# Normal Forms for Priority Graphs [*]

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**Abstract**

This brief note compares a few ways of deriving pre-orders over sets of objects from strict partial orders over properties of those objects— so-called *priority graphs* that have recently been used by logicians as a rich explicit model of preference merge, belief revision, norm change, and other aspects of agency. Our note establishes equivalence results between three popular formats for priority graphs, and defines a normal form for them.

## 1   Introduction

Pre-orders over a given finite set $S$ of objects can be derived from strict partial orders over properties of these objects, forming a richer medium for showing how these orderings arise. Properties in such *priority graphs* are expressed by simple propositional formulae and partial orders over these: cf. [1, 4] for concrete examples concerning social preference merge, and reason-based preference of single agents. In [6], the authors apply the format to deontic reasoning, contrary-to-duty norms, and norm change.

The present note investigates three typical formats of priority graphs and proves their equivalence with respect to two different 'recipes' for deriving pre-orders, where equivalence is understood as the definition of the same class of underlying order structures for objects. This type of equivalence reasoning allows us to suggest two normal-form formats for priority graphs. What we will see, in particular, is how inclusion orders of predicates can mimic the effects of abstract priority orders.

### 1.1   Priority graphs

**Definition 1** (P-graphs). *Let $\mathcal{L}(\mathbf{P})$ be a propositional language built on the set of atoms $\mathbf{P}$. A P-graph is a tuple $\mathcal{G} = \langle \Phi, < \rangle$ such that:*

- ▸ $\Phi \subset \mathcal{L}(\mathbf{P})$ *with* $|\Phi| < \omega$;

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► $\prec$ *is a strict order on* $\Phi$.[1]

*The set of all P-graphs for* $\mathcal{L}(\mathbf{P})$ *is denoted* $SO_{\mathbf{P}}$ *("Strict Orders").*[2]

Given a P-graph $\mathcal{G}$, we denote with $\uparrow_{\mathcal{G}} \varphi = \{\psi \in \Phi \mid \varphi \prec \psi \text{ or } \psi = \varphi\}$ the *upset* of $\varphi$ in $\mathcal{G}$. We also denote with $\uparrow_{\mathcal{G}}^{+} \varphi = \{\psi \in \Phi \mid \varphi \prec \psi\}$ the *strict upset* of $\varphi$ in $\mathcal{G}$. When clear from the context, we will drop the index and write simply $\uparrow \varphi$ and $\uparrow^{+} \varphi$.

## 1.2   Derivation functions

Derivation functions for P-graphs are functions of type:

$$\delta : SO_{\mathbf{P}} \times V_{\mathbf{P}} \longrightarrow PO_S \tag{1}$$

where $V_{\mathbf{P}}$ is the set of valuation functions $\mathcal{V}_{\mathbf{P}} : \mathbf{P} \longrightarrow 2^S$, and $PO_S$ the set of pre-orders over $S$. That is, given a strict order on propositional formulae from $\mathcal{L}(\mathbf{P})$ and a valuation for $\mathbf{P}$ over $S$, $\delta$ outputs a preorder on the set $S$. We will work in particular with two functions of the type given in Formula 1: the lexicographic derivation (borrowed from [1]), and the subsumption-based derivation (borrowed from [4]).

be strict orders. will simplify the presentation of our result considerably

**Definition 2** (Lexicographic derivation function)**.** *Let* $\mathcal{G} = \langle \Phi, \prec \rangle$ *be a P-graph, S a non-empty set of states and* $\mathcal{V} : \mathbf{P} \longrightarrow 2^S$ *a valuation function. The preference relation* $\preceq_{\mathcal{G}}^{\text{lex}} \subseteq S^2$ *is defined as follows:*

$$s \preceq_{\mathcal{G}}^{\text{lex}} s' \iff \forall \varphi \in \Phi : [s \in [\![\varphi]\!] \Rightarrow s' \in [\![\varphi]\!]$$
$$\text{or } \exists \varphi' : [\varphi \prec \varphi' \text{ and } s \notin [\![\varphi']\!] \text{ and } s' \in [\![\varphi']\!]]]. \tag{2}$$

