Eleusis: Complexity and Interaction in Inductive Inference

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Abstract

This paper analyzes the computational complexity of the inductive inference game *Eleusis*. Eleusis is a card game in which one player constructs a secret rule which has to be discovered by the other players. We determine the complexity of various decision problems that arise in Eleusis. We show that on the one hand, many of the problems are solvable in polynomial time whereas on the other hand, the rules of Eleusis allow for secret rules that can force players to face intractable and undecidable problems. Our results show that computational complexity plays a crucial role in the game and has to be taken into account by the players in their strategic considerations. As Eleusis can be seen as a simulation of inductive inference using membership queries, our results also have relevance for interactive approaches to formal learning theory.

1 Introduction

Board and card games have been widely studied in computer science and artificial intelligence; key aspects here are the computational complexity of the games and the strategic abilities of the players. In this paper, we want to put forward the complexity theoretical analysis of a particular class of games, called *inductive inference games*, which are not only interesting from a game theoretical and computational perspective but also from a philosophical and learning theoretical point of view as they provide a simulation of scientific discovery. The general idea of inductive inference games is that players try to infer a general rule from the feedback they get to their moves. One designated player has to come up with a rule about which moves of the other players are accepted and which are rejected. The goal of the other players is then to discover the rule. They make their moves (which can e.g. be of the form of playing cards [1, 8, 14] or building configurations with objects [17]) and the first player gives feedback as to whether a move was accepted or rejected. Then the players use this information to inductively infer the rule.

In the card game Eleusis, Player 1 – who in the game is referred to as *God* or *Nature* – comes up with a rule about sequences of cards. Then the other players – called *Scientists* – take turn in each playing a card in a sequence. After each move, Player 1 announces whether the card was accepted. Rejected cards are moved out of the sequence and stay below the position at which they were played. This way during the whole game, all players can see which cards have been accepted or rejected at which positions. The setting of Eleusis allows for a formal and game theoretical analysis of interactive processes involving inductive inference.

Eleusis has received attention within the philosophy of science literature, since it nicely illustrates scientific inquiry [21]: Playing the cards can be seen as performing experiments, and the feedback given by Player 1 (i.e. the acceptance or rejection of the cards played) can be thought of as the outcomes of the experiments. The players form hypotheses about the rule and choose to perform experiments accordingly, after each move updating their information state with the outcome of the experiment, and then revising their hypotheses. The game Eleusis can thus be seen as a nice simulation of scientific inquiry in which players employ two kinds of strategies: *selection strategies*, which determine what experiment to perform (i.e. what cards to play), and *reception strategies* for using the results of the experiments (i.e. the acceptance and rejection of the cards) for constructing and choosing hypotheses about the rule. Eleusis has also been investigated within the computer science and artificial intelligence literature since there is a close relationship to pattern recognition as discovering a rule essentially means to discover a pattern in the sequence of accepted cards. Several algorithms have been developed taking the role of the scientist in Eleusis [3, 6, 18]. Some sample secret rules have been classified informally with respect to the difficulty for the scientist players to discover them [14]. However, to the best of our knowledge, there has not been done any complexity theoretical analysis of Eleusis. In this work, we show that computational complexity plays a crucial role in Eleusis, and give complexity results with a practical relevance for the actual play of the game. Player 1's choice of rule not only determines the difficulty of the tasks of the other players during the game but also has an impact for herself since as we show there are secret rules that Player 1 can choose that make it impossible for herself to give feedback to the other players since she is faced with undecidable problems during the play.

The remainder of this paper is structured as follows: Section 2 describes the rules of the game Eleusis and its version that we investigate in the current work.

Section 3 gives our main results about the complexity of Eleusis. This is done for several classes of secret rules, showing the impact of the allowed secret rules on the complexity of the game. Section 4 concludes this work and gives directions for further work.

2 Eleusis: The Game

In this section, we will describe the rules of Eleusis and the version of it considered in this work. There are several versions of the rules of the card game Eleusis [1, 8, 14]. In this paper, we will focus on *Eleusis Express* [14]. We first briefly give the rules in order to give the reader an idea of the actual game, as it is played in practice, and then give our version of the game.

2.1 Eleusis Express

Beginning of the Game. One player (we call her *Player 1*) has the designated role of *God* or *Nature*. She starts the game by coming up with a secret rule determining which sequences of cards are accepted. An example of such a rule is the following: "*every black card has to be followed by a card with an even value*". Player 1 writes down the rule without any other player seeing it.

Secret Rule. The only constraints on the secret rule are that it can only take into account the sequence of cards previously accepted and the card currently played. So, whether a particular card is accepted can only depend on the cards previously accepted and the card itself. External factors, such as who played the card whether the player uses his left or right hand to play the card, have to be irrelevant.

