ON THE GAP BETWEEN TRIVIAL AND NONTRIVIAL INITIAL SEGMENT PREFIX-FREE COMPLEXITY

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Abstract. An infinite sequence X is said to have trivial (prefix-free) initial segment complexity if $K(X \restriction_n) \leq^+ K(0^n)$ for all n, where K is the prefix-free complexity and $\leq^+$ denotes inequality modulo a constant. In other words, if the information in any initial segment of it is merely the information in a sequence of 0s of the same length. We study the gap between the trivial complexity $K(0^n)$ and the complexity of a non-trivial sequence, i.e. the functions f such that

$$(\star) \quad K(X \restriction_n) \leq^+ K(0^n) + f(n) \text{ for all } n$$

for a non-trivial (in terms of initial segment complexity) sequence X. We show that given any Δ^0_2 unbounded non-decreasing function f there exist uncountably many sequences X which satisfy $(\star)$. On the other hand there exists a Δ^0_3 unbounded non-decreasing function f which does not satisfy $(\star)$ for any X with non-trivial initial segment complexity. This improves the bound Δ^0_4 that was known from [CM06]. Finally we give some applications of these results.

1. Introduction

It is an interesting idea to try to express computability, or equivalently definability, in terms of initial segment complexity. Chaitin [Cha76] did exactly this, when he proved that a set X is computable iff its initial segments have minimal (plain) Kolmogorov complexity. Let C denote the plain Kolmogorov complexity and $\leq^+$ denote inequality modulo a constant. For all strings σ we have $C(\sigma) \geq^+ C(0^{|\sigma|}) =^+ C(|\sigma|)$ (where $n =^+ m$ if $n \leq^+ m$ and $m \leq^+ n$). Chaitin showed that a set X is computable iff $C(X \restriction_n) \leq^+ C(n)$ for all $n \in \mathbb{N}$. Can we express or at least 'approximate' computability in terms of the prefix-free complexity K? Chaitin [Cha76] showed that every set X which has minimal prefix-free initial segment complexity, i.e. $K(X \restriction_n) \leq^+ K(n)$ for all $n \in \mathbb{N}$, is Δ^0_2. The sets that satisfy this condition are called K-trivial and are known to form an interesting proper subclass of Δ^0_2. Moreover there are non-computable K-trivial sets, so it seems impossible to characterize computability in terms of prefix-free complexity.

A natural question is, how large a margin can we allow above the minimal complexity $K(n)$ for the first n bits of a set in the above condition so that we still get a considerable restriction on the class of sets that satisfies it? For example, we may ask if there is an unbounded non-decreasing function f such that any set satisfying

$$K(X \restriction_n) \leq^+ K(n) + f(n) \quad \text{for all } n \in \mathbb{N} \tag{1.1}$$

is Δ^0_2, or even K-trivial.

Key words and phrases. Kolmogorov complexity, initial segment prefix-free complexity, K-triviality, low for Ω.

There is a constant c such that $K(\sigma) + c \geq K(0^{|\sigma|}) = K(|\sigma|)$ for all strings σ. Hence, modulo a constant, $K(n)$ is a lower bound of the complexity of any string of length n. A set X is K-trivial if it has the lowest possible initial segment complexity, namely $K(X \upharpoonright_n) \leq^+ K(n)$. Suppose that we are given a non-decreasing unbounded function $f : \mathbb{N} \to \mathbb{N}$. It is plausible that based on f one can construct a set X which is not K-trivial but $K(X \upharpoonright_n) \leq^+ K(n)+f(n)$ for all $n \in \mathbb{N}$. Intuitively, we would try to construct X such that the complexity of its first n bits increases when f is sufficiently large. Since $\lim_s f(s) = \infty$ one would hope that we can achieve $\lim(K(X \upharpoonright_n) - K(n)) = \infty$ so that X is not K-trivial.

Surprisingly, this is not the case. This was shown in [CM06] where an unbounded non-decreasing function f was constructed such that for each set X,

(1.2) If $K(X \upharpoonright_n) \leq^+ K(n) + f(n)$ for all $n \in \mathbb{N}$, then X is K-trivial.

Following [DH10, End of Section 10.12] an analysis of the proof shows that the function f is Δ^0_4. In [BV] it was shown that f cannot be Δ^0_2. In fact, it was shown that if f is Δ^0_2, unbounded and non-decreaing, then there exists a c.e. set X which is not K-trivial but $K(A \upharpoonright_n) \leq^+ K(n) + f(n)$ for all $n \in \mathbb{N}$. In Section 3 we use a result from [BS] in order to show that there is a Δ^0_3 unbounded non-decreasing function f satisfying (1.2).

Theorem 1.1. *There exists a Δ^0_3 unbounded non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ such that if $K(X \upharpoonright_n) \leq^+ K(n) + f(n)$ for all $n \in \mathbb{N}$ and some set X then X is K-trivial.*

The diagonalization employed in the proof of Theorem 1.1 (originally from [CM06]) is particularly interesting since it deals with all possible sequences X. A discussion for cases where the oracles X are restricted in a certain arithmetical class can be found in [BV, Section 5]. For example, the following facts were shown.

(a) There is a Δ^0_2 function f with $\lim_n f(n) = \infty$ such that (1.2) holds for all Σ^0_1 sets.

(b) There is no Δ^0_2 unbounded nondecreasing function f such that (1.2) holds for all Σ^0_1 sets.

(c) If $\lim_n(f(n) - K(n)) = \infty$ then there are uncountably many infinite sequences X satisfying (1.1).

Notice that it is not clear whether (a) holds for an arithmetical class which is larger than Σ^0_1. In general, one can ask if given a Δ^0_2 unbounded function f one can construct a set X which is not K-trivial but $K(X \upharpoonright_n) \leq^+ K(n) + f(n)$ holds for all $n \in \mathbb{N}$.

In Section 4 we show that if the unbounded nondecreasing function which gives an upper bound on the excess complexity that the first n bits of a sequence can have is Δ^0_2, then there are continuum many sequences meeting this condition. This result contrasts (a) above.

Theorem 1.2. *Let g be a Δ^0_2 unbounded nondecreasing function. There are uncountably many sets X such that $K(X \upharpoonright_n) \leq^+ K(n) + g(n)$ for all $n \in \mathbb{N}$. In fact, there is a non-empty perfect Π^0_1 class which consists entirely of such sets X.*

We would like to strengthen Theorem 1.2 so that the constructed Π^0_1 class does not have any K-trivial members. The reason for this is various applications that are based on basis theorems for Π^0_1 classes, as we explain below. Before we do this, it's worth considering if it is possible to obtain this strengthening without extra effort. In other words, if every perfect Π^0_1 class has a Π^0_1 subclass without K-trivial members. The answer is strongly negative. The following can be shown using standard methods in computability theory.