The procedure works as a form of lexicographic ordering. The relation $s \preceq_{\mathcal{G}}^{\text{lex}} s'$ holds if $s$ satisfies all formulae in the sequence that also $s'$ satisfies or, if that is not the case, if $s'$ satisfies a formula which $s$ does not satisfy and which is ranked higher in the P-graph. In a P-graph where all properties happen to be disjoint the effect of this latter clause is clear. will be ranked according to the ranking of the proposition whose truth-So we can recognize in the lexicographic derivation two components: a set-inclusion component that determines the equivalence classes of the derived pre-order by essentially looking at the relative logical strength of the properties each element satisfies; and a "compensation" component which ranks element that would be left unrelated by the set-inclusion component by resorting to the strict order information present in the P-graph.

**Definition 3** (Subsumption-based derivation)**.** *Let* $\mathcal{G} = \langle \Phi, \prec \rangle$ *be a P-graph, S a non-empty set of states and* $\mathcal{V} : \mathbf{P} \longrightarrow 2^S$ *a valuation function. The preference relation* $\preceq_{\mathcal{G}}^{\text{sub}} \subseteq S^2$ *is defined as follows:*

$$s \preceq_{\mathcal{G}}^{\text{sub}} s' \iff \forall \varphi \in \Phi : s \in [\![\varphi]\!] \Longrightarrow s' \in [\![\varphi]\!]. \tag{3}$$

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[1]That is, an irreflexive and transitive binary relation.
[2]We will often drop the index referring to the current language.

Intuitively, according to Definition 3, a state $s$ is ranked above a state $s'$ if and only if any of the properties in the graph satisfied by $s'$ it is also satisfied by $s$. In other words, $s$ enjoys at least all the properties of $s'$. So, subsumption-based derivation consists of the sole set-inclusion component of the lexicographic derivation.

in such a way that no inclusion relation holds among them, then the valuation interprets the properties in the graph as an ascending chain of below), as if a state satisfies a property in the sequence, then it also satisfies all

**Remark 1.** *It should be noted that in "flat" P-graphs, i.e., P-graphs $\mathcal{G} = \langle \Phi, \emptyset \rangle$ where the strict order is empty, the two derivation methods are equivalent, as the second part of the lexicographic method (see Formula 2) cannot be called in. More precisely, for any P-graph $\mathcal{G}$ with empty orders, and any valuation $\mathcal{V}$ it is the case that:*

$$\texttt{lex}(\mathcal{G}, \mathcal{V}) = \texttt{sub}(\mathcal{G}, \mathcal{V}).$$

## 2   Classes of P-graphs

The section deals with two specific classes of P-graphs whose elements exhibit logical structure and which we call: inclusive and, respectively, exclusive P-graphs.

**Definition 4** (Inclusive P-graphs)**.** *An inclusive P-graph $\mathcal{G} = \langle \Phi, \prec \rangle$ for language $\mathcal{L}(\mathbf{P})$ is a P-graph $\mathcal{G}$ such that, for every $\varphi \in \Phi$:*

$$\varphi \quad \leftrightarrow \quad \bigvee \uparrow \varphi$$

*The set of all inclusive P-graphs for language $\mathcal{L}(\mathbf{P})$ is denoted $SO_{in}(\mathbf{P})$.*

In other words, each property in an inclusive graph is equivalent to the big disjunction of all elements in the upset of that property. So, an inclusive P-graph is a strict order over disjunctions built in such a way that if a property occurs as a disjunct at some point in the graph, it occurs in all lower points. Two examples of such graphs are depicted in Figure 1.

