CHAITIN'S HALTING PROBABILITY AND THE COMPRESSION OF STRINGS USING ORACLES

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ABSTRACT. If a computer is given access to an oracle—the characteristic function of a set whose membership relation may or may not be algorithmically calculable—this may dramatically affect its ability to compress information and to determine structure in strings which might otherwise appear random. This leads to the basic question, "given an oracle A, how many oracles can compress information at most as well as A?"

This question can be formalized using Kolmogorov complexity. We say that $B \leq_{LK} A$ if there exists a constant c such that $K^A(\sigma) < K^B(\sigma) + c$ for all strings σ, where K^X denotes the prefix-free Kolmogorov complexity relative to oracle X. The formal counterpart to the previous question now is, "what is the cardinality of the set of $\leq_{LK}$-predecessors of A?"

We completely determine the number of oracles that compress at most as well as any given oracle A, by answering a question of Miller [Mil10, Section 5] which also appears in Nies [Nie09, Problem 8.1.13]; the class of $\leq_{LK}$-predecessors of a set A is countable if and only if Chaitin's halting probability Ω is Martin-Löf random relative to A.

1. Introduction

Kolmogorov complexity is a fundamental notion which has found applications in topics as diverse as combinatorics, language recognition, information distance, thermodynamics and chaos theory. The basic idea behind this approach to quantifying the degree of randomness of a finite binary string, is that a string is simple or non-random if it has a short description relative to its length. Kolmogorov [Kol65] formalized this idea using the theory of computation. In this context, Turing machines play the role of our idealized computing devices, and we assume that there are Turing machines capable of simulating any calculational process which proceeds in a precisely defined and algorithmic manner. Programs can be identified with binary strings. A string τ is said to be a description of a string σ with respect to a Turing machine M if this machine halts when given program τ and outputs σ. Then the complexity of σ with respect to M is the length of its shortest description with respect to M.

When we come to consider randomness for infinite strings it becomes important to consider machines whose domain satisfies a certain condition which is true of the words in any reasonable language. The machine M is called *prefix-free* if it has prefix-free domain (which means that no program for which the machine halts and gives output is an initial segment of another). It can be shown that there

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exist *universal* prefix-free machines U, i.e. machines which give optimal complexity for all strings, modulo a constant. This means that for each prefix-free machine M there exists a constant c such that $K_U(\sigma) < K_M(\sigma) + c$ for all finite strings σ. Hence the choice of the underlying optimal machine does not change the complexity distribution significantly and the theory of prefix-free complexity can be developed without loss of generality, based on a fixed underlying optimal prefix-free machine U.

In order to define randomness for infinite sequences, we consider the complexity of all finite initial segments. A finite string σ is said to be *c-incompressible* if $K(\sigma) \geq |\sigma| - c$, where $K = K_U$. Chaitin and Levin defined an infinite binary sequence X to be random if there exists some constant c such that all of its initial segments are c-incompressible.[1] This definition of randomness for infinite sequences is then independent of the choice of underlying universal machine, and coincides with other definitions of randomness given in terms of computable betting strategies and also the definition given by Martin-Löf in [ML66] (a result of Schnorr, see Chaitin [Cha75]). Strings which are random in this sense are called Martin-Löf random. The coincidence of the randomness notions resulting from these different approaches may be seen as evidence of a robust and natural theory.

If we allow the underlying optimal prefix-free machine access to an oracle A, the resulting complexity K^A will often be reduced. Thus the use of external information often allows for better compression of strings and can be used in order to determine structure in sequences that would otherwise appear random. The following basic question then arises:

Informal question. Given an oracle A, how many oracles can compress strings at most as well as A?

Formally, an oracle X compresses strings at most as well as A if there exists some constant c such that $K^A(\sigma) \leq K^X(\sigma) + c$ for all strings σ. This relation was introduced by Nies [Nie05] and is denoted $X \leq_{LK} A$. The formal counterpart to the informal question above now becomes:

(1.1) Given an oracle A, what is the cardinality of $\{X \mid X \leq_{LK} A\}$?

It's also a natural question as to whether the ability of an oracle to compress random strings is essentially the same as its ability to compress strings in general. In the same paper that he introduced the $\leq_{LK}$ relation, Nies also discussed a simple variation; $X \leq_{LR} A$ if all sets which are Martin-Löf random relative to A are also Martin-Löf random relative to X. In [KHMS10] it was shown that $\leq_{LR}$ is identical to $\leq_{LK}$, and so our solution to Question 1.1 also gives the solution to the corresponding question for the $\leq_{LR}$ relation.

A short history of the literature surrounding Question 1.1 can be found in [Bar10]. The special case when $A = \emptyset$ was Question 4.4 in [ASK00] (stated in terms of $\leq_{LR}$). The main motivation for asking this question at the time was the then recent discovery that there are non-computable oracles X such that $K(\sigma) \leq^+ K^X(\sigma)$ for all strings σ (where $\leq^+$ denotes $\leq$ modulo a constant). Such sets X (identifying sets, their characteristic functions and infinite binary strings) are of no use in the task of compressing information and are known as *low for K* (see [Nie09, Section

[1] It is at this point that it becomes important that we restrict to the case of prefix-free machines—if we did not then it can be shown that according to this definition there would be *no* random sequences.