Playing Procedure. Then each of the other players receives a number of cards (usually 12). Player 1 takes a card from the deck, called the *starter card*. This card will be the first card of what is called the *mainline*. Then the other players take turns in each playing one of their cards by appending it on the right to the mainline. After each move, Player 1 announces whether this card is accepted according to the secret rule. If it is rejected, it is moved from the mainline to the *sideline*, directly below the position at which it was played in the mainline, and the player who played the card has to draw an additional card from the deck. In case the card played is accepted, it stays in the mainline and the player does not need to draw a card. If a player thinks that none of the cards on his hand would be accepted, he can declare "*no play*". In this case, his hand of cards has to be shown to everyone, and Player 1 has to check whether indeed none of the cards would have been accepted. If this is the case, Player 1 gives him a new hand of cards, which should be one card less than the hand he had before. If Player 1 finds a card that could have been played, he plays it and the player has to draw a card from the deck.

Guessing the Rule. If a player has made a correct play, i.e. he played a card that was accepted or he correctly declared *no play*, he can make a guess about the rule and say it out loud. If the guess is correct, the game ends.

End of the Game. The game ends if a player has discovered the rule or gotten rid of all his cards.

Scoring. The player with the highest score wins, where the score is calculated as follows. Each player gets twelve points minus the number of cards on his hand. Having no cards gives a bonus of three points and having guessed the rule correctly gives a bonus of six points. Player 1 cannot score in the game. The idea is to play several times so that each player once takes the role of Player 1, and then add up the points of the different plays.

Eleusis Express as given above turns out to be rather difficult to formally analyze from a game theoretical perspective. Features that contribute to these difficulties are the chance factor arising from the hand of cards a player receives in the beginning, and the scoring rule. In practice, the communication between the players also has a great influence on their strategic reasoning. As a first step towards an analysis of Eleusis Express, we will start by investigating a simplified version of the game.

2.2 Eleusis Express Simplified

In this work, we consider a game between two players Payer 1, and Player 2. With respect to Eleusis Express as explained above, we make the following changes, which simplify the formal analysis. We eliminate the chance factor by supposing that Player 2 has an infinite supply of cards at hand, so that at any point of the game, he has any of the different cards available for playing. Coming back to viewing Eleusis as a simulation of scientific inquiry, this means that there are unlimited resources for conducting experiments, i.e. any experiment can be conducted. In case a card is rejected, it is put in the sideline but Player 2 does not need to draw a new card from the deck. Thus, all players have perfect information about the moves that have taken place after the secret rule was constructed. We let the game end as soon as Player 2 has guessed the rule.

The main reason for the adaptations of the original game are that we want to make the discovery of the secret rule the aim of Player 2. Note that in the original game, even though discovering the rule gives a considerable bonus, it can still be that a player who has discovered the rule is not the winner because he might have a lot more cards than some other player. Thus, in the original game, in the strategic considerations of Player 2, conflicts might arise between choosing to play cards as to get rid of them as soon as possible, and playing cards as to maximize the information about the secret rule.

With respect to its winning conditions, the version of Eleusis we consider is closely related to the game *Zendo* [17], a game in which the goal is to inductively infer a secret rule about the configuration of pyramid shaped pieces. After we have adapted the rules of Eleusis, now leaving Player 2 with the only objective of finding out the secret rule, the reader familiar with the game game *Mastermind* might also see some similarities between our version of Eleusis and Mastermind.

Mastermind is a *deductive* inference game which has received a lot of attention within computer science [15, 16, 23] and also psychology [4, 24]. In this game, one player constructs a code consisting of four pegs that can each have one of six different colors. The other player starts by guessing the code and gets feedback from the first player saying how many colors were at the correct position, and how many were at wrong positions. The game continues until Player 2 has inferred the code. Whereas the roles of the players seem similar in Mastermind and Eleusis, there are some substantial differences. For Mastermind, there are strategies that allow a player to infer the secret code with certainty within a small number of rounds (e.g. five) [16]. In Eleusis, in general this is not possible as there are rules that cannot be identified with certainty at a finite stage of the game. Speaking in terms of formal learning theory, there are thus rules which are not finitely identifiable [19]. Another difference is the impact of the chosen code or rule on the difficulty of the subsequent play. In Mastermind, the difficulty for Player 2 to infer the code and for Player 1 to check the guesses of Player 2 are similar for all the codes that Player 1 could choose. As we illustrate in Section 3, in Eleusis on the other hand, the choice of secret rule has a great influence on the difficulty of the game for both players.

3 Complexities in Eleusis

In this section, we will give a complexity analysis of different decision problems and tasks involved in Eleusis. One motivation behind this is to investigate the complexity involved in scientific inquiry, trying to determine what features of rules contribute to the difficulty of their inductive discovery. We are interested in the complexity that agents face in interactive processes involving inductive inference. Thus, we examine the complexity of the game Eleusis from an agent-oriented perspective focussing on different tasks the players face during the game rather than taking an external perspective examining the complexity of determining which player has a winning strategy. There are several levels of complexity in the game of Eleusis. On the one hand, there is the complexity or difficulty of playing the game itself, as there is the challenge for Player 1 to choose a rule of an adequate level of complexity. Note that there is a close relationship between the complexity/difficulty of playing the game and the complexity of the secret rule.