**Remark 2.** *It is useful to see how Definition 4 simplifies in the case of $\mathcal{G} = \langle \varphi_1, \dots, \varphi_n \rangle$ being a strict linear order, i.e., a chain:*

$$\varphi_i \quad \leftrightarrow \quad \bigvee \uparrow \varphi_i$$
$$\leftrightarrow \quad \bigvee_{1 \leq j \leq i} \varphi_j$$

*with $1 \leq i \leq n$. In this case, each element is equivalent to the disjunction of itself with the union of all the preceding elements.*

**Definition 5** (Exclusive P-graphs)**.** *An exclusive P-graph $\mathcal{G} = \langle \Phi, \prec \rangle$ for language $\mathcal{L}(\mathbf{P})$ is a P-graph $\mathcal{G}$ such that, for every $\varphi \in \Phi$:*

$$\varphi \quad \leftrightarrow \quad \varphi \wedge \neg \bigvee \uparrow^+ \varphi$$

*The set of all inclusive P-graphs for language $\mathcal{L}(\mathbf{P})$ is denoted $SO_{ex}(\mathbf{P})$.*

In other words, an exclusive graph is such that all its elements are equivalent to the complementation of themselves with the disjunction of all the elements in their strict upset. Examples of such graphs are given in Figure 2.
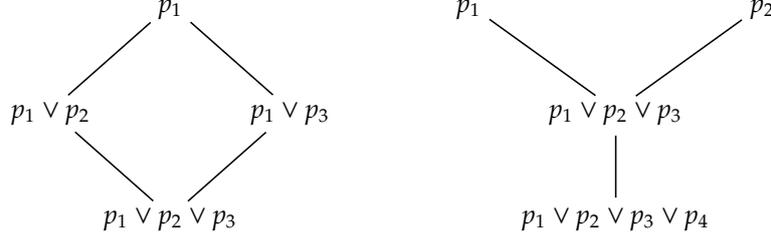
Figure 1: Hasse diagrams of inclusive P-graphs

# 3    Normal forms for P-graphs

Let us first stress an aspect of Definition 2.

**Remark 3.** *It is interesting to see how the strict part $\prec_{\mathcal{G}}^{\text{lex}}$ and the indifference part $\sim_{\mathcal{G}}^{\text{lex}}$ of $\preceq_{\mathcal{G}}^{\text{lex}}$ can be independently defined. We start with the very natural definition of $\sim_{\mathcal{G}}^{\text{lex}}$:*

$$s \sim_{\mathcal{G}}^{\text{lex}} s' \iff \forall \varphi \in \Phi : [s \in [\![\varphi]\!] \iff s' \in [\![\varphi]\!]]$$

*As to the strict part:*

$$s \prec_{\mathcal{G}}^{\text{lex}} s' \iff \exists \varphi \in \Phi : [s \notin [\![\varphi]\!] \text{ and } s' \in [\![\varphi]\!]$$
$$\text{and } \forall \varphi' : [s \in [\![\varphi']\!] \text{ and } s' \notin [\![\varphi']\!] \implies \varphi' \prec \varphi]].$$

*It is then easy to see that the following is the case:*

$$s \preceq_{\mathcal{G}}^{\text{lex}} s' \iff s \sim_{\mathcal{G}}^{\text{lex}} s' \text{ or } s \prec_{\mathcal{G}}^{\text{lex}} s'.$$

    In other words, the lexicographic derivation builds a pre-order consisting of equivalence classes—of relation $\sim_{\mathcal{G}}^{\text{lex}}$—strictly ordered according to $\preceq_{\mathcal{G}}^{\text{lex}}$. Now, these equivalence classes consist of elements that satisfy precisely the same properties. It becomes then evident that such classes could themselves be used as properties in a P-graph, with the same order. Such new P-graph will obviously be equivalent. The next section makes this intuition precise.
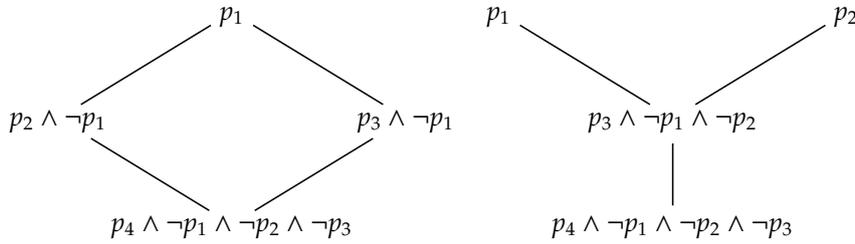


