CONSTRUCTIVE AGENTS

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Abstract Brouwer's ideas of construction, proof, and inquiry in mathematics are more widely applicable. On a well-known philosophical view, intuitionistic logic is a general account of meaning and reasoning for natural language and epistemology. In this brief discussion piece, I go one step further, and discuss how intuitionistic semantics fits with information update and belief revision in agency. In the process, I define a number of new logical systems that give rise to several open problems.

1 Introduction: agency in intuitionistic logic

Brouwer's idealized mathematician keeps adding proofs and constructions extending the realm of what we know. The Creative Subject never retracts. But retracting, learning from errors and revising a current theory, rather than infallible correctness and monotonic growth, is an equally striking intellectual talent of rational agents – including mathematicians. Is there a foothold in intuitionism for such a richer picture of rational human agency? Is there a foothold for agency at all, in addition to truth and proof?

Constructivism is about methods for establishing truth (proofs) and constructing objects (algorithms). Now proof and construction are activities, and one might think this is where agency belongs in logic: in methods of proof and construction, based on fine-grained syntax. The logical constants then get a precise operational meaning in the BHK interpretation, which has even been recast in two-player dialogue games (Lorenzen 1955). In contrast, the realm of pure truth is in the semantics, abstracting from the details of these activities, and giving the logical constants another role, that of creating compound ontological structure.

But this border is too rigid. Proof theory is mainly about proofs: static records produced by the activity of proving, not that activity itself. Also, crucial aspects of agency such as the earlier learning by trial and error do not really occur in constructive proofs. And on the other hand, basic informational activities of agents such as observation have quite appealing models in the realm of semantics. Either way, much current literature since the 1990s has shown that agency can be studied and clarified in logic, with topics such as beliefs, issues, goals, and social interaction – van Benthem 2011 pursues one such paradigm.

Against this background, in this brief exploratory paper, I will discuss semantic models for intuitionistic logic and its ally of modal logic as an arena for information and agency, showing how three important aspects might be incorporated.

2 Semantic models for intuitionistic logic

Historically, intuitionistic logic has been analyzed with a wide variety of semantic models, algebraic, topological, graph-theoretic, or category-theoretic; which have been introduced by Tarski, Beth, Kripke, and others starting from the 1930s. Many of these approaches to constructivism have been discussed and compared in van Dalen & Troelstra 1988, but the book is by no means closed: the present publication will add even more.

In this article, I start with standard relational models, that is, partial orders $(S, \leq)$ with a valuation for proposition letters recording at which points in S they hold, with upward-closed sets assigned to proposition letters. For future reference, here is the truth definition for intuitionistic propositional logic (my discussion will ignore predicate logic, for lack of space, though object handling is indeed a crucial aspect of both intuitionism and agency):

$\boldsymbol{M}, s \models p$	iff	$s \in V(p)$
$\boldsymbol{M}, s \models \varphi \wedge \psi$	iff	$\boldsymbol{M}, s \models \varphi$ *and* $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s \models \varphi \vee \psi$	iff	$\boldsymbol{M}, s \models \varphi$ *or* $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s \models \neg\varphi$	iff	*for no* $t \geq s$, $\boldsymbol{M}, t \models \varphi$
$\boldsymbol{M}, s \models \varphi \rightarrow \psi$	iff	*for all* $t \geq s$, *if* $\boldsymbol{M}, t \models \varphi$, *then* $\boldsymbol{M}, t \models \psi$

As is well known, this semantics makes all intuitionistic formulas persistent:

if $\boldsymbol{M}, s \models \varphi$ and $s \leq t$, then also $\boldsymbol{M}, t \models \varphi$

The intuition behind these models is one of stages of a process of construction and inquiry, where we gradually get to know mathematical structures matching completed histories. We will return in more detail to the issue of how to view these models presently.

But these same models also support another language, now with a universal modality

$\Box\varphi$ φ is true in all successors in the ordering.

As is well known, Gödel showed that intuitionistic propositional logic IL embeds faithfully into the well-known modal logic S4 by a simple translation of the above truth conditions. Starting with the earlier intuitive interpretation of the models, S4 then describes a richer epistemic setting where propositions can also be non-persistent – for instance, existential statements $\Diamond\varphi$ about what is still possible in the future from the current point. Thus, our models come to provide for co-existence of intuitionistic views of growing knowledge and more general epistemic views of knowledge entangled with ignorance, or even knowledge of ignorance. This asymmetry is the intuitive picture, but there is a less known converse faithful embedding of S4 into IL (Fernandez 2006), so the exact conceptual relation of the two systems is a delicate matter. However, my discussion does not depend on an answer.