5.1]). Nies [Nie05] answered this question by showing that if A is computable then the class of (1.1) is contained in Δ^0_2 and hence is countable (a set is Δ^0_2 iff it is computable in Turing's halting problem—for an introduction to the effective hierarchies we refer the reader to the excellent book of Hinman [Hin78]). On the other hand, in [BLS08] it was shown this class is uncountable if A is the halting problem. Miller [Mil10] used a variation of the notion of low for K sets to exhibit a large class of oracles A for which the class of (1.1) is countable. He called an oracle A *weakly low for K* if $K(\sigma) \leq^+ K^X(\sigma)$ for infinitely many strings σ. In this paper he showed that if A is weakly low for K then the class of (1.1) is countable. He also gave a characterization of the class of weakly low for K sets in terms of the halting probability $\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$. In [Mil10] it was shown that a set B is weakly low for K if and only if Ω is Martin-Löf random relative to B. The sets B with the latter property are known as *low for Ω*. These results prompted the following question.

(1.2) **Question in** [Mil10]**:** If A is not low for Ω, is the class of (1.1) uncountable?

In this paper we give an affirmative answer, thus providing a characterization of the sets with uncountably many $\leq_{LK}$-predecessors—a set has uncountably many (and so continuum many) predecessors iff it is not low for Ω.

Theorem 1.1. *Let $A \subseteq \mathbb{N}$. Then the following are equivalent.*

(a) $\lim_n(K(n) - K^A(n)) = \infty$.

(b) There are uncountably many $X \subseteq \mathbb{N}$ for which there exists $c \in \mathbb{N}$ such that $K^A(n) \leq K^X(n) + c$ for all $n \in \mathbb{N}$.

Hence a set is weakly low for K iff the set of its $\leq_{LK}$-predecessors is countable.

Notice that the first clause of Theorem 1.1 (under a standard identification of strings with numbers) means that A is not weakly low for K (or equivalently, A is not low for Ω). This theorem unifies a number of older results. For example, in [Bar10] the first author showed the following.

(1.3) If A is Δ^0_2 and not low for K then the class of (1.1) uncountable. Moreover it has a Π^0_1 perfect subclass.

A class is perfect if it does not contain any isolated points according to the Cantor topology. Since every Δ^0_2 weakly low for K set is already low for K, (see [Nie09, Exercise 8.1.11]) the first part of (1.3) can be seen as a special case of Theorem 1.1. We note, however, that the latter (or its proof) does not imply the second clause of (1.3). In retrospect, (1.3) from [Bar10] can be seen as an 'effectivization' of Theorem 1.1, in the same way that the construction of a maximal set is an 'effectivization' of the construction of a cohesive set.

The advantage of the effective nature of (1.3) (the fact that we obtain an effectively closed uncountable set) is that it lends itself to the application of basis theorems for Π^0_1 classes. For example, the low for Ω basis theorem (from [RS10] and independently [DHMN05]) says that every non-empty Π^0_1 class contains a low for Ω path. As was demonstrated in [BB10], the proof of (1.3) can be augmented so as to establish the following.

(1.4) If A is Δ^0_2 and not low for K then the class of (1.1) contains a perfect Π^0_1 class without low for K paths.

Another result from [BB10] is the following generalization of the low for Ω theorem.

(1.5) Every nonempty Π^0_1 class contains a nonempty $\Pi^0_1[\emptyset']$ subclass which consists entirely of low for Ω sets.

This implies that that every nonempty Π^0_1 class without any low for K members contains uncountably many low for Ω paths. Indeed, in that case, the $\Pi^0_1[\emptyset']$ class that is given by (1.5) cannot have isolated paths since these would be Δ^0_2 and so low for K (given that they are also low for Ω). We can now use these observations to deduce the following fact about the LK-degrees, the degree structure that is induced by the pre-order $\leq_{LK}$. Notice that $A \equiv_{LK} B$ (denoting $A \leq_{LK} B$ and $B \leq_{LK} A$) informally means that A and B have the same strength with respect to compressing strings. An LK-degree is Δ^0_2 if it contains a Δ^0_2 set.

Corollary 1.2. *Every Δ^0_2 non-zero LK-degree bounds uncountably many LK-degrees with countably many predecessors.*

The proof is a straightforward combination of (1.4), (1.5) and the result from [Mil10] that if A is low for Ω then the class $\{X \mid \forall\sigma\ K^A(\sigma) \leq^+_{LK} K^X(\sigma)\}$ is countable.

Given that there have been a number of previous attempts by different people to answer Question 1.2, it seems appropriate that we outline the ideas behind the proof of Theorem 1.1, especially the new ingredient that made it possible. We do this in Section 2, as well as providing some preliminaries and notation for the proof that is given in Section 3.

2. About the proof of Theorem 1.1

2.1. **Preliminaries.** Let U be the universal prefix-free oracle machine which underlies the prefix-free complexity K^X relative to oracles X. Hence $K^X = K^X_U$ for all sets X. This machine is optimal in the sense that given any other prefix-free oracle machine M there is a constant c such that $K^X(\sigma) \leq K^X_M(\sigma) + c$ for all strings σ and oracles X. We let $\subseteq$, $\subset$ denote the prefix or the subset relation (with equality or not) depending on the context. The oracle machine U can be seen as a computably enumerable (c.e.) set of triples $\langle\tau,\rho,\sigma\rangle$ which indicate that U with τ written on the oracle tape, and with input program ρ, halts and produces σ. We also write $U^\tau(\rho) = \sigma$ in order to denote that this relation holds. If X is a set we let $U^X = \{\langle\rho,\sigma\rangle \mid \exists\tau \subset X,\ \langle\tau,\rho,\sigma\rangle \in U\}$ and for a string η we let $U^\eta = \{\langle\rho,\sigma\rangle \mid \exists\tau \subseteq \eta,\ \langle\tau,\rho,\sigma\rangle \in U\}$. The fact that U^X is prefix-free for all X can be expressed by the following condition;

(2.1) If $\langle\tau_i,\rho_i,\sigma_i\rangle \in U$ for $i = 0, 1$ and $\tau_0 \subseteq \tau_1$ then $\rho_0|\rho_1$

where $\rho_0|\rho_1$ denotes the incomparability of the two strings with respect to the prefix relation. Without loss of generality we can fix a computable enumeration of U such that any triple $\langle\tau,\rho,\sigma\rangle$ appearing in U at stage s has $|\tau| = s$. In this way for each string η the set U^η is finite and the map $\eta \to U^\eta$ is computable.