Figure 2: Hasse diagrams of exclusive P-graphs

$$p \wedge q \wedge r$$

$$p \wedge q \wedge \neg r$$

$$p \qquad\qquad q$$

$$p \wedge \neg q \wedge r \qquad\qquad \neg p \wedge q \wedge r$$

$$r$$

$$p \wedge \neg q \wedge \neg r \qquad\qquad \neg p \wedge q \wedge \neg r$$

$$\neg p \wedge \neg q \wedge r$$

$$\neg p \wedge \neg q \wedge \neg r$$
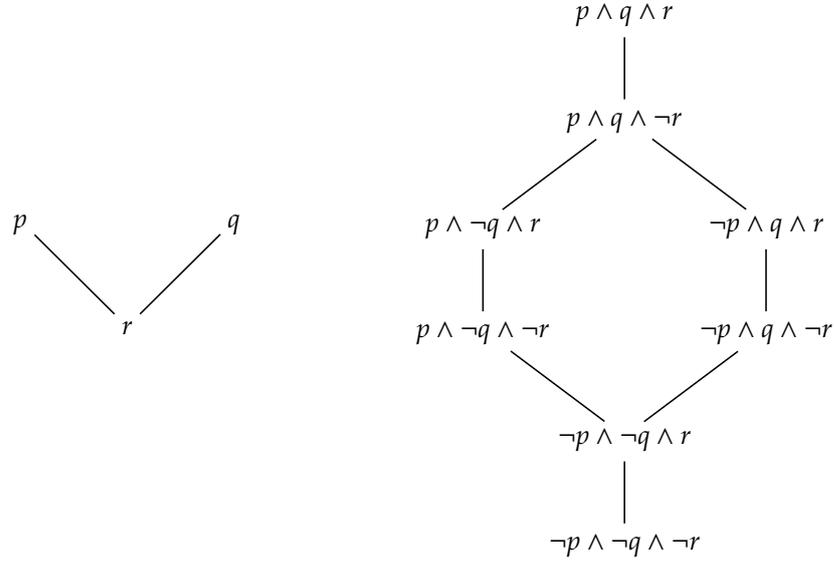
Figure 3: Hasse diagrams of a P-graph (left) and its exclusive normal form (right)

### 3.1   Exclusive normal form

We introduce the first type of normal form.

**Definition 6** (Exclusive normal form for P-graphs)**.** *Let* $\mathcal{G} = \langle \Phi, \prec \rangle$ *be a P-graph. The exclusive normal form of* $\mathcal{G}$ *is a P-graph* $\mathcal{G}_{ex} = \langle \Phi_{ex}, \prec_{ex} \rangle$ *such that:*

▸ $\Phi_{ex} = 2^{\Phi}$. *Each element* $\Psi \in \Phi_{\wedge \neg}$ *has to be read as a finite conjunction:*

$$\bigwedge \Psi \wedge \bigwedge \neg(\Phi - \Psi)$$

*that is, the conjunction of all properties in* $\Psi$ *and all the negations of the properties not in* $\Psi$.

▸ *The relation* $\prec_{ex}$ *is defined as follows:*

$$\Psi \prec_{ex} \Psi' \iff \exists \varphi \in \Phi : [\varphi \in \Psi' \text{ and } \varphi \notin \Psi$$
$$\text{and } \forall \varphi' : [\varphi' \notin \Psi' \text{ and } \varphi \in \Psi \implies \varphi' \prec \varphi]].$$

An example of the exclusive normal form of a graph is given in Figure 3.[3] It should be clear by the construction of the exclusive normal forms that these graphs are indeed exclusive in the sense of Definition 5. Notice also that the number of elements in the exclusive normal form is bound by the cardinality of $\Phi$ being equal to $2^{|\Phi|}$.