(1.3) There is a perfect Π^0_1 class such that every Π^0_1 subclass of it has computable members.

However, with considerable effort, it is possible to modify the proof of Theorem 1.2 so that we get the following stronger result.

Theorem 1.3. *The Π^0_1 class P of Theorem 1.2 can be chosen such that it has no K-trivial members.*

The value of this strengthening of Theorem 1.2 lies on the use of basis theorems for Π^0_1 classes in order to get sequences with certain computational properties with very low but non-trivial prefix-free complexity. We discuss this direction in Section 2, with a special attention on low for Ω sequences. Recall that Ω is the halting probability of a universal prefix-free machine. Also, given a Martin-Löf random sequence Y we say that X is low for Y if Y is Martin-Löf random relative to X.

Before we embark into a detailed discussion of our results, it seems appropriate to note a possible connection between our work and work on a different type of gap functions that were studied in [BD09]. For example they study functions $h : \mathbb{N} \to \mathbb{N}$ such that $K(X \upharpoonright_n) \geq^+ n - h(n)$ is a sufficient condition for a set X to be Martin-Löf random. These gap functions may refer to randomness or triviality, but are different than the ones that we study in this paper. Although we have not found a direct relation between these notions, there is an analogy in the two lines of research.

We finish this introduction with a word on terminology. In Sections 3, 4 and 5 we use the notion of a tree in the Cantor space. This can be defined in the following two different ways:

(i) As a downward $\subseteq$-closed set of strings.

(ii) As a partial map from strings to strings, which preserves compatibility and incompatibility relations.

Perfect trees correspond to total maps in clause (ii). For convenience, in Section 3 we refer to the first formulation while in Section 4 we refer to the latter one. Level n in a tree under (i) is the collection of strings of length n which belong to the tree. On the other hand, if $T : 2^{<\omega} \to 2^{<\omega}$, $\sigma \to T_\sigma$ is a tree under (ii), level n of T refers to the collection of the strings T_τ such that T_τ is defined and τ has length n. If in any level of T the strings have the same length (as will be the case in Section 4), this length is said to be *the height* of this level.

The *weight* of a prefix-free set S of strings is defined to be the sum $\sum_{\sigma \in S} 2^{-|\sigma|}$. The *weight* of a prefix-free machine M is defined to be the weight of its domain. Prefix-free machines are most often built in terms of *request sets*. A request set L is a set of tuples $\langle \rho, \ell \rangle$ where ρ is a string and ℓ is a positive integer. A 'request' $\langle \rho, \ell \rangle$ represents the intention of describing ρ with a string of length ℓ. We say that L is a *bounded request set* if $\sum \{2^{-|\ell|} \mid \exists \rho,\ \langle \rho, \ell \rangle \in L\} < 1$. This sum is the *weight of the request set L*. The Kraft-Chaitin theorem (see e.g. [DH10, Section 2.6]) says that for every bounded request set L which is c.e., there exists a prefix-free machine M such that for each $\langle \rho, \ell \rangle \in L$ there exists a string τ of length ℓ such that $M(\tau) = \rho$.

Finally, it is appropriate to view the (prefix-free) *initial segment complexity* of a sequence X as the function $n \to K(X\restriction_n)$. In this way, for example, Theorem 1.1 can be concisely stated as 'There exists a Δ^0_3 unbounded non-decreasing function such that any sequence whose initial segment complexity is $\leq^+ K(n) + f(n)$ is K-trivial'.

2. Applications of Theorems 1.2 and 1.3

2.1. **A gap between the finite and the uncountable.** We are interested in the cardinality of the sequences with initial segment complexity $\leq c \cdot K(n) + d$, where $c \geq 1$ is a real number and $d \in \mathbb{N}$. By the coding theorem (see e.g. [Nie09, Theorem 2.2.26]) if $c = 1$ and d is any integer there are finitely many such sequences. In this section we use Theorem 1.2 to show that for any $c > 1$ there exists $d \in \mathbb{N}$ such that there are continuum many sequences with initial segment complexity $\leq c \cdot K(n) + d$. First we need the following.

Lemma 2.1. *Let $c > 1$ be a real number. There exists a Δ^0_2 unbounded non-decreasing function f such that $K(n) + f(n) \leq c \cdot K(n)$ for all $n \in \mathbb{N}$.*

Proof. Let $q > 0$ be a rational number such that $q + 1 < c$. Then $K(n) + q \cdot K(n) \leq c \cdot K(n)$ for each $n \in \mathbb{N}$. Let $f(n)$ be the largest $t \in \mathbb{N}$ such that $t \leq q \cdot K(i)[s]$ for all $i \geq n$ and all $s \in \mathbb{N}$. Clearly f is computable from $\emptyset'$, hence Δ^0_2. Moreover since $K(i)[s]$ is non-increasing in s, for each $n \in \mathbb{N}$ the number $f(n)$ is the largest t such that $t \leq q \cdot K(i)$ for all $i \geq n$. Hence f is non-decreasing. Since $\lim_i K(i) = \infty$ it is also unbounded. By the definition of f we have $K(n) + f(n) \leq K(n) + q \cdot K(n) \leq c \cdot K(n)$ for all $n \in \mathbb{N}$. $\square$

By combining Lemma 2.1 and Theorem 1.2 we have the following.

Corollary 2.2. *Let $c > 1$ be a real number. For some $d \in \mathbb{N}$ there exist uncountably many sequences X with initial segment complexity $\leq c \cdot K(n) + d$.*

Notice that Corollary 2.2 improves the basic fact (c) that was mentioned in Section 1.

2.2. **Complexity of low for Ω sequences.** One advantage of obtaining an effective uncountable class in Theorems 1.2 and 1.3 is that we can use a variety of basis theorems in order to obtain sequences with certain computational properties and low but not trivial prefix-free complexity. For example, Theorem 1.3 shows that given any non-decreasing unbounded Δ^0_2 function g there is a computably dominated set whose initial segment prefix-free complexity is $\leq^+ K(n) + g(n)$ but not $\leq^+ K(n)$. Here we use the computably dominated basis theorem for Π^0_1 classes. Perhaps a more striking example is the following result about low for Ω sequences.

The applications in this section are focussed on the class of low for Ω sequences. Hence it is appropriate here to recall a few of the basic properties of this class. The low for Ω basis theorem says that every non-empty Π^0_1 class has a low for Ω member. It is an easy consequence of compactness and was shown in [RS10] and independently in [DHMN05]. Note that Theorem 2.4 below is a generalized version of the low for Ω basis theorem. All K-trivial sets are low for Ω by [HNS07]. In fact the K-trivial sets are the only Δ^0_2 low for Ω sets. This follows from the fact that all K-trivial sets are computable from Ω and the results in [HNS07]. For a presentation see [Nie09, Theorem 8.1.18].