The *weight* of a prefix-free set S of strings, denoted $\mathtt{wgt}(S)$, is defined to be the sum $\sum_{\sigma\in S} 2^{-|\sigma|}$. The *weight* of a prefix-free machine M^X is defined to be the weight of its domain and is denoted $\mathtt{wgt}(M^X)$. Without loss of generality we assume that $\mathtt{wgt}(U^X) < 2^{-4}$ and that all strings in the domain of U^X begin with 1, for all sets X.

Prefix-free machines are most often built in terms of *request sets*. A request set L is a set of tuples $\langle\rho,\ell\rangle$ where ρ is a string and ℓ is a positive integer. A 'request'

$\langle\rho,\ell\rangle$ represents the intention of describing ρ with a string of length ℓ. We say that L is a *bounded request set* if $\sum\{2^{-|\ell|} \mid \exists\rho,\ \langle\rho,\ell\rangle\in L\} < 1$. This sum is the *weight of the request set* L and is denoted by $\mathtt{wgt}(L)$. The Kraft-Chaitin theorem (see e.g. [DH10, Section 2.6]) says that for every bounded request set L which is c.e., there exists a prefix-free machine M such that for each $\langle\rho,\ell\rangle\in L$ there exists a string τ of length ℓ such that $M(\tau)=\rho$. The same holds when L is c.e. relative to an oracle X, giving a machine M^X. In Section 3 we freely use this method of construction without explicit reference to the Kraft-Chaitin theorem.

A real number $0\le r<1$ is called c.e. if it is the limit of a non-decreasing computable sequence of rational numbers. For each set X we define $\Omega^X := \mathtt{wgt}(U^X)$. Notice that this definition is compatible with the definition of the halting probability Ω that was discussed above since $\Omega=\Omega^{\emptyset}$. Similarly we let $\Omega^\eta := \mathtt{wgt}(U^\eta)$ for any string η. The map $X\to\Omega^X$ is called the Ω operator and plays a crucial role in Section 3. By the conventions we adopted earlier concerning U and its computable enumeration, we have that the map $\eta\to\mathtt{wgt}(U^\eta)$ is computable. Our assumptions about U^X also mean that $\Omega^X<2^{-4}$ for all oracles X.

Finally, a *tree* T is a partial map $\sigma\to T_\sigma$ from strings to strings which preserves compatibility and incompatibility relations between strings, and which has downward closed domain. For any σ, the image T_σ is called a *node* of the tree. The *level* of a node T_σ is $|\sigma|$. An infinite binary sequence is a path through a tree if infinitely many of its initial segments are nodes of the tree. The set of infinite paths through a tree T is denoted by $[T]$. A tree which is a total function may also be called *perfect*.

2.2. **Informal ideas behind the proof.** In [Mil10] it was shown that a set is weakly low for K if and only if it is low for Ω, and that weakly low for K sets have only countably many $\le_{LK}$-predecessors. Therefore it suffices to show that if a set A is not low for Ω then it has uncountably many $\le_{LK}$-predecessors. The first construction of an uncountable lower $\le_{LK}$-cone was presented in [BLS08] (in terms of $\le_{LR}$). The proof of (1.3) in [Bar10] used new ideas in order to implement such a construction below any Δ^0_2 set which is not low for K. This proof relied entirely, however, on computable approximations. The argument that pointed to the possibility of implementing (a version of) the construction from [BLS08] below an arbitrary set which is not low for Ω, was the proof in [Mil10] that the class of low for Ω sets coincides with the class of weakly low for K sets. In this argument Miller showed how to use short descriptions of Ω in order to improve the overall compression of programs by any given constant. This is why Question (1.2) was asked, sometimes in the form of a conjecture.

Given an oracle A which is not low for Ω, the basic plan is to use an A-computable construction to build a prefix-free machine M^A and an approximation to a perfect tree T such that $K^A_M(\sigma)\le K^X(\sigma)$ for all strings σ and all $X\in[T]$. Since any perfect tree has continuum many paths, this certainly suffices to give the result. The basic obstacle is also clear—the machine M^A has to simulate *all* machines U^X for $X\in[T]$, but we must keep the weight of the domain of M^A bounded. In order to achieve this, we wish to ensure that where M^A has to simulate the descriptions given by a large number of strings in T (corresponding to a high level in T), the weight of these descriptions is relatively small. Why should we be able to achieve such a goal? For each m and each string ρ there exists $\sigma\supset\rho$ such that $\mathtt{wgt}(U^\tau)-\mathtt{wgt}(U^\sigma)<2^{-m}$ for all $\tau\supseteq\sigma$ (see [BLS08, Section 4]). If we were armed with an oracle for $\emptyset'$ then we could simply find the string σ when required and the construction of T would be relatively simple. Now we do not have an oracle for $\emptyset'$ but we still wish to use the fact that A is not low for Ω in order to try and identify strings ρ such that:

$$\max\{\texttt{wgt}(U^{\tau}) - \texttt{wgt}(U^{\rho}) \mid \rho \subseteq \tau\} \text{ is appropriately small for all } \tau \supseteq \rho. \tag{2.2}$$

How can we make use of the fact that A is not low for Ω? If A can compress initial segments of Ω, then in fact it can do the same for the initial segments of any c.e. real. Indeed, by [Sol75] (see [DH10, Sections 8.1, 8.2]) if B is a c.e. real then $K(B\restriction_n) \leq^+ K(\Omega\restriction_n)$ for all numbers n. So if we approximate X (which is a potential path through T) in such a way that Ω^X is a c.e. real, then A will be able to compress initial segments of Ω^X. We shall see in a moment how this is useful to us in constructing T. Here it is important to note that the apparent obstacle to such an approach is that we cannot allow the approximation to X to make any use of the oracle A. If it were to make use of this oracle then we would have no guarantee that Ω^X will be a c.e. real, and we would need A to be able to provide short descriptions of Ω^A rather than Ω, which clearly is not possible. Thus the *key new idea in the argument of Section 3* is to incorporate into the A-computable construction auxiliary procedures which proceed in a computable fashion and do not make any use of the oracle A.