We can now prove a simple normal form theorem guaranteeing that every P-graph has an equivalent exclusive normal form.

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[3]It might be instructive to compare the definition of $\prec_{ex}$ with the definition of the strict part of the preorders derivable via `lex` (Remark 3).

**Lemma 1** (Adequacy of exclusive normal forms). *Each graph $\mathcal{G}$ is* lex*-equivalent to its exclusive normal form $\mathcal{G}_{ex}$.*

*Proof.* Let us restate the lemma in a more extensive way:

$$\forall \mathcal{G} \in SO : [\forall \mathcal{V} \in V : \mathtt{lex}(\mathcal{G}, \mathcal{V}) = \mathtt{lex}(\mathcal{G}_{ex}, \mathcal{V})].$$

We prove the claim by showing that, for any valuation $\mathcal{V}$:

  (i) $|S|_{\preceq^{\mathtt{lex}}_{\mathcal{G}}} = |S|_{\preceq^{\mathtt{lex}}_{\mathcal{G}_{ex}}}$, i.e., the two preorders give rise to the same equivalence classes;

  (ii) $\prec^{\mathtt{lex}}_{\mathcal{G}} = \prec^{\mathtt{lex}}_{\mathcal{G}_{ex}}$, i.e., the strict part of the two preorders is the same.

As to (i) it suffices to observe that all equivalence classes $|s|_{\preceq^{\mathtt{lex}}_{\mathcal{G}}}$ yielded by $\preceq^{\mathtt{lex}}_{\mathcal{G}}$ must be the truth-sets—under $\mathcal{V}$—of some conjunctions of length $|\Phi|$ of formulae of the form $\bigwedge \Psi \wedge \bigwedge \neg(\Phi \setminus \Psi)$ with $\Psi \subseteq \Phi$. But these are precisely the disjoint elements of the normal form. As to (ii), it is proven by the following series of equivalences:

$$
\begin{aligned}
s \prec^{\mathtt{lex}}_{\mathcal{G}} s' \iff & \ \exists \varphi \in \Phi : s \notin [\![\varphi]\!] \text{ and } s' \in [\![\varphi]\!] \\
& \text{and } \forall \varphi' : (s \in [\![\varphi']\!] \text{ and } s' \notin [\![\varphi]\!] \implies \varphi' \prec \varphi] && (4) \\
\iff & \ \forall \Psi, \Psi' \in \Psi_{ex} : [\text{ if } [\![\Psi]\!] = |s|_{\sim^{\mathtt{lex}}_{\mathcal{G}}} \text{ and } [\![\Psi']\!] = |s'|_{\sim^{\mathtt{lex}}_{\mathcal{G}}} \\
& \text{then } \exists \varphi \in \Phi : [\varphi \in \Psi' \text{ and } \varphi \notin \Psi \\
& \text{and } \forall \varphi' : [\text{ if } \varphi' \notin \Psi' \text{ and } \varphi \in \Psi \text{ then } \varphi' \prec \varphi]]] && (5) \\
\iff & \ \forall \Psi, \Psi' \in \Psi_{ex} : [\text{ if } [\![\Psi]\!] = |s|_{\sim^{\mathtt{lex}}_{\mathcal{G}}} \text{ and } [\![\Psi']\!] = |s'|_{\sim^{\mathtt{lex}}_{\mathcal{G}}} && (6) \\
& \text{then } \Psi \prec_{ex} \Psi'] \\
\iff & \ \exists \Xi \in \Phi_{ex} : [s \notin [\![\Xi]\!] \text{ and } s' \in [\![\Xi]\!] \\
& \text{and } \forall \Xi' : [s \in [\![\Xi']\!] \text{ and } s' \notin [\![\Xi']\!] \implies \Xi' \prec_{ex} \Xi]] && (7) \\
\iff & \ s \prec^{\mathtt{lex}}_{\mathcal{G}_{ex}} s' && (8)
\end{aligned}
$$

