KOLMOGOROV COMPLEXITY OF INITIAL SEGMENTS OF SEQUENCES AND ARITHMETICAL DEFINABILITY

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Abstract. The structure of the K-degrees provides a way to classify sets of natural numbers or infinite binary sequences with respect to the level of randomness of their initial segments. In the K-degrees of infinite binary sequences, X is below Y if the prefix-free Kolmogorov complexity of the first n bits of X is less than the complexity of the first n bits of Y, for each n. Identifying infinite binary sequences with subsets of $\mathbb{N}$, we study the K-degrees of arithmetical sets and explore the interactions between arithmetical definability and prefix free Kolmogorov complexity.

We show that in the K-degrees, for each $n > 1$ there exists a Σ^0_n nonzero degree which does not bound any Δ^0_n nonzero degree. An application of this result is that in the K-degrees there exists a Σ^0_2 degree which forms a minimal pair with all Σ^0_1 degrees. This extends work of Csima/Montalbán [CM06] and Merkle/Stephan [MS07]. Our main result is that, given any Δ^0_2 family $\mathcal{C}$ of sequences, there is a Δ^0_2 sequence of non-trivial initial segment complexity which is not larger than the initial segment complexity of any non-trivial member of $\mathcal{C}$. This general theorem has the following surprising consequence. There is a $\mathbf{0}'$-computable sequence of nontrivial initial segment complexity which is not larger than the initial segment complexity of any nontrivial computably enumerable set.

Our analysis and results demonstrate that, examining the extend to which arithmetical definability interacts with the K reducibility (and in general any 'weak reducibility') is a fruitful way of studying the induced structure.

1. Introduction

The desire to compare the randomness 'degree' of two infinite binary sequences has led to the introduction of randomness reducibilities. An infinite sequence is called random if the prefix-free complexity K of its initial segments is very high, namely equal to the very length of the segment (modulo a constant). Therefore a straightforward way to compare two sequences with respect to randomness is to compare the prefix-free complexity of their initial segments. Let $K(\sigma)$ denote the prefix free complexity of string σ and say that $A \leq_K B$ if $K(A \restriction_n) \leq^+ K(B \restriction_n)$ for all $n \in \mathbb{N}$.[1] This measure of randomness is called K-reducibility and was introduced in [DHL04], along with its plain Kolmogorov complexity counterpart. The induced structure of K-degrees has been a subject of interest in the last 5 years or so, though in terms of development this area is still in its infancy.

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[1] By $\leq^+$ we mean that the inequality holds modulo a constant that does not depend on n.

Miller and Yu studied the K-degrees of random sets in [MY08, MY10]. Csima and Montalbán constructed a minimal pair of K-degrees in [CM06]. Their method was highly non-constructive, making the pair merely Δ^0_4 (i.e. definable with four quantifiers), as noticed in [DH09, Section 10.13]. Merkle and Stephan (motivated by a number of related questions in [MN06]) studied the interaction between the Turing and the K reducibility in [MS07], along with its plain complexity counterpart, the C reducibility. Amongst many other results they showed that there is a pair of Σ^0_2 sets which form a minimal pair in the K-degrees.

The study of the K reducibility is part of a larger study of the so-called 'weak reducibilities'. These are preorders that measure various notions related to randomness (of sets), as opposed to computational complexity. Such reducibilities, like K, do not have an underlying map i.e. an algorithm mapping (reducing) the second set to the first one. The existence of such maps is a vital feature in the Turing or stronger reducibilities.

In the Turing degrees, Post's theorem gives an important link between reducibility (computability) and definability. For example, if a set is Turing reducible to a Σ^0_1 set then it is Δ^0_2. A lot of the methods that underly the theory of Turing degrees rest on this link with definability. This breaks down when one considers weak reducibilities. For example, a feature that one finds in most weak reducibilities is that they can have uncountable lower cones. That is, there are uncountable classes, all of whose elements are reducible to a single set. Consider a related weak reducibility that was defined in [Nie05], the LK reducibility (see Table 1).[2] We say that $A \le_{LK} B$ if $K^B(\sigma) \le^+ K^A(\sigma)$ for all strings σ, where K^X denotes the prefix free complexity relative to X. That is, if B compresses more than (or at least as well as) A. It was shown in [BLS08] that for sufficiently 'strong' oracles B, the $\le_{LK}$-cone below B is uncountable. Such properties also affect the study of local structures of the degrees, for example restricted to the Σ^0_1 or the Δ^0_2 sets. To illustrate this, consider a Δ^0_2 set B and $A \le_T B$; then A is Δ^0_2. However by [Bar10b] there are uncountably many A such that $A \le_{LK} B$ (unless $B \le_{LK} \emptyset$). This means that there is no hope to derive any definability of A from B when $A \le_{LK} B$.[3]

TABLE 1. Equivalence relations with respect to various weak reducibilities and their meaning.

$\equiv_K$	Same prefix-free complexity of the corresponding initial segments.
$\equiv_C$	Same plain complexity of the corresponding initial segments.
$\equiv_{LK}$	Same relativized prefix free complexity.
$\equiv_{LR}$	Same notion of relativized randomness.

As a result, a number of methods that we use in the study of the Turing degrees do not have a counterpart in the study of 'weaker' degrees. Following up such differences sometimes lead to elementary differences between classical structures like the Turing degrees and related structures based on weaker reducibilities. For a number of such examples we refer to [Bar10a].

[2] Miller (see [Nie09, Theorem 5.6.5]) showed that $\le_{LR}$ equals $\le_{LK}$.

[3] However in the special case when B is low for Ω (i.e. the halting probability Ω is random relative to it) $A \le_{LK} B$ implies that A is Δ^0_2 relative to B. This was shown in [Mil09].

But why is it useful to look for definability in weaker reducibilities? The presence of definability in a weak reducibility indicates that methods from the classical theory of the Turing degrees may be applicable to the study of it. We illustrate this by an analysis of definability in the K-degrees which, amongst other things, gives new ways to obtain minimal pairs in this structure. In Section 2 we study the class of *infinitely often (i.o.) K-trivial* sequences. These are sequences that have infinitely many initial segments σ with the property that $K(\sigma) \leq_+ K(|\sigma|)$. We note that the class of sets that are $\leq_K$-below such sequences is very well behaved; in particular it is countable. Therefore such sequences locally generate good definability conditions. We also show that these sequences are rather common. Every truth-table degree contains an i.o. K-trivial set; in particular they are uncountably many. Also every (weakly) 1-generic set is i.o. K-trivial, so they form a co-meager class.

In Section 3 we study how arithmetical complexity interacts with the structure of K-degrees. Given a degree structure, the Σ^0_1 degrees are the ones which contain a Σ^0_1 set. The same applies to other classes of the arithmetical hierarchy. We show that, in the K-degrees, for each $n > 1$ there exists a nonzero Σ^0_n degree which does not bound any nonzero Δ^0_n degree. The particular case $n = 2$, combined with the basic properties of the i.o. K-trivial sets from Section 2, gives a Σ^0_2 degree which forms a minimal pair with every non-zero Σ^0_1 degree. This extends the work of Csima/Montalbán [CM06] and Merkle/Stephan [MS07] on minimal pairs in the K-degrees. However their methods are entirely different than ours.

In fact it is possible in the K-degrees to construct a Δ^0_2 non-zero degree which does not bound any Σ^0_1 non-zero degrees. This result requires more effort and is rather surprising as Σ^0_1 sets have relatively low initial segment complexity. It also shows a contrast between the local structures of the K and the LK degrees, since in [Bar10a] it was shown that in the LK degrees every non-zero Δ^0_2 degree bounds a non-zero Σ^0_1 degree. Our method shows that, more generally, given any uniformly $\mathbf{0}'$-computable family of sets there exists a $\mathbf{0}'$-computable set of non-zero K-degree such that no set in the family is $\leq_K$-reducible to it, unless it is reducible to $\emptyset$. The proof of this main result is presented in Section 4.

The first construction of a minimal pair in the K-degrees was given in [CM06] through a brute-force argument. The proof relied on the construction of a non-decreasing unbounded function f such that for each set X

$$\forall n\ [K(X \upharpoonright_n) \leq^+ K(n) + f(n)] \iff X \leq_K \emptyset. \tag{1.1}$$

We refer to functions that satisfy (1.1) as *gap functions for K-triviality* and study them in Section 5. For example, we show that there is no Δ^0_2 unbounded non-decreasing gap function for K-triviality. This shows that the method used in [CM06] cannot be used in order to produce minimal pairs in the K-degrees of arithmetical complexity less than Σ^0_2. Gap functions for K-triviality are interesting in their own right and are also related to the so-called Solovay functions that were studied in [BD09, HKM09]. In Section 5 we study their arithmetical complexity and discuss the role they play in the K-degrees.

2. Infinitely often K-trivial sets

Recall that a set A is low for K if the compression of strings is not improved when A is used as an oracle. In other words, if $K^A =^+ K$.[4] Hirschfeldt and Nies showed

[4] We say that $f =^+ g$ for two functions f, g if $f \leq^+ g$ and $g \leq^+ f$.

in [Nie05] that lowness of K is equivalent to K-triviality. In [Mil09] Miller defined a weak version of lowness for K by requiring that $K^A(n) =^+ K(n)$ for infinitely many n (instead of all n). This variation turned out to be a fruitful characterization of lowness for Ω.[5] In particular, within Δ^0_2 it coincides with lowness for K. Consider the following analogous weakening of the notion of K-triviality.

Definition 2.1. A set A is called K-trivial on a set $M \subseteq \mathbb{N}$ with constant c if $K(A\restriction_n) \leq K(n) + c$ for all $n \in M$. If it is K-trivial on an infinite set then we call it infinitely often K-trivial with constant c.

A simple argument in [Nie09, Exercise 5.2.9] shows that K-triviality on an infinite computable set coincides with K-triviality. In the following we show that the class of infinitely often K-trivial sets is rather large, and quite different to the class of weakly low for K sets. Recall that given an enumeration of a set in stages, there are infinitely many n, s such that n is enumerated at stage s and no $i < n$ is enumerated at any stage $r \geq s$. Given a c.e. set A (and a computable enumeration of it with no repetitions), let us call the set of all such n (which are part of a pair n, s as above) *set of minimal enumerations* of A. The following proposition was shown for plain complexity in [HKM09] using the same argument. Moreover it has been known to a number of researchers, although we are not aware of any explicit reference in the literature.

Proposition 2.2. *Every c.e. set is infinitely often K-trivial (on the set of its minimal enumerations).*

Proof. Fix a computable enumeration (A_s) of A without repetitions and a universal prefix free machine U. Machine M does the following for each $n \in \mathbb{N}$. It waits for a stage s were n is enumerated in A and assigns to $A_s\restriction_n$ all U-descriptions of 0^n. Since each number is enumerated in A at most once, M is prefix free. If n is a minimal enumeration of A it is clear that $K_M(A\restriction_n) \leq K(0^n)$. Hence $K(A\restriction_n) \leq K(n) + c$ for some constant c and all n in the set of minimal enumerations of A. □

The following results show that the sets that are $\leq_K$-below an infinitely often K-trivial set Y are Δ^0_2 definable in Y.

Proposition 2.3. *Suppose that Y is infinitely often K-trivial. Then each set in the lower cone $\{X \mid X \leq_K Y\}$ is computable in $Y \oplus \emptyset'$.*

Proof. Suppose that Y is infinitely often K-trivial via constant c_0 and $X \leq_K Y$ via c_1. Let $c = c_0 + c_1$ and $F_c(n) := \{\sigma \mid |\sigma| = n \ \wedge \ K(\sigma) \leq K(n) + c\}$. By the coding theorem we have that there is some constant b such that $|F_c(n)| < 2^{c+b}$ for all $n \in \mathbb{N}$. Let F_c denote the downward closure of the set of strings $\cup_{n\in\mathbb{N}} F_c(n)$. Since the prefix free complexity function K is computable from $\emptyset'$, the infinite set M on which Y is K-trivial (via constant c_0) is computable from $Y \oplus \emptyset'$. Hence the downward closure of the set of strings $\cup_{n\in M} F_c(n)$ is computable from $Y \oplus \emptyset'$. Let us denote this subtree of the tree F_c by L_c. The cardinality of the levels of L_c have the same constant bound 2^{c+b}. By the choice of c, the set X is an infinite path through L_c. Since L_c is a $Y \oplus \emptyset'$-computable tree with a constant bound on the cardinality of its levels, its infinite paths are computable in $Y \oplus \emptyset'$. □

[5] Recall that a set is low for Ω if the latter is Martin-Löf random relative to it.

Proposition 2.4. *If Y is K-trivial on an infinite set M, then it is computable from $\emptyset' \oplus M$.*

Proof. The tree L_c from the proof of Proposition 2.3 is also computable in $\emptyset' \oplus M$. Since there is a constant bound on the cardinality of its levels, its paths (including Y) are computable in $\emptyset' \oplus M$. $\square$

By Proposition 2.3, every set that is $\leq_K$-below an infinitely often K-trivial Δ^0_2 set Y is Δ^0_2. However we do not know if *the class* of sets that are $\leq_K$-below Y is (uniformly) Δ^0_2. To be more precise, we recall the following definition from computability theory. Let (Φ_e) be an effective list of all Turing functionals.

Definition 2.5. A class $\mathcal{C}$ of subsets of $\mathbb{N}$ is called a Δ^0_2 family (or uniformly $\emptyset'$-computable) if it can be written in the form $\{C_e \mid e \in \mathbb{N}\}$ where $C_e = \{n \mid \psi(e,n)\}$ and ψ is a Δ^0_2 property (i.e. a property that can be expressed in arithmetic with equivalent Σ^0_2 and Π^0_2 formulas). Equivalently, if there is a computable function f such that $\mathcal{C} = \{\Phi^{\emptyset'}_{f(e)} \mid e \in \mathbb{N}\}$, where $\Phi^{\emptyset'}_{f(e)}$ is total for each $e \in \mathbb{N}$.

Recall that a set is ω-c.e. if it has a computable approximation where, for each $n \in \mathbb{N}$ the number of changes of the nth digit is bounded by the value of a computable function on n. It is not hard to see that the ω-c.e. sets form a Δ^0_2 family while the Δ^0_2 sets do not. A basic fact about the K-trivial c.e. sets is that they form a uniformly c.e. family of sets (e.g. see [Nie09, Fact 5.2.6]). Perhaps more interestingly, *the K-trivial sets form a Δ^0_2 family.* This follows from the fact that the ω-c.e. K-trivial sets form a Δ^0_2 family (see [Nie09, Theorem 5.3.28]) and the deeper fact that K-trivial sets are ω-c.e. (see [Nie09, Corollary 5.5.4]). In particular, the lower cone in the K-degrees below $\emptyset$ is a Δ^0_2 family. We do not know if there are non-trivial lower cones in the K-degrees with the same property. The notions introduced in Definition 2.5 will play an important role in Sections 3 and 4.

In terms of Lebesgue measure the class of infinitely often K-trivial sets is small (i.e. it has measure 0). Indeed, no infinitely often K-trivial set is Martin-Löf random. However in most other respects it is rather large, as we demonstrate below. We first need the following fact.

Lemma 2.6. *Let V be an infinite c.e. set with the property that for each $n \in \mathbb{N}$ there is at most one string of length n in V. Then $K(\sigma) \leq^+ K(|\sigma|)$ for all $\sigma \in V$.*

Proof. Let U be the universal prefix free machine. Consider a prefix-free machine which, given $\sigma \in V$ it assigns to σ the U-descriptions of $0^{|\sigma|}$. By the properties of V such a machine exists, and $K(\sigma) \leq^+ K(0^\sigma) \leq^+ K(|\sigma|)$ for each $\sigma \in V$. $\square$

Theorem 2.7. *There is a computable pruned perfect tree such that every path in it is infinitely often K-trivial. In particular there are $2^{\aleph_0}$ infinitely often K-trivial sets.*

Proof. Consider a computable tree $T : 2^{<\omega} \to 2^{<\omega}$ such that $|T(\sigma)| \neq |T(\tau)|$ if $\sigma \neq \tau$. Then the range of T is a c.e. set V which satisfies the properties of Lemma 2.6. Hence any path $\cup_n T(X \restriction_n)$ through the tree is K-trivial on the infinite set of numbers $|T(X \restriction_n)|$, $n \in \mathbb{N}$. $\square$

The following Corollary is implicit in [MS07], although it is obtained using different methods.

Corollary 2.8. *Every truth table degree contains an infinitely often K-trivial set.*

Proof. Let A be any set and let T be the tree of Theorem 2.7. Notice that T can be viewed as a total Turing functional $\Phi : 2^\omega \to 2^\omega$ via $X \to \cup_n T(X\restriction_n)$. Moreover we can define a total Turing functional $\Psi : 2^\omega \to 2^\omega$ which is the inverse of Φ on the paths of T and some finite set on other paths. Now let $\Phi^A = B$, so that $\Psi^B = A$. Then $B \leq_{tt} A$ and $A \leq_{tt} B$. Moreover B is infinitely often K-trivial since it is on T. $\square$

Theorem 2.9. *Every (weakly) 1-generic set is infinitely often K-trivial. In particular, the class of infinitely often K-trivial sets is co-meager.*

Proof. Every weakly 1-generic set meets every infinite dense set of strings infinitely often. This is because every co-finite subset of a dense set of strings is dense. Hence by Lemma 2.6 it suffices to define a dense set V of strings σ such that no two strings in V have the same length. Indeed, in that case we have $K(\sigma) \leq^+ K(|\sigma|)$ for each string $\sigma \in V$. Hence any sequence that intersects V infinitely often is infinitely often K-trivial.

The set V is defined recursively as follows, based on a computable enumeration (σ_s) of all strings. At stage $s+1$ put in V the least string[6] which extends σ_s and its length is larger than the lengths of all the strings in $V[s]$. Clearly V has the desired properties. $\square$

The following theorem can be combined with various basis theorems for Π^0_1 classes to give infinitely often K-trivial sets with additional properties.

Theorem 2.10. *There is a nonempty Π^0_1 class which consists of infinitely often K-trivial sets but does not contain any K-trivial sets.*

Proof. We only give a sketch of the proof since it does not involve new ideas. Let V be a c.e. dense set of strings σ such that $K(\sigma) \leq^+ K(|\sigma|)$. This is obtained as in the proof of Theorem 2.9. Let c be a constant such that $K(n) \leq 2\log n + c$ for all n, where $\log n$ is the largest number k such that $2^k \leq n$. To avoid the K-trivial sets in the class we use a computable function f such that for all n and all strings σ of length n, there is an extension τ of σ of length $f(n)$ such that $K(\tau\restriction_k) > 2\log k + c + n$ for some $k < |\tau|$.

At step 1 we put all strings of length $f(1)$ in our tree and promise to remove any of them σ such that $K(\sigma\restriction_k) \leq 2\log k + c + 1$ for all $k \leq |\sigma|$ (this is a Π^0_1 event). Notice that by this action we also remove those σ such that $K(\sigma \restriction k) \leq K(k) + 1$ for all $k \leq |\sigma|$. Let $\ell_1 = f(1)$. At step 2, for each of the chosen strings of step 1 we choose an extension τ in V. If ℓ is the length of the largest such extension, we let $\ell_2 = \ell$. We put on the tree each such extension τ concatenated with $\ell - |\tau|$ zeros. We also declare any other extension of σ that is incompatible with τ not to be part of the tree. We continue in the same way for the rest of the steps, where at step $2n+1$ we put on T the extensions of the strings of step $2n$ of length $f(\ell_{2n})$. Also, we promise to remove those strings such that $K(\sigma\restriction_k) \leq 2\log k + c + n$ for all $k \leq |\sigma|$.

This procedure defines a Π^0_1 tree T such that the set $[T]$ of its paths is nonempty. Any real in $[T]$ intersects V infinitely often by the construction of T (in particular the even steps). Hence it is infinitely often K-trivial. Moreover it does not contain K-trivial reals by the odd steps of the construction. $\square$

[6] We order the set of binary strings first by length and then lexicographically.

If we combine Theorem 2.10 with the computably dominated basis theorem (e.g. see [Nie09, Theorem 1.8.42]) we get that there are non-computable i.o. K-trivial computably dominated sets. This contrasts the fact that every computably dominated K-trivial set is computable. We close this section with two more subclasses of the infinitely often K-trivial sets. Recall that f is a DNC (or diagonally non-computable) function if $f(e) \neq \varphi_e(e)$ for all e such that $\varphi_e(e)\downarrow$ (where (φ_e) is an effective enumeration of all partially computable functions).

Theorem 2.11. *If a set A does not compute a DNC function then it is i.o. K-trivial.*

Proof. Suppose that every function that is computed by A fails to be diagonally non-computable. Consider the function f which, given n it outputs $\langle A\restriction_n\rangle$ (i.e. a code of the first n bits of the characteristic function of A). Since $f \leq_T A$, it is not DNC. Therefore $\varphi_e(e)\downarrow = \langle A\restriction_e\rangle$ for infinitely many $e \in \mathbb{N}$. Consider the c.e. set V of strings which is defined as follows. For each $e \in \mathbb{N}$ wait until $\varphi_e(e)\downarrow$ and if the output is (a code of) a string of length e, enumerate it into V. Clearly for each $e \in \mathbb{N}$, the set V contains at most one string of length e. Hence by Lemma 2.6, $K(\sigma) \leq^+ K(|\sigma|)$ for all $\sigma \in V$. Also V contains infinitely many segments of A. Therefore A is i.o. K-trivial. □

Recall that by [DK87] (also see [DH09, Theorem 1.23.18]), if a set that is computed by a 1-generic then it does not compute a DNC function. Therefore Theorem 2.11 implies the following.

Corollary 2.12. *Every set that is computed by a 1-generic is i.o. K-trivial.*

Theorem 2.11 shows that there non-trivial lower cones in the Turing degrees that consist entirely of i.o. K-trivial sets. However i.o. K-trivial sets are not closed downward under Turing reducibility. Indeed, the halting set is i.o. K-trivial but it computes random sets.

3. Arithmetical complexity in the K-degrees

In this section we explore the definability restrictions in $\leq_K$-lower cones. A consequence of this analysis is that there is a Σ^0_2 set which forms a minimal pair with any (non-trivial) c.e. set in the K-degrees. We start with the following, which has an analog in the Turing degrees. Moreover, the proofs in the two cases are similar.

Theorem 3.1. *There exists a Σ^0_2 set $A >_K \emptyset$ such that $X \not\leq_K A$ for all Δ^0_2 sets $X >_K \emptyset$.*

Proof. We will enumerate A in a $\emptyset'$-computable construction, so that A is $\Sigma^0_1(\emptyset')$, hence Σ^0_2. To ensure that $A \not\leq_K \emptyset$ we need to meet the following requirements:

$$R_e:\ \exists n\ [K(A\restriction_n) > K(n) + e].$$

To ensure that A does not K-bound any non-trivial Δ^0_2 sets we meet the following:[7]

$$N_e:\ [\Phi^{\emptyset'}_e \text{ is total and } \Phi^{\emptyset'}_e \not\leq_K \emptyset] \Rightarrow \exists n\ [K(\Phi^{\emptyset'}_e\restriction_n) \not\leq K(A\restriction_n) + e].$$

[7]Notice that if $K(\Phi^{\emptyset'}_k\restriction_n) \leq K(A\restriction_n) + t$ for all n and some $k, t \in \mathbb{N}$, then there is some $e > t$ such that $\Phi_e = \Phi_t$. Hence $K(\Phi^{\emptyset'}_e\restriction_n) \leq K(A\restriction_n) + e$ for all n. So requirements N_e are sufficient.

Suppose that we only wish to satisfy a single N_e (and all R_i). We can compute a constant c such that $K(1^n) \leq K(n) + c$ for all $n \in \mathbb{N}$. Fix a Martin-Löf random sequence $Y \leq_T \emptyset'$. We can proceed by defining $A \restriction_s [s] = Y \restriction_s$ while constantly searching for some n such that $\Phi_e^{\emptyset'} \restriction_n \downarrow$ and $K(\Phi_e^{\emptyset'} \restriction_n) > K(n) + c + e$. If we find such a number n at stage s, we enumerate all numbers $\leq n$ into A thus meeting N_e. In this case, for the positions $> n$ of A we copy the corresponding digits of Y If the search does not halt during the stages s, N_e is satisfied trivially. In any case all R_i are met as A will be equal to Y apart from finitely many positions.

We combine these strategies for $N_e, e \in \mathbb{N}$ in order to construct A which satisfies all of these requirements. Each strategy N_e imposes a restraint $r_e[s]$ on A at stage s. Let $c_e[s]$ be a constant such that $K(Z_e[s] \restriction_n) \leq K(n) + c_e[s]$ for all $n \in \mathbb{N}$, where $Z_e[s] = (A \restriction_{r_{e-1}})[s] * 1^\omega$. Here '$*$' denotes concatenation and $r_{-1}[s] := 0$ for all s. Notice that $c_e[s]$ is computable from $(A \restriction_{r_e})[s]$. Set $r_e[0] = 0$ for all e. If $r_e[s+1]$ is not defined explicitly in the construction we have $r_e[s+1] = r_e[s]$. We say that N_e *requires attention at stage* $s+1$ if there exists $n \leq s$ such that $\Phi_e^{\emptyset'} \restriction_n \downarrow$ and $K(\Phi_e^{\emptyset'} \restriction_n) > K(n) + c_e[s] + e$.

Construction. At stage $s+1$ let $m_s = \max\{s, \max A[s]\}$, where $\max A[s]$ denotes the largest element in the finite set $A[s]$. Also, find the least $e < s$ such that N_e requires attention and is not currently declared *satisfied*. Enumerate all numbers n with $r_{e-1}[s] < n \leq m_s$ into A. Define

$$A[s+1] = (A \restriction_{r_{e-1}})[s] * 1^{m_s - r_{e-1}[s]} * Y \restriction_k$$

where k is the least number such that $K(A[s+1] \restriction_{m_s+k}) > K(m_s+k) + e$. Finally set $r_e[s+1] = m_s + k$, declare N_e *satisfied* and all $N_i, i > e$ *not satisfied*. If no $N_e, e < s$ requires attention let $A[s+1] = A[s] * Y(s)$, where $Y(s)$ is the sth digit of Y.

Verification. If only finitely many N_e require attention during the construction, $A = \sigma * Y$ for some finite string σ and r_i, c_i, $i \in \mathbb{N}$ reach a limit. Hence all R_e are satisfied. Moreover, if some N_e was not satisfied, it would require attention at some stage of the construction. Hence almost all (therefore, by the padding lemma, all) N_e are satisfied.

If infinitely many N_e require attention during the construction, infinitely many of them receive attention. If at some stage s requirement N_e receives attention and no $N_i, i < e$ receives attention after s, requirement R_e is satisfied (and remains so for the rest of the stages). Indeed, in that case $(A \restriction_{r_e})[s]$ will not change after stage s. Hence infinitely many R_e are met, which implies that all R_e are met.

Finally we show by induction on e that each N_e is satisfied and r_e, c_e reach a limit. Suppose that this holds for $e < k$ and let s_0 be a stage after which the values of r_e, c_e remain constant for all $e < k$. If N_k does not require attention after stage s_0, it is satisfied and r_e, c_e remain constant after s_0. Otherwise N_k will receive attention at some stage $s_1 > s_0$ and will be satisfied according to the action taken in the construction (the definition of $(A \restriction_{r_k})[s_1]$) and the fact that $A \restriction_{r_k}$ will be preserved from then on. Notice that N_k will not receive attention after stage s_1, thus r_k, c_k reach a limit at that stage. This concludes the induction step and the proof. $\square$

The following result improves the complexity of the minimal pair of K-degrees that was constructed in [MS07].

Corollary 3.2. *There is a Σ^0_2 set whose greatest lower bound with every Σ^0_1 set is* $\mathbf{0}$ *in the K-degrees.*

Proof. Let A be the Σ^0_2 set of Theorem 3.1 and B any c.e. set. By Propositions 2.2, 2.3 every set $X \leq_K B$ is Δ^0_2. Therefore, if $X \leq_K A$ by the choice of A the set X has to be K-trivial. $\square$

Notice that the argument we gave in the proof of Theorem 3.1 relativizes to $\emptyset^{(n)}$ for all $n > 0$, giving analogs on each level of arithmetical complexity. Hence the following.

Theorem 3.3. *Let $n > 1$. There exists a Σ^0_n set $A >_K \emptyset$ such that $X \not\leq_K A$ for all Δ^0_n sets $X >_K \emptyset$.*

As above, this gives the following application to the study of minimal pairs in the K-degrees.

Corollary 3.4. *Let $n > 1$. There exists a Σ^0_n set $A >_K \emptyset$ whose greatest lower bound in the K-degrees with any Δ^0_n infinitely often K-trivial set is* $\mathbf{0}$.

Proof. By Propositions 2.2, 2.3 the lower cone below an i.o. K-trivial Δ^0_n set consists entirely of Δ^0_n sets. Hence the Σ^0_n set of Theorem 3.3 has the desired properties. $\square$

Theorem 3.3 can be seen as a strong separation of the Σ^0_n classes from their predecessors Δ^0_n in the K-degrees. An immediate question is whether we can also separate Δ^0_n from Σ^0_{n-1} in the same way. In Section 4 we show the following.

Theorem 3.5. *Given any Δ^0_2 family of sets there exists a Δ^0_2 set whose K-degree is non-zero and does not bound any non-zero K-degree of a set in the family.*

Since the class of Σ^0_1 sets is a Δ^0_2 family, we get the following.

Corollary 3.6. *In the K-degrees, there is a Δ^0_2 non-zero degree that does not bound any Σ^0_1 nonzero degree.*

The above result is rather surprising as Σ^0_1 sets have relatively low initial segment complexity.

4. Proof of Theorem 3.5

Suppose that (X_e) is a uniformly $\emptyset'$-computable family of sets. To simplify the requirements, assume without loss of generality that each set in the family has infinitely many indices in this list. Let $X_e[s]$ be a computable system of approximations to the sets in the family. Then $K(X_e \restriction_n)[s]$ is a computable system of approximations to their initial segment complexities. For each e we will make sure that the following requirements are satisfied:

$$R_e: \quad \exists n \left[K(X_e \restriction_n) \not\leq K(A \restriction_n) + e\right] \quad \vee \quad \forall k \left[K(X_e \restriction_k) \leq^+ K(k)\right].$$

To test the K-triviality of X_e the construction will enumerate a c.e. set of strings V and use Lemma 2.6. We will make sure that for each n there is at most one string in V of length n. Hence by Lemma 2.6, the satisfaction of R_e follows from the satisfaction of the following modified requirement.

$$N_e: \begin{cases}\text{There exists a c.e. set } V \text{ as in Lemma 2.6 such that either for} \\ \text{some } n \text{ we have } K(X_e \upharpoonright_n) \not\leq K(A \upharpoonright_n) + e, \text{ or for all } \sigma \in V \text{ we} \\ \text{have } K(X_e \upharpoonright_{|\sigma|}) \leq K(\sigma) + e.\end{cases}$$

In the next section we give an atomic construction which, given e it uniformly produces $A \leq_T \emptyset'$ which is not K-trivial and, if $X_e \not\leq_K \emptyset$ then $X_e \not\leq_K A$. Although this is not used explicitly in the main construction of Section 4.2, it helps understanding the ideas involved.

4.1. **Strategy for one N_e.** To increase the complexity of A we use a Δ^0_2 random set Y with computable approximation $Y[s]$. Define $V = \{Y_s \upharpoonright_s \ \mid s \in \mathbb{N}\}$.

Let σ_s be the shortest string $\sigma \in V[s]$ such that $K(X \upharpoonright_{|\sigma|})[s] > K(\sigma)[s] + e$. If this does not exist, let $\sigma_s = Y_s \upharpoonright_s$. The *witness* of the strategy at stage s is defined to be the string $\sigma_s * (Y_s \upharpoonright_s)$. Below we show that the witnesses of the strategy in the various stages s converge to a unique inifinite binary sequence. We define A to be this very sequence.

The set A converges. One of the following must occur.

(a) The string σ_s reaches a (finite) limit τ.
(b) The length of σ_s tends to infinity.

Indeed, if (a) does not hold we have that $K(X \upharpoonright_{|\sigma|}) \leq K(\sigma) + e$ for all $\sigma \in V$. In this case each $\sigma \in V$ can only be chosen as σ_s finitely often. Therefore (b) must occur.

In the first case there exists some stage s_0 such that the witness of the strategy is $\tau * (Y_s \upharpoonright_s)$ for all $s > s_0$. In this case A converges to $\tau * Y$. In the second case the witnesses converge to Y. Therefore A is well defined in any case.

The set A satisfies N_e and is not K-trivial. Clearly A is defined uniformly from the index e, a Δ^0_2 index of X and $\emptyset'$. As explained above, in any case Y is a tail of A. Therefore A is not K-trivial. Finally, in case (a) we have $K(X \upharpoonright_{|\tau} |) > K(A \upharpoonright_{|\tau|}) + e$, since $\tau \subset A$. In case (b) we have that $K(X \upharpoonright_{|\sigma|}) \leq K(\sigma) + e$ for all $\sigma \in V$. By Lemma 2.6 we have that $K(\sigma) \leq^+ K(|\sigma|)$ for all $\sigma \in V$. Hence X is K-trivial. Therefore in any case the sets A, X satisfy N_e.

4.2. **Satisfying all N_e.** We will use a priority tree (the full binary tree) in order to construct A which meets all requirements. To make sure that A is not K-trivial we need to meet the following requirements.

$$P_e : \exists n \, [K(A \upharpoonright_n) > K(n) + e]$$

Strategies are identified with nodes on the tree. Each node on the tree is 2-branching with outcomes $0 < 1$. For a node that is associated with N_e, the outcome 0 corresponds to the belief that X_e is K-trivial while outcome 1 corresponds to the negation of this belief. Along with the (current) outcome, each node will have a primary and a secondary witness. The primary witness will be as in Section 4.1, associated with the satisfaction of N_e. The secondary witness will be an extension of the primary witness that is associated with the satisfaction of P_e. The secondary witnesses will play the role that Y played in Section 4.1, i.e. they will increase the initial segment complexity of the constructed set A. In the following whenever we refer to 'witnesses' of a strategy, we always mean both the primary and the secondary witness of it. Consider a computable partition of $\mathbb{N}$ into sets $\mathbb{N}^{[\alpha]}$ indexed by the strategies α. A node α will enumerate a c.e. set V_α containing strings of length in $\mathbb{N}^{[\alpha]}$. A strategy of length e on the leftmost infinitely often visited path (also called the true path) will run successfully and satisfy N_e, P_e.

In the following we define the outcomes and witnesses of the strategies during the stages of the construction. Fix a $\emptyset'$-computable function (in both arguments) $p_e(\sigma)$ which gives some $\tau \supset \sigma$ such that $K(\tau) > K(|\tau|) + e$. Also let $p_e(\sigma)[s]$ be a computable approximation to it.

At stage s a path δ_s of length s through the tree will be defined inductively, determining the 'visited nodes' at stage s. A β-stage is a stage s where β was visited, i.e. $\beta \subseteq \delta_s$. The root is the first visited node at each stage and the other visited nodes are determined by the current outcomes and witnesses of their predecessors. The outcome of a visited node α at stage s is 0 if $K(X_e \restriction_{|\sigma|})[s] \leq K(\sigma)[s] + e$ for all strings $\sigma \in V_\alpha[s-1]$ which extend the current witnesses of each $\beta \subset \alpha$. In this case the primary witness of α is equal to the union of the current witnesses of each $\beta \subset \alpha$. Otherwise the outcome is 1 and the primary witness of α is the shortest string $\sigma \in V_\alpha[s-1]$ which extends the current witnesses of all $\beta \subset \alpha$ and $K(X_e \restriction_{|\sigma|})[s] > K(\sigma)[s] + e$. In any case the secondary witness of α is defined to be $p_e(\sigma)[s]$, where σ is its primary witness. Finally, the parameters of a node α are only updated at the α-stages.

4.3. **Construction.** At stage s calculate the path δ_s of length s, starting from the root and following the current outcomes of the nodes. Pick the least number $n < s$ which is in some $\mathbb{N}^{[\alpha]}$ for $\alpha * 0 \subset \delta_s$ and such that there is no string of length n in V_α. Enumerate into V_α the least string of length n which is compatible with the current witnesses of δ_s. If such n does not exist, go to the next stage.

4.4. **Verification.** Since the branching of the priority tree is finite, there exists a leftmost infinitely often visited infinite path δ. Moreover, it follows from the definition of the outcomes and witnesses that at each stage s the witnesses of the initial segments of δ_s are linearly ordered.

Lemma 4.1. *Suppose that $\beta \subset \delta$ and α is the immediate predecessor of β. The witnesses of α reach a limit in the β-stages.*

Proof. The secondary witnesses are just the images of the primary witnesses under the Δ^0_2 function p. Therefore it suffices to show the lemma for primary witnesses. We do this by induction on the length of α. Suppose that it holds for all $\alpha \subset \delta$ of length $< n$ and σ is the union of the final values of the witnesses of these nodes in the $\delta \restriction_n$-stages. We show that it holds for $\alpha = \delta \restriction_n$. Let $\beta = \delta \restriction_{n+1}$. If $\alpha * 0 \subset \delta$ the primary witness of α in the β-stages has limit σ. Otherwise $\alpha * 1 \subset \delta$ which means that the primary witness of α will reach a limit τ (over all stages) such that $K(X_e \restriction_{|\tau|}) > K(\tau) + e$ for some $\tau \in V_\alpha$ and $e = |\alpha|$. $\square$

Given $\beta \subset \delta$, the *true witnesses of* the immediate predecessor of α are defined to be the limits of its witnesses in the α-stages. In Lemma 4.3 we will define A to be the union of these true witnesses. Moreover, the *true outcomes* of the nodes on δ are the outcomes that lie on δ.

Lemma 4.2. *Suppose that $|\alpha| = e$. If $\alpha * 1 \subset \delta$ then V_α is finite and $K(X_e \upharpoonright_{|\sigma|}) > K(\sigma) + e$, where σ is the final witness of α. If $\alpha * 0 \subset \delta$ then V_α contains a string of each length in $\mathbb{N}^{[\alpha]}$ and X_e is K-trivial.*

Proof. For the first clause notice that if $\alpha * 1 \subset \delta$ then the construction will stop enumerating into V_α after some stage. Therefore V_α is finite. Moreover, after some stage the primary witness of α will settle on the shortest string σ in V_α which extends the true witnesses of its predecessors and $K(X_e \upharpoonright_{|\sigma|}) > K(\sigma) + e$.

For the second clause suppose that $\alpha * 0 \subset \delta$. By the construction the set V_α contains a string of each length in $\mathbb{N}^{[\alpha]}$. By Lemma 4.1 the witnesses of the predecessors of α reach a limit in the α-stages. Let σ be the union of these witnesses. By construction, almost all strings in V_α will be extensions of σ. Hence, the fact that $\alpha * 0 \subset \delta$ implies that for almost all $\tau \in V_\alpha$ (in particular, all that extend σ) we have $K(X_e \upharpoonright_{|\tau|}) \le K(\tau) + e$. By Lemma 2.6 we have $K(\tau) \le^+ K(|\tau|)$ for all $\tau \in V_\alpha$. Hence $K(X_e \upharpoonright_k) \le^+ K(k)$ for almost all $k \in \mathbb{N}^{[\alpha]}$ and X_e is K-trivial. □

The following lemma is crucial, in that it enables us to define the set A and more importantly to ensure that it is Δ^0_2.

Lemma 4.3. *The strings enumerated into the sets V_α during the construction converge to a unique sequence A, which is the union of the true witnesses of the nodes on δ. In other words, for all $n \in \mathbb{N}$ there exists a stage s_0 such that all strings enumerated by the construction after stage s_0 are extensions of $A \upharpoonright_n$.*

Proof. Let $\beta \subset \delta$ be a node with true secondary witness σ which reaches a limit in the $\delta \upharpoonright_{|\beta|+1}$ stages at stage s_*. In the following all stages are assumed to be larger than s_* and the last stage where a node to the left of β was visited. Since β is an arbitrary initial segment of δ, the lemma is a consequence of the following.

Claim: There is a stage s_0 after which the only strings enumerated in the sets V_α (where α is a node on the tree) are extensions of σ. (4.1)

Claim 4.1 clearly holds for the nodes α that lie on the left of β. Indeed, in this case V_α is finite. By Lemma 4.1 it also holds for the nodes α that extend β and its true outcome. Indeed, in this case the strings enumerated in V_α must extend the current secondary witness of β, which reaches limit σ in the α stages. Finally it holds for the nodes $\alpha \subset \beta$ such that $\alpha * 1 \subset \delta$ since in this case by Lemma 4.2 V_α is finite. Hence it remains to show Claim 4.1 for the case where $\alpha \subseteq \beta$ and $\alpha * 0 \subset \delta$, or α is to the right of the true outcome of β. The latter case holds when α extends some $\eta * 1$ where $\eta \subseteq \beta$ and $\eta * 0 \subset \delta$.

In the latter case, the choice of these η implies that the length of their witnesses at stages s where $\delta_s \supset \eta * 1$ tends to infinity. So if we show that almost all strings of V_η extend σ, we have that at almost all stages s such that $\delta_s \supset \eta * 1$ the witnesses of η extend σ. From this it follows that beyond some stage, any string enumerated to some V_α for $\alpha \supset \eta * 1$ must extend σ.

Hence it remains to show Claim 4.1 for the particular case where $\alpha \subseteq \beta$ and $\alpha * 0 \subset \delta$. We prove this by finite induction. Let $\eta_0 \supset \cdots \supset \eta_t$ be the descending sequence of all strings $\eta \subseteq \beta$ such that $\eta * 0 \subset \delta$. Fix $i < t$, suppose that the claim holds for all η_j, $j < i$ and let ρ_j be the union of the true witnesses of the predecessors of η_j (for each $j < i$). Also let s_i be a stage beyond which we have $K(X_{e_j} \upharpoonright_{|\tau|}) \le K(\tau) + e_j$ (where $e_j = |\eta_j|$) for each $j < i$ and each string τ in V_{η_j}

which is an extension of ρ_j but not an extension of σ. By induction hypothesis there are finitely many such strings τ, so s_i exists.

If a string is enumerated in V_{η_i} at a stage $s > s_i$ then either δ_s extends the true outcome of β or $\delta_s \supset \eta_j * 1$ for some $j < i$. In the first case the enumerated string must be an extension of the witness σ of β. In the second case it must extend the a current witness of some η_j, $j < i$ where $\delta_s \supset \eta_j * 1$. According to the choice of s_j the current witness of η_j at stage s must extend σ. Hence in either case the enumerated string is an extension of σ. This concludes the induction, the proof of Claim 4.1 and the proof of the Lemma. □

Lemma 4.4. *The set A is Δ^0_2 and satisfies all N_e, P_e for $e \in \mathbb{N}$.*

Proof. Lemma 4.3 shows how to calculate A by asking Σ^0_1 questions. Hence A is Δ^0_2. Let $e \in \mathbb{N}$ and let α be the unique node on δ of length e. Also let σ be the true primary witness of α.

For P_e it suffices to show that there is some string $\tau \subset A$ such that $p_e(\tau) \subset A$. Clearly $\sigma \subset A$ and $p_e(\sigma)$ is the secondary witness of α. Hence $p_e(\sigma) \subset A$ and P_e is satisfied. For N_e suppose that X_e is not K-trivial. By Lemma 4.2 we have that $\alpha * 1 \subset \delta$ and $K(X_e \restriction_{|\sigma|}) > K(\sigma) + e$. By the definition of A we have $\sigma \subset A$ so N_e is satisfied. □

5. Gap functions for K-triviality

An interesting fact from [CM06] is the existence of a non-decreasing unbounded function that can replace the constant in the definition of K-triviality. In this section we isolate this notion and exhibit its role in the structure of the K-degrees. It is instructive to compare the results of this section with [MY10, Sections 3, 5] where a different notion of a 'gap function' plays a crucial role in analyzing the downward and upward oscillations of the initial segment prefix free complexity of random sets.

Definition 5.1. We say that $f : \mathbb{N} \to \mathbb{N}$ is a *gap function for K-triviality* if for each set X we have

$$\forall n\ [K(X \restriction_n) \leq^+ K(n) + f(n)] \iff X \text{ is } K\text{-trivial.} \tag{5.1}$$

Moreover, f is a *gap function for K-triviality of Δ^0_2 sets* if (5.1) holds for all Δ^0_2 sets X. An analogous definition holds for the other arithmetical classes.

If a set X satisfies the left hand side of (5.1), we say that it *obeys* f. Clearly the '$\Leftarrow$' of the equivalence in Definition 5.1 holds always. Csima and Montalbán [CM06] showed the following.

> There is a Δ^0_4 unbounded and non-decreasing gap function for K-triviality. (5.2)

The following result[8] from [CM06] shows a connection between gap functions of K-triviality and minimal pairs in the K-degrees. It also shows why the particular case of unbounded and non-decreasing gap functions is of special interest.

> Let f be any unbounded and non-decreasing gap function for K-triviality. Then $f \oplus \emptyset'$ computes two sets that form a minimal pair in the K-degrees. (5.3)

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[8] Statement (5.3) is not explicitly stated in [CM06], but can be extracted from a simple analysis of their argument.

Moreover we have the following converse of (5.3).

If X, Y form a minimal pair in the K-degrees, then

$$f(n) := \min\{K(X \restriction_n), K(Y \restriction_n)\} - K(n) \tag{5.4}$$

is a gap function for K-triviality.

The following fact is useful in Theorem 5.2.

If f is a Δ^0_2 non-decreasing unbounded function, then there is an unbounded non-decreasing function g which is approximable from above and such that $g(n) \leq f(n)$ for all $n \in \mathbb{N}$.[9] (5.5)

Proof of (5.5). Let $f(n)[s]$ be a computable approximation to f. Without loss of generality we can assume that for all stages s and all $n \leq m \leq s$ we have $f(n)[s] \leq f(m)[s]$. Let $g(n)[s] = \min\{f(n)[t] \mid n \leq t \leq s\}$ for each $n \leq s$. Clearly $g(n) = \lim_s g(n)[s]$ is Δ^0_2 and $g(n) \leq f(n)$ for all $n \in \mathbb{N}$. Also, g is non-decreasing. To show that it is unbounded, let $c, n, s_0 \in \mathbb{N}$ such that $f(n)[s] > c$ for all $s \geq s_0$. Clearly $g(s_0) > c$. $\square$

Case (a) in Theorem 5.2 is due to Frank Stephan (see [Nie09, Theorem 5.2.25]).

Theorem 5.2. *Suppose that $f : \mathbb{N} \to \mathbb{N}$ is Δ^0_2 and $\lim_n f(n) = \infty$. If f satisfies one of the following*

(a) *It can be computably approximated from above.*
(b) *It is non-decreasing.*

then there is a Turing complete c.e. set which obeys f. In particular, f is not a gap function for K-triviality of c.e. sets.

Proof. If g satisfies (a) and the assumptions of the theorem then the non-decreasing function $h(n) = \min\{g(i) \mid i \geq n\}$ also does and $h(n) \leq g(n)$ for all $n \in \mathbb{N}$. So g bounds a function satisfying (a), (b) and the assumptions of the theorem. Moreover by (5.5), any function satisfying (b) and the assumption of the theorem bounds a function with the same properties which also satisfies (a). Hence to prove the theorem it suffices to show that given any Δ^0_2 unbounded non-decreasing function which has an approximation from above, there is a Turing complete c.e. set which obeys it. Let f be such a function with an approximation $f(n)[s]$ from above. Construct a c.e. set A as follows.

At stage $s + 1$ find the least $n < s$ such that the number k of 0s in $A[s] \restriction_n$ is larger than $f(n)[s]$, and switch the $k - f(n)[s]$ largest 0-positions of $A \restriction_n [s]$ into 1s. Moreover, if m is the least number enumerated in $\emptyset'$ at this stage, switch the mth 0-position A into 1.

First, notice that the number of 0s in $A \restriction_n$ is $\leq f(n)$ for all $n \in \mathbb{N}$. Therefore $K(A \restriction_n) \leq^+ K(n) + f(n)$ since to describe $A \restriction_n$ we only need to know n and a string of length $f(n)$ indicating the digits in $A \restriction_n$ that are not 1 at the point of the construction where the number of 1s in the string $A \restriction_n$ is $f(n)$. Second, we show that $\mathbb{N} - A$ is infinite so that the standard coding of $\emptyset'$ into A that was performed above is valid. Given any $c \in \mathbb{N}$ let m_c be such that there are at least c 0s in $\emptyset' \restriction m_c$. Also let n_c be such that $f(n_c) > m_c$. Clearly the number of 1s in the string $A \restriction_{n_c}$ is at most $n_c - c$. $\square$

The following shows that the conditions in Theorem 5.2 are essential.

Proposition 5.3. *There is a Δ^0_2 function f such that $\lim_n f(n) = \infty$ and f is a gap function for K-triviality of Σ^0_1 sets.*

Proof. Let (W_e) be an effective list of all c.e. sets. We meet the following requirements.

$$R_e : \text{If } W_e \text{ obeys } f \text{ then } K(W_e \restriction_n) \le K(n) + e \text{ for almost all } n.$$

We say that R_e requires attention at stage s if $K(W_e \restriction_s) > K(s) + e$. Clearly this property is decidable in $\emptyset'$.

At stage s find the least $e \le s$ such that R_e requires attention and is not satisfied. Let $f(s) = e$ and say that R_e is satisfied. If there is no such e, let $f(s) = s$. It is easy to verify that all R_e are met, and from some stage on they are either satisfied or do not require attention. Moreover, since each R_e 'receives attention' at most once, $\lim_n f(n) = \infty$. □

The following contrasts Theorem 5.2.

Theorem 5.4. *There is a gap function for K-triviality of Δ^0_2 sets which is unbounded, non-decreasing and has Σ^0_2 degree.*

Proof. This is similar to the proof of Theorem 3.1, so we just give a sketch. To make f of Σ^0_2 degree it suffices to define a $\emptyset'$-computable approximation to it from above. We meet the following conditions:

$$N_e : \ [\Phi_e^{\emptyset'} \text{ is total and } \Phi_e^{\emptyset'} \not\le_K \emptyset] \Rightarrow \exists n\, [K(\Phi_e^{\emptyset'} \restriction_n) \not\le K(n) + f(n) + e].$$

If we had only one N_e to satisfy, we would define $f(n)[s] = n$ while searching (recursively in $\emptyset'$) for some $m > e$ and a stage t such that

$$\Phi_e^{\emptyset'}[t] \restriction_m\downarrow \quad \text{and} \quad K(\Phi_e^{\emptyset'} \restriction_m) > K(m) + 2e.$$

If and when m, t are found, we let $f(i)[t] = e$ for all $i \in [e, m]$ and continue as before, defining $f(n)[s] = n$ for $n \in (m, s]$ and $s > t$. In this case m is called a witness for N_e. In the global construction we make sure that N_e can only modify f on arguments that are larger than the largest witness that any $N_i, i < e$ may have. This ensures that f is approximated monotonically from above. So if $k_e[s]$ is the least number which is larger than any witness of $N_i, i < e$ at stage s and larger than e, strategy N_e at stage s looks for $m \in (k_e[s], s]$ such that

$$\Phi_e^{\emptyset'}[s] \restriction_m\downarrow \quad \text{and} \quad K(\Phi_e^{\emptyset'} \restriction_m) > K(m) + 2e. \tag{5.6}$$

If it finds such, it sets $f(i)[s] = e$ for all $i \in (k_e, m]$. Notice that there is no injury amongst different strategies. Moreover f is unbounded as each N_e acts at most once and never sets the values of f below e. For the same reason each N_e is 'receives attention' at some stage, or is trivially satisfied. If it lowers the values of f, it is satisfied by (5.6). □

Problem 5.5. Is there an unbounded non-decreasing gap function for K-triviality of Σ^0_2 or Δ^0_3 Turing degree?

Proposition 2.3 shows that if a function f has finite $\liminf$ then there is a bound for the complexity of all sets that obey it (they are computable from $f \oplus \emptyset'$). In particular, the class of sets that obey it is countable. By (5.2) the converse does not hold. However we have the following, which can be seen as a generalization of the fact from [YDD04, MY10] that the $\le_K$-lower cone below any random set is uncountable.

Theorem 5.6. *Suppose that for some $f : \mathbb{N} \to \mathbb{N}$ we have $\lim_n(f(n) - K(n)) = \infty$. Then the class of sets that obey f is uncountable. In particular, it contains a perfect pruned $\Delta^0_2(f \oplus \emptyset')$ tree.*

Proof. We use an oracle argument to construct a tree T as above. For each string σ there is a constant q_σ such that

$$K(\sigma 0^{m+n+1}) < K(m+n+1+|\sigma|) + q_\sigma \tag{5.7}$$
$$K(\sigma 0^n 1 0^m) < K(m+n+1+|\sigma|) + q_\sigma + K(n) \tag{5.8}$$

for all $n, m \in \mathbb{N}$. Indeed, q_σ codes the command 'make the first $|\sigma|$ digits of the output identical to those of σ' and the $K(n)$ in the second inequality is from a program instructing to make digit $n+1$ a 1.

If we look at T as a map from $2^{<\omega}$ to $2^{<\omega}$ (preserving comparability and incomparability relations), level k of the tree consists of the strings $T(\rho)$ for ρ of length k. The strings of level k will have the same length ℓ_k.

Suppose inductively that level k of the tree has already been defined, and for each string σ on that level the sequence $\sigma 0^\omega$ obeys f. Now find $n > \ell_k$ such that $f(i) > q_\sigma + K(i)$ for all $i \geq n$ and each σ on level k of T. For each such σ define its two successors in level $k+1$ to be $\sigma 0^n j$ for $j = 0, 1$. By (5.7), (5.8) we have that for each string τ of level $k+1$, the sequence $\sigma 0^\omega$ obeys f.

Since $\lim_n(f(n) - K(n)) = \infty$ all levels of T will be defined. By induction, all paths of T obey f. Moreover the construction was computable in $(f \oplus \emptyset')'$. □

A well known open problem in the LK degrees is a characterization of the oracles which have uncountable lower cones with respect to $\leq_{LK}$ (see [Bar10b, Mil09]). The same question can be asked about the K-degrees. Notice that by Propositions 2.2, 2.3 the cone below a c.e. set is always countable, while by Theorem 5.6 there are many sets with uncountable lower cone (including all random sets). Another way to ask the same question is the following.

Problem 5.7. Characterize the functions $f : \mathbb{N} \to \mathbb{N}$ with the property that the class of sets that obey them is countable.

Finally we would like to suggest that it may be interesting to study the connection between the functions we discussed in this section and the so-called Solovay functions that were studied in [BD09].

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