In order to help us define the paths through T we wish to computably approximate sets X with the property that:

$$\text{For all } m \text{ there exists } n \text{ such that } \texttt{wgt}(U^{\tau}) - \texttt{wgt}(U^{X\restriction_n}) < 2^{-m} \text{ for all } \tau \supset X\restriction_n. \tag{2.3}$$

First of all let us consider a simplified way of approximating sets X of this kind. Then we shall have to modify this method slightly in order to ensure that our approximation satisfies some further conditions.

Definition 2.1. Given a finite or infinite sequence X, the Ω-sequence of X is the sequence (n_i), where n_i is the least number such that $\Omega^X - \Omega^{X\restriction_{n_i}} < 2^{-i}$.

So suppose that we wish to approximate X extending τ. At stages $s \leq |\tau|$ let $X_s = \tau\restriction_s$. At stage $s+1 > |\tau|$ let $(n_i[s])$ be the Ω-sequence of X_s. Check to see if there is some $i < s$ and a string $\tau' \supseteq X_s\restriction_{n_i[s]}$ of length $\leq s+1$ extending τ such that $\Omega^{\tau'} - \Omega^{X_s\restriction_{n_i[s]}} \geq 2^{-i}$. If so, then pick the least such i, and for this choice the least such τ'. Define X_{s+1} to be the concatenation of τ' with $s+1-|\tau'|$ zeros. Otherwise let $X_{s+1} = X_s * 0$.

Since we shall subsequently modify the details of this approximation, we will not yet verify the details precisely. It should be clear, however, that the approximation to X given by this construction converges, that Ω^X is a c.e. real, and that if we let (n_i) be the Ω-sequence of X then for all $B \supseteq X_s\restriction_{n_i[s]}$, $\Omega^B - \Omega^{X_s\restriction_{n_i[s]}} < 2^{-i}$. So now suppose that at some point in the process of approximating T, we have defined $T_{\sigma'}$ for all $\sigma' \subset \sigma$. Imagine that we wish to define T_σ to be some initial segment of a set X which is approximated according to a construction like the one above. We have to decide how long this initial segment should be, i.e. where we should aim to start putting further branching in T. Since A is not low for Ω, for any b we can find ρ and t such that $U^A(\rho) = \Omega^X\restriction_t$ and $|\rho| \leq t-b$. So, as we computably approximate X, we also use the oracle for A to try and search for a string ρ of this kind. When we find ρ which compresses the initial segment of Ω^{X_s} of length t, we can temporarily define T_σ to be $X_s \restriction_{n_t[s]}$. If $U^A(\rho)$ is not an initial segment of Ω^X then we will eventually realize this, we can change our mind about T_σ and then there is no harm done—ρ simply corresponded to an incorrect guess as regards Ω^X. If on the other hand $U^A(\rho)$ *is* an initial segment of Ω^X, then $X_s \restriction_{n_t[s]}$ is an initial segment of X and for all $B \supseteq X_s \restriction_{n_t[s]}$, $\Omega^B - \Omega^{X_s \restriction_{n_t[s]}} < 2^{-t}$. Roughly speaking then, since $|\rho| \leq t-b$ there is sufficient room above ρ to simulate the machines corresponding to b-many paths extending $X_s \restriction_{n_t[s]}$. So it is reasonable to but a branching into T here.

While this provides the basic idea, what we have said so far is not quite correct. In the situation above, when $U^A(\rho)$ is an initial segment of Ω^X, it *will* be the case that $X_s \restriction_{n_t[s]}$ is an initial segment of X and that for all $B \supseteq X_s \restriction_{n_t[s]}$, $\Omega^B - \Omega^{X_s \restriction_{n_t[s]}} < 2^{-t}$, while the measure of all strings extending ρ is at least 2^{b-t}. The slight complication is that just as we had to approximate T_σ, all the values $T_{\sigma'}$ for $\sigma' \supset \sigma$ will also have to be approximated. As we approximate T we do not know which nodes we shall subsequently have to change our mind about, and thus in practice we have to simulate the machines corresponding to all strings which appear to be in T at any stage, and not just those corresponding to the nodes in the final version of T. We therefore need to approximate X in such a way that we successfully coordinate our need to satisfy (2.3), while at the same time limiting the cost incurred by our changing approximation to X and the corresponding T_σ. We now formally describe the computable subroutine which defines the appropriate approximation.

2.3. **The computable subroutine of the construction.** Given inputs $\sigma \in 2^{<\omega}$ and $e \in \omega$, the following lemma provides an algorithm which produces a computable approximation (X_s) which converges to an infinite extension X of σ such that the 'low for K' cost

$$\begin{array}{ll} \sum_s \{c_K(n, s+1) \mid n \text{ is the least such that } X_s(n)\downarrow \neq X_{s+1}(n)\downarrow\} \\ \text{where } c_K(n, s+1) = \Omega^{X_s\restriction_s} - \Omega^{X_s \restriction_n} \end{array} \tag{2.4}$$

that is associated with (X_s), is at most 2^{-e}. According to the standard 'cost function method' of constructing low for K sets (e.g. see [Nie09, Proposition 5.3.34]) the set X is low for K. In Lemma 2.2 we establish some additional details concerning (X_s), which play a crucial role in the proof of Theorem 1.1.