One way to determine the complexity of the secret rules would of course be empirically, by determining how difficult it is for human subjects to discover them. This would lead to an interesting study identifying the features of rules about (finite) sequences that make their discovery easy or difficult. For the moment, we leave such an analysis to future work, and in this paper we focus on a more theoretical analysis of the complexity of the secret rules in Eleusis.

Another perspective from which we can investigate the complexity in Eleusis is to capture the complexity of the secret rules using methods from descriptive complexity by specifying the formal languages in which the rules can be expressed. This way, the complexity of a rule is captured by the expressive power required to express it in a formal language.

Example 1 (Rules of different descriptive complexity)**.** *As examples, consider the following two rules*

1. "At even positions accept a card iff it is red, and at odd positions accept a card iff it is black."

2. "First accept two black cards, then three red, then five black, . . . then p_{2k} red, then p_{2k+1} black cards, etc." *Where p_n is the n-th prime number.*

Then, it is easy to see that Rule 1 can be expressed by a regular expression whereas Rule 2 cannot.

The complexity of the secret rules can also be analyzed by investigating the computational complexity of different decision problems arising from the secret rules. We will now present some of these decision problems informally and explain their motivation, before we will investigate them in more detail and more formally. Consider the following decision problems related to Eleusis.

1. Given a class of rules, a configuration of the game (i.e. a finite sequence of cards (accepted/rejected)), is there a rule in the class such that the play so far has been consistent with the rule (i.e. a rule that could have been the secret rule)?

2. Given a rule and a configuration of the game, is the play consistent with the rule?

3. Given a rule, a finite sequence of previously accepted cards, and a card c, is c accepted by rule?

Problem 1 is the Eleusis analogue of the problem that has been investigated for Mastermind and has been shown to be NP-complete [23]. However, an important difference of the problem for Eleusis is that we restrict the class of secret rules. The reason for this is the following. Suppose we are given a sequence of cards that have been accepted/rejected in the game so far. Now, if we ask whether there is some rule that could be the secret rule that Player 1 constructed, we only need to check if no card has been both rejected and accepted at the same position. If there is no such card, then the answer to the question is *yes* because there are rules that are consistent with the play so far (e.g. the rule that explicitly says for each position to accept the card that actually has been accepted and to reject the cards that have been rejected).

Problem 2 is a problem that Player 2 encounters when analyzing the current situation in the game and deliberating whether a certain rule might be the secret rule. Problem 3 is relevant in the game because it describes the very task that Player 1 has to solve in each round. This problem is of course very relevant in practice and should be kept in mind by Player 1 when constructing the rule.

A closer investigation of these decision problems requires that we first formalize some aspects of Eleusis. Let start by fixing some notation.

Notation 1.

- *We let Card to be a finite set, representing the set of cards; alternatively we could also represent cards as a pair* $(value, suit)$.
- $Card^*$ *is the set of finite sequences of elements of Card.*
- *For* $s \in Card^*$, $|s|$ *denotes the length of the sequence* s*, defined in the standard way.*
- *For* X *being a set* $|X|$ *denotes the cardinality of* X.
- s_i *denotes the ith element of the sequence of cards* s*, and* $s_{<i}$ *denotes the initial subsequence of* s *of length* i*, i.e. if* $s = s_0 s_1 \dots s_i \dots s_n$*, then* $s_{<i} = s_0 s_1 \dots s_{i-1}$
- *For* $s, t \in Card^*$, st *is the sequence of cards resulting from the concatenation of* s *and* t.
- *By* $\overline{C}_i$*, we denote the set of cards that have been rejected at position* i.

Next, we want to formalize the secret rules. Considering Eleusis in practice, human players mostly define rules in terms of certain properties or attributes that the cards have, such as *color, suit* and *value* but also properties of having a face (of some gender) and certain numerical properties of the value, such as being even/odd, greater/smaller than some number or being prime. Analyzing the reasoning involved in humans playing Eleusis requires a cognitively adequate representation of the rules in terms of the attributes and properties of the cards. Technically speaking however, all of the rules can of course also be expressed in terms of the cards itself. This is what we will do in this paper.

An Eleusis rule says which sequences of cards are accepted and which are not. The way in which the rules are used in the game is that in each round Player 1 has to check whether it is accepted to extend the current sequence with a certain card. Thus, we represent rules as functions that tell us for every pair of sequence of cards and a single card whether appending the sequence with the card is allowed.

Definition 3.1. *Eleusis rules ρ are functions $\rho : Card^{*} \times Card \to \{0, 1\}$.*

Note that with this definition, whether a card is accepted is fully determined by the sequence of cards that have been accepted so far; the previously rejected cards are irrelevant here. In practice, it can probably be observed that most rules chosen by human players have the property that accepted sequences are closed under taking prefixes, i.e. for any $s \in Card^{*}$, if $\rho(s, c) = 1$, then also for every $0 \leq i < |s|$, $\rho(s_{<i}, s_i) = 1$. However, in the rules of Eleusis Express, this is not required, and therefore neither will we do here.

3.1 Easy Problems in Eleusis

In the following we will focus on several restricted classes of rules.

Definition 3.2 (Periodic Rules)**.** *We call a secret Eleusis rule ρ **periodic** if it satisfies the following condition: There is some $p \in \mathbb{N}$ such that for all $s, s' \in Card^{*}, c \in Card$, if $|s| = |s'| = n$ and for all $0 \leq l < |s|$, if $l \mod p = n \mod p$, then $s_l = s'_l$, then $\rho(s, c) = \rho(s', c)$. We call the smallest such p the **number of phases** of ρ. A periodic rule ρ with p phases can then be written as a sequence of rules $(\rho_0, \ldots \rho_{p-1})$, where $\rho(s, c) = \rho_i(s, c)$ if $|s| \mod p = i$.*

Periodic rules are thus rules that can be split into different phases, each following some rule which is independent of the other phases. Let us give some examples of periodic rules.

Example 2 (Periodic Rules)**.**

1. *1 Phase:* "At every position, accept all the red cards and the black ones with a male face. The other black cards are only accepted if they are preceded by two red cards."
2. *2 Phases:* "On even positions only accept cards that have a face or whose value is greater than or equal to the one of the card at the previous even position. At odd positions, accept any card."
3. *3 Phases:* "Two cards of even value, then one with an odd value, then two even ones again, etc."

Comparing these rules, we see that in Rule 3 we only need to look at the current position in order to determine whether a card is accepted. In Rule 1, on the other hand, if a black card without a male face is played, then Player 1 has to look at the two previously accepted cards in order to determine if the card is accepted. In Rule 2 on even positions we also have to look at the card that is placed at the previous even position, in order to check if a card is accepted.

This leads us to the concept of *lookback*, which is the length of the sequence of previously accepted cards that are relevant when deciding whether a card should be accepted.

Definition 3.3 (Lookback)**.** *Let ρ be an Elesis rule ρ. Now if*

$$\min\{l \in \mathbb{N} \mid \textit{for all } c \in \mathit{Card}, s, s', s'' \in \mathit{Card}^* \textit{ with } |s''| \leq l, \rho(ss'', c) = \rho(s's'', c)\}$$

is defined, we call it the ***lookback*** *of ρ. If the minimum does not exist, we say that the lookback of ρ is ∞.*

Example 3. *The following are example rules with lookback.*

- *Lookback 0:* "Accept all black cards and all red cards that have a face; reject all the others."
- *Lookback 1:* "If the previous card had a female face, accept only aces."
- *Lookback 2:* "Whenever two cards whose values are prime have been played in a row, the next card has to be red."

Definition 3.4 (Perodic Rules with Lookback)**.** *We define $\mathbf{P}_l^p$ to be the class of periodic rules ρ of p phases, such that the maximum lookback of $\rho_0, \dots, \rho_{p-1}$ is l.*

Intuitively speaking, the simplest secret rules of Eleusis are those that accept a card only on the basis of the card itself, and neither take into account previously played cards nor the position at which a card is played. These are the rules in the class $\mathbf{P}_0^1$.

Fact 1. *For every $\rho \in \mathbf{P}_0^1$, the following condition is satisfied: For all $s, s' \in \mathit{Card}^*$, and $c \in \mathit{Card}$, $\rho(s, c) = \rho(s', c)$. Every rule $\rho \in \mathbf{P}_0^1$ can thus be expressed as function $\rho' : \mathit{Card} \to \{0, 1\}$.*

Example 4 (Rules in $\mathbf{P}_0^1$)**.**

1. "Accept all red cards, reject all black cards."
2. "Accept all cards with a value ≤ 7, reject all the others."
3. "Accept all cards of clubs and all the ones of hearts that have an even value, reject all the others."

After we have introduced some formal notation and defined some classes of Eleusis rules, we will now start investigating the complexity of decision problems related to Eleusis. We start with the Eleusis-Satisfiability problem ESAT, which can be seen as an analogue to the problem investigated for Mastermind in [23]. For Mastermind, the problem asks given a configuration of the game, whether there is any secret code that is consistent with the play so far. For Eleusis, the problem ESAT is to determine whether, given a configuration of the game, there is some rule which is consistent with the play so far. If we do not make any restrictions onto the class of rules under consideration, this problem becomes easy as it boils down to just checking whether the same card has been both rejected and accepted at the same position.[1]

[1] This is the case because whenever there is no card accepted and rejected at the same position, any function $\rho : \mathit{Card}^* \times \mathit{Card} \to \{0, 1\}$ that extends the current play (i.e. that accepts the accepted cards at the correct positions and rejects the rejected ones) is an Eleusis rule consistent with the game so far.

Definition 3.5. *For R being a class of Eleusis rules, the decision problem* $\mathsf{ESAT}(R)$ *is defined as follows.*
Decision Problem $\mathsf{ESAT}(R)$*.*

INPUT: *A sequence of cards $s \in Card^*$, and for each $i, 0 \leq i \leq |s|$ a set $\overline{C}_i \subseteq Card$ (representing the cards rejected at position i).*

QUESTION: *Is there a $\rho \in R$ such that for all i with $0 \leq i < |s|$, $\rho(s_{<i}, s_i) = 1$ and for all $c' \in \overline{C}_i$, $\rho(s_{<i}, c') = 0$?*

The first class of rules for which we investigate this problem is the class of very simple rules $\mathbf{P}_0^1$. Given a configuration of the game Eleusis, it is quite easy to check whether it is possible that the secret rule that Player 1 has in mind is in $\mathbf{P}_0^1$. These rules are so simple because whether a card is accepted does not depend on the current position of the sequence on the table, and neither on the cards played so far. If during the play one card has ever once been accepted and once been rejected, then the secret rule cannot be in $\mathbf{P}_0^1$. On the other hand, if no card has been both accepted and rejected, then it is indeed possible that the secret rule is in $\mathbf{P}_0^1$. Any rule that accepts all the cards that have previously been accepted and rejects those who have not is a candidate.

Proposition 1. *The problem* $\mathsf{ESAT}(\mathbf{P}_0^1)$ *can be solved in polynomial time.*

Proof. Going through the sequence of cards, for each position, we check whether the card accepted at the current position is rejected at the same position or at a further position, and then for each card rejected at the current position, we check if this card is accepted at any future position. As soon as we find a card where any of this is the case, we can stop and the answer is *no*. If we reach the end of the sequence, the answer is *yes*. Since in this procedure each card in the input is compared to at most all the other cards, this takes time at most n^2 in the worst case, for n being the size of the input. □

We now look at ESAT for periodic rules without lookback.

Proposition 2. *For any $p \in \mathbb{N}$, the problem* $\mathsf{ESAT}(\mathbf{P}_0^p)$ *can be solved in polynomial time.*

Proof. First of all, if $p \geq |s|$, then we only need to check if there is some position $i \leq |s|$ such that $s_i \in \overline{C}_i$. If this is the case, the answer is *no*, otherwise the answer is *yes*. If $p < |s|$, then for all i, j such that $i \leq j < |s|$ and $j \mod p = i \mod p$, we check if $s_i \in C_j$ or $s_j \in C_i$. If this is the case for any such i, j, then we can stop, and the answer is *yes*. If there are no such i, j, then the answer is *no*. □

We will now move to rules that do take into account previously accepted cards. Let us first consider $\mathsf{ESAT}(\mathbf{P}_1^1)$. Rules in $\mathbf{P}_1^1$ have the property that whether a card is accepted is completely determined by the card accepted at the previous position. So, we have to look for an instance where the same card has been accepted at two positions, and at the immediate respective successors of these positions the same card has been rejected in one case and accepted in the other. If we find such an instance, then we know that the secret rule cannot be in $\mathbf{P}_1^1$.

Proposition 3. $\mathsf{ESAT}(\mathbf{P}_1^1)$ *can be solved in polynomial time.*

Proof. In order to solve this problem, we can go through the sequence of cards, and for all $0 \leq i < |s|$, we check if there are i, j, $i \leq j < |s| - 1$ such that $s_i = s_j$ and $s_{i+1} \in \overline{C}_{j+1}$ or $s_{j+1} \in \overline{C}_{i+1}$. If we find such i, j, then the answer is no. Otherwise, the answer is yes. $\square$

Solving $\mathsf{ESAT}(\mathbf{P}^1_k)$, and in general $\mathsf{ESAT}(\mathbf{P}^p_k)$ can be done analogously; instead of looking for positions where the same card has been accepted, for each phase we have to look for two sequences of positions where the same k cards have been accepted, and then check if it is the case that at the next positions one card has been once accepted and once rejected.

Corollary 1. $\mathsf{ESAT}(\mathbf{P}^p_k)$ *can be solved in polynomial time.*

Thus, we have seen that for several classes of rules, it can be decided in polynomial time whether there is a rule in the class that is consistent with the play so far. In actual play, we can think of Player 2 solving this problem for various classes of rules, trying to restrict the set of rules that are still possible. In other words, coming back to Eleusis as a simulation of scientific inquiry, this is the problem describing the scientist checking whether there is some hypothesis in a certain class that is consistent with the experimental results so far.

3.2 A Hard Eleusis Problem

After having discussed various tractable decision problems in Eleusis, we will now show that Eleusis also gives rise to hard problems. We give a secret Eleusis rules that has the property that checking whether the sequence of cards on the table is consistent with the rule is NP-complete.

We now show that Eleusis allows secret rules that force Player 1 to solve NP-complete problems when checking if a sequence is according to the rule. We use the Partition Problem [9], which is the problem of deciding whether a multi-set can be partitioned into two subsets that add up to the same sum.

Decision Problem *Partition*.

INPUT: *A multiset of positive integers $\mathcal{S}$.*

QUESTION: *Is there a way to partition $\mathcal{S}$ into two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that*

$$\sum \mathcal{S}_1 = \sum \mathcal{S}_2\ ?$$

The following is the task Player 1 has to solve in each round when giving feedback to Player 2. In the strategic considerations of Player 1, the complexity of this task plays of course a crucial role since in practice she should be able to solve it in reasonable time.

Decision Problem $\mathsf{E} - \mathsf{Check}(\rho)$.

INPUT: *A sequence of cards $s \in Card^*$, and a card $c \in Card$.*

QUESTION: *Is it the case that $\rho(s, c) = 1$?*

We now show that in the generalized version of Eleusis where we have an infinite but countable deck of cards, and a function $value : Card \to \mathbb{N}$, there are Eleusis rules that make it NP-hard for Player 1 to check if a sequence is according to the rule. One such example is a rule that forces Player 1 to solve the problem *Partition*, because she has to check if the values of the cards played so far can be partitioned into two subsets that give the same sum.

$$\rho_{part}(s, c) := \begin{cases} 1 & \text{if there is a partition of the sequence } sc \text{ into sequences } s' \text{ and} \\ & s'' \text{ such that } \sum_{i=1}^{i=|s'|} value(s'_i) = \sum_{i=1}^{i=|s''|} value(s''_i), \\ 0 & \text{otherwise.} \end{cases}$$

Now, the following fact follows immediately.

Proposition 4. $\mathsf{E} - \mathsf{Check}(\rho_{part})$ *is NP-complete.*

Proof. NP membership is straightforward. NP-hardness follows by reduction from *Partition*. □

Note that in practice, the procedure of the game allows Player 1 to use information from previous rounds about which initial segments of the sequence can be partitioned to determine whether a card is accepted.

Our result shows that when Player 1 is constructing a secret rule, she should be aware of its complexity to ensure that she won't be faced with intractable problems when she has to give feedback to the other players, as it happens with the rule ρ_{part}.

3.3 An Undecidable Eleusis Problem

We can now show that Eleusis allows for even harder rules: we give an Eleusis rule such that it is undecidable whether a sequence of cards is consistent with it. We will first introduce some notation for this.

Notation 2.

- *We now use standard decks of cards, and let* $Card = Value \times Suit$, *for* $Value = \{1, \ldots, 13\}$ *and* $Suit = \{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$.
- *For* $c \in Card$, *we let* $Value(c)$ *denote its value and* $Suit(c)$ *its suit.*
- *For a sequence of cards* $s \in Card^*$, $Value(s)$ *denotes the sequence of the values, i.e.* $Value(s) = Value(s_0) \ldots Value(s_{|s|-1})$.
- *We define a function* $color : Card \to \{b, r\}$, *assigning to each card its color (black or red), defined as follows.*

$$color(c) = \begin{cases} b & \text{if } Suit(c) \in \{\spadesuit, \clubsuit\} \\ r & \text{if } Suit(c) \in \{\heartsuit, \diamondsuit\} \end{cases}$$

In Definition 3.1, we defined Eleusis rules as functions $\rho : Card^* \times Card \to \{0, 1\}$. It is clear that we can give an equivalent definition of Eleusis rules as functions $\rho' : Card^+ \to \{0, 1\}$, where $Card^+$ is the set of nonempty finite sequences over *Card*. For technical convenience, we will use this latter definition in the remainder of this section.

We now define the set of *black (red) words* in a sequence of cards s. The set of black (red) words in a sequence of cards is simply the set of maximal subsequences of black (red) cards in the sequence, i.e. the set of subsequences of black (red) cards that are separated by red (black) cards. Let us illustrate this with an example. Given the sequence $s = (4, \spadesuit)(3, \clubsuit)(9, \diamondsuit)(8, \spadesuit)$, its set of red words is the singleton $\{(9, \diamondsuit)\}$, and its set of black words is $\{(4, \spadesuit)(3, \clubsuit), (8, \spadesuit)\}$.

Definition 3.6. *For a sequence $s \in Card^*$, we define the* ***set of black words*** *of s $BW(s)$ to be the set of all those $w \in Card^*$ with $|s| \geq 1$ that satisfy the following conditions.*

1. *$\forall i$ such that $0 \leq i < |w|$ it holds that $color(w_i) = b$ and*
2. *$\exists i$ such that $0 \leq i < |s|$ and $\forall j$ with $0 \leq j < |w|$ it holds that $s_{i+j} = w_j$ and*
   (i) if $i > 0$, then $color(s_{i-1}) = r$ and
   (ii) if $i + |w| < |s|$, then $color(s_{(i+j)+1}) = r$.

The ***set of red words*** *of a sequence of cards s, $RW(s)$, is defined analogously by swapping r and b in the above definition.*

For constructing a rule that gives rise to an undecidable problem, we will use the above definition and view a sequence of cards as a sequence of black and red words. Our proof of undecidability is by reduction from Post's Correspondence Problem [20].

Post's Correspondence Problem.

INPUT: *A finite set of pairs of non-empty strings over a finite alphabet Σ, $P = \{(x_1, y_1), \ldots (x_n, y_n)\}$.*

QUESTION: *Is there a sequence $(i_1, \ldots i_m)$ for some $m \in \mathbb{N}$, with $1 \leq i_j \leq n$ such that for all $1 \leq j \leq m$*

$$x_{i_1} \ldots x_{i_m} = y_{i_1} \ldots y_{i_m}?$$

Note that even if Σ is small ($|\Sigma| = 2$), the problem is undecidable [22].

We define an Eleusis rule that has the property that checking whether a sequence of cards is consistent with the rule is at least as hard as solving Post's Correspondence Problem. Before giving the formal definition, let us explain the intuition. The idea is that all sequences of cards starting and ending with cards of the same color are accepted. If the first and last card are of different colors (i.e. the sequence has many red words as black words), then the sequence is accepted if and only if it has the following property: If we view it as a sequence of pairs each consisting of a red word and a black word, then it is possible to rearrange the order of these pairs (possibly using a pair more than once or not at all) such that the resulting string of red values is the same as the resulting sequence of black values. Let us illustrate this with an example showing a positive instance. Consider the sequence $(9, \spadesuit)(3, \heartsuit)(9, \diamondsuit)(10, \clubsuit)(3, \clubsuit)(3, \diamondsuit)(10, \diamondsuit)(3, \spadesuit)(3, \clubsuit)(3, \heartsuit)$. Reading it as a sequence of red and black words, gives

$$\underbrace{(9, \spadesuit)}_{w_1^b}\ \underbrace{(3, \heartsuit)(9, \diamondsuit)}_{w_1^r}\ \underbrace{(10, \clubsuit)(3, \clubsuit)}_{w_2^b}\ \underbrace{(3, \diamondsuit)(10, \diamondsuit)}_{w_2^r}\ \underbrace{(3, \spadesuit)(3, \clubsuit)}_{w_3^b}\ \underbrace{(3, \heartsuit)}_{w_3^r}.$$

Now, (3,2,2,1) is a solution since $(Value(w_3^b)\ Value(w_2^b)\ Value(w_2^b)\ Value(w_1^b)) = (3\ 3\ 10\ 3\ 10\ 3\ 9) = (Value(w_3^r)\ Value(w_2^r)\ Value(w_2^r)\ Value(w_1^r))$. Similarly, $(3, 2, 1)$ is a solution.

$$\rho_{Post}(s) := \begin{cases} 1 & \text{if } |BW(s)| \neq |RW(s)| \text{ or} \\ & |BW(s)| = |RW(s)| \text{ and } s = w_1^r w_1^b w_2^r w_2^b \dots w_k^r w_k^b \text{ with} \\ & w_l^b \in BW(s) \text{ and } w_l^r \in RW(s) \text{ then } \exists (i_1 \dots i_m) \text{ with } 1 \leq i_j \leq k \\ & \text{and } (Value(w_{i_1}^r) \dots Value(w_{i_m}^r)) = (Value(w_{i_1}^b) \dots Value(w_{i_m}^b)) \text{ or} \\ & |BW(s)| = |RW(s)| \text{ and } s = w_1^b w_1^r w_2^b w_2^r \dots w_k^b w_k^r \text{ with} \\ & w_l^b \in BW(s) \text{ and } w_l^r \in RW(s) \text{ then } \exists (i_1 \dots i_m) \text{ with } 1 \leq i_j \leq k \\ & \text{and } (Value(w_{i_1}^r) \dots Value(w_{i_m}^r)) = (Value(w_{i_1}^b) \dots Value(w_{i_m}^b)); \\ 0 & \text{otherwise.} \end{cases}$$

We now show undecidability of $\mathsf{E-Check}(\rho_{Post})$, the problem of deciding whether a for a given sequence $s \in Card^+$, $\rho_{Post}(s) = 1$.

Theorem 1. $\mathsf{E-Check}(\rho_{Post})$ *is undecidable.*

Proof. By reduction from Post's Correspondence Problem with alphabet $\Sigma = Value = \{1, \dots 13\}$. Given $P = \{(x_1, y_1), \dots (x_n, y_n)\}$ with $x_j, y_j \in Value^*$, we transform it into a sequence of cards. We define a (partial) function $g : Value^* \to (Value \times Suit)^*$ as follows: For each $(x_i, y_i) \in P$, we define $g(x_i) = (x_{i0}, \heartsuit)(x_{i1}, \heartsuit) \dots (x_{i|x_i|-1}, \heartsuit)$ and $g(y_i) = (y_{i0}, \spadesuit)(x_{i1}, \spadesuit) \dots (y_{i|y_i|-1}, \spadesuit)$. Then, let $f(P) = g(x_1)g(y_1) \dots g(x_n)g(y_n)$. First of all note that f can be computed in polynomial time since g can be computed in polynomial time. Now, we have to show that f is indeed a proper reduction.

Assume that for $P = \{(x_1, y_1), \dots (x_n, y_n)\}$, there is a sequence $(i_1 \dots i_m)$, with $1 \leq i_j \leq n$ such that $x_{i_1} \dots x_{i_m} = y_{i_1} \dots y_{i_m}$. Now, we have to show that $\rho_{Post}(f(P)) = 1$. First of all note that by construction $|BW(f(P))| = |RW(f(P))|$. Moreover, $(Value(w^r{}_{i_1}) \dots Value(w^r{}_{i_m})) = (Value(w^b{}_{i_1}) \dots Value(w^b{}_{i_m}))$. Thus $\rho_{Post}(f(P)) = 1$.

For the other direction, assume that $\rho_{Post}(f(P)) = 1$. By construction of $f(P)$, it has to be the case that $|BW(f(P))| = |RW(f(P))|$, and $f(P)$ has to start with a red card. Thus, $f(P) = w^r{}_1 w^b{}_1 w^r{}_2 w^b{}_2 \dots w^r{}_k w^b{}_k$ with $w^b{}_l \in BW(f(P)), w^r{}_l \in RW(f(P))$ and $\exists (i_1 \dots i_m)$ with $1 \leq i_j \leq k$ such that $(Value(w^r{}_{i_1}) \dots Value(w^r{}_{i_m})) = (Value(w^b{}_{i_1}) \dots Value(w^b{}_{i_m}))$. But then it must also be the case that $x_{i_1} \dots x_{i_m} = y_{i_1} \dots y_{i_m}$. This concludes the proof. □

We have thus shown that whereas there are various tractable problems in Eleusis, the game also gives rise to NP-complete problems and problems that are undecidable even when played with a standard deck of cards.

This section has shown that the inductive inference game Eleusis is interesting from a complexity theoretical point of view as it gives rise to decision problems of various complexities. On the other hand, we also showed that there are hard decision problems that are relevant for the actual play of the game, as they are not about deciding which player has a winning strategy as the usual complexity results about games, but describe the tasks the players face during the game. Considering the problem $\mathsf{E-Check}(\rho)$, we have seen that as opposed to Mastermind, in Eleusis the complexity for Player 1 crucially depends on her choice at the beginning of the game, as some choices can make it impossible for her to make a move, i.e. to give feedback to Player 2. Therefore, we have shown that the first player has a very active role in

Eleusis, and as opposed to the literature where the difficulty of Eleusis is only discussed with respect to the difficulty to discover certain rules, our results show that Player 1's first move has crucial complexity implications also for herself. Coming back to Eleusis as a simulation of scientific inquiry, our work thus fits with approaches putting forward an interactive view on learning, with the environment or teacher having an active role [12].

4 Conclusion and Further Work

Our work brings together complexity theory, game theory and learning theory, as we investigate interactive processes involving inductive inferences by using methods from computational complexity for giving a complexity analysis of various tasks that players face during the inductive inference game Eleusis.

In this work, we take a formal perspective, investigating a generalized version of Eleusis Express. We formalize the key aspects of the game and show that a variety of interesting decision problems arises. We have shown that for several classes of secret rules it can be decided in polynomial time whether there is a rule in the class that is consistent with the current state of the game. Moreover, our results show that Eleusis also gives rise to intractable problems. We have constructed a rule that requires the players to solve the NP-complete Partition Problem in order to decide if a card should be accepted. Finally, using Post's Correspondence Problem, we showed that – even when played with standard decks of cards – Eleusis allows for rules that make it undecidable to check if a sequence of cards is consistent with the rule.

Thus, the current work promotes the computational complexity analysis of inductive inference games, showing that a variety of interesting problems arise, ranging from very easy to undecidable. This complexity theoretical perspective gives us new insights into the strategic abilities of agents engaged in interactive processes that involve inductive inference and also highlights the special role complexity plays in inductive inference games, distinguishing them from deductive inference games such as Mastermind. Our work also promotes a categorization of secret Eleusis rules not only with respect to their difficulty of being discovered but also with respect to how difficult it is for the first player to give feedback to the other players. Our work thus fits with approaches to formal learning theory that consider the learning process as an interaction between learner and teacher [12, 13].

The current work gives rise to a wide range of directions for further work. On the one hand, we suggest an investigation of *dynamic* Eleusis, a version of the game in which Player 1 can at each round change his mind with respect to the secret rule as long as the new rule is consistent with the play up to the current stage. In this version, Player 1 has a more active role and we can analyze the impact of her helpfulness on the abilities of the players taking the role of Scientist, similarly as has been done in [13].

For further work, we also suggest to investigate the relationship between Eleusis and learning theory frameworks such as learning with membership queries [2, 10]

In Eleusis, the concepts of knowledge, belief and information change play a crucial role. In the current work, we took a purely computational perspective and did not go into the exact processes that describe how the information state of Player 2 changes throughout the game. For further work, we propose to apply dynamic epistemic logic [7] and dynamic doxastic logic to model these processes in Eleusis [5, 11].

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