Corollary 2.3. *Let g be an unbounded non-decreasing Δ^0_2 function. Then there exists a low for Ω sequence X which is not K-trivial and $K(X\restriction_n) \leq^+ K(n)+g(n)$ for all $n \in \mathbb{N}$.*

Proof. Consider the Π^0_1 class of Theorem 1.3 for the given g. By the low for Ω basis theorem it has a low for Ω member X. By the properties of the class, X is not K-trivial and satisfies the desired inequality. □

To obtain uncountably many sets as in Corollary 2.3 we need the following generalized version of the low for Ω basis theorem.

Theorem 2.4. *Let Z be a set and X be Z-random. Every nonempty $\Pi^0_1(Z)$ class contains a nonempty $\Pi^0_1(Z \oplus X)$ subclass class which consists of low for X sets.*

Proof. Let P be a $\Pi^0_1(Z)$ class and let (U_i) be a universal oracle Martin-Löf test. For each $i \in \mathbb{N}$ let V_i be the set of reals A which are in U_i^Y for all $Y \in P$. Clearly $\mu(V_i) < 2^{-i}$ and by compactness V_i is a $\Sigma^0_1(Z)$ class (uniformly in i). Therefore (V_i) is a Martin-Löf test relative to Z. Since X is random relative to Z, there is some $i_0 \in \mathbb{N}$ such that $X \not\in V_{i_0}$. This means that there are paths Y in P such that $X \not\in U^Y_{i_0}$. Let us denote the collection of these sets by Q. Clearly Q is a nonempty $\Pi^0_1(X \oplus Z)$ subclass of P. Since (U_i) was chosen universal, for any path $Y \in Q$ the set X is random relative to Y. □

If we let $Z = \emptyset$ and $X = \Omega$ in Theorem 2.4 we get the following. Notice that by the basic facts that we discussed above, if the given Π^0_1 class does not have any K-trivial members then the $\Pi^0_1[\emptyset']$ subclass given by Corollary 2.5 has no Δ^0_2 members.

Corollary 2.5. *Every nonempty Π^0_1 class contains a nonempty $\Pi^0_1[\emptyset']$ subclass which consists entirely of low for Ω sets.*

Now we are ready to argue for the following generalized version of Corollary 2.3.

Corollary 2.6. *Let g be an unbounded non-decreasing Δ^0_2 function. There exist uncountably many low for Ω sequences X which are not K-trivial and $K(X\restriction_n) \leq^+ K(n) + g(n)$ for all $n \in \mathbb{N}$.*

Proof. Consider the Π^0_1 class of Theorem 1.3 for the given g. Then use Corollary 2.5 to obtain a non-empty subclass P of it, which is $\Pi^0_1[\emptyset']$. Since the original class does not contain K-trivial sequences, so does P. As we recalled above, every Δ^0_2 low for Ω sequence is K-trivial. Hence P does not have Δ^0_2 members. Since it is a $\Pi^0_1[\emptyset']$ class it follows that it is perfect, hence uncountable. Finally, its members satisfy the desired properties since it is a subclass of the original class. □

We now wish to obtain a version of Corollary 2.3 for sequences that are not low for Ω. We need a basis theorem for Π^0_1 classes that establishes the existence of sequences that are not low for Ω. The following is the first step towards this basis theorem. We say that a countable class $\mathcal{C} =\subseteq 2^\omega$ is uniformly $\mathbf{0}'$-*computable* if it can be presented as $\{\Phi^{\emptyset'}_{f(e)} \mid e \in \mathbb{N}\}$, where f is a computable sequence of indices which correspond to total $\emptyset'$-computable functions.

Lemma 2.7. *Let $T : 2^{<\omega} \to 2^{<\omega}$ be a perfect Δ^0_2 tree and let $\mathcal{C} \subseteq 2^\omega$ be a uniformly $\mathbf{0}'$-computable class. Then there is a Δ^0_2 path of T which is not in $\mathcal{C}$.*

Proof. Let $\mathcal{C} = \{\Phi^{\emptyset'}_{f(e)} \mid e \in \mathbb{N}\}$, where f is a computable function and $\Phi^{\emptyset'}_{f(e)}$ is total for every $e \in \mathbb{N}$. Define a path A inductively as follows. If $A \restriction_n = \sigma$ is defined, let $A \restriction_{n+1}$ be $\sigma * i$, where i is chosen such that $T_{\sigma * i} \not\subseteq \Phi^{\emptyset'}_{f(n)}$. Clearly A and $T_A := \cup_{\sigma \subset A} T_\sigma$ are Δ^0_2 and $T_A \neq \Phi^{\emptyset'}_{f(e)}$ for all $e \in \mathbb{N}$. □

Now we can state a strong version of the promised basis theorem. Observe the contrast with the low for Ω basis theorem.

Corollary 2.8. *Every perfect Δ^0_2 tree contains a path which is not low for Ω. In particular, there is no perfect Π^0_1 class containing only low for Ω paths.*

Proof. Let $T \subseteq 2^{<\omega}$ be a perfect Δ^0_2 tree. The K-trivial sets form a $\emptyset'$-computable class by [Nie09, Theorem 5.3.28]. Apply Lemma 2.7 to get a Δ^0_2 path X of T which is not K-trivial. As we discussed above, every Δ^0_2 low for Ω set is K-trivial. Therefore the path X of T is not low for Ω. □

Finally we can show the following analogue of Corollary 2.3.

Corollary 2.9. *Let g be an unbounded non-decreasing Δ^0_2 function. Then there exists a sequence X that is not low for Ω and $K(X \restriction_n) \leq^+ K(n) + g(n)$ for all $n \in \mathbb{N}$.*

Proof. Consider the Π^0_1 class of Theorem 1.1 for the given g. Then use Corollary 2.8 to obtain a member X of it which is not low for Ω. Since X is a member of the class, it satisfies the desired inequality. □

3. Proof of Theorem 1.1

3.1. **Preliminary facts.** In this section we give a basic fact about the K-trivial sets, which is largely a consequence of the work done in [BS]. First, we need the following 'uniformity' lemma.

Lemma 3.1. *Given a $\emptyset'$-computable sequence (T_i) of trees with finitely many paths such that $T_i'' \leq_T \emptyset''$ uniformly in i, there is a $\emptyset''$-computable function f such that for each i the number $f(i)$ is a code for a finite set of indices t_j, $j < k_i$ such that there are exactly k_i paths through T_i and these are $\Phi^{\emptyset'}_{t_j}$ for $j < k_i$.*

Proof. Given i we show how to define $f(i)$ computably in $\emptyset''$. First we ask the cardinality k_i of $[T_i]$. This can be decided in $\emptyset''$, see [BS, Corollary 2.10]. Then we can search for k_i incomparable strings $\sigma_j, j < k_i$ of the same length, such that for each $j < k_i$ the subtree of T_i below σ_j has a unique infinite path. By the definition of k_i such strings exist. Moreover the check amounts to asking for a given string σ if for all levels ℓ above $|\sigma|$ there exists a level $n > \ell$ such that there exists exactly one string of level ℓ which extends σ and has an extension at level n.

This is a Π^0_2 question. Hence the condition can be checked computably in T_i''. Since $T_i'' \leq_T \emptyset''$ (uniformly in i) the strings can be found computably in $\emptyset''$. Once we determine σ_j, $j < k_i$ we can effectively obtain the indices $t_j, j < k_j$ as follows. Given $j < k_i$ we let t_j be the program that defines the unique path of T_i extending σ_j. This definition is sound since given a Δ^0_2 tree with a unique path we can effectively get a Δ^0_2 definition of it from the tree. □

For each $e \in \mathbb{N}$ fix T_e to be the set of strings σ such that $K(\sigma \restriction_i) \leq K(i) + e$ for each $i \leq |\sigma|$. Clearly the trees T_e, $e \in \mathbb{N}$ are uniformly Δ^0_2. Moreover for each $e \in \mathbb{N}$

the set $[T_e]$ consists of the finitely many K-trivial infinite sequences with constant e. By [BS, Corollary 3.4] there exists a uniformly c.e. sequence (Q_e) of trees and a constant c such that $[Q_e] = [T_e]$ and Q_e (as a set of strings) is K-trivial with constant $2e + c$ for each $e \in \mathbb{N}$. Moreover, given a constant via which a set Q is K-trivial one can $\emptyset''$-effectively obtain a reduction $Q'' \leq_T \emptyset''$ (see [BS, Proposition 3.6]). If we combine these facts with Lemma 3.1 we obtain the following.

Proposition 3.2. *There is a $\emptyset''$-computable function f such that for each i the number $f(i)$ is a code for a finite set of indices t_j, $j < k_i$ such that the K-trivial sequences with constant e are exactly the ones given by $\Phi_{t_j}^{\emptyset'}$ for $j < k_i$.*

Using Proposition 3.2 one can revisit the argument given in [CM06] and explicitly make sure that the function f of (1.2) is Δ^0_3. Instead we give a different, more direct presentation of this argument in the following section.

3.2. **Construction of f of Theorem 1.1.** Let us denote by $\mathcal{K}_e$ the class of K-trivial sequences with constant e. In the argument below we freely use the fact that:

$$\text{Given } X \leq_T \emptyset' \text{ and } e \in \mathbb{N} \text{ we can } \emptyset''\text{-computably decide if } X \in \mathcal{K}_e. \tag{3.1}$$

Here the set X is given in the sense of a reduction of it to $\emptyset'$. We define an increasing sequence (n_k) and let $f(t)$ be the least k such that $n_k \geq t$. Given k define $n_k > n_{k-1}$ to be the least number such that for each $e \leq k$:

- For all $X \in \mathcal{K}_{e+k+2} - \mathcal{K}_e$ there exists $i < n_k$ such that $K(X \upharpoonright_i) > K(i) + e$.
- If $k > e + 1$ and for some set X the least number i such that $K(X \upharpoonright_i) > K(i) + e$ is in $[n_{k-2}, n_{k-1})$ then there exists $j < n_k$ such that $K(X \upharpoonright_j) > K(j) + e + k$.

By Proposition 3.2 and (3.1) using $\emptyset''$ we can determine a large enough n_k satisfying the first condition. For the second condition, notice that by the previous step (the definition of n_{k-2}) if the least number i such that $K(X \upharpoonright_i) > K(i) + e$ is in $[n_{k-2}, n_{k-1})$ then we have that $X \notin \mathcal{K}_{e+k}$. Hence for each such set, the string $X \upharpoonright_{n_{k-1}}$ is not extendible in the tree T_{e+k}. Hence by König's lemma there exists a level ℓ in T_{e+k} at which no extendible string has $K(\sigma \upharpoonright_i) > K(i) + e$ for $i \in [n_{k-2}, n_{k-1})$. This level ℓ can be calculated using $\emptyset'$ and is lower bound for n_k satisfying the second condition. This concludes the definition of (n_k) and shows that $f \leq_T \emptyset''$.

Now suppose that some set X satisfies $K(X \upharpoonright_n) \leq K(n) + f(n) + e$ for some $e > 1$ and all $n \in \mathbb{N}$. For a contradiction, suppose that X is not K-trivial. So let t be the least number $> n_e$ such that $K(X \upharpoonright_t) > K(t) + e$. Let k be such that $t \in [n_{k-2}, n_{k-1})$. Then by the second condition of the definition of n_k there exists some $j < n_k$ such that $K(X \upharpoonright_j) > K(j) + e + k$. But this contradicts the fact that $K(X \upharpoonright_j) \leq K(j) + f(j) + e$ since $f(j) < k$. This concludes the proof that f satisfies (1.2) and is Δ^0_3.

4. Proof of Theorem 1.2

It was observed in [BV, Section 5] that if g is a Δ^0_2 non-decreasing unbounded function, then there exists an unbounded non-decreasing function f such that $f(n) \leq g(n)$ for all $n \in \mathbb{N}$ and it is approximable from above in the following way.

(4.1) There is a computable approximation $f[s] \to f$ with $n \to f(n)[0]$ being the identity, $f(i)[s] \leq f(j)[s]$ for all $i < j$, and for each s there exists a unique n such that $f(n)[s] \neq f(n)[s+1]$, in which case $f(n)[s+1] = f(n)[s] - 1$.

Therefore we may replace g with f in the statement of the Theorem 1.2. The parameters $f(n)[s]$ can be viewed as movable markers that can only move from right to left and their initial position is n. Moreover by 4.1 at most one marker can move at each stage, and each marker can only move by one position (i.e. decrease its value by 1). Now it suffices to define a perfect Π^0_1 tree T and a prefix-free machine M such that $K_M(X \restriction_n) \leq^+ K(n) + f(n)$ for all $n \in \mathbb{N}$ and all $X \in [T]$. Equivalently, its suffices to ensure that at each stage s

$$K_M(\sigma) \leq K(|\sigma|)[s] + f(|\sigma|)[s] \text{ for all } \sigma \text{ on } T[s] \text{ with } |\sigma| \leq s. \tag{4.2}$$

4.1. **Building the Π^0_1 class of Theorem 1.2.** We define an effective sequence of 1–1 maps $T[s] : 2^{<\omega} \to 2^{<\omega}$ which preserve the ordering and compatibility relations. These can be viewed as uniformly computable perfect trees, and we can consider the set of infinite paths through them:

$$[T[s]] = \{X \mid \forall n \exists \sigma \ (|\sigma| = n \ \wedge \ T_\sigma[s] \supseteq X \restriction n)\}$$

which is a Π^0_1 class. We will also ensure that $[T[s+1]] \subseteq [T[s]]$ for each $s \in \mathbb{N}$ and that $T_\sigma = \lim_s T_\sigma[s]$ exists for each $\sigma \in 2^{<\omega}$. Then the downward closure of the range of the map T is a Π^0_1 tree T and $[T] = \cap_s [T[s]]$ is a perfect Π^0_1 class, where T is the limit map $\sigma \to T_\sigma$. Level n of tree $T[s]$ consists of the nodes $T_\sigma[s]$ for $\sigma \in 2^n$.

Intuitively, the above formal description amounts to starting from a certain map $\sigma \to T_\sigma[0]$ and at each stage $s > 0$ moving the markers $T_\sigma[s-1]$ to possibly new positions (i.e strings) $T_\sigma[s]$. The movement of the markers T_σ will satisfy the following conditions at each stage.

(i) The map $\sigma \to T_\sigma[0]$ is the identity.
(ii) Compatibility and incompatibility relations are preserved.
(iii) Each new position is the position of some marker at the previous stage.
(iv) All nodes of a level of T have the same length.
(v) If some node T_σ moves, then all nodes of the same or larger levels move.
(vi) If a level of T moves at stage s, it moves to a number $\geq s$.

In order to meet (4.2) it suffices to control the enumeration of M-descriptions at stage s of the construction by the following clause.

(4.3) For all strings σ on $T[s]$ of length $\leq s$ such that $K_M(\sigma)[s-1] > K(|\sigma|)[s] + f(|\sigma|)[s]$ request an M-description of σ of length $K(|\sigma|)[s] + f(|\sigma|)[s]$.

Let $n_{-1}[s] = 0$ and for each $k \in \mathbb{N}$ let $n_k[s]$ be the least number such that $f(n_k)[s] > 2k$ and $n_k[s] > n_{k-1}[s]$. By (4.1) we have $n_k[0] = 2k + 1$.

4.2. **Ideal scenario.** Suppose that the approximation $f[s]$ to f was constant (hence f was computable). Then each $n_k[s]$ would also be constant in s and $k \to n_k$ would be computable. In that case it would suffice to let T be any computable tree such that T_σ has length $n_{|\sigma|}$ (in particular, the markers T_σ do not move). Indeed, in that case the weight of the domain of M would be at most

$$\sum_k \sum_{|\sigma|=k} \Big(\sum_{i=|T_{\sigma^-}|}^{|T_\sigma|-1} u_i \cdot 2^{-f(i)} \Big) \leq \sum_k 2^k \Big(\sum_{i=n_{k-1}}^{n_k-1} u_i \cdot 2^{-2k} \Big) \leq \sum_i u_i < 1$$

where σ^- is the predecessor of σ and u_i is the weight of the descriptions of i of the universal machine.[1] Also if σ is the empty sequence then (by convention) T_{σ^-} is the empty sequence. In this ideal scenario, each node T_σ is responsible for the weight of the M-descriptions that are issued for strings between T_{σ^-} and T_σ. By the choice of $k \to n_k$ and since $|T_\sigma| = n_{|\sigma|}$ we have that the weight for which T_σ is responsible is at most $2^{-|\sigma|}q_\sigma$, where q_σ is the weight of the U-descriptions of numbers between $|T_{\sigma^-}|$ and $|T_\sigma|$. Since for each k there are 2^k nodes T_σ with $|\sigma| = k$ it follows that the total weight of all M-descriptions is bounded. This way of making nodes responsible for certain weight of M-descriptions will be useful in Section 4.3.

4.3. **Real scenario.** We will modify the argument of Section 4.2 to deal with the real possibility that the 'markers' $f(k)[s]$ may move to smaller numbers during the stages s. We will allow the revision of the positions of the markers T_σ which define the tree T as discussed above. The movement of the markers T_σ and the enumeration of M will follow the prescriptions given in Section 4.1, as well as the following condition.

(4.4) At each stage s the height of the kth level of $T[s]$ is $\geq n_k[s]$ for $k < s$.

Notice that (4.4) was implicit in Section 4.2. Now the main challenge is to bound the weight of the domain of M. The weight of the requested M-descriptions will be distributed to the markers T_σ as follows. Each T_σ is responsible for

(a) The M-descriptions of strings between the final position of T_{σ^-} and the final position of T_σ.

(b) The M-descriptions of strings ρ which were on $T[s]$ at some stage s, $\rho \supset T_\sigma[s]$ but at $s+1$ level $|\sigma|$ was the least to move, and moved so that ρ is no longer on $T[s+1]$.

Before we state the actual construction, we show that in any construction of $\sigma \to T_\sigma[s]$ which satisfies (i)-(vi) of Section 4.1, condition (4.4) and any enumeration of a prefix-free machine M which is defined according to (4.3) the weight of the requests for M-descriptions is bounded.

Lemma 4.1. *If a computable sequence of maps $\sigma \to T_\sigma[s]$ meets conditions* (i)-(vi) *of Section* 4.1 *and condition* (4.4)*, then a prefix-free machine M that is enumerated according to* (4.3) *has bounded domain.*

[1] Notice that the final bound holds even if we merely defined n_k to be the least number such that $f(n_k) > k$. The stronger condition $f(n_k) > 2k$ that we required in the definition of n_k will be useful in the main argument, in Section 4.3.

Proof. Let A_σ contain the M-descriptions under clause (a) above. Notice that A_σ is empty unless T_{σ^-} reaches a limit. Also notice that the sets A_σ may not be uniformly (in σ) computably enumerable, although $T_\sigma[s]$ is uniformly computable in σ and s. Let B_σ be the c.e. set of M-descriptions that are attributed to T_σ under clause (b). Notice that any M-description that is issued must either fall under clause (a), or under clause (b). Hence it suffices to show that the sets $\cup_\sigma A_\sigma$ and $\cup_\sigma B_\sigma$ (where σ ranges over all strings) have bounded weight.

By (4.4), the argument of Section 4.2, the definition of $k \to n_k[s]$ and the definition of M by (4.3) shows that the weight of $\cup_\sigma A_\sigma$ is bounded. For $\cup_\sigma B_\sigma$ fix a string σ. As in Section 4.1 let u_i be the weight of the descriptions of i of the universal machine. Let $s_1, s_2, \dots$ be the stages (finitely many or infinitely many) where enumerations into B_σ occurred and $s_0 = 0$. These are typically the stages where level $|\sigma|$ was the least to move. Fix $i > 0$. Following the calculation of Section 4.2 adapted to the snapshot at stage s of the restriction of T on extensions of σ, the weight of the descriptions that where enumerated in B_σ at s_i, $i > 0$ is at most $2^{-2|\sigma|} \cdot \sum_{j=s_{i-1}}^{s_i-1} u_j$. In this calculation we use (4.4), property (vi) and the fact that at stage s only strings of length $\leq s$ may have M-descriptions. Hence

$$\mathtt{wgt}(B_\sigma) \leq \sum_i \Big(2^{-2|\sigma|} \cdot \sum_{j=s_{i-1}}^{s_i-1} u_j\Big) \leq 2^{-2|\sigma|} \cdot \sum_i u_i < 2^{-2|\sigma|}.$$

Since there are 2^i strings of length i we have

$$\mathtt{wgt}(\cup_\sigma B_\sigma) \leq \sum_i \Big(\sum_{\sigma\in 2^i} \mathtt{wgt}(B_\sigma)\Big) \leq \sum_i \Big(\sum_{\sigma\in 2^i} 2^{-2|\sigma|}\Big) \leq \sum_i 2^{-i} \leq 2.$$

Since both $\cup_\sigma A_\sigma$ and $\cup_\sigma B_\sigma$ have bounded weight, so does the domain of M. $\square$

Now we are ready to give the formal construction of $\sigma \to T_\sigma[s]$ and M and verify the desired properties.

4.4. **Construction.** At stage $s+1$ do the following:

(I) If there is some $k \leq s$ such that $n_k[s+1] > n_k[s]$, pick the least one and let t be the least number $> n_k[s+1]$ such that the t-th level of $T[s]$ is $> s$ (i.e. $|T_\rho[s]| > s$ if $|\rho| = t$). Then move level k of T to the current level t as follows. For each σ of length k let $T_\sigma[s+1]$ equal to $T_{\sigma^*}[s]$ where $\sigma^* = \sigma * 0^{t-k}$. Also, let $T_{\sigma*\eta}[s+1] = T_{\sigma^* *\eta}[s]$ for all strings η.

(II) For all strings σ on $T[s+1]$ of length $< s$, if $K_M(\sigma)[s] > K(|\sigma|)[s+1] + f(|\sigma|)[s+1]$ request an M-description of σ of length $K(|\sigma|)[s+1] + f(|\sigma|)[s+1]$.

4.5. **Verification.** It is clear that the sequence of maps $\sigma \to T_\sigma[s]$ that we define in the construction meets conditions (i)-(vi) of Section 4.1. Moreover it satisfies (4.4) and the machine M is enumerated according to (4.3). By Lemma 4.1 the requested M-descriptions have bounded weight. Hence by the Kraft-Chaitin theorem there is a prefix-free machine that gives descriptions as requested in the construction. We express this fact by saying that M is a prefix-free machine.

Clause (II) of the construction explicitly ensures (4.2). It remains to show that the markers $T_\sigma[s]$ reach a limit. i.e. they are eventually permanently defined. We do this by induction on the levels of the trees $T[s]$. By the hypothesis on f and the definition of $n_k[s]$, for each k the sequence $(n_k[s])_{s\in\mathbb{N}}$ converges. Let t_k be the modulus of convergence of $(n_k[s])_{s\in\mathbb{N}}$. We show that for each $n \in \mathbb{N}$, level n of $T[s]$ reaches a limit with respect to s. By construction level 0 of T will reach a limit by stage t_0. Assume that all levels $< k$ have reached a limit by stage m and $m_* = \max\{m, t_k\}$. By construction level k of T will reach a limit by stage m_*. This concludes the induction step and the verification of the construction.

5. Proof of Theorem 1.3

A perfect Π^0_1 class is constructed by introducing 'splits' along every path in the class. That is, we make sure that each path splits into two paths at infinitely many lengths. In order to construct a Π^0_1 class which does not contain any K-trivial paths one has to introduce 'clumps' in the tree instead of mere splits. By choosing the 'clumps' large enough, we can be sure that they contain strings of appropriately high Kolmogorov complexity. This is based on the following well known and widely used fact (e.g. see [Bar10a, Theorem 2.6] or [DG08]).

(5.1) There exists a computable function $f(e,n)$ such that for all $e,n \in \mathbb{N}$ and any string σ of length n there exists an extension τ of σ of length $f(e,n)$ such that $K(\tau\restriction_i) > K(i) + e$ for some $i < |\tau|$.

Hence removing the ones that have low complexity (a computably enumerable event) leaves us with a non-empty class with the desired property. However this rather crude method is not compatible with the argument of Section 4. Indeed, in that argument the nodes T_σ of the constructed Π^0_1 tree T were given as limits of their computable approximations $T_\sigma[s]$. If we tried to implement a strategy based on (5.1) we would have to ask for larger and larger 'clumps' above T_σ, each time this marker moves. Hence the sums in the calculations of Section 4 would no longer be bounded, even if one considers modifications of the function $k \to n_k$.

The solution is a more dynamic approach which is compatible with keeping the size of the 'clumps' above each movable node *constant*. We present the argument as an extension of the proof of Section 4.[2] In particular, we define a computable sequence of trees $\sigma \to T_\sigma[s]$ that satisfy (i)-(vi) of Section 4. Moreover in order to ensure (4.2) we build a machine M which operates according to (4.3). Since we wish to obtain a Π^0_1 class with no K-trivial paths, we will enumerate a c.e. set Q of strings such that

(5.2) $[T]-[Q]$ is non-empty and does not contain any K-trivial sequence

where for a set of strings S we let $[S] = \{X \mid \exists\sigma \in S,\ \sigma \subset X\}$. As we explained in Section 4 the set of infinite paths $[T]$ of T will be a Π^0_1 class. Hence $[T] - [Q]$ is a Π^0_1 class.

5.1. **Additional requirements and strategy.** To make sure that $[T] - [Q]$ does not contain any K-trivial sequences we add a set of parameters ℓ_e to the construction of Section 4 and satisfy the following additional requirements for all $e \in \mathbb{N}$.

(5.3) If a string σ on level ℓ_{e+1} of T has an extension in $[T] - [Q]$ then $K(\sigma\restriction_i) > K(i) + e$ for some $i < |\sigma|$.

Intuitively, (5.3) says that by level ℓ_{e+1} all paths of T have been 'revealed' to be not K-trivial with constant e. In order to satisfy these conditions we will need to

[2] Alternatively one can construct a suitable Π^0_1 class as an effectively closed set in the Cantor space (e.g. see [BLS08, Theorem 7]) or as the set of extensions of a 0-1 partial computable function (e.g. see [BLS08, Theorem 9]).

build an additional prefix-free machine N in order to gain partial control of $K(t)$, $t \in \mathbb{N}$ (i.e. establish certain inequalities of the type $K(t) < m$). By the recursion theorem we may use the index $c > 0$ of N in our construction. Then

$$K(t) \leq K_N(t) + c \text{ for all } t \in \mathbb{N}. \tag{5.4}$$

For each e we have a strategy such that all these strategies together will make sure that the requirements of (5.3) are fulfilled. These strategies enumerate strings into Q and requests into N as follows. Let $\ell_0 = 0$ and $\ell_{e+1} = \ell_e + 2e + c + 3$. Thus every string at level ℓ_e of the tree T has 2^{2e+c+3} extensions at level ℓ_{e+1}. Moreover this holds at every stage s. A basic feature of the construction (that was not present in the argument of Section 4) is that only levels ℓ_e can cause changes in the approximation to the tree.

Each strategy for (5.3) works in cycles. A cycle of the strategy corresponding to e will be interrupted upon a movement of a marker-node T_η for $|\eta| \leq \ell_e$. Notice that this is an interaction of the strategy with the original construction of Section 4. Such events may be considered as *injuries* of the strategy. We will make sure that for each strategy for (5.3) they occur finitely often. The strategy corresponding to e is committed to keep the weight of the N-requests that it issues to at most 2^{-e-2}. In order to keep track of the weight that it adds in the domain of N with its requests, the strategy has a parameter $b_e[s]$. We let $b_e[0] = e + 4$. Each time the strategy is injured, the value of b_e increases by 1. Let $t_e[s]$ be the height of level ℓ_e on $T[s]$ plus $2e + c + 3$. Notice that the height of level ℓ_{e+1} is $\geq t_e[s]$.

A cycle of the strategy corresponding to e starts at a stage $s + 1$ by ensuring that $K(t_e[s]) < b_e[s] + c$. It does this by enumerating an N-description of $t_e[s]$ of length $b_e[s]$. It continues as follows, as long as the strategy is not injured. If $K(\eta)[t] \leq b_\sigma[s] + c + e$ at some later stage t for some η of length $t_e[s]$ on $T[s]$, it enumerates η into Q unless it is the last extension on $T[s]$ with that length of a node of $T[s]$ of level ℓ_e such that $[\eta] \cap [T[s]] - [Q[s]] \neq \emptyset$. In the latter event the cycle finishes. When the cycle finishes at some stage k, the strategy starts a new cycle by moving the nodes T_σ with $|\sigma| = \ell_e$ to strings on $T[k]$ of length $> k$ that do not have an initial segment in $Q[k]$ (unless $T_\sigma[s]$ already had a prefix in $Q[s]$). This is possible since at each stage s during the cycle, each string of level ℓ_e that does not have a prefix in $Q[s]$ has an infinite extension in $[T[s]] - [Q[s]]$.

Notice that there are three possible outcomes for a cycle of a sub-strategy.

(a) The cycle may be interrupted by injury.

(b) The cycle may finish.

(c) The cycle may never finish or be injured.

5.2. **Calculating the N-weight that is produced by a strategy.** We wish to obtain an upper bound on the weight of the N-requests that a strategy issues in the course of the construction. Every such request is issued during a cycle of the strategy. Moreover exactly one request is issued within a cycle of the strategy. First we consider the requests that were issued in a cycle that was either injured or never finished. In the latter case, note that no more cycles will occur. Since the parameter b_e increases by 1 upon each injury, we have the following bound on the N-weight that is attributed to the cycles that were either injured or never finished.

$$\sum_j 2^{-b_e[0]-j} = \sum_j 2^{-e-4-j} = 2^{-e-3}.$$

For the calculation of the weight of the requests that were issued during a cycle which finished we have to argue in a different way. If such a cycle begins at stage $s+1$, it adds weight $2^{-b_e[s]}$ to the weight of the domain of N and by the end of it at least 2^{2e+c+3} strings of length $t_e[s]$ have U-descriptions of length $\leq b_e[s]+c+e$. In other words, for each such increase on the weight of N we can count an increase in the domain of U which is 2^{e+3} times larger. Since $\mathtt{wgt}(U)<1$ the total weight of the requests that are issued during such cycles is bounded by 2^{-e-3}. Adding the two kinds of N-weight increases, we get the following.

(5.5) The weight of the N-requests of the strategy corresponding to e is at most 2^{-e-2}.

Hence $\sum_e 2^{-e-2}=2^{-1}$ is a bound for the total weight of the requests of N that are produced in the construction.

5.3. **Construction.** We use the definitions and conventions of the argument of Section 4. For example, the map $\sigma \to T_\sigma[0]$ is the identity. Let $b_e[0]=e+4$ and $t_e[0]$ be the height of level ℓ_e in $T[0]$ plus $2e+c+3$. At stage $s+1$, by "move level ℓ_e to level t" we mean the following.

(5.6) For each σ of length ℓ_e choose the leftmost extension ρ_σ of σ of length t such that either $T_\sigma[s]$ has a prefix in $Q[s+1]$ or $T_{\rho_\sigma}[s]$ does not have a prefix in $Q[s+1]$. Let $T_\sigma[s+1]=T_{\rho_\sigma}[s]$. Also, let $T_{\sigma*\eta}[s+1]=T_{\rho_\sigma*\eta}[s]$ for all strings η. For all $i>e$ let $b_i[s+1]=b_i[s]+1$ and for all $i\geq e$ let $t_i[s+1]$ be the height of level ℓ_e in $T[s+1]$ plus $2e+c+3$.

The strategy corresponding to e *requires attention* at some stage $s+1$ when there is an extension η of length $t_e[s]$ of $T_\sigma[s]$ for some σ of length ℓ_e with the property $K(\eta)[s]\leq b_\sigma[s]+c+e$ and $\eta\notin Q[s]$.

At even stages $s+1$ do the following:

(EI) *Ensure* (4.4). If there is some $k\leq s$ such that $n_k[s+1]>n_k[s]$, pick the least one and let t be the least number $\geq n_k[s+1]$ such that $|T_\rho[s]|>s$ for all ρ of length t. Let e be the largest number such that $\ell_e\leq k$. Then move level ℓ_e to level t and set $b_e[s+1]=b_e[s]+1$.

(EII) *Enumerate requests into M.* For all strings σ on $T[s+1]$ of length $\leq s$, if $K_M(\sigma)[s]>K(|\sigma|)[s+1]+f(|\sigma|)[s+1]$ request an M-description of σ of length $K(|\sigma|)[s+1]+f(|\sigma|)[s+1]$.

At odd stages $s+1$ do the following

(OI) *Enumerate requests into N.* For all e such that $t_e[s]\leq s$ and no N-description of $t_e[s]$ of length $b_e[s]$ has been enumerated yet, enumerate an N-description of $t_e[s]$ of length $b_e[s]$.

(OII) *Enumerate into $Q[s+1]$.* Let e be the least number such that $t_e[s]\leq s$ for which the strategy corresponding to e requires attention. If no such e exists, end this stage. Otherwise let Q_* be the union of $Q[s]$ and the strings η of length $t_e[s]$ in $T[s]$ with the property $K(\eta)[s]\leq b_e[s]+c+e$. If for each σ of length ℓ_e

$$\text{either } T_\sigma[s] \text{ has a prefix in } Q_* \text{ or } [T_\sigma[s]]\cap[T[s]]-[Q_*]\neq\emptyset$$

let $Q[s+1]=Q_*$. If not, let P be the set of strings ν of level ℓ_{e+1} that are the leftmost extension of $T_\sigma[s]$ for some σ with $|\sigma|=\ell_e$ such that either $T_\sigma[s]$ has a prefix in Q_* or $[T[s]]\cap[\nu]-[Q_*]\neq\emptyset$. Let $Q[s+1]=Q_*-P$, let t be the least number such that the height of level t is $> s$ and move level ℓ_e to level t.

5.4. **Verification.** In order to make a precise application of the recursion theorem, at this point we may view c as an arbitrary parameter of the construction (not necessarily an index of N).

A basic feature of the construction is that for all e, all σ of length ℓ_e and all stages s, either $T_\sigma[s]$ has a prefix in $Q[s]$ or $[T_\sigma[s]] \cap [T[s]] - [Q[s]] \neq \emptyset$. This property holds at stage 0 since $Q[0] = \emptyset$ and is preserved inductively throughout the construction via Step OII, Step EI and (5.6). In particular, (5.6) is justified as a way of moving the nodes of the tree (i.e. the conditions that it asks for the new positions of the nodes can be satisfied). As a consequence, since we never enumerate any prefix of $T_\emptyset$ into Q, we have $[T] - [Q] \neq \emptyset$.

Now one may view the construction as a computable function which takes c and returns a program for N (or rather the request set associated with N). By the recursion theorem we may choose c to be an index of N. After these necessary justifications we verify the desired properties of the set nonempty $[T] - [Q]$ in a series of lemmas.

Lemma 5.1. *There is a prefix-free machine M with the specification given in the construction.*

Proof. According to the justification above, the construction defines a computable sequence of maps $\sigma \to T_\sigma[s]$. A simple inspection of the construction shows that this sequence meets conditions (i)-(vi) of Section 4.1. The same holds for condition (4.4) restricted to the even stages (since only at even stages descriptions are enumerated into M this restriction is allowed). The request set for the prefix-free machine M that is enumerated in the construction follows (4.3). By Lemma 4.1 the requests for M have bounded weight. This shows that the specification of M given in the construction corresponds to an actual prefix-free machine. $\square$

Lemma 5.2. *There is a prefix-free machine N with the specification given in the construction.*

Proof. The argument of Section 5.2 applies to the construction and shows that the weight of the requests for N is finite. The lemma follows by the Kraft-Chaitin theorem. $\square$

Notice that the markers T_σ may move for two reasons. One is the original strategies of Section 4 (Clause (EI) of the construction) and the other is the additional strategies (Clause (OII) of the construction). The reason that the first movement stops is that the approximations to f converge. The second movement stops because the additional strategy corresponding to a level can only conclude a certain number of cycles, as we argued in Section 5.2. The proof of the following fact requires the combination of these arguments, in an induction.

Lemma 5.3. *Each movable marker T_σ reaches a limit.*

Proof. If levels move then the least level to move is level ℓ_e for some e, so it suffices to show that all levels ℓ_e reach a limit. Inductively assume that by stage s_0 all markers T_η with $|\eta| < \ell_e$ have reached a final value. We show that all markers T_ρ for $|\rho| = \ell_e$ reach a limit by some later stage. Let $s_1 > s_0$ be a stage after which n_{ℓ_e} remains constant.

After stage s_1 the strategy corresponding to e will not be 'injured'. In terms of the analysis of Section 5.1, each time it moves after stage s_1 it completes a cycle, while b_e remains constant. According to the same analysis, each time it completes a cycle after stage s_1, at least $2^{e+3-b_e[s_1]}$ additional weight can be counted in the domain of U. Alternatively, at least $2^{-b_e[s_1]}$ additional weight can be counted in the domain of N. Since the weight of the domain of a prefix-free machine is < 1, after stage s_1 the marker T_σ can only move at most $2^{b_e[s_1]}$ times. This concludes the induction step. □

Notice that (4.2) holds by the explicit action of step (EII) of the construction. Hence it remains to show that the additional strategies succeed in eliminating the K-trivial paths from $[T] - [Q]$.

Lemma 5.4. *There are no K-trivial paths in $[T] - [Q]$.*

Proof. It suffices to show that for each $X \in [T] - [Q]$ and for each $e \in \mathbb{N}$ there exists some $i \in \mathbb{N}$ such that $K(X \upharpoonright_i) > K(i) + e$. Let σ be a string of length ℓ_e such that $T_\sigma \subset X$ (where T_σ refers to the final value of $T_\sigma[s]$). Also using Lemma 5.3, let s_0 be a stage after which level ℓ_e does not move. This means that $t_e[s]$ reaches a limit t_e after s_0. If $K(X \upharpoonright_{t_e}) \leq K(t_e) + e$, by the choice of c (as a code for machine N, see (5.4)) we would also have $K(X \upharpoonright_{t_e})[s] \leq b_e[s] + c + e$ at some even stage $s \geq s_0$. At that stage the strategy corresponding to e would require attention. Since $X \upharpoonright_{t_e}$ was not enumerated into Q, according to Step (EII) of the construction $X \upharpoonright_{t_e}$ was the last extension of $T_\sigma[s]$ that was not in Q and thus T_σ would move. This contradicts the choice of stage s_0. This contradiction shows that $K(X \upharpoonright_{t_e}) > K(t_e) + e$. □

This concludes the proof of Theorem 1.3. We note that the ideas that were elaborated in this section can be used in a number of other arguments in order to obtain Π^0_1 classes which are free of K-trivials. This is especially useful in cases where the crude strategy that is based on (5.1) is not applicable. As an example we mention the main result in [Bar10b] about the LK degrees. We say that $A \leq_{LK} B$ if $K^B(\sigma) \leq^+ K^A(\sigma)$ for all strings σ. In other words, if oracle B can compress at least as efficiently as A. In [Bar10b] it was shown that for every Δ^0_2 set B which is not K-trivial, there exist uncountably many sets X such that $X \leq_{LK} B$; in fact, a perfect Π^0_1 class of such sequences. Using the strategy that was elaborated in Section 5.1 one can ensure the existence of such a class with no K-trivial members. The value of such a strengthening is again the use of basis theorems. For example, by the low for Ω basis theorem that we discussed in Section 2.2, we have that for every Δ^0_2 set $X >_{LK} \emptyset$ there exists a (in fact, uncountably many) low for Ω set Z such that $\emptyset <_{LK} Z <_{LK} X$.

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