Lemma 2.2. *For each string σ and each $e > 0$ there exists an infinite binary extension X of σ and a sequence of numbers (n_t) such that Ω^X is a c.e. real, $\Omega^X - \Omega^{X\restriction_{n_t}} < 2^{-t}$ and $\Omega^\rho - \Omega^{X\restriction_{n_t}} < 2^{e-t}$ for all $\rho \supset X \restriction_{n_t}$ with $\rho \supset \sigma$ and $t \in \mathbb{N}$. In fact, there is a computable function $h : 2^{<\omega} \times \mathbb{N} \times \mathbb{N} \to 2^{<\omega}$ such that $|h(\sigma, e)[s]| = s$ for all $s \in \mathbb{N}$, and such that if $(n_i[s])$ denotes the Ω-sequence of X_s then:*

- *the strings $h(\sigma, e)[s] := X_s$ tend to a set X as $s \to \infty$;*
- *(Ω^{X_s}) is a non-decreasing sequence tending to Ω^X;*
- *For each $i \in \mathbb{N}$ the sequence $(n_i[s])_{s\in\mathbb{N}}$ is non-decreasing and tends to a number n_i as $s \to \infty$ such that $\Omega^X - \Omega^{X\restriction_{n_i}} < 2^{-i}$.*
- *$\Omega^\rho - \Omega^{X\restriction_{n_t}} < 2^{e-t}$ for all $\rho \supset X \restriction_{n_t}$ with $\rho \supset \sigma$.*

Also, the low for K cost of (X_s) as this is defined in (2.4) is $< 2^{-e}$.

Proof. Given a string σ and a number e we show how to define the computable function $h(\sigma,e)[s] := X_s$ for all $s \in \mathbb{N}$. The basic idea is as follows. At stage $s+1$ we make sure that if k is the least number such that $X_s(k)\downarrow \neq X_{s+1}(k)\downarrow$ (should there exist any such) and if t is the greatest number such that $n_t[s] \leq k$, then $\Omega^{X_{s+1}} - \Omega^{X_s\restriction_k} \geq 2^{e-t}$. Hence, we only allow the this change to our approximation of X at stage $s+1$ if this change adds at least $2^{e-t} - 2^{-t}$ to the current approximation to Ω^X. Thus, every time we pay cost $c_K(k,s+1)=\epsilon$, the monotone approximation to Ω^X increases by at least $(2^e-1)\cdot\epsilon$. Since $\Omega^X < 1/2^4$ and $e>0$, this guarantees that the overall cost corresponding to (X_s) is less than 2^{-e}.

The precise instructions are as follows. At stages $s \leq |\sigma|$ let $X_s = \sigma\restriction_s$. At stage $s+1 > |\sigma|$ let $n_i[s]$ be the Ω-sequence of X_s. Check to see whether there exists some $i<s$ and a string $\tau \supseteq X_s\restriction_{n_i[s]}$ of length $\leq s+1$ with $\sigma \subset \tau$ such that $\Omega^\tau - \Omega^{X_s\restriction_{n_i[s]}} \geq 2^{e-i}$. If so, pick the least such i, and for this choice the least such τ. Define X_{s+1} to be the concatenation of τ with $s+1-|\tau|$ zeros. If not, then let $X_{s+1} = X_s * 0$.

Clearly the function $h(\sigma,e)[s]$ is computable. Whenever $n_t[s] \neq n_t[s+1]$ or $X_s\restriction_{n_t[s]} \neq X_{s+1}\restriction_{n_t[s+1]}$, we have that $\Omega^{X_{s+1}\restriction_{n_t[s+1]}} - \Omega^{X_s\restriction_{n_t[s]}} \geq 2^{-t}$. Since $\Omega^Y<1$ for all Y, it follows that for each t the sequence $(n_t[s])$ converges monotonically to a final value n_t and the sequence $(X_s\restriction_{n_t[s]})$ reaches a limit. From the fact that Ω^Y is irrational for all Y, it follows that $\lim_t n_t = \infty$. Hence, the strings $X_s, s\in\mathbb{N}$ converge to an infinite sequence X, such that $\Omega^X - \Omega^{X\restriction_{n_t}} < 2^{-t}$ for each $t\in\mathbb{N}$.

The sequence Ω^{X_s} is non-decreasing and computable (since $\rho \to \Omega^\rho$ is computable). On the other hand, for each $t\in\mathbb{N}$ and for all stages s after which $X_s\restriction_{n_t[s]}$ stabilizes, we have $\Omega^X < \Omega^{X_s\restriction_{n_t[s]}} + 2^{-t}$. Hence Ω^X is the limit of the non-decreasing computable sequence (Ω^{X_s}) and is a c.e. real.

Now let s be such that $n_t[r]=n_t$ and $X_r\restriction_{n_t[r]} = X\restriction_{n_t}$ for all $r\geq s$. Then $\Omega^\rho - \Omega^{X\restriction_{n_t}} < 2^{e-t}$ for all $\rho \supset X\restriction_{n_t}$ with $\rho\supset\sigma$, since otherwise $n_t[r+1]\neq n_t[r]$ for the first stage $r+1>s$ at which we find ρ violating this condition.