How clear is the intuitive interpretation of the above models? One can find textual evidence for several interpretations. For a start, points might stand for 'information pieces'. This view fits substructural resource-based logics (Restall 2000) – but even so, I find it not so easy to say what the 'piece' metaphor means. A second interpretation views points as possible information 'states', or epistemic states, that an agent can be in, ordered by some suitable relation, extension in the intuitionistic case, something slightly more complex in the modal case. And a third view sees points as temporal 'stages' in a process of inquiry unfolding over time. These perspectives make a difference, as I will show below.

But cannot we get the right guiding intuition from the standard canonical model for intuitionistic or modal logic? Its points are deductively closed prime sets, or classically maximally consistent sets of formulas, and the ordering is inclusion in the intuitionistic case, and accessibility of a standard sort in the modal case. Such models best fit the above state account, where it should be noted that the states in this setting are rich. They do not just contain facts, but also information about what agents know about the facts, and so on. Or, sneaking in a temporal reading after all, states contain factual information plus ‘procedural information’ about how the further process of inquiry might develop.

In what follows, I explore the state-based approach in two different ways, proposing richer languages that bring out features beyond standard intuitionistic or modal views. I will use the temporal perspective occasionally for contrast. A point of clarification may be needed here. When people discuss ‘semantics’ of intuitionistic or modal logics, they may mean two things. One stance takes some formal language and logic as an unchanging given, and searches for models that fit. The other stance starts from some class of models as a conceptual framework of independent interest, and then asks which languages and logics would fit best. Of course, in the history of logic, both movements occur intertwined.

3 Information states and modal logic

Language Let us view partial orders as universes of information states. The modal language allows us to talk about upward structure from a given point. Let us highlight this for present purposes in the notation $[\uparrow]\varphi$. This is important, but there are more things to say.

The downward direction, too, makes sense in discussing weaker information, and to do this, we also need a modality $[\downarrow]\varphi$. The resulting system is temporal S4. There are also good mathematical reasons for this move, as it gives us natural adjunction principles such as

$$\langle\downarrow\rangle\varphi \to \psi \text{ implies } \varphi \to [\uparrow]\psi, \text{ and its obvious dual.}$$

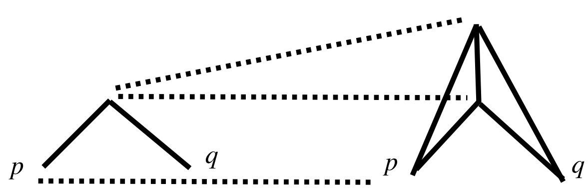
But there is more natural structure to information states. Two states s, t may have a *supremum*, a smallest state extending both, intuitively, their sum. We need not assume that all sums exist in the partial order, but where they do, the logic should encode their behavior. And vice versa, there may be informational *infima,* again a natural structure. To deal with suprema and infima, we can introduce two binary modalities:

$$\boldsymbol{M}, s \models \langle sup\rangle\varphi\psi \quad \text{iff} \quad \text{there exist } t, u \text{ with } s = sup(t, u), \boldsymbol{M}, t \models \varphi \text{ and } \boldsymbol{M}, u \models \psi$$

and likewise for $\langle inf\rangle\varphi\psi$. These are truly new operators.

Fact $\langle sup\rangle pq$ is not definable in the temporal modal language.

To see this, consider the following two models. The dotted lines form a bisimulation for the temporal language, but $\langle sup\rangle pq$ holds only in the top node on the left, not that on the right.



Note that $\langle sup\rangle\varphi\psi$ does not define the set of lowest upper bounds of the denotations of φ and ψ in the current model. Defining the latter involves non-standard inverted modalities true in points who stand in the accessibility order to all φ–points.

Our rich modal language of information states comes with a more discerning notion of bisimulation that we do not formulate here (cf. Okholikov 2016 for a general approach).

Logic The logic of the new modalities comes from various sources. One simple observation is that we can define the standard tense modalities for S4:

$$\langle\uparrow\rangle\varphi \leftrightarrow \langle inf\rangle\varphi T, \qquad \langle\downarrow\rangle\varphi \leftrightarrow \langle sup\rangle\varphi T$$

Like the existential modality in the minimal logic, $\langle sup\rangle\varphi\psi$ distributes over disjunction, in both its arguments. Given the definition of supremum, we also get the following validities:

$$\langle sup\rangle\varphi\psi \rightarrow \langle sup\rangle\psi\varphi, \quad \varphi \rightarrow \langle sup\rangle\varphi\varphi$$

Moreover, there is an obvious connection with the standard modalities in the validities

$$\langle sup\rangle\varphi\psi \rightarrow (\langle\downarrow\rangle\varphi \wedge \langle\downarrow\rangle\psi), \qquad \langle inf\rangle\varphi\psi \rightarrow (\langle\uparrow\rangle\varphi \wedge \langle\uparrow\rangle\psi)$$

However, one prima facie attractive principle that fails in general is associativity:

$$\langle sup\rangle\langle sup\rangle\varphi\psi\alpha \rightarrow \langle sup\rangle\varphi\langle sup\rangle\psi\alpha$$

and its right-to-left converse. The reason is that we do not demand existence of all suprema in our partial orders, and such failures readily yield counterexamples to the recombination of states that would be needed to validate associativity.

To see what the axioms say we can also work a bit more abstractly, and interpret $\langle sup\rangle$ as a standard existential binary modality interpreted via a ternary accessibility relation C:

$$\boldsymbol{M}, s \models \langle sup\rangle\varphi\psi \quad \text{iff} \quad \text{there exist } t, u \text{ such that } Cstu, \boldsymbol{M}, t \models \varphi \text{ and } \boldsymbol{M}, u \models \psi$$

Modal frame correspondence methods apply to this combined modal logic of C and $\leq$. For instance, the axiom $\langle sup\rangle\varphi\psi \rightarrow \langle\downarrow\rangle\varphi$ expresses that $\forall yz\ (Cxyz \rightarrow y \leq x)$. In particular, imposing the associativity axiom matches first-order associativity conditions on partial orders that can be computed by standard algorithms:

$$\forall yzuv\ ((Sxyz \wedge Syuv) \rightarrow \exists w\ (Sxuw \wedge Swvz))$$

Similar observations can be made about the modality $\langle inf\rangle\varphi\psi$.

As for the combined logic of information structures, I have not been able to find interesting general axioms linking $\langle inf\rangle\varphi\psi$ to $\langle sup\rangle\varphi\psi$, but I suspect there should be. One meta-property of the combined logic that is easy to prove runs just as for temporal logic:

Fact Every validity of modal information logic remains a validity when we interchange $[\uparrow]$ with $[\downarrow]$ and $\langle sup\rangle$ with $\langle inf\rangle$.

What is the complexity of the total system of validities?

Fact The modal logic of information structures is recursively axiomatizable.

This is a consequence of the fact that the above truth conditions translate effectively into the first-order logic of partial orders, making the modal language a first-order fragment. However, the more pressing issue about our framework, open as far as I know, is this:

Open problem Is our modal logic of information models *decidable*?

Dropping associativity seems essential to a positive answer, since modal logics with associative binary modalities can encode undecidable word problems.

Further interesting issues arise with specific information models. What is the modal logic of complete partial orders, satisfying a second-order frame condition? And what of grid structures like IN x IN, with entangled suprema and infima? The latter are known to have Π^1_1–complete bimodal logics with *<north>, <east>* if we add a universal modality.

Comments This section is based on van Benthem 1989A which explores a view of modal logic as a high-level theory of information. The new structure found in this way has no direct counterpart in intuitionistic logic, as persistence clearly fails for $<sup>\varphi\psi$. Moreover, $<sup>[\uparrow]\varphi[\uparrow]\psi$ collapses in our logic into the equivalent standard conjunction $[\uparrow]\varphi \wedge [\uparrow]\psi$. (This equivalence can even be derived formally from the valid principles stated above.) But the distinction in our logic between what might be called 'sum' conjunction and Boolean conjunction is also pervasive in categorical and relevant logics of information (Dunn 1991). More recently, it has also come up in newly proposed logics for metaphysics (Fine 2014) where points stand for parts or states of the world. Various notions of truth making, false making and matching varieties of consequence can be defined in our modal language, with a parallel recursion for truth making and false making. Our modal logic of information structures then faithfully embeds reasoning with such notions (van Benthem 2017).

4 Update and modal conditional logic

The intuitionistic focus on inquiry also suggests a dynamic view of the above models, with growth along the ordering taking place by update steps. Now information update has many logical implementations today, but one pervasive intuition is *minimality*: we move to a new state closest to the current one where some new proposition holds. This idea has a folklore history – for a recent manifestation in the theory of belief revision, see Hansson 2017.

Minimality has no place in the standard semantics of intuitionistic or modal logic. But it occurs in the proof-theoretic understanding of implications $\varphi \rightarrow \psi$: we just assume φ in a conditional proof for ψ, no more. So, minimality is worth exploring – and in fact, it is easy to add to the semantic setting that we already have.

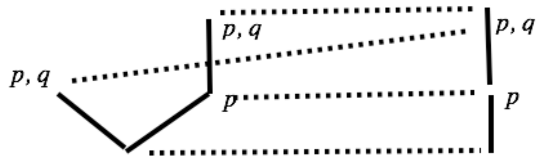
Update language Let us define new update modalities $[+\varphi]\psi$ as follows:

$$\boldsymbol{M}, s \models [+\varphi]\psi \quad \text{iff} \quad \text{for each } t \text{ that is minimal in } \{ t \mid s \leq t \text{ and } \boldsymbol{M}, s \models \varphi \},\ \boldsymbol{M}, t \models \psi$$

Again this is a genuine language extension.

Fact $[+\varphi]\psi$ is not definable in the language of temporal S4.

Again, this is easy to see with a temporal bisimulation, marked by the dotted lines between the two models depicted in the following figure:



The formula *[+p]¬q* is true in the root on the right, but not on the left.

One striking feature of our semantics is that update steps can be *non-deterministic*. There may be different upward jumps for an action $+\varphi$. I find this natural, but many people modeling updates have stated strong intuitions of functionality. This may be restored by lifting the above models from states to sets of states, making updates transformations of sets, but such a move would diminish the concrete appeal of our framework.

Logic The valid principles for the new operator, on top of modal S4, are natural and interesting. They include the obvious ones for the minimal modal logic, in particular

$$[+\varphi](\psi \wedge \alpha) \leftrightarrow ([+\varphi]\psi \wedge [+\varphi]\alpha)$$

In addition, there are principles reflecting properties of the truth definition such as

$$[+\varphi]\varphi, \qquad\qquad \langle +\varphi \rangle \psi \rightarrow \langle +(\varphi \wedge \psi) \rangle T$$

There are also simple links with the S4 modalities, such as

$$\langle +\varphi \rangle T \rightarrow \langle \uparrow \rangle \varphi$$

The most striking source of valid laws for update is the analogy with *conditional logic*. Defining relative closeness *Cstu* from a vantage point *s* as the preorder $s \leq t \leq u$, we have:

Fact $[+\varphi]\psi$ satisfies all principles of the minimal conditional logic.

We do not spell out these laws (cf. Burgess 1981), but merely observe that modal update logic poses interesting issues for conditional logic. In general conditional semantics, the closeness order at different points *s* can be completely independent. However, in our case, while there is dependence on *s* (evaluation only looks at subdomains $\{t \mid s \leq t\}$), the underlying binary order is uniform. This shows in some valid principles, such as the following:

Fact $\langle \uparrow \rangle (\langle \uparrow \rangle \neg\varphi \wedge \langle +[\uparrow]\varphi \rangle \psi) \rightarrow \langle +[\uparrow]\varphi \rangle \psi$ is valid in our semantics.

The reason is this. Let the antecedent hold in any point *x* of a model. Any later point *y* with $\langle \uparrow \rangle \neg\varphi \wedge \langle +[\uparrow]\varphi \rangle \psi$ has a still later point *z* satisfying $[\uparrow]\varphi$ for the first time as well as ψ. But z is also a first point after *x* where $[\uparrow]\varphi$ is true. Note that this reasoning does not hold for arbitrary φ instead of $[\uparrow]\varphi$. Also, the simpler-looking principle $\langle \uparrow \rangle \langle +[\uparrow]\varphi \rangle \psi \rightarrow \langle +[\uparrow]\varphi \rangle \psi$ is not valid: if the point *y* itself is the witness for $\langle +[\uparrow]\varphi \rangle \psi$, nothing follows for the initial *x*.

There are issues of completeness and representation here, but we forego these, except for the following general remark. Like modal information logic, modal update logic is a richer language over the models that we normally use for S4. This means, in particular, that we can ask what becomes, in this natural richer language, of known modal completeness theorems for special (classes of) partial orders, and perhaps, whether some sort of general transfer is possible of completeness theory from the basic language to this richer setting.

Downdate Next, as with modal information logic, the reverse direction of our models makes sense. In belief revision, there is not just update to richer or at least further states, but also revision to incomparable states, and contraction to weaker information states. In our models, this can be done simply as 'downdate' on partial orders by moving backward:

$$\boldsymbol{M}, s \models [-\varphi]\psi \quad \text{iff} \quad \text{for each } t \text{ that is maximal in } \{ t \mid t \leq s \text{ and } \boldsymbol{M}, s \models \neg\varphi \}, \boldsymbol{M}, t \models \psi$$

This has the same logic as $[+\varphi]\psi$, by a duality like in the information models of Section 3.

The total language is of interest as it combines two conditionals, something not often done in conditional logic (but cf. Ryan & Schobbens 1997 on combined conditional modalities for update, revision, and world change treated in a dynamic logic style). There are valid laws connecting the two directions. However, on combinations of update and downdate, this logic is often at odds with the laws of standard belief revision theories.

Fact $+\varphi\,;\,-\varphi$ is not equivalent to the identity update, and neither is $-\varphi\,;\,+\varphi$.

Comments The systems presented in this section are based on van Benthem 1989B. They were triggered by the belief revision theory of Gärdenfors 1988, in an attempt to see how far existing modal logics, suitably extended to include minimality, could do similar jobs. This comparison also included investigating algebraic laws for update, such as

$$+\varphi\,;\,+\varphi \;=\; +\varphi$$

which fails on our models, as is easy to see. Likewise, the following equality is not valid:

$$+\varphi\,;\,+\psi \;=\; +(\varphi \wedge \psi)$$

This failure is typical for dynamic logics of update, which tend to distinguish sequential conjunction *;* and parallel conjunction $\wedge$. Still, one can salvage valid special cases, such as

$$+[\uparrow]\varphi\,;\,+\psi \;\leq\; +([\uparrow]\varphi \wedge \psi)$$

where a sequential composition does get absorbed into a Boolean conjunction. Here a full equality would only hold modulo some well-foundedness assumptions on the ordering.

The analogy between belief revision and conditional logic that we have pointed out here is not new in itself: it has shown up in many different settings since the 1980s, cf. the handbook chapter Gärdenfors & Rott 1995. Also, it should be noted that other models exist for belief revision using the minimality intuition, in terms of minimal distance to an original information state (cf. Hansson 2017, Klein & Rendsvig 2017).

Coda There are also different approaches to information update, such as dynamic-epistemic logic with detailed instructions for changing a current model to a definable new one (Baltag, Moss & Solecki 1998). Dynamic-epistemic updates, too, suggest minimal change in a model to incorporate new information. A public announcement $!\varphi$ of a factual proposition φ changes the current model to its largest sub-model where φ is com-mon knowledge. Still, there are subtle differences. To show this, for simplicity, consider the minimal modal logic.

Compare two bisimilar modal models, $\boldsymbol{M_1}$ consisting of one reflexive point where p holds, and $\boldsymbol{M_2}$ consisting of a 2-cycle having p true in both points. Now consider an update modality $<+\Box\bot>\top$ in our style. This formula is true in both points of $\boldsymbol{M_2}$, but not in $\boldsymbol{M_1}$.

Thus, the update modalities of this section fail to respect standard bisimulation invariance. It would be of interest to study dynamic-epistemic logics with this extended repertoire.

5 Digression: inquiry over time

In this section, we briefly discuss the other possible interpretation of intuitionistic models, as temporal universes with histories standing for possible courses of inquiry.

Coexistence This view is not at odds with the earlier state view. We can view any modal model as a set of states with possible transitions – and there is a natural unraveling to a tree of possible histories, that is, finite sequences of states with suitable transitions. More precisely, the unraveling can yield a 'forest model' consisting of many such trees. In such models, a history can cycle through states, or show other kinds of informational behavior.

Some people even take the two views to be the same, as unraveling is a bisimulation. But differences remain. A basic bisimulation will not preserve the richer modal languages of the preceding sections. For instance, our initial model may have suprema which disappear in trees, which have no non-trivial suprema, and conversely, trees always have infima, that need not be infima in the original model. Another difference is the choice of optimal languages for a temporal perspective. Over trees, we may want a standard branching temporal language interpreting formulas on pairs of histories and points, which makes comparison with our earlier point- or stage-based intuitionistic or modal languages harder.

Specializing to trees What happens when we interpret our earlier languages on trees? We collect a few observations. In modal update logic on trees the following holds:

$$<\uparrow>\varphi \rightarrow <+\varphi>T$$

This implication reflects the well-foundedness of trees. Next, downdate logic looks different from update logic, as the closeness order in the downward direction is now linear. This asymmetry has well-known reflections in the temporal logic, but I have not been able to find pure laws of conditional logic for downdate that would reflect this special setting.

Next, we can bring a general result to bear.

Conjecture On trees, the combined update–downdate temporal S4 logic is decidable.

The idea is this. All our truth conditions are first-order in the tree order. Moreover, completeness holds with respect to truth in one single frame: the binary branching countable tree. (Here we must mimic arbitrary branching by means of iterated binary branching, like one does for *S4*.) Then we can embed validity in our system into monadic second-order logic, and apply Rabin's Theorem. This general argument still works if we strengthen the update language to include analogues of temporal Until and Since on the ordering.

Uncovering the epistemic dynamics of inquiry Yet one can doubt if trees in backward-loo-king mode really model inquiry over time. A retraction event seems a forward step in the tree, just like an update, but toward a later stage associated with a weaker information state. Therefore, a natural question in a temporal setting is what drives the process forward. What triggers upward steps? Intuitionistic or modal semantics does not say.

Van Benthem 2009 distinguishes two kinds of step. One is observing facts about the eventual structure found on complete histories, as in dynamic-epistemic logics of announcements $!\varphi$ of the truth of propositions φ which exclude future states. The other step is 'awareness raising' $\#\varphi$: becoming aware of a proposition φ that was already inevitable implicitly. This, too, can be studied in dynamic-epistemic logic, modifying stocks of propositions one is explicitly aware of. Consider two models $\boldsymbol{M_1}$, $\boldsymbol{M_2}$, where the first refutes the double negation law $\neg\neg p \rightarrow p$, and the second the law of 'weak excluded middle' $\neg p \vee \neg\neg p$:

The two steps in $\boldsymbol{M_2}$ may be seen as public announcements of which endpoints (classical valuations) the process can get to. But the step in $\boldsymbol{M_1}$ performs no elimination, and we merely get more proposition letters true at the next stage. However, there can be further dynamic triggers for epistemic state changes along a tree, and we will discuss one below.

We end our discussion of the temporal perspective here. It should be noted, however, that there are many epistemic-temporal frameworks around, each with their own take on information dynamics: cf. Fagin, Halpern, Moss & Vardi 1995, Parikh & Ramanujam 2003.

6 Retraction and belief

Our logic of update and revision over pre-orders still lacks a driver. What reason can Brouwer's infallible mathematician have for going back to earlier information states? Just to reminisce about the good old days? More serious reasons apply to ordinary agents, who make mistakes and must correct them. Then we need to distinguish between infallible knowledge which can only grow, and belief, which is fallible and may have to be revised.

Belief occurs even in mathematics, as we have both theorems and conjectures, or more generally, expectations about where a theory is going to go. Can we model this phenomenon staying close to our earlier semantic style? There are in fact various ways of doing this, but I will mention just one that is particularly easy to implement.

Conjectures Conjectures can be viewed as hypotheses about what the actual world, or actual history, will turn out to be. A simple way of modeling this already yields interesting semantic structure and logical laws. We let the current conjectures be a set C of propositions, and we define an order on points in intuitionistic or modal models as follows:

> a point y is at least as plausible as another point x if y has verified all the conjectures that x has verified.

