MACNEILLE TRANSFERABILITY AND STABLE CLASSES OF HEYTING ALGEBRAS

GURAM BEZHANISHVILI, JOHN HARDING, JULIA ILIN, FREDERIK M. LAURIDSEN

ABSTRACT. A lattice P is transferable for a class of lattices $\mathcal{K}$ if whenever P can be embedded into the ideal lattice $\mathfrak{I}K$ of some $K \in \mathcal{K}$, then P can be embedded into K. There is a rich theory of transferability for lattices. Here we introduce the analogous notion of MacNeille transferability, replacing the ideal lattice $\mathfrak{I}K$ with the MacNeille completion $\overline{K}$. Basic properties of MacNeille transferability are developed. Particular attention is paid to MacNeille transferability in the class of Heyting algebras where it relates to stables classes of Heyting algebras, and hence to stable intermediate logics.

1. INTRODUCTION

A lattice P is *transferable* if whenever there is a lattice embedding $\varphi : P \to \mathfrak{I}K$ of P into the ideal lattice of a lattice K, then there is a lattice embedding $\varphi' : P \to K$. It is *sharply transferable* if φ' can be chosen so that $\varphi'(x) \in \varphi(y)$ iff $x \leq y$. If we restrict K to belong to some class of lattices $\mathcal{K}$, we say P is *transferable in $\mathcal{K}$*. The notion has a long history, beginning with [16, 13], and remains a current area of research [25]. For a thorough account of the subject, see [17], but as a quick account, among the primary results in the area are the following.

Theorem 1.1. (see [17, pp. 502–503],[25]) *A finite lattice is transferable for the class of all lattices iff it is projective, and in this case it is sharply transferable.*

Theorem 1.2. (see [13],[23]) *Every finite distributive lattice is sharply transferable for the class of all distributive lattices.*

Our purpose here is to introduce, and make the first steps towards, an analogous study of MacNeille transferability. Characterizations as in Theorems 1.1 and 1.2 are beyond our scope. But we do provide a number of results, both positive and negative, that begin to frame the problem. Among easy results are that no non-distributive lattices are MacNeille transferable for any class that contains distributive lattices, and that every finite projective distributive lattice is MacNeille transferable for the class of lattices whose MacNeille completions are distributive.

Among further results, we show that the 7-element distributive lattice D, formed by placing two 4-element Boolean lattices on top of one another and identifying the top of the lower with the bottom of the upper (see Figure 2), is MacNeille transferable for the class of distributive lattices, but not sharply transferable. We also give an example of a finite distributive lattice that is not MacNeille transferable for even the class of lattices whose MacNeille completions are distributive. If MacNeille transferability is extended to include

bounds, we show that no finite distributive lattice with a complemented element is boundedly MacNeille transferable for the class of Heyting algebras.

Positive results are also obtained by considering the class of Heyting algebras whose dual spaces have finite width. Here we provide an infinite family of finite non-projective distributive lattices that are MacNeille transferable for the class. Stable classes of Heyting algebras are roughly those obtained by omitting a chosen family of finite distributive lattices from occurring within them. They correspond to stable intermediate logics [5, 6]. Our results provide an infinite family of stable universal classes of Heyting algebras that are closed under MacNeille completions. Due to the result of [15], this implies that these stable universal classes are also closed under canonical extensions.

We obtain stronger positive results for bi-Heyting algebras. We prove that each finite distributive lattice is MacNeille transferable for the class of bi-Heyting algebras. This is established by showing that the MacNeille completion of a bi-Heyting algebra of finite width is a bounded sublattice of its ideal lattice.

We also consider versions of MacNeille transferability where the maps preserve $\wedge, 0, 1$. Here our results have some overlap with those of [9], however the two approaches are different. While our technique is mostly algebraic, that of [9] is of a syntactic nature and mainly focuses on residuated lattices. Restricted to Heyting algebras, [9] shows that all universal clauses in the $\{\wedge, 0, 1\}$-language of Heyting algebras are preserved by MacNeille completions. The two approaches are related in the following way: for a finite distributive lattice P, there is a universal clause $\rho(P)$ in the $\{\wedge, 0, 1\}$-language of Heyting algebras expressing the property of not having a $\{\wedge, 0, 1\}$-subalgebra isomorphic to P. By [9], $\rho(P)$ is preserved under MacNeille completions of Heyting algebras, and hence P is $\{\wedge, 0, 1\}$-MacNeille transferable for the class of Heyting algebras. We note that neither the results here, nor those of [9], subsume the results of the other.

2. Basic definitions and MacNeille transferability for lattices

For a lattice K, we use $\overline{K}$ for the MacNeille completion of K. This will be viewed either as the set of normal ideals of K, or abstractly as a complete lattice that contains K as a join and meet dense sublattice. We also consider lattices with one or both bounds as part of their basic type. For $\tau \subseteq \{\wedge, \vee, 0, 1\}$, a τ-*lattice* is a lattice, or lattice with one or both bounds, whose basic operations are of type τ. A τ-*homomorphism* is a homomorphism with respect to this type, and a τ-*embedding* is a one-one τ-homomorphism.

Definition 2.1. Let $\tau \subseteq \{\wedge, \vee, 0, 1\}$, P be a τ-lattice, and $\mathcal{K}$ a class of τ-lattices. Then P is τ-*MacNeille transferable* for $\mathcal{K}$ if for any τ-embedding $\varphi : P \to \overline{K}$ where $K \in \mathcal{K}$, there is a τ-embedding $\varphi' : P \to K$. We say P is *sharply τ-transferable* for $\mathcal{K}$ if φ' can be chosen so that $\varphi'(x) \leq \varphi(y)$ iff $x \leq y$ for all $x, y \in P$. We use the terms *transferable* and *sharply transferable* when $\tau = \{\wedge, \vee\}$.

There are obvious examples of lattices P that are MacNeille transferable for the class of all lattices. Any finite chain, and the 4-element Boolean lattice provide examples. Infinite chains can be problematic, as is seen by a simple cardinality argument for P the chain of real numbers and K the chain of rational numbers. Such difficulties arise also with the traditional study of transferability using ideal lattices. Here we restrict our attention to the case where P is finite. Our first result follows immediately from [18] where it was shown that any lattice can be embedded into the MacNeille completion of a distributive lattice.

Theorem 2.2. *A lattice that is MacNeille transferable for the class of all lattices is distributive.*

For a bounded lattice K, let K^σ be the canonical completion of K [14]. For a lattice K, we let K^+ be the result of adding a bottom to K if it does not have one, and adding a top to K if it does not have one. Note that for a finite lattice P, there is a lattice embedding of P into K iff there is a lattice embedding of P into K^+. Following the convention that $\varnothing$ is not an element of the ideal lattice, but can be an element of the MacNeille completion when viewed as normal ideals, it is easily seen that $\overline{K^+} = \overline{K}$ and $\mathfrak{I}(K^+) = (\mathfrak{I}K)^+$.

Theorem 2.3. *If a finite lattice P is MacNeille transferable for a class $\mathcal{K}$ of lattices that is closed under ultrapowers, then P is transferable for $\mathcal{K}$.*

Proof. Suppose that P is isomorphic to a sublattice of the ideal lattice $\mathfrak{I}K$ of some $K \in \mathcal{K}$. Then P is isomorphic to a sublattice of $\mathfrak{I}(K^+)$. The ideal lattice $\mathfrak{I}(K^+)$ is isomorphic to the sublattice of open elements of the canonical extension $(K^+)^\sigma$ [14, Lem. 3.3]. In [15] it was shown that the canonical extension $(K^+)^\sigma$ is isomorphic to a sublattice of the MacNeille completion $\overline{(K^+)^*}$ of an ultrapower $(K^+)^*$ of K^+. Clearly $(K^+)^* = (K^*)^+$. So P is isomorphic to a sublattice of $\overline{(K^*)^+} = \overline{K^*}$. Since $\mathcal{K}$ is closed under ultrapowers, $K^* \in \mathcal{K}$. Then as P is MacNeille transferable for $\mathcal{K}$, there is a sublattice of K^* that is isomorphic to P. Therefore, K^* satisfies the diagram Δ_P of the finite lattice P (see, e.g., [8, pp. 68–69]), hence K satisfies Δ_P by Łoś' Theorem. Thus, K has a sublattice that is isomorphic to P, and this shows that P is transferable. $\qquad\square$

Combining the above two results with Theorem 1.1, it follows that if a finite lattice is MacNeille transferable for the class of all lattices, then it is projective in the class of all lattices and distributive. Using the result of Kostinsky [21] that a finite lattice is projective in the class of all lattices iff it is a sublattice of a free lattice, and the result of Galvin and Jónsson [12] characterizing the finite sublattices of free lattices that are distributive as exactly those that do not contain a doubly reducible element (one that is both a join and meet), gives the following.

Corollary 2.4. *If a finite lattice P is MacNeille transferable for the class of all lattices, then it is distributive and has no doubly reducible elements.*

Galvin and Jónsson [12] further characterize all (not just finite) distributive lattices that have no doubly reducible elements. This will be of use in later considerations for us as well.

Theorem 2.5. [12] *A distributive lattice has no doubly reducible elements iff it is a linear sum of lattices each of which is isomorphic to an eight-element Boolean algebra, a one-element lattice, or $2 \times C$ for a chain C.*

Problem 1. Is each finite distributive lattice with no doubly reducible elements MacNeille transferable for the class of all lattices?

If we move from the setting of MacNeille transferability with both lattice operations $\wedge, \vee$ involved, to a setting where at most one lattice operation is involved, the situation opens considerably. The key result here relates to the projectivity of a semilattice in the class of all semilattices. A consequence of [20] gives that if P is a semilattice that is the $\wedge$-reduct of a finite distributive lattice, then P is projective in semilattices. We will also need the following slight extension of a result of Baker and Hales [1].

Lemma 2.6. *For a lattice K, there is a sublattice S of an ultrapower K^* of K and an onto lattice homomorphism $\mu : S \to \mathfrak{I}K$. If K has a 0, then the 0 of K^* belongs to S and is the only element of S mapped by μ to 0. Similar comments hold if K has a 1.*

Proof. We follow [1]. Let X be the set of all finite subsets of K partially ordered by set inclusion. The principle upsets of X form a filter base. Let $\mathcal{U}$ be an ultrafilter extending this filter base. Let M be the set of order preserving maps from X to K. Then M is a sublattice of K^X. Let S be the image of M in $K^* = K^X/\mathcal{U}$. So the elements of S are equivalence classes $\sigma/\mathcal{U}$ of order preserving functions $\sigma : X \to K$. Define $\mu : S \to \mathfrak{I}K$ by letting $\mu(\sigma/\mathcal{U})$ be the ideal generated by the image of σ. In [1] it is shown that μ is a well defined onto lattice homomorphism.

If K has a 0, then the constant map from X to K taking value 0 belongs to K^X, and the corresponding element of K^* is its 0. Since μ is onto, it must preserve 0. If $\sigma : X \to K$ is order preserving and $\mu(\sigma/\mathcal{U}) = 0$, then the ideal generated by the image of σ is the zero ideal $\{0\}$, and that implies that σ is the zero function. So the 0 of S is the only element mapped to 0 in K. Similarly, if K has a 1, then the 1 of K^* is the equivalence class of the constant function 1, it belongs to S, and since μ is onto it preserves 1. Suppose $\mu(\sigma/\mathcal{U}) = 1$. Then 1 is in the ideal of K generated by the image of σ, and since σ is order preserving and X is a lattice, 1 is in the image of σ. Then there is a finite subset $A \subseteq K$ with $\sigma(A) = 1$. Since σ is order preserving, σ takes value 1 on the upset generated by A, hence σ is in the equivalence class of the constant function 1. $\square$

Theorem 2.7. *Let P be a finite distributive lattice and $\tau \subseteq \{\wedge, \vee, 0, 1\}$ that does not contain both $\wedge, \vee$. Let $\mathcal{K}_\tau$ be the class of all lattices if τ does not contain $0, 1$, the class of all lattices with 0 if τ contains 0 and not 1, the class of all lattices with 1 if τ contains 1 and not 0, and the class of all bounded lattices if τ contains $0, 1$. Then P is τ-MacNeille transferable for $\mathcal{K}_\tau$.*

Proof. Assume $K \in \mathcal{K}_\tau$ and that $\varphi : P \to \overline{K}$ is a τ-embedding. By symmetry, we may assume that τ does not contain $\vee$. We consider first the case that τ does not contain $\wedge$. If K is finite, then $\overline{K} = K$ and there is nothing to show. Otherwise, since P is finite, there is a τ-embedding of P into K.

Assume that τ contains $\wedge$. Viewing the MacNeille completion $\overline{K}$ as a set of normal ideals, the identical embedding $\iota : \overline{K} \to \mathfrak{I}K$ is a τ-embedding, hence so is the composite $\iota \circ \varphi : P \to \mathfrak{I}K$. Let $S \leq K^*$ and $\mu : S \to \mathfrak{I}K$ be as in the proof of the previous lemma. Since P is projective as a semilattice, there is a $\wedge$-semilattice homomorphism $\gamma : P \to S$ with $\mu \circ \gamma = \iota \circ \varphi$. Since the composite is an embedding, so is γ. Since the composite preserves the bounds belonging to τ, the result of the previous lemma about the uniqueness of elements of S mapped to bounds of $\mathfrak{I}K$ shows that γ preserves bounds in τ. Thus, S has a τ-subalgebra isomorphic to P, and hence so does K^*. So K^* satisfies the diagram of P in the τ-language, and by Łós' Theorem, K satisfies this diagram, hence has a τ-subalgebra isomorphic to P. $\square$

We conclude this section with comments related to adding bounds to the type.

Definition 2.8. For a finite lattice P, let $0 \oplus P$ be the result of adding a new bottom 0 to P, $P \oplus 1$ be the result of adding a new top 1 to P, and $0 \oplus P \oplus 1$ be the result of doing both.

For a lattice K and $k \in K$, we can consider the principal ideals $(\downarrow k)_K$ and $(\downarrow k)_{\overline{K}}$. Using the abstract characterization of MacNeille completions (see, e.g., [2, p. 237]), it is easily seen

that $\overline{(\downarrow k)_K} = (\downarrow k)_{\overline{K}}$. We say that a class $\mathcal{K}$ of lattices is *closed under principal ideals* if for each $K \in \mathcal{K}$ and each $k \in K$, the lattice $(\downarrow k)_K$ belongs to $\mathcal{K}$. Similarly, $\mathcal{K}$ is closed under principal filters if each $(\uparrow k)_K$ belongs to $\mathcal{K}$.

Proposition 2.9. *Let P be a finite lattice, $\tau \subseteq \{\wedge, \vee, 0\}$, and $\mathcal{K}$ be a class of $\tau \cup \{1\}$-lattices that is closed under principal ideals. If P is τ-MacNeille transferable for $\mathcal{K}$, then $P \oplus 1$ is $(\tau \cup \{1\})$-MacNeille transferable for $\mathcal{K}$. Similar results hold for $0 \oplus P$ and $\tau \cup \{0\}$ when $\mathcal{K}$ is closed under principal filters, and $0 \oplus P \oplus 1$ and $\tau \cup \{0, 1\}$ when $\mathcal{K}$ is closed under both.*

Proof. We prove the result for $\tau \cup \{1\}$, the result for $\tau \cup \{0\}$ is by symmetry, and the result for $\tau \cup \{0, 1\}$ follows from these. Suppose $K \in \mathcal{K}$ and $\varphi : P \oplus 1 \to \overline{K}$ is a $\tau \cup \{1\}$-embedding. Let $\top$ be the top of P and 1 be the top of $P \oplus 1$. Then $\varphi(\top) = x$ for some $x < 1$ in $\overline{K}$. Since the MacNeille completion is meet dense, there is $k \in K$ with $x \leq k < 1$. Then $\varphi|P : P \to (\downarrow k)_{\overline{K}} = \overline{(\downarrow k)_K}$ is a τ-embedding. Since $(\downarrow k)_K \in \mathcal{K}$ and P is τ-MacNeille transferable for $\mathcal{K}$, there is a τ-embedding $\psi : P \to (\downarrow k)_K$. Since $k < 1$, there is a $(\tau \cup \{1\})$-embedding of $P \oplus 1$ into K. $\qquad \square$

Remark 2.10. We note that $\{\wedge, \vee, 0, 1\}$-MacNeille transferability is an elusive concept when lattices are not of the form $0 \oplus P$ or $P \oplus 1$. For the real unit interval $[0, 1]$, the bounded lattice $K = ([0, 1] \times [0, 1]) \smallsetminus \{(1, 0), (0, 1)\}$ has no complemented elements other than the bounds, while its MacNeille completion $\overline{K} = [0, 1] \times [0, 1]$ does. Thus, the 4-element bounded lattice 2×2 is not $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for lattices. We return to this matter in the section on Heyting algebras where it takes particular significance.

3. MacNeille transferability for distributive lattices

One can consider MacNeille transferability for the class of distributive lattices. However, since distributive lattices are not closed under MacNeille completions, one would leave distributive lattices to do so. Instead, we consider MacNeille transferability for the class $\mathcal{K}$ of lattices whose MacNeille completions are distributive. This class $\mathcal{K}$ includes such interesting classes as (the lattice reducts of) Heyting algebras, co-Heyting algebras, and bi-Heyting algebras. Since MacNeille transferability for $\mathcal{K}$ holds vacuously for any non-distributive lattice, we consider only the case when a finite distributive lattice is MacNeille transferrable for $\mathcal{K}$. In contrast to Theorem 1.2, which says that every finite distributive lattice is transferrable for all distributive lattices, we have the following.

Theorem 3.1. *There is a finite distributive lattice P that is not MacNeille transferable for the class $\mathcal{K}$ of lattices whose MacNeille completions are distributive.*

Proof. Consider the lattices P, in Figure 1 at left, and K, in Figure 1 at right. Here the shaded middle portion of K is $([0, 1] \times [0, 1]) \smallsetminus \{(0, 1), (1, 0)\}$, the product of two copies of the unit interval with the "corners" removed. The MacNeille completion $\overline{K}$ simply reinserts the missing "corners", and there is a lattice embedding of P into $\overline{K}$. Using the fact that $[0, 1] \times [0, 1]$ does not contain a sublattice isomorphic to the 8-element Boolean algebra and $([0, 1] \times [0, 1]) \smallsetminus \{(0, 1), (1, 0)\}$ does not have any complemented elements, it is easily seen that P is not isomorphic to a sublattice of K. $\qquad \square$

Before moving to some positive results, we note that a lattice being projective in the class of distributive lattices is not the same as it being distributive and projective in the class of all lattices. The finite lattices that are projective in the class of distributive lattices are
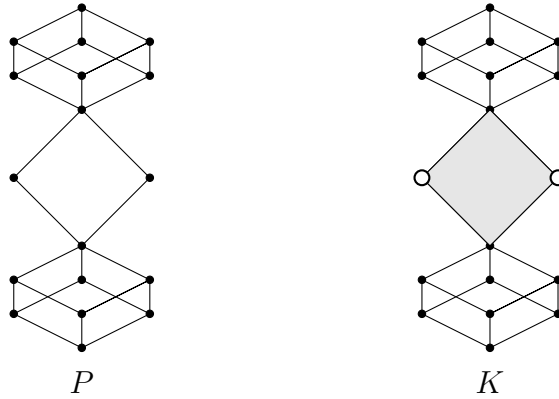
FIGURE 1

characterized in [3] as exactly those where the meet of two join irreducible elements is join irreducible (and 0 is considered join irreducible).

Theorem 3.2. *Every finite lattice that is projective in the class of distributive lattices is MacNeille transferable, $\{\wedge, \vee, 0\}$-MacNeille transferable, and $\{\wedge, \vee, 1\}$-MacNeille transferable for the class of lattices, resp. lattices with 0 or with 1, whose MacNeille completions are distributive.*

Proof. We begin with the result for MacNeille transferability. Let P be a finite distributive lattice that is projective in the class of distributive lattices. Let K be a distributive lattice whose MacNeille completion is distributive, and let $\varphi : P \to \overline{K}$ be a lattice embedding. By Theorem 2.7, there is a $\wedge$-embedding $\chi : P \to K$. Since P is projective in distributive lattices, the set of join irreducibles J in P forms a $\wedge$-sub semilattice of P, and $\chi|J : J \to K$ is a $\wedge$-embedding. Define $\chi' : P \to K$ by setting $\chi'(x) = \bigvee\{\chi(a) : a \in J \text{ and } a \leq x\}$. It is shown in [3, Thm. 4] that χ' is a lattice homomorphism, and since χ is order preserving, $\chi'(x) \leq \chi(x)$ for each $x \in P$. To see that χ' is an embedding, suppose $x, y \in P$ with $\chi'(x) \leq \chi'(y)$ and $a \in J$ with $a \leq x$. Then $\chi(a) = \chi'(a) \leq \chi'(x) \leq \chi'(y) \leq \chi(y)$, and as χ is an embedding $a \leq y$. Since x is the join of the members of J beneath it, $x \leq y$. So $\chi' : P \to K$ is a lattice embedding, showing P is MacNeille transferable.

Showing that P is $\{\wedge, \vee, 0\}$-MacNeille transferable for the class of lattices with a 0 whose MacNeille completion is distributive is nearly identical. One uses Theorem 2.7 to obtain a $\{\wedge, 0\}$-embedding $\chi : P \to K$, and notes that $0 \in J$. Since the lattice embedding $\chi' : P \to K$ produced agrees with χ on J, it follows that χ' preserves 0. The result for $\{\wedge, \vee, 1\}$-MacNeille transferability follows by symmetry. $\qquad \square$

Remark 3.3. The above theorem can be obtained using the transferability of finite distributive lattices in the class of all distributive lattices rather than using Theorem 2.7.

We next provide notation and detail for the linear sums in Theorem 2.5 of Galvin and Jónsson [12]. This theorem will be used to show that there are finite non-projective distributive lattices that are MacNeille transferable for distributive lattices.

Definition 3.4. Let A be a chain, and for each $a \in A$ let $(K_a, \leq_a)$ be a lattice. Then the linear sum $\bigoplus_A K_a$ is the disjoint union of the K_a with the ordering $\leq$ given by setting $x \leq y$ iff $x \in K_a$ and $y \in K_b$ for some $a < b$, or $x, y \in K_a$ for some a and $x \leq_a y$.

Theorem 3.5. *Let D be the seven-element distributive lattice that has a doubly reducible element shown in Figure 2. Then D is MacNeille transferable for the class of distributive lattices.*
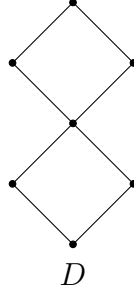

D

FIGURE 2

Proof. Suppose that K is a distributive lattice that does not contain D as a sublattice. We must show that $\overline{K}$ does not contain D as a sublattice. By Theorem 2.5, K is a linear sum of lattices $\bigoplus_A K_a$ with each K_a isomorphic to either an 8-element Boolean algebra, a 1-element lattice, or $2 \times C$ for a chain C. We describe the MacNeille completion of K.

For $x \in \overline{A}$, let $M_x = \overline{K_x}$ if $x \in A$, and let $M_x = \{x\}$ be a 1-element lattice otherwise. Let 0_x be the least element of M_x and 1_x be the greatest element of M_x. Set $M = \bigoplus_{\overline{A}} M_x$.

If $S \subseteq M$, let $S' = \{x \in \overline{A} : S \cap M_x \neq \varnothing\}$, and let $z = \bigvee_{\overline{A}} S'$. Then it is not difficult to see that $\bigvee_M S = \bigvee_{M_z}(S \cap M_z)$ with it understood that the join of the emptyset in M_z is 0_z. Similar remarks hold for $\bigwedge_M S$. Thus, M is complete and clearly K is a sublattice of M.

We consider the matter of K being join dense in M. Suppose $p = 0_z$ where $z \in \overline{A}$ is the join of the elements of A strictly beneath it. This includes the case when $p \in M_z$ for any $z \in \overline{A} \smallsetminus A$ since then M_z is a singleton lattice. Suppose that $p \in M_z$ where $z \in A$. If $\downarrow p \cap K_z \neq \varnothing$, then $p = \bigvee_M \downarrow p \cap K_z$. If $\downarrow p \cap K_p = \varnothing$, then it must be that $p = 0_z$ and $0_z \notin K_z$. The first case still covers this situation if z is the join of the elements in A strictly beneath it. Otherwise $z \in A$ covers an element $y \in A$, $p = 0_z$, and K_z has no least element. In this case, p is not the join in M of the elements of K beneath it. Similar remarks apply to meet density.

Define a special covering pair in M to be an ordered pair formed from elements $1_x, 0_y$ where $x, y \in A$ with x covered by y and either $1_x \notin K_x$ or $0_y \notin K_y$. This implies that 1_x is join irreducible in M, or 0_y is meet irreducible in M, or both. Let θ be the set of all special covering pairs union the diagonal relation on M. It is easy to see that special covering pairs cannot overlap, and hence θ is a congruence relation. The quotient M/θ is the MacNeille completion $\overline{K}$.

It remains to show that $\overline{K} = M/\theta$ has no doubly reducible elements, and hence does not have a sublattice that is isomorphic to D. Along the way, we will show that M/θ is distributive. Let $z \in \overline{A}$. Then M_z is either a 1-element lattice, the MacNeille completion of an 8-element Boolean algebra, which is an 8-element Boolean algebra, or $\overline{2 \times C}$ for some chain C. The MacNeille completion $\overline{2 \times C}$ depends on whether C is bounded. If it is, then $\overline{2 \times C} = 2 \times \overline{C}$. If C has a top, but no bottom, then $\overline{2 \times C} = (2 \times \overline{C}) \smallsetminus \{(1, 0_{\overline{C}})\}$, and so forth. But in any case $\overline{2 \times C}$ is a sublattice of $2 \times \overline{C}$. Therefore, each summand M_z is distributive and has no doubly reducible elements. Thus, M is distributive and has no doubly reducible

elements. In forming the quotient M/θ we only collapse covering pairs where either the lower half is not a proper join, or the upper half is not a proper meet, and hence introduce no doubly reducible elements into the quotient. $\square$

Problem 2. Classify the finite distributive lattices that are MacNeille transferable for the class of lattices whose MacNeille completions are distributive.

4. MacNeille transferability for Heyting algebras

In this section we consider τ-MacNeille transferability for the class $\mathcal{H}$ of Heyting algebras. Here we treat members of $\mathcal{H}$ as bounded lattices and note that the notion of MacNeille transferability does not involve the Heyting implication. Since the MacNeille completion of a Heyting algebra is distributive, results of the previous section give the following.

Theorem 4.1. *If P is a finite distributive lattice that is projective in distributive lattices, then P is MacNeille transferable for $\mathcal{H}$, and the 7-element distributive lattice D of Figure 2 is MacNeille transferable for $\mathcal{H}$.*

We turn attention to $\{\wedge, \vee, 0, 1\}$-MacNeille transferability for $\mathcal{H}$. Using Theorem 3.2, Proposition 2.9, and the fact that for a Heyting algebra K each principle ideal $(\downarrow k)_K$ and filter $(\uparrow k)_K$ are Heyting algebras, gives the following.

Theorem 4.2. *For a finite distributive lattice P that is projective in distributive lattices, $0 \oplus P$, $P \oplus 1$, and $0 \oplus P \oplus 1$ are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$, and for the 7-element distributive lattice D of Figure 2, $0 \oplus D \oplus 1$ is $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$.*

We consider a closely related result, but one that to the best of our knowledge requires a completely different approach.

Theorem 4.3. *For the 7-element distributive lattice D with a doubly reducible element shown in Figure 2, $D \oplus 1$ is $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$.*

Proof. Let K be a Heyting algebra and suppose there is a bounded sublattice of $\overline{K}$ isomorphic to $D \oplus 1$. Then there are normal ideals P, Q, R, S of K situated as in Figure 3. We will show that K has a bounded sublattice isomorphic to $D \oplus 1$.
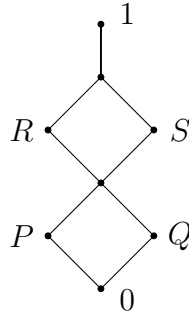

Figure 3

If there are $0 < p_1 < p \in P$ and $0 < q_1 < q \in Q$, then since $P \wedge Q = 0$ implies that $p \wedge q = 0$, it follows from applications of the distributive law that $\{0, p_1, q_1, p_1 \vee q_1, p \vee q_1, q \vee p_1, p \vee q, 1\}$ is a bounded sublattice of K isomorphic to $D \oplus 1$. So one of P, Q must be an atom of $\overline{K}$ and hence an atom of K. We assume $P = {\downarrow}p$ where p is an atom of K.

Choose $r \in R \smallsetminus S$ and $s \in S \smallsetminus R$ with $p \leq r, s$. Since K is a Heyting subalgebra of $\overline{K}$ (see, e.g., [19, Sec. 2]), the pseudocomplement p^* of p in K is the pseudocomplement of p in $\overline{K}$. Therefore, $p^* \geq Q$, giving $p \vee p^* \geq P \vee Q = R \wedge S \geq r \wedge s$. It follows that $p \vee (r \wedge s \wedge p^*) = r \wedge s$. Thus, $\{0, p, r \wedge s \wedge p^*, r \wedge s, r, s, r \vee s, 1\}$ is a bounded sublattice of K isomorphic to $D \oplus 1$. $\square$

To make further progress with positive results, we consider more restrictive classes of Heyting algebras. We assume the reader is familiar with Esakia duality for Heyting algebras [11, 4].

Definition 4.4. Let K be a Heyting algebra, X the Esakia space of K, and $n \geq 1$ a natural number. We say that K and X have *width* n if n is the maximal cardinality of an antichain in X and that K and X have *top width* n if n is the cardinality of the maximum of X. Let $\mathcal{H}_w$ be the class of Heyting algebras of finite width, and $\mathcal{H}_t$ the class of Heyting algebras of finite top width.

Let K be a Heyting algebra. We call $S \subseteq K$ *orthogonal* if $x \wedge y = 0$ for any distinct $x, y \in S$. An element x of K is *regular* if $x = x^{**}$ where x^* is the pseudocomplement of x. It is well known (see, e.g., [24, Sec. IV.6]) that the regular elements form a Boolean algebra B that is a $\{\wedge, 0\}$-subalgebra of K. So an orthogonal set in B is an orthogonal set in K. The following lemmas are easily proved.

Lemma 4.5. *Let K be a Heyting algebra and $n \geq 1$ a natural number. Then K has width at most n iff 2^{n+1} is not isomorphic to a sublattice of K.*

Lemma 4.6. *Let K be a Heyting algebra, B the Boolean algebra of regular elements of K, and $n \geq 1$ a natural number. The following are equivalent.*

   (1) *K has top width at most n.*
   (2) *$2^{n+1} \oplus 1$ is not isomorphic to a bounded sublattice of K.*
   (3) *The maximal cardinality of an orthogonal set in K is at most n.*
   (4) *$|B| \leq 2^n$.*

If a finite distributive lattice P is MacNeille transferable for $\mathcal{H}$, then by definition it is MacNeille transferable for $\mathcal{H}_w$. The converse holds as well, since P is a sublattice of finite Boolean algebra, hence of any Heyting algebra that is not of finite width. Similar reasoning shows that $P \oplus 1$ is MacNeille transferable for $\mathcal{H}_t$ iff $P \oplus 1$ is MacNeille transferable for $\mathcal{H}$. However, the notion of $\{\wedge, \vee, 0, 1\}$-MacNeille transferability for $\mathcal{H}_w$ and $\mathcal{H}_t$ differs from that of $\mathcal{H}$. We begin with the following.

Lemma 4.7. *If K is a Heyting algebra of finite top width, then the complemented elements of the MacNeille completion $\overline{K}$ belong to K.*

Proof. Let x be a complemented element of $\overline{K}$. For any $z \in \overline{K}$ we have $z = (z \wedge x) \vee (z \wedge x^*)$. Since K is a Heyting subalgebra of $\overline{K}$, for $a \in K$, we have that a^* is the pseudocomplement of a in both K and $\overline{K}$. Suppose $a \leq x$. Then $a \leq a^{**} \leq x^{**} = x$. So the normal ideal $N = \{a \in K : a \leq x\}$ is generated by regular elements of K, and for regular elements $a, b \in N$ we have $a \sqcup b = (a \vee b)^{**} \in N$. Since K has finite top width, by Lemma 4.6, there are finitely

many regular elements of K. Therefore, there is a largest regular element in N, so N is a principal ideal of K. Since K is join dense in $\overline{K}$, we have $x = \bigvee N$, hence $x \in K$.   □

Remark 4.8. Lemma 4.7 may be false without the assumption of finite top width as is seen by considering the MacNeille completion of an incomplete Boolean algebra.

Theorem 4.9. *The class of finite lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for the class $\mathcal{H}_t$ of Heyting algebras of finite top width is closed under finite products, as is the class of finite lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for the class $\mathcal{H}_w$ of Heyting algebras of finite width.*

Proof. Let P, Q be finite lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}_t$. Suppose that $K \in \mathcal{H}_t$ and $\varphi : P \times Q \to \overline{K}$ is a bounded lattice embedding. Let $\varphi(1, 0) = x$ and $\varphi(0, 1) = y$. Then x, y are complemented elements of $\overline{K}$, and since K is of finite top width, Lemma 4.7 implies $x, y \in K$.

The restrictions $\varphi|{\downarrow}(1, 0) : {\downarrow}(1, 0) \to ({\downarrow}x)_{\overline{K}}$ and $\varphi|{\downarrow}(0, 1) : {\downarrow}(0, 1) \to ({\downarrow}y)_{\overline{K}}$ are bounded lattice embeddings. But ${\downarrow}(1, 0)$ and ${\downarrow}(0, 1)$ are isomorphic to P and Q respectively, while $({\downarrow}x)_{\overline{K}}$ and $({\downarrow}y)_{\overline{K}}$ are isomorphic to $\overline{({\downarrow}x)_K}$ and $\overline{({\downarrow}y)_K}$ respectively. Since P and Q are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}_t$ and the Heyting algebras $({\downarrow}x)_K$ and $({\downarrow}y)_K$ belong to $\mathcal{H}_t$, there are bounded sublattices of $({\downarrow}x)_K$ and $({\downarrow}y)_K$ isomorphic to P and Q respectively. Since K is isomorphic to $({\downarrow}x)_K \times ({\downarrow}y)_K$, K has a bounded sublattice isomorphic to $P \times Q$. The argument for $\mathcal{H}_w$ is identical, using that $\mathcal{H}_w \subseteq \mathcal{H}_t$.   □

As we will see below, the class of finite lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$ is not closed under binary products. In fact, Theorem 4.12 shows that the product of any two finite distributive lattices is not $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$. Thus, the results of Theorem 4.9 fail if we replace $\mathcal{H}_t$ or $\mathcal{H}_w$ by $\mathcal{H}$. To establish this we require a preliminary definition and lemma.

Definition 4.10. For a natural number $n \geq 1$ let S_n be the finite connected poset shown in Figure 4.
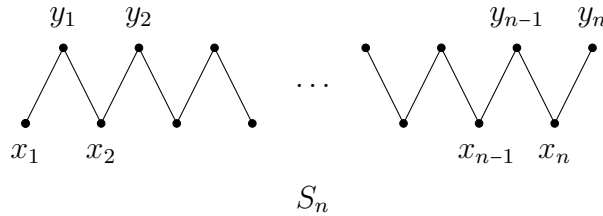

FIGURE 4

Lemma 4.11. *For each finite connected poset F, there is a natural number $n \geq 1$ and an order preserving map from S_n onto F.*

Proof. Let $T_1, \ldots, T_k$ be trees with respective roots $r_1, \ldots, r_k$. Suppose for each T_i there are two distinct maximal nodes t_i^l and t_i^r such that for $2 \leq i \leq k$ we have $t_{i-1}^r = t_i^l$ and $T_{i-1} \cap T_i = \{t_{i-1}^r\}$, and $T_i \cap T_j = \varnothing$ for $j \notin \{i-1, i, i+1\}$, as shown in Figure 5.
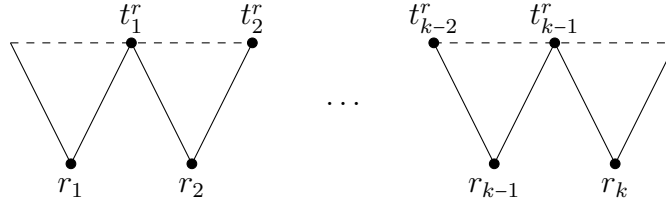
FIGURE 5

Let T be the union of the T_i. By [7, Lem. 16], for every finite connected poset F there is a T as above such that F is a p-morphic image of T. Since p-morphisms are order preserving maps, without loss of generality we may assume that F is $T = \bigcup_{i=1}^{k} T_i$. We then define recursively

$$n_1 = |T_1| - 1$$
$$n_i = n_{i-1} + |T_i| - 2 \quad \text{for } 2 \leq i \leq k.$$

Let $n = n_k$ and define a map $h : S_n \to F$ as follows. Let h map $y_1, \ldots, y_{n_1-1}$ bijectively onto $T_1 \smallsetminus \{r_1, t_1^r\}$. The exact nature of this bijection is irrelevant. Let h map $x_1, \ldots, x_{n_1}$ to r_1 and y_{n_1} to t_1^r. For each $2 \leq i < k$ let h map $y_{n_{i-1}+1}, \ldots, y_{n_i-1}$ bijectively onto $T_i \smallsetminus \{r_i, t_i^l, t_i^r\}$, map $x_{n_{i-1}+1}, \ldots, x_{n_i}$ to r_i, and map y_{n_i} to t_i^r. Finally, let h map $y_{n_{k-1}+1}, \ldots, y_n$ bijectively onto $T_k \smallsetminus \{r_k, t_k^l\}$ and map $x_{n_{k-1}+1}, \ldots, x_n$ to r_k. Then h is the desired map. $\qquad\square$

Theorem 4.12. *If P is a finite distributive lattice with a complemented element $a \neq 0, 1$, then P is not $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$.*

Proof. Let X be the Esakia space whose domain is $\{x_i, y_i, w_i, z_i \mid i \geq 1\} \cup \{\infty\}$ and that is topologized by the one-point compactification of the discrete topology on $\{x_i, y_i, w_i, z_i \mid i \geq 1\}$ with compactification point ∞ and whose ordering is as shown in Figure 6.
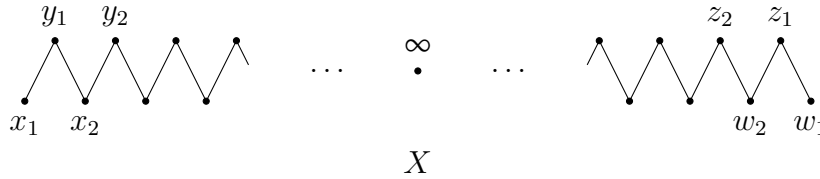

FIGURE 6

Let K be the Heyting algebra of clopen upsets of X. Since $\varnothing, X$ are the only clopen upsets of X that are also downsets, the only complemented elements of K are $0, 1$.

For an Esakia space Y, we let $\overline{Y}$ be the Esakia space of the MacNeille completion of the Heyting algebra corresponding to Y. Since K is in fact a bi-Heyting algebra, by [19, Thm. 3.8], the elements of the MacNeille completion of K are the regular open upsets of X. Therefore, $\{x_i, y_i \mid i \geq 1\}$ and $\{w_i, z_i \mid i \geq 1\}$ are the new complemented elements of $\overline{K}$. Thus, $\overline{X}$ has domain $\{x_i, y_i, w_i, z_i \mid i \geq 1\} \cup \{\infty_1, \infty_2\}$ that is topologized by the two-point compactification of $\{x_i, y_i, w_i, z_i \mid i \geq 1\}$. Consequently, $\overline{X}$ is the disjoint union of two Esakia spaces X_1 and X_2, each of which carries the topology of the one-point compactification of the discrete topology on $\{x_i, y_i \mid i \geq 1\}$ and $\{w_i, z_i \mid i \geq 1\}$, respectively; see Figure 7.
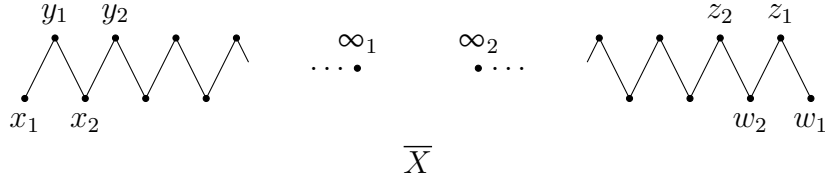
FIGURE 7

To show that P is not $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$, by Esakia duality, it is sufficient to construct an Esakia space Y such that the Esakia dual E of P is a continuous order preserving image of $\overline{Y}$ but not of Y. Since P has a complemented element $a \neq 0, 1$, the finite poset E is disconnected. Let $E = F_1 \cup \cdots \cup F_m$ be the decomposition of E into connected components.

First suppose that $m = 2$, so $E = F_1 \cup F_2$. If there were a continuous order preserving onto map $f : X \to E$, then the inverse image of the upset of E corresponding to a would be a complemented element of K different from $0, 1$. Since this is a contradiction, there is no continuous order preserving map from X onto E.

We show that there is a continuous order preserving map from $\overline{X}$ onto E. By Lemma 4.11, there are n_1 and n_2 and order preserving onto maps $h_1 : S_{n_1} \to F_1$ and $h_2 : S_{n_2} \to F_2$. We have that $\overline{X}$ is the disjoint union of X_1 and X_2 and we can regard S_{n_1} as a subposet of X_1 and S_{n_2} as a subposet of X_2 in the obvious way. Let m_1 be a minimal element of F_1 that is below $h_1(y_{n_1})$ and let m_2 be a minimal element of F_2 that is below $h_2(y_{n_2})$. Define

$$h(s) = \begin{cases} h_1(s) & \text{if } s \in S_{n_1}, \\ h_2(s) & \text{if } s \in S_{n_2}, \\ m_1 & \text{if } s \in X_1 \smallsetminus S_{n_1}, \\ m_2 & \text{if } s \in X_2 \smallsetminus S_{n_2}. \end{cases}$$

Then h is the desired continuous order preserving onto map.

If $m > 2$, then define Y to be the disjoint union of X and $F_3, \ldots, F_m$. Then $\overline{Y}$ is the disjoint union of $\overline{X}$ and $F_3, \ldots, F_m$. Since E has m components and Y has $m - 1$ components, there is no continuous order preserving map from Y onto E. But an obvious modification of the above argument provides a continuous order preserving map from $\overline{Y}$ onto E. $\qquad\square$

To conclude this section, we give an example to show that even in the setting of Heyting algebras, sharp MacNeille transferability is not the norm.

Proposition 4.13. *The lattice D of Figure 2 is not sharply MacNeille transferable for $\mathcal{H}$.*

Proof. Let $L = K \oplus 2^2$, where K is the sublattice of $\mathbb{R}^2$ with domain

$$\{(x, y) \mid 0 \le x < 1, 0 \le y \le x\} \cup \{(x, y) \mid 1 < x \le 2, 0 \le y \le 1\}.$$

It is routine to verify that K and L are Heyting algebras. The MacNeille completion $\overline{L}$ of L inserts the "missing" line $\{(1, y) \in \mathbb{R}^2 \mid 0 \le y \le 1\}$. We can visualize L and $\overline{L}$ as follows in Figure 8. The black circles in the picture of $\bar{L}$ define a lattice embedding φ from D into $\overline{L}$.

Suppose there is a lattice embedding $\varphi' : D \to L$ that is sharp, meaning that $\varphi'(x) \le \varphi(y)$ iff $x \le y$. Recall that $D = 2^2 \oplus 2^2$ is built from two 4-element Boolean algebras. Since φ' is
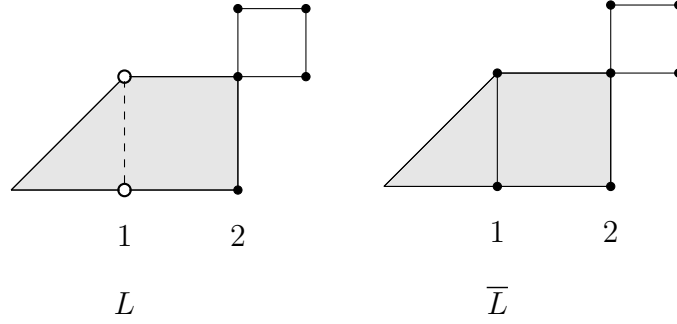
FIGURE 8

sharp, φ and φ' must agree on the upper copy of 2^2 comprising D. So φ' maps the doubly reducible element of D to $(2,1)$. Since φ maps the bottom element of D to $(1,0)$, φ' must map the bottom element of D to some $(a,0)$ with $a < 1$. But there do not exist elements in L whose meet is $(a,0)$ where $a < 1$ and whose join is $(2,1)$.   □

Among many open problems in this area, the following seems to us the most pressing. We expect the answer is negative, but that a counterexample is quite involved.

Problem 3. Is every finite distributive lattice P MacNeille transferable for $\mathcal{H}$?

5. Bi-Heyting algebras

Restricting attention to the class of bi-Heyting algebras, stronger results of a positive nature are possible. Recall that a Heyting algebra A is a bi-Heyting algebra if the order dual of A is also a Heyting algebra. Note that the property of being a bi-Heyting algebra is preserved under MacNeille completions (see, e.g., [19]).

Lemma 5.1. *Let A be a Heyting algebra of finite width and X the Esakia space of A. If U, V are regular open upsets of X, then so is $U \cup V$.*

Proof. Suppose the width of A is n. Then the width of X is n, and by [10, p. 3], X can be covered by n maximal chains $C_1, \ldots, C_n$. By [11, Lem. III.2.8], each of these maximal chains is closed in X. To show that $U \cup V$ is regular open, it is enough to show that $\mathsf{IC}(U \cup V) \subseteq U \cup V$ where I and C are the interior and closure operators.

Let $x \in \mathsf{IC}(U \cup V)$. Then there is a clopen set K with $x \in K \subseteq \mathsf{C}(U \cup V) = \mathsf{C}(U) \cup \mathsf{C}(V)$. Let $S = \{i : x \in C_i\}$ and $T = \{i : x \notin C_i\}$. Consider the following statement

(A) $U \cap C_i \subseteq \mathsf{C}(V)$ for each $i \in S$.

Suppose (A) holds. Then since $D := \bigcup\{C_i : i \in T\}$ is closed and $x \notin D$, there is clopen K' with $x \in K' \subseteq K \subseteq \mathsf{C}(U) \cup \mathsf{C}(V)$ and K' disjoint from D. Therefore, $K' \subseteq (K' \cap \mathsf{C}(U)) \cup \mathsf{C}(V)$. Since $C_1 \cup \cdots \cup C_n$ covers X and K' is disjoint from D, and hence disjoint from each $C_i = \mathsf{C}(C_i)$ for $i \in T$, we have $K' \cap \mathsf{C}(U) = \bigcup\{K' \cap \mathsf{C}(U \cap C_i) : i \in S\}$. Then condition (A) gives that $K' \cap \mathsf{C}(U) \subseteq \mathsf{C}(V)$, hence $K' \subseteq \mathsf{C}(V)$. Thus, $x \in \mathsf{IC}(V) = V$.

Suppose that (A) does not hold. Then there is i with $x \in C_i$ and $U \cap C_i \nsubseteq \mathsf{C}(V)$. Let $y \in U \cap C_i$ with $y \notin \mathsf{C}(V)$. Note that if $x \in U$, then there is nothing in showing that $x \in U \cup V$, and if $x \notin U$, then since x, y belong to the chain C_i and U is an upset, we must have $x < y$. Since $y \notin \mathsf{C}(V)$, there is clopen G with $y \in G$ and G disjoint from V. So G is disjoint from

$C(V)$. Since X is an Esakia space and V is an upset, $C(V)$ is an upset. Therefore, G disjoint from $C(V)$ implies that the clopen set $\downarrow G$ is disjoint from $C(V)$. Thus, $K' = K \cap \downarrow G$ is a clopen neighborhood of x disjoint from $C(V)$. Since $K' \subseteq K \subseteq C(U) \cup C(V)$, we have $K' \subseteq C(U)$. Consequently, $x \in IC(U) = U$. $\square$

Proposition 5.2. *If A is a bi-Heyting algebra of finite width, then $\overline{A}$ is a bounded sublattice of $\mathfrak{I}A$.*

Proof. Let X be the Esakia space of A. It is well known that $\mathfrak{I}A$ is isomorphic to the open upsets of X, and since A is a bi-Heyting algebra, by [19, Thm. 3.8], $\overline{A}$ is isomorphic to the regular open upsets of X. By Lemma 5.1, regular open upsets of X form a bounded sublattice of open upsets of X. The result follows. $\square$

As a consequence, we obtain the following.

Theorem 5.3. *Every finite distributive lattice is sharply MacNeille transferable for the class of bi-Heyting algebras of finite width, and MacNeille transferable for the class of all bi-Heyting algebras.*

Proof. Let P be a finite distributive lattice. If A is a bi-Heyting algebra of finite width and $\varphi : P \to \overline{A}$ is a lattice embedding, then since $\overline{A}$ is a sublattice of $\mathfrak{I}A$, we may consider $\varphi : P \to \mathfrak{I}A$. Then sharp MacNeille transferability follows from Theorem 1.2. That P is MacNeille transferable for the class of all bi-Heyting algebras follows since P is a sublattice of a finite Boolean algebra, and hence of any bi-Heyting algebra that is not of finite width. $\square$

6. Stable universal classes and stable intermediate logics

In this section we show how to produce stable universal classes of Heyting algebras [5, 6] closed under MacNeille completions and draw implications for stable intermediate logics. This was the original motivation for this study.

Definition 6.1. A class $\mathcal{K}$ of Heyting algebras is *stable* if whenever A, B are Heyting algebras with $A \in \mathcal{K}$ and B isomorphic to a bounded sublattice of A, we have $B \in \mathcal{K}$.

It is known [5, 6] that a universal class $\mathcal{K}$ of Heyting algebras is stable iff there is a set $\mathcal{P}$ of finite distributive lattices such that $\mathcal{K}$ consists of those Heyting algebras K such that no member of $\mathcal{P}$ is isomorphic to a bounded sublattice of K. It is convenient to introduce the following notation

$$\mathcal{H}(\mathcal{P}) = \{K \in \mathcal{H} : K \text{ has no bounded sublattice isomorphic to any } P \in \mathcal{P}\}.$$

The definition of τ-MacNeille transferability for $\tau = \{\wedge, \vee, 0, 1\}$ gives the following.

Proposition 6.2. *Let $\mathcal{P}$ be a set of finite lattices where each $P \in \mathcal{P}$ is $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$. Then the stable class $\mathcal{H}(\mathcal{P})$ is closed under MacNeille completions.*

Theorems 4.2 and 4.3 entail:

Theorem 6.3. *Let P be a finite distributive lattice that is projective in distributive lattices.*
  (1) *The stable classes $\mathcal{H}(0 \oplus P)$, $\mathcal{H}(P \oplus 1)$, and $\mathcal{H}(0 \oplus P \oplus 1)$ are closed under MacNeille completions.*
  (2) *The stable classes $\mathcal{H}(0 \oplus D \oplus 1)$ and $\mathcal{H}(D \oplus 1)$, with D as in Figure 2, are closed under MacNeille completions.*

Remark 6.4. Theorem 6.3(1) is based on Theorem 2.7, which says that the class of lattices that do not contain a $\{\wedge, 0, 1\}$-subalgebra isomorphic to P is closed under MacNeille completions. Using syntactic methods, an analogous result for residuated lattices was given in [9]. These results overlap in the setting of Heyting algebras (see [22]), and can be used to provide examples of stable classes that are closed under MacNeille completions based on omitting finite projective distributive lattices. However, since the lattices $D \oplus 1$ and $0 \oplus D \oplus 1$ are not projective in distributive lattices, the results of Theorem 6.3(2) do not fall in the scope of the results of [9].

There is a related method to create stable classes of Heyting algebras that are closed under MacNeille completions. Note that for a finite lattice P and bounded lattice K, that P is isomorphic to a sublattice of K iff at least one of $P, 0 \oplus P, P \oplus 1, 0 \oplus P \oplus 1$ is a bounded sublattice of K. The following result for a finite lattice P that is MacNeille transferable for $\mathcal{H}$ is immediate, and has obvious generalization to a set of such MacNeille transferable P.

Theorem 6.5. *Let P be a finite lattice that is MacNeille transferable for $\mathcal{H}$. Then for $\mathcal{P} = \{P, 0 \oplus P, P \oplus 1, 0 \oplus P \oplus 1\}$, the stable class $\mathcal{H}(\mathcal{P})$ is closed under MacNeille completions.*

Recall from Section 4 that by $\mathcal{H}_t$ and $\mathcal{H}_w$ we denote the classes of Heyting algebras of finite top-width and of finite width, respectively.

Theorem 6.6. *Let $\mathcal{P}$ be a set of finite lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}_t$ and let $n \geq 1$ be a natural number. Then for $\mathcal{P}' = \mathcal{P} \cup \{2^n \oplus 1\}$, the stable class $\mathcal{H}(\mathcal{P}')$ is closed under MacNeille completions. Similarly, if $\mathcal{P}$ is a set of finite lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}_w$, then for $\mathcal{P}'' = \mathcal{P} \cup \{2^n, 0 \oplus 2^n, 2^n \oplus 1, 0 \oplus 2^n \oplus 1\}$, the stable class $\mathcal{H}(P'')$ is closed under MacNeille completions.*

Proof. Suppose $K \in \mathcal{H}(\mathcal{P}')$. Then K has no bounded sublattice isomorphic to $2^n \oplus 1$, so by Lemma 4.6, K has finite top width, hence belongs to $\mathcal{H}_t$. Since every finite Boolean algebra is projective in distributive lattices [3], Theorem 6.3 provides that $2^n \oplus 1$ is $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$. Thus, $\overline{K}$ has no bounded sublattice isomorphic to $2^n \oplus 1$. Also, since each $P \in \mathcal{P}$ is $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}_t$ and $K \in \mathcal{H}_t$, we have that $\overline{K}$ contains no bounded sublattice isomorphic to any $P \in \mathcal{P}$. So $\overline{K} \in \mathcal{H}(\mathcal{P}')$. The result for $\mathcal{H}_w$ is similar, using that each 2^n is MacNeille transferable for $\mathcal{H}$, Lemma 4.5, and Theorem 6.5. $\square$

Remark 6.7. Our primary examples of finite distributive lattices that are $\{\wedge, \vee, 0, 1\}$-MacNeille transferable for $\mathcal{H}$ are those of the shape $0 \oplus P$, $P \oplus 1$, and $0 \oplus P \oplus 1$ for some finite projective distributive lattice P, and the lattices $0 \oplus D \oplus 1$ and $D \oplus 1$. We can make stable classes that are closed under MacNeille completions directly from these lattices as in Theorems 6.3 and 6.5.

Since $\{\wedge, \vee, 0, 1\}$-MacNeille transferability for $\mathcal{H}$ implies that for $\mathcal{H}_t$ and $\mathcal{H}_w$, by Theorem 4.9, we can take finite products of these lattices, and then use Theorem 6.6 to produce stable classes that are closed under MacNeille completions. This provides a range of examples of stable universal classes that are closed under MacNeille completions that are not within the scope of [9].

Finally, we discuss the impact of our results on the stable intermediate logics from [5, 6]. Recall that an intermediate logic is *stable* if its corresponding variety is generated by a stable universal class of Heyting algebras. Thus, the above results produce stable intermediate logics generated by stable universal classes that are closed under MacNeille completions. Due to

[15] such logics are always canonical. Two stable universal classes may generate the same variety and hence give rise to the same stable intermediate logic while one of them is closed under MacNeille completions and the other is not. For instance, $\mathcal{H}(2^2)$ and $\mathcal{H}(0 \oplus 2^2)$ both generate the variety of all Heyting algebras $\mathcal{H}$, but by Theorems 6.3 and 4.12, respectively, $\mathcal{H}(0 \oplus 2^2)$ is closed under MacNeille completions while $\mathcal{H}(2^2)$ is not. The following is open.

Problem 4. Is every stable intermediate logic generated by a stable universal class that is closed under MacNeille completions?

The following provides an infinite family of stable intermediate logics each of which is determined by a stable universal class closed under MacNeille completions. None of these logics is covered by the results of [9].

Proposition 6.8. *For a natural number $n \geq 1$, let L_n be the logic of the stable universal class $\mathcal{H}_n := \mathcal{H}(\{D \oplus 1, 2^n, 0 \oplus 2^n, 2^n \oplus 1, 0 \oplus 2^n \oplus 1\})$. Then $\{L_n \mid n \geq 1\}$ is an infinite family of stable intermediate logics each of which is determined by a stable universal class of Heyting algebras closed under MacNeille completions.*

Proof. By Theorem 6.6, $\mathcal{H}_n$ is closed under MacNeille completions. Therefore, each L_n is a stable intermediate logic determined by a stable universal class closed under MacNeille completions. By [6], a subdirectly irreducible Heyting algebra K validates L_n iff none of the algebras $D \oplus 1, 2^n, 0 \oplus 2^n, 2^n \oplus 1$, or $0 \oplus 2^n \oplus 1$ is isomorphic to a bounded sublattice of K. If $m > n$, then $0 \oplus 2^n \oplus 1$ validates L_m but refutes L_n, so $L_m \neq L_n$. □

Acknowledgements

References

1. K. A. Baker and A. W. Hales, *From a lattice to its ideal lattice*, Algebra Universalis **4** (1974), 250–258.
2. R. Balbes and P. Dwinger, *Distributive lattices*, University of Missouri Press, Columbia, Mo., 1974.
3. R. Balbes and A. Horn, *Projective distributive lattices*, Pacific J. Math. **33** (1970), 273–279.
4. G. Bezhanishvili (ed.), *Leo Esakia on duality in modal and intuitionistic logics*, Dordrecht: Springer, 2014.
5. G. Bezhanishvili and N. Bezhanishvili, *Locally finite reducts of Heyting algebras and canonical formulas*, Notre Dame J. Form. Log. **58** (2017), no. 1, 21–45.
6. G. Bezhanishvili, N. Bezhanishvili, and J. Ilin, *Cofinal stable logics*, Studia Logica **104** (2016), no. 6, 1287–1317.
7. G. Bezhanishvili and M. Gehrke, *Completeness of S4 with respect to the real line: revisited*, Ann. Pure Appl. Logic **131** (2005), no. 1-3, 287–301.
8. C. C. Chang and H. J. Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland Publishing Co., Amsterdam, 1990.
9. A. Ciabattoni, N. Galatos, and K. Terui, *MacNeille completions of FL-algebras*, Algebra Universalis **66** (2011), no. 4, 405–420.
10. P. Crawley and R. P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, 1973.
11. L. Esakia, *Heyting algebras I. Duality theory (Russian)*, "Metsniereba", Tbilisi, 1985.
12. F. Galvin and B. Jónsson, *Distributive sublattices of a free lattice*, Canad. J. Math. **13** (1961), 265–272.
13. H. S. Gaskill, *On the relation of a distributive lattice to its lattice of ideals*, Bull. Austral. Math. Soc. **7** (1972), 377–385; corrigendum, ibid. 8 (1973), 317–318.
14. M. Gehrke and J. Harding, *Bounded lattice expansions*, J. Algebra **238** (2001), no. 1, 345–371.
15. M. Gehrke, J. Harding, and Y. Venema, *MacNeille completions and canonical extensions*, Trans. Amer. Math. Soc. **358** (2006), no. 2, 573–590 (electronic).

16. G. Grätzer, *Universal algebra*, Trends in Lattice Theory (Sympos., U.S. Naval Academy, Annapolis, Md., 1966), Van Nostrand Reinhold, New York, 1970, pp. 173–210.
17. _____, *Lattice theory: foundation*, Birkhäuser/Springer Basel AG, Basel, 2011.
18. J. Harding, *Any lattice can be regularly embedded into the MacNeille completion of a distributive lattice*, Houston J. Math. **19** (1993), no. 1, 39–44.
19. J. Harding and G. Bezhanishvili, *MacNeille completions of Heyting algebras*, Houston J. Math. **30** (2004), no. 4, 937–952.
20. A. Horn and N. Kimura, *The category of semilattices*, Algebra Universalis **1** (1971), no. 1, 26–38.
21. A. Kostinsky, *Projective lattices and bounded homomorphisms*, Pacific J. Math. **40** (1972), 111–119.
22. F.M. Lauridsen, *Intermediate logics admitting a structural hypersequent calculus*, Studia Logica, To appear.
23. E. Nelson, *The embedding of a distributive lattice into its ideal lattice is pure*, Algebra Universalis **4** (1974), 135–140.
24. H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Monografie Matematyczne, Tom 41, Państwowe Wydawnictwo Naukowe, Warsaw, 1963.
25. F. Wehrung, *Relative projectivity and transferability for partial lattices*, arXiv:1612.04189, 2016.

Guram Bezhanishvili: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA, guram@nmsu.edu

John Harding: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA, hardingj@nmsu.edu

Julia Ilin: Institute for Logic, Language and Computation, University of Amsterdam, 1090 GE Amsterdam, The Netherlands, ilin.juli@gmail.com

Frederik M. Lauridsen: Institute for Logic, Language and Computation, University of Amsterdam, 1090 GE Amsterdam, The Netherlands, f.m.lauridsen@uva.nl