(I, J, \lambda)$-generic filter over W then $\bigcup G$ is a function as any two elements in G are compatible. Vice versa, given a function g which is $\bigcup G$ for some $\mathrm{Fn}(I, J, \lambda)$-generic filter G over W, we can reconstruct G: $G = \{p \in \mathrm{Fn}(I, J, \lambda) : p \subseteq g\}$. In this situation $W[G] = W[g]$, the smallest model of ZFC extending W and containing g. Thus, we will also call such a function g an $\mathrm{Fn}(I, J, \lambda)$-generic function over W keeping in mind that we have this one-to-one correspondence between $\mathrm{Fn}(I, J, \lambda)$-generic functions and $\mathrm{Fn}(I, J, \lambda)$-generic filters. An $\mathrm{Fn}(I, J, \lambda)$-generic function f has the property that for each dense subset $D \subseteq \mathrm{Fn}(I, J, \lambda)$ there is a $p \in D$ with $p \subseteq f$.

Lemma 2.2. *Let I and J be sets and $\lambda \leq |J|$ an infinite cardinal (in W). Let g be a $\mathrm{Fn}(I \times J, 2, \lambda)$-generic function over W. Then there is an injection from I to $\mathcal{P}(J)$ in $W[g]$.*

Proof. For each $x \in I \times J$ the set

$$\{p \in \mathrm{Fn}(I \times J, 2, \lambda) : x \in \mathrm{dom}(p)\}$$

is dense. Hence, $\mathrm{dom}(g) = I \times J$.

This allows us to define the function $f : I \to \mathcal{P}(J)$ via

$$f(y) = \{z \in J : g(y, z) = 1\} \text{ for all } y \in I.$$

We show that f is injective: Let $y_1, y_2 \in I$ with $y_1 \neq y_2$. The following set is dense as for each $p \in \mathrm{Fn}(I \times J, 2, \lambda)$ we have that $|\mathrm{dom}(p)| < |J|$:

$$\begin{aligned}\{p \in \mathrm{Fn}(I \times J, 2, \lambda) :& \exists z \in J(\langle y_1, z\rangle \in \mathrm{dom}(p) \wedge \langle y_2, z\rangle \in \mathrm{dom}(p) \\ & \wedge p(\langle y_1, z\rangle \neq p(\langle y_2, z\rangle))\}.\end{aligned}$$

Hence, we get that $f(y_1) \neq f(y_2)$, and as y_1 and y_2 were arbitrary, we conclude that f is injective. □

Lemma 2.3. *If λ is regular, then $\mathrm{Fn}(I, J, \lambda)$ is λ-closed.*

Proof. Let $\gamma < \lambda$ be an ordinal and $\{f_\alpha : \alpha < \gamma\}$ be a decreasing sequence of elements of $\mathrm{Fn}(I, J, \lambda)$. Then clearly $\bigcup\{f_\alpha : \alpha < \gamma\}$ is a function as all elements of the sequence are pairwise compatible. Further, $\mathrm{dom}(\bigcup\{f_\alpha : \alpha < \gamma\}) \subseteq I$, $\mathrm{ran}(\bigcup\{f_\alpha : \alpha < \gamma\}) \subseteq J$ and $|\bigcup\{f_\alpha : \alpha < \gamma\}| < \lambda$ since for all $\alpha < \gamma$ we have $|f_\alpha| < \lambda$, $\gamma < \lambda$ and λ is regular. So, $\bigcup\{f_\alpha : \alpha < \gamma\}$ is in $\mathbb{P}$ and extends f_α for all $\alpha < \gamma$. □

The following lemma can be shown by a standard Δ-system argument. For a proof see [22, VII, Lemma 6.10].

Lemma 2.4. *The partial order* $\mathrm{Fn}(I, J, \lambda)$ *has the* $(|J|^{<\lambda})^+$*-c.c.*

The most prominent example of a forcing with partial functions is the partial order $\mathrm{Fn}(\omega, 2, \omega)$. This partial order is forcing equivalent to the Cohen forcing introduced by Cohen in 1963 to prove the independence of the continuum hypothesis in [6]. $\mathrm{Fn}(\omega, 2, \omega)$-generic functions are also called Cohen reals. Furthermore, the partial order $\mathrm{Fn}(\omega, 2, \omega)$ is countable and later on we will see that all countable forcing notions $\mathbb{P}$ satisfying the splitting condition

$$\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}(q \leq p \wedge r \leq p \wedge q \perp r)$$

are forcing equivalent.

2.2.2 Perfect Set Forcing

In 1971, Sacks [28] introduced perfect set forcing, now also known under the name Sacks forcing. A main importance of this forcing notion for our investigation later on is that it produces minimal extensions, i.e. extensions such that there are no proper intermediate models between the original model and the extension. In this section, we present the generalized version of this forcing for uncountable regular cardinals as it was introduced by Kanamori in [21].

Our presentation uses perfect trees while Sacks originally used perfect sets. Also Sacks original forcing partial order is now often presented in terms of perfect trees (see e.g. [10]). The reason is that trees are easier to deal with and that there is a one-to-one-correspondence between perfect subsets of 2^ω and perfect subtrees of $2^{<\omega}$.

Throughout this section, let κ be a regular uncountable cardinal with $2^{<\kappa} = \kappa$. Under the generalized continuum hypothesis, GCH, this condition is met by any regular uncountable cardinal.

Definition 2.5. Let ${}^{<\kappa}2 := \bigcup_{\alpha<\kappa} {}^{\alpha}2$. For $p \subseteq {}^{<\kappa}2$ and $s \in p$ we say that s splits in p iff $s^\frown 0 \in p$ and $s^\frown 1 \in p$.
We say that $p \subseteq {}^{<\kappa}2$ is a *perfect κ-tree* iff:

1. If $s \in p$, then $s{\restriction}\alpha \in p$ for all ordinals α.
2. If $\alpha < \kappa$ is a limit ordinal, $s \in {}^{\alpha}2$, and $s{\restriction}\beta \in p$ for all $\beta < \alpha$ then $s \in p$.
3. If $s \in p$, then there is a $t \in p$ with $t \supseteq s$ such that t splits in p.
4. If $\alpha < \kappa$ is a limit ordinal, $s \in {}^{\alpha}2$, and $s{\restriction}\beta$ splits in p for arbitrarily large $\beta < \alpha$, then s splits in p.

We also refer to condition 2 by saying that p is closed, and to condition 4 by saying that the splitting nodes of p are closed. For a perfect κ-tree p we define

$$p_s := \{t \in p : s \subseteq t \text{ or } t \subseteq s\}.$$

Finally, the partial order $\mathbb{S}_\kappa$ is defined as the set of all perfect κ-trees, ordered by inclusion.

Note that p_s is a perfect κ-tree for any perfect κ-tree p and $s \in p$.

Lemma 2.6. *The partial order $\mathbb{S}_\kappa$ is κ-closed.*

Proof. Let $\gamma < \kappa$ and let $\langle p_\alpha : \alpha < \gamma \rangle$ be a decreasing sequence in $\mathbb{S}_\kappa$, i.e. for any $\alpha < \beta < \gamma$ we have $p_\beta \subseteq p_\alpha$. Let $p := \bigcap_{\alpha<\gamma} p_\alpha$. We claim that p is a perfect κ-tree. Conditions 1,2 and 4 clearly hold for p.

To see that condition 3 also holds let $s \in p$. By the closedness of the p_α it follows that we can find a branch $f \in {}^\kappa 2$ through p with $s \subseteq f$: We recursively define sequences $f_\beta \in p$ of length lenght$(s) + \beta$ for $\beta < \kappa$, starting with $f_0 = s$. Once we have defined $f_\beta \in p$ for some $\beta < \kappa$ we know that all p_α, $\alpha < \gamma$, contain $f_\beta^\frown 0$ or $f_\beta^\frown 1$ and as the p_α form a descending chain one of the two has to be in p as well and we can choose one of the two as $f_{\beta+1}$. For a limit ordinal $\delta < \kappa$ we simply take the union $f_\delta := \bigcup_{\beta<\delta} f_\beta$ which is in p as p is closed and $\delta < \kappa$. Then we put $f = \bigcup_{\beta<\kappa} f_\beta$.

Now we recursively define an increasing sequence $\langle \eta_\beta : \beta \leq \gamma \rangle$ of ordinals in κ: Let η_0 be such that $s \subseteq f{\restriction}\eta_0$. If η_β is defined we know that $f{\restriction}\eta_\beta + 1$ has an extension in p_β which splits and the least such extension is a subset of f. So, there is a $\eta_{\beta+1}$ such that $f{\restriction}\eta_{\beta+1}$ splits in p_β. For limit ordinals $\delta < \gamma$, let $\eta_\delta = \bigcup_{\beta<\delta} \eta_\beta$. As κ is regular $\eta_\delta < \kappa$.

Now, we claim that $f{\restriction}\eta_\gamma$ splits in p. For any $\alpha < \gamma$ we know that $f{\restriction}\eta_\beta$ splits in p_α for all successor ordinals $\gamma > \beta > \alpha + 1$. As $\eta_\gamma = \bigcup_{\gamma>\beta>\alpha+1} \eta_{\beta+1}$ this means that by $f{\restriction}\eta_\gamma$ splits in p_α condition 4. Hence, $f{\restriction}\eta_\gamma$ is a splitting node of p extending s. □

Clearly, the size of ${}^{<\kappa}2$ is $2^{<\kappa} = \kappa$. Hence the size of $\mathbb{S}_\kappa$ is at most 2^κ and therefore $\mathbb{S}_\kappa$ satisfies the $(2^\kappa)^+$-chain condition. So, under GCH we get that $\mathbb{S}_\kappa$ satisfies the κ^{++}-chain condition.

An important notion often used in the context of tree-like forcings is the notion of a fusion sequence. While it is clear that $\mathbb{S}_\kappa$ is not κ^+-closed, the intersection of a decreasing sequence of length κ in $\mathbb{S}_\kappa$ is again in $\mathbb{S}_\kappa$ if we add an additional requirement that ensures that consecutive members of the sequence are not too different. This is made precise by the following definition and lemma.

Definition 2.7. For an ordinal $\alpha < \kappa$ and two perfect κ-trees p and q we say $p \leq_\alpha q$ iff $p \leq q$ and $p \cap {}^{\alpha+1}2 = q \cap {}^{\alpha+1}2$. We call a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ in $\mathbb{S}_\kappa$ a *fusion sequence* if $p_{\alpha+1} \leq_\alpha p_\alpha$ for all $\alpha < \kappa$ and $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$ for all limit ordinals $\delta < \kappa$..

Lemma 2.8 (Fusion Lemma). *Let $\langle p_\alpha : \alpha < \kappa \rangle$ be a fusion sequence in $\mathbb{S}_\kappa$. Then,*

$$p := \bigcap_{\alpha<\kappa} p_\alpha \in \mathbb{S}_\kappa.$$

Proof. Again, we only have to check condition 3 to prove that p is a perfect κ-tree. Given $s \in p$ we can take a branch $f \in {}^{\kappa}2$ of p which extends s as in lemma 2.6. Analogously to lemma 2.6, we will also define an increasing sequence of ordinals, this time of length ω. So, we start with η_0 s.t. $s \subseteq f{\restriction}\eta_0$ and given η_n we choose η_{n+1} such that $f{\restriction}\eta_{n+1}$ splits in p_{η_n}. Now, we set $\eta = \bigcup_{n<\omega} \eta_n$ and get that $f{\restriction}\eta$ splits in p_η. For $\gamma > \eta$ we have that $p_\gamma \cap {}^{\eta+1}2 = p_\eta \cap {}^{\eta+1}2$ and hence $f{\restriction}\eta$ splits in p_γ. Therefore $f{\restriction}\eta$ extends s and splits in p. $\square$

The rest of this section is devoted to showing that $\mathbb{S}_\kappa$ produces minimal extensions. The proof is a generalization of the proof for the minimality of forcing with perfect sets of reals, or equivalently with the set of perfect ω-trees, as it was given by Sacks in [28]. This proof is also presented in [10] by Geschke and Quickert and we will follow their notation and generalize the proof of lemma 28 in that paper to the uncountable case.

Whenever G is a $\mathbb{S}_\kappa$-generic filter over a model W, there is a unique $f \in {}^{\kappa}2$ s.t. $f{\restriction}\beta \in p$ for all $\beta < \kappa$ and $p \in G$. In other words, $f = \bigcup \bigcap G$. We call this f the Sacks subset added by G. The filter G can be reconstructed from the corresponding Sacks subset f as it contains exactly those perfect κ-trees p for which $f{\restriction}\beta \in p$ for all $\beta < \kappa$.

Given a perfect κ-tree p we let p^α denote the nodes of p after the αth splitting, i.e. $s \in p^\alpha$ iff s is minimal w.r.t. $\subseteq$ among the elements $t \in p$ for which the set

$$\{r : r \text{ is a proper initial segment of } t \text{ and splits in } p\}$$

has order type α. We now get a natural bijection between ${}^{\alpha}2$ and p^α as both are the set of paths through a tree with α-levels of binary splitting. This bijection gives us an element $t_\sigma \in p^\alpha$ for each $\sigma \in {}^{\alpha}2$ and we define the subtree $p * \sigma$ as $p_{t_\sigma} = \{s \in p : t_\sigma \subseteq s \text{ or } s \subseteq t_\sigma\}$.

Theorem 2.9. *Let W be a transitive model of* ZFC *and let G be a $\mathbb{S}_\kappa$-generic filter over W. Then $W[G]$ is a minimal extension of W, i.e. for any model U of* ZFC *with $W \subseteq U \subseteq W[G]$ we have that $W = U$ or $U = W[G]$.*

Proof. Theorem A.22 tells us that U is a forcing extension of W in this situation. So, there is a partial order $\mathbb{Q}$ and a $\mathbb{Q}$-generic filter H over W such that $U = W[H]$. We now code the filter H by a set of ordinals A. This can be done by letting (θ, E) be isomorphic to $(\mathrm{trcl}(\{H\}), \in)$ for some ordinal θ and $E \subseteq \theta \times \theta$. Then E can be coded as a set of ordinals A using a usual coding function $\mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$. The Mostowski collapse lemma then allows us to reconstruct $\mathrm{trcl}(\{H\})$ and hence H from A. As a result, we get that $U = W[A]$, the smallest model of ZFC extending W and containing A.

So, let $\dot{A} \in W^{\mathbb{S}_\kappa}$ be a name for a set of ordinals. So, $1_{\mathbb{S}_\kappa} \Vdash$ "$\dot{A}$ is a set of ordinals". Hence there is an ordinal α such that $1_{\mathbb{S}_\kappa}$ forces $\dot{A}$ to be a subset of α. So, there is a name $\dot{z}$ for the characteristic function of $\dot{A}$ from α to 2. We say that a condition $p \in \mathbb{S}_\kappa$ decides all of $\dot{z}$ if there is a function $y : \alpha \to 2$ in W such that $p \Vdash \dot{z} = \check{y}$.

We want to show that $\mathrm{val}(\dot{z}, G) \in W$ or that the Sacks subset f added by G is definable from $\mathrm{val}(\dot{z}, G)$. Clearly, the set

$$\{p \in \mathbb{S}_\kappa : p \text{ decides all of } \dot{z}\} \cup \{p \in \mathbb{S}_\kappa : \text{ no } q \leq p \text{ decides all of } \dot{z}\}$$

is dense. If G contains a condition from $\{p \in \mathbb{S}_\kappa : p \text{ decides all of } \dot{z}\}$ then $\mathrm{val}(\dot{z}, G)$ is just a function in W and we are done. So, let us assume that G contains a condition r such that no condition below r decides all of $\dot{z}$. Our goal now is to show that below any $p \leq r$ there is a $q \leq p$ such that $q \in G$ implies that f is definable from $\mathrm{val}(\dot{z}, G)$.

So, take a such a p and define z_q to be the longest initial segment of $\dot{z}$ that is decided by q for $q \leq p$. So, z_q is maximal among the sequences y for which $q \Vdash \check{y} \subseteq \dot{z}$. This maximal sequence exists as for any q there is a least β such that q does not decide the value of $\dot{z}$ at position β.

We now recursively define a fusion sequence below p: Let $p_0 = p$. Assume we have defined p_α and let $\sigma \in^\alpha 2$. Then $p_\alpha * (\sigma^\frown 0)$ and $p_\alpha * (\sigma^\frown 1)$ are conditions below p and hence do not decide all of $\dot{z}$. So we can find stronger conditions $q_{\sigma^\frown o} \leq p_\alpha * (\sigma^\frown 0)$ and $q_{\sigma^\frown 1} \leq p_\alpha * (\sigma^\frown 1)$ such that $z_{q_{\sigma^\frown o}}$ and $z_{q_{\sigma^\frown 1}}$ are incompatible with respect to $\subseteq$. Now, we just let $p_{\alpha+1} = \bigcup_{\tau \in^{\alpha+1} 2} q_\tau$. This ensures that $p_{\alpha+1} \leq_\alpha p_\alpha$ as $p_{\alpha+1}$ and p_α even coincide up to the αth splitting level, so in particular $p_{\alpha+1} \cap {}^{\alpha+1}2 = p_\alpha \cap {}^{\alpha+1}2$. For limit ordinals $\delta < \kappa$ we let $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$. Now, we can define $q := \bigcap_{\alpha<\kappa} p_\alpha$ to get a condition $q \leq p$ by the fusion lemma.

Suppose now that $q \in G$. We want to show that then f is definable from $\mathrm{val}(\dot{z}, G)$ to conclude that $f \in W[\mathrm{val}(\dot{A}, G)]$. We know that f is a path through q. Furthermore, it is the unique path g s.t. for all $\sigma \in {}^{<\kappa}2$ we have that if g is a path through $q * \sigma$ then $z_{q*\sigma} \subseteq \mathrm{val}(\dot{z}, G)$. Therefore, f and thus G are definable from $\mathrm{val}(\dot{z}, G)$ and so $W[\mathrm{val}(\dot{A}, G)] = W[G]$. $\square$

3 The Multiverse

In this chapter, we take a structural perspective on the ways the forcing construction links models of ZFC. Let $\mathcal{W}$ be the collection of all countable transitive models of ZFC which we will call the *global multiverse*. In our meta-universe, $\mathcal{W}$ is a non-empty set. On this set, we can define the binary relation $\mathcal{F}$ expressing "is a forcing extension of", i.e. for countable transitive models W, U of ZFC we say $\mathcal{F}WU$ if U is a forcing extension of W. This structure captures how the countable transitive models are related via forcing, and hence the patterns of truth of sentences in the global multiverse is the basis for our analysis of the principles according to which forcing allows us to change set-theoretic truth.

We begin this chapter by introducing the concept of grounds corresponding to the inverse relation $\mathcal{F}^{-1}$ which we denote as $\mathcal{G}$. We will then discuss the structure of the global multiverse in view of the two relations $\mathcal{F}$ and $\mathcal{G}$ and draw special attention to the connected components of this structure because they do not interact with each other via forcing.

3.1 Grounds

The forcing construction not only lets us extend models of set theory by adding new objects, but it also relates a model W to some of its inner models as W can itself be a forcing extension of an inner model. We call such an inner model a ground:

Definition 3.1 (Ground)**.** Let $U \subseteq W$ be countable transitive models of ZFC. We say U is a *ground* of W if there is a partial order $\mathbb{P} \in U$ and a $\mathbb{P}$-generic filter G over U such that $W = U[G]$, i.e. if W is a forcing extension of U.

So, the relation "is a ground of", $\mathcal{G}$, is just the inverse of the relation $\mathcal{F}$. We know that $\mathcal{F}$ is reflexive as any model is a forcing extension of itself by trivial forcing, and transitive because of the two-step-iteration of forcing. So, a first obvious observation is that the same is true for the relation $\mathcal{G}$.

The first important result about grounds is a result by Laver (see [24]) and independently by Woodin which states that any ground W' of a model W is first-order definable with parameters in W'. Fuchs, Hamkins and Reitz showed the following uniform formulation in [9]:

Theorem 3.2 (Laver-Woodin Theorem)**.** *There is a formula $\phi(x, r)$ without parameters such that for any countable transitive model W of* ZFC

1. *for any ground W' of W, there is an $r \in W$ such that $W' = \{x : \phi(x, r)^W\}$, and*
2. *for any $r \in W$, the class $\{x : \phi(x, r)^W\}$ is a ground of W.*
3. *for any $r \in W$, we have $r \in \{x : \phi(x, r)^W\}$.*

We do not give a detailed proof here. To give an idea of how this formula looks like, nevertheless, we need the following concept, introduced by Hamkins in [12].

Definition 3.3. Let $M \subseteq W$ be countable transitive models of (a suitable fragment of) ZFC with the same ordinals. Let κ be an infinite cardinal.

Then, we say that M satisfies the *κ-covering property* for W if every set of ordinals A in W with $(|A| < \kappa)^W$ is covered by a set of ordinals B in M with $(|B| < \kappa)^W$, i.e. there is a set of ordinals $B \in M$ of size less than κ with $A \subseteq B$.

We say that M satisfies the *κ-approximation property* for W if for any set of ordinals $A \in W$ for which $A \cap x \in M$ for every set of ordinals $x \in M$ with $(|x| < \kappa)^W$ we have that $A \in M$.

The following simplified version of a result by Hamkins [12, lemma 13] connects these notions to forcing:

Lemma 3.4. *Let U and W be countable transitive models of* ZFC *and suppose that $W = U[G]$ for a $\mathbb{P}$-generic filter G over U for some partial order $\mathbb{P} \in U$. Let $\delta := |\mathbb{P}|^W$. Then U satisfies the δ^+-covering and the δ^+-approximation property for W.*

Now, the following result by Laver [24] is the key to the definability result.

Theorem 3.5. *Suppose U, U' and W are countable transitive models of a suitable fragment of* ZFC*, δ a regular cardinal in W, the extensions $U \subseteq W$ and $U' \subseteq W$ have the δ-cover and δ-approximation properties, $\mathcal{P}(\delta)^U = \mathcal{P}(\delta)^{U'}$, and $(\delta^+)^U = (\delta^+)^{U'} = (\delta^+)^W$. Then, $U = U'$.*

Given a ground U of W witnessed by a forcing of size δ, then inside W we can define $U_\gamma := (V_\gamma)^U$ for any $\beth$-fixed point $\gamma > \delta$ as the unique model M of the axioms of ZFC except for replacement (but including separation) of height γ satisfying the δ^+-approximation and δ^+-cover property such that $M_{\delta^+ +1} = U_{\delta^+ +1}$. This only requires the parameter $U_{\delta^+ +1}$ (see [24, theorem 3]). Of course, being able to define U up to an arbitrarily high rank suffices to define U in W.

The importance of the definability of grounds for our purposes is that it shows that a model W of ZFC can internally refer to its grounds, and hence, truth in the grounds of the model can be expressed within the model W itself. This mirrors the expressibility of truth in forcing extensions by means of the forcing relation.

3.2 The Structure of the Multiverse

We know that we can build non-trivial forcing extensions over any countable transitive model of ZFC. For grounds on the other hand, there are models without a non-trivial ground: Given a model W, we know that L^W has no proper inner models and so in particular no non-trivial grounds. In this sense, L^W is quite different from many other models and to draw a clear distinction we state the ground axiom introduced by Hamkins and Reitz (see [26]).

Definition 3.6. The *ground axiom* (GA) is the assertion that there is no proper ground.

So, models satisfying the ground axiom are minimal nodes in the structure $\langle \mathcal{W}, \mathcal{F} \rangle$. A natural question to ask about a model W in $\mathcal{W}$ is whether it has a ground satisfying GA. We call such a ground a bedrock. Since "being a ground of" is transitive, we can equivalently define:

Definition 3.7. If U is a minimal ground of W, i.e. if there is no ground U' of W with $U' \subsetneq U$, we call U a *bedrock* of W.

Later, in chapter 5, we will present a construction by Reitz [26] resulting in a model without a bedrock. So, the relation $\mathcal{G}$ on the global multiverse $\mathcal{W}$ is not atomic, i.e. there are models $W \in \mathcal{W}$ which have no $\mathcal{G}$-maximal $\mathcal{G}$-successor – in other words, no minimal ground below them.

Furthermore, the uniform formula defining grounds allows us to talk about the size of the collection of all grounds of a model W of ZFC from the perspective of W:

Definition 3.8. Given a model W of ZFC and a collection $\mathcal{U}$ of grounds of W we say that $\mathcal{U}$ is *set-like with respect to* W if there is a set $X \in W$ such that $\mathcal{U} = \{W_r : r \in X\}$ where W_r is the ground $\{x : \phi(x,r)^W\}$ of W with ϕ given by theorem 3.2. We say that W *has set-many grounds* if the collection of all grounds of W is set-like with respect to W. Otherwise, we say that W *has proper class-many grounds*.

Reitz [26, p. 57], as well as Hamkins and Löwe [17, question 4], raised the question whether two grounds of a model always have a common ground. Furthermore, Fuchs, Hamkins and Reitz [9] discussed various strengthenings of this weak directedness. In 2016, Usuba [31] then showed the following strong version of the so called downwards directedness of grounds providing an important structural property of $\langle \mathcal{W}, \mathcal{G}\rangle$:

Theorem 3.9 (Strong Downwards Directedness of Grounds). *Let W be a model of* ZFC. *If $\mathcal{U}$ is a collection of grounds of W which is set-like with respect to W then there is a common ground of all $U \in \mathcal{U}$. That means if $R \in W$ is a set then there is a ground W' of W which is a ground of $W_r := \{x \in W : \phi(x,r)^W\}$ for all $r \in R$.*

We give an exposition of Usuba's proof in appendix B. This theorem settles many open questions about the ground structure below models of ZFC. In particular, the following two results are immediate consequences.

Corollary 3.10. *If a model W of* ZFC *has a bedrock U then U is the least ground of W, i.e. for any ground U' of W we have $U \subseteq U' \subseteq W$.*

Corollary 3.11. *If a model W of* ZFC *has only set many grounds it has a bedrock.*

For the upwards direction, we do not get directedness. In the early 1990s, Woodin observed that it is possible that two forcing extensions of a model W do not have a common extension.

Proposition 3.12 (Woodin). *Let W be a countable transitive model of* ZFC. *Then there are two Cohen reals r and s over W such that there is no transitive model of* ZFC *with the same ordinals as W containing r and s. In particular, $W[r]$ and $W[s]$ do not have a common forcing extension.*

Proof. The idea behind the proof is to code a well-order of order type $> \mathrm{Ord}(W)$, the order type of the ordinals of W, into the pair $\langle r, s\rangle$. Then, the forcing extensions $W[r]$ and $W[s]$

cannot have a common extension W' as this extension would have the same ordinals as W but also access to a well-order of greater order type. We start by taking some countable ordinal α greater than $\mathrm{Ord}(W)$. Then we find a relation $A \subseteq \omega \times \omega$ which is a well-order of order type α. By taking a constructible bijection between ω and $\omega \times \omega$ we can encode A as a real c.

Let $\langle D_n : n \in \omega \rangle$ be an enumeration of the dense subsets of the Cohen forcing

$$\mathbb{P} := \{f : f \text{ is a function from some natural number } n \to \omega\}$$

which are in W. Note that $\mathbb{P}$ is forcing equivalent to $\mathrm{Fn}(\omega, 2, \omega)$ as we will prove later in theorem 5.12. We can construct the two Cohen reals r and s while encoding c in the pair $\langle r, s \rangle$: We start by letting r_0 be an element of D_0 with a natural number k_0 as domain. Then we let s_0' be the function which has domain $k_0 + 2$ and which maps each element of k_0 to 0, k_0 itself to 1 and $k_0 + 1$ to $c(0)$. Then we extend s_0' to an element of D_0 with a natural number ℓ_0 as domain. Likewise we extend r_0 to a function r_1' with domain $\ell_0 + 2$ by letting r_1' map any element of ℓ_0 which is not in the domain of r_0 to 0, then ℓ_0 itself to 1 and $\ell_0 + 1$ to $c(1)$. Then we extend r_1' to an element r_1 of D_1 of length k_1. Recursively, we proceed by adding 0s to r_n and s_n, respectively, up to the length of s_{n-1} and r_n, respectively, followed by a 1 and the next digit of c to obtain r_{n+1}' or s_{n+1}'. Then we extend the sequence to an element r_{n+1} or s_{n+1} of D_{n+1}.

Finally let $r = \bigcup_{n \in \omega} r_n$ and $s = \bigcup_{n \in \omega} s_n$. Then r and s correspond to the filter of all finite initial segments of these reals. By construction those two filters are generic and hence $W[r]$ and $W[s]$ are Cohen extensions of W. However, any model U of ZFC with the same ordinals as W containing r and s would also contain the well-order A as the blocks of 0s allow us to reconstruct the lengths of r_n and s_n, respectively, for all $n \in \omega$ and hence also the real c. But then the order type of A is not an ordinal in U, a contradiction. $\square$

Hamkins [14] generalizes this non-amalgamation phenomenon to other forcing notions and also shows that some forcing notions always admit amalgamation.

Although the generic multiverse is not directed in the upwards direction, we will see that this does not have an impact on the change of the truth values of sentences via forcing. As the Cohen forcing is weakly homogeneous, it is in particular clear that this proposition does not influence the truth patterns in the generic multiverse as the extensions $W[r]$ and $W[s]$ are elementarily equivalent. So, also the truth patterns in their forcing extensions coincide as the models can internally refer to the theory of their extensions.

3.3 The Generic Multiverse

If we consider the global multiverse $\mathcal{W}$ together with the relation $\mathcal{F} \cup \mathcal{G}$, we obtain an undirected graph. For any model $W \in \mathcal{W}$, the connected component of this graph containing W is of great interest to us as this connected component contains exactly those models which can be obtained from W by iteratively going to a ground or a forcing extension. Regarding our overall goal of studying the principles of how forcing can change set-theoretic truth, these connected components hence do not directly influence each other. These connected components are the generic multiverses introduced by Woodin [32]:

Definition 3.13 (Generic Multiverse)**.** Let W be a countable transitive model of ZFC. The *generic multiverse* of W, Mult_W, is the smallest collection of countable transitive models of ZFC containing W which is closed under taking forcing extensions and ground models.

Proposition 3.14. *Let W be a countable transitive model of* ZFC*. Then,* Mult_W *is the collection of all forcing extensions of all grounds of W.*

Proof. As usual for closure properties, the generic multiverse of a model W can be obtained by the following iterative construction:

1. Put $\mathrm{Mult}_W^0 := \{W\}$.
2. Let $\mathrm{Mult}_W^{2k+1} = \{U \in \mathcal{W} : \text{ there is a } U' \text{ in } \mathrm{Mult}_W^{2k} \text{ such that } U \text{ is a ground of } U'\}$ for any $k \in \mathbb{N}$.
3. Let $\mathrm{Mult}_W^{2k+2} = \{U \in \mathcal{W} : \text{ there is a } U' \text{ in } \mathrm{Mult}_W^{2k+1} \text{ s.t. } U \text{ is a forcing extension of } U'\}$ for any $k \in \mathbb{N}$.

Then, $\mathrm{Mult}_W = \bigcup_{k\in\mathbb{N}} \mathrm{Mult}_W^k$. We claim that $\mathrm{Mult}_W = \mathrm{Mult}_W^2$. The set Mult_W^2 consists of all forcing extensions of all grounds of W. Now, let $U \in \mathrm{Mult}_W^3$. So, there is a ground U_1 of W and a forcing extension U_2 of U_1 such that U is a ground of U_2. Now, U_1 and U are grounds of U_2 and hence they have a common ground U_3 by the downwards directedness of grounds. But as the ground relation is transitive, U_3 is also a ground of W and so $U \in \mathrm{Mult}_W^3$.

Now, it is easy to see that also $\mathrm{Mult}_W^4 = \mathrm{Mult}_W^2$. We now know that Mult_W^4 consists of all forcing extensions of forcing extensions of grounds of W. As the forcing extension relation is also transitive, we conclude that $\mathrm{Mult}_W^4 = \mathrm{Mult}_W^3 = \mathrm{Mult}_W^2$. But then it is clear, that the iterative construction stabilized and hence $\mathrm{Mult}_W = \mathrm{Mult}_W^2$. □

The core of the argument in the proof of proposition 3.14 goes back to Woodin's study of his concept of *multiverse truth* [32] and its syntactic version was established before Usuba's proof by Löwe and Hamkins (unpublished).

In particular, this result also shows that any two models in a generic multiverse have a common ground. Furthermore, if a model W has a bedrock W', then W' is a bedrock of all models $U \in \mathrm{Mult}_W$. So, we can draw a clear distinction between two different types of generic multiverses: generic multiverses with a bedrock and generic multiverses without a bedrock. This distinction will play a crucial role for our main results in chapter 7.

3.4 The Mantle

An important concept for the study of generic multiverses is the mantle, introduced by Fuchs, Hamkins and Reitz [9].

Definition 3.15 (Mantle)**.** The *mantle* of a model of ZFC is the intersection of all its grounds.

Lemma 3.16 (Definability of the Mantle)**.** *The mantle of any model is a parameter-free uniformly first-order-definable transitive class containing all ordinals in this model.*

Proof. Let $\phi(x, r)$ be the formula uniformly defining grounds, given by theorem 3.2. Then $\forall r \phi(x, r)$ defines the mantle. $\square$

We fix this parameter-free first order definition of the mantle and refer to the class $\{x : \forall r \phi(x, r)\}$ as $\mathbb{M}$. If W has a bedrock, then it is clear that this least ground is the intersection of all grounds of W. So then, the bedrock is $\mathbb{M}^W$.

Conversely, if the mantle of a model W is also a ground of that model, it is a bedrock.

It was already proved by Fuchs, Hamkins and Reitz that the mantle is always a model of ZFC under the assumption that grounds are strongly downwards directed (see [9, theorem 22]). Usuba's result about the strong downwards directedness of grounds hence leads to the following result (see also [31, corollary 5.5])

Theorem 3.17 (Forcing Invariance, ZFC in the Mantle)**.** *The mantle of any model of* ZFC *is a model of* ZFC*. Furthermore, the mantle is a forcing-invariant class, i.e. if U and W are models of* ZFC *such that U is a forcing extension of W then $\mathbb{M}^U = \mathbb{M}^W$.*

Of course, that also means that the mantle stays constant whenever we go to a ground of a model. So, for any model W of ZFC and any U in the generic multiverse of W we have that $\mathbb{M}^W = \mathbb{M}^U$ because W and U have a common ground. Hence, the mantle offers the powerful possibility of referring to a class which is constant throughout a generic multiverse, but still much closer related to the models in the multiverse than the constructible universe L for example.

The following remarkable result by Fuchs, Hamkins and Reitz [9, main theorem 4, theorem 66] shows that the only thing we know about the mantle in general is that it is a model of ZFC.

Theorem 3.18. *Any model of* ZFC *is the mantle of another model of* ZFC*.*

Note that whenever U is the mantle of W and U is not a model of the ground axiom then W does not have a bedrock.

The proofs of the last two theorems are outside the scope of this thesis. Nevertheless, we want to remark that the proof of theorem 3.18 uses a class forcing construction with a progressively closed product similar to Reitz's construction mentioned earlier. We will come back to this kind of construction of models without a bedrock in chapter 5 and chapter 7.

4 Modal Logic

4.1 Preliminaries

4.1.1 Basic Modal Logic

In this section, we introduce a few basic concepts of modal logic. Again, some familiarity with the topic is presupposed. Unless stated otherwise, all results and definitions in this thesis concerning modal logic can be found in [3].

We fix a countable set $\mathcal{A}$ of proposition letters and we define:

Definition 4.1 (Modal Formula). A *modal formula* is a formula built from proposition letters in $\mathcal{A}$ using the binary connective $\wedge$ and the unary connectives $\Box$ and $\neg$. Further, we define the following abbreviations: $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\phi \to \psi := \neg\phi \vee \psi$ and $\Diamond\phi := \neg\Box\neg\phi$. We call the set of all modal formulas $\mathcal{L}_\Box$.

Definition 4.2 (Kripke frame and model). A *Kripke frame* is a pair $\langle W, R\rangle$ where W is a set and R is a binary relation on W. We call the elements of W worlds and we call R the *accessibility relation*. If for $v, w \in W$ we have Rvw we also say that v sees w.

A *Kripke model* is a Kripke frame $F = \langle W, R\rangle$ together with a *valuation* $V : \mathcal{A} \to \mathcal{P}(W)$. For a proposition letter p we call $V(p)$ the *extension* of p.

A *general Kripke frame* is a triple $\langle W, R, U\rangle$ where W and R are as above and $U \subseteq \mathcal{P}(W)$ is a set of subsets of W closed under union, intersection, complementation and such that if $A \in U$ then also $\{w \in W : \forall v \in W(Rwv \to v \in A)\} \in U$, i.e. if $A \in U$ then also the set of all worlds which only see worlds in A is in U.

A *generalized general Kripke frame* is a triple $\langle W, R, U\rangle$ where W and R are as above and $U \subseteq \mathcal{P}(W)$ is a set of subsets of W.

A Kripke model based on a (generalized) general Kripke frame $\langle W, R, U\rangle$ is a Kripke model $\langle W, R, V\rangle$ on the frame $\langle W, R\rangle$ with a valuation $V : \mathcal{A} \to U$. That means that U specifies the allowed extensions of proposition letters.

So, a Kripke frame $\langle W, R\rangle$ can be regarded as the general Kripke frame $\langle W, R, \mathcal{P}(W)\rangle$.

In this thesis, the restrictions on a general Kripke frame in contrast to a generalized general Kripke frame are not necessary. We will only work with generalized general Kripke frames and to improve the readability we will from now on simply refer to them as general Kripke frames or general frames.

Definition 4.3 (Kripke Semantics). Let $\mathcal{M} = \langle W, R, V\rangle$ be a Kripke model, $w \in W$ and $\phi \in \mathcal{L}_\Box$. The relation $\mathcal{M}, w \Vdash \phi$ expresses that ϕ is true in $\mathcal{M}$ at the world w and is recursively defined:

- For $p \in \mathcal{A}$: $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$,
- $\mathcal{M}, w \Vdash \phi \wedge \psi$ iff $\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$,

- $\mathcal{M}, w \Vdash \neg\phi$ iff $\mathcal{M}, w \nVdash \phi$, and
- $\mathcal{M}, w \Vdash \Box\phi$ iff for all $v \in W$ with Rwv we have $\mathcal{M}, v \Vdash \phi$.

Definition 4.4. Let $F = \langle W, R \rangle$ be a Kripke frame and $w \in W$. A modal formula is said to be *valid on F in w*, $F, w \vDash \phi$, if for all valuations $V : \mathcal{A} \to \mathcal{P}(W)$ we have $\langle F, V \rangle, w \Vdash \phi$. The formula ϕ is said to be *valid on F*, $F \vDash \phi$, if for all $v \in W$ we have $F, v \vDash \phi$.

There are many well-studied modal axioms whose validity corresponds to certain frame conditions. Each of the modal formulas in the table is valid on a frame $\langle W, R \rangle$ if and only if $\langle W, R \rangle$ satisfies the frame condition listed below (see [3, chapter 3]):

	Formula	Frame Condition	First Order Condition on $\langle W, R \rangle$
(K)	$\Box(p \to q) \to (\Box p \to \Box q)$	all Kripke frames	-
(T)	$\Box p \to p$	reflexive frames	$\forall x(Rxx)$
(4)	$\Box p \to \Box\Box p$	transitive frames	$\forall x, y, z(Rxy \wedge Ryz \to Rxz)$
(.2)	$\Diamond\Box p \to \Box\Diamond p$	directed frames	$\forall w, x, y \exists z(Rwx \wedge Rwy \to Rxz \wedge Ryz)$
(5)	$\Diamond p \to \Box\Diamond p$	Euclidean frames	$\forall w, v, u(Rwv \wedge Rwu \to Rvu)$

Definition 4.5. A *normal modal logic* is a set of formulas which contains all classical tautologies and (K) and which is closed under modus ponens and necessitation, i.e. if ϕ is in the set then so is $\Box\phi$, as well as uniform substitution. Whenever we refer to a set of modal formulas as a modal logic we mean the smallest modal logic containing this set of formulas.

Important normal modal logics are for example:

$$\begin{aligned} \text{K} &:= \text{(K)} \\ \text{T} &:= \text{(K)} + \text{(T)} \\ \text{S4} &:= \text{(K)} + \text{(T)} + (4) \\ \text{S4.2} &:= \text{(K)} + \text{(T)} + (4) + (.2) \\ \text{S5} &:= \text{(K)} + \text{(T)} + (4) + (5) \end{aligned}$$

Alternatively, one can add $\Diamond\Box p \to p$ which is valid on symmetric frames to S4 to obtain S5 as the two formulas are equivalent over S4.

Definition 4.6. Let $F = \langle W, R \rangle$ and $F' = \langle W', R' \rangle$ be Kripke frames. F' is a *bounded morphic image* of F if there is a surjection $b : W \to W'$ such that for all $w, v \in W$ we have that Rwv implies $R'b(w)b(v)$ and that whenever there is a $u \in W'$ with $R'b(w)u$ then there is an $x \in W$ with Rwx and $b(x) = u$.

Lemma 4.7. *Let $F = \langle W, R \rangle$ and $F' = \langle W', R' \rangle$ be Kripke frames and let F' be a bounded morphic image of F. Then $F \vDash \phi$ implies $F' \vDash \phi$ for all modal formulas ϕ.*

Definition 4.8. Let $\mathcal{C}$ be a class of Kripke frames. A modal logic $\mathcal{L}$ is said to be *sound with respect to* $\mathcal{C}$ if all formulas $\phi \in \mathcal{L}$ are valid on all frames $F \in \mathcal{C}$. The logic $\mathcal{L}$ is said to be *complete*[1] *with respect to* $\mathcal{C}$ if any formula ϕ which is valid on all frames $F \in \mathcal{C}$ is in $\mathcal{L}$. We say that the modal logic $\mathcal{L}$ is *characterized* by the class $\mathcal{C}$ if $\mathcal{L}$ is sound and complete with respect to $\mathcal{C}$.

In the following, we summarize some characterization results for modal logics.

Lemma 4.9. *The modal logic K is characterized by the class of all finite Kripke frames.*

A proof can be found in [3, section 4.1], and the following lemma is shown in [5, corollary 5.19].

Lemma 4.10. *The modal logic* S5 *is characterized by the class of finite equivalence relations with one equivalence class.*

Definition 4.11. A *pre-Boolean algebra* is a transitive, reflexive frame $F = \langle W, R\rangle$ such that $F/\sim$ is a Boolean algebra with the equivalence relation $\sim$ defined as $w \sim v$ iff Rwv and Rvw. Here $F/\sim$ is the pair $\langle W/\sim, \bar{R}\rangle$ and for $V, U \in W/\sim$ the relation $\bar{R}VU$ holds iff there are $v \in V$ and $u \in U$ with Rvu.

So, a pre-Boolean algebra is a Boolean algebra in which the nodes are replaced with clusters of nodes which all see each other. Hamkins and Löwe [16, theorem 11] showed:

Lemma 4.12. *The modal logic* S4.2 *is characterized by the class of finite pre-Boolean algebras.*

4.1.2 The Modal Logic S4.2Top

In this section, we shortly present the less well-known modal logic S4.2Top which we will encounter in the context of the modal logic of grounds. The modal logic S4.2Top is obtained by adding the Top-axiom, introduced by Inamdar [18], to S4.2. TheTop-axiom is the formula

$$\Diamond((p \leftrightarrow \Box p) \wedge (\neg p \leftrightarrow \Box\neg p)).$$

It is valid on, among others, any frame $F = \langle W, R\rangle$ with a top element, i.e. an element $w \in W$ such that for all $v \in W$ we have Rvw and such that Rwu implies $u = w$.

Remark. Over T the Top axiom is equivalent to the more well-known McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$. To see this, first note that we can reformulate the implication in the McKinsey axiom to

$$\neg(\Box\Diamond p \wedge \neg\Diamond\Box p)$$

which leads to

$$\neg(\Box\Diamond p \wedge \Box\Diamond\neg p)$$

[1] Note that in [3] this is called weakly complete.

by spelling out abbreviation $\Diamond$. Again using the abbreviation and De Morgan's law, this is equivalent to

$$\Diamond\Box\neg p \vee \Diamond\Box p.$$

Now, $\Diamond p \vee \Diamond q \leftrightarrow \Diamond(p \vee q)$ is clearly valid on any Kripke frame and hence in $\mathrm{K} \subseteq \mathrm{T}$. So, the formula above is equivalent to

$$\Diamond(\Box p \vee \Box\neg p).$$

Further, T contains the axiom $\Box q \to q$ and together with the fact that $p \vee \neg p$ is a tautology we can conclude that $\Box p \vee \Box\neg p$ is equivalent to $(p \leftrightarrow \Box p) \wedge (\neg p \leftrightarrow \Box\neg p)$. This finishes the proof that the McKinsey and the Top-axiom are equivalent over T. So, alternatively we can obtain S4.2Top by adding the McKinsey axiom to S4.2.

As S4 together with the McKinsey axiom is usually referred to as S4.1, some authors call the extension of S4.2 with the McKinsey axiom S4.2.1 or S4.1.2. Nevertheless, we use Inamdar's terminology in the sequel.

Definition 4.13. An *inverted lollipop* is a frame with a top element such that after removing the top element the frame becomes a pre-Boolean algebra.

In his Master's thesis, Inamdar proved the following characterization theorem [18, theorem 93] (see also [19]):

Proposition 4.14. *The modal logic* S4.2Top *is characterized by the class of finite inverted lollipops.*

4.2 The Modal Logic of Model Constructions

4.2.1 Set-Up

In various areas of mathematics, we encounter the following situation: We have some first order theory $\mathcal{T}$ in a language $\mathcal{L}$ and are interested in a class $\mathcal{C}$ of models of this theory. In addition, there is construction method taking a model of the theory and leading to another model, or a notion of extension or submodel of models of this theory. For example, we can think of the theory of fields and the notion of field extension, the theory of $\mathbb{Q}$-vector spaces and the notion of subspace, or ZFC and forcing.

If the theory $\mathcal{T}$ is not complete, then starting with some model M, constructing a model or moving to an extension N could change the truth of sentences in the language $\mathcal{L}$. Now, we want to employ modal logic in order to analyze this change of truth. The idea for the general set-up is not far to seek: We take the class $\mathcal{C}$ of models and represent our notion connecting different models as a binary relation R on $\mathcal{C}$, just as we defined $\mathcal{F}$ on the global multiverse in chapter 3. Then, we define a general frame on the structure $\langle \mathcal{C}, R\rangle$ by allowing the extension of a proposition letter under a valuation to be in

$$E := \{A \subseteq \mathcal{C} : \text{ there is a sentence } \phi \text{ in the language } \mathcal{L} \text{ s.t. for all } M \in \mathcal{C},\ M \vDash \phi \text{ iff } M \in A\}.$$

For the rest of this chapter, let $\mathcal{L}$, $\mathcal{T}$, $\mathcal{C}$, and R be as above, and to avoid meta-mathematical complications, we assume that $\mathcal{C}$ is a set. In the following, we will use R also to refer to the model construction or notion of relation between models of $\mathcal{T}$ that it represents and simply call it a model construction for sake of legibility.

Definition 4.15. The *modal logic of the model construction* R, ML^R, is the modal logic of the frame $\langle \mathcal{C}, R, E\rangle$, i.e. the set of modal formulas valid on this frame. For any $M \in \mathcal{C}$, the *modal logic of* R *of the model* M, $\mathrm{ML}^R(M)$, is the set of modal formulas valid on $\langle \mathcal{C}, R, E\rangle$ in (the world) M.

Any formula which is valid on a Kripke frame F is clearly valid on any general frame based on F. Hence, results about the structure $\langle \mathcal{C}, R\rangle$ often suffice to provide lower bounds for ML^R. For instance, many natural notions of extension are transitive and reflexive guaranteeing that the modal logics of such notions contain S4.

4.2.2 Control Statements

To find upper bounds for ML^R, very helpful tools have been developed. The most important tool for our purposes are the following two types of control statements:

Definition 4.16. A *button* is a sentence ϕ in the language $\mathcal{L}$ such that $\langle \mathcal{C}, R, V\rangle, M \Vdash \Box\Diamond\Box p$ for any world M in any model on $\langle \mathcal{C}, R, E\rangle$ with a valuation V assigning $\{N \in \mathcal{C} : N \vDash \phi\}$ to p. We call the button ϕ *pure* if $\langle \mathcal{C}, R, V\rangle, M \Vdash p \leftrightarrow \Box p$ in any world M in any such model. We say that the button is *pushed* in M if $\langle \mathcal{C}, R\rangle, M \Vdash \Box p$.

A *switch* is a sentence ψ such that $\Box\Diamond q$ and $\Box\Diamond\neg q$ hold in any world in any model on $\langle \mathcal{C}, R, E\rangle$ with a valuation V assigning $\{N \in \mathcal{C} : N \vDash \psi\}$ to q.

We call a family of buttons and switches *independent* over $M \in \mathcal{C}$ if all the buttons are unpushed and in any R-extension it is possible to change the truth value of any of the switches by going to a further R-extension without affecting any truth value of the other switches or buttons, and it is possible to push any still unpushed button by going to an R-extension without affecting any other button or switch.

So, a button is a statement that can be made true at any point by going to an R-extension and afterwards stays true in all further extensions while a switch can be turned on and off at will.

The following results provide useful connections between the existence of independent families of these control statements in a model and its modal logic of the model construction R. In [16, section 2 and 3], Hamkins and Löwe show the following two theorems.

Theorem 4.17. *Let $M \in \mathcal{C}$. If for any natural number n there is an independent family of n switches over M, then* $\mathrm{ML}^R(M) \subseteq \mathrm{S5}$.

Theorem 4.18. *Let $M \in \mathcal{C}$. If for any natural numbers n and m there is an independent family of n buttons and m switches over M, then* $\mathrm{ML}^R(M) \subseteq \mathrm{S4.2}$.

We conclude this section by defining the notion of labeling of a Kripke frame for a model M in $\langle \mathcal{C}, R, E\rangle$ which was introduced under this name in [15] by Hamkins, Leibman and Löwe. In [16] the idea of labelings was already used, for example in the proof of the previous two theorems, as well.

Definition 4.19. Let $F = \langle W, S\rangle$ be a Kripke frame, $w_0 \in W$ an initial element of F, and $M \in \mathcal{C}$. An *F-labeling* for M in $\langle \mathcal{C}, R, E\rangle$ assigns a sentence ϕ_w in the language $\mathcal{L}$ to each node $w \in W$ such that:

1. For each R-extension N of M, there is exactly one $w \in W$ such that $N \vDash \phi_w$,
2. If a R-extension N of M satisfies ϕ_w, then it has an R-successor N' satisfying ϕ_v if and only if Swv, and
3. the model M satisfies ϕ_{w_0}.

The following lemma shows how we can use labelings to prove upper bounds for the modal logic of forcing and of grounds, respectively, using labelings:

Lemma 4.20. *Let $F = \langle W, S\rangle$ be a finite Kripke frame, $w_0 \in W$ an initial element of F, and $M \in \mathcal{C}$. Suppose that $w \mapsto \phi_w$ is an F-labeling for M. Then, any modal formula ψ which is not valid on F in w_0 is not in $\mathrm{ML}^R(M)$.*

Proof. Let $A : \mathcal{A} \to \mathcal{P}(W)$ be a valuation on F such that $\langle W, S, A\rangle, w_0 \nvDash \psi$. Then we let V be a valuation on $\langle \mathcal{C}, R, E\rangle$ with $V(p) = \{N \in \mathcal{C} : N \vDash \bigvee_{w\in A(p)} \phi_w\}$. Now it is easy to check that $\langle \mathcal{C}, R, V\rangle, M \nVdash \psi$. □

Intuitively, we could also think of F as a bounded morphic image of the frame of all forcing R-extensions M assuming that for each $w \in W$ the formula ϕ_w holds in some R-extension of M: We simply map an R-extension N of M to the unique $w \in W$ with $N \vDash \phi_w$. Condition 2 for labelings guarantees that this mapping is a surjective bounded morphism.

5 The Forcing and the Grounds Interpretation of Modal Logic

Now, we are ready to start the analysis of the modal logic of forcing and the modal logic of grounds. In this chapter, we transfer the set-up from chapter 4 to this set-theoretic setting. Further, we will demonstrate the advantages of the fact that truth in forcing extensions or in grounds can be expressed in the language of set theory itself.

5.1 Set-Up

We simply apply the general methods of the previous chapter to the global multiverse $\mathcal{W}$ of all countable transitive models of set theory together with either the relation $\mathcal{R}$, "is a forcing extension of", or $\mathcal{G}$, "is a ground of". We will refer to the first case as the *upwards direction.* We call the corresponding modal logic the *modal logic of forcing*, MLF, and refer to the *modal logic of forcing of a model $W \in \mathcal{W}$* by MLF(W). Furthermore, we call sentences ϕ in the language of set theory *upwards buttons* and *upwards switches*, respectively, if they are buttons or switches with respect to this first relation. For labelings in this setting we analogously use the term *upwards labeling.*

Likewise, for the relation "is a ground of", we call the corresponding modal logic the *modal logic of grounds*, MLG, and the *modal logic of grounds of a model W*, MLG(W). For this *downwards direction*, we use the terms *downwards buttons*, *downwards switches*, and *downwards labelings.*

5.1.1 Translations

A main feature of forcing is the possibility of expressing truth in a forcing extension in the language of set theory using the forcing relation (see section A.1 in appendix A). This allows us to directly translate the modalities into first-order expressions. Then, $\Diamond\phi$ can just be considered as an abbreviation for "ϕ holds in some forcing extension". By the forcing theorem this is first-order expressible as

$$\text{"}\exists \mathbb{P} \exists p \in \mathbb{P}(\mathbb{P} \text{ is a partial order} \wedge p \Vdash_{\mathbb{P}} \phi)\text{."}$$

Likewise, we translate $\Box\phi$ to "ϕ holds in all forcing extensions" or

$$\text{"}\forall \mathbb{P} \forall p \in \mathbb{P}(\mathbb{P} \text{ is a partial order} \rightarrow p \Vdash_{\mathbb{P}} \phi)\text{."}$$

Note that these two interpretations also behave dually in the sense that $\Box$ and $\neg\Diamond\neg$ abbreviate equivalent statements.

By the uniform definability of grounds, theorem 3.2, we can express the statements "ϕ holds in some ground" and "ϕ holds in all grounds" in first-order logic as well. So, also in the downwards direction we can use the modalities as abbreviations in the language of set theory. We will make sure that it is always clear whether we work in the upwards or the downwards setting.

Remark. Note that the set

$$E := \{A \subseteq \mathcal{W} : \text{ there is a } \phi \text{ in the language } \mathcal{L}_\in \text{ s.t. for all } W \in \mathcal{W},\, W \models \phi \text{ iff } W \in A\}$$

naturally satisfies the closure conditions for general frames which we disregarded in the previous chapter in the upwards and the downwards setting: If $A \in E$ and this is witnessed by a sentence ϕ, then also $\{W \in \mathcal{W} : \forall U \in \mathcal{W}(RWU \rightarrow U \in A)\} \in E$ where R can be either of the two relations under discussion because this is now witnessed by $\Box\phi$ where the abbreviation is interpreted according to the relation R.

We now also can take a different perspective on valuations on the general frame $\langle \mathcal{W}, R, E\rangle$:

Definition 5.1 (Translation)**.** Let $\mathcal{L}_\Box$ be the set of modal formulas and $\mathrm{Sent}(\mathcal{L}_\in)$ the set of sentences in the language of set theory. We call a function $H : \mathcal{L}_\Box \rightarrow \mathrm{Sent}(\mathcal{L}_\in)$ an *upwards translation* if

1. $H(\phi \wedge \psi) = H(\psi) \wedge H(\phi)$,
2. $H(\neg\phi) = \neg H(\phi)$, and
3. $H(\Box\phi) = \forall \mathbb{P} \forall p \in \mathbb{P}(\mathbb{P} \text{ is a partial order } \rightarrow p \Vdash_{\mathbb{P}} H(\phi))$.

Analogously, we call a function $H : \mathcal{L}_\Box \rightarrow \mathrm{Sent}(\mathcal{L}_\in)$ a *downwards translation* if

1. $H(\phi \wedge \psi) = H(\psi) \wedge H(\phi)$,
2. $H(\neg\phi) = \neg H(\phi)$, and
3. $H(\Box\phi) =$ In all grounds $H(\phi)$ holds.

Now, in either direction, valuations on the general frame $\langle \mathcal{W}, R, E\rangle$ as well as translations are determined by a mapping from proposition letters to sentences in the language of set theory. It is easy to check, that now a modal formula ψ is in $\mathrm{MLF}(W)$ for some $W \in \mathcal{W}$ iff $W \models H(\psi)$ for all upwards translations H, and analogously $\psi \in \mathrm{MLG}(W)$ iff $W \models H(\psi)$ for all downwards translations H. We will also say that ψ is *upwards valid* and *downwards valid*, respectively, in W in this situation.

Also, a translation or valuation will always assign exactly the same proposition letters to elementarily equivalent models in the multiverse. By the first-order expressibility of the modalities, elementarily equivalent models in fact satisfy exactly the same modal formulas under each translation.

5.1.2 Uniform Buttons and S4.2Top

In the setting of set theory, we have the possibility to give further connections between modal logics and control statements. The following theorem links the modal logic S4.2Top to the existence of certain control statements. A difference to the characterization theorem for S4.2 or S5 in terms of buttons and switches is that we need a *uniform* infinite family of buttons this time:

Definition 5.2. We call an infinite family $\langle b_i : i \in \omega \rangle$ of buttons *uniform* if there is a formula $\psi(n)$ with one free variable such that b_i is equivalent to $\psi(i)$ for each $i \in \omega$.

The uniformity allows us to use these buttons to construct new statements by referring to the least n such that the nth button is not pushed, for example. In the general setting this was not possible as the internal language and theory of the models under discussion might not allow such expressions.

Theorem 5.3. *If there is a uniform family of infinitely many independent pure upwards (downwards) buttons over a countable transitive model W of* ZFC *then the modal logic of forcing (of grounds) of W is contained in the modal logic* S4.2Top.

Proof. We use the upwards terminology in the proof. The downwards direction works in the exact same way.

Let χ be a modal formula not contained in S4.2Top. By proposition 4.14 there is a finite inverted lollipop K on which ϕ is not valid in some bottom node. This inverted lollipop is the bounded morphic image of an inverted lollipop L with an underlying pre-Boolean algebra in which each cluster contains exactly 2^m worlds and in which there are n atom clusters for some $m, n \in \mathbb{N}$: we simply map several points of one cluster of L to the same point of K if some clusters of K are smaller and keep the same underlying Boolean algebra structure. This means that ϕ is also not valid in some bottom node of L. Moreover, L can be represented as $\mathcal{P}(n) \times 2^m \cup \{\star\}$ where $\star$ is greater than any other element and for $\langle I, k\rangle, \langle J, \ell\rangle \in \mathcal{P}(n) \times 2^m$ we define $\langle I, k\rangle \leq \langle J, \ell\rangle$ iff $I \subseteq J$. The uniform family of independent buttons can be written as $\{\psi(k) : k \in \omega\}$ for some formula ψ. Now we define a labeling of L for W. Let $\phi_\star$ be the formula

$$\forall k \in \omega \exists \ell \in \omega(\ell > k \wedge \psi(2\ell + 1)).$$

So, $\phi_\star$ expresses that infinitely many of the buttons with odd number are pushed. Let $\phi_{\langle I,k\rangle}$ be the formula

$$\neg\phi_\star \wedge \bigwedge_{i\in I} \psi(2i) \wedge \bigwedge_{j\in n-I} \neg\psi(2j) \wedge \begin{array}{l}\text{the greatest } h \in \omega \text{ such that } \psi(2h+1) \text{ holds}\\ \text{is of the form } a \cdot 2^m + k \text{ for some } a \in \omega.\end{array}$$

The formula $\phi_{\langle I,k\rangle}$ says that there is a largest odd number b for which the corresponding button is pushed, that this number $b \equiv k \bmod 2^m$, and that I expresses exactly which among the first n

even-numbered buttons are pushed. Every model of ZFC satisfies exactly one of these formulas or $\phi_\star$. Further, if $W' \vDash \phi_\star$ clearly every extension also satisfies $\phi_\star$. On the other hand, if $W' \vDash \langle I, k\rangle$ for some $\langle I, k\rangle \in L$ there is a forcing extension in which $\phi_\star$ holds. This can be achieved e.g. by pushing all the odd-numbered buttons by a product forcing (in the downwards case by applying the strong downwards directedness of grounds). If $\langle I, k\rangle \leq \langle J, \ell\rangle$, then $\phi_{\langle J,\ell\rangle}$ can be forced by pushing $\psi(2i)$ for $i \in J - I$ as well as a button of the form $\psi(2(a \cdot 2^m + \ell) + 1)$ with a big enough to ensure that $\neg\psi(2b + 1)$ holds for all $b > a \cdot 2^m + \ell$. Note that this only requires to push finitely many buttons. Finally, if $\langle I, k\rangle \not\leq \langle J, \ell\rangle$ then it is impossible to force $\phi_{\langle J,\ell\rangle}$ as there is an $h \in I - J$ and that means $\psi(2h)$ is pushed in W' but it would have to be unpushed in a model satisfying $\phi_{\langle J,\ell\rangle}$. By the results from [15] about labelings, this proves that χ is not contained in the modal logic of forcing of W. Hence, $\mathrm{MLF}(W) \subseteq \mathrm{S4.2Top}$. $\square$

5.2 Lower Bounds

On the way to the determination of the modal logic of forcing and the modal logic of grounds we start with the following lemmata which give lower bounds for both cases. The modal logic of forcing was determined by Hamkins and Löwe in [16].

Lemma 5.4. *The modal logic of forcing contains all* S4.2 *formulas, i.e.* $\mathrm{S4.2} \subseteq \mathrm{MLF}$.

Proof. It is enough to check that all S4.2 axioms are upwards valid in any model of ZFC. So, let $W \vDash \mathrm{ZFC}$. The first axiom $p \to \Diamond p$ (T) is valid because for any formula ϕ we have that if $W \vDash \phi$ then ϕ can be forced by a trivial forcing. The one element partial order yields W itself as a forcing extension of W and hence there is a forcing extension in which ϕ holds.

To check the next axiom $\Diamond\Diamond p \to \Diamond p$ (4), suppose there is a forcing extension U of a forcing extension of W in which ϕ holds. But we know that we can build one partial order in W that leads to the same forcing extension by taking a two-step-iteration. So, U is in fact a forcing extension of W and $U \vDash \phi$.

For the last axiom $\Diamond\Box\phi \to \Box\Diamond\phi$ (.2), note that in Kripke frames this axiom corresponds to the directedness of the frame. We have seen that the forcing extensions of a model are not necessarily directed. However, the validity of the axiom does not contradict those facts as our translations do not correspond to arbitrary valuations on the frame.

So, let $W[G]$ be a forcing extension with a $\mathbb{P}$-generic filter G over W for some partial order $\mathbb{P} \in W$ and suppose that $W[G] \vDash$ "In all forcing extensions ϕ holds". So, there is a condition $p \in G$ forcing this sentence. Now let $W[H]$ be an arbitrary extension of W with some $\mathbb{Q}$-generic filter H over W. We have to show that $W[H]$ has a forcing extension in which ϕ holds. So, we take an $\mathbb{P}$-generic filter K over $W[H]$ with $p \in K$ and go to the extension $W[H][K]$. In fact this is an extension via the product forcing $\mathbb{Q} \times \mathbb{P}$ and by the product lemma this means that $W[H][K] = W[K][H]$ and as $p \in K$ we know that $W[K] \vDash$ "In all forcing extensions ϕ holds". So, $W[H][K]$ satisfies ϕ. $\square$

Due to the downwards directedness of grounds, the situation is easier in the downwards direction. The following lemma is part of the results about the modal logic of grounds by Hamkins and Löwe in [17].

Lemma 5.5. *The modal logic of grounds contains all* S4.2 *formulas, i.e.* $\mathrm{S4.2} \subseteq \mathrm{MLF}$.

Proof. Analogously to the previous lemma, the fact that any model is a ground of itself and that grounds of grounds are grounds by the two-step-iteration (T) and (4) are valid. The directedness of grounds directly gives us (.2). $\square$

We want to remark here that this result only requires (weak) downwards directedness of grounds: Any two grounds of a model have a common ground.

5.3 General Results

The modal logic of forcing and of grounds have been exactly determined. In 2008, Hamkins and Löwe ([16]) showed:

Theorem 5.6. *The* ZFC*-provable modal principles of forcing,* MLF*, are exactly those in the modal logic* S4.2*.*

By lemma 5.4, we know that $\mathrm{S4.2} \subseteq \mathrm{MLF}$. To show that $\mathrm{MLF} = \mathrm{S4.2}$, it is enough to provide a model whose modal logic of forcing is exactly S4.2. By theorem 4.18, we only have to give a model in which for any natural numbers m and n there is an independent family of n buttons and m switches. Given any countable transitive model W, the model L^W suffices and several independent families of buttons and switches have been constructed. Instead of giving an exposition of the proof of the independence of any of these buttons, we refer to chapter 6 and its main result, theorem 6.8, which shows that in any generic multiverse there is a model whose modal logic of forcing is S4.2.

Up to now, it is unknown whether the buttons b_n stating

$$\text{“}\omega_n^L \text{ is not a cardinal”}$$

for $n \in \omega$, originally proposed by Hamkins and Löwe (see [16]), are independent. However, in his Master's thesis, Inamdar showed that b_2 and b_n with $n \neq 2$ are independent (see [18, corollary 194]).

Hamkins and Löwe also mentioned an alternative family which has been shown to be independent (see [15]) based on a result by Baumgartner, Harrington and Kleinberg [1]. For these buttons, let $\omega_1^L = \bigcup_{n\in\omega} S_n$ be the L-least partition of ω_1^L into ω many disjoint stationary sets. For each $n \in \omega$, we then get the button

$$\text{“}S_n \text{ is not stationary in } \omega_1^L\text{”.}$$

Furthermore, Hamkins, Leibman, and Löwe [15, p. 12-13] gave an overview over other independent families of buttons which had subsequently been established: Rittberg provided an independent family of infinitely many buttons and infinitely many switches over L in his Master's thesis [27], and Friedman, Fuchino and Sakai [8] gave two different such families over L as well. We will generalize one of the latter families in chapter 6. For the buttons and switches we define in that chapter we can find a ground below any model such that the buttons and switches are independent over this ground. Of course, that means that they are in particular independent over L.

Besides showing that there are models with modal logic of forcing S4.2, Hamkins and Löwe [16, theorem 17] provided the following bounds for the modal logic of forcing of a model:

Theorem 5.7. *For any model W of* ZFC*, the modal logic of forcing is bounded by* $\mathrm{S4.2} \subseteq \mathrm{MLF}(W) \subseteq \mathrm{S5}$.

To prove this theorem it is enough to provide arbitrarily large, finite independent families of switches for any model of ZFC by theorem 4.17. Again, we want to refer to the switches we define in chapter 6 which are independent over any model of ZFC.

The upper bound S5 is moreover the best possible bound: Hamkins's maximality principle which expresses that $\Diamond\Box p \to p$ is upwards valid implies that all S5 formulas are upwards valid. Since Hamkins [13] showed the consistency of this maximality principle, we know that there are models N of ZFC with $\mathrm{MLF}(N) = \mathrm{S5}$. Recently, Hamkins and Block showed that there are also models of ZFC with a modal logic of forcing strictly between S4.2 and S5 (personal communication, April 11, 2017)[2].

In 2013, Hamkins and Löwe [17] showed that the modal logic of grounds is S4.2 as well under the assumption of the downwards directedness of grounds. So, Usuba's result about the downwards directedness allows us to formulate the analogue of theorem 5.6 for the modal logic of grounds in theorem 5.9. The proof uses the model without a bedrock constructed by Reitz (see [26]).

An important step in the proof is to show that any ground of an extension with a progressively closed class product (see definition A.31) contains a model obtained by forcing with a tail of the class forcing. Reitz used the fact that grounds satisfy the λ-approximation property for sufficiently large λ and the fact that initial segments of a $\mathrm{Fn}(\kappa, 2, \kappa)$-generic function are in the ground model to show this result in the case of the forcing used in his construction.

In the following lemma, we show this result in general which will be very useful in the last

[2] The idea of the proof is to show the consistency of the theory T stating that V is a set forcing extension of L, that ω_1^L is a cardinal, that every definable cardinal of L above ω_1 is collapsed to ω_1, and that any button which can be pushed without collapsing ω_1 is already pushed. The consistency is established by a compactness argument. Now, pushing any unpushed button collapses ω_1^L. Since then every definable L-cardinal is collapsed, all buttons are pushed by an argument from [13]. So, there cannot be two independent buttons. As arbitrarily many independent switches still exist, the modal logic of forcing of a model M of T is the modal logic of a chain of two infinite clusters which lies strictly between S4.2 and S5.

chapter of this thesis where we show that any such class forcing allows for the definition of downwards switches.

Lemma 5.8. *Let W be a countable transitive model of* ZFC *and let $\langle P_\alpha : \alpha \in Ord\rangle$ be a definable class in W. Assume that $\mathbb{P} = \prod_{\alpha\in Ord} P_\alpha$ is a progressively closed product in W. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over W and U is a ground of $W[G]$ with $W \subseteq U$, then U contains $W[G^{>\eta}]$ for some η where $G^{>\eta}$ is the projection of G on $\prod_{\alpha>\eta} P_\alpha$.*

Proof. As $\mathbb{P}$ is a progressively closed product, it factors as $\mathbb{P}^1 \times P^2$ where P^2 is a set and $\mathbb{P}^1$ is δ-closed for arbitrarily large regular δ. In this situation, P^2 completely embeds into $\prod_{\alpha\leq\eta} P_\alpha$ for some η. As $\mathbb{P}^1$ is δ-closed, this then implies that also $\prod_{\alpha>\eta} P_\alpha$ is δ-closed. So, for arbitrarily large regular δ there is an η such that P_α is δ-closed for all $\alpha > \eta$.

We know that $W[G] = U[h]$ for some R-generic filter h over U and $R \in U$. Let κ be a regular cardinal bigger than $(|R|^+)^{W[G]}$ such that $\mathbb{P}$ factors as $P^1 \times \mathbb{P}^2$ where P^1 has the κ-chain condition and $\mathbb{P}^2$ is κ-closed. Let η be such that P_α is κ-closed for all $\alpha > \eta$. We claim that $G^\alpha \in U$ for all $\alpha > \eta$. Here, G^α denotes the projection of G on the αth component.

Suppose $G^\beta \notin U$ for some $\beta > \eta$. We know that $g := G^\beta$ has an R-name $\dot{g}$ in U such that $g = \mathrm{val}(\dot{g}, h)$.

First, we work in $W[g]$. We construct a descending sequence of conditions in g such that the sequence eventually lies below any element of g. We start with an arbitrary condition $p_0 \in g$. Once a descending sequence $\langle p_\alpha : \alpha < \gamma\rangle$ in g has been defined for some γ, check whether there are conditions below all p_α, $\alpha < \gamma$, in g. If this is not the case, we are done. Otherwise, take an element $p_\gamma \in g$ below all p_α with $\alpha < \gamma$.

Call the length of the resulting sequence λ. So, $\langle p_\alpha : \alpha < \lambda\rangle$ is such that for each $p \in g$ there is an α such that $p \geq p_\alpha$ and

$$g = \{p \in P_\beta : \exists\alpha < \lambda(p \geq p_\alpha)\}.$$

In particular, g can be defined from any cofinal subsequence of this sequence. We claim that $\mathrm{cf}(\lambda)^W \geq \kappa$. Suppose there is an increasing cofinal function $f : \theta \to \lambda$ in W with $\theta < \kappa$. In $W[g]$, we can then build the sequence $\langle p_{f(\alpha)} : \alpha < \theta\rangle$ from which g can be defined. Furthermore, as P_β is κ-closed this sequence is in W by theorem A.19. So, $g \in W$ and hence $g \in U$ contradicting our assumption.

Next, we claim that also $\mathrm{cf}(\lambda)^{W[G]} \geq \kappa$. We know that the forcing $\mathbb{P} \cong P^1 \times \mathbb{P}^2$. Forcing with κ-closed forcing on the one hand does not add any sequences of length $< \kappa$ and hence the cofinality of λ is still at least κ after forcing with $\mathbb{P}^2$. As P^1 has the κ-chain condition, on the other hand, it preserves cofinalities $\geq \kappa$. So, $\mathrm{cf}(\lambda)^{W[G]} \geq \kappa$.

Now, we work in $W[G]$. Recall that $W[G] = U[h]$. We now show that conditions in h cannot decide whether $\check{p_\alpha} \in \dot{g}$ for unboundedly many $\alpha < \lambda$. So, suppose a condition $r \in h$ forces $\check{p_\alpha} \in \dot{g}$ for all α in $A \subseteq \lambda$ where A is unbounded in λ. But then

$$g = \{p \in P_\beta : (r \Vdash \check{p} \in \dot{g})^U\}^\uparrow,$$

where the arrow denotes the upwards closure in P_β. Note that $\{p \in P_\beta : (r \Vdash \check{p} \in \dot{g})^U\} \subseteq g$ follows from $r \in h$. So, g is in U contradicting the assumption.

Thus, for each $r \in h$ the set

$$A_r := \{\alpha < \lambda : (r \Vdash \check{p_\alpha} \in \dot{g})^U\}$$

is bounded in λ. As $\mathrm{cf}(\lambda)^{W[G]} \geq \kappa$ and $|h| < \kappa$ that means that also $\bigcup_{r \in h} A_r$ is bounded in λ. So, there is an $\alpha \in \lambda - \bigcup_{r\in h} A_r$. For this α, we have that $p_\alpha \in g$ but also that for all $r \in h$, $r \nVdash \check{p_\alpha} \in \dot{g}$. This contradicts the forcing theorem.

Hence, we can conclude that $G_\beta \in U$ and therefore that $W[G^{>\eta}] \subseteq U$. $\square$

Now, we can present the proof for the general result on the modal logic of grounds which was only sketched by Hamkins and Löwe [17, theorem 6].

Theorem 5.9. *The* ZFC*-provable modal principles of grounds,* MLG*, are exactly those in the modal logic* S4.2*.*

Proof. Lemma 5.5 states that S4.2 $\subseteq$ MLG. So again, it is enough to give a model of ZFC in which the modal logic of grounds is S4.2. In order to obtain such a model, we follow Reitz's construction (see [26]) to obtain a model without bedrock. Then, we define arbitrarily large independent finite families of buttons and switches and apply theorem 4.18.

We start with a countable transitive model W of $\mathrm{ZFC} + V = L$. We force with the product $\mathbb{P} := \prod_{\kappa \in \mathrm{Reg}} \mathrm{Fn}(\kappa, 2, \kappa)$ where Reg is the class of regular cardinals. This product factors as $\mathbb{P}^1 \times P^2$ where $\mathbb{P}^1$ is δ^+-closed and P^2 has the δ^+-c.c. for any regular δ as $\prod_{\kappa \in \mathrm{Reg}, \kappa \leq \delta} \mathrm{Fn}(\kappa, 2, \kappa)$ has size δ and hence the δ^+-c.c. while $\prod_{\kappa \in \mathrm{Reg}, \kappa > \delta} \mathrm{Fn}(\kappa, 2, \kappa)$ is δ^+-closed. So, $\mathbb{P}$ is a progressively closed product. So, let G be $\mathbb{P}$-generic over W and consider the model $W[G]$.

In $W = L^W$, let $A_0, A_1, \ldots$ be a partition of the regular cardinals above $\aleph_\omega$ in ω many proper classes. For the switches we define the auxiliary sentences $\phi(n)$ for $n \in \omega$ stating

> "the least regular L-cardinal κ above $\aleph_\omega^L$ for which there is a $\mathrm{Fn}(\kappa, 2\kappa)$-generic filter over L is in A_n."

For the buttons, we use the sentences $\psi(n)$ for $n \in \omega$ stating

"there is no $\mathrm{Fn}(\aleph_n^L, 2, \aleph_n^L)$-generic filter over L."

Clearly, the statements $\psi(n)$ are pure buttons: Once they are true they stay true in any deeper ground.

For the independence of the buttons and switches, we use that any ground of W has a ground of the form $W[G^{>\gamma}]$ where $G^{>\gamma}$ is the projection of G on $\prod_{\kappa \in \mathrm{Reg}, \kappa > \gamma} \mathrm{Fn}(\kappa, 2, \kappa)$ by lemma 5.8. In particular, $G^{>\gamma}$ is $\prod_{\kappa \in \mathrm{Reg}, \kappa > \gamma} \mathrm{Fn}(\kappa, 2, \kappa)$-generic over W. This also implies that in each ground of $W[G]$ exactly one of the sentences $\phi(n)$ is true.

Let U be a ground of $W[G]$ and let k be a natural number. Let $A = \{n \in k : \psi(n)\}$. We show that for any $m \in \omega$ and any $B \subseteq A$ there is a ground U' of U in which $\phi(m)$ holds and for which $B = \{n \in k : U' \vDash \psi(n)\}$. First, let $\gamma > \omega$ be such that $W[G^{>\gamma}]$ is a ground of U. Let $\delta > \gamma$ be an ordinal such that $\aleph_\delta^L \in A_m$. Then also $W[G^{\geq\delta}]$ is a ground of U as it is clearly a ground of $W[G^{>\gamma}]$. For each $n \in B$ there is a $\mathrm{Fn}(\aleph_n^L, 2, \aleph_n^L)$-generic filter $g_n \in U$. Each g_n with $n \in B$ is $\prod_{n\in B} \mathrm{Fn}(\aleph_n^L, 2, \aleph_n^L)$-generic over $W[G^{\geq\delta}]$ as $W[G^{\geq\delta}]$ is obtained from $W = L^W$ by $\aleph_\delta^L$-closed forcing for $\delta > \omega$ and $|\mathrm{Fn}(\aleph_n^L, 2, \aleph_n^L)|^L = \aleph_n^{L<\aleph_n^L} = \aleph_n^L$ by GCH. So, $W[G^{\geq\delta}]$ contains no new dense subsets of $\mathrm{Fn}(\aleph_n^L, 2, \aleph_n^L)$. Furthermore, g_n and g_l are mutually generic for distinct $n < l$ because $\mathrm{Fn}(\aleph_l^L, 2, \aleph_l^L)$ is $\aleph_l^L$-closed and so forcing with $\mathrm{Fn}(\aleph_l^L, 2, \aleph_l^L)$ over $W[G^{\geq\delta}]$ also does not add dense subsets to $\mathrm{Fn}(\aleph_n^L, 2, \aleph_n^L)$. So, g_n is still generic over $W[G^{\geq\delta}][g_l]$ and by the commutativity of products g_l is also generic over $W[G^{\geq\delta}][g_n]$.

Now, we simply add all these filters to $W[G^{\geq\delta}]$ to obtain $U' := W[G^{\geq\delta}][\prod_{n\in B} g_n]$. By the mutually genericity, U' does also not contain any $\mathrm{Fn}(\aleph_\alpha^L, 2, \aleph_\alpha^L)$-generic filter for $\alpha < \delta$ and $\alpha \notin B$.

We can now use the statements $\phi(n)$ to define finitely many switches. Let i be a natural number and for a $j < i$ let $\phi'(j)$ be the sentence

"the jth last digit in binary representation of the unique n for which $\phi(n)$ holds is 1".

So, we have shown that there is an independent family of k downwards buttons and i downwards switches for $W[G]$. As k and i were arbitrary we can conclude that $\mathrm{MLG}(W[G]) = \mathrm{S4.2}$ by theorem 4.18. □

5.4 Examples

In this section, we want to give different examples of models with a modal logic of grounds which is not contained in S5. This illustrates that the situation is somewhat different to the upwards setting.

The most obvious difference is the following: While every model of ZFC has many non-trivial forcing extensions, there are models which have no proper grounds or for example exactly one proper ground. Consider the following two examples:

1. Let W be a model of ZFC. Then the constructible universe as it is defined in W, L^W, is a model of ZFC without proper inner models and hence without non-trivial grounds. Therefore, $\mathrm{MLG}(L^W)$ is the modal logic of a single reflexive point. In particular, the formulas $p \leftrightarrow \Box p$ and $p \leftrightarrow \Diamond p$ are downwards valid in L^W.

2. If we again take L^W for a countable transitive model W of ZFC, we can build forcing extensions with the generalized Sacks forcing. We have seen there that for any uncountable regular cardinal $\kappa \in L^W$ and any $\mathbb{S}_\kappa$-generic filter G over L^W the extension $L^W[G]$ is minimal. Hence, $L^W[G]$ has exactly one proper ground, namely L^W. On the other hand the formula $V = L$ is valid in L^W but not in $L^W[G]$. So, the translations of proposition letters can hold in any subset of this two element frame, and therefore, the modal logic of grounds of L^W is the modal logic of a two element chain.

In the following, we want to illuminate these simple examples by two further examples. First, we will look at a Cohen extensions of models of the ground axiom, another example for a model with the modal logic of a two element chain as its modal logic of grounds. What makes this model remarkable is that the structure of its grounds is by far not a two element chain but that all of the grounds except for one are elementarily equivalent.

5.4.1 Cohen Extensions of Models of ZFC + GA

Before we can analyze the modal logic of a Cohen extension of a model of ZFC + GA, we have to do some work on results about Boolean algebras. We will see how useful the Boolean algebra perspective on forcing is to get a hand on the intermediate models between a model and one of its forcing extensions.

Definition 5.10. We define the *density* of a Boolean algebra B to be the least cardinal κ such that there is a dense subset $D \subseteq B - \{0\} =: B^+$ of cardinality κ. We denote the density of B as $d(B)$.

Lemma 5.11. *Let B be a complete Boolean algebra and let $A \subseteq B$ be a complete subalgebra. Then $d(A) \leq d(B)$.*

Proof. Let $D \subseteq B^+$ be dense. For each $d \in D$ we define $a_d := \bigwedge\{a \in A : a \geq d\} \in A$. Since A and B agree on all meets, we get that $a_d \geq d$, and so in particular that $a_d \neq 0$.

Now, for any $a \in A^+$ there is a $d \in D$ with $d \leq a$ and hence $a_d \leq a$. So, the set $\{a_d : d \in D\} \subseteq A^+$ is dense in A. $\square$

The following theorem is well-known. The proof presented here follows the proof of [7, theorem 8.6].

Theorem 5.12. *Any two atomless complete Boolean algebras with a countable dense subset are isomorphic.*

We use the following lemma for the proof of this theorem:

Lemma 5.13. *Let A and B be complete Boolean algebras and let $S \subseteq A^+$ and $T \subseteq B^+$ be dense subsets and assume there is an order isomorphism $f : S \to T$. Then, f can be extended to an isomorphism from A to B.*

Proof. For any $a \in A$ let $S{\restriction}a := \{x \in S : x \leq a\}$. We claim that for any $a \in A$ we have that $\sup(S{\restriction}a) = a$. Suppose that $\sup(S{\restriction}a) < a$. Then $a \wedge \neg\sup(S{\restriction}a) > 0$ and hence there is $x \in S$ with $x \leq a \wedge \neg\sup(S{\restriction}a)$. But then on the one hand, we have that $x \in S{\restriction}a$ and $x \leq \sup(S{\restriction}a)$, and on the other hand, $x \leq \neg\sup(S{\restriction}a)$. This implies $x = 0$, a contradiction.

Now, we define an extension of f:

$$\begin{aligned} g : A &\to B \\ a &\mapsto \sup(f[S{\restriction}a]). \end{aligned}$$

To see that g is indeed an extension of f, let $a \in S$. Then,

$$f(a) = f(\sup(S{\restriction}a)) = \sup(f[S{\restriction}a]) = g(a)$$

as f is an order isomorphism. Next, we show that g is surjective. Let $b \in B$. Choosing $a = \sup(f^{-1}[T{\restriction}b])$ leads to

$$\begin{aligned} g(a) &= \sup(f[S{\restriction}\sup(f^{-1}[T{\restriction}b])]) \\ &= \sup(f[f^{-1}[T{\restriction}b]]) \\ &= \sup(T{\restriction}b) = b. \end{aligned}$$

For the injectivity, we begin by showing that for $x, y \in S$ we have that $x \wedge y = 0$ iff $f(x) \wedge f(y) = 0$. If $x \wedge y > 0$, then there is $z \in S$ with $z \leq x \wedge y$, i.e. $z \leq x, y$. But then we have that $0 < f(z) \leq f(x), f(y)$ and hence $f(x) \wedge f(y) > 0$. The other direction follows by considering f^{-1}.

Now, let $a \neq b$ be two elements of A. W.l.o.g. we have that $a \wedge \neg b > 0$, and so there is an $x \in S$ with $x \leq a \wedge \neg b$. So, $x \in S{\restriction}a$ and therefore $f(x) \leq g(a)$. Further, for any $y \leq b$ we have that $x \wedge y = 0$. Therefore, also $f(x) \wedge f(y) = 0$ and so $f(x) \wedge sup\{f(y) : y \leq b\} = f(x) \wedge g(b) = 0$. Thus, $f(x) \leq g(a) \wedge \neg g(b)$ and so $g(a) \neq g(b)$.

Finally, we show that g preserves the order. Let $a \leq b$ be elements of A. Then $S{\restriction}a \subseteq S{\restriction}b$ and so $g(a) \leq g(b)$. Hence, g is an isomorphism between A and B. $\square$

Proof of Theorem 5.12. Let A be a complete atomless Boolean algebra with a countable dense subset $D = \{d_n : n \in \omega\}$. We show that A has a dense subset isomorphic to the Cantor tree $\langle 2^{<\omega}, \supseteq \rangle$.

We will define this subset recursively. We will say that $b, c \in A$ split $a \in A$ if $b, c < a$ and $b \vee c = a$.

Let $a_\emptyset = 1$. Now, suppose that we have defined a_s for all $s \in \{0,1\}^n$ with $\bigvee\{a_s : s \in 0, 1^n\} = 1$. If there is a $t \in \{0,1\}^n$ such that $a_t \leq d_n$ then we pick arbitrary $a_{s^\frown 0}, a_{s^\frown 1} > 0$ splitting a_s for each $s \in \{0,1\}^n$. Otherwise, as $\bigvee\{a_s : s \in 0, 1^n\} = 1$ there is a $t \in 0, 1^n$ such that $0 < d_n \wedge a_t < a_t$. Then we pick $a_{t^\frown 0} = d_n \wedge a_t$ and $a_{t^\frown 1} = \neg d_n \wedge a_t$ and for all other $s \in \{0,1\}^n$ we pick an arbitrary splitting $a_{s^\frown 0}, a_{s^\frown 1}$ of a_s.

In this way, we recursively obtain elements a_s for all $s \in \{0,1\}^{<\omega}$ such that $s \mapsto a_s$ is an order isomorphism between the Cantor tree and $S := \{a_s : s \in \{0,1\}^{<\omega}\}$ and such that S is dense as for each $d \in D$ there is an $a \in S$ with $a \leq d$. So, A has a dense subset isomorphic to the Cantor tree and therefore all complete atomless Boolean algebras with a countable dense subset are isomorphic by lemma 5.13. $\square$

Now, we are ready to determine the modal logic of grounds of a Cohen extension of a model of the Ground Axiom. Let Coh be the partial order $\mathrm{Fn}(\omega, 2, \omega)$. The previous theorem shows that the distinction between the partial order we called Cohen forcing earlier and the partial order $\mathrm{Fn}(\omega, 2, \omega)$ is not necessary, as both are forcing equivalent to the unique complete atomless Boolean algebra with a countable dense subset.

Theorem 5.14. *Let W be a countable transitive model of* $\mathrm{ZFC} + \mathrm{GA}$*, i.e. a model of* ZFC *without non-trivial grounds. Let G be a* Coh*-generic filter over W. Then the modal logic of grounds of the model $W[G]$ is the modal logic of a two element chain.*

Proof. The algebra B of regular open subsets of Coh, i.e. the complete algebra given by theorem A.21 into which Coh densely embeds, is up to isomorphism the only complete atomless Boolean algebra with a countable dense subset by theorem 5.12.

Now, let U be a model of ZFC with $W \subsetneq U \subseteq W[G]$. Then by theorem B.5, there is a complete subalgebra A of B and an $A - \{0\}$-generic filter H over W with $U = W[H]$.

Since $W \subsetneq U$. The filter H does not contain any atoms of A. Let $C \subseteq A$ be the set of atoms of A and let $c := \bigvee C$. So, $\neg c \in H$ and more importantly $H{\restriction}\neg c$ is $A{\restriction}\neg c$-generic over W. Further, $A{\restriction}\neg c$ is a complete subalgebra of $B{\restriction}\neg c$, and by lemma 5.11, $d(A{\restriction}\neg c) \leq d(B{\restriction}\neg c) \leq d(B) = \aleph_0$. So, $A{\restriction}\neg c$ is a complete atomless Boolean algebra with a countable dense subset and hence isomorphic to B and forcing equivalent to Coh. As the partial order Coh is weakly homogeneous, any two extensions over an algebra isomorphic to B are elementarily equivalent. In particular, U and $W[G]$ are elementarily equivalent.

On the other hand, W is a model of the ground axiom and hence not elementarily equivalent to $W[G]$. Therefore, any downwards translation of a proposition letter is valid on all grounds $W \subsetneq U \subseteq W[G]$ iff it is valid in one of them. So, in the modal perspective, all these grounds look the same and can be treated as one reflexive point. Thus, the modal logic of grounds of $W[G]$ is the modal logic of a two element chain. $\square$

5.4.2 Finite Boolean Algebras as Ground Patterns

Another example we want to give here shows how we can generalize the idea of a model with exactly one proper ground to a model with an arbitrary finite Boolean algebra as its ground structure.

Proposition 5.15. *Let W be a countable transitive model of ZFC. In L^W define $\mathbb{P} := \prod_{k=1}^{n} \mathbb{S}_{\aleph_{2k}}$. Let G be $\mathbb{P}$-generic over L^W and call the projections to the components $G_1, \ldots, G_n$. Let U be a transitive model of ZFC with $L^W \subseteq U \subseteq L^W[G]$. Then, there is a set $A \subseteq \{1, \ldots, n\}$ such that $U = L^W[H]$ where $H := \prod_{k \in A} G_k$ is $\prod_{k \in A} \mathbb{S}_{\aleph_{2k}}$-generic over L^W.*

Proof. By lemma A.24, $G = G_1 \times \cdots \times G_n$. We use the following observation by Solovay (see [29, lemma 2.5]):

> Let M be a model of ZFC, let $\mathbb{Q}$ and $\mathbb{O}$ partial orders in M, let J and K be $\mathbb{Q}$-generic and $\mathbb{O}$-generic filters over M, respectively, and assume that J and K are mutually generic, i.e. J is $\mathbb{Q}$-generic over $M[K]$ and vice versa. Then $M[J] \cap M[K] = M$.

To prove this observation, suppose there are a $\mathbb{Q}$-name τ and an $\mathbb{O}$-name σ such that there is an $x \notin M$ with $x = \mathrm{val}(\tau, J) = \mathrm{val}(\sigma, K)$. By $\in$-induction on x we can assume that $x \subseteq M$. There must be a condition $\langle p, q\rangle \in \mathbb{Q} \times \mathbb{O}$ which forces $\tau = \sigma$. Here, we consider τ and σ as $\mathbb{Q} \times \mathbb{O}$ names which in fact can be done by exchanging all conditions r occurring in τ by the condition $\langle r, 1\rangle$ and similarly in σ. Now, suppose p does not decide for all $y \in M$ whether $\check{y} \in \tau$. Then, there is a $y \in M$ and two conditions p_1 and p_2 stronger than p s.t. $p_1 \Vdash \check{y} \in \tau$ and $p_2 \Vdash \check{y} \notin \tau$. Pick a condition $q_1 \leq q$ which decides whether $\check{y} \in \sigma$. W.l.o.g. assume $q_1 \Vdash \check{y} \in \sigma$. The condition $\langle p_2, q_1\rangle$ is stronger than $\langle p, q\rangle$ and forces $\tau \neq \sigma$, a contradiction.

Now, let U be a transitive model with $L^W \subseteq U \subseteq L$. Then define A by $k \in A$ if and only if $U \cap (L^W[G_k] - L^W) \neq \emptyset$ for any $k \in \{1, \dots, n\}$. In the proof of the minimality of extensions by $\mathbb{S}_{\aleph_{2k}}$ for such k, we have seen that G_k is constructible from any set in $L^W[G_k] - L^W$. So for any $k \in A$, we indeed get that $G_k \in U$. But then, it is clear that $U = L^W[\prod_{k \in A} G_k]$. □

Corollary 5.16. *Let $L^W[G]$ be as in proposition 5.15. The grounds of the model $L^W[G]$ are exactly all models of the form $L^W[\prod_{k\in A} G_k]$ for an arbitrary subset $A \subseteq \{1, \dots, n\}$. Further, for any two distinct subsets A and B of $\{1, \dots, n\}$ the models $L^W[\prod_{k\in A} G_k]$ and $L^W[\prod_{k\in B} G_k]$ are not elementarily equivalent.*

Proof. We know that any ground of $L^W[G]$ is of this form by proposition 5.15. Further, it is clear that $L^W[\prod_{k\in A} G_k]$ is a ground of $L^W[G]$ for any $A \subseteq \{1, \dots, n\}$. For distinct $A, B \subseteq \{1, \dots, n\}$, $L^W[\prod_{k\in A} G_k]$ and $L^W[\prod_{k\in B} G_k]$ are also distinguishable by first order formulas as for any set $A \subseteq \{1, \dots, n\}$ the fact that a model contains Sacks subsets over L of exactly the regular cardinals $\aleph_{2k}$ with $k \in A$ is expressible. □

Hence, the ground structure of $L^W[G]$ is isomorphic to $\mathcal{P}(\{1, \dots, n\})$ and no two grounds have the same first-order theory. So, we can conclude:

Corollary 5.17. *The modal logic of grounds of $L^W[G]$ is the modal logic of a finite Boolean algebra with n atoms.*

6 The Modal Logic of Forcing of Generic Multiverses

In the previous chapter, we presented various results about the modal logic of forcing MLF and the modal logic of grounds MLG, as well as the modal logic of forcing and of grounds of a model W, MLF(W) and MLG(W). However, in chapter 3, we have seen that the global multiverse is divided in various connected components, the generic multiverses, which do not reach each other via forcing.

We know that the modal logic of forcing MLF is S4.2. Or even stronger, we have seen that there are models of ZFC with modal logic of forcing S4.2. The prime example for such a model was L, the constructible universe, for which various independent families of buttons and switches have been proposed. However, there are many models W which are not a forcing extension of L, for example the bottomless model constructed by Reitz and used in the proof of theorem 5.9. Furthermore, models with a modal logic of forcing beyond S4.2 are known.

Given a model $W \in \mathcal{W}$ we call the modal logic of the general frame $(\mathrm{Mult}_W, \mathcal{F})$ the *modal logic of forcing of the generic multiverse* Mult_W and write $\mathrm{MLF}(\mathrm{Mult}_W)$. In other words, $\mathrm{MLF}(\mathrm{Mult}_W) = \bigcap_{U \in \mathrm{Mult}_W} \mathrm{MLF}(U)$. The question we answer in this chapter is whether it is possible that a generic multiverse does not contain a model with modal logic of forcing S4.2 or that the modal logic of forcing of the generic multiverse even exceeds S4.2.

To both these questions, the answer is negative. We prove that the modal logic of forcing of any generic multiverse is S4.2 because any countable transitive model W has a ground U with $\mathrm{MLF}(U) = \mathrm{S4.2}$. We will define an infinite family of buttons and switches and show that below every model there is a model where these buttons and switches form a sufficiently large independent family to show that the modal logic of forcing of this ground is S4.2.

6.1 The Buttons

In [8], Friedman, Fuchino and Sakai showed the independence of the following buttons over L (cf. p. 32): For each $n \in \omega$ let ψ_n state

$$\text{“there is an injection from } \aleph_{n+2}^L \text{ to } \mathcal{P}(\aleph_n^L)\text{”.}$$

We modify these buttons. Our definition of buttons refers to $\mathbb{M}$ instead of L which will allow us to find a ground in which all these buttons are unpushed below any model. In a generic multiverse, we can find models which coincide with the mantle up to an arbitrarily large rank using the strong downwards directedness of grounds. This fact is the key step in finding a ground in which the buttons we construct are unpushed. The use of the strong downwards directedness of grounds is interesting as for previous results on the modal logic of forcing or of grounds, e.g. lemma 5.5, the fact that two grounds of a model always have a common ground was sufficient.

Before we proceed we want to point out why the buttons ψ_n cannot be used in an arbitrary generic multiverses:

Proposition 6.1. *There is a generic multiverse in which all the buttons ψ_n, $n \in \omega$, are pushed in any model, and hence not independent.*

Proof. First, we build a forcing extension W of L in which all the sentences are true by adding all the wanted injections as in proposition 6.2 below. Then, we can rely on theorem 3.18 stating that any model of ZFC is the mantle of another model. If we take a model U such that $\mathbb{M}^U = W$, then it is clear that all models in Mult_U also contain these injections and hence all ψ_n are true in all models in Mult_U. $\square$

The fact that the mantle is uniformly definable and absolute within a generic multiverse guarantees that it behaves somewhat similar to L for our purpose. However, GCH does not hold in the mantle in general by theorem 3.18.

Recall the definition of the beth-sequence: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for any α, and $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$ for any limit ordinal λ. For any $n \in \omega$ let ϕ_n be the statement

$$\text{“there is an injection from } (\beth_{3n+2}^+)^{\mathbb{M}} \text{ to } \mathcal{P}(\mathcal{P}(\beth_{3n}^{\mathbb{M}}))\text{”}.$$

Each of these statements is a pure button. Once ϕ_n holds in some W, it also holds in any forcing extension W' as $\mathcal{P}(\mathcal{P}(\beth_n^{\mathbb{M}}))^W \subseteq \mathcal{P}(\mathcal{P}(\beth_n^{\mathbb{M}}))^{W'}$ and hence the injection in W also witnesses ϕ_n in W'.

The advantage of these new buttons is that the mantle is not absolute if we switch generic multiverses, but still it is constant throughout any particular generic multiverse. Hence, as long as we stay in a given generic multiverse, the statements of the buttons are interpreted in the same way in any model. As soon as we switch the multiverse, the statements refer to the cardinals in the new interpretation of the mantle. We will see how we can exploit this fact, given any model W, to find a ground of W in which all the buttons are unpushed in section 6.3.

That it is possible to force ϕ_n over any model of ZFC without affecting any other button ϕ_m with $m \neq n$ is shown in the next proposition.

Proposition 6.2. *Let W be a countable transitive model of* ZFC *and let $A := \{n \in \omega : \phi_n^W\}$. For any set $B \subseteq \omega$ in W with $A \subseteq B$ there is a forcing extension of W such that ϕ_n holds in the extension for all $n \in B$ and such that $\neg\phi_n$ holds for all $n \in \omega - B$.*

Proof. For any $n \in \omega - A$, we have that $\neg\phi_n$ holds in W. So, we have:

$$\left|(\beth_{3n+2}^+)^{\mathbb{M}}\right| > 2^{2^{\left|\beth_{3n}^{\mathbb{M}}\right|}} > 2^{\left|\beth_{3n}^{\mathbb{M}}\right|} > \left|\beth_{3n}^{\mathbb{M}}\right|.$$

Further, $\left|(\beth_{3n+2}^+)^{\mathbb{M}}\right| \leq (2^{2^{\left|\beth_{3n}^{\mathbb{M}}\right|}})^+$ as $\left|\beth_{3n+2}^{\mathbb{M}}\right| \leq 2^{2^{\left|\beth_{3n}^{\mathbb{M}}\right|}}$ and as every cardinal in W is also a cardinal in $\mathbb{M}^W$. So, we get $\left|(\beth_{3n+2}^+)^{\mathbb{M}}\right| = (2^{2^{\left|\beth_{3n}^{\mathbb{M}}\right|}})^+$. Finally, since $\left|(\beth_{3n+2}^+)^{\mathbb{M}}\right| \leq \left|\beth_{3n+2}^{\mathbb{M}}\right|^+$ we can conclude that $\left|\beth_{3n+2}^{\mathbb{M}}\right| = 2^{2^{\left|\beth_{3n}^{\mathbb{M}}\right|}}$ and $\left|(\beth_{3n+2}^+)^{\mathbb{M}}\right| = \left|\beth_{3n+2}^{\mathbb{M}}\right|^+$. In summary,

$$\left|\beth_{3n+2}^{\mathbb{M}}\right|^+ = \left|(\beth_{3n+2}^+)^{\mathbb{M}}\right| > \left|\beth_{3n+2}^{\mathbb{M}}\right| = 2^{2^{\left|\beth_{3n}^{\mathbb{M}}\right|}} > 2^{\left|\beth_{3n}^{\mathbb{M}}\right|} > \left|\beth_{3n}^{\mathbb{M}}\right|.$$

That also means that $\left|\beth_{3n}^{\mathbb{M}}\right|^{+} < \left|(\beth_{3n+2}^{+})^{\mathbb{M}}\right| = \left|(\beth_{3n+2}^{+})^{\mathbb{M}} \times \mathcal{P}((\beth_{3n})^{\mathbb{M}})\right|$. Hence the following definition of partial orders has non-trivial factors for all $n \in B - A$ as for such n we have that $\neg\phi_n$ holds in W:

$$\mathbb{P}_n := \begin{cases} \mathrm{Fn}((\beth_{3n+2}^{+})^{\mathbb{M}} \times \mathcal{P}((\beth_{3n})^{\mathbb{M}}), 2, \left|\beth_{3n}^{\mathbb{M}}\right|^{+}) & \text{, if } n \in B - A, \\ 1 & \text{, otherwise.} \end{cases}$$

We claim that forcing with the full support product $\mathbb{P} := \prod_{n<\omega} \mathbb{P}_n$ leads to an extension in which exactly the buttons ϕ_n with $n \in B$ are pushed. So, let $G = \prod_{n<\omega} G_n$ be a $\mathbb{P}$-generic filter over W. By lemma 2.2, the function $\bigcup G_n$ allows us to define an injection from $(\beth_{3n+2}^{+})^{\mathbb{M}}$ to $\mathcal{P}(\mathcal{P}((\beth_{3n})^{\mathbb{M}}))$ as $\left|\mathcal{P}((\beth_{3n})^{\mathbb{M}})\right| \geq \left|\beth_{3n}^{\mathbb{M}}\right|^{+}$.

So, ϕ_n holds in $W[G]$ for all $n \in B$. Note that for $n \in A$ the button ϕ_n is pushed in $W[G]$ because it was true in W and the buttons are pure buttons.

Now, we want to show that for any $n \in \omega - B$ the button ϕ_n is still unpushed in $W[G]$. For such an n the product $\mathbb{P}$ factors as $\mathbb{P}_{<n} \times \mathbb{P}_{>n}$ where $\mathbb{P}_{<n} := \prod_{k<n} \mathbb{P}_k$ and $\mathbb{P}_{>n} := \prod_{k>n} \mathbb{P}_k$.

First, we show that forcing with $\mathbb{P}_{>n}$ does not add an injection from $(\beth_{3n+2}^{+})^{\mathbb{M}}$ to $\mathcal{P}(\mathcal{P}(\beth_{3n}^{\mathbb{M}}))$. We know that each component of $\mathbb{P}_{>n}$ is $\left|\beth_{3m}^{\mathbb{M}}\right|^{+}$-closed for some $m > n$ as this successor cardinal is regular (cf. [22, lemma 6.13]). So in particular, each component is $|\beth_{3(n+1)}^{\mathbb{M}}|^{+}$-closed hence the product $\mathbb{P}_{>n}$ is also $|\beth_{3(n+1)}^{\mathbb{M}}|^{+}$-closed (cf. [20, lemma 15.12]). We know that $|\beth_{3(n+1)}^{\mathbb{M}}| > |\mathcal{P}((\beth_{3n})^{\mathbb{M}})|$ as $\neg\phi_n$ holds in W. So, forcing with $\mathbb{P}_{>n}$ adds no new subset of $\mathcal{P}((\beth_{3n})^{\mathbb{M}})$.

Thus, the only way how forcing with $\mathbb{P}_{>n}$ could add an injection is by collapsing $|\beth_{3n+2}^{\mathbb{M}}|^{+} = |(\beth_{3n+2}^{+})^{\mathbb{M}}|$. But as the forcing is $|\beth_{3(n+1)}^{\mathbb{M}}|^{+}$-closed it preserves cardinals $\leq |\beth_{3(n+1)}^{\mathbb{M}}|^{+}$ and in particular, it does not collapse $|\beth_{3n+2}^{\mathbb{M}}|^{+}$.

Now, we show that the factor $\mathbb{P}_{<n}$ does not add such an injection. We work in $W^{\mathbb{P}_{>n}}$. By the $|\beth_{3(n+1)}^{\mathbb{M}}|^{+}$-closedness of $\mathbb{P}_{>n}$ and the fact that we know that $|\beth_{3n}^{\mathbb{M}}|^{+} < |\beth_{3(n+1)}^{\mathbb{M}}|^{+}$ we know that

$$\begin{aligned} &\mathrm{Fn}((\beth_{3m+2}^{+})^{\mathbb{M}} \times \mathcal{P}((\beth_{3m})^{\mathbb{M}}), 2, \left|\beth_{3m}^{\mathbb{M}}\right|^{+})^{W} \\ =&\mathrm{Fn}((\beth_{3m+2}^{+})^{\mathbb{M}} \times \mathcal{P}((\beth_{3m})^{\mathbb{M}}), 2, \left|\beth_{3m}^{\mathbb{M}}\right|^{+})^{W^{\mathbb{P}_{>n}}} \end{aligned}$$

for $m < n$.

Let $m < n$ be the greatest number less than n in $B - A$. If there is no such number then $\mathbb{P}_{<n}$ is trivial. Each factor of the finite product $\mathbb{P}_{<n}$ has the $(2^{<\left|\beth_{3m}^{\mathbb{M}}\right|^{+}})^{+}$-chain condition. As $\neg\phi_m$ holds in $W^{\mathbb{P}_{>n}}$ because it even still holds in $W^{\mathbb{P}_{>m}}$, we have that

$$(2^{<\left|\beth_{3m}^{\mathbb{M}}\right|^{+}})^{+} = (2^{\left|\beth_{3m}^{\mathbb{M}}\right|})^{+} < \left|\beth_{3m+2}^{\mathbb{M}}\right| \leq \left|(\beth_{3n})^{\mathbb{M}}\right|.$$

So, $\mathbb{P}_{<n}$ has the $\left|(\beth_{3n})^{\mathbb{M}}\right|$-c.c. and forcing with $\mathbb{P}_{<n}$ does thus not collapse $\left|(\beth_{3n+2}^{+})^{\mathbb{M}}\right|$. Further, we see that $\mathbb{P}_{<n}$ has at most cardinality

$$(\left|\beth_{3m+2}^{\mathbb{M}}\right|^{+})^{<\left|\beth_{3m}^{\mathbb{M}}\right|^{+}} \leq (2^{\left|\beth_{3n}^{\mathbb{M}}\right|})^{\left|\beth_{3n}^{\mathbb{M}}\right|}.$$

Hence, there are at most $(2^{|\beth_{3n}^{\mathbb{M}}|})^{|\beth_{3n}^{\mathbb{M}}|} \leq 2^{|\beth_{3n}^{\mathbb{M}}|}$ many anti-chains. So, forcing with $\mathbb{P}_{<n}$ adds at most $(2^{|\beth_{3n}^{\mathbb{M}}|})^{|\beth_{3m}^{\mathbb{M}}|} = 2^{|\beth_{3n}^{\mathbb{M}}|}$ new subsets to $|\beth_{3m}^{\mathbb{M}}|$ by a nice name argument (cf. [22], VII, Lemma 5.12): We consider the canonical name $|\check{\beth}_{3m}^{\mathbb{M}}|$ which has size $|\beth_{3m}^{\mathbb{M}}|$ and then each new subset of $|\beth_{3m}^{\mathbb{M}}|$ has a nice name of the form $\bigcup\{\{\pi\} \times A_\pi\}$ where $\pi \in \mathrm{dom}(|\check{\beth}_{3m}^{\mathbb{M}}|)$ and A_π is an anti-chain and there are at most $(2^{|\beth_{3n}^{\mathbb{M}}|})^{|\beth_{3m}^{\mathbb{M}}|}$ such nice names. Hence, forcing with $\mathbb{P}_{<n}$ does not increase the size of $2^{|\beth_{3n}^{\mathbb{M}}|}$. Further, it also adds at most

$$(2^{|\beth_{3n}^{\mathbb{M}}|})^{(2^{|\beth_{3n}^{\mathbb{M}}|})} = 2^{2^{|\beth_{3n}^{\mathbb{M}}|}}$$

new subsets to $2^{|\beth_{3n}^{\mathbb{M}}|}$. So, it does not increase the size of $\mathcal{P}(\mathcal{P}(\beth_{3n}^{\mathbb{M}}))$. Therefore, $\mathbb{P}_{<n}$ also adds no injection from $(\beth_{3n+2}^{+})^{\mathbb{M}}$ to $\mathcal{P}(\mathcal{P}(\beth_{3n}^{\mathbb{M}}))$ and ϕ_n stays unpushed after forcing with $\mathbb{P}$ for $n \in \omega - B$. □

6.2 The Switches

Now, we add switches to our new family of buttons. Let α be the ordinal number defined by the formula $(\beth_\omega = \aleph_\alpha)^{\mathbb{M}}$. We define the switches ψ_n for $n \in \omega$ as

$$2^{\aleph_{\alpha+n+1}} = \aleph_{\alpha+n+2}.$$

We show that for any $n \in \omega$ the collection $\{\psi_k : k < n\}$ is an independent collection of switches. Further, we will see that it is possible to obtain any constellation of these switches over any model of set theory by $\aleph_\alpha$-closed forcing. Such a forcing does not affect the status of the buttons defined above because $|\aleph_n^{\mathbb{M}}| < \aleph_\alpha$ for all $n \in \omega$.

First, we will show that we can force $\bigwedge_{k<n} \psi_k$ over any model of ZFC:

Lemma 6.3. *Let W be a model of* ZFC*. Then there is an $\aleph_\alpha$-closed partial order $\mathbb{P}$ such that for any $\mathbb{P}$-generic filter G we have $W[G] \vDash \bigwedge_{k<n} \psi_k$.*

Proof. The idea is to simply collapse $2^{\aleph_{\alpha+k+1}}$ to $\aleph_{\alpha+k+2}$ step by step. Assume we already obtained an extension W' by $\aleph_\alpha$-closed forcing in which $\bigwedge_{k<m} \psi_k$ holds. Then we force over W' with

$$\mathbb{P}_m := Fn(2^{\aleph_{\alpha+m+1}}, \aleph_{\alpha+m+2}, \aleph_{\alpha+m+2}).$$

We already know that this partial order is $\aleph_{\alpha+m+2}$-closed. Further, it adds a surjection from $(2^{\aleph_{\alpha+m+1}})^{W'}$ to $\aleph_{\alpha+m+2}^{W'}$. Since this forcing preserves cardinals $\leq \aleph_{\alpha+m+2}$, it is enough to show that it does not increase the size of $2^{\aleph_{\alpha+m+1}}$. But as the forcing is $\aleph_{\alpha+m+2}$-closed it does not add any bounded subsets to $\aleph_{\alpha+m+2}$, so in particular, it does not change $2^{\aleph_{\alpha+m+1}^{W}}$. Therefore, if G is $\mathbb{P}_m$-generic over W' we have that in $W'[G]$ the conjunction $\bigwedge_{k<m+1} \psi_k$ holds.

The two step iteration of two $\aleph_\alpha$-closed forcings is $\aleph_\alpha$-closed (see [20], Lemma 16.7). So, $W'[G]$ can be obtained form W by $\aleph_\alpha$-closed forcing. □

Now, we will show that we can reach any finite pattern of these switches from a model in which they are all turned on by $\aleph_\alpha$-closed forcing.

Lemma 6.4. *Suppose $W \vDash \bigwedge_{k<n} \psi_k$ and let $A \subseteq n$. Then there is an extension of W by $\aleph_\alpha$-closed forcing in which $\bigwedge_{k \in n-A} \psi_k \wedge \bigwedge_{k \in A} \neg\psi_k$ holds.*

Proof. We claim that forcing over W with

$$\mathbb{P} := \prod_{k \in A} \mathrm{Fn}(\aleph_{\alpha+k+3} \times \aleph_{\alpha+k+1}, 2, \aleph_{\alpha+k+1})$$

leads to the desired model. Let G be $\mathbb{P}$-generic over W. First, we show that for $k \in n - A$ we have that $W[G] \vDash \psi_k$. As in the proof of proposition 6.2, $\mathbb{P}$ factors as $\mathbb{P}_{<k} \times \mathbb{P}_{>k}$. Again, we have that $\mathbb{P}_{>k}$ is $\aleph_{\alpha+k+2}$-closed and hence preserves cardinals $\leq \aleph_{\alpha+k+2}$ and also $2^{\aleph_{\alpha+k+1}}$. Hence, $W^{\mathbb{P}_{>k}} \vDash \psi_k$. Further, $\mathbb{P}_{<k}^W = \mathbb{P}_{<k}^{W[G^{>k}]}$. So, we can work in $W[G^{>k}]$ and show that forcing with $\mathbb{P}_{<k}$ does not change GCH at $\aleph_{\alpha+k+1}$. We know that $\mathbb{P}_{<k}$ has the $(2^{<\aleph_{\alpha+k}})^+$-c.c. and since GCH holds at $\aleph_\alpha, \dots, \aleph_{\alpha+k}$ we have $(2^{<\aleph_{\alpha+k}})^+ \leq \aleph_{\alpha+k+1}$. So, $\mathbb{P}_{<k}$ preserves cardinals $\geq \aleph_{\alpha+k+1}$. Thus, it is enough to show that it does not increase the size of $2^{\aleph_{\alpha+k+1}}$. Again, we use a nice name argument. $\mathbb{P}_{<k}$ has at most cardinality

$$(\aleph_{\alpha+k+2})^{<\aleph_{\alpha+k}} = (2^{\aleph_{\alpha+k+1}})^{<\aleph_{\alpha+k}} = \aleph_{\alpha+k+2}.$$

So, there are at most

$$(\aleph_{\alpha+k+2})^{\aleph_{\alpha+k+1}} = \aleph_{\alpha+k+2}$$

anti-chains, and hence it adds at most

$$(\aleph_{\alpha+k+2})^{\aleph_{\alpha+k+1}} = \aleph_{\alpha+k+2}$$

many subsets to $\aleph_{\alpha+k+1}$. Therefore, ψ_n holds in $W[G]$.

For $k \in A$, we know that all factors of $\mathbb{P}$ except for $\mathrm{Fn}(\aleph_{\alpha+k+3} \times \aleph_{\alpha+k+1}, 2, \aleph_{\alpha+k+1})$ preserve $\aleph_{\alpha+k+1}$. Further, $2^{\aleph_{\alpha+k+1}} = \aleph_{\alpha+k+2}$ by the above argument. Finally, forcing with $\mathrm{Fn}(\aleph_{\alpha+k+3} \times \aleph_{\alpha+k+1}, 2, \aleph_{\alpha+k+1})$ adds an injection from $\aleph_{\alpha+k+3}$ to $2^{\aleph_{\alpha+k+1}}$ and hence $\neg\psi_k$ holds in $W[G]$. $\square$

So, we can always arrange any finite pattern of the switches over any model of ZFC without affecting the state of the buttons defined above.

6.3 A Ground with Modal Logic of Forcing S4.2

Let W be a countable transitive model of ZFC. Now, we show that there is a ground of W in which all buttons from section 6.1 are unpushed such that this ground has modal logic of forcing $S4.2$ by theorem 4.18. We proceed in two steps. First, we show that for any natural number n we can find a ground in which ϕ_n doesn't hold. As the buttons are pure, this means that ϕ_n also doesn't hold in any further ground. Then, we go to a common ground of all these grounds to find a ground where all these buttons are unpushed.

Lemma 6.5. *Let W be a model of* ZFC *and let $n \in \omega$. There is a ground W' of W where $\neg\phi_n$ holds.*

Proof. We have to find a ground in which there is no injection from $(\beth^+_{3n+2})^{\mathbb{M}}$ to $\mathcal{P}(\mathcal{P}(\beth^{\mathbb{M}}_{3n}))$. If W is not itself such a ground, let

$$\begin{aligned}\mathcal{I} = \{f \in \mathcal{P}((\beth^+_{3n+2})^{\mathbb{M}} \times \mathcal{P}(\mathcal{P}(\beth^{\mathbb{M}}_{3n}))) :\\ f \text{ is an injective function with domain } (\beth^+_{3n+2})^{\mathbb{M}}\}.\end{aligned}$$

For each $f \in \mathcal{I}$, we know that $f \notin \mathbb{M}$ holds in W. By the definition of the mantle this means that $\exists r \neg\phi(f, r)$ holds in W where $\phi(x, r)$ is the formula from theorem 3.2. By looking only at the witnesses of minimal degree and using the axiom of choice, we know that there is a set R such that for each $f \in \mathcal{I}$ there is an $r \in R$ such that $\neg\phi(f, r)$ holds in W. In other words, $f \notin W_r$ where W_r is the ground $\{x : \phi(x, r)\}$ of W.

By the strong downwards directedness of grounds, we know that there is a ground W' of W which is also a ground of W_r for all $r \in R$. This W' hence does not contain any element of $\mathcal{I}$. Since being an injection from $(\beth^+_{3n+2})^{\mathbb{M}}$ to $\mathcal{P}(\mathcal{P}(\beth^{\mathbb{M}}_{3n}))$ is absolute between W' and W, there is no such injection in W' and hence $\neg\phi_n$ holds in W'. $\square$

Proposition 6.6. *Let W be a model of* ZFC. *Then there is a ground W' of W where all the buttons ϕ_n with $n \in \omega$ are unpushed.*

Proof. By the previous lemma, we know that $\forall n < \omega \exists r (\neg{\phi_n}^{W_r})$ holds in W. So, we can again find a set R such that for each n there is an $r \in R$ such that $\neg\phi_n$ holds in W_r. By the strong downwards directedness, there is a ground W' of W which is also a ground of W_r for all $r \in R$. Since the buttons are pure, $\neg\phi_n$ holds in W' for all $n \in \omega$. $\square$

Corollary 6.7. *For each $n \in \omega$, each model of* ZFC *has a ground with an independent family of infinitely many upwards buttons and n upwards switches.*

By theorem 4.18 this leads to the main result of this chapter:

Theorem 6.8. *Each model W of* ZFC *has a ground W' with* $\mathrm{MLF}(W') = \mathrm{S4.2}$. *Therefore, the modal logic of forcing of any generic multiverse is* S4.2.

As an immediate consequence we also get the following:

Corollary 6.9. *Let W be a countable transitive model of* ZFC. *If W satisfies the ground axiom, then* $\mathrm{MLF}(W) = \mathrm{S4.2}$.

7 The Modal Logic of Grounds of Generic Multiverses

In this chapter, we switch directions and try to analyze the modal logic of grounds in a generic multiverse. So, we try to determine the modal logic of the frame $(\mathrm{Mult}_W, \mathcal{G})$, the *modal logic of grounds of the generic multiverse* Mult_W, for a given $W \in \mathcal{W}$. As for the modal logic of forcing, we again have that $\mathrm{MLG}(\mathrm{Mult}_W) = \bigcap_{U \in \mathrm{Mult}_W} \mathrm{MLG}(U)$.

One main observation is that models with a bedrock certainly do not have any downwards switches as the truth of sentences cannot be changed anymore by going to a ground once we reached the bedrock. Nevertheless, we can provide an infinite uniform family of downwards buttons which all are unpushed in some model in any generic multiverse. This allows us to conclude that the modal logic of that model is contained in S4.2Top. As the Top-axiom is valid in any model in a generic multiverse with a bedrock, this is also the best general upper bound possible, and the modal logic of generic multiverses with a bedrock is exactly S4.2Top.

7.1 Upper Bound

In this section, we prove that the modal logic of grounds of any generic multiverse is contained in S4.2Top.

Let W be a countable transitive model of ZFC for the rest of this section. We will first go to a ground of W which has the same cardinals and power sets of cardinals up to a certain height as the mantle. Then, we add generic subsets to $\aleph_0$ many of these cardinals and define downwards buttons, each stating that there is no $\mathbb{M}$-generic subset of one of these cardinals. After showing that these buttons are independent, theorem 5.3 gives the desired result.

We begin by recursively defining a sequence of cardinals similar to the $\beth$-sequence which gives us the cardinals of which we add generic subsets:

Definition 7.1. Recursively, we define

$$\begin{aligned} \Gamma_0 &= \aleph_0 \ , \\ \Gamma_{\alpha+1} &= (2^{\Gamma_\alpha})^+ \text{ for } \alpha \in \mathrm{Ord}, \\ \Gamma_\lambda &= \bigcup_{\alpha<\lambda} \Gamma_\alpha \text{ for all limit ordinals } \lambda. \end{aligned}$$

The recursive definition ensures that there is a formula $\phi(x, \alpha)$ defining Γ_α with the parameter α.

Lemma 7.2. *There is a ground U of W such that in U*

1. *for all $\kappa \leq \Gamma_\omega^{\mathbb{M}}$ we have that κ is a cardinal iff κ is a cardinal in $\mathbb{M}$,*
2. *$\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^{\mathbb{M}}$ for all cardinals $\kappa \leq \Gamma_\omega^{\mathbb{M}}$, and hence $\Gamma_n = \Gamma_n^{\mathbb{M}}$ for all $n \leq \omega$,*
3. *the partial order $\mathrm{Fn}(\Gamma_n, 2, \Gamma_n) = \mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^{\mathbb{M}}$, and*

4. $\mathcal{P}(\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)) = \mathcal{P}(\mathrm{Fn}(\Gamma_n, 2, \Gamma_n))^{\mathbb{M}}$

Proof. The proof is similar to the proof of lemma 6.5. First, for each cardinal $\kappa \leq \Gamma_\omega^{\mathbb{M}}$ of the mantle we find a ground W_κ of W in which κ is a cardinal: For each injection $f : \kappa \to \alpha$ with $\alpha < \kappa$ in W, we find a ground without this injection as there is no such injection in $\mathbb{M}$. In a common ground of these set-many grounds, κ is a cardinal.

Likewise, in a common ground W' of all W_κ the cardinals up to $\Gamma_\omega^{\mathbb{M}}$ are the same as in the mantle.

Further, starting from W' for each $\kappa \leq \Gamma_\omega^{\mathbb{M}}$ we find a ground W'_κ in which $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^{\mathbb{M}}$: For each subset of κ which is not in $\mathbb{M}$, there is a ground of W' which does not contain this subset. A common ground W'_κ of all these grounds satisfies $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^{\mathbb{M}}$.

Conditions 3. and 4. can be achieved in the same way and so we find the desired ground U. □

Note that all the conditions for the ground U also hold in any ground of U. More generally, this proof technique allows us to find e.g. a ground of W in which $V_\kappa = V_\kappa^{\mathbb{M}}$, or for any set X in W with $X \cap \mathbb{M} = \emptyset$, to find a ground W' with $W' \cap X = \emptyset$.

Let $\phi(n)$ be the statement

"There is no $\mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$-generic filter over $\mathbb{M}$."

In the ground U given by the lemma,

$$\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^{\mathbb{M}} = \mathrm{Fn}(\Gamma_n, 2, \Gamma_n)$$

and

$$\mathcal{P}(\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)) = \mathcal{P}(\mathrm{Fn}(\Gamma_n, 2, \Gamma_n))^{\mathbb{M}}.$$

So in particular, $\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)$ has the same dense sets in U and $\mathbb{M}$ and therefore a subset $G \subseteq \mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^{\mathbb{M}}$ is $\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^{\mathbb{M}}$-generic over $\mathbb{M}$ if and only if it is $\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^U$-generic over U. The same holds for any ground of U.

Lemma 7.3. *Let U be as in lemma 7.2. Let $\mathbb{P}$ be the partial order*

$$\prod_{n\in\omega} \mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}} = \prod_{n\in\omega} \mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^U,$$

let G be $\mathbb{P}$-generic over U and let $V := U[G]$. Then, the family $\{\phi(n) : n \in \omega\}$ is an independent family of downwards buttons in V.

Proof. Clearly, all buttons are unpushed in V. Let V' be a ground of V and let

$$A = \{n \in \omega : \neg\phi(n)^{V'}\}.$$

Further, let U' be a common ground of V' and U. So, V' is a forcing extension of U' and for each $n \in A$ it contains an $Fn(\Gamma_{n+1}, 2, \Gamma_{n+1})^{U'}$-generic filter over U' by our previous observation. We show that these filters are mutually generic. So, let $n < m$ and let G be $\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^{U'}$-generic filter over $U'$and H an $\mathrm{Fn}(\Gamma_{m+1}, 2, \Gamma_{m+1})^{U'}$-generic filter over U'. Since Γ_{m+1} is regular (in U') we know that $Fn(\Gamma_{m+1}, 2, \Gamma_{m+1})^{U'}$ is Γ_{m+1}-closed. The partial order $\mathrm{Fn}(\Gamma_n, 2, \Gamma_n)^{U'}$ has size

$$\Gamma_{n+1}^{<\Gamma_{n+1}} = ((2^{\Gamma_n})^+)^{2^{\Gamma_n}} \leq (2^{2^{\Gamma_n}})^{2^{\Gamma_n}} = 2^{2^{\Gamma_n}} < \Gamma_{n+2} \leq \Gamma_{m+1}.$$

So, in $U'[H]$ we still have that

$$\mathcal{P}(\mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}) = \mathcal{P}(Fn(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}})^{U'}$$

by theorem A.19. Thus, G intersects all dense subsets of the partial order $\mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$. So, we get that G is $\mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$-generic over $U'[H]$. Therefore, the filter $G \times H$ is $\mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}} \times \mathrm{Fn}(\Gamma_{m+1}, 2, \Gamma_{m+1})^{\mathbb{M}}$-generic over U' by theorem A.25 and so G and H are mutually generic.

Thus, V' contains a $\prod_{n \in A} \mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$-generic filter K over U'. So, $U'[K]$ is a ground of V' and there is an $\prod_{n \in B} \mathrm{Fn}(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$-generic filter K' in $U'[K]$ for any set $B \subseteq A$ in U' and $U'[K']$ is a ground of V'. Clearly, $\neg\phi(n)$ holds in $U'[K']$ for all $n \in B$.

It remains to show that $\phi(n)$ holds in $U'[K']$ for $n \in \omega - B$. Fix such an n. We know that an $Fn(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$-generic filter L over $\mathbb{M}$ and hence over U' is generic over all projections of K' and hence also over the product, i.e. L is $Fn(\Gamma_{n+1}, 2, \Gamma_{n+1})^{\mathbb{M}}$-generic over $U'[K']$ and hence not in $U'[K']$. $\square$

We can summarize those results as follows:

Theorem 7.4. *For any model W of* ZFC*, there is a model U in the generic multiverse of W in which there is a uniform family of infinitely many independent buttons. So,* $\mathrm{MLG}(U) \subseteq \mathrm{S4.2Top}$ *and hence* $\mathrm{MLG}(\mathrm{Mult}_U) \subseteq \mathrm{S4.2Top}$.

7.2 On the Connection Between the Existence of a Bedrock and the Downwards Validity of Top

As discussed above, if Mult_W has a bedrock, the Top-axiom is downwards valid in every model in the generic multiverse as the corresponding frame is topped. On the one hand, that means that S4.2Top is the best general upper bound for the modal logic of grounds of a generic multiverse. In particular, the following theorem is immediate from the previous section.

Theorem 7.5. *The modal logic of grounds in any generic multiverse with bedrock is exactly* S4.2Top.

On the other hand, we have seen that there are countable transitive models of ZFC with modal logic of grounds S4.2 in section 5.3. Nevertheless, these results raise the question whether the Top-axiom is only downwards valid in models with a bedrock.

If there is a model W of ZFC which has no bedrock but all whose grounds are elementarily equivalent, then the Top-axiom is downwards valid in any model in the generic multiverse of W as in W the formulas $\Box p \leftrightarrow p \wedge \Box\neg p \leftrightarrow \neg p$ is downwards valid. In fact, even a weaker condition is sufficient for the validity of Top: If W is a model of ZFC such that for any formula ϕ there is a ground of W below which the truth value of ϕ does not change anymore then Top is downwards valid. So, the validity of the Top-axiom is equivalent to the condition that there are no downwards switches. In terms of the McKinsey axiom, this becomes even more obvious: $\Box\Diamond p \to \Diamond\Box p$ is equivalent to $\Box\Diamond p \to \neg\Box\Diamond\neg p$ by duality. The validity of the latter formula then just states that anything that satisfies the first half of the definition of a switch does not satisfy the second half.

7.2.1 On models with class-many grounds which are all elementarily equivalent

We first want to investigate the possibility of the existence model without a bedrock, but with only elementarily equivalent grounds. Recall our example from section 5.4.1. There, we started with a countable transitive model W of ZFC + GA and added a Cohen real g. Then, the model $W[g]$ has the bedrock W. We showed that $W[g]$ has infinitely many grounds and that all of them except for the bedrock W are elementarily equivalent. As a bedrock satisfies GA, it cannot be elementarily equivalent to any other model in its generic multiverse. But, it is conceivable that a model without a bedrock has only elementarily equivalent grounds.

The following proposition provides a condition under which this cannot be the case:

Proposition 7.6. *Let W be a model of* ZFC *without bedrock. Further, assume that the ordinals which are first-order definable without parameters in $\mathbb{M}^W$ are unbounded. Then the grounds of W are not all elementarily equivalent.*

Proof. Let U be a ground of W. Let α be the least ordinal such that $V_\alpha \neq V_\alpha^{\mathbb{M}}$ holds in U. Since $U \neq \mathbb{M}^U = \mathbb{M}^W$ as W does not have a bedrock, there is such an α. Now, let $\beta > \alpha$ be an ordinal which is definable in $\mathbb{M}^W = \mathbb{M}^U$. Then, the sentence $V_\beta = V_\beta^{\mathbb{M}}$ can be expressed as a first-order sentence without parameters which is not true in U but true in some ground of U. Since the rank and the mantle are absolute through the generic multiverse, we can simply proceed as in earlier proofs: For each $x \in V_\beta^U - \mathbb{M}^U$ let U_x be a ground of U with $x \notin U_x$. Since V_β^U is a set in U, there is a common ground of all the U_x by the strong downwards directedness of grounds. And in this ground $V_\beta = V_\beta^{\mathbb{M}}$ holds by the mentioned absoluteness of the occurring notions. $\Box$

We can even take this a bit further. Suppose that W is a model without a bedrock such that all its grounds are elementarily equivalent. Now, define the class

$$A := \{\alpha \in \mathrm{Ord} : \text{ there is a ground } W_r \text{ of } W \text{ such that } \alpha \text{ is the least ordinal with } V_\alpha^{W_r} \neq V_\alpha^{\mathbb{M}}\}.$$

By the argument in the proof of the previous proposition, we get that A is a proper class. As the mantle $\mathbb{M}^W$ is definable without parameters, we have that $\phi(\alpha)^{\mathbb{M}^W} \leftrightarrow \phi(\beta)^{\mathbb{M}^W}$ for all $\alpha, \beta \in A$

and for all formulas ϕ. Otherwise, the corresponding grounds U and U' in which α and β, respectively, are the least ordinals with $V_\alpha^U \neq V_\alpha^{\mathbb{M}}$ and $V_\alpha^{U'} \neq V_\alpha^{\mathbb{M}}$ would not be elementarily equivalent. So in particular, each element of A is bigger than any ordinal definable in the mantle. That also means that $V_\alpha^W = V_\alpha^{\mathbb{M}^W}$ for all α smaller than an ordinal definable in the mantle.

7.2.2 Certain class forcings cannot create models without bedrock and without downwards switches

So far, models of ZFC without bedrock have been constructed via class forcing. The standard way in which also Reitz showed that the non-existence of a bedrock is consistent is to take a model of the ground axiom W and to conduct a class forcing with a progressively closed product. We will show that such a construction will always allow us to define downwards switches in the extension showing that the Top axiom is not valid there.

The following definition makes precise which constructions we cover.

Definition 7.7. Let W be a countable transitive model of ZFC. Let $\langle P_\alpha : \alpha \in \mathrm{Ord}\rangle$ be a definable class in W. Assume that $\mathbb{P} = \prod_{\alpha\in\mathrm{Ord}} P_\alpha$ is a progressively closed product in W and that P_α is a forcing notion which only generates non-trivial extensions, i.e. it satisfies the splitting condition of lemma A.4, for all α (hence $\mathbb{P}$ is a proper class). Then we call $\mathbb{P}$ a *reasonable progressively closed product*.

Theorem 7.8. *Let W be a countable transitive model of* $\mathrm{ZFC} + \mathrm{GA}$ *and* $\mathbb{P} = \prod_{\alpha\in\mathrm{Ord}} P_\alpha$ *be a reasonable progressively closed product. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over W, then there are arbitrarily large finite families of independent downwards switches for $W[G]$. Therefore,* $\mathrm{MLG}(W[G]) \subseteq \mathrm{S5}$.

Proof. Note that for arbitrarily large regular δ there is a γ such that P_α is δ^+-closed for all $\alpha > \gamma$ as whenever $\mathbb{P}$ factors as $\mathbb{P}_1 \times P_2$ where P_2 is a set then P_2 completely embeds into $\prod_{\alpha\leq\gamma} P_\alpha$ for some γ.

First, in W, we recursively define a subsequence $\langle Q_\beta : \beta \in \mathrm{Ord}\rangle$ of partially ordered sets such that for all $\beta < \alpha$ the partial order Q_α is $(2^{<|Q_\beta|})^+$-closed. We start by putting $Q_0 = P_0$. Then, whenever Q_β has been defined for all $\beta < \alpha$ let $\kappa = (\sup\{2^{<|Q_\beta|} : \beta < \alpha\})^+$. Let γ be the least ordinal such that P_γ is κ-closed. Such a γ exists as $\mathbb{P}$ is progressively closed. Then simply take $Q_\alpha := P_\gamma$.

Fix a partition of the class of ordinals in ω-many disjoint proper classes $A_0, A_1, \ldots$ which is definable in L. We define the sentence $\phi(n)$ to express:

"The least α such that there is a Q_α-generic filter over $\mathbb{M}$ is in A_n."

We start by showing that $\mathbb{M}^{W[G]} = W$. For arbitrarily large regular δ the partially ordered class factors as $\mathbb{P}_1 \times P_2$ where P_2 is a set and $\mathbb{P}_1$ is δ-closed. So there is a class H definable in $W[G]$ which is a $\mathbb{P}_1$-generic filter over W. Then, $W[G] = W[H][g]$ for some P_2-generic filter g over

$W[H]$. So, $W[H]$ is a ground of $W[G]$ and as $\mathbb{P}_1$ is δ-closed the bounded subsets of δ in $W[H]$ and W are the same. As δ can be chosen arbitrarily large, we conclude that $\mathbb{M}^{W[G]} \subseteq W$. On the other hand, $\mathbb{M}^{W[G]} \subsetneq W$ would contradict the ground axiom in W as then there would be a ground U of $W[G]$ with $U \subsetneq W$ which would also be a proper ground of W.

We claim that the sentences $\phi(n)$, $n \in \omega$, have the property that in any ground of $W[G]$ exactly one of the sentences is true and that in any ground one can switch to the truth of any other of the sentences by going to a ground of this ground.

To check that exactly one of the sentences is true in any ground U of $W[G]$ requires to show that there always is an α such that there is a Q_α-generic filter over W in U. We show that for any $n \in \omega$ there are class many $\alpha \in A_n$ such that there is a Q_α-generic filter over W in U. It is enough to show that there is a $\gamma \in Ord$ such that U contains a tail $G^{>\gamma}$ of the filter G, where $G^{>\gamma}$ is the projection to the components bigger than γ, i.e. the projection onto $\prod_{\alpha>\gamma} P_\alpha$. But this is exactly the statement of lemma 5.8 which we proved earlier.

Now, let U be a ground of $W[G]$ and let n be a natural number. We want to find a ground of U in which $\phi(n)$ holds. We know that U has a ground of the form $W[G^{>\gamma}]$ for some $\gamma \in \mathrm{Ord}$. But now we can simply take the least ordinal η in A_n such that there is a P_α which is forcing equivalent to Q_η with $\alpha > \gamma$. Then for all $\zeta < \eta$ we let $D_\zeta = \{\alpha : P_\alpha \text{ adds a } Q_\zeta\text{-generic filter}\}$. Since there is a $\delta > 2^{<|Q_\zeta|}$ such that only set many of the partial orders P_α are not δ-closed D_ζ is a set because any δ-closed partial order cannot add a Q_ζ-generic filter over W as forcing with Q_ζ adds a subset to $2^{<|Q_\zeta|}$. Let $D = \bigcup_{\zeta<\eta} D_\zeta$ and consider the partially ordered class $\mathbb{Q} := \prod_{\alpha>\gamma,\alpha\notin D} P_\alpha$. Then $W[G^{>\gamma}]$ has a ground which is obtained by adding a $\mathbb{Q}$-generic filter to W as only set many factors of $\prod_{\alpha>\gamma} P_\alpha$ have been removed in $\mathbb{Q}$. By construction it is clear that this ground satisfies $\phi(n)$.

Finally, for any natural number m we define the independent family of switches $\psi(k)$ for $k < m$ stating that the kth digit of the unique n such that $\phi(n)$ holds in binary representation is 1. As we can change this unique n completely freely by going to a ground it is clear that these statements form independent switches. Therefore, the modal logic of grounds of $W[G]$ is contained in S5. $\square$

Corollary 7.9. *Let W be a countable transitive model of* ZFC *and let $\langle P_\alpha : \alpha \in \mathrm{Ord}\rangle$ be a definable class in W. Assume that $\mathbb{P} = \prod_{\alpha\in\mathrm{Ord}} P_\alpha$ is a progressively closed product in W. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over W such that $W[G]$ has no bedrock and $\mathbb{M}^{W[G]} = W$ then there are arbitrarily large finite families of independent downwards switches for $W[G]$. Therefore,* $\mathrm{MLG}(W[G]) \subseteq \mathrm{S5}$.

Proof. In the proof of the theorem, we only needed the assumption that W is a model of the ground axiom to show that $W = \mathbb{M}^{W[G]}$. Furthermore, if $W[G]$ has no bedrock, but W is its mantle it is clear that $\mathbb{P}$ has to be a proper class and hence class many of the factors P_α have to be non-trivial. Then, the corollary follows by the same proof. $\square$

The theorem shows that Reitz's original construction to obtain a model without bedrock and all variations of this construction forcing with a class product over a model of the ground axiom cannot lead to a model without a bedrock but satisfying Top. The corollary shows that also the method in [9] Fuchs, Hamkins and Reitz used to construct a model with mantle W for arbitrary models W, i.e. in particular for models not satisfying the ground axiom, also does not yield models without a bedrock but with downwards validity of Top.

8 Outlook

In this thesis, we have been able to give a complete answer to the question about the modal logic of forcing of generic multiverses. The main result, theorem 6.8, in chapter 6 settles this question by stating that the modal logic of forcing of any generic multiverse is S4.2.

For the modal logic of grounds of a generic multiverse, we established S4.2Top as an upper bound in chapter 7. For multiverses with a bedrock, this implies that their modal logic of grounds is exactly S4.2Top which also shows that S4.2Top is the best upper bound possible without further knowledge about the generic multiverse.

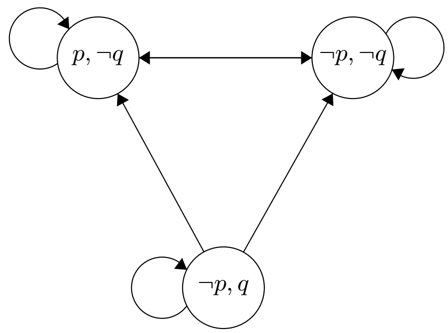
The remaining case concerns the modal logic of grounds of generic multiverses without a bedrock. Theorem 7.8 gives a partial answer for these cases. It states that all models W obtained by forcing with a reasonable progressively closed class product over their mantle, the only method we know to construct models without a bedrock, have a modal logic of grounds contained in S5. That also means that $\mathrm{MLG}(\mathrm{Mult}_W) \subseteq \mathrm{S4.2Top} \cap \mathrm{S5}$.

Remark. We have that $\mathrm{S4.2} \subsetneq \mathrm{S4.2Top} \cap \mathrm{S5} \subsetneq \mathrm{S4.2Top}$. The inclusions are clear. The second inequality is immediate as the Top-axiom is not a theorem of S5 as it is for example not valid on a two element cluster.

For the first inequality, consider the following formula ϕ:

$$\begin{aligned}&(\Diamond((p \leftrightarrow \Box p) \land (\neg p \leftrightarrow \Box \neg p)))\\ \lor&(\Diamond q \to \Box\Diamond q)\end{aligned}$$

This formula is in S4.2Top as the first disjunct is the Top-axiom, and it is in S5 as the second disjunct is the (5)-axiom. However, it is not valid on all finite pre-Boolean algebras as the following Kripke model shows:



In the bottom node, ϕ does not hold. So, $\phi \notin \mathrm{S4.2}$ by theorem 4.12.

On the other hand, we know generic multiverses with modal logic of grounds S4.2. Reitz's construction presented in the proof of theorem 5.9 provides a model U with $\mathrm{MLG}(U) = \mathrm{S4.2}$ and hence $\mathrm{MLG}(\mathrm{Mult}_U) = \mathrm{S4.2}$. This raises the following question:

Question 1a: Is there a model W without a bedrock such that $\mathrm{MLG}(\mathrm{Mult}_W) \neq \mathrm{S4.2}$?

In such a model, the Top-axiom might be downwards valid. So, a next question to ask is:

Question 1b: Is there a model W such that $\mathrm{S4.2} \subsetneq \mathrm{MLG}(\mathrm{Mult}_W) \subsetneq \mathrm{S4.2Top}$?

The modal logic $\mathrm{S4.2Top} \cap \mathrm{S5}$ comes to mind as a candidate for the modal logic of grounds of a generic multiverse in light of the above discussion.

Further, generic multiverses without a bedrock which cannot be obtained by forcing with a class product satisfying the conditions of theorem 7.8 could provide examples for generic multiverses with a modal logic of grounds strictly between S4.2 and S4.2Top. As we do not know any examples of such generic multiverses, an answer to the following questions would be a crucial step in understanding in which cases this theorem is applicable:

Question 2a: Under which circumstances is a countable transitive model W without a bedrock a class forcing extension of its mantle by a forcing satisfying the conditions of theorem 7.8?

Question 2b: Are there models without a bedrock which cannot be obtained in this way?

Our results point out that a model W which is elementarily equivalent to all its grounds and does not have a bedrock would provide an example for question 2b as the modal logic of grounds of such a model is the modal logic of a single reflexive point. Hence, the modal logic of grounds of W would not be contained in S5 and could not be obtained by forcing with a reasonable progressively closed product over its mantle.

Furthermore, the use of the strong downwards directedness of grounds in the proofs of our main results is remarkable as in previous work on the modal logic of grounds the directedness in the sense that two grounds of a model always have a common ground was sufficient. For example, this weaker form of the downwards directedness was the missing part to Hamkins's and Löwe's result that the modal logic of grounds is S4.2 (see [17]).

However, in our results, the strong downwards directedness of grounds is essential in two different ways: First, the strong downwards directedness of grounds implies that the mantle of any model of ZFC is itself a model of ZFC. Hence, we could refer to the mantle instead of the constructible universe in the definition of new buttons. Second, the strong downwards directedness of grounds allowed us to find grounds below any model which coincide with the mantle up to a certain rank. This was crucial to prove the independence of the various control statements we used.

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Appendices

A Forcing Theory

In this appendix, we give a brief summary of forcing theory.

A.1 Basics and Notation

We begin by defining the basic notions in forcing theory and state standard results. We follow the exposition in [22, chapter VII] where the proofs to the results in this section can be found.

Definition A.1. A *partial order* is a triple $\langle \mathbb{P}, \leq, 1 \rangle$ such that $\leq$ is a reflexive, transitive and anti-symmetric relation on $\mathbb{P}$ and $1 \in \mathbb{P}$ is the largest element of $\mathbb{P}$, i.e. $\forall p \in \mathbb{P}(p \leq 1)$.

We will often refer to a partial order by its underlying set $\mathbb{P}$ when the context allows for that. Further, we will also refer to $p, q \in \mathbb{P}$ as conditions and we will say that p is stronger than q if $p \leq q$.

Definition A.2. Let $\langle \mathbb{P}, \leq, 1 \rangle$ be a partial order. We call a subset $D \subseteq \mathbb{P}$ *dense* if for every $p \in \mathbb{P}$ there is a $q \in D$ with $q \leq p$.

We call two elements $p, q \in \mathbb{P}$ *compatible*, in symbols $p \parallel q$, if there is $r \in \mathbb{P}$ with $r \leq p, q$. Otherwise, we call them *incompatible* and write $p \perp q$.

We call a subset $A \subseteq \mathbb{P}$ an *anti-chain* if for any $p, q \in A$ we have $p \perp q$.

For a cardinal θ, we say that $\langle \mathbb{P}, \leq, 1 \rangle$ satisfies the *θ-chain condition* (θ-c.c.) if every anti-chain $A \subseteq \mathbb{P}$ has size less than θ. If $\langle \mathbb{P}, \leq, 1 \rangle$ satisfies the $\aleph_1$-c.c. we also say it satisfies the *countable chain condition* (c.c.c.).

We call a subset $F \subseteq \mathbb{P}$ a *filter* if $1 \in F$, for any $p, q \in F$ there is an $r \in F$ with $r \leq p, q$ and for all $p \in F$ and $q \geq p$ we have that $q \in F$. In particular, all elements of a filter are pairwise compatible.

Let W be a model of ZFC and assume $\langle \mathbb{P}, \leq, 1 \rangle \in W$. Then we call a filter $F \subseteq \mathbb{P}$ a *$\mathbb{P}$-generic filter over W* if it intersects every dense subset D of $\mathbb{P}$ in W.

A reason why we work with countable models instead of arbitrary set models is given by the following lemma.

Lemma A.3. *Let W be a countable model of* ZFC*, let $\mathbb{P} \in W$ be a partial order and let $p \in \mathbb{P}$. Then there is a $\mathbb{P}$-generic filter G over W with $p \in G$.*

If any $\mathbb{P}$-generic filter G can be found in W itself, we call $\mathbb{P}$ a trivial forcing partial order. But there is a simple combinatorial property ensuring that the filters do not lie in W:

Lemma A.4. *Let W be a transitive model of* ZFC*, let $\mathbb{P} \in W$ be a partial order such that*

$$\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}(q \leq p \wedge r \leq p \wedge q \perp r),$$

and let G be a $\mathbb{P}$-generic filter over W. Then $G \notin W$.

The class of $\mathbb{P}$-names $W^{\mathbb{P}}$ together with the valuation function val forms the core of the construction of extensions of the model W.

Definition A.5. Let W be a model of ZFC. Given a partial order $\langle \mathbb{P}, \leq, 1\rangle \in W$ we recursively define the class of $\mathbb{P}$-names in W:

$$\begin{aligned}
\mathrm{Name}_0^{\mathbb{P}} &:= \{\emptyset\},\\
\mathrm{Name}_{\alpha+1}^{\mathbb{P}} &:= \{\tau : \tau \text{ consists of ordered pairs} \wedge \forall x, y(\langle x, y\rangle \in \tau \to x \in \mathrm{Name}_\alpha^{\mathbb{P}} \wedge y \in \mathbb{P}\},\\
\mathrm{Name}_\lambda^{\mathbb{P}} &:= \bigcup_{\alpha<\lambda} \mathrm{Name}_\alpha^{\mathbb{P}}, \text{ for limit ordinals } \lambda.
\end{aligned}$$

The class of $\mathbb{P}$-names in W is now defined as $W^{\mathbb{P}} := \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{Name}_\alpha^{\mathbb{P}}$. For any $x \in W$ we can define the canonical name $\check{x} := \{\langle \check{y}, 1\rangle : y \in x\}$, again by recursion.

Definition A.6. Let $W \subseteq V$ be a model of ZFC and let $\mathbb{P}$ be a partial order in W. Given a $G \subseteq \mathbb{P}$ in V we define the valuation of $\mathbb{P}$-names in W by G as

$$\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \exists p \in G(\langle \sigma, p\rangle \in \tau)\},$$

where $\tau \in W^{\mathbb{P}}$.

This allows us to define the extension $W[G] := \{\mathrm{val}(\tau, G) : \tau \in W^{\mathbb{P}}\}$. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over W we call the extension $W[G]$ a *generic extension*. Of course, this definition is a shorthand for a recursive definition on the hierarchy of $\mathbb{P}$-names.

Theorem A.7 (Generic Model Theorem)**.** *Let W be a transitive model of* ZFC*, $\mathbb{P}$ a partial order in W and $G \subseteq \mathbb{P}$ a $\mathbb{P}$-generic filter over W. Then $W \subseteq W[G]$, and $W[G]$ is a transitive model of* ZFC *with the same ordinals as W.*

Lemma A.8. *Let W, U be transitive models of* ZFC *with $W \subseteq U$ and let G be a $\mathbb{P}$-generic filter over W such that $G \in U$. Then $W[G] \subseteq U$.*

A key feature of the theory of forcing is the forcing relation $\Vdash$. Given a partial order $\mathbb{P}$ in a model W of ZFC this relation is a relation between conditions in $\mathbb{P}$ and sentences in the so called forcing language. The forcing language contains the binary predicate $\in$ as well as all $\mathbb{P}$-names in $W^{\mathbb{P}}$ as constants. For a formal definition of the forcing relation, consult [22, VII, definition 3.3].

The important property of the forcing relation is that it is definable within W. The following theorem states that the truth of formulas in a generic extension is captured inside the original model via the forcing relation.

Theorem A.9 (Forcing Theorem)**.** *Let W be a transitive model of* ZFC*, $\mathbb{P} \in W$ a partial order and $G \subseteq \mathbb{P}$ a $\mathbb{P}$-generic filter over W. Let $\tau^1, \ldots, \tau^n \in W^{\mathbb{P}}$ and let $\phi(x_1, \ldots, x_n)$ be a formula. Then the following are equivalent:*

1. $W[G] \vDash \phi(\mathrm{val}(\tau^1, G), \ldots, \mathrm{val}(\tau^n, G))$.

2. $\exists p \in G((p \Vdash \phi(\tau^1, \ldots, \tau^n))^W)$.

For our later analysis of the modal logic of forcing and of grounds, it is of importance under which circumstances forcing extensions are elementarily equivalent. Hence, the notion of a weakly homogeneous partial order is very useful.

Definition A.10. A partial order $\mathbb{P}$ is called *weakly homogeneous* if for any two conditions $p, q \in \mathbb{P}$ there is an order automorphism f of $\mathbb{P}$ such that $f(p)$ and q are compatible.

The reason for the importance of this notion is the following standard result. A proof can be found in [2, lemma 3.5] for example .

Lemma A.11. *If $\mathbb{P} \in W$ is weakly homogeneous and G and H are two $\mathbb{P}$-generic filters over W then $W[G]$ and $W[H]$ are elementarily equivalent. In other words, if $\mathbb{P} \in W$ is weakly homogeneous, then for any sentence ϕ in the language of set theory either $1_{\mathbb{P}} \Vdash \phi$ or $1_{\mathbb{P}} \Vdash \neg\phi$.*

Furthermore, we are interested in the relation of extensions with two different partial orders. In particular, we do not have to distinguish partial orders which produce the same extensions.

Definition A.12. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders in W. If for any $\mathbb{P}$-generic filter G over W there is an $\mathbb{Q}$-generic filter H over W such that $W[G] = W[H]$, and vice versa, then we call $\mathbb{P}$ and $\mathbb{Q}$ *forcing equivalent.*

The main tools to prove that two partial orders are forcing equivalent or that extensions with one partial order $\mathbb{P}$ are always contained in an extension with a partial order $\mathbb{Q}$ are complete and dense embeddings.

Definition A.13. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. We call a function $i : \mathbb{P} \to \mathbb{Q}$ a *complete embedding* iff the following three conditions hold:

1. $\forall p, r \in \mathbb{P}(p \leq r \to i(p) \leq i(r))$,
2. $\forall p, r \in \mathbb{P}(p \perp r \leftrightarrow i(p) \perp i(r))$,
3. $\forall q \in \mathbb{Q} \exists p \in \mathbb{P} \forall r \in \mathbb{P}(r \leq p \to (i(r)$ and q are compatible in $\mathbb{Q}))$.

We call $i : \mathbb{P} \to \mathbb{Q}$ a *dense embedding* iff:

1. $\forall p, r \in \mathbb{P}(p \leq r \to i(p) \leq i(r)$,
2. $\forall p, r \in \mathbb{P}(p \perp r \leftrightarrow i(p) \perp i(r))$,
3. $i[\mathbb{P}]$ is dense in $\mathbb{Q}$.

It is easy to check that any dense embedding is also a complete embedding. Complete and dense embeddings between partial orders yield the following strong connections between the corresponding forcings:

Lemma A.14. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders in W and $i : \mathbb{P} \to \mathbb{Q}$ be a complete embedding in W. Further, let H be a $\mathbb{Q}$-generic filter over W. Then $i^{-1}(H)$ is a $\mathbb{P}$-generic filter over W and $W[i^{-1}(H)] \subseteq W[H]$.*

Lemma A.15. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders in W and $i : \mathbb{P} \to \mathbb{Q}$ be a dense embedding in W. Further, let $\tilde{i}(G) := \{q \in \mathbb{Q} : \exists p \in G(i(p) \leq q)\}$ for any $G \subseteq \mathbb{P}$. Then:*

1. *If $H \subseteq \mathbb{H}$ is $\mathbb{Q}$-generic over W then $i^{-1}(H)$ is $\mathbb{P}$-generic over W and $H = \tilde{i}(i^{-1}(H))$.*
2. *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over W then $\tilde{i}(G)$ is $\mathbb{P}$-generic over W and $G = i^{-1}(\tilde{i}(G))$.*

In both cases, if $G = i^{-1}(H)$ or $H = \tilde{i}(G)$ then $W[G] = W[H]$.

In particular, the previous lemma shows that two partial orders $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent if there is a dense embedding between them, or if they both densely embed into the same partial order.

A.2 Preservation Theorems

The notion of a cardinal is not absolute between transitive models of set theory. In particular, it is possible to collapse cardinals via forcing by adding a bijection between two distinct cardinals κ and λ. Preservation theorems connect combinatorial properties of partial orders to the preservation of cardinals in forcing extensions and are a crucial tool to control the behavior of the extension. We start by making precise what preserving cardinals means.

Definition A.16. Let W be a countable transitive model of ZFC, let $\mathbb{P} \in W$ be a partial order and let θ be an infinite cardinal in W.

We say that $\mathbb{P}$ *preserves cardinals* $\leq \theta$ (and $\geq \theta$, respectively) iff for any $\mathbb{P}$-generic filter G over W and for any ordinal $\alpha \leq \theta$ (or $\alpha \geq \theta$) in W we have that $W \models$ "α is a cardinal" iff $W[G] \models$ "α is a cardinal".

Likewise, we say that $\mathbb{P}$ *preserves cofinalities* $\leq \theta$ (and $\geq \theta$, respectively) iff for any $\mathbb{P}$-generic filter G over W and for any limit ordinal α in W with $\mathrm{cf}(\alpha)^W \leq \theta$ (or $\mathrm{cf}(\alpha)^W \geq \theta$) we have that $\mathrm{cf}(\alpha)^W = \mathrm{cf}(\alpha)^{W[G]}$.

The first preservation theorem connects the θ-chain condition of partial orders to the preservation of cardinals.

Theorem A.17. *Let W be a countable transitive model of* ZFC*, $\mathbb{P} \in W$ be a partial order, θ a regular cardinal in W and assume that $W \models$ "$\mathbb{P}$ has the θ-c.c.". Then $\mathbb{P}$ preserves cardinals $\geq \theta$.*

In particular, $\mathbb{P}$ preserves all cardinals if it satisfies the countable chain condition. Another combinatorial property allowing for a preservation theorem is κ-closedness for a cardinal κ.

Definition A.18. Let κ be a cardinal. A partial order $\mathbb{P}$ is said to be *κ-closed* if for any $\gamma < \kappa$ and any decreasing sequence $\langle p_\alpha : \alpha < \gamma \rangle$ of elements of $\mathbb{P}$, i.e. $p_\alpha \in \mathbb{P}$ for all $\alpha < \gamma$ and for any $\alpha < \beta < \gamma$ we have $p_\alpha \geq p_\beta$, there is a $p \in \mathbb{P}$ below the whole sequence, i.e. $p \leq p_\alpha$ for all $\alpha < \gamma$.

Theorem A.19. *Let W be a countable transitive model of* ZFC*, $\mathbb{P} \in W$ a partial order, $A, B \in W$, λ a cardinal in W such that $W \models$ "$\mathbb{P}$ is λ-closed" and $W \models |A| < \lambda$. Let G be a $\mathbb{P}$-generic filter over W and $f : A \to B$ a function in $W[G]$. Then $f \in W$.*

So, λ-closed forcing does not add bounded subsets to λ. Further, we get the following preservation theorem as a corollary.

Corollary A.20. *Let W be a countable transitive model of* ZFC *and let $\mathbb{P} \in W$ be a partial order. Let λ be a cardinal in W and let $W \models$ "$\mathbb{P}$ is λ-closed". Then $\mathbb{P}$ preserves cofinalities and cardinals $\leq \lambda$.*

A.3 Forcing with Boolean Algebras

Any complete Boolean algebra B is a partial order, but since it has a least element, it does not satisfy the requirements of Lemma A.4. If we remove the bottom element of B, then $B - 0$ is also a partial order, and whenever we talk about forcing with a Boolean algebra, we mean forcing with the partial order of non-zero elements of that algebra.

In order to analyze the structure of intermediate models between a model and a fixed generic extension, forcing with a complete Boolean algebra B is often more fruitful because the intermediate models are generated by complete subalgebras of B. The following result shows that any forcing partial order is forcing equivalent to a complete Boolean algebra. Thus, we can restrict our analysis to forcing with complete Boolean algebras whenever this is more convenient.

Theorem A.21. *For any partial order $\mathbb{P}$ there is a complete Boolean algebra B s.t. there is a dense embedding $i : \mathbb{P} \to B - \{0\}$ and B is unique up to isomorphism.*

The construction to obtain this Boolean algebra goes as follows: For any partial order $\mathbb{P}$, we can call downwards closed sets open and consider the complete Boolean algebra of regular open sets in this topological space. It is well-known that the collection of regular open sets of any topological space forms a complete Boolean algebra. Then, $\mathbb{P}$ densely embeds into that Boolean algebra by mapping $p \in \mathbb{P}$ to the interior of the closure of $\{q \in \mathbb{P} : q \leq p\}$.

For a detailed proof we refer to [20] where this theorem is stated as corollary 14.12.

The following result, also known as Grigoriev's theorem, is the reason why forcing with Boolean algebras is often more convenient if we are interested in intermediate models between a model and a generic extension. A proof can be found in [20, lemma 15.43].

Theorem A.22. *Let B be a complete Boolean algebra in a model W of* ZFC*, let G a B-generic filter over W and let U be a model of* ZFC *with $W \subseteq U \subseteq W[G]$. Then there is a complete subalgebra $D \subseteq B$ such that $U = W[G \cap D]$. In particular, U is a forcing extension of W. Further, $W[G]$ is a forcing extension of U.*

A.4 Products and Iterations

Products and iterations of forcings are important methods to add various generic objects to a given model W. Products allow us to simultaneously add $\mathbb{P}_i$-generic filters over W for a family $\langle \mathbb{P}_i : i \in I \rangle$ in W. In particular, we can also take all the $\mathbb{P}_i$ to be the same partial order and then add arbitrarily many distinct generic objects. Iterations, on the other hand, are used to add further generic objects over forcing extensions of W. So, if we already added a generic filter G to W we can just take some partial order $\mathbb{Q} \in W[G]$ and add a $\mathbb{Q}$-generic filter H over $W[G]$ to obtain $W[G][H]$. The important result here is that we can define a partial order $\mathbb{P} * \dot{\mathbb{Q}}$ in W that can generate this extension in one step using a $\mathbb{P}$-name for $\mathbb{Q}$.

Throughout this section let W be a model of ZFC. Proofs to the results of this section are given in [22, chapter VIII].

Definition A.23. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$, $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ be partial orders. The *product* $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle \times \langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ is the partial order $\langle \mathbb{P} \times \mathbb{Q}, \leq, \langle 1_{\mathbb{P}}, 1_{\mathbb{Q}} \rangle \rangle$, where $\langle p, q \rangle \leq \langle p', q' \rangle$ iff $p \leq_{\mathbb{P}} p'$ and $q \leq_{\mathbb{Q}} q'$. We will usually refer to the product by its underlying set $\mathbb{P} \times \mathbb{Q}$.

Lemma A.24. *Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders in W, let $i : \mathbb{P} \to \mathbb{P} \times \mathbb{Q}$ be defined by $i(p) = \langle p, 1_{\mathbb{Q}} \rangle$ for all $p \in \mathbb{P}$ and let likewise $j : \mathbb{Q} \to \mathbb{P} \times \mathbb{Q}$ be defined by $j(q) = \langle 1_{\mathbb{P}}, q \rangle$ for all $q \in \mathbb{Q}$. Then i and j are complete embeddings. So, if G is $\mathbb{P} \times \mathbb{Q}$-generic over W then $i^{-1}(G)$ is $\mathbb{P}$-generic over W and $j^{-1}(G)$ is $\mathbb{Q}$-generic over W. Furthermore, $G = i^{-1}(G) \times j^{-1}(G)$.*

Theorem A.25. *Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders in W and let $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Then the following are equivalent:*

1. *$G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over W.*
2. *G is $\mathbb{P}$-generic over W and H is $\mathbb{Q}$-generic over $W[G]$.*
3. *H is $\mathbb{Q}$-generic over W and G is $\mathbb{P}$-generic over $W[H]$.*

If 1 – 3 hold then $W[G][H] = W[H][G] = W[G \times H]$.

This theorem highlights the commutativity as a main feature of forcing with products. More general, we can also define a product of arbitrarily many partial orders. Recall that an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$ for some set X is a non-empty collection of sets which is closed under subsets and finite unions.

Definition A.26. Let $\langle \mathbb{Q}_\beta : \beta < \alpha \rangle$ be a sequence of partial orders in W. Then the *product* of the partial orders $\prod_{\beta<\alpha} \mathbb{Q}_\beta$ with *support* in an ideal $\mathcal{I} \subseteq \mathcal{P}(\alpha)$ consists of all functions f with domain α such that for all $\beta < \alpha$ we have $f(\beta) \in \mathbb{Q}_\beta$ and $\operatorname{spt}(f) := \{\beta < \alpha : f(\beta) \neq 1_{\mathbb{Q}_\beta}\} \in \mathcal{I}$ ordered componentwise, i.e. $f \leq g$ iff $f(\beta) \leq g(\beta)$ for all $\beta < \alpha$.

If $\mathcal{I}$ is the ideal of all finite subsets of α, we call the product a *finite support product*. Likewise, we define *countable support products*. If $\mathcal{I} = \mathcal{P}(\alpha)$, we call the product a *full support* *product*. Finally, we say that the product uses *Easton support* if $\mathcal{I} = \{A \subseteq \alpha : |A \cap \lambda| < \lambda$ for all weakly inaccessible cardinals $\lambda\}$.

In this thesis the, different kinds of support do not play a major role. We just want to remark that the use of different supports can assure different properties of the resulting product. For example, the finite support product of c.c.c.-forcings is necessarily again a c.c.c.-forcing (see [22, VIII, lemma 5.12]).

The commutativity as given by theorem A.25 applies for arbitrary products as well. In particular, forcing with $\prod_{\beta<\alpha} \mathbb{Q}_\beta$ adds $\mathbb{Q}_\beta$-generic filters for all $\beta < \alpha$ and these filters are mutually generic.

To conclude this section, we define the two-step-iteration of forcing which allows for a useful structural observation: It shows that "is a forcing extension of" is transitive.

Definition A.27. Let $\mathbb{P} \in W$ be a partial order and let $\dot{\mathbb{Q}} \in W^{\mathbb{P}}$ be a name for a partial order. Then the *two-step-iteration* $\mathbb{P} * \dot{\mathbb{Q}}$ has the underlying set

$$\{\langle p, \dot{q}\rangle : p \in \mathbb{P} \text{ and } 1_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}\}.$$

The order is defined as follows:

$$\langle p, \dot{q}\rangle \leq \langle p', \dot{q}'\rangle \text{ iff } p \leq p' \text{ and } p \Vdash \dot{q} \leq \dot{q}'.$$

Theorem A.28. *Let $\mathbb{P}$ be a partial order in W. Let G be a $\mathbb{P}$-generic filter over W, $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a partial order, $\mathbb{Q} = \mathrm{val}(\dot{\mathbb{Q}}, G)$ and H a $\mathbb{Q}$-generic over $W[G]$. Then,*

$$G * H = \{\langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G \textit{ and } \mathrm{val}(\dot{q}, G) \in H\}$$

is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over W. Further, $W[G * H] = W[G][H]$.

On the other hand, let K be $\mathbb{P} * \dot{\mathbb{Q}}$-generic over W. Then, $G' := \{p \in \mathbb{P} : \exists \dot{q}(\langle p, \dot{q}\rangle \in K)\}$ is $\mathbb{P}$-generic over W, $H' := \{\mathrm{val}(\dot{q}, G') : \exists p(\langle p, \dot{q}\rangle \in K)\}$ is $\mathrm{val}(\dot{\mathbb{Q}}, G)$-generic over $W[G']$ and $K = G' * H'$.

A.5 Class Forcing

The forcing theory build up so far in this appendix always requires the partial order to be a set. However, it is possible to drop this requirement under certain conditions. If $\mathbb{P}$ is a definable, partially ordered proper class then one can generalize the notion of $\mathbb{P}$-names, define a forcing relation and for any subclass $G \subseteq \mathbb{P}$ which forms a $\mathbb{P}$-generic filter over W one can construct the extension $W[G]$. However, the generic model theorem does not generalize in full strength. In fact, this is not surprising if one considers for example the partially ordered class of finite partial functions from $\omega \to \mathrm{Ord}$, $\mathbb{P} := \{f \subseteq \omega \times \mathrm{Ord} : f \text{ is a function and } |f| < \omega\}$, ordered by $\supseteq$. A $\mathbb{P}$-generic filter G over W would now yield a surjection $\bigcup G : \omega \to \mathrm{Ord}^W$ by the usual density argument. Hence, $W[G]$ clearly cannot be a model of ZFC with the same ordinals as W.

A detailed exposition of the theory of class forcing is outside the scope of this thesis. In most standard references the theory of class forcing is sketched and a comprehensive account can be found in the appendix of Reitz's PhD thesis [26]. Note that Reitz works with models of Bernays-Gödel set theory (BGC), a conservative extension of ZFC. It is well-known that any model of ZFC can be regarded as a model of BGC by letting the collection of classes be the collection of definable classes. We state the results here in terms of models of ZFC.

The special case of class forcing we are going to use in this thesis is the case of progressively closed products. Following Reitz we define:

Definition A.29. A partially ordered class $\mathbb{P}$ is a *chain of complete subposets* if and only if $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} P_\alpha$ where each P_α is a partially ordered set such that $\alpha \leq \beta$ implies that $P_\alpha \subseteq P_\beta$ and that the inclusion is a complete embedding and, further, the sequence $\{\langle \alpha, P_\alpha \rangle : \alpha \in \mathrm{Ord}\}$ a definable class.

The following lemma shows that forcing with a chain of complete subposets adds generic filters to all the stages P_α in the chain. For a proof consult [26, lemma 41].

Lemma A.30. *If $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} P_\alpha$ is a chain of complete subposets then $P_\alpha \subseteq \mathbb{P}$ is a complete subposet for each α, i.e. the inclusion $P_\alpha \to \mathbb{P}$ is a complete embedding.*

Definition A.31. A partial order $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} P_\alpha$ is a *progressively closed product* if and only if $\mathbb{P}$ is a chain of complete subposets and for arbitrarily large regular δ the partially ordered class $\mathbb{P}$ factors as

$$\mathbb{P} \cong \mathbb{P}^1 \times P^2$$

such that P^2 is a partially ordered set, P^2 has the δ^+-c.c., $\mathbb{P}^1$ is a chain of complete subposets, and $\mathbb{P}^1$ is δ^+-closed.

Theorem A.32. *If $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} P_\alpha$ is a progressively closed product and the class $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over W then $W[G]$ is a transitive model of* ZFC.

This generalized generic model theorem for progressively closed products is proved in [26, theorem 98].

Many important applications of class forcing use progressively closed products. These forcings are usually presented as $\mathbb{P} = \prod_{\alpha \in \mathrm{Ord}} P_\alpha$ where $\langle P_\alpha : \alpha \in \mathrm{Ord} \rangle$ is a definable sequence in W and where the support is in some class $\mathcal{I}$. The class long product is meant to mean the following: For any $\alpha \leq \beta$ we can consider $\prod_{\gamma < \alpha} P_\gamma$ with support in $\mathcal{I}$ as a complete subposet of $\prod_{\gamma < \beta} P_\gamma$ with support in $\mathcal{I}$ in a natural way. Then, we mean $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \prod_{\gamma < \alpha} P_\gamma$. So, whenever we present a class forcing in this way we get that the forcing is a chain of complete subposets for free.

To ensure that $\mathbb{P}$ factors as in definition A.31, the sequence $\langle P_\alpha : \alpha \in \mathrm{Ord} \rangle$ usually satisfies that for arbitrarily large regular δ there is an η such that $\prod_{\gamma < \eta} P_\gamma$ has the δ^+-c.c., or even is of size less than δ^+, and such that P_α is δ^+-closed for all $\alpha \geq \eta$.

As in the case of products of partially ordered sets, the following commutativity result for forcing with the product of a partially ordered set and a partially ordered class holds.

Theorem A.33. *Let δ be a regular cardinal and let Q be a partially ordered set which satisfies the δ^+-c.c. Further, let $\mathbb{P}$ be a chain of complete subposets and assume that $\mathbb{P}$ is δ^+-closed. For $H \subseteq Q$ and $\mathbb{G} \subseteq \mathbb{P}$ the following are equivalent:*

1. *$H \times \mathbb{G}$ is $Q \times \mathbb{P}$-generic over W.*
2. *$\mathbb{G} \times H$ is $\mathbb{P} \times Q$-generic over W.*
3. *H is Q-generic over W and $\mathbb{G}$ is $\mathbb{P}$-generic over $W[H]$.*
4. *$\mathbb{G}$ is $\mathbb{P}$-generic over W and H is Q-generic over $W[\mathbb{G}]$.*

In addition, if the four conditions hold (if one of them holds) then $W[H \times \mathbb{G}] = W[\mathbb{G} \times H] = W[H][\mathbb{G}] = W[\mathbb{G}][H]$.

Reitz proved this theorem in [26, lemma 121].

B Usuba's Proof of the Strong Downwards Directedness of Grounds

In this appendix, we give an exposition of Usuba's proof of the strong downwards directedness of grounds (see [31]), mainly because this proof is still unpublished and not available online. Furthermore, the notes provided by Usuba are brief in some steps of the proof, and hence we provide some more detail in this exposition, in particular in the proofs of lemma B.4, lemma B.12, and theorem B.13.

Theorem (Strong Downwards Directedness of Grounds)**.** *Below set many grounds of any model of set theory there is a common ground. That means if W is a model of* ZFC *and $R \in W$ is a set then there is a ground W' of W which is a ground of $W_r := \{x \in W : \phi(x, r)^W\}$ for all $r \in R$.*

For the proof of the strong DDG, we will first transfer definition B.3 to models of $\mathrm{ZFC} - \mathrm{P}$, following Usuba [31]. By $\mathrm{ZFC} - \mathrm{P}$ we mean the axioms of ZF without the power set axiom and using collection rather than replacement + the well-ordering theorem instead of the axiom of choice. The use of collection instead of replacement is not necessary for the following proofs and Usuba does not explicitly specify which of the two axioms he uses. Nevertheless, Gitman, Hamkins and Johnstone [11] point out that the list of axioms presented here is of the appropriate strength when thinking of ZFC without power set. For a regular uncountable cardinal κ the set of sets hereditarily of cardinality $< \kappa$ forms a natural model of $\mathrm{ZFC} - \mathrm{P}$. One well-known fact about $\mathrm{ZFC} - \mathrm{P}$ is that it implies that every set can be coded as a set of ordinals. This uses the fact that the Mostowski collapse lemma is a theorem of $\mathrm{ZFC} - \mathrm{P}$ (see [22, III, theorem 5.14]). Given a set X, the well-ordering theorem allows us to find a bijection $f : \alpha \to \mathrm{trcl}(X)$, the transitive closure of X. Using f, we can define a relation $E \subseteq \alpha \times \alpha$ such that (α, E) is isomorphic to $(\mathrm{trcl}(X), \in)$: We define $\langle \beta, \gamma \rangle \in E$ iff $f(\beta) \in f(\gamma)$. The subset of E of $\alpha \times \alpha$ can then be coded as a subset A of an ordinal using some coding function $\mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$. The Mostowski collapse lemma then ensures that we can reconstruct X in any model of $\mathrm{ZFC} - \mathrm{P}$ from A. In particular, when we want to show that two models of $\mathrm{ZFC} - \mathrm{P}$ are equal we will use this and show that the models contain the same sets of ordinals.

Definition B.1. Let κ be an infinite cardinal. We define $\mathcal{H}(\kappa) := \{x : |\mathrm{trcl}(x)| < \kappa\}$, the set of sets hereditarily of cardinality $< \kappa$.

The following is well-known (see [22, IV, theorem 6.5]):

Lemma B.2. *For any regular uncountable cardinal κ the set $\mathcal{H}(\kappa)$ is a transitive model of* $\mathrm{ZFC} - \mathrm{P}$.

As the proof of the strong DDG heavily relies on the notion of κ-approximation and κ-cover property. We recall the definitions here and additionally introduce the κ-global covering property.

Definition B.3. Let $M \subseteq W$ be transitive models of ZFC with the same ordinals. Let κ be an infinite cardinal.

Then, we say that M satisfies the *κ-covering property* for W if every set of ordinals A in W with $|A| < \kappa$ is covered by a set of ordinals B in M with $|B| < \kappa$, i.e. there is a set of ordinals $B \in M$ of size less than κ with $A \subseteq B$.

We say that M satisfies the *κ-approximation property* for W if for any set of ordinals $A \in W$ for which $A \cap x \in M$ for every set of ordinals $x \in M$ with $|x| < \kappa$ we have that $A \in M$.

We say that M satisfies the *κ-global covering property* for W if for any ordinal $\alpha \in W$ and any function $f : \alpha \to Ord$ in W there is a function $F \in M$ with $\mathrm{dom}(F) = \alpha$ and $f(\beta) \in F(\beta)$ as well as $|F(\beta)| < \kappa$ for all $\beta < \alpha$.

Lemma B.4. *If κ is regular then the κ-global covering property also implies the following stronger property:*

For any function $f : \alpha \to M$ such that $f(\beta)$ is a set of ordinals of size $< \kappa$ for all $\beta < \alpha$ there is a function $F \in M$ with $\mathrm{dom}(F) = \alpha$, $f(\beta) \subseteq F(\beta)$ and $|F(\beta)| < \kappa$ for all $\beta < \alpha$.

Proof. We fix a bijection $b : \gamma \to \alpha \times \kappa$ in L where γ is some ordinal and define a function $g : \gamma \to \mathrm{Ord}$ by defining $g(\delta)$ to be the unique element x of $f(b(\delta)_0)$ such that the order type of $x \cap f(b(\delta)_0)$ is $f(b(\delta)_1)$ if there is such an element and $g(\delta) = 0$, otherwise, for all $\delta < \gamma$. By the κ-global covering property we find a function $G \in M$ with domain γ such that $g(\delta) \in G(\delta)$ and $|G(\delta)| < \kappa$ for all $\delta < \gamma$. Now we can define $F(\beta) := \bigcup_{\eta \in b^{-1}[\{\beta\} \times \kappa]} G(\eta)$. Since $b^{-1}[\{\beta\} \times \kappa] \subseteq f(\beta) \cup \{0\}$ this is a union of less than κ many sets of size less than κ. So, $F(\beta)$ has size less than κ. Furthermore, it is clear that $f(\beta) \subseteq F(\beta)$. $\square$

The following theorem by Bukovský states that grounds of W always satisfy the κ-global covering property for W for sufficiently large κ (see [4]).

Theorem B.5. *Let $M \subseteq W$ be transitive models of* ZFC *with the same ordinals and let κ be a regular uncountable cardinal. Then the following are equivalent:*

1. *M satisfies the κ-global covering property for W.*
2. *There is a partial order $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter over M such that $W = M[G]$ and $\mathbb{P}$ satisfies the κ-c.c. in M.*

But also the κ-covering and the κ-approximation property can be related to the forcing partial order. The following theorem is due to Hamkins (see [12]) and Mitchell gave a simpler proof of a more general statement (cf. [25]).

Theorem B.6 (Hamkins)**.** *Let W be a transitive model of* ZFC *and let M be a ground of W. So, there is $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter G over M such that $M[G] = W$. For any infinite cardinal κ with $|\mathbb{P}|^M < \kappa$ the model M satisfies the κ-covering and the κ-approximation property for W.*

We now transfer the definition of the approximation and covering properties to transitive submodels of $\mathcal{H}(\kappa)$.

Definition B.7. Let $\kappa < \chi$ be regular uncountable cardinals and let $\chi \subseteq M \subseteq \mathcal{H}(\chi)$ be a transitive model of $\mathrm{ZFC} - \mathrm{P}$.

Then, we say that M satisfies the *κ-covering property* for $\mathcal{H}(\chi)$ if every set of ordinals $A \subseteq \chi$ with $|A| < \kappa$, i.e. A in $\mathcal{H}(\chi)$, is covered by a set of ordinals B in M with $|B| < \kappa$, i.e. there is a set of ordinals $B \in M$ of size less than κ with $A \subseteq B$.

We say that M satisfies the *κ-approximation property* for $\mathcal{H}(\chi)$ if for any bounded set of ordinals $A \subseteq \chi$ for which $A \cap x \in M$ for every set of ordinals $x \in M$ with $|x| < \kappa$ we have that $A \in M$.

We say that M satisfies the *κ-global covering property* for $\mathcal{H}(\chi)$ if for any ordinal $\alpha < \chi$ and any function $f : \alpha \to \chi$, i.e. $f \in \mathcal{H}(\chi)$, there is a function $F \in M$ with $\mathrm{dom}(F) = \alpha$ and $f(\beta) \in F(\beta)$ as well as $|F(\beta)| < \kappa$ for all $\beta < \alpha$.

We start the proof of the DDG by establishing some results about the covering and approximation properties for model of $\mathrm{ZFC} - \mathrm{P}$.

Lemma B.8 (Usuba). *Let κ and χ be regular uncountable cardinals with $\kappa^+ < \chi$. Further, let M and N be transitive models of* $\mathrm{ZFC} - \mathrm{P}$ *with $\chi \subseteq M, N \subseteq \mathcal{H}(\chi)$. Assume that M and N satisfy the κ-global covering and the κ-approximation property for $\mathcal{H}(\chi)$ and that $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$. Then $M = N$.*

Proof. First, we show that $(\kappa^+)^M = \kappa^+$. Suppose there is a surjection $f : \kappa \to (\kappa^+)^M$. Then by the κ-global covering property there is a function $F \in M$ with $f(\beta) \in F(\beta)$ and $|F(\beta)| < \kappa$ for all $\beta < \kappa$. But that means that $(\kappa^+)^M \subseteq \bigcup_{\beta<\kappa} F(\beta)$ and this union of κ sets of size less than κ has size κ in M, a contradiction. Hence, $\kappa^+ = (\kappa^+)^M = (\kappa^+)^N$. This argument also allows to conclude that M and N satisfy the κ-covering property for $\mathcal{H}(\chi)$.

Now, we claim that $\mathcal{P}(\alpha) \cap M = \mathcal{P}(\alpha) \cap N$ for any $\alpha < \kappa^+$. As $|\alpha|^M = \kappa$ there is a bijection $f : \kappa \to \alpha$ which allows us to obtain the set $X := \{\langle \beta, \gamma \rangle \in \kappa \times \kappa : f(\beta) < f(\gamma)\} \in M$. Taking a definable bijection between κ and $\kappa \times \kappa$ we can identify X with an element of $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$ and see that X is thus also in N. As we can define f from X we also have that $f \in N$ and using that $f \in M, N$ we can map subsets of α to subsets of κ and conclude that $\mathcal{P}(\alpha) \cap M = \mathcal{P}(\alpha) \cap N$.

As remarked above, it is enough to show that M and N contain the same sets of ordinals. We first show that they contain the same sets of size $< \kappa$. So, let $A \subseteq \chi$ be a set in M with $|A| < \kappa$.

We will first show that there is a set $B \supseteq A$ of size less than κ which is in M and N. Recursively, we define a sequence $\langle x_\beta, y_\beta : \beta < \kappa \rangle$ of sets of ordinals such that $A \subseteq x_0, y_0$ and:

1. $x_\beta \in M$, $y_\beta \in N$ and $|x_\beta|, |y_\beta| < \kappa$ for all $\beta < \kappa$.

2. $\langle x_\beta : \beta < \kappa \rangle$ and $\langle y_\beta : \beta < \kappa \rangle$ are $\subseteq$-increasing.

3. $x_\beta \cup y_\beta \subseteq x_{\beta+1} \cap y_{\beta+1}$ for all $\beta < \kappa$.

We can choose $x_0 = A$ and by the κ-covering property we find a $y_0 \supseteq A$ in N with $|y_0| < \kappa$. Now, suppose we already defined $\langle x_\beta, y_\beta : \beta < \gamma \rangle$ for some $\gamma < \kappa$. Then $\bigcup_{\beta<\gamma} x_\beta \cup y_\beta \subseteq \chi$ has size less than κ. Again by the κ-approximation property we can choose $x_\gamma, y_\gamma \subseteq \chi$ of size $< \kappa$ with $x_\gamma \in M$ and $y_\gamma \in N$ such that $\bigcup_{\beta<\gamma} x_\beta \cup y_\beta \subseteq x_\gamma, y_\gamma$.

We set $B := \bigcup_{\beta<\kappa} x_\beta \cup y_\beta = \bigcup_{\beta<\kappa} x_\beta = \bigcup_{\beta<\kappa} y_\beta$. We use the κ-approximation property to show that $B \in M$. Let $c \in M$ be a set of ordinals of size $< \kappa$. Since $B = \bigcup_{\beta<\kappa} x_\beta$ and $|B \cap c| < \kappa$ we know that there must be a $\gamma < \kappa$ such that $B \cap c \subseteq x_\gamma$ because the numbers of the stages of the union at which the elements of $B \cap c$ occur cannot be cofinal in κ. But then $B \cap c = x_\gamma \cap c$ which is in M. Therefore, $B \in M$ by the κ-approximation property. By the same argument, we also get that $B \in N$.

Now let $\pi : B \to \delta$ be the transitive collapse of B (so δ is the order type of B). As $|B| \leq \kappa$ we know that $\delta < \kappa^+$. As B is in M and N the same holds for π. But then the image $\pi[A]$ of A under π is in $\mathcal{P}(\delta) \cap M$ and therefore also in N as $\mathcal{P}(\delta) \cap M = \mathcal{P}(\delta) \cap N$. But then $\pi^{-1}(\pi[A]) = A \in N$. So, M and N contain the same sets of ordinals of size less than κ.

Finally, let $C \subseteq \chi$ be in M. We show that $C \in N$. For any $x \subseteq \chi$ in M of size less than κ we know that $x \cap A \in M$. But by the previous claim that also means $x \cap A \in N$. So, by the κ-approximation property $A \in N$. $\square$

Lemma B.12 below provides conditions under which the κ-global covering property already implies the κ^+-approximation property. For the proof we need a few results about the combinatorics of trees. We start with a well-known theorem by Kurepa from 1935 (see [23]):

Theorem B.9. *Let $\kappa < \lambda$ be regular cardinals. Let T be a tree of height λ such that each level T_α, $\alpha < \lambda$, has size less than κ. Then T has a cofinal branch.*

On the other hand, the following lemma shows that the number of cofinal branches is small:

Lemma B.10 (Usuba). *Let $\kappa < \lambda$ be regular cardinals. Let T be a tree of height λ such that each level T_α, $\alpha < \lambda$, has size less than κ. Then T has less than κ many cofinal branches.*

Proof. Towards a contradiction, suppose that $\langle B_\alpha : \alpha < \kappa \rangle$ is a one-to-one enumeration of cofinal branches of T. For any two $\alpha < \beta < \kappa$ there is a least level at which B_α and B_β differ, i.e. there is a least γ such that $B_\alpha \cap T_\gamma \neq B_\beta \cap T_\gamma$. Define $d(\alpha, \beta)$ to be this least γ for all $\alpha < \beta < \kappa$. Then $\delta := \sup\{d(\alpha, \beta) : \alpha < \beta < \kappa\} < \lambda$ as $\kappa < \lambda$ and λ is regular. But as $B_\alpha \cap T_\delta \neq B_\beta \cap T_\delta$ for $\alpha \neq \beta$, T_δ contains at least κ many distinct elements of $B_\alpha \cap T_\delta$ for $\alpha < \kappa$. This contradicts $|T_\delta| < \kappa$. $\square$

Lemma B.11 (Usuba). *Let κ be a regular cardinal and let μ be an ordinal with $\mathrm{cf}(\mu) > \kappa$. Let $W \subseteq V$ be a transitive model of ZFC containing all ordinals. Let $T \in W$ be a tree of height μ all whose levels have size less than κ. Then every cofinal branch of T (in V) is in W.*

Proof. Fix a cofinal subset $X \subseteq \mu$ in W of order type $\mathrm{cf}(\mu)^W$. Note that $\mathrm{cf}(\mu)^W \geq \mathrm{cf}(\mu)^V > \kappa$. Consider the tree $T' := \bigcup_{\alpha \in X} T_\alpha$. This is a tree of height $\mathrm{cf}(\mu)^W$ in W. Furthermore, any cofinal branch B through T yields a unique cofinal branch $B' = B \cap T'$ through T' as X is cofinal and conversely for any cofinal branch C' in T' there is a unique cofinal branch C with $C' = C \cap T'$.

Now, let $B \in V$ be a cofinal branch in T and let $B' = B \cap T'$. Suppose $B \notin W$ and thus $B' \notin W$. By lemma B.10 we know that in W there are less than κ many cofinal branches through T' as $\mathrm{cf}(\mu)^W$ is regular and bigger than κ in W. Let $\langle B'_\alpha : \alpha < \lambda \rangle$ be an enumeration of all cofinal branches of T' in W for some $\lambda < \kappa$ and let $\langle B_\alpha : \alpha < \lambda \rangle$ enumerate the corresponding cofinal branches in T. Working in V, we know that for each $\alpha < \lambda$ there is a $d(\alpha)$ such that $B \cap T_{d(\alpha)} \neq B_\alpha \cap T_{d(\alpha)}$. As in the previous proof we take $\delta := \sup\{d(\alpha) : \alpha < \lambda\} < \mu$ as $\lambda < \mathrm{cf}(\mu)^V$. So, there is a unique t in $T_\delta \cap B$. This t is by construction not contained in any cofinal branch of T in W. The tree $S := \{s \in T : s \leq t \text{ or } t \leq s\}$ in W still has height μ as B is a branch through it and each of its levels has size less than κ. Going to $S' = \bigcup_{\alpha \in X} S_\alpha$, we can apply Kurepa's theorem and get a cofinal branch C' in W. But then C' is also cofinal in T' and the corresponding cofinal branch C of T in W with $C \cap T' = C'$ contains t which is a contradiction. $\square$

Lemma B.12 (Usuba)**.** *Let θ be a strong limit cardinal and $\kappa < \theta$ a regular uncountable cardinal. Define $\chi := \theta^+$. Assume $W \subseteq V$ is a transitive model of* ZFC *and define $M := \mathcal{H}(\chi) \cap W$. If M satisfies the κ-global covering property for $\mathcal{H}(\chi)$ then M satisfies the κ^+-approximation property for $\mathcal{H}(\chi)$.*

Proof. We start with two observations:

1. For $\kappa \leq \alpha < \chi : \alpha$ is regular in M iff α is regular in V.

2. For $\kappa \leq \alpha < \chi : \alpha$ is a cardinal in M iff α is a cardinal in V.

Both observations follow from the κ-global covering property. Suppose that α is singular in V. So, there is a cofinal function $f : \beta \to \alpha$ for some $\beta < \alpha$. By the κ-global covering property there is now a function $F : \beta \to \mathcal{P}(\alpha) \cap M$ in M such that $f(\gamma) \in F(\gamma)$ and $|F(\gamma)| < \kappa$ for all $\gamma < \beta$. But then $\bigcup_{\gamma < \beta} F(\gamma)$ is an unbounded subset of α if size at most $|\beta| \cdot \kappa < \alpha$. Note that $\alpha \neq \kappa$ as κ is regular. If α is regular in V, on the other hand, then clearly α is regular in M by the downwards absoluteness between models of $\mathrm{ZFC} - \mathrm{P}$.

Similarly, for observation 2 if α is not a cardinal in V then there is a surjection $f : \beta \to \alpha$ for some $\beta < \alpha$ and the same procedure as above shows that then α has size $|\beta| \cdot \kappa$. Again, if α is a cardinal in V then α is also a cardinal in M.