The fact that the low for K cost of (X_s) is $< 2^{-e}$ follows from the argument given above. □

With the function h now defined according to the lemma, there is just one more consideration to be had before we can specify the entire construction precisely. When we defined T_σ in the discussion above, there was a string ρ *associated with* T_σ, which compressed an initial segment of Ω^{T_σ}, and the measure of the set of strings extending ρ was seen to give a certain amount of room for simulating machines with oracle input extending T_σ. There is nothing to stop there being multiple strings σ, however, for which which ρ is the string associated with T_σ in this way. This is easily dealt with using some simple accounting. Corresponding to each σ we shall have values a_σ and b_σ, and ρ is chosen so as to compress by a margin which depends upon these values in such a way that these sums work out as they should.

3. Proof of Theorem 1.1

3.1. **The machine M.** In Section 3.3 the machine M^A is defined in terms of a uniformly A-c.e. family of bounded request sets L_ρ, indexed by strings in the domain of U^A (and the extra 1-bit string 0, see below). This family gives a uniformly A-computable sequence of machines M^A_ρ such that for each request $\langle\tau,\ell\rangle$ in L_ρ there exists a string η of length ℓ such that $M^A_\rho(\eta)=\tau$. The main machine M^A is defined as follows.

$$M^A(\rho * \eta) = M^A_\rho(\eta). \tag{3.1}$$

Since each machine M^A_ρ is prefix-free (and all strings in the domain of U^A are incomparable with the 1-bit string 0, according to the conventions of Section 2.1) it follows from (3.1) that M^A is prefix-free.

3.2. **Parameters of the construction.** Let $b_\sigma[0]$ be a computable sequence of numbers such that

$$\textstyle\sum_\sigma 2^{-b_\sigma[0]} < 2^{-3} \qquad \text{and} \qquad b_\emptyset[0] = 4 \tag{3.2}$$

where σ ranges over all strings and $\emptyset$ denotes the empty sequence. The parameter b_σ will be used in the construction to help make sure that there is room for the descriptions that are allocated to T_σ, and will be updated upon an 'injury' of this node. A second parameter a_σ will indicate our belief as regards an upper bound to $\sup_\rho(\Omega^\rho - \Omega^{T_\sigma})$ where ρ runs over all extensions of T_σ.

We order the strings first by length and then lexicographically. Define $T_\emptyset[s] = \emptyset$ for all stages s, where $\emptyset$ denotes the empty string. This means that in the approximation $T[s]$ to T, the root of the tree will always be the empty string. Let σ be a nonempty string and let j be its last digit, so that $\sigma = \eta * j$ for some string η. Also let $(n_t[s])_{t\in\mathbb{N}}$ denote the Ω-sequence of $h(T_\eta[s] * j, a_\eta[s])[s]$. Following a standard convention, the latter expression is simplified to $h(T_\eta * j, a_\eta)[s]$. We say that T_σ *requires attention at stage* $s+1$ if $|\sigma| > 0$ and one of the following conditions holds.

(a) $T_\sigma[s]$ is undefined and there exists a string ρ of length $< s$ such that $U^A(\rho)$ is defined, is a prefix of the binary expansion of Ω^τ where $\tau = h(T_\eta * j, a_\eta)[s] \restriction_{n_t[s]}$, $t = |U^A(\rho)|$, $n_t[s] > |T_\eta * j|$ and $|U^A(\rho)| - |\rho| > a_\eta[s] + b_\sigma[s]$.

(b) $T_\sigma[s]$ is defined but $T_\sigma[s] \neq h(T_\eta * j, a_\eta)[s+1] \restriction_{n_t[s+1]}$, where $t = |U^A(\rho)|$ and ρ is the string associated with T_σ.

So a node requires attention either when it is undefined and is ready to be defined (corresponding to case (a)), or else is defined and should be made undefined (corresponding to case (b)).

As discussed previously, each T_σ will be associated with a string in the domain of U^A. We trivially let $T_\emptyset$ be permanently associated with the 1-bit string 0. In order to cover this trivial case (and since the string 0 is incomparable with all strings in the domain of U^A, see Section 2.1) we use the special set L_0 for the enumeration of requests corresponding to definitions of T_0 and T_1, and we let $\alpha_\emptyset[0] = 4$. Thus every definition of some T_σ entails an enumeration of requests into L_ρ, where ρ is the string associated with T_η, and η is the immediate predecessor of σ.

3.3. **Construction.** At stage $s + 1$ let σ be the least string such that T_σ requires attention (or if there exists no such then proceed to the next stage). Let j be the last digit of σ so that $\sigma = \eta * j$ for some string η. If $T_\sigma[s]$ is defined, let $b_\tau[s + 1] = b_\tau[s] + 1$ for all $\tau \supset \sigma$. Also let $T_\tau[s + 1]$, a_τ be undefined for all $\tau \supseteq \sigma$ and disassociate the strings in the domain of U^A that were associated with them. Declare that the nodes T_τ for $\tau \supset \sigma$ are *injured*.

Otherwise let $(n_t[s])_{t\in\mathbb{N}}$ be the Ω-sequence of $h(T_\eta * j, a_\eta)[s]$. Also let ρ be the least string such that, if t denotes $|U^A(\rho)|$ and τ denotes $h(T_\eta * j, a_\eta)[s] \restriction_{n_t[s]}$,

- $U^A(\rho)$ is a prefix of the binary expansion of Ω^τ
- $n_t[s] > |T_\eta * j|$ and $|U^A(\rho)| - |\rho| > a_\eta[s] + b_\sigma[s]$.

Put $T_\sigma[s+1]$ equal to $h(T_\eta * j, a_\eta)[s] \restriction_{n_t[s]}$, let $a_\sigma[s+1] = |U^A(\rho)| - a_\eta[s]$ and say that $T_\sigma[s+1]$ is associated with ρ. Also if μ is the string that is associated with $T_\eta[s]$ then for each $\langle \nu, \xi, \upsilon \rangle \in U[s]$ such that $T_\eta[s] \subset \nu \subseteq T_\sigma[s+1]$ enumerate $\langle \upsilon, |\xi| - |\mu| \rangle$ into L_μ.

3.4. **Verification.** The following lemmas establish the required properties of the construction of Section 3.3.

Lemma 3.1. *All nodes $T_\sigma[s]$ and parameters $a_\sigma[s], b_\sigma[s]$ reach limit values as $s \to \infty$. The function T which maps σ to $\lim_s T_\sigma$ is a perfect tree.*

Proof. We argue the first sentence in the statement of the lemma by induction on the strings σ. The values $T_\emptyset[s]$, $a_\emptyset[s]$ and $b_\emptyset[s]$ are the same for all $s \in \mathbb{N}$. This concludes the base case of the induction.

Suppose that $|\sigma| > 0$ and s_0 is a stage such that for all $s \geq s_0$ and all strings τ less than σ the parameters $T_\tau[s]$, $b_\tau[s]$, $a_\tau[s]$ always equal their final values T_τ, b_τ and a_τ respectively. In particular, if η is the immediate predecessor of σ (say $\sigma = \eta * j$), the parameters $T_\eta[s]$, $b_\eta[s]$, $a_\eta[s]$ take their final values T_η, b_η, a_η at all stages $s \geq s_0$. This immediately means that the same holds for b_σ since this value can only change when T_σ is injured (which happens only when T_η is redefined). By Lemma 2.2 the sequence $h(T_\eta * j, a_\eta)[s] := X_s$ converges to an infinite sequence X such that Ω^X is a c.e. real. By the same lemma, each term $n_i[s]$ of the Ω-sequence of X_s reaches a limit n_i as $s \to \infty$, and since Ω^X is irrational $\lim_i n_i = \infty$. Since A is not low for Ω, there exist infinitely many $n \in \mathbb{N}$ such that $K^A(\Omega^X \restriction_n) < n - a_\eta - b_\sigma$. Choose the least string υ such that $U^A(\upsilon) = \Omega^X \restriction_t$ for some t with $n_t > |T_\eta * j|$ and $|\upsilon| < t - a_\eta - b_\sigma$. Let $s_1 > s_0$ be sufficiently large that $n_t[r] = n_t$ and $X_r \restriction_{n_t} = X \restriction_{n_t}$ for all $r \geq s_1$ and $U^A(\upsilon)[s_1] = \Omega^X \restriction_t$. Let $s_2 > s_1$ be a stage such that for any string κ less than υ and all stages $s \geq s_2$ one of the following holds:

- $U^A(\kappa)[s]$ is not a prefix of the binary expansion of Ω^{X_s};
- $|\kappa| \not< |U^A(\kappa)[s]| - a_\eta - b_\sigma$;
- If $t' = |U^A(\kappa)[s]|$ then $n_{t'} \leq |T_\eta * j|$.

Such a stage exists by the minimality of υ and the fact that Ω^{X_s} is a computable non-decreasing sequence of rational numbers which tends to Ω^X.

Now if T_σ ever became undefined after stage s_2, it would require attention (by the choice of υ and s_2). No T_ρ with ρ less than σ would require attention at such a stage, since these nodes have reached their limit values. Therefore T_σ would receive attention and would be defined based on υ (i.e. defined with associated string υ) at some stage $s_3 > s_2$. The parameter $T_\sigma[s_3]$ would be given the value $X_{s_3} \restriction_{n_t} = X \restriction_{n_t}$ (for $t = |U^A(\upsilon)|$), a_σ would be given the value $|U^A(\upsilon)| - a_\eta$ and these values would never subsequently be redefined. Thus T_σ, a_σ and b_σ reach limit values, as required.

Let $T_\sigma := \lim_s T_\sigma[s]$ for each string σ. Notice that for each η and stage s the nodes $T_{\eta*0}[s]$, $T_{\eta*1}[s]$ are extensions of T_η and are incomparable with each other (unless one of them is undefined). Hence their final values will have the same properties and the function $\sigma \to T_\sigma$ is a perfect tree. □

Lemma 3.2. *Let $\sigma = \eta * j$ for some $\eta \neq \emptyset$ and suppose that at stage $s+1$ we newly define T_σ. Then $\Omega^{T_\sigma} - \Omega^{T_\eta} < 2^{-a_\eta}$.*

Proof. Let $\eta = \zeta * z$, put $X_s := h(T_\zeta * z, a_\zeta)[s]$ and let $n_i[s]$ be the Ω-sequence of X_s. At stage $s+1$, let μ be the string associated with T_η, and let $t = |U^A(\mu)|$. Then, at stage $s+1$, since T_η does not require attention, we must have that $T_\eta[s] = X_{s+1} \restriction_{n_t[s+1]} = X_s \restriction_{n_t[s]}$. Now if we define $T_\sigma = \tau$ and $\Omega^\tau - \Omega^{X_s \restriction n_t[s]} \geq 2^{-a_\eta}$ then $\Omega^\tau - \Omega^{X_s \restriction n_t[s]} \geq 2^{-(|U^A(\mu)| - a_\zeta)} = 2^{a_\zeta - t}$. According to the definition of h this would cause us to define X_{s+1} so that $\tau \subseteq X_{s+1}$ and $n_t[s+1] \neq n_t[s]$, a contradiction. □

Lemma 3.3. *For each string μ the weight of L_μ is < 1. Hence M is a prefix-free machine.*