More formally, for each point s in a model $\boldsymbol{M}$, the following stipulation defines a binary order $x \propto_s y$ on the set $\{ t \mid s \leq t \}$:

$$\text{For all } \gamma \in C: \quad \text{if } \boldsymbol{M}, x \models [\uparrow]\gamma, \text{ then } \boldsymbol{M}, y \models [\uparrow]\gamma$$

This way of introducing an ordering from sets is analogous to the well-known construction of ordering models from premise semantics in conditional logic (cf. Lewis 1981).

Fact The order $\propto_s$ satisfies the following two properties:
(a) $\propto_s$ contains $\leq$ restricted to successors of s, (b) $\propto_s$ is transitive.

These two properties immediately imply that is $\propto_s$ reflexive, and that $x \propto_s y \leq z$ only if $x \propto_s z$.

Fact Conditions (a), (b) characterize conjecture models completely.

The proof rests on a simple observation. Given any two abstract orders $\leq$, $\propto_s$ satisfying (a), (b), we can define a set C of conjectures generating precisely these orders by taking C to be the family of all $\propto_s$ –closed subsets. This construction resembles a representation theorem in Liu 2011 for reason-based preference. Another analogy for the proposal made here are the models for evidence-based beliefs in van Benthem & Pacuit 2011.

A final point to note is that the orderings defined at different points cohere as follows:

$$\text{if } s \leq t, \text{ then } x \propto_t y \ \text{ iff } \ x \propto_s y, \quad \text{for all } x, y \geq t$$

Logic of belief Now we can define a logic of plausibility-based belief in the usual style (cf. van Benthem & Smets 2015), on top of our base logic for $[\uparrow]$. We say that

$$\boldsymbol{M}, s \models B\varphi \quad \text{iff} \quad \text{for all } t \geq s \text{ that are maximal in the } \propto_s\text{–order, } \boldsymbol{M}, t \models \varphi$$

To make sure that such maximal points exist, thereby avoiding more complex truth definitions in the infinite case, we will assume that the set C is finite, as seems reasonable in our setting. We will also assume that there is at least one conjecture, say the statement 'True'. The effect of this is that each state has most plausible states above it in the inclusion order where maximal families of conjectures have been verified, and stay true from then on. This may be interpreted as agents having a sort of expectation about which states are most plausible – or in a temporal interpretation, about how inquiry will develop.

The combined logic of intuitionistic knowledge and belief has some interesting features.

Fact The following principles are valid:

(a) $[\uparrow]\varphi \to B\varphi$, (b) $B\varphi \to B[\uparrow]\varphi$, (c) $B(B\varphi \to [\uparrow]\varphi)$

Here (b) says that if the agent believes φ, he believes that he has verified φ (he 'knows' it), while (c) says roughly that the agent believes that his beliefs are knowledge. These statements follow from the above description of most plausible states in a model as having settled a maximal subset of the given finite set of conjectures.

These laws are not quite like in standard logics of knowledge and belief, but they make sense in our semantics. Taken as axioms, these principles also have standard derivable consequences, such as positive introspection $B\varphi \to BB\varphi$. But not everything goes.

Fact The following principles are invalid in our semantics:

(a) $B\varphi \to [\uparrow]\varphi$, (b) $B\varphi \to [\uparrow]B\varphi$

For instance, (b) can fail for an interesting reason. Some later points may have a poor epistemic future that only establishes sets of conjectures properly contained in larger such subsets verified elsewhere. Then $B\varphi$ does not tell us that φ holds in these 'local optima'.

Finally, absolute belief may not be the most interesting notion eventually. Given the plausibility order, we can also define conditional belief in the usual style. Given the above properties of $\propto_s$, this again yields a minimal conditional logic with standard laws.

Combining perspectives It would be of interest to explore how the preceding model of intuitionistic modality plus belief meshes with our earlier themes of update and retraction dynamics and of information structure. In particular, with points as information states, we can introduce a distinction between two components: 'hard information' tied to knowledge and 'soft information' tied to belief. We may then need various inclusion orderings to reflect this twofold structure, but this and other system combinations are beyond the scope of this article. It should also be noted that adding belief is not new per se: it has been studied extensively in the setting of epistemic temporal logics: cf. Bonanno 2007, Kelly 2015.

Dynamics of conjectures Our model of conjecture-based belief is simplistic. In particular, conjectures stay the same throughout a model. But conjectures can be added, or dropped. In fact, some dropping occurs in our models, since the above definition of plausibility order above a point s only uses those conjectures that might still become verified at some stage. The others play no active role – which can be seen as a conservative form of belief revision.

But what about adding new conjectures? How does this affect complex statements involving our two modalities $[\uparrow]$ and B? We could study these effects with dynamic-epistemic logics, and then interesting issues arise about valid reduction axioms. However, a simpler approach would simply assign a set of conjectures to each point in our models, and impose some cumulativity constraints.

In this case, the earlier coherence between orderings $\propto_s$, $\propto_t$ for states $s \leq t$ fails, as the later order may be a proper refinement of the earlier one using additional properties. Of the principles stated earlier, this will invalidate $B(B\varphi \to [\uparrow]\varphi)$ and $B\varphi \to BB\varphi$. However, some valid principles remain for a combined logic, of which we mention

$$B(B\varphi \to \langle\uparrow\rangle[\uparrow]\varphi)$$

Informally: it is possible to know what one believes. This still follows from our earlier description of most plausible states, even when the plausibility order has become refined.

We will not pursue this extended setting further here, but we submit that it is a natural concrete logic of knowledge growth and belief change.

Even with the present material, we hope to have shown that adding beliefs explicitly can be done in a natural manner, leading to perspicuous bimodal logics over our earlier models.

7 Conclusion

We have discussed how three basic features of agency might enter our understanding of intuitionistic logic, viz. information, update and retraction, and belief. The models we proposed for this came from a standard modal perspective on intuitionism, and they fit well with existing models and languages in the modal literature. The resulting three systems raised some interesting questions of completeness and decidability, while also suggesting broader questions in modal and intuitionistic logic.

Still, this is only a very first exploration. Many crucial aspects of agency were left out, such as issues, goals, and social interaction. These, too, fit well with intuitionism. Mathematical inquiry is driven by issues and goals just as general agency, and indeed, an old interpretation of constructivism as a logic of 'problems' points in this direction. I suspect that bringing these in will connect intuitionistic semantics with formal learning theory. Likewise, the long-standing dialogical interpretations of intuitionism going back to Lorenzen suggest a natural place for multi-agent interaction, but again I must leave this open here.

Another and yet more severe limitation lies on the side of constructivism. Brouwer's views were not about mere propositional knowledge, but entangled with that, about construction of mathematical objects. At the very least, this would require extending the propositional logics of this article to predicate-logical versions. This raises some very interesting issues of mixing incomplete information and partial objects, which I have had to forego here.

But to me, the most intriguing challenge is this. Intuitionism is perennially intriguing as it lies at the intersection of two natural motivations. On the one hand there is the grand proof-theoretic view of the BHK interpretation, which leads to deep connections with lambda calculus and category theory. On the other hand, there are the semantic models used in this article, which lead to modal logic and topology. How are the two related?

Standard semantic models do not encode proof steps, and there may be no simple reductions. Still, it would be of interest to find a common ground between Kripke models and BHK constructions, as happens in the semantics proposed in Fine 2014. But one can also opt for keeping the two intuitions distinct, and explore *combinations* of proof-theoretic and semantic views, as first proposed for epistemic logic in van Benthem 1993, and developed in a deeper and sophisticated way in Artemov 2008. And this is not just speculation. The semantic topics in this paper do have proof-theoretic analogues. Dynamic upward steps in intuitionistic models included 'awareness raising', a major function of drawing conclusions.

We also noted that minimality of update fits with the minimality of assumptions in hypothetical proof (when based on the right structural rules). And finally, it is well known in the literature that a semantic perspective of belief and retraction fits well with the proof-theoretic engine of non-monotonic default logics, a connection which keeps coming up in different flavors (cf. Velazquez-Quesada 2011, Roy & Shi 2017). This connection is also reflected in our bimodal logic of knowledge and belief, where the former might refer to classical or intui-tionistic deduction, while the latter is based on some form of default logic.

This leads us to an intriguing question going back to the proof-theoretic origins of intuitionism. Could there be an extension of the BHK interpretation to encompass default logic, that would match the semantic explorations offered in this article?

Both with what I have discussed, and what I have left open, I hope to have made a case for the attractions of interfacing intuitionistic logic with a general perspective on agency.

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