We prove the κ^+-approximation property by induction: We have to show that for any bounded set $A \subseteq \chi$ such that $A \cap x \in M$ for all $x \subseteq \chi$ in M with $|x| \leq \kappa$ we have $A \in M$. For any bounded subset A of χ, there is some $\alpha < \chi$ such that $A \subseteq \alpha$ and we do an induction on this α.

So, let $\alpha < \chi$ and suppose that for any $\beta < \alpha$ and $B \subseteq \beta$ we know that $B \in M$ if $B \cap x \in M$ for all $x \subseteq \beta$ in M with $|x| \leq \kappa^+$. Let $A \subseteq \alpha$ be such that $A \cap x \in M$ for every $x \in M$ with $|x| \leq \kappa$. We distinguish three cases:

Case 1: α is not a cardinal (in M).

By observation 2, we know that α is not a cardinal in M. Hence, there is a bijection $\pi : \beta \to \alpha$ in M for some $\beta < \alpha$. But then $\pi^{-1}(A) \subseteq \beta$ and for any $x \subseteq \beta$ in M we have that $\pi^{-1}(A) \cap x = \pi^{-1}(A \cap \pi[x])$. So, if $|x| \leq \kappa$ then $A \cap \pi[x] \in M$ by assumption and therefore also $\pi^{-1}(A) \cap x \in M$. By induction hypothesis, we can conclude that $\pi^{-1}(A) \in M$ and thus $A \in M$.

Case 2: α is a cardinal and $\mathrm{cf}(\alpha) < \kappa^+$.

By observation 2, we know that $\alpha \leq \theta$ as $\alpha < \theta^+$. Let $[\alpha]^{<\kappa^+}$ denote the set of all subsets of α of size less than κ^+. The idea of the proof is the observation that for any unbounded $X \subseteq \alpha$ the set A is the union of the sets $A \cap \beta$ with $\beta \in X$ and by induction hypothesis $A \cap \beta \in M$. As $cf(\alpha) < \kappa^+$ we can choose X of size less than κ^+. Then we construct a certain elementary submodel N of $\mathcal{H}(\chi)$ which is closely related to M in the sense that $N \cap \alpha \in M$. This model will allow us to define the sequence $\langle A \cap \beta : \beta \in X \rangle$ in M.

To obtain such a model N with $N \cap \alpha \in M$ we start by showing that $[\alpha]^{<\kappa^+} \cap M$ is stationary in $[\alpha]^{<\kappa^+}$. By [20, lemma 8.26], it is enough to show that $[\alpha]^{<\kappa^+} \cap M$ contains a closure point of any function $f : [\alpha]^{<\omega} \to [\alpha]^{<\kappa}$. Here, $x \subseteq \alpha$ is a closure point of f if $f(e) \subseteq x$ for all $e \subseteq x$.

So, let $f : [\alpha]^{<\omega} \to [\alpha]^{<\kappa}$ be a function. By the κ-global covering property of M we can find a function $F : [\alpha]^{<\omega} \to M \cap \mathcal{P}([\alpha]^{<\kappa^+})$ such that $f(e) \in F(e)$ and $|F(e)| < \kappa$ for all $e \in [\alpha]^{<\omega}$. In particular, that means that $f(e) \subseteq \bigcup F(e) \cap \alpha$ and furthermore $|\bigcup F(e)| \leq \kappa$ as this is a union of less than κ many sets of size less or equal to κ. Let $G(d) = \bigcup F(d) \cap \alpha$ for all $d \in [\alpha]^{<\omega}$.

We recursively construct a closure point of f in $[\alpha]^{<\kappa^+} \cap M$. Let $x_0 = \emptyset$. Once x_n has been defined, let $x_{n+1} = \bigcup\{G(d) : d \in [x_n]^{<\omega}\}$. By induction, it is easy to check that $|x_n| \leq \kappa$ for all $n \in \omega$: Suppose x_n has size $\leq \kappa$. Then $[x_n]^{<\omega}$ has size $\leq \kappa$ and hence x_{n+1} is the union of $\leq \kappa$ many sets of size $< \kappa$. Let $x = \bigcup_{n \in \omega} x_n \in [\alpha]^{<\kappa^+} \cap M$. We show that x is a closure point of f. So, let $e \subseteq x$ be finite. That means that there is an $n \in \omega$ such that $e \subseteq x_n$. But then $f(e) \subseteq x_{n+1} \subseteq x$ by construction. So, $[\alpha]^{<\kappa^+} \cap M$ is stationary in $[\alpha]^{<\kappa^+}$.

As $\alpha \leq \theta$ we know that $\{x \subseteq \alpha : x \text{ is bounded in } \alpha\}$ has size $\leq \theta^{<\theta} = \theta$ and hence is in $\mathcal{H}(\chi)$. So, $M \cap \{x \subseteq \alpha : x \text{ is bounded in } \alpha\} \in M$. Let $\langle B_i : i < \mu \rangle$ be an enumeration of $M \cap \{x \subseteq \alpha : x \text{ is bounded in } \alpha\}$ in M for some cardinal μ. Further, let X be an unbounded subset of α in M with $|X| < \kappa^+$. By observation 1 such an X exists.

Now, for each $\beta < \alpha$ let $i(\beta)$ be the least ordinal such that $A \cap \beta = B_{i(\beta)}$. In V this defines a function and we let $f : X \to \alpha$ be the restriction to X. Then by the κ-global covering property we find $F : X \to \mathcal{P}(\alpha) \cap M$ in M such that $f(\beta) \in F(\beta)$ and $|F(\beta)| < \kappa$ for all $\beta \in X$. Then $Y := \bigcup_{\beta \in X} F(\beta) \in M$ has size $\leq \kappa$ and $\{i(\beta) : \beta \in X\} \subseteq Y$.

We know that $A = \bigcup_{\beta \in X} B_{i(\beta)}$. So, it is enough to show that $\{i(\beta) : \beta \in X\} \in M$. Let $\mathcal{N}$ be the collection of all elementary submodels N of $\mathcal{H}(\chi)$ such that

1. $\{\langle B_i : i < \mu\rangle, X, Y, A\} \cup \kappa \subseteq N$.

2. $|N| = \kappa$.

By the Skolem hull construction, we know that $\mathcal{N}$ is not empty as $|\{\langle B_i : i < \mu\rangle, X, Y, A\} \cup \kappa| = \kappa$ and the language has only one symbol.

We claim that there is an $N \in \mathcal{N}$ with $N \cap \alpha \in M$. We will prove this by showing that $\{N \cap \alpha : N \in \mathcal{N}\}$ is closed and unbounded in $[\alpha]^{<\kappa^+}$. Then, we find such a model as $[\alpha]^{<\kappa^+} \cap M$ is stationary in $[\alpha]^{<\kappa^+}$.

That $\{N \cap \alpha : N \in \mathcal{N}\}$ is unbounded is easy to see as for any $x \subseteq \alpha$ of size at most κ we can build the Skolem hull of $\{\langle B_i : i < \mu\rangle, X, Y, A\} \cup \kappa \cup x$ in $\mathcal{H}(\chi)$ and obtain an model $N \prec \mathcal{H}(\chi)$ of size κ with $N \cap \alpha \supseteq x$.

Now, let $\gamma < \kappa$ be a limit ordinal and $\langle N_i : i < \gamma\rangle$ be a sequence in $\mathcal{N}$ such that for $i < j < \gamma$ we have $N_i \cap \alpha \subseteq N_j \cap \alpha$. We modify the sequence such that the models themselves also form an increasing sequence. For any $i < \gamma$ let N_i' be the Skolem hull of

$$\{\langle B_i : i < \mu\rangle, X, Y, A\} \cup \kappa \cup (N_i \cap \alpha)$$

in N_{i+1}. Then, we get that $N_i' \subseteq N_{i+1}'$ for all $i < \mu$. Furthermore,

$$\bigcup_{i<\gamma}(N_i \cap \alpha) = \bigcup_{i<\gamma}(N_i' \cap \alpha)$$

as $N_i \cap \alpha \subseteq N_i' \cap \alpha \subseteq N_{i+1} \cap \alpha$ for each $i < \gamma$. Now, the direct limit D of $\langle N_i' : i < \gamma\rangle$ is an elementary supermodel of all N_i' and hence clearly $D \prec \mathcal{H}(\chi)$. Furthermore, as in the chain $\langle N_i' : i < \gamma\rangle$ the elementary embeddings are the inclusions we get that $D = \bigcup_{i<\gamma} N_i'$. So, N has size κ as $\gamma < \kappa$ and each N_i' has size κ. But now, $\bigcup_{i<\gamma}(N_i \cap \alpha) = D \cap \alpha$ and as $D \in \mathcal{N}$ we conclude that $\{N \cap \alpha : N \in \mathcal{N}\}$ is closed and unbounded in $[\alpha]^{<\kappa^+}$.

So, we can find a model $N \prec \mathcal{H}(\chi)$ such that $\{\langle B_i : i < \mu\rangle, X, Y, A\} \cup \kappa \subseteq N$, $|N| = \kappa$ and $N \cap \alpha \in M$. Note that by the elementarily equivalence there is a surjection from κ to X in N and since $\kappa \subseteq N$ and the fact that the surjection is also a surjection in $\mathcal{H}(\chi)$ we get that $X \subseteq N$. Likewise, $Y \subseteq N$.

For $\beta \in X$ and $i \in Y$, we know that $i = i(\beta)$ iff $A \cap \beta = B_i$. As $X, Y \subseteq N$ and by the elementarity of N we get that $i = i(\beta)$ iff $A \cap \beta \cap N = B_i \cap N$. As $N \cap \alpha \in M$ and has size less than κ^+ we now by assumption that $N \cap \alpha \cap A \in M$. So in M, we can decide whether $i = i(\beta)$ for $i \in Y$ and $\beta \in X$ as this is the case iff $(N \cap \alpha \cap A) \cap \beta = B_i \cap N \cap \alpha$. So, $\{i(\beta) : \beta \in X\} \in M$ and therefore also $A = \bigcup_{i \in \{i(\beta):\beta\in X\}} B_i \in M$.

Case 3: α is a cardinal with $\mathrm{cf}(\alpha) \geq \kappa^+$ (in M).

Again, we know that $\alpha \leq \theta$. We will show that the characteristic function of $A \subseteq \alpha$ is in M by representing that function $f : \alpha \to 2$ as a cofinal branch through a tree in M so that we can apply the previous lemma.

By induction hypothesis and assumption, we know that $A \cap \beta \in M$ for all $\beta < \alpha$. Hence, also $f \upharpoonright \beta \in M$ for all $\beta < \alpha$.

We claim that $M \cap {}^{<\alpha} 2 \in M$ as well. We know that $M \cap {}^{<\alpha} 2 = \mathcal{H}(\chi) \cap {}^{<\alpha} 2 \cap W$. But since $\alpha \leq \theta$ and θ is a strong limit we know that ${}^{<\alpha}2 \cap \mathcal{H}(\chi) \in \mathcal{H}(\chi)$. Further,

$$({}^{<\alpha}2 \cap \mathcal{H}(\chi))^W = {}^{<\alpha} 2 \cap \mathcal{H}(\chi) \cap W \in W.$$

So, we can conclude that ${}^{<\alpha}2 \cap M \in \mathcal{H}(\chi) \cap W = M$.

Let $\langle g_\gamma : \gamma < \mu \rangle$ be a bijective enumeration of ${}^{<\alpha}2 \cap M$ in M for some $\mu < \chi$. Choose a function $h : \alpha \to \mu$ in V such that $h(\beta) = \gamma$ iff $f \upharpoonright \beta = g_\gamma$. By the κ-global covering property of M we find a function $F : \alpha \to M$ in M such that $h(\beta) \in F(\beta)$ and $|F(\beta)| < \kappa$ for all $\beta < \alpha$. As we know that for any $\beta < \alpha$ the function $g_{h(\beta)}$ has domain β we can require the function F to satisfy that for any $\gamma \in F(\beta)$ the domain of g_γ is β because all of this is definable in M. We define a tree from this function F by first defining

$$T' := \{g_\gamma : \gamma \in F(\beta) \text{ for some } \beta < \alpha\}.$$

Note that $f \upharpoonright \beta \in T'$ for all $\beta < \alpha$. Now, we simply remove all functions from T' whose initial segments are not all in T'. So we define:

$$T := \{g \in T' : g \upharpoonright \gamma \in T' \text{ for all } \gamma \in \mathrm{dom}(g)\}.$$

Then, T ordered by $\subseteq$ is a tree and the βth level of T is contained in $\{g_\gamma : \gamma \in F(\beta)\}$ and hence has size less than κ. Further, we know that $f \upharpoonright \beta \in T$ for all $\beta < \alpha$ and as $T \subseteq {}^{<\alpha} 2 \cap M$ we conclude that T has height α. Thus, the tree T satisfies the conditions for lemma B.11. As the set $\{f \upharpoonright \beta : \beta < \alpha\}$ is a cofinal branch through T we can conclude that this set is in W and hence also $\bigcup \{f \upharpoonright \beta : \beta < \alpha\} = f \in W$. As $f \in \mathcal{H}(\chi)$ as well we get $f \in \mathcal{H}(\chi) \cap W = M$. $\square$

Now, we are ready for the proof of the strong downwards directedness of grounds.

Theorem B.13 (Usuba)**.** *The strong downwards directedness of grounds holds.*

Proof. We will work in V and show that for any set X the collection of grounds defined by the parameters in X has a common ground. That means we will construct an inner model $W \subseteq \bigcap_{r \in X} W_r$ and show that it is a ground of V.

So, fix a set X. For each $r \in X$ we know that V is a forcing extension of W_r. So there is a partial order $\mathbb{P}_r \in W_r$ and a $\mathbb{P}_r$-generic filter G_r over W_r such that $V = W_r[G_r]$. Let κ be a regular uncountable cardinal with $|X| < \kappa$ and for each $r \in X$ also $|\mathbb{P}_r| < \kappa$. By Bukovský's theorem, theorem B.5 we know that each W_r satisfies the κ-global covering property for V and by theorem B.6 it satisfies the κ-approximation property.

First, we will show that for any strong limit cardinal $\theta > \kappa$ there is a transitive model M of $\mathrm{ZFC} - \mathrm{P}$ with $\theta^+ \subseteq M \subseteq \mathcal{H}(\theta^+)$ which satisfies the κ^{++}-global covering and the κ^{++}-approximation property for $\mathcal{H}(\theta^+)$ and which is contained in $\bigcap_{r \in X} W_r$. The idea is to construct a function G with two arguments in $\bigcap_{r\in X} W_r$ which uniformly provides the functions required for the κ-global covering property for all functions in $f \in \mathcal{H}(\theta^+)$. Then, we will take the smallest inner model of V containing G, namely the constructible universe $L[G]$ constructed from G and choose M to be $\mathcal{H}(\chi)^{L[G]}$. As $G \in W_r$ for all $r \in X$, we get that $M \subseteq \bigcap_{r\in X} W_r$ and the existence of G will prove the κ^+-global covering property and lemma B.12 shows the κ^{++}-approximation property.

Let $\chi := \theta^+$. First, let $\gamma = \chi^{<\chi}$ and fix an enumeration $\langle f_\zeta : \zeta < \gamma\rangle$ of ${}^{<\chi}\chi$. We define a function $g : \chi \times \gamma \to \chi$ by

$$g(\alpha,\zeta) = \begin{cases} f_\zeta(\alpha) & \text{if } \alpha \in \mathrm{dom}(f_\zeta), \\ 0 & \text{otherwise.} \end{cases}$$

Then, we recursively define functions $G_{\beta,r}$ for $\beta < \kappa$ and $r \in X$ such that

1. $G_{\beta,r} \in W_r$.
2. $\mathrm{dom}(G_{\beta,r}) = \chi \times \gamma$, $h(\alpha,\zeta) \in G_{\beta,r}(\alpha,\zeta) \subseteq \chi$, and $|G_{\beta,r}(\alpha,\zeta)| < \kappa$ for all $\alpha < \chi$ and $\zeta < \gamma$.
3. $\bigcup_{\beta'<\beta, s\in X} G_{\beta',s}(\alpha,\zeta) \subseteq G_{\beta,r}(\alpha,\zeta)$ for all $\alpha < \chi$ and $\zeta < \gamma$.

For all $r \in G$ we let $G_{0,r} \in W_r$ be a function such that

1. $\mathrm{dom}(G_{0,r}) = \chi \times \gamma$,
2. $g(\alpha,\zeta) \in G_{0,r}(\alpha,\zeta) \subseteq \chi$,
3. and $|G_{0,r}(\alpha,\zeta)| < \kappa$ for all $\alpha < \chi$ and $\zeta < \gamma$. Such a function exists by the κ-global covering property of W_r for V.

Let $0 < \beta < \kappa$ now and suppose now that $G_{\beta',s}$ has been defined for all $\beta' < \beta$ and $s \in X$. Let

$$G'(\alpha,\zeta) = \bigcup_{\beta'<\beta, s\in X} G_{\beta',s}(\alpha,\zeta)$$

for all $\alpha < \chi$ and $\zeta < \gamma$. As $\beta < \kappa$ and $|X| < \kappa$ we know that $|\bigcup_{\beta'<\beta,s\in X} G_{\beta',s}(\alpha,\zeta)| < \kappa$ as each of the sets $G_{\beta',s}(\alpha,\zeta)$ has size less than κ. By lemma B.4, there is a function $G_{\beta,r} \in W_r$ with $G'(\alpha,\zeta) \subseteq G_{\beta,r}(\alpha,\zeta) \subseteq \chi$ and $|G_{\beta,r}(\alpha,\zeta)| < \kappa$ for all $\alpha < \chi$ and $\zeta < \gamma$.

Now, we define the function G via

$$G(\alpha,\zeta) = \bigcup_{\beta'<\kappa, s\in X} G_{\beta',s}(\alpha,\zeta).$$

Then clearly $g(\alpha,\zeta) \in G(\alpha,\zeta)$ and $|G(\alpha,\zeta)| < \kappa^+$ for all $\alpha < \chi$ and $\zeta < \gamma$.

We want to show that $G \in W_r$ for any $r \in X$. We use the κ-approximation property. Let

$$\Gamma := \{\langle \alpha,\zeta,\eta\rangle : \eta \in G(\alpha,\zeta)\}.$$

As G can easily be defined from Γ, it is enough to show that $\Gamma \in W_r$. Let $x \subseteq \chi \times \gamma \times \chi$ be a set of cardinality less than κ in W_r. Let e be the projection of $\Gamma \cap x$ to the first two components, i.e.

$$e = \{\langle \alpha, \zeta \rangle \in \chi \times \gamma : \text{ there is an } \eta < \chi \text{ with } \langle \alpha, \zeta, \eta \rangle \in \Gamma \cap x\}.$$

As $|x| < \kappa$ we also have $|e| < \kappa$. For each pair $\langle \alpha, \zeta \rangle \in e$, we know that

$$E(\alpha, \zeta) := \{\eta : \langle \alpha, \zeta, \eta \rangle \in \Gamma \cap x\} \subseteq G(\alpha, \zeta)$$

and that this set has cardinality $< \kappa$. By condition 3 in the definition of G, we know that

$$G(\alpha, \zeta) = \bigcup_{\beta < \kappa} G_{\beta,r}(\alpha, \zeta).$$

As κ is regular and the sets $G_{\beta,r}(\alpha, \zeta)$ form an increasing sequence, there is a $\delta(\alpha, \zeta) < \kappa$ such that $E(\alpha, \zeta) \subseteq G_{\delta(\alpha,\zeta),r}(\alpha, \zeta)$. Let $\delta = \sup\{\delta(\alpha, \zeta) : \langle \alpha, \zeta \rangle \in e\}$. As $|e| < \kappa$ and κ is regular we get that $\delta < \kappa$. So,

$$x \cap \Gamma = \{\langle \alpha, \zeta, \eta \rangle \in x : \eta \in G_{\delta,r}(\alpha, \zeta)\} \in W_r.$$

By the κ-approximation property, we conclude that $\Gamma \in W_r$. Therefore, $G \in \bigcap_{r \in X} W_r$.

By fixing a bijection $\pi : \chi \times \gamma \times \chi \to \gamma$ in L, we can represent Γ and hence G as a set of ordinals $A := \pi[\Gamma]$. Now, the constructible universe constructed from A can be defined in V and we get the transitive model $L[A]$ of ZFC containing all ordinals. Further, any inner model of V containing A contains all of $L[A]$. So, $L[A] \subseteq \bigcap_{r \in X} W_r$. Now, we let

$$M = \mathcal{H}(\chi)^{L[A]} = \mathcal{H}(\chi) \cap L[A] \subseteq \bigcap_{r \in X} W_r.$$

To see that M satisfies the κ^+-global covering property for $\mathcal{H}(\chi)$, let $f : \alpha \to \chi$ be a function in V for some $\alpha < \chi$. Since $f \in^{<\chi} \chi$, we can find a ζ such that $f = f_\zeta$ in the fixed enumeration of ${}^{<\chi}\chi$. Now, in $L[A]$ we can define the function $F : \alpha \to L[A]$ by $F(\beta) = G(\beta, \zeta)$ for all $\beta < \zeta$. We know that $|F(\beta)| < \kappa^+$ and $f(\beta) = g(\beta, \zeta) \in F(\beta)$ for all $\beta < \alpha$. Furthermore, $F \in \mathcal{H}(\chi)$ as it consists of pairs of ordinals $< \chi$ and bounded subsets of χ. So, $F \in \mathcal{H}(\chi) \cap L[A] = M$ which shows the κ^+-global covering property.

So far, we have shown that, given a successor $\chi = \theta^+$ of a strong limit cardinal $\theta > \kappa$, we can construct a transitive model $\chi \subseteq M_\theta \subseteq \mathcal{H}(\chi)$ which satisfies the κ^+-global covering property for $\mathcal{H}(\chi)$. As the κ^{++}-global covering property is weaker, the model M_θ also satisfies this property for $\mathcal{H}(\chi)$. By lemma B.12, we get that M_θ satisfies the κ^{++}-approximation property for $\mathcal{H}(\chi)$ because we constructed M_θ such that $M_\theta = \mathcal{H}(\chi) \cap L[A]$. In the light of lemma B.8, the model M is in fact uniquely determined by χ and $\mathcal{P}(\kappa^{++}) \cap M_\theta$.

To obtain a common ground of all W_r with $r \in X$, we now construct an increasing sequence $\langle N_\alpha : \alpha \in \mathrm{Ord} \rangle$ of such models of ZFC $-$ P such that the union of the sequence is the desired model. However, we have to make sure that the sequence is coherent in the sense that for any

$\alpha < \beta$ we find a χ such that $N_\alpha = N_\beta \cap \mathcal{H}(\chi)$. The key is lemma B.8 and the fact that there are only set many options for the value $\mathcal{P}(\kappa^{++}) \cap N_\alpha$.

First, let Θ be the class of strong limit cardinals above κ. For each $p \subseteq \mathcal{P}(\kappa^{++})$ define

$$\Theta_p := \{\theta \in \Theta : \mathcal{P}(\kappa^{++}) \cap M_\theta = p\}.$$

As the Θ_p with $p \in \mathcal{P}(\mathcal{P}(\kappa^{++}))$, form a partition of the proper class Θ there must be a $q \subseteq \mathcal{P}(\kappa^{++})$ such that Θ_q is a proper class. Let $\langle \theta_\alpha : \alpha \in \mathrm{Ord}\rangle$ enumerate Θ_q by size. Now, for each α the model M_{θ_α} is the unique model $\theta_\alpha^+ \subseteq M_{\theta_\alpha} \subseteq \mathcal{H}(\theta_\alpha^+)$ which satisfies the κ^{++}-global covering and κ^{++}-approximation property for $\mathcal{H}(\theta_\alpha^+)$ and satisfies $\mathcal{P}(\kappa^{++}) \cap M_{\theta_\alpha} = q$. Furthermore, $M_{\theta_\alpha} \subseteq \bigcap_{r \in X} W_r$.

For each $\alpha < \beta$, we show that $M_{\theta_\beta} \cap \mathcal{H}(\theta_\alpha^+) = M_{\theta_\alpha}$. By construction, $M_{\theta_\beta} = W \cap \mathcal{H}(\theta_\beta^+)$ for some inner model $W \subseteq V$ and it satisfies the κ^+-global covering property for $\mathcal{H}(\theta_\beta^+)$. But then, $M_{\theta_\beta} \cap \mathcal{H}(\theta_\alpha^+) = W \cap \mathcal{H}(\theta_\alpha^+)$ and this model satisfies the κ^+-global covering property for $\mathcal{H}(\theta_\alpha^+)$ because any covering function in W for a function $f : \alpha \to \theta_\alpha^+$ with $\alpha < \theta_\alpha^+$ yields a covering function $F \in W \cap \mathcal{H}(\theta_\alpha^+)$. Further, $\mathcal{P}(\kappa^{++}) \cap M_{\theta_\beta} \cap \mathcal{H}(\theta_\alpha^+) = q$ still holds and so by lemma B.12 and lemma B.8 we get that $M_{\theta_\alpha} = M_{\theta_\beta} \cap \mathcal{H}(\theta_\alpha^+)$. Furthermore, $\theta_\alpha \in M_{\theta_\beta}$ and $M_{\theta_\alpha} = \mathcal{H}(\theta_\alpha^+)^{M_{\theta_\beta}}$. So, $M_{\theta_\alpha} \in M_{\theta_\beta}$.

Now, let $W := \bigcup_{\theta \in \Theta_q} M_\theta$. As W is the union of transitive sets it is transitive and as $\theta \subseteq M_\theta$ for all $\theta \in \Theta_q$ it is also clear that W contains all ordinals. We cite [20, theorem 13.9]:

> A transitive class M is an inner model of ZF if and only if it is closed under the Gödel operations and it is almost universal, i.e. every subset $X \subseteq M$ is included in some $Y \in M$.

For a definition of the Gödel operations see [20, definition 13.6]. It is clear that each M_θ is closed under the Gödel operations and thus also W is closed under the operations as the M_θ form an increasing sequence. That W is almost universal is clear from the fact that $M_\theta \in M_{\theta'}$ for $\theta' > \theta$ in Θ_q. So, W is an inner model of ZF.

Given any $x \in W$ there is $\theta \in \Theta_q$ with $x \in M_\theta$ and as M_θ is a model of $\mathrm{ZFC} - \mathrm{P}$ it contains a well-order on x. So, also in W the set x has a well-order. Therefore, W is an inner model of ZFC and $W \subseteq \bigcap_{r \in X} W_r$.

Finally, we claim that W satisfies the κ^+-global covering property for V which finishes the proof by theorem B.5. So, let $f : \alpha \to \mathrm{Ord}$ be a function in V for some ordinal α. Then the range of f is bounded by some β. Let $\theta \in \Theta_q$ be bigger than β. Then f is a function from $\alpha \to M_\theta$ and by the κ^+-global covering property there is a function $F \in M_\theta \subseteq W$ with $\mathrm{dom}(F) = \alpha$ and for all $\alpha' < \alpha$, $f(\alpha') \in F(\alpha')$ and $|F(\alpha')| < \kappa^+$. Hence, W satisfies the κ^+-global covering property and is hence a ground of V. So by lemma A.22 it is a common ground of all W_r. $\square$