Proof. Fix a string μ. Enumerations into L_μ occur when a node T_σ with $\sigma = \eta * j$ becomes newly defined and T_η has the string μ currently associated with it. Hence every enumeration into L_μ during the construction can be *allocated* to the unique node T_η whose immediate successor became defined at the stage where the enumeration occurred. Let $L_\mu(T_\eta)$ be the set of tuples $\langle \upsilon, \ell \rangle$ in L_μ which are assigned to T_η in this way. Then $\texttt{wgt}(L_\mu) = \sum_\eta \texttt{wgt}(L_\mu(T_\eta))$. By (3.2) it suffices to show that

$$\texttt{wgt}(L_\mu(T_\eta)) < 2^{-b_\eta[0]+3} \qquad \text{for all strings } \eta. \tag{3.3}$$

In the special case where μ is the 1-bit string 0, this weight is 0 unless η is the empty string. In the latter case we have $b_\eta[0] = 4$ by (3.2) so it suffices to show that $\texttt{wgt}(L_0(T_\eta)) < 2^{-1}$. The requests that are enumerated into $L_0(T_\eta)$ come from various definitions of the nodes T_0, T_1 during the construction. Let $X_s := h(T_\eta * 0, a_\eta)[s]$ and let X be the limit of this sequence. Notice that T_η is the empty sequence and $a_\eta[s] = 4$ for all stages s, so that $X_s = h(0,4)[s]$. Clearly the weight of the requests that are enumerated into $L_0(T_\eta)$ and which are caused by the various definitions of T_0 is bounded by $2 \cdot (\Omega^X + c)$, where c is the 'low for K cost' of the approximation $(X_s) \to X$ (the factor 2 here comes from the fact that in the last line of the construction we enumerate the request $\langle \upsilon, |\xi| - |\mu| \rangle$ rather than the request $\langle \upsilon, |\xi| \rangle$ into L_μ). By Lemma 2.2 we have $c < 2^{-4}$ and by the conventions established in Section 2.1 we have $\Omega^X < 2^{-4}$. Hence the weight of the requests that are enumerated into $L_0(T_\eta)$ as we define T_0 is bounded by 2^{-2}. Similarly the weight that is caused by the various definitions of T_1 is bounded by 2^{-2}. Hence $\texttt{wgt}(L_0(T_\eta))$ is bounded by 2^{-1}. This concludes the proof of (3.3) in the special case when μ is the 1-bit string 0.

Now suppose that μ is in the domain of U^A and let η be any string. Notice that if η is the empty string then the weight in (3.3) is 0. So without loss of generality we may assume that η is nonempty, say $\eta = \zeta * z$ for a string ζ and some $z \in \{0,1\}$. Notice that each time T_η is declared *injured*, the current value of b_η increases by 1. Consider a partition of all stages into maximal intervals where T_η remains uninjured (in other words, T_ζ remains constantly defined). Let $L^*_\mu(T_\eta)$ denote the requests that are allocated to T_η and are enumerated into L_μ during one of these intervals I. To establish (3.3) it suffices to show that

$$\texttt{wgt}(L^*_\mu(T_\eta)) < 2^{-b^*_\eta+2} \tag{3.4}$$

where b^*_η denotes the value of b_η during the interval I. Notice that during the stages in I the node T_η may be redefined many times, but each time it is redefined during this interval, it will be associated with a different string than at all previous stages during the interval. Thus we may consider a maximal subinterval J, during which η is always always associated with μ, and during which the values a_η and T_η are fixed (in the following discussion we let a_η, a_ζ and T_η refer to their fixed values during this interval J).

During the interval J, some enumerations allocated to T_η are caused by definitions of $T_{\eta*0}$ and others are caused by definitions of $T_{\eta*1}$. Fix $j = 0, 1$. Let $X_s := h(T_\eta * j, a_\eta)[s]$ and let X be the limit of (X_s). By combining Lemma 3.2 with the fact that the low for K cost corresponding to the approximation (X_s) is at most 2^{-a_η}, we get that the weight that is enumerated into $L^*_\mu(T_\eta)$ as we give the various definitions of $T_{\eta*j}$ during the stages in J is bounded by $2^{|\mu|} \cdot 2 \cdot 2^{-a_\eta}$ (the factor $2^{|\mu|}$ comes from the fact that in the last line of the construction we enumerate the request $\langle \upsilon, |\xi| - |\mu| \rangle$ rather than the request $\langle \upsilon, |\xi| \rangle$ into L_μ). Then μ was chosen so that

$$|U^A(\mu)| - |\mu| > a_\zeta + b^*_\eta \qquad \text{and} \qquad a_\eta = |U^A(\mu)| - a_\zeta.$$

Therefore

$$a_\eta > |\mu| + b^*_\eta \qquad \text{and} \qquad 2^{-a_\eta} < 2^{-|\mu|} 2^{-b^*_\eta}$$

so $2^{|\mu|} \cdot 2 \cdot 2^{-a_\eta} < 2^{-b^*_\eta + 1}$. Since the same argument holds for $T_{\eta*(1-j)}$, the overall weight that is enumerated into $L^*_\mu(T_\eta)$ during the stages in J is $< 2^{-b^*_\eta + 2}$, as required. □

Let T denote the tree $\sigma \to T_\sigma$.

Lemma 3.4. *Given any path X on the tree T we have $K^A_M(\sigma) \leq K^X(\sigma)$ for all strings σ.*

Proof. Suppose that $K^X(\sigma) = n$ and $U^X(\nu) = \sigma$ for some string ν of length n. Then there is $\tau \subset X$ such that $\langle \tau, \nu, \sigma \rangle \in U$. Let η be a string such that $T_\eta \subset \tau \subseteq T_{\eta*j}$ for some $j = 0, 1$. When $T_{\eta*j}$ was permanently defined at a stage s, the construction enumerated the request $\langle \sigma, |\nu| - |\mu| \rangle$ to L_μ where μ is the string that is cofinally associated with η. Hence $K^A_M(\sigma) \leq |\mu| + (|\nu| - |\mu|) = K^X(\sigma)$. □

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