where, to simplify notation, by $\Psi, \Xi$ we still also denote the finite conjunction of the elements of $\Psi, \Xi$ and of the negations of the members of its complement. Equivalence 4 holds by definition. Equivalence 5 reformulates 4 by resorting to elements of the exclusive graph corresponding to the equivalence classes of $s$ and $s'$. By the construction of the exclusive normal form we know that such elements exist. Equivalence 6 holds by the definition of exclusive normal form. Equivalence 7 holds because of the fact that exclusive graphs consist of disjoint elements. Finally, 8 holds by definition. $\square$

**Remark 4.** *It is worth noticing that the exclusive normal form $\mathcal{G}_{ex}$ of a P-graph $\mathcal{G}$ is order-isomorphic to the quotient $Q(\mathcal{P})$ of the pre-order $\mathcal{P} = \langle S, \prec^{\mathtt{lex}}_{\mathcal{G}} \rangle$ obtained by lexicographic derivation from $\mathcal{G}$. The following diagram depicts this observation.*
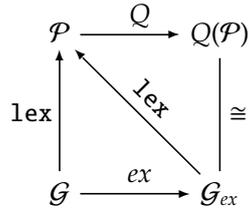
We proceed now to the second type of normal form dealt with in the paper.

Figure 4: Hasse diagrams of an exclusive P-graph (left) and its inclusive normal form (right)

## 3.2 Inclusive normal form

**Definition 7** (Inclusive normal forms). *Let $\mathcal{G} = \langle \Phi, \prec \rangle$ be an exclusive P-graph. The inclusive normal form of $\mathcal{G}$ is a P-graph $\mathcal{G}_{in} = \langle \Phi_{in}, \prec_{in} \rangle$ such that:*

- ► *$\Phi_{in} = \uparrow \mathcal{G}$, that is, the set of upsets of $\mathcal{G}$. To simplify notation, each element $\Psi \in \Phi_{in}$ will be read also as the finite disjunction: $\bigvee \Psi$.*

- ► *The relation $\prec_{in}$ is defined as follows:*

$$\Psi \prec_{in} \Psi' \iff \Psi' \subset \Psi.$$

An example of inclusive normal form is given in Figure 4. Notice that this form is defined only for graphs in exclusive form.

Like for the case of exclusive normal forms, we obtain a simple theorem guaranteeing that every exclusive P-graph has an equivalent inclusive normal form.

**Lemma 2** (Adequacy of inclusive normal forms). *Each graph $\mathcal{G}$, used as argument of* `lex`*, is equivalent to its inclusive normal form $\mathcal{G}_{in}$, used as argument of* `sub`*.*

*Proof.* Let us restate the lemma in a more extensive way:

$$\forall \mathcal{G} \in SO : [\forall \mathcal{V} \in V : \mathtt{lex}(\mathcal{G}, \mathcal{V}) = \mathtt{sub}(\mathcal{G}_{in}, \mathcal{V})].$$

We prove the claim by the following series of equivalences:

$$
\begin{aligned}
s \leq^{\mathtt{lex}}_{\mathcal{G}} s' \iff & \exists \varphi \in \Phi : (s \notin [\![\varphi]\!] \text{ and } s' \in [\![\varphi]\!] \\
& \text{and } \forall \varphi' : (s \in [\![\varphi']\!] \text{ and } s' \notin [\![\varphi']\!] \implies \varphi' \prec \varphi) \quad (9) \\
\iff & \forall \Psi, \Psi' \in \Phi_{in} : [\text{ if } [\![\Psi]\!] = \uparrow |s|_{\sim^{\mathtt{lex}}_{\mathcal{G}}} \text{ and } [\![\Psi']\!] = \uparrow |s'|_{\sim^{\mathtt{lex}}_{\mathcal{G}}} \\
& \text{then } \Psi' \subseteq \Psi] \quad (10) \\
\iff & \forall \Xi \in \Phi_{in} : s \in [\![\Xi]\!] \implies s' \in [\![\Xi]\!] \quad (11) \\
\iff & s \leq^{\mathtt{sub}}_{\mathcal{G}_{in}} s' \quad (12)
\end{aligned}
$$

where $\uparrow_{\mathcal{G}_{ex}} |s|_{\sim^{\mathtt{lex}}_{\mathcal{G}}}$ denotes the upset of the equivalence class of $s$ in the exclusive normal form of $\mathcal{G}$. Equivalence 9 holds by definition. Equivalence 10 holds under the assumption that $\mathcal{G}$ is exclusive, and hence that it has yields pre-orders with just as many equivalence classes as the elements of its domain. It just says that for any two elements of the inclusive normal form, if they coincide with the upsets of the equivalence classes of $s$ and $s'$ respectively, then the latter is included in the former. But this is to say—Equivalence 11—that for any property, if $s$ satisfies it, so does $s'$. Equivalence 12 holds then by definition. □

So, any exclusive graph has an equivalent inclusive normal form. What about general graphs? In that case an inclusive normal form can be obtained from the exclusive normal form of the graph. To make it clear, the inclusive normal form of the graph given in Figure 3 (left) is not the graph given in Figure 4 (right), but a graph obtained from Figure 3 (right) which consists of all the upsets of the latter, ordered by set inclusion.

**Remark 5.** *What discussed in this section is strictly related to a well-known result in order theory:*

> *Every pre-order is order-isomorphic to the set of upsets of its elements ordered by set-inclusion [2, Theorem 5.9].*

*In fact, the disjunctive normal form of an exclusive P-graph is the set of upsets of the graph, ordered by (strict) set-inclusion. Unlike exclusive normal forms, inclusive normal forms are therefore order-isomorphic to their original graphs. However, notice that the inclusive normal form of an exclusive normal form will not be order-isomorphic any more with respect to the original graph.*

## 3.3 Graph equivalences

We can now pull together Lemmata 1 and 2 into one characterization theorem showing that the class of P-graphs defines, by lexicographic derivation, the same class of pre-orders that can be derived via the exclusive normal forms of those graphs by lexicographic derivation, or by subsumption-based derivation from the inclusive normal form of the exclusive normal form of the graph.

**Theorem 1** (Equivalence of classes of graphs)**.** *For any P-graph $\mathcal{G}$ and valuation $\mathcal{V}$ it holds that:*

$$\texttt{lex}(\mathcal{G}, \mathcal{V}) = \texttt{lex}(\mathcal{G}_{ex}, \mathcal{V}) = \texttt{sub}((\mathcal{G}_{ex})_{in}, \mathcal{V}). \tag{13}$$

*Proof.* It follows from Lemmata 1 and 2. □

# 4 Conclusions

In the light of Theorem 1, the following two equations summarize the findings of this note:

$$\texttt{lex} = ex \circ \texttt{lex} \tag{14}$$

$$\texttt{lex} = ex \circ in \circ \texttt{sub} \tag{15}$$

Putting the same point a bit differently, the lexicographic derivation is the function consisting of: the composition of the function extracting the exclusive normal form with the lexicographic derivation itself (Formula 14); but the lexicographic derivation is also the composition of the function extracting the exclusive normal form, the function extracting the inclusive normal form (on exclusive graphs), and the subsumption-based derivation (Formula 15). This

is illustrated by the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{G}_{ex} & \xrightarrow{\ in\ } & (\mathcal{G}_{ex})_{in} \\[2pt]
\Big\uparrow{ex} & \diagdown^{\mathtt{lex}} & \Big\downarrow{\mathtt{sub}} \\[2pt]
\mathcal{G} & \xrightarrow[\ \mathtt{lex}\ ]{} & \mathcal{P}
\end{array}
$$

To put it yet otherwise, given a graph $\mathcal{G}$, the pre-order obtained from it via lexicographic derivation is the same pre-order that is obtained via lexicographic derivation from the exclