Intuitionistic implication without disjunction

Gerard R. Renardel de Lavalette
Lex Hendriks
Dick de Jongh

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Abstract

We investigate fragments of intuitionistic propositional logic containing implication but not disjunction. These fragments are finite, but their size grows superexponentially with the number of generators. Exact models are used to characterize the fragments.

1 Introduction

Intuitionistic propositional logic IpL, envisaged as the free Heyting algebra over a nonempty collection of generators P, is infinite. When P is a singleton, we obtain the well-known Rieger-Nishimura lattice. For larger P, however, the free Heyting algebra is very complex and little is known about it. This is unlike the situation for classical logic: Boolean algebras over finitely many generators are finite, and their structure is well known.

Closer inspection learns that the combination of disjunction and implication causes the free Heyting algebras to become infinite. When we only consider formulae of IpL without disjunction, the corresponding algebras are finite; idem if we drop implication instead of disjunction.

In this paper, we investigate *fragments* of IpL, i.e. sublogics defined by restricting the set of atomic formulae and the set of connectives. We focus on several fragments that contain $\rightarrow$ and not $\vee$: see Fig. 1 (observe that we treat double negation $\neg\neg$ as a connective on its own).

We denote a fragment by listing the generators and connectives between square brackets, so $[p, q, \wedge, \rightarrow]$ is the fragment consisting of formulae that only contain the propositional variables p, q and the connectives $\wedge$ and $\rightarrow$. When the identity of the propositional variables is not relevant but only their number, we may write e.g. $[\wedge, \rightarrow]^n$ for the fragment with n propositional variables.

The *diagram* $F_{\equiv}$ of fragment F is the set of the equivalence classes of its formulae, partially ordered by the derivability relation. Some small diagrams are drawn in Fig. 2. We shall see that the size of these diagrams grows superexponentially with the number of generators.

We shall use exact and quasi-exact models to study the diagrams of fragments. A finite model $M = \langle W, \leqslant, \mathsf{atom}\rangle$ is an *exact model* for fragment L whenever diagram $F_{\equiv}$ is isomorphic to $\wp^u(W)$, the collection of upward closed

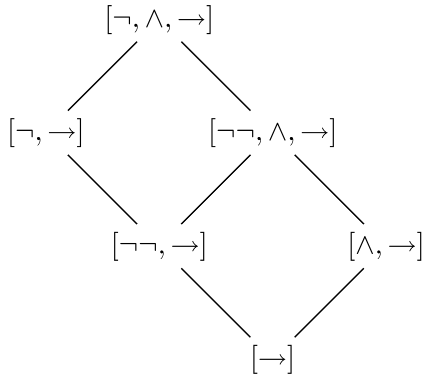

Figure 1: Inclusion diagram of the fragments studied in this paper.

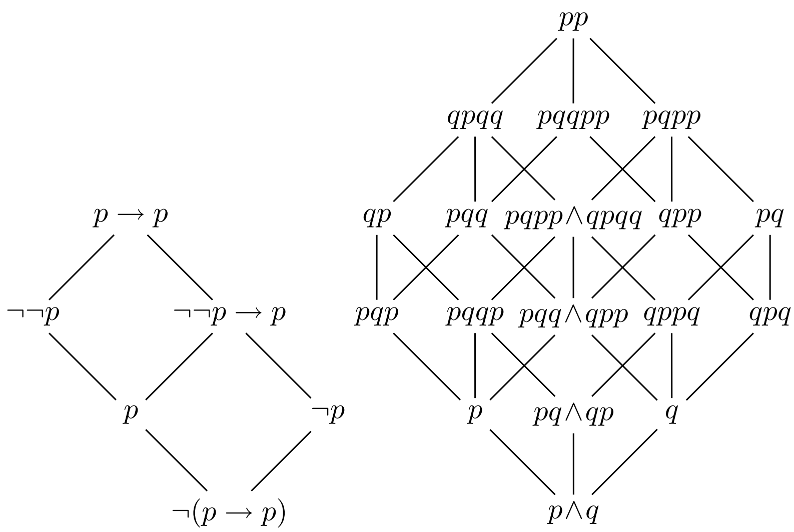

Figure 2: The diagrams of $[p, \neg, \wedge, \rightarrow]$ and $[p, q, \wedge, \rightarrow]$. To reduce the size of formulae, the implication arrows are omitted in the right hand diagram: so e.g. pqp abbreviates $(p \rightarrow q) \rightarrow p$.

subsets of W. So an exact model is also a minimal universal model: it is (modulo isomorphism) the smallest model such that equivalence in the model implies provable equivalence in the fragment. As we shall see, from the fragments considered here only the fragments $[P, \neg, \wedge, \rightarrow]$ and $[P, \wedge, \rightarrow]$ have exact models. Other fragments will be characterized by *quasi-exact models*, where the diagram $F_{\equiv}$ is not isomorphic with the full $\wp^u(W)$, but only with a subset of it.

1.1 The historical perspective

The study of fragments of propositional logics may have remained somewhat at the backstage of logic research, but the subject has always fascinated both logicians and algebraists. In this subsection, we give an overview of the main developments from a historical perspective.

As a forerunner, one may consider Th. Skolem's 1919 paper [35] on the application of concepts from (what we now call) lattice theory to non-classical logics. But it really started with A. Heyting's groundbreaking formalisation [16] of intuitionistic logic in 1930. This led to the notion of Heyting algebras, and to the natural question: what is the structure of the free Heyting algebra, i.e. the algebra of the equivalence classes of formulae of IpL? In 1932 K. Gödel proved in [12] that this algebra is infinite, in other words: IpL does not have a finite set of 'truth values'. N. Rieger [33] discovered in 1949 that the fragment of IpL with only one propositional variable has already infinitely many equivalence classes. This free Heyting algebra over one generator is a nice lattice, rediscovered by I. Nishimura [29] in 1960.

In 1952, Skolem [36] showed that in the intuitionistic algebra of pure implication, every formula containing not more than two variables is equivalent to one of a collection of 14 formulas. This diagram of the fragment $[\to]^2$ must have been rediscovered many times since (and maybe even before) by several logic students. The same will undoubtedly be true for the 18-point diagram of $[\wedge, \to]^2$ (see Fig. 2), first published by R. Balbes [1] in 1973.

A first systematic study of the algebras corresponding to the $[\to]$ and $[\wedge, \to]$ fragments of IpL, called *Hilbert algebras*[1] and *Brouwerian* (or *implicative*) *semi-lattices* respectively, was published by A. Monteiro [27] in 1955.

In 1955, E. Beth developed in [3] a semantics for intuitionistic logic based on semantic tableaus, emerging from systematic attempts to disprove the derivability of a formula (see also [20]). Similar ideas for possible world semantics for modal logic were investigated by S. Kanger and J. Hintikka. These developments culminated in S. Kripke's famous paper in 1965 on the semantics of intuitionistic logic [22] in terms of partial orders of possible worlds, now known as Kripke models.

In this period, the interest in the connections between logic and (universal) algebra increased, as can be deduced from the popularity of the book *The mathematics of metamathematics* [31] by H. Rasiowa and R. Sikorski, published in 1963. Several attempts were made to connect the algebraic and the partial order approach to the semantics of IpL and the intermediate logics between IpL and classical propositional logic, e.g. in 1966 by Heyting's and Beth's students A. Troelstra and one of the authors (De Jongh) in [37].

In 1965 A. Diego, a student of Monteiro, proved in [9] the basic result for the area of our research: *finitely generated Hilbert algebras are finite.* G. McKay seems to have been the first to observe in 1968 in [26] that Diego's result can be extended quite easily to implicative semi-lattices. Independently, Urquhart [39] proved the finiteness of the diagrams of $[\to]^n$ and $[\wedge, \to]^n$ in 1974.

Finite implicative semi-lattices are bounded: for all x we have $b \leqslant x \leqslant \top$, where $b = \bot$ if the fragment contains negation and otherwise $b = \bigwedge P$ with P the set of atomic formulae in the fragment. They are also lattices, as can be seen by taking $\bigwedge\{x \mid a \leqslant x \text{ and } b \leqslant x\}$ for $a \vee b$. Using an early result from Skolem [35], implicative lattices are distributive; and by a well known

[1] The term *Hilbert algebra* is also used for unitary algebras, related to Hilbert spaces and first introduced by V. Rohlin in [34].

theorem of G. Birkhoff [4], distributive lattices are isomorphic to the lattice of upward closed subsets of the partially ordered set of their (join-)irreducible elements. Based on these insights and Diego's result, the following question comes up naturally: what is the structure of the underlying partial orders of the diagrams of $[\to]^n$, $[\wedge,\to]^n$ and $[\neg,\wedge,\to]^n$? N. de Bruijn was the first to address this question in [6] (a shorter version appeared as [8]). He baptised these underlying partial orders *exact models*: they turn out to be Kripke models. These exact models are finite, which gives another proof for Diego's result. De Bruijn discovered the 61-point exact model of $[\wedge,\to]^3$ (see Fig. 7) and used it to compute the size of its diagram: 623 662965 552330. In [7], De Bruijn developed an Algol60 computer program to test formulas in $[\wedge,\to]^3$ based on the exact model. Such programs using model checking are in general much faster than, e.g., tableau-based testers.

Due to their somewhat obscure publication medium, De Bruijn's results were not immediately noticed in the algebraic or logic communities. Building upon the pioneering work of Rasiowa and Sikorski [31] on algebraic semantics and Nemitz [28] on implicative algebras, Landolt and Whaley [25] and Krzystek [23] studied free implicative semilattices and duplicated independently De Bruijn's results. Köhler [21] also studies implicative semilattices, and refers to De Bruijn's work. He computed the size of the exact model of $[\wedge,\to]^4$ to be 2494 651862 209437, using a formula that improves upon a result in [6] (it is the first formula in Theorem 9 of the present paper).

In the late 1960s, one of the authors (De Jongh) and H. Kamp started the investigation of the structure of the diagrams of fragments of IpL using a tableau-based computer program to compute equivalence classes. Due to the limitations of computer power at that time, only small fragments could be investigated. Later on, De Jongh stimulated one of the authors (Hendriks), H. van Riemsdijk and J. Tromp to continue this research, now using more powerful computers. This subsequently led to a research project by the present three authors, focusing on the study of diagrams of fragments of IpL, combining tableau-based testers and fast computer programs using exact models, as previously developed by De Bruijn. A first publication of their results is [19], which describes exact models for $[\wedge,\to]$ and $[\neg,\wedge,\to]$ as minimal complete Kripke models for these fragments, and provides characteristic formulas for the nodes in exact models, using methods that were developed in De Jongh's Ph.D. thesis [18] and are akin to those of Jankov in [17]. This research was extended to exact models of $[\neg,\wedge,\vee]$ and some subfragments, and resulted in the Ph.D. thesis of Hendriks [14]. In this thesis, the notion of *semantic type* was introduced, inspired by the notion of *character* introduced in modal logic by K. Fine [10] and G. Boolos [5] (see also [11]). Some of the techniques developed for fragments of IpL have proved applicable in modal logic and provability logic (see [13]). The computer programs developed by Hendriks were used to find a counterexample for the interpolation property of the fragment $[\leftrightarrow]$: see [15] and the reference there to [30] by Porębska with an earlier discovery of a counterexample.

We end this historical survey by mentioning two recent publications. In [2], J. Berman and W. Blok study free Hilbert and related algebras. They compute 25 165802 as the size of $[\to]^3$ which is isomorphic with the free Hilbert algebra over three elements, and refer to [14] for the size of $[\to]^4$. In [40], F. Yang studies several fragments of IpL and points out a flaw in the inductive reasoning in proofs of [19] and [14].

The current paper recapitulates the research on fragments of IpL with implication and without disjunction in a uniform and perspicuous way; moreover, it introduces new semantical concepts, repairs a flawed proof and presents new results, e.g. the characterisation of fragments with double negation, and the computation of $[\to]^4$. In a forthcoming paper, we plan to do the same for the other class of finite IpL-fragments, viz. those without implication.

1.2 How to obtain exact models

Let us indicate how we will construct exact models of fragments, using the fragment $[P, \neg, \wedge, \to]$ as an example.

1. First of all, we need a way to reduce arbitrary (possibly infinite) models to finite models, in a way that is invariant for the formulae in $[P, \neg, \wedge, \to]$. For this, we use the semantical property of inductiveness: a node w in a model M is inductive iff $\forall v > w(p \in \mathsf{atom}(v))$ implies $p \in \mathsf{atom}(w)$, for all propositional variables $p \in P$. In words: if p is true in all nodes v above w, then it is true in w. It appears that this inductive property extends to all formulae φ in $[P, \neg, \wedge, \to]$: if $\forall v > w(v \models \varphi)$ then $w \models \varphi$.

2. This suggests that inductive nodes are not needed to distinguish non-equivalent formulae in $[P, \neg, \wedge, \to]$. To make this explicit, we define a reduction operation $M^{-\mathsf{i}}$ on models M which eliminates all inductive nodes in M, and we show that all formulae in $[P, \neg, \wedge, \to]$ are invariant wrt. this reduction, i.e. $M \models \varphi \Leftrightarrow M^{-\mathsf{i}} \models \varphi$ for all $\varphi \in [P, \neg, \wedge, \to]$.

3. Furthermore, we identify a 'maximal' model E consisting of only noninductive nodes that contains $M^{-\mathsf{i}}$ for every M. It is not hard to see that E is finite, due to the fact that the maximal depth of a node in E is bounded by the cardinality of P. Moreover, we prove: if $\varphi, \psi \in [P, \neg, \wedge, \to]$ are equivalent in E, then they are equivalent in *all* models, so E is universal for $[P, \neg, \wedge, \to]$.

4. Finally, to show that E is an exact model, we define for every upward closed subset X in E a formula φ_X in $[P, \neg, \wedge, \to]$ that characterises X in the sense that we have $w \in X \Leftrightarrow E, w \models \varphi_X$.

For other fragments, we use other semantical properties, based on full and hybrid nodes (Definition 5). For the characterisation of fragments without conjunction, we use the property $J(\varphi \to \psi) \subseteq J(\psi)$ of the semantical mapping J, first described by De Bruijn in [6].

1.3 Survey of the rest of the paper

In Section 2, we introduce the notions to be used in the paper: partial orders, models, the semantical mapping J and bisimulation. Section 3 contains definitions of the semantical notions that we use to characterize the fragments studied here. The universal model is presented in Section 4, followed by the definition of exact models as finite submodels of the universal model in Section 5, where we also introduce characteristic formulae and prove the main results about (quasi-)exact models. In Section 6, we have a closer look at the structure of the models and the diagrams, and we derive several formulae about their size and its asymptotic behaviour. The final Section 7 contains some concluding remarks.

For reasons of readability, some proofs are delegated to the Appendix.

2 Preliminaries

Definition 1 (Partial orders and related notions)
Models are constructed as usual from partially ordered sets $\langle W, \leqslant \rangle$, where $\leqslant$ is a reflexive, antisymmetric and transitive relation on W. The *one-step* order $<_1$ is defined by

$$v <_1 w \text{ iff } v < w \; \& \; \neg \exists u \in W \; v < u < w.$$

The *cover* $\overline{w}$ of w is the set $\{v \mid v >_1 w\}$ of one-step successors of w. The *upward closure* $X{\uparrow}$ of $X \subseteq W$ is defined by

$$X{\uparrow} = \{w \mid \exists v \in X (w \geqslant v)\}$$

and the *strict upward closure* $X^{\wedge}$ is defined as $X{\uparrow} - X$. For closures of singleton sets, we drop the parentheses and write $w{\uparrow}$ and $w^{\wedge}$. Also

$$\mathsf{min}(X) = \{v \in X \mid \neg \exists w \in X (w < v)\}$$

We adopt analogous definitions for the downward closure $X{\downarrow}$, the strict downward closure $X_{\vee}$ and for $\mathsf{max}(X)$. We say that $X, Y \subseteq W$ are *incomparable* if $\neg \exists x \in X \exists y \in Y (x \leqslant y \vee y \leqslant x)$. The *depth* $d(w)$ of $w \in W$ is defined inductively by

$$d(w) = \sup\{d(v) + 1 \mid v > w\}, \text{ where } \sup(\emptyset) = 0$$

We put

$$\begin{array}{rcl} \wp^u(W) & = & \{X \in \wp(W) \mid \forall x \in X \forall y \geqslant x \; y \in X\} \\ \wp^a(W) & = & \{A \in \wp(W) \mid \forall ab \in A (a \leqslant b \Rightarrow a = b)\} \end{array}$$

so $\wp^u(W)$ is the collection of upward closed subsets of W, and $\wp^a(W)$ the collection of *antichains*, i.e. subsets where no two elements are comparable. We shall use the isomorphism

$$i : \wp^u(W) \to \wp^a(W), \text{ defined by } i(X) = \mathsf{max}(W - X) \tag{1}$$

with inverse $i^{-1}(A) = W - A{\downarrow}$. i induces a partial order $\preceq$ on $\wp^a(W)$, defined by

$$A \preceq B \text{ iff } B \subseteq A{\downarrow}.$$

Definition 2 (IpL and its fragments)
The language of IpL is defined as usual from propositional variables $p, q, r, \dots \in \mathsf{PV}$, the constant $\top$ and the connectives $\neg, \wedge, \vee, \rightarrow$. We shall use P, Q, R, for finite subsets of PV, and we write P_n for the subset $\{p_1, \dots, p_n\}$ of PV. $\bot$ is defined as $\neg\top$.
We write $\mathsf{IpL}(P)$ for the collection of IpL-formulae containing only propositional variables in $P \subseteq \mathsf{PV}$. Moreover, when C is a collection of connectives (where we also allow $\neg\neg$ and $\bot$), the fragment $[P, C]$ is the collection of IpL-formulae containing as propositional variables only elements of P and as connectives only elements of C. Since $\top$ is part of the language, $\mathsf{IpL}(P)$ and $[P, C]$ are always nonempty, even if $P = C = \emptyset$.

Observe that every fragment contains the constant $\top$, and that $\neg$ and $\bot$ are interchangeable in fragments containing $\rightarrow$ (for $\neg\varphi \equiv (\varphi \rightarrow \bot)$ and $\bot \equiv \neg\top$).

Definition 3 (Models and validity)
A model for P is a triple $M = \langle W, \leqslant, \mathsf{atom}\rangle$ where $\langle W, \leqslant\rangle$ is a nonempty partial order and $\mathsf{atom} : W \rightarrow \wp(P)$ is a mapping that indicates where the propositional variables $p \in P$ are valid. atom is monotonic, i.e. $v \leqslant w \Rightarrow \mathsf{atom}(v) \subseteq \mathsf{atom}(w)$. We extend atom to sets of nodes by $\mathsf{atom}(X) = \bigcap\{\mathsf{atom}(x) \mid x \in X\}$ if $X \neq \emptyset$, and $\mathsf{atom}(\emptyset) = P$. In this paper, we only consider *locally finite* partially ordered sets $\langle W, \leqslant\rangle$, where $w\uparrow$ is always finite. $\mathsf{MOD}(P)$ denotes the collection of locally finite models for P.

Validity of a formula in a node in a model is defined as usual:

$$
\begin{array}{lcl}
M, w \models p & \Leftrightarrow & w \in V(p) \\
M, w \models \neg\varphi & \Leftrightarrow & \forall v \geqslant w (M, v \not\models \varphi) \\
M, w \models \varphi \wedge \psi & \Leftrightarrow & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \varphi \vee \psi & \Leftrightarrow & M, w \models \varphi \text{ or } M, w \models \psi \\
M, w \models \varphi \rightarrow \psi & \Leftrightarrow & \forall v \geqslant w (M, v \models \varphi \Rightarrow M, v \models \psi) \\
M, w \models \varphi \leftrightarrow \psi & \Leftrightarrow & \forall v \geqslant w (M, v \models \varphi \Leftrightarrow M, v \models \psi)
\end{array}
$$

As is well known (see e.g. [38]), we have:

> IpL is sound for all models, and complete wrt. the collection of finite models.

An alternative (but equivalent) definition of models uses valuation mappings $V : P \rightarrow \wp^u(W)$ instead of atom. V can be extended to a semantical mapping $V : \mathsf{IpL} \rightarrow \wp^u(W)$ by

$$V(\varphi) = \{w \in W \mid M, w \models \varphi\}$$

When we combine V with the isomorphism i defined in (1), we obtain the mapping $J : \mathsf{IpL} \rightarrow \wp^a(W)$ with

$$J(\varphi) = \max\{w \in W \mid w \not\models \varphi\}$$

So $J(\varphi)$ is the collection of so-called border points that lie outside $V(\varphi)$. The main reason for working with J instead of the more usual mapping V is the property, first mentioned by De Bruijn in [6]:

$$J(\varphi \rightarrow \psi) \subseteq J(\psi) \tag{2}$$

We shall use this property (which follows from (8) in the next lemma) when we investigate fragments not containing conjunction.

Lemma 1 (Main properties of J)
For all formulae φ, ψ we have:

$$\begin{aligned}
J(p) &= \{w \mid p \in \mathsf{atom}(\overline{w}) - \mathsf{atom}(w)\} && (3)\\
J(\top) &= \emptyset && (4)\\
J(\bot) &= \mathsf{max}(W) && (5)\\
J(\neg\varphi) &= \mathsf{max}(W) - J(\varphi) && (6)\\
J(\varphi \wedge \psi) &= (J(\varphi) \cap J(\psi)) \cup (J(\varphi) - J(\psi){\downarrow}) \cup (J(\psi) - J(\varphi){\downarrow}) && (7)\\
J(\varphi \rightarrow \psi) &= J(\psi) - J(\varphi){\downarrow} && (8)
\end{aligned}$$

Proof (3), (4) and (5) are verified easily, and (6) follows from (5) and (8). For (7), we reason as follows:

$$\begin{aligned}
& J(\varphi \wedge \psi) \\
= \\
& \mathsf{max}\{w \mid w \not\models (\varphi \wedge \psi)\} \\
= \\
& \{w \mid (w \not\models \varphi \text{ or } w \not\models \psi) \;\&\; \forall v > w (v \models \varphi \;\&\; v \models \psi)\} \\
= \\
& \{w \mid (w \not\models \varphi \;\&\; w \not\models \psi \;\&\; \forall v > w(v \models \varphi \;\&\; v \models \psi)) \\
& \qquad \& \; (w \not\models \varphi \;\&\; w \models \psi \;\&\; \forall v > w(v \models \varphi)) \\
& \qquad \& \; (w \models \varphi \;\&\; w \not\models \psi \;\&\; \forall v > w(v \models \psi))\} \\
= \\
& (J(\varphi) \cap J(\psi)) \cup (J(\varphi) - J(\psi){\downarrow}) \cup (J(\psi) - J(\varphi){\downarrow})
\end{aligned}$$

Finally we prove (8).

$$\begin{aligned}
& J(\varphi \rightarrow \psi) \\
= \\
& \mathsf{max}\{w \mid w \not\models (\varphi \rightarrow \psi)\} \\
= \\
& \{w \mid w \not\models (\varphi \rightarrow \psi) \;\&\; \forall v > w(v \models \varphi \Rightarrow v \models \psi)\} \\
= \\
& \{w \mid w \models \varphi \;\&\; w \not\models \psi \;\&\; \forall v > w(v \models \varphi \Rightarrow v \models \psi)\} \\
= \\
& \{w \mid w \models \varphi\} \cap \{w \mid w \not\models \psi \;\&\; \forall v > w \; v \models \psi\} \\
= \\
& \mathsf{max}\{w \mid w \not\models \psi\} - \{w \mid w \not\models \varphi\} \\
= \\
& J(\psi) - J(\varphi){\downarrow}
\end{aligned}$$

□

The following properties of J are verified in a similar manner:

$$\begin{aligned}
J(\neg\neg\varphi) &= \mathsf{max}(W) \cap J(\varphi) && (9)\\
J(\neg\neg\varphi \to \varphi) &= J(\varphi) - \mathsf{max}(W) && (10)\\
J((\varphi \to \psi) \to \varphi) &= J(\varphi) - J(\psi)_{\vee} && (11)\\
J((\psi \to \varphi) \to \varphi) &= J(\varphi) \cap J(\psi){\downarrow} && (12)\\
J(((\varphi \to \psi) \to \varphi) \to \varphi) &= J(\varphi) \cap J(\psi)_{\vee} && (13)\\
J(((\varphi \to \psi) \to \psi) \to \varphi) &= J(\varphi) - J(\psi) && (14)\\
J((\varphi \leftrightarrow \psi) \to \varphi) &= J(\varphi) \cap J(\psi) && (15)\\
J((\psi \wedge \chi) \to \varphi) &= J(\psi \to \varphi) \cap J(\chi \to \varphi) && (16)
\end{aligned}$$

Definition 4 (Bisimulation)
A relation B between two models $M = \langle W, \leqslant, \mathsf{atom}\rangle$ and $M' = \langle W', \leqslant', \mathsf{atom}'\rangle$ is a *bisimulation* if it satisfies the following three conditions (where the $\cdot$ denotes relational composition):

$$\begin{aligned}
&B \subseteq \{(w, w') \mid \mathsf{atom}(w) = \mathsf{atom}'(w')\}\\
&(\geqslant \cdot B) \subseteq (B \cdot \geqslant')\\
&(B \cdot \leqslant') \subseteq (\leqslant \cdot B)
\end{aligned}$$

A functional bisimulation is also called a *p-morphism*[2].
Two elements w and w' are called *bisimilar* if there is a bisimulation B between M and M' with wBw'. Notation: $w \underline{\leftrightarrow} w'$.

Since the union of bisimulations is again a bisimulation, $\underline{\leftrightarrow}$ is the largest bisimulation. We have as a well-known fact, for all formulae φ:

if $v \underline{\leftrightarrow} w$ and $v \models \varphi$, then $w \models \varphi$.

3 Some semantical properties

In this section, we define some semantical properties that are related to the fragments we consider here. First a definition of properties of nodes in a model.

Definition 5 (inductive, full and hybrid nodes)
Let $M = \langle W, \leqslant, \mathsf{atom}\rangle \in \mathsf{MOD}(P)$ with $w \in W$.

1. w is *inductive* or an *i-node* if it is not maximal and $\mathsf{atom}(w) = \mathsf{atom}(w^{\wedge})$ (i.e. if an atom holds in all worlds above w, then it also holds in w);

2. w is *full* or an *f-node* if $\mathsf{atom}(w) = P$ (i.e. all propositional variables of P hold in w);

3. w is *hybrid* or an *h-node* if there are u, v with $w <_1 u$, $w <_1 v$, u is full and v is not full (i.e. w has both full and non-full immediate successors).

See Fig. 3 for examples.

We have the following simple properties of inductive and full nodes:

[2]For modal logic this notion was first invented by K. Segerberg. It was used previously, however, on intuitionistic frames in [37] under the name *strongly isotonic function*.

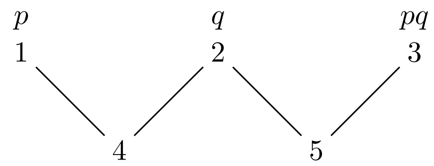

Figure 3: A model where node 4 is inductive, 3 is full and 5 is hybrid.

Lemma 2

1. If $\varphi \in [\neg, \wedge, \rightarrow]$, w inductive and $w^\wedge \models \varphi$, then $w \models \varphi$.

2. If $\varphi \in [P, \neg\neg, \wedge, \vee, \rightarrow]$ and w full, then $w \models \varphi$.

The proofs proceed via straightforward induction.

We want to know more about x-nodes (x = f, i or h) than Lemma 2 tells us: what is the class of formulae invariant under the operation of eliminating x-nodes? Therefore we define some reductions of models. In $M^{-\mathsf{f}}$ we leave out the full nodes, and in $M^{-\mathsf{i}}$ the inductive nodes. In $M^{-\mathsf{h}}$, we do not leave out nodes but we take away links in the accessibility relation in such a way that hybrid nodes lose their link with full nodes.

Definition 6
Let $M = \langle W, \leqslant, \mathsf{atom} \rangle \in \mathsf{MOD}(P)$.

1. $M^{-\mathsf{i}} = \langle W^{-\mathsf{i}}, \leqslant^{-\mathsf{i}}, \mathsf{atom}^{-\mathsf{i}} \rangle$, the *i-reduct* of M, is defined by $W^{-\mathsf{i}} = \{w \in W \mid w \text{ is not inductive}\}$, and $\leqslant^{-\mathsf{i}}$ and $\mathsf{atom}^{-\mathsf{i}}$ are the restrictions of $\leqslant$ and atom to $W^{-\mathsf{i}}$.

2. $M^{-\mathsf{f}} = \langle W^{-\mathsf{f}}, \leqslant^{-\mathsf{f}}, \mathsf{atom}^{-\mathsf{f}} \rangle$, the *f-reduct* of M, is defined by $W^{-\mathsf{f}} = \{w \in W \mid w \text{ is not full}\}$, and $\leqslant^{-\mathsf{f}}$ and $\mathsf{atom}^{-\mathsf{f}}$ are the restrictions of $\leqslant$ and atom to $W^{-\mathsf{f}}$.

3. $M^{-\mathsf{h}} = \langle W, \leqslant^{-\mathsf{h}}, \mathsf{atom} \rangle$, the *h-reduct* of M, is defined by: $\leqslant^{-\mathsf{h}}$ is the reflexive transitive closure of $<_1^{-\mathsf{h}}$, where

   $$v <_1^{-\mathsf{h}} w \text{ iff } v <_1 w \ \& \ \text{ not}(v \text{ hybrid } \& \ w \text{ full})$$

   or, equivalently,

   $$v <_1^{-\mathsf{h}} w \text{ iff } v <_1 w \ \& \ (\mathsf{atom}(w) \neq P \text{ or } \mathsf{atom}(v^\wedge) = P)$$

When M contains only full nodes, $M^{-\mathsf{f}}$ is empty and hence not a model. In that case, we interpret $M^{-\mathsf{f}} \models \varphi$ as vacuously true. For $M^{-\mathsf{i}}$ this does not apply, since every locally finite model has maximal nodes, and they are by definition not inductive.

Lemma 3
For x equals i, f or h, we have

$M^{-\mathsf{x}}$ has no x-nodes

Proof For $M^{-\mathsf{f}}$ this is evident. To see that $M^{-\mathsf{i}}$ has no inductive nodes: observe that if w were inductive in $M^{-\mathsf{i}}$, then it would also be inductive in M, so it cannot be in $M^{-\mathsf{i}}$. Finally, $M^{-\mathsf{h}}$ has no hybrid nodes, for a hybrid node in M has no full immediate $\geqslant^{-\mathsf{h}}$-successors, hence it is no longer hybrid in $M^{-\mathsf{h}}$. □

Now we can define the main semantical properties.

Definition 7 (Invariance)
Let x equal f, i or h. INV_x, the collection of x-*invariant* formulae, is defined by

$$\mathsf{INV}_\mathsf{x} = \{\varphi \mid \text{for all models } M: \quad M \models \varphi \;\Leftrightarrow\; M^{-\mathsf{x}} \models \varphi\}$$

Furthermore we define

$$\mathsf{INV}_{\mathsf{fi}} = \mathsf{INV}_\mathsf{f} \cap \mathsf{INV}_\mathsf{i}$$
$$\mathsf{INV}_{\mathsf{hi}} = \mathsf{INV}_\mathsf{h} \cap \mathsf{INV}_\mathsf{i}$$
$$\mathsf{VAL}_\mathsf{f} = \{\varphi \mid \varphi \text{ holds in all full nodes }\}$$

We write $\mathsf{INV}_\mathsf{i}(P)$ for $\mathsf{INV}_\mathsf{i} \cap \mathsf{IpL}(P)$, and similarly for other formula collections.

We shall show that these notions of invariance characterize the fragments considered in this paper. The following theorem is a step in that direction:

Theorem 1

1. INV_i contains $\bot$ and PV, and is closed under $\neg$, $\wedge$ and $\rightarrow$.

2. INV_f contains PV and is closed under $\wedge$, $\vee$ and $\rightarrow$.

3. $\mathsf{INV}_\mathsf{h} \cap \mathsf{VAL}_\mathsf{f}$ contains PV and is closed under $\neg\neg$, $\wedge$, $\vee$ and $\rightarrow$.

Proof See the Appendix. □

Observe that $\bot$ is not f-invariant: if M contains only full nodes, then $M^{-\mathsf{f}}$ is empty and by convention $M^{-\mathsf{f}} \models \bot$, while of course $M \not\models \bot$.

As a direct consequence of Theorem 1, we have one half of the characterisation of three fragments:

1. $[\neg, \wedge, \rightarrow] \subseteq \mathsf{INV}_\mathsf{i}$.

2. $[\wedge, \rightarrow] \subseteq \mathsf{INV}_{\mathsf{fi}}$.

3. $[\neg\neg, \wedge, \rightarrow] \subseteq \mathsf{INV}_{\mathsf{hi}} \cap \mathsf{VAL}_\mathsf{f}$.

We shall prove the other half of the characterisation in Theorem 6.

4 Types and the universal model

Types are objects of the form $\langle P, X\rangle$ where $P \subseteq \mathsf{PV}$ is a collection of propositional variables and X is a finite collection of types. They were introduced as semantic types in [14]. We shall construct models from types as nodes: the first component of a type indicates which propositional variables are valid, and the second components contains its direct successors. So $\mathsf{atom}(\langle P, X\rangle) = P$, and the partial order $\leqslant$ on types is the reflexive transitive closure of the one-step order $<_1$, defined by $\langle P, X\rangle <_1 \langle Q, Y\rangle$ iff $\langle Q, Y\rangle \in X$.

A collection X of types is called *closed* when we have $Y \subseteq X$ for all $\langle Q, Y\rangle \in X$. A closed collection of types X can be seen as a model $\langle X, \leqslant, \mathsf{atom}\rangle$, where $\leqslant$ and atom are as defined above.

We define the universal model as a collection of types, in such a way that no two types in the universal model are bisimilar. This is realized as follows.

Definition 8 (the universal model)
The universal model UM of IpL is defined inductively as the smallest collection of types satisfying the following condition:

$$\begin{array}{lll} \text{if} & X \text{ finite and } X \in \wp^a(\mathsf{UM}), & \\ & Q \subseteq \mathsf{atom}(X) \text{ and} & \\ & |X| = 1 \Rightarrow Q \subset \mathsf{atom}(X), & \text{then } \langle Q, X\rangle \in \mathsf{UM} \end{array}$$

Here $\subset$ denotes strict inclusion: $X \subset Y$ iff $X \subseteq Y$ and $X \neq Y$.

Observe that $\langle Q, \emptyset\rangle \in \mathsf{UM}$ for all $Q \subseteq \mathsf{PV}$, since $\emptyset \in \wp^a(\mathsf{UM})$ and $\mathsf{atom}(\emptyset) = \mathsf{PV}$ (by convention). In general, $\langle Q, X\rangle$ is a node in UM if X is a finite antichain in UM with $Q \subseteq \mathsf{atom}(X)$ and Q is a proper subset of $\mathsf{atom}(X)$ in the case that X is a singleton set. The last condition is added to exclude types of the form $\langle Q, \{\langle Q, X\rangle\}\rangle$ in UM, which are bisimilar with $\langle Q, X\rangle$.

In order to embed a locally finite model into the universal model, we define a reduction mapping that maps nodes of the model to types in the universal model.

Definition 9 (reduction mapping)
Let M be a locally finite model: we define the mapping $\rho = \rho_M : M \to \mathsf{UM}$. $\rho(w)$ is defined with induction over the depth of w by

$$\begin{array}{rcll} \rho(w) & = & t & \text{if } \mathsf{min}\{\rho(v) \mid v >_1 w\} = \{t\} \\ & & & \text{and } \forall v >_1 w\ \mathsf{atom}(v) = \mathsf{atom}(w) \\ & = & \langle \mathsf{atom}(w), \mathsf{min}\{\rho(v) \mid v >_1 w\}\rangle & \text{otherwise} \end{array}$$

So $\rho(w) = \rho(v)$ if $\rho(v)$ is the unique element of $\mathsf{min}\{\rho(v) \mid v >_1 w\}$ and $\mathsf{atom}(w) = \mathsf{atom}(v)$; otherwise $\rho(w) = \langle \mathsf{atom}(w), X\rangle$ with $X = \mathsf{min}\{\rho(v) \mid v >_1 w\}\rangle$. It is evident that always $\mathsf{atom}(w) = \mathsf{atom}(\rho(w))$, and also that $v \leqslant w$ implies $\rho(v) \leqslant \rho(w)$.

The following theorem summarises the main properties of types, the universal model UM and the reduction mapping ρ.

Theorem 2

1. Bisimilar types in UM are equal: $\forall st \in \mathsf{UM}(s \underline{\leftrightarrow} t \;\Rightarrow\; s = t)$.
2. ρ_M is a bisimilarity (and hence a p-morphism).
3. $\rho_M(w) \in \mathsf{UM}$ for all models $M \in \mathsf{MOD}$ and all w in M.
4. UM is a locally finite universal model for IpL, and ρ_{UM} is the identity.

Proof See the Appendix. □

5 Exact models

In this section we shall define, for finite sets $P \subseteq \mathsf{PV}$, exact models $\mathsf{EM}_\neg(P)$ for $[P, \neg, \wedge, \rightarrow]$ and $\mathsf{EM}(P)$ for $[P, \wedge, \rightarrow]$. Moreover, we shall define quasi-exact models $\mathsf{QEM}(P)$ for $[P, \neg\neg, \wedge, \rightarrow]$. All (quasi-)exact models are finite submodels of the universal model. In the definition, we shall use the modification atom_P of atom, defined by

$$\mathsf{atom}_P(X) = \mathsf{atom}(X) \cap P,$$

so $\mathsf{atom}_P(\emptyset) = P$, and $\mathsf{atom}_P(X) = \mathsf{atom}(X)$ if X is a nonempty set of nodes with atoms in P.

Definition 10 (exact and quasi-exact models)
$\mathsf{EM}(P)$, $\mathsf{EM}_\neg(P)$ and $\mathsf{QEM}(P)$ are defined inductively as follows.

1. If $X \in \wp^a(\mathsf{EM}(P))$ and $Q \subset \mathsf{atom}_P(X)$, then $\langle Q, X\rangle \in \mathsf{EM}(P)$.
2. $\langle P, \emptyset\rangle \in \mathsf{EM}_\neg(P)$;
   if $X \in \wp^a(\mathsf{EM}_\neg(P))$ and $Q \subset \mathsf{atom}_P(X)$, then $\langle Q, X\rangle \in \mathsf{EM}_\neg(P)$.
3. $\langle P, \emptyset\rangle \in \mathsf{QEM}(P)$;
   if $Q \subset P$ then $\langle Q, \{\langle P, \emptyset\rangle\}\rangle \in \mathsf{QEM}(P)$;
   if $X \in \wp^a(\mathsf{QEM}(P) - \{\langle P, \emptyset\rangle\})$ and $Q \subset \mathsf{atom}_P(X)$, then $\langle Q, X\rangle \in \mathsf{QEM}(P)$.

It is evident that $\mathsf{EM}(\emptyset) = \emptyset$, and that $\mathsf{EM}_\neg(\emptyset) = \mathsf{EM}(\{p\}) = \mathsf{QEM}(\{p\}) = \{\langle\emptyset, \emptyset\rangle\}$. $\mathsf{EM}_\neg(\{p\})$ has 3 nodes and equals $\mathsf{QEM}(\{p\})$, and $\mathsf{EM}(\{p, q\})$ has 5 nodes: see Fig. 4. Observe that $\wp^a(\mathsf{EM}_\neg(\{p\})$, the collection of antichains in $\mathsf{EM}_\neg(\{p\})$, is isomorphic to the diagram of $[p, \neg, \wedge, \rightarrow]$ given in Fig. 2; idem for $\wp^a(\mathsf{EM}(\{p, q\})$ and $[p, q, \wedge, \rightarrow]$. $\mathsf{EM}_\neg(\{p, q\})$ has 15 nodes and differs from $\mathsf{QEM}(\{p, q\})$, which has 13 nodes: see Fig. 5 and 6. $\mathsf{EM}(\{p, q, r\})$ with 61 nodes is given in Fig. 7.

We observe that in all these models the nodes with empty atom set are the most frequent, while nodes with larger atom sets are increasingly rare.

In general, $\mathsf{EM}(P)$ is a submodel of $\mathsf{QEM}(P)$ which is a submodel of $\mathsf{EM}_\neg(P)$, which is a finite submodel of UM. Moreover, if $P \subseteq Q$ then $\mathsf{EM}(P) \subseteq \mathsf{EM}(Q)$, $\mathsf{EM}_\neg(P) \subseteq \mathsf{EM}_\neg(Q)$ and $\mathsf{QEM}(P) \subseteq \mathsf{QEM}(Q)$. We also observe that, in all these models, $v < w$ implies $\mathsf{atom}(v) \subset \mathsf{atom}(w)$.

Figure 4: The exact models $\mathsf{EM}_\neg(\{p\})$ and $\mathsf{EM}(\{p,q\})$

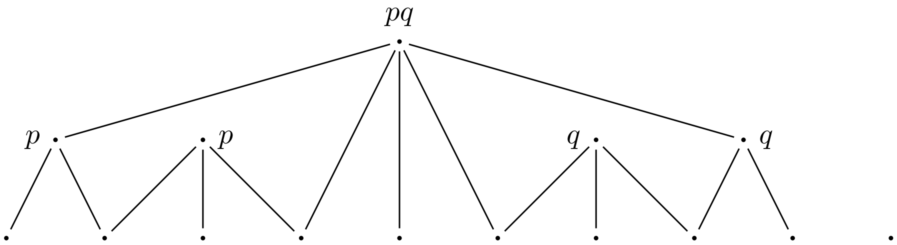

Figure 5: The exact model $\mathsf{EM}_\neg(\{p,q\})$

We shall study the structure of these models more closely later on. For now we establish the link between these models and semantical notions introduced earlier.

Theorem 3

1. $\mathsf{EM}_\neg(P)$ is universal for $\mathsf{INV}_\mathsf{i}(P)$
2. $\mathsf{EM}(P)$ is universal for $\mathsf{INV}_\mathsf{fi}(P)$
3. $\mathsf{QEM}(P)$ is universal for $\mathsf{INV}_\mathsf{hi}(P)$

Proof See the Appendix. □

Using the (quasi-)exact models, we define the diagrams of the fragments studied here.

Definition 11 (diagrams)
The diagrams $\mathsf{D}_\mathsf{ci}(P)$ (c for conjunction, i for implication), $\mathsf{D}_\mathsf{nci}(P)$ (n for negation), $\mathsf{D}_\mathsf{dci}(P)$ (d for double negation), etc., are defined by

$$\begin{array}{lcll}
\mathsf{D}_\mathsf{ci}(P) & = & \wp^a(\mathsf{EM}(P)) & \\
\mathsf{D}_\mathsf{nci}(P) & = & \wp^a(\mathsf{EM}_\neg(P)) & \\
\mathsf{D}_\mathsf{dci}(P) & = & \wp^a(\mathsf{QEM}(P) - \{\langle P, \emptyset\rangle\}) & \\
\mathsf{D}_\mathsf{i}(P) & = & \bigcup_{p\in P} \wp(J_{\mathsf{EM}(P)}(p)) & \subseteq \ \mathsf{D}_\mathsf{ci}(P) \\
\mathsf{D}_\mathsf{ni}(P) & = & \bigcup_{p\in P} \wp(J_{\mathsf{EM}_\neg(P)}(p)) \cup \wp(J_{\mathsf{EM}_\neg(P)}(\bot)) & \subseteq \ \mathsf{D}_\mathsf{nci}(P) \\
\mathsf{D}_\mathsf{di}(P) & = & \bigcup_{p\in P} \wp(J_{\mathsf{QEM}(P)}(p)) & \subseteq \ \mathsf{D}_\mathsf{dci}(P)
\end{array}$$

By Theorems 5 and 8, we have that $[P, \wedge, \rightarrow]_\equiv$ is isomorphically embedded in $\mathsf{D}_\mathsf{ci}(P)$, $[P, \neg, \wedge, \rightarrow]_\equiv$ in $\mathsf{D}_\mathsf{nci}(P)$ to $[P, \neg, \wedge, \rightarrow]_\equiv$, etc. In the next section, we shall prove that these embeddings are surjective.

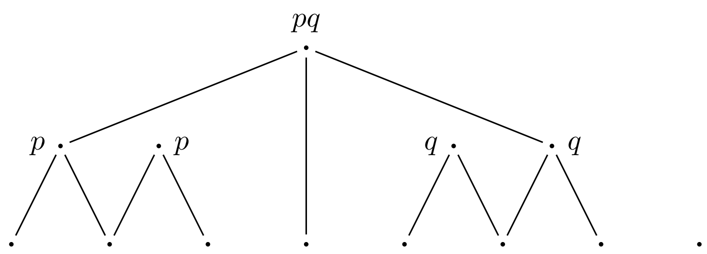

Figure 6: The quasi-exact model $\mathsf{QEM}(\{p,q\})$

5.1 Characteristic formulae for nodes in exact models

We shall show that $\mathsf{EM}(P)$, $\mathsf{EM}_\neg(P)$ are indeed exact models for $[P,\wedge,\to]$ and $[P,\neg,\wedge,\to]$ respectively, i.e. that for every antichain X in the model there is a formula ψ_X in the corresponding fragment with $J(\psi_X)=X$. For $\mathsf{QEM}(P)$ and the corresponding fragment $[P,\neg\neg,\wedge,\to]$, we will do this for almost all upward closed subsets. For this purpose, we define (for E equals $\mathsf{EM}(P)$, $\mathsf{EM}_\neg(P)$ or $\mathsf{QEM}(P)$) characteristic formulae $\chi_{E,w}$ and prove in Theorem 4 that $J_E(\chi_{E,w}) = \{w\}$.

Definition 12 (Characteristic formulae)
For $E = \mathsf{EM}(P), \mathsf{EM}_\neg(P)$ or $\mathsf{QEM}(P)$ and $w = \langle Q, X\rangle \in E$, we define $\chi_{E,w}$ with downward induction over $|\mathsf{atom}(w)|$ as follows.

1. $E = \mathsf{EM}(P)$. We know that $\mathsf{atom}_P(X) - Q$ is not empty, so let p be an element of this set. (We shall see in the proof of Theorem 4 that the specific choice of p does not matter.) Now

   $$\chi_{E,w} = (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4) \;\to\; p$$

   where

   $$\begin{array}{rcl} \varphi_1 & = & \bigwedge Q \\ \varphi_2 & = & \bigwedge\{p \leftrightarrow q \mid q \in \mathsf{atom}_P(X) - Q - \{p\}\} \\ \varphi_3 & = & \bigwedge\{(p \to \chi_{E,x}) \to p \mid x \in X\} \\ \varphi_4 & = & \bigwedge\{p \to \chi_{E,v} \mid v \in Y\} \end{array}$$

   and $Y = \max\{v \in W \mid v \notin X\!\uparrow, \mathsf{atom}(v) \supseteq \mathsf{atom}_P(X)\}$.

2. $E = \mathsf{EM}_\neg(P)$. As for EM, with the additional case $w = \langle P, \emptyset\rangle$, for which we define $\chi_{E,w} = \neg(\bigwedge P)$.

3. $E = \mathsf{QEM}(P)$. Here we define $\chi_{E,w}$ only for $w \neq \langle P, \emptyset\rangle$. We proceed as for EM, with the additional case $w = \langle Q, \{\langle P, \emptyset\rangle\}\rangle$ with $Q \subset P$, for which we define (recall that $p \in P - Q$)

   $$\chi_{E,w} = (\bigwedge Q \wedge \bigwedge\{p \leftrightarrow q \mid q \in P - Q - \{p\}\} \wedge \neg\neg p) \;\to\; p$$

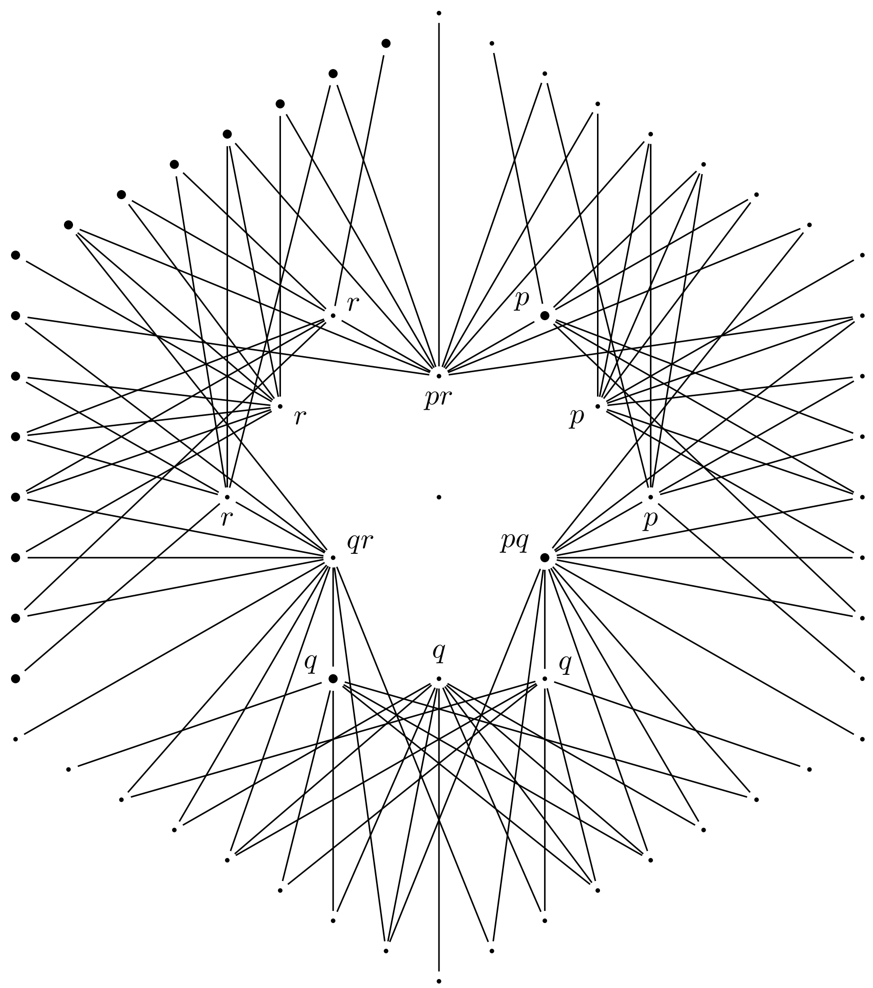

Figure 7: The exact model $\mathsf{EM}(\{p,q,r\})$. The fat nodes indicate the embedding of $\wp^a(\mathsf{EM}(\{p,q\}))$ by $e_{\{p,q,r\}}$, as described in the proof of Lemma 5.

Observe that the definition of φ_3 is correct, since $x \in X$ implies $|\mathsf{atom}(x)| > |\mathsf{atom}(w)|$, so $\chi_{E,x}$ is defined in an earlier stage. Idem for φ_4, since $v \in Y$ implies $|\mathsf{atom}(v)| > |\mathsf{atom}(w)|$, and moreover $\langle P, \emptyset\rangle \notin Y$ (which is relevant in the case $E = \mathsf{QEM}(P)$). Observe also that $\chi_{\mathsf{EM}(P),w}$ is indeed a formula in $[P, \wedge, \rightarrow]$, $\chi_{\mathsf{EM}_\neg(P),w}$ in $[P, \neg, \wedge, \rightarrow]$ and $\chi_{\mathsf{QEM}(P),w}$ in $[P, \neg\neg, \wedge, \rightarrow]$.

Theorem 4
For $E = \mathsf{EM}(P), \mathsf{EM}_\neg(P)$ or $\mathsf{QEM}(P)$ and $w = \langle Q, X\rangle \in E$, we have

$$J_E(\chi_{E,w}) = \{w\}$$

unless $E = \mathsf{QEM}(P)$ and $w = \langle P, \emptyset\rangle$, in which case $\chi_{E,w}$ is not defined.

Proof See the Appendix. □

Now we can characterize the structure of the diagrams of the $\wedge$-fragments.

Theorem 5

1. $[P, \neg, \wedge, \rightarrow]_\equiv$ and $\mathsf{D_{nci}}(P)$ are isomorphic.
2. $[P, \wedge, \rightarrow]_\equiv$ and $\mathsf{D_{ci}}(P)$ are isomorphic.
3. $[P, \neg\neg, \wedge, \rightarrow]_\equiv$ and $\mathsf{D_{dci}}(P)$ are isomorphic.

Proof 1. We shall show that $J = J_{\mathsf{EM}_\neg(P)} : [P, \neg, \wedge, \rightarrow] \rightarrow \wp^a(\mathsf{EM}_\neg(P))$ is an isomorphism (recall that $\mathsf{D_{nci}}(P) = \wp^a(\mathsf{EM}_\neg(P)))$. J is injective: if $\varphi, \psi \in [P, \neg, \wedge, \rightarrow]$ and $J(\varphi) = J(\psi)$ then $\varphi \equiv \psi$, for φ, ψ are in $\mathsf{INV_i}(P)$ and $\mathsf{EM}_\neg(P)$ is universal wrt. $\mathsf{INV_i}(P)$. To show that J is surjective, too, let $X \in \wp^a(E)$ be arbitrary and define $\psi_X = \bigwedge\{\chi_{E,w} \mid w \in X\}$: we shall show that $J(\psi_X) = X$. By (7), we have that $J(\varphi \wedge \psi) = J(\varphi) \cup J(\psi)$ whenever $J(\varphi)$ and $J(\psi)$ are incomparable. All different $v, w \in X$ are incomparable and $J(\chi_{E,w}) = \{w\}$, so indeed $J(\psi_X) = \bigcup_{w \in X}\{w\} = X$.

2. Similar.

3. Since $\wp^u(\mathsf{QEM}(P) - \{\langle P, \emptyset\rangle\})$ is isomorphic to $\{X \in \wp^u(\mathsf{QEM}(P)) \mid \langle P, \emptyset\rangle \in X\}$, it suffices for the third part to show that $[P, \neg\neg, \wedge, \rightarrow]_\equiv$ and $\{X \in \wp^u(\mathsf{QEM}(P)) \mid \langle P, \emptyset\rangle \in X\}$ are isomorphic. Now every $\varphi \in [P, \neg\neg, \wedge, \rightarrow]$ holds in all full nodes, so $\langle P, \emptyset\rangle \in V_E(\varphi)$ (where $E = \mathsf{QEM}(P)$). On the other hand, if $X \in \wp^u(\mathsf{QEM}(P))$ and $\langle P, \emptyset\rangle \in X$, then ψ_X is defined and in $[P, \neg\neg, \wedge, \rightarrow]$. This proves the last part of the theorem.

□

So indeed $\mathsf{EM}_\neg(P)$ is an exact model for $[P, \neg, \wedge, \rightarrow]$, $\mathsf{EM}(P)$ for $[P, \wedge, \rightarrow]$, and $\mathsf{QEM}(P)$ is a quasi-exact model for $[P, \neg\neg, \wedge, \rightarrow]$.

We complete the characterisation of the $\wedge$-fragments:

Theorem 6

1. $\mathsf{INV_i} = [\neg, \wedge, \rightarrow]_\equiv$.
2. $\mathsf{INV_{fi}} = [\wedge, \rightarrow]_\equiv$.
3. $\mathsf{INV_{hi}} \cap \mathsf{VAL_f} = [\neg\neg, \wedge, \rightarrow]_\equiv$.

Proof The inclusions from right to left were formulated as a direct consequence of Theorem 1. For the other direction, we argue as follows.

Let $\varphi \in \mathsf{INV_i} \cap \mathsf{IpL}(P)$, $E = \mathsf{EM}_\neg(P)$ and $X = V_E(\varphi)$. Now $\psi = \psi_X$ as defined in the proof of Theorem 5 is a formula in $[P, \neg, \wedge, \rightarrow]$ that is equivalent with φ on $\mathsf{EM}_\neg(P)$, i.e. $\mathsf{EM}_\neg(P) \models \varphi \leftrightarrow \psi$. By Theorem 1, we have that $\varphi \leftrightarrow \psi$ is i-invariant, so with Theorem 3 we now get $\models \varphi \leftrightarrow \psi$, i.e. φ and ψ are equivalent.

The proof for the second and third part of the theorem is similar. □

5.2 Fragments without conjunction

We now look at fragments without conjunction, and introduce two classes of formulae.

Definition 13
DIMP and DNEG are defined by

$$\begin{array}{lcl} \mathsf{DIMP} & = & \{\varphi \mid \varphi \equiv (\varphi \to p) \to p \text{ for some } p \in \mathsf{PV}\} \\ \mathsf{DNEG} & = & \{\varphi \mid \varphi \equiv \neg\neg\varphi\} \end{array}$$

We have

Lemma 4

1. $[\to]_\equiv \ = \ \mathsf{DIMP} \ \cap \ [\wedge, \to]_\equiv$
2. $[\neg, \to]_\equiv \ = \ (\mathsf{DIMP} \cup \mathsf{DNEG}) \ \cap \ [\neg, \wedge, \to]_\equiv$
3. $[\neg\neg, \to]_\equiv \ = \ \mathsf{DIMP} \ \cap \ [\neg\neg, \wedge, \to]_\equiv$

Proof 1. First we prove that $[\to]_\equiv \subseteq \mathsf{DIMP}$, so let $\varphi \in [\to]$. We define $\mathsf{head}(\varphi)$ inductively by: $\mathsf{head}(p) = p$ and $\mathsf{head}(\varphi \to \psi) = \mathsf{head}(\psi)$. We claim: $(\varphi \to \mathsf{head}(\varphi)) \to \mathsf{head}(\varphi) \equiv \varphi$. This is proved with induction over φ, using the logical laws

$$\begin{array}{c} (p \to p) \to p \equiv p \\ (\psi \to p) \to p \equiv \psi \quad \Rightarrow \quad ((\varphi \to \psi) \to p) \to p \equiv \varphi \to \psi \end{array}$$

Now let $\varphi \in [\wedge, \to]$ with $(\varphi \to p) \to p \equiv \varphi$ for some $p \in \mathsf{PV}$: we shall give a formula $\psi \in [\to]$ with $\varphi \equiv \psi$. Using the logical laws

$$\begin{array}{c} \varphi \to (\psi \wedge \chi) \equiv (\varphi \to \psi) \wedge (\varphi \to \chi) \\ (\varphi \wedge \psi) \to \chi \equiv \varphi \to (\psi \to \chi) \end{array}$$

we observe that φ is equivalent to a conjunction $\varphi_0 \wedge \ldots \wedge \varphi_n$ of elements of $[\to]$. Now define

$$\psi = (\varphi_0 \to (\varphi_1 \to \ldots \to (\varphi_n \to p) \ldots)) \to p$$

then $\psi \equiv ((\varphi_0 \wedge \ldots \wedge \varphi_n) \to p) \to p \equiv (\varphi \to p) \to p \equiv \varphi$.

2. As the previous case, using that $\neg\neg\varphi$ and $(\varphi \to \bot) \to \bot$ are equivalent.

3. As the first case. We extend the definition of head with $\mathsf{head}(\neg\neg\varphi) = \mathsf{head}(\varphi)$. In the proof of $(\varphi \to \mathsf{head}(\varphi)) \to \mathsf{head}(\varphi) \equiv \varphi$, the induction step for $\varphi = \neg\neg\psi$ follows from

$$(\neg\neg\psi \to \chi) \to \chi \ \Rightarrow \ \neg\neg((\psi \to \chi) \to \chi)$$

reading $\mathsf{head}(\psi)$ for χ. For the other direction, we now also use the property

$$\neg\neg(\varphi \wedge \psi) \equiv \neg\neg\varphi \wedge \neg\neg\psi.$$

□

As a direct consequence of Theorem 6 and Lemma 4, we have the characterisation of the $\wedge$-free fragments:

Theorem 7

1. $[\neg, \rightarrow]_{\equiv} \; = \; \mathsf{INV_i} \cap (\mathsf{DIMP} \cup \mathsf{DNEG})$
2. $[\rightarrow]_{\equiv} \; = \; \mathsf{INV_{fi}} \cap \mathsf{DIMP}$
3. $[\neg\neg, \rightarrow]_{\equiv} \; = \; \mathsf{INV_{hi}} \cap \mathsf{VAL_f} \cap \mathsf{DIMP}$

Finally, we characterize the structure of the diagrams of the $\wedge$-free fragments.

Theorem 8
$[P, \rightarrow]_{\equiv}$ and $\mathsf{D_i}(P)$ are isomorphic.
$[P, \neg, \rightarrow]_{\equiv}$ and $\mathsf{D_{ni}}(P)$ are isomorphic.
$[P, \neg\neg, \rightarrow]_{\equiv}$ and $\mathsf{D_{di}}(P)$ are isomorphic.

Proof Follows directly from Theorems 5 and 7 and the fact that $J(\varphi) \subseteq J(p)$ whenever $(\varphi \rightarrow p) \rightarrow p \equiv \varphi$, which follows from Lemma (8).

□

6 Structure of the models and the diagrams

In this section, we study the structure and the size of the (quasi-)exact models and the diagrams of the fragments. For this purpose, we define two operators.e

Definition 14 ($\oplus$ and $\ominus$)
The operators $\oplus$ and $\ominus$ on (sets of) types and sets of atoms are defined inductively by:

$$\begin{array}{ll} \langle P, X\rangle \oplus Q & = \langle P \cup Q, X \oplus Q\rangle \\ X \oplus Q & = \{t \oplus Q \mid t \in X\} \\ & \\ \langle P, X\rangle \ominus Q & = \langle P - Q, X \ominus Q\rangle \\ X \ominus Q & = \{t \ominus Q \mid t \in X\} \end{array}$$

So $\oplus Q$ adds the elements of Q in appropriate places, and $\ominus Q$ takes them away. They satisfy

$$\begin{array}{lll} \mathsf{atom}(X \oplus Q) = \mathsf{atom}(X) \cup Q & & \\ \mathsf{atom}(X \ominus Q) = \mathsf{atom}(X) - Q & & \\ P \cap Q = \emptyset & \Rightarrow & (\langle P, X\rangle \oplus Q) \ominus Q = \langle P, X\rangle \\ t \in \mathsf{EM}(P) \;\&\; P \cap Q = \emptyset & \Rightarrow & t \oplus Q \in \mathsf{EM}(P \cup Q) \\ Q \subseteq \mathsf{atom}(X) & \Rightarrow & (X \ominus Q) \oplus Q = X \\ X \subseteq \mathsf{EM}(P) \;\&\; Q \subset \mathsf{atom}_P(X) & \Rightarrow & X \ominus Q \subseteq \mathsf{EM}(P - Q) \\ X \text{ antichain} & \Rightarrow & X \oplus Q \text{ antichain} \end{array}$$

Lemma 5
For every finite $P \subseteq \mathsf{PV}$, there is an injective mapping e_P with

$$e_P(Q, X) = \langle \mathsf{atom}_Q(X), X \oplus (P - Q) \rangle$$

for $Q \subset P$ and $X \in \mathsf{D_{ci}}(Q)$, such that

$$\begin{array}{lcl} \mathsf{EM}(P) & = & \bigcup_{Q \subset P} \{e_P(Q, X) \mid X \in \mathsf{D_{ci}}(Q)\} \\ \mathsf{EM}_\neg(P) & = & \bigcup_{Q \subset P} \{e_P(Q, X) \mid X \in \mathsf{D_{nci}}(Q)\} \cup \{\langle P, \emptyset \rangle\} \\ \mathsf{QEM}(P) & = & \bigcup_{Q \subset P} \{e_P(Q, X) \mid X \in \mathsf{D_{dci}}(Q)\} \cup \{\langle Q, \{\langle P, \emptyset \rangle\}\rangle \mid Q \subset P\} \cup \{\langle P, \emptyset \rangle\} \end{array}$$

The indicated unions are partitions: all sets involved are mutually disjoint.

Proof Is is verified easily that $e_P(Q, X) \in \mathsf{EM}(P)$, and also that the mapping e_P^{-1} defined by

$$e_P^{-1}(\langle R, Y \rangle) = ((P - (\mathsf{atom}_P(Y) - R)), Y \ominus (\mathsf{atom}_P(Y) - R))$$

is an inverse of e_P. As a consequence, we have that $\bigcup_{Q \subset P} \{e_P(Q, X) \mid X \in \mathsf{D_{ci}}(Q)\}$ is a partition of $\mathsf{EM}(P)$. For the partitions of $\mathsf{EM}_\neg(P)$ and $\mathsf{QEM}(P)$, the reasoning is similar. □

See Fig. 7 for an illustration of the embedding of $\mathsf{D_{ci}}(\{p, q\})$ into $\mathsf{EM}(\{p, q, r\})$ by $e_{\{p,q,r\}}$. So $\mathsf{EM}(P)$ consists of copies of the diagrams $\mathsf{D_{ci}}(Q)$ for $Q \subset P$, and analogously for $\mathsf{EM}_\neg(P)$ and $\mathsf{QEM}(P)$.

To determine the size of the models and diagrams, we define

Definition 15

$$\begin{array}{lcl} \varepsilon(n) & = & |\mathsf{EM}(P_n)| \\ \delta_{\mathsf{ci}}(n) & = & |\mathsf{D_{ci}}(P_n)| \end{array}$$

and similarly $\varepsilon_\neg, \varepsilon_{\neg\neg}, \delta_{\mathsf{nci}}, \delta_{\mathsf{dci}}, \delta_{\mathsf{i}}, \delta_{\mathsf{ni}}, \delta_{\mathsf{di}}$.

We have the following formulae for the size of models and diagrams:

Theorem 9

1. $\varepsilon(n) = \sum_{m=0}^{n-1} \binom{n}{m} \delta_{\mathsf{ci}}(m)$
2. $\varepsilon_\neg(n) = 1 + \sum_{m=0}^{n-1} \binom{n}{m} \delta_{\mathsf{nci}}(m)$
3. $\varepsilon_{\neg\neg}(n) = 2^n + \sum_{m=0}^{n-1} \binom{n}{m} \delta_{\mathsf{dci}}(m)$
4. $\delta_{\mathsf{i}}(n) = \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{\varepsilon(n-m) + \delta_{\mathsf{ci}}(n-m)}$
5. $\delta_{\mathsf{ni}}(n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} 2^{2^{n-m}} + \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{\varepsilon_\neg(n-m) + \delta_{\mathsf{nci}}(n-m) - 1}$
6. $\delta_{\mathsf{di}}(n) = \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{\varepsilon_{\neg\neg}(n-m) + \delta_{\mathsf{dci}}(n-m)}$

Proof The first three formulae directly follow from Lemma 5.
To compute $\delta_{\mathsf{i}}(n)$, we use a generalisation of the property $|X \cup Y| = |X| + |Y| - |X \cap Y|$ to arbitrary finite collections of sets. Let I and X_i ($i \in I$) be finite, then

$$|\bigcup_{i\in I} X_i| = \sum_{m=1}^{|I|} (-1)^{m-1} \sum_{J\subseteq I, |J|=m} |\bigcap_{j\in J} X_j| \tag{17}$$

For $n > 0$, we have (by the definition of δ_{i} and D_{i}, and by (17)):

$$\delta_{\mathsf{i}}(n) = \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{|\bigcap_{i\leqslant m} J_{\mathsf{EM}(P_n)}(p_i)|}$$

Now

$$\begin{array}{ll} & \bigcap_{q\in Q} J_{\mathsf{EM}(P)}(q) \\ = & \\ & \bigcap_{q\in Q}\{\langle R, X\rangle \in \mathsf{EM}(P) \mid q \notin R, q \in \mathsf{atom}_P(X)\} \\ = & \\ & \{\langle R, X\rangle \in \mathsf{EM}(P) \mid R \cap Q = \emptyset, Q \subseteq \mathsf{atom}_P(X)\} \\ = & \\ & \{\langle R, X\rangle \mid X \in \wp^a(\mathsf{EM}(P)), R \subseteq P - Q, R \cup Q \subseteq \mathsf{atom}_P(X)\} \\ = & \\ & \{\langle R, Y \oplus Q\rangle \mid Y \in \wp^a(\mathsf{EM}(P-Q)), R \subseteq \mathsf{atom}_{P-Q}(Y)\} \\ = & \quad \text{(split cases: } R \subset \mathsf{atom}_{P-Q}(Y) \text{ or } R = \mathsf{atom}_{P-Q}(Y)) \\ & \{\langle R, Y \oplus Q\rangle \mid Y \in \wp^a(\mathsf{EM}(P-Q)), R \subset \mathsf{atom}_{P-Q}(Y)\} \\ & \quad \cup\{\langle \mathsf{atom}_{P-Q}(Y), Y \oplus Q\rangle \mid Y \in \wp^a(\mathsf{EM}(P-Q))\} \\ = & \\ & \{\langle R, Y \oplus Q\rangle \mid \langle R, Y\rangle \in \mathsf{EM}(P-Q)\} \\ & \quad \cup\{\langle \mathsf{atom}_{P-Q}(Y), Y \oplus Q\rangle \mid Y \in \mathsf{D}_{\mathsf{ci}}(P-Q)\} \end{array}$$

so $|\bigcap_{i\leqslant m} J_{\mathsf{EM}(P_n)}(p_i)| = |\mathsf{EM}(P_n - P_m)| + |\mathsf{D}_{\mathsf{ci}}(P_n - P_m)| = \varepsilon(n-m) + \delta_{\mathsf{ci}}(n-m)$, and hence formula 4.
For $\delta_{\mathsf{ni}}(n)$, we argue as follows. We have (writing J for $J_{\mathsf{EM}_\neg(P_n)}$)

$$\begin{array}{ll} & \delta_{\mathsf{ni}}(n) \\ = & \quad \text{(definition)} \\ & |\wp(J(\bot)) \cup \bigcup_{i\leqslant n} \wp(J(p_i))| \\ = & \\ & |\wp(J(\bot))| + |\bigcup_{i\leqslant n} \wp(J(p_i))| - |\wp(J(\bot)) \cap \bigcup_{i\leqslant n} \wp(J(p_i))| \\ = & \\ & |\wp(J(\bot))| + \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{|\bigcap_{i\leqslant m} J(p_i)|} - \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{|J(\bot)\cap\bigcap_{i\leqslant m} J(p_i)|} \end{array}$$

Now $|\wp(J(\bot))| = |\wp(\{\langle Q, \emptyset\rangle \mid Q \subseteq P\})| = 2^{2^n}$, and

$$\begin{array}{rcl} \bigcap_{q\in Q} J(q) & = & \{\langle R, Y \oplus Q\rangle \mid \langle R, Y\rangle \in \mathsf{D}_{\mathsf{nci}}(P-Q) - \{\langle P, \emptyset\rangle\}\} \\ & & \cup\{\langle \mathsf{atom}_{P-Q}(Y), Y \oplus Q\rangle \mid Y \in \wp^a(\mathsf{EM}(P-Q))\} \end{array}$$

so $|\bigcap_{i\leqslant m} J(p_i)| = \delta_{\mathsf{nci}}(n-m) - 1 + \varepsilon_\neg(n-m)$. Moreover,

$$J(\bot) \cap \bigcap_{q\in Q} J(q) = \{\langle R, \emptyset\rangle \mid R \subseteq P - Q\}$$

hence $|J(\bot) \cap \bigcap_{i\leqslant m} J(p_i)| = 2^{n-m}$. Summing this up, we get formula 5.
Finally we consider δ_{di}. We have (writing J for $J_{\mathsf{QEM}(P_n)}$)

$$\delta_{\mathsf{di}}(n) = |\bigcup_{i\leqslant n} \wp(J(p_i))| = \sum_{m=1}^{n} (-1)^{m-1} \binom{n}{m} 2^{|\bigcap_{i\leqslant m} J(p_i)|}$$

and

$$\begin{aligned} \textstyle\bigcap_{q\in Q} J(q) \quad = \quad & \{\langle P-Q, \{\langle P, \emptyset\rangle\}\rangle\} \\ & \cup \{\langle R, Y \oplus Q\rangle \mid \langle R, Y\rangle \in (\mathsf{QEM}(P-Q) - \{\langle P-Q, \emptyset\rangle\})\} \\ & \cup \{\langle \mathsf{atom}_{P-Q}(Y), Y \oplus Q\rangle \mid Y \in \mathsf{D}_{\mathsf{dci}}(P-Q)\} \end{aligned}$$

so $|\bigcap_{i\leqslant m} J(p_i)| = |\mathsf{QEM}(P_n - P_m)| + |\mathsf{D}_{\mathsf{dci}}(P_n - P_m)| = \varepsilon_{\neg\neg}(n-m) + \delta_{\mathsf{dci}}(n-m)$, and hence we have formula 6. □

So we can compute ε and δ_{i} from δ_{ci}, $\varepsilon_\neg$ and δ_{ni} from δ_{nci}, and $\varepsilon_{\neg\neg}$ and δ_{di} from δ_{dci}. For δ_{ci}, δ_{nci} and δ_{dci}, however, we have no easy way to compute them other than counting the number of antichains in their generating (quasi-)exact models. For $n \leqslant 2$, this is rather straightforward, since the generating models are small (at most 15 elements).

We present some values for the functions treated here:

	0	1	2	3	4
ε	0	1	5	61	2494 651862 209437
$\varepsilon_\neg$	1	3	15	6423	
$\varepsilon_{\neg\neg}$	1	3	13	2049	
δ_{ci}	1	2	18	623 662965 552330	
δ_{nci}	2	6	2134		
δ_{dci}	1	4	676		
δ_{i}	1	2	14	25 165802	$2^{623\,662965\,552393} - 50\,331618$
δ_{ni}	2	6	518	$3 \cdot 2^{2148} - 546$	
δ_{di}	1	4	252	$3 \cdot 2^{689} - 380$	

$\varepsilon_\neg(2)$ and $\delta_{\mathsf{nci}}(2)$ have first been computed in [6].
$\delta_{\mathsf{ci}}(3)$, the number of antichains in the 61-element model $\mathsf{EM}(P_3)$ (see Fig. 7) has been computed in [6], [25] and [23].
An upper bound of 10^{27} for $\delta_{\mathsf{i}}(3)$ is given by Diego in [9]; Urquhart [39] found that $2^{23} < \delta_{\mathsf{i}}(3) < 3 \cdot 2^{23}$. The exact value of $\delta_{\mathsf{i}}(3)$ is given in [19], [14] and [2].
The value of $\delta_{\mathsf{i}}(4)$ is mentioned without computation in [14], and in [2] (with reference to [14]).
The value of $\delta_{\mathsf{nci}}(3)$ has been computed by one of the authors (Renardel de

Lavalette) and appeared in [14], but it has not been confirmed yet. It is slightly larger that 2^{6386}, where 6386 equals the size of the largest antichain in $\mathsf{EM}_{\neg}(3)$. When the value D of $\delta_{\mathsf{nci}}(3)$ is known, we can compute $\varepsilon_{\neg}(4)$ and $\delta_{\mathsf{ni}}(4)$:

$$\varepsilon_{\neg}(4) = 4D + 12831 \qquad \delta_{\mathsf{ni}}(4) = 2^{D+6424} - 3 \cdot 2^{2149} + 65614$$

The value E of $\delta_{\mathsf{dci}}(3)$ has not been computed yet: since the largest antichain in $\mathsf{QEM}(3)$ has 2018 elements, we know that E is slightly larger than 2^{2018}, and we have

$$\varepsilon_{\neg\neg}(4) = 4E + 4089 \qquad \delta_{\mathsf{di}}(4) = 2^{E+2051} - 3 \cdot 2^{690} + 508$$

6.1 Asymptotic behaviour

In the table in the previous section, we see that these functions grow fast. More precisely, we have:

Theorem 10

$$\begin{array}{lcl} 2^{\delta_{\mathsf{ci}}(n)} & \leqslant \ \delta_{\mathsf{ci}}(n+1) \ \leqslant & 2^{(n+2)\delta_{\mathsf{ci}}(n)} \\ (n+1)\delta_{\mathsf{ci}}(n) & \leqslant \ \varepsilon(n+1) \ \leqslant & (n+2)\delta_{\mathsf{ci}}(n) \\ n2^{\varepsilon(n)+\delta_{\mathsf{ci}}(n)} & \leqslant \ \delta_{\mathsf{i}}(n+1) \ \leqslant & (n+1)2^{\varepsilon(n)+\delta_{\mathsf{ci}}(n)} \end{array}$$

Proof The first two lines follow from

$$2^{\delta_{\mathsf{ci}}(n)} \leqslant \delta_{\mathsf{ci}}(n+1) \tag{18}$$
$$\delta_{\mathsf{ci}}(n) \leqslant 2^{\varepsilon(n)} \tag{19}$$
$$(n+1)\delta_{\mathsf{ci}}(n) \leqslant \varepsilon(n+1) \tag{20}$$
$$\varepsilon(n+1) \leqslant (n+2)\delta_{\mathsf{ci}}(n) \tag{21}$$

which we prove now. For (18), we consider the mapping

$$\lambda X.\langle \emptyset, X \oplus (P - Q)\rangle$$

which embeds any diagram $\mathsf{D}_{\mathsf{ci}}(Q)$ with $Q \subset P$ into $\mathsf{EM}(P)$. Since nodes with equal atom sets are incomparable, the image $\{\langle \emptyset, X \oplus (P - Q)\rangle \mid X \in \mathsf{D}_{\mathsf{ci}}(Q)\}$ of this embedding is an antichain. As a consequence, $\mathsf{EM}(P_{n+1})$ has antichains with length $\delta_{\mathsf{ci}}(n)$. Since any subset of an antichain is again an antichain, we see that $\mathsf{D}_{\mathsf{ci}}(P_{n+1}) = \wp^a(\mathsf{EM}(P_{n+1}))$ has $> 2^{\delta_{\mathsf{ci}}(n)}$ elements, i.e. (18).
(19) follows from $\mathsf{D}_{\mathsf{ci}}(P_n) = \wp^a(\mathsf{EM}(P_n)) \subseteq \wp(\mathsf{EM}(P_n))$.
By Theorem 9.1, we have $\varepsilon(n+1) = \sum_{m=0}^{n} \binom{n+1}{m}\delta_{\mathsf{ci}}(m) \leqslant \binom{n+1}{n}\delta_{\mathsf{ci}}(n) = (n+1)\delta_{\mathsf{ci}}(n)$, i.e. (20).
Finally (19). For $n \leqslant 3$, this is easily verified, and for $n \geqslant 4$, we argue as follows. First we observe

$$n \geqslant 2 \ \Rightarrow \ \delta_{\mathsf{ci}}(n) > 2^{n+2} \tag{22}$$

which is proved with induction, using (18) and the fact that $2^{n+2} > n+3$. Now, for $n \geqslant 4$:

$$
\begin{array}{ll}
& \varepsilon(n+1) \\
= & \qquad \text{(definition)} \\
& \sum_{m=0}^{n} \binom{n+1}{m} \delta_{\mathsf{ci}}(m) \\
< & \\
& \binom{n+1}{n} \delta_{\mathsf{ci}}(n) + \delta_{\mathsf{ci}}(n-1) \sum_{m=0}^{n-1} \binom{n+1}{m} \\
< & \\
& (n+1)\delta_{\mathsf{ci}}(n) + 2^{n+1}\delta_{\mathsf{ci}}(n-1) \\
< & \qquad (n \geqslant 4, \text{ so with (22) } 2^{n+1}\delta_{\mathsf{ci}}(n-1) < \delta_{\mathsf{ci}}(n-1)^2 \leqslant 2^{\delta_{\mathsf{ci}}(n-1)}) \\
& (n+1)\delta_{\mathsf{ci}}(n) + 2^{\delta_{\mathsf{ci}}(n-1)} \\
\leqslant & \\
& (n+2)\delta_{\mathsf{ci}}(n)
\end{array}
$$

and we conclude that (19) holds.
The last line of the theorem follows from Theorem 9.4. □

Similar inequalities hold for δ_{nci}, $\varepsilon_{\neg}$ and δ_{ni} (with an additional -1 in the exponents in the inequalities involving δ_{ni}), and for δ_{dci}, $\varepsilon_{\neg\neg}$ and δ_{di} (with the exception $\varepsilon_{\neg\neg}(2) = 13 > 12 = 3 \cdot \delta_{\mathsf{dci}}(1)$). We conclude that the size of all (quasi-)exact models and diagrams considered here grows superexponentially in the number of propositional variables.

7 Concluding remarks

We investigated the structure and size of several finite fragments of IpL, making fruitful use of (quasi-)exact models. As a side result, we obtained semantical characterisations of these fragments. There are some open questions, however, which we discuss here shortly.

First of all, there is the other class of finite fragments of IpL: fragments without implication, i.e. subfragments of $[\neg, \wedge, \vee]$. The interesting fragments in this class are $[\neg, \wedge], [\neg\vee], [\neg\neg\wedge, \vee], [\neg\neg, \wedge], [\neg\neg, \vee]$. We intend to investigate these fragments in a subsequent publication.

Secondly, we observe that characteristic formulae (see Definition 12) may be more complex than needed. To give an example: in the exact model $\mathsf{EM}(\{p, q\})$ (see Fig. 4), the characteristic formula for $J(p)$ is by Theorem 4 equivalent to p, but it reads $(((p \to q) \to p) \to p) \wedge (((p \to q) \wedge (q \to p)) \to p) \wedge (q \to p)$. The question arises: is there an alternative definition of characteristic formulae where the result is as simple as possible? The notion 'simple' may be defined in terms of number of logical symbols, or in terms of nesting of implications.

Finally, a deeper question: how may the results presented here help us in gaining more insight in the structure of the full fragment $[\{p_1, \ldots, p_n\}, \neg, \wedge, \vee, \to]$, i.e. the free Heyting algebra of n generators? (Recall that for $n = 1$ this is the Rieger-Nishimura lattice.) We admit that we see no direct application of our results in that direction, going further than describing certain finite substructures of the free Heyting algebra. But we think that the methods developed and used here may be applied fruitfully on different classes of fragments that have the free Heyting algebra as a limit case. More specifically, we expect that fragments with restricted nesting of implications are good candidates for this purpose. Some initial results were presented in [14]. We hope to come back on this issue.

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A Proofs

Theorem 1

1. $\mathsf{INV_i}$ contains $\bot$ and PV, and is closed under $\neg$, $\wedge$ and $\rightarrow$.

2. $\mathsf{INV_f}$ contains PV and is closed under $\wedge$, $\vee$ and $\rightarrow$.

3. $\mathsf{INV_h} \cap \mathsf{VAL_f}$ contains PV and is closed under $\neg\neg$, $\wedge$, $\vee$ and $\rightarrow$.

Proof 1. The i-invariance of $\perp$ is evident, due to the fact that $M^{-\mathsf{i}}$ is always a proper model. Closure under conjunction is straightforward.

For the proof of the i-invariance of p and of closure under implication, we use the following equivalent definition of i-invariance:

$$M, w \models \varphi \Leftrightarrow \forall v \in W^{-\mathsf{i}}(v \geqslant w \Rightarrow M^{-\mathsf{i}}, v \models \varphi) \tag{23}$$

For $\varphi = p$, (23) comes down to the equivalence of $p \in \mathsf{atom}(w)$ and $\forall v \in W^{-\mathsf{i}}(v \geqslant w \Rightarrow p \in \mathsf{atom}(v))$. When $w \in W^{-\mathsf{i}}$, this is obvious, and when $w \notin W^{-\mathsf{i}}$ then w is inductive, which implies the equivalence.

For closure under implication, we argue as follows. Let $\varphi = \psi \to \chi$ and assume that (23) holds for ψ and χ. We demonstrate (23) for $\psi \to \chi$:

$$\begin{aligned}
& M, w \models \psi \to \chi \\
\Leftrightarrow \quad & \\
& \forall v \geqslant w(M, v \models \psi \;\Rightarrow\; M, v \models \chi) \\
\Leftrightarrow \quad & \quad ((23) \text{ for } \psi \text{ and } \chi) \\
& \forall v \geqslant w(\forall u \in W^{-\mathsf{i}}(u \geqslant v \Rightarrow M^{-\mathsf{i}}, u \models \psi) \;\Rightarrow\; \forall x \in W^{-\mathsf{i}}(x \geqslant v \Rightarrow M^{-\mathsf{i}}, x \models \chi)) \\
\Leftrightarrow \quad & \quad (\text{logic}) \\
& \forall x \in W^{-\mathsf{i}}(\exists v(x \geqslant v \geqslant w \;\&\; \forall u \in W^{-\mathsf{i}}(u \geqslant v \Rightarrow M^{-\mathsf{i}}, u \models \psi)) \;\Rightarrow\; M^{-\mathsf{i}}, x \models \chi) \\
\Leftrightarrow \quad & \quad (\text{monotonicity of } M^{-\mathsf{i}}, u \models \psi) \\
& \forall x \in W^{-\mathsf{i}}(x \geqslant w \;\&\; M^{-\mathsf{i}}, x \models \psi \;\Rightarrow\; M^{-\mathsf{i}}, x \models \chi) \\
\Leftrightarrow \quad & \\
& \forall x \in W^{-\mathsf{i}}(x \geqslant w \;\&\; \forall y \in W^{-\mathsf{i}}(y \geqslant x \;\&\; M^{-\mathsf{i}}, y \models \psi \;\Rightarrow\; M^{-\mathsf{i}}, y \models \chi)) \\
\Leftrightarrow \quad & \\
& \forall x \in W^{-\mathsf{i}}(x \geqslant w \;\&\; M^{-\mathsf{i}}, x \models (\psi \to \chi))
\end{aligned}$$

Finally, closure under negation follows from closure under implication and the i-invariance of $\perp$.

2. We use the following equivalent definition of f-invariance:

$$M, w \models \varphi \Leftrightarrow (\mathsf{atom}(w) = P \text{ or } M^{-\mathsf{f}}, w \models \varphi) \tag{24}$$

For $\varphi = p$, (24) is obvious, and closure under conjunction and disjunction is straightforward. For closure under implication, let $\varphi = \psi \to \chi$ and assume that (24) holds for ψ and χ. Now we demonstrate (24) for φ:

$$\begin{aligned}
& M, w \models \psi \to \chi \\
\Leftrightarrow \quad & \\
& \forall v \geqslant w(M, v \models \psi \;\Rightarrow\; M, v \models \chi) \\
\Leftrightarrow \quad & \quad ((24) \text{ for } \psi \text{ and } \chi) \\
& \forall v \geqslant w(\mathsf{atom}(v) = P \text{ or } (M^{-\mathsf{f}}, v \models \psi \;\Rightarrow\; M^{-\mathsf{f}}, v \models \chi)) \\
\Leftrightarrow \quad & \quad (\mathsf{atom}(w) = P \text{ implies } \forall v \geqslant w \; \mathsf{atom}(v) = P) \\
& \mathsf{atom}(w) = P \text{ or } \forall v \geqslant w(\mathsf{atom}(v) = P \text{ or } (M^{-\mathsf{f}}, v \models \psi \;\Rightarrow\; M^{-\mathsf{f}}, v \models \chi)) \\
\Leftrightarrow \quad & \quad (v \geqslant^{-\mathsf{f}} w \text{ iff } \mathsf{atom}(w) \neq P \;\&\; v \geqslant w \;\&\; \mathsf{atom}(v) \neq P) \\
& \mathsf{atom}(w) = P \text{ or } \forall v \geqslant^{-\mathsf{f}} w(M^{-\mathsf{f}}, v \models \psi \;\Rightarrow\; M^{-\mathsf{f}}, v \models \chi)) \\
\Leftrightarrow \quad & \\
& \mathsf{atom}(w) = P \text{ or } M^{-\mathsf{f}}, w \models (\psi \to \chi)
\end{aligned}$$

3. We saw in Lemma 2 that $\mathsf{VAL_f}$ satisfies the closure properties. Furthermore it is evident that all propositional variables are in $\mathsf{INV_h}$, and it is straightforward that $\mathsf{INV_h}$ is closed under conjunction. For the other closure properties, we use

$$M, w \models \varphi \Leftrightarrow M^{-\mathsf{h}}, w \models \varphi \tag{25}$$

as an equivalent formulation of $\varphi \in \mathsf{INV_h}$. It is not hard to see that (25) and hence $\mathsf{INV_h}$ is closed under disjunction.

To prove that $\mathsf{INV_h} \cap \mathsf{VAL_f}$ is closed under implication, we use the implications

$$v \geqslant^{-\mathsf{h}} w \ \Rightarrow\ v \geqslant w \ \Rightarrow\ (\mathsf{atom}(v) = P \text{ or } v \geqslant^{-\mathsf{h}} w) \tag{26}$$

which follows directly from the definition of $\geqslant^{-\mathsf{h}}$. Now assume that $\psi, \chi \in \mathsf{INV_h} \cap \mathsf{VAL_f}$, then we demonstrate $\psi \to \chi \in \mathsf{INV_h}$ as follows (using (25)):

$$\begin{array}{ll} & M, w \models \psi \to \chi \\ \Leftrightarrow & \\ & \forall v \geqslant w\ (M, v \models \psi \ \Rightarrow\ M, v \models \chi) \\ \Leftrightarrow & \qquad ((25) \text{ for } \psi \text{ and } \chi) \\ & \forall v \geqslant w\ (M^{-\mathsf{h}}, v \models \psi \ \Rightarrow\ M^{-\mathsf{h}}, v \models \chi) \\ \Leftrightarrow & \qquad (\text{by } (26) \text{ and } \psi, \chi \in \mathsf{VAL_f}) \\ & \forall v \geqslant^{-\mathsf{h}} w\ (M^{-\mathsf{h}}, v \models \psi \ \Rightarrow\ M^{-\mathsf{h}}, v \models \chi) \\ \Leftrightarrow & \\ & M^{-\mathsf{h}}, w \models (\psi \to \chi) \end{array}$$

For closure of $\mathsf{INV_h}$ under double negation, we use the following consequence of (26):

$$\mathsf{max}^{-\mathsf{h}}(w\uparrow) \subseteq \mathsf{max}(w\uparrow) \subseteq \mathsf{max}^{-\mathsf{h}}(w\uparrow) \cup \{v \mid \mathsf{atom}(v) = P\} \tag{27}$$

where $\mathsf{max}^{-\mathsf{h}}(X)$ denotes the collection of $\geqslant^{-\mathsf{h}}$-maximal elements in X. Now assume that $\varphi \in \mathsf{INV_h}$, then

$$\begin{array}{ll} & M, w \models \neg\neg\varphi \\ \Leftrightarrow & \\ & \forall v \in \mathsf{max}(w\uparrow)\ M, v \models \varphi \\ \Leftrightarrow & \qquad (\text{by } (26) \text{ and } (27)) \\ & \forall v \in \mathsf{max}^{-\mathsf{h}}(w\uparrow)\ M, v \models \varphi \\ \Leftrightarrow & \qquad ((25) \text{ for } \varphi) \\ & \forall v \in \mathsf{max}^{-\mathsf{h}}(w\uparrow)\ M^{-\mathsf{h}}, v \models \varphi \\ \Leftrightarrow & \\ & M^{-\mathsf{h}}, w \models \neg\neg\varphi \end{array}$$

so (25) holds for $\neg\neg\varphi$, hence $\neg\neg\varphi \in \mathsf{INV_h}$. This completes the proof.

□

Theorem 2

1. Bisimilar types in UM are equal: $\forall st \in \mathsf{UM}(s \underline{\leftrightarrow} t \;\Rightarrow\; s = t)$.
2. ρ_M is a bisimilarity.
3. $\rho_M(w) \in \mathsf{UM}$ for all models $M \in \mathsf{MOD}$ and all w.
4. UM is a locally finite universal model for IpL, and ρ_{UM} is the identity.

Proof 1. We prove $\forall st \in \mathsf{UM}(s \underline{\leftrightarrow} t \Rightarrow s = t)$ with double induction over the order of UM. So assume $s \underline{\leftrightarrow} t$: we shall show that $s = t$ using the induction hypotheses

$$\forall s' > s \forall t'(s' \underline{\leftrightarrow} t' \Rightarrow s' = t') \tag{28}$$

$$\forall t'' > t(s \underline{\leftrightarrow} t'' \Rightarrow s = t'') \tag{29}$$

$s \underline{\leftrightarrow} t$ implies $\mathsf{atom}(s) = \mathsf{atom}(t)$, so we may assume $s = \langle P, X\rangle$ and $t = \langle PY\rangle$ with X, Y finite antichains in UM. We shall show that $X = Y$. $s \underline{\leftrightarrow} t$ implies that, for every $x \in X$, there is a $y \in \{t\} \cup Y{\uparrow}$ with $x \underline{\leftrightarrow} y$, so (by (28)) $x = y$, i.e. $x \in \{t\} \cup Y{\uparrow}$. Now $x = t$ implies $x \underline{\leftrightarrow} s$, so (via (28)) that $x = s$, contradicting $s = \langle P, X\rangle$ and $x \in X$. We conclude that all $x \in X$ are in $Y{\uparrow}$, i.e. $X \subseteq Y{\uparrow}$. On the other hand, $s \underline{\leftrightarrow} t$ also implies that, for every $y \in Y$, there is an $x \in \{s\} \cup X{\uparrow}$ with $x \underline{\leftrightarrow} y$. Now $x = s$ implies $s \underline{\leftrightarrow} y$ so (with (29)) $s = y$; we shall show that this leads to $Y = \{s\}$ which contradicts the definition of UM. For assume that $y' \in Y$, then there is an $x' \in \{s\} \cup X$ with $x' \underline{\leftrightarrow} y'$, so (by (28)) $x' = y'$ and $y' \in \{s\} \cup X$. $y' \in X$ is impossible, for then $y' >_1 t$ while $y' > s > t$ and contradiction. So $y' = s$. This proves indeed that $Y = \{s\}$. We conclude that $Y \subseteq X{\uparrow}$. But now we have $X \subseteq Y{\uparrow}$ and $Y \subseteq X{\uparrow}$, hence $X{\uparrow} = Y{\uparrow}$, so $X = \mathsf{min}(X{\uparrow}) = \mathsf{min}(Y{\uparrow}) = Y$ and we have that $s = \langle P, X\rangle = t$.

2. Induction over the order $<$ in M. We assume as induction hypothesis that $\forall v > w\; v \underline{\leftrightarrow} \rho_M(v)$ and we shall verify $w \underline{\leftrightarrow} \rho_M(w)$. We distinguish two cases, according to the definition of $\rho_M(w)$.

   (a) $\mathsf{min}\{\rho_M(v) \mid v >_1 w\}) = \{t\}$, $\forall v >_1 w\; \mathsf{atom}(v) = \mathsf{atom}(w)$ and $\rho_M(w) = t$: so $\rho_M(w) = \rho_M(v)$ for some $v >_1 w$ Since $v \underline{\leftrightarrow} \rho(v)$ by the induction hypothesis, it suffices to show that $v \underline{\leftrightarrow} w$. Now the first bisimilarity condition $\mathsf{atom}(v) = \mathsf{atom}(w)$ follows directly. For the second bisimilarity condition, we must find for any $u > v$ a node $u' \geqslant w$ with $u \underline{\leftrightarrow} u'$: now $u' := u$ works, for $u > v > w$. For the third bisimilarity condition, we must find for any $u > w$ a node $u' \geqslant v$ with $u' \underline{\leftrightarrow} u$. Let v' satisfy $u \geqslant v' >_1 w$ (i.e. v' is the first node $\neq w$ in an ascending path from w to u, possibly u itself), then $\rho_M(v') = t = \rho_M(v)$, so by the induction hypothesis $v' \underline{\leftrightarrow} v$. Hence there is a $u' \geqslant v$ with $u' \underline{\leftrightarrow} u$, and we are done.

(b) $\rho(w) = \langle \mathsf{atom}(w), \mathsf{min}\{\rho(v) \mid v >_1 w\}\rangle$ and $(|\mathsf{min}\{\rho(v) \mid v >_1 w\}| \neq 1$ or $\exists v >_1 w\ \mathsf{atom}(v) \neq \mathsf{atom}(w))$. So we have $\mathsf{atom}(w) = \mathsf{atom}(\rho_M(w))$, the first bisimilarity condition for $w \underline{\leftrightarrow} F_M(w)$. For the second bisimilarity condition, we must find for any $v > w$ a node $t \geqslant \rho_M(w)$ with $t \underline{\leftrightarrow} v$. Let u satisfy $v \geqslant u >_1 w$ (i.e. u is the first node $\neq w$ in an ascending path from w to v, possibly v itself), then $\rho_M(u) > \rho_M(w)$. By the induction hypothesis, we have $u \underline{\leftrightarrow} \rho_M(u)$, so there is a $t \underline{\leftrightarrow} v$ with $t \geqslant \rho_M(u) > \rho_M(w)$, and we have found our t. For the other direction, we must find for any $t > \rho_M(w)$ a $v \geqslant w$ with $v \underline{\leftrightarrow} t$. Let s satisfy $t \geqslant s >_1 \rho_M(w)$ (i.e. s is the first node $\neq \rho_M(w)$ in an ascending path from $\rho_M(w)$ to t, possibly t itself), then there is a $u >_1 w$ with $\rho_M(u) = s$. By the induction hypothesis, we have $u \underline{\leftrightarrow} \rho_M(u)$, so there is a $v \underline{\leftrightarrow} t$ with $v \geqslant u > w$, and we have found v.

(c) Local finiteness of UM is proved straightforwardly with induction.
It follows from the completeness of locally finite models for IpL that UM is a universal model for IpL, i.e. if $\mathsf{UM} \models \varphi$ then $\models \varphi$. For if w is a node in model M then $w \underline{\leftrightarrow} \rho_M(w)$, so $M, w \models \varphi$ iff $\mathsf{UM}, \rho_M(w) \models \varphi$.
The previous parts of this theorem directly imply that ρ_{UM} is the identity on UM.

3. Induction over the order $<$ in M. Let w be a node in M and assume as induction hypothesis that $\rho_M(v) \in \mathsf{UM}$ for all $v > w$. We consider two cases, according to the two clauses in the definition of $\rho_M(w)$.
In the first case, $\rho_M(w) = t$ with $\mathsf{min}\{\rho_M(v) \mid v >_1 w\} = \{t\}$, then $\rho_M(w) = \rho_M(v)$ for some $v > w$, so $\rho_M(w) \in \mathsf{UM}$ by the induction hypothesis.
In the second case, $\rho_M(w) = \langle \mathsf{atom}(w), \mathsf{min}(Y)\rangle$ with $Y = \{\rho_M(v) \mid v >_1 w\}\rangle$, $|X| \neq 1$ or $\exists v >_1\ w\ \mathsf{atom}(v) \supset \mathsf{atom}(w)$. We claim that the condition of the inductive definition of types holds. By the induction hypothesis, Y is a finite subset of UM, so $\mathsf{min}(Y)$ is a finite antichain of UM; $\mathsf{atom}(w) \subseteq \mathsf{atom}(\mathsf{min}(Y))$ since $\mathsf{atom}(\mathsf{min}(Y)) = \mathsf{atom}(Y)$; and if $|\mathsf{min}(Y)| = 1$ then there is a $v >_1 w$ with $w\ \mathsf{atom}(v) \supset \mathsf{atom}(w)$, so $\mathsf{atom}(\mathsf{min}(Y)) = \mathsf{atom}(v)$ and indeed $\mathsf{atom}(w) \subset \mathsf{atom}(\mathsf{min}(Y))$. So the condition in the inductive definition of types in UM is satisfied, and we have indeed that $\rho_M(w) \in \mathsf{UM}$.

□

Theorem 3

1. $\mathsf{EM}_\neg(P)$ is universal for $\mathsf{INV}_\mathsf{i}(P)$;

2. $\mathsf{EM}(P)$ is universal for $\mathsf{INV}_\mathsf{fi}(P)$;

3. $\mathsf{QEM}(P)$ is universal for $\mathsf{INV}_\mathsf{hi}(P)$;

Proof 1. Let $\varphi \in \mathsf{INV_i}(P)$ with $\mathsf{EM}_\neg(P) \models \varphi$, and let M be some model in $\mathsf{MOD}(P)$. We must show $M \models \varphi$. Since $\varphi \in \mathsf{INV_i}$, it suffices to show $M^{-\mathsf{i}} \models \varphi$. We claim:

for all w in $M^{-\mathsf{i}}$: $\rho_M(w) \in \mathsf{EM}_\neg(P)$.

Since ρ_M is a bisimulation and $\mathsf{EM}_\neg(P) \models \varphi$, it follows that $M^{-\mathsf{i}} \models \varphi$. We prove the claim with induction over the ordering in M, so we may assume that $\forall v > w \rho_M(w) \in \mathsf{EM}_\neg(P)$. If the first clause of the definition of ρ_M applies, then $\rho_M(w) = \rho_M(v)$ for some $v >_1 w$ and the induction hypothesis yields $\rho_M(w) \in \mathsf{EM}_\neg(P)$. If the second clause applies, then $|\mathsf{min}\{\rho_M(v) \mid v >_1 w\}| \neq 1$ or $\mathsf{atom}(v) \subset \mathsf{atom}(w)$ for some $v >_1 w$, and $\rho_M(w) = \langle \mathsf{atom}(w), \mathsf{min}\{\rho_M(v) \mid v >_1 w\}\rangle$. We distinguish three cases.

(a) w is full: then $\rho_M(w) = \langle P, \emptyset\rangle$ which is in $\mathsf{EM}_\neg(P)$.

(b) w is maximal and not full: then $\rho_M(w) = \langle Q, \emptyset\rangle$ with $Q \subset P = \mathsf{atom}_P(\emptyset)$ so it is in $\mathsf{EM}_\neg(P)$.

(c) w is not maximal: then $\mathsf{atom}(w) \subset \mathsf{atom}(w^\wedge)$, since w is not inductive. Since $\mathsf{atom}_P(\mathsf{min}\{\rho_M(v) \mid v >_1 w\}) = \mathsf{atom}(w^\wedge)$, we have $\mathsf{atom}(w) \subset \mathsf{atom}_P(\mathsf{min}\{\rho_M(v) \mid v >_1 w\})$. By the induction hypothesis, $\mathsf{min}\{\rho_M(v) \mid v >_1 w\}$ is a finite antichain in $\mathsf{EM}_\neg(P)$. So indeed $\rho_M(w) \in \mathsf{EM}_\neg(P)$ by the second clause of the definition of $\mathsf{EM}_\neg(P)$.

2. The same reasoning as for (1), but now without the case that w is full.

3. The same reasoning as for (1), except the third case, which runs now as follows.
If w is not maximal: then $\mathsf{atom}(w) \subset \mathsf{atom}(w^\wedge)$ since w is not inductive. Moreover, w is not hybrid, so all $v >_1 w$ are full or none of them is. When all are full, we have $\rho_M(w) = \langle \mathsf{atom}(w), \{\langle P, \emptyset\rangle\}\rangle$ with $\mathsf{atom}(w) \subset P$, so $\rho_M(w) \in \mathsf{QEM}(p)$. When no $v >_1 w$ is full, we have that $\langle P, \emptyset\rangle \notin \mathsf{min}\{\rho_M(v) \mid v >_1 w\}$, so $\mathsf{min}\{\rho_M(v) \mid v >_1 w\}$ is an antichain in $\mathsf{QEM}(P) - \{\langle P, \emptyset\rangle\}$, and we have $\rho_M(w) \in \mathsf{QEM}(p)$ via the third clause of the definition of $\mathsf{QEM}(P)$.

□

Theorem 4

For $E = \mathsf{EM}(P), \mathsf{EM}_\neg(P)$ or $\mathsf{QEM}(P)$ and $w = \langle Q, X\rangle \in E$, we have

$$J_E(\chi_{E,w}) = \{w\}$$

unless $E = \mathsf{QEM}(P)$ and $w = \langle P, \emptyset\rangle$, in which case $\chi_{E,w}$ is not defined.

Proof First the case $E = \mathsf{EM}(P)$. Downward induction over $|\mathsf{atom}(w)|$, so we may assume as induction hypothesis that $J_E(\chi_{v,E}) = \{v\}$ for all v with

$|\mathsf{atom}(v)| > |\mathsf{atom}(w)|$. We claim

$$J_E(\varphi_1 \to p) = J_E(p) \cap \{\langle R, Y\rangle \mid R \supseteq Q\} \tag{30}$$
$$J_E(\varphi_2 \to p) = \{\langle R, Y\rangle \in \mathsf{EM}(P) \mid \mathsf{atom}(Y) - R \supseteq \mathsf{atom}(X) - Q\} \tag{31}$$
$$J_E(\varphi_3 \to p) = J_E(p) \cap \{\langle R, Y\rangle \mid Y{\uparrow} \supseteq X\} \tag{32}$$
$$J_E(\varphi_4 \to p) = J_E(p) \cap \{u \mid \forall v > u\ (\mathsf{atom}(v) \supseteq \mathsf{atom}(X) \ \Rightarrow\ v \in X{\uparrow})\} \tag{33}$$

(30) follows from (16), (31) follows from (15), (16), and (32) follows from (13) and the induction hypothesis. For (33), we argue as follows.

$$\begin{aligned}
& J_E(\varphi_4 \to p) \\
= \quad & \quad \text{(definition of } \varphi_4\text{, induction hypothesis)} \\
& J_E(p) - \bigcup\{v_\vee \mid \mathsf{atom}(v) \supseteq \mathsf{atom}(X) \ \&\ v \notin X{\uparrow}\} \\
= \quad & \\
& J_E(p) - \{u \mid \exists v > u\ (\mathsf{atom}(v) \supseteq \mathsf{atom}(X) \ \&\ v \notin X{\uparrow})\} \\
= \quad & \\
& J_E(p) \cap \{u \mid \forall v > u\ (\mathsf{atom}(v) \supseteq \mathsf{atom}(X) \ \Rightarrow\ v \in X{\uparrow})\}
\end{aligned}$$

Now we argue

$$\begin{aligned}
& J_E(\chi_{E,w}) \\
= \quad & \\
& J_E((\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4) \to p) \\
= \quad & \quad (16) \\
& J_E(\varphi_1 \to p) \cap J_E(\varphi_2 \to p) \cap J_E(\varphi_3 \to p) \cap J_E(\varphi_4 \to p) \\
= \quad & \quad ((30), (31), (32)) \\
& \{\langle R, Y\rangle \mid Q \subseteq R, \mathsf{atom}(X) - Q \subseteq \mathsf{atom}(Y) - R, X \subseteq Y{\uparrow}\} \cap J_E(\varphi_4 \to p) \\
= \quad & \\
& \{\langle Q, Y\rangle \mid \mathsf{atom}(X) \subseteq \mathsf{atom}(Y), X \subseteq Y{\uparrow}\} \cap J_E(\varphi_4 \to p) \\
= \quad & \quad (33) \\
& \{\langle Q, Y\rangle \mid \mathsf{atom}(X) \subseteq \mathsf{atom}(Y), X{\uparrow} = Y{\uparrow}\} \\
= \quad & \\
& \{\langle Q, X\rangle\} \\
= \quad & \\
& \{w\}
\end{aligned}$$

This ends the proof for $E = \mathsf{EM}(P)$. Observe that the value of $J_E(\chi_{E,w})$ does not depend on the choice of p in $\mathsf{atom}_P(X) - Q$, as we claimed in Definition 12.

For the case $E = \mathsf{EM}_\neg(P)$, we also have to check that $J_E(\chi_{E,\langle P,\emptyset\rangle}) = \{\langle P, \emptyset\rangle\}$. Now $\chi_{E,\langle P,\emptyset\rangle} = \neg(\bigwedge P)$ and $J_E(\neg(\bigwedge P)) = \{v \in E \mid v \text{ maximal } \& \ E, v \models \bigwedge P\} = \{\langle P, \emptyset\rangle\}$.

For the case $E = \mathsf{QEM}(P)$, we have to check that $J_E(\chi_{E,w}) = \{w\}$ for $w = \langle Q, \{\langle P, \emptyset\rangle\}\rangle$. This runs as follows:

$$\begin{aligned}
& J_E(\chi_{E,w}) \\
= \quad & \\
& J_E((\bigwedge Q \wedge \bigwedge\{p \leftrightarrow q \mid q \in P - Q - \{p\}\} \wedge \neg\neg p) \to p) \\
= \quad &
\end{aligned}$$

$$J_E(\bigwedge Q \to p) \cap J_E(\bigwedge\{p \leftrightarrow q \mid q \in P - Q - \{p\}\} \to p) \cap J_E(\neg\neg p \to p)$$

$$=$$

$$\{\langle R, X\rangle \in \mathsf{QEM}(P) \mid Q \subseteq R, p \in \mathsf{atom}(X) - R\} \cap$$
$$\{\langle R, X\rangle \in \mathsf{QEM}(P) \mid P - Q \subseteq \mathsf{atom}(X) - R\} \cap$$
$$\{\langle R, X\rangle \in \mathsf{QEM}(P) \mid X \neq \emptyset, p \in \mathsf{atom}(X) - R\}$$

$$=$$

$$\{\langle R, X\rangle \in \mathsf{QEM}(P) \mid X \neq \emptyset, Q \subseteq R, P - Q \subseteq \mathsf{atom}(X) - R\}$$

$$=$$

$$\{\langle R, X\rangle \in \mathsf{QEM}(P) \mid Q = R, X \neq \emptyset, \mathsf{atom}(X) = P\}$$

$$=$$

$$\{\langle Q, \{\langle P, \emptyset\rangle\}\rangle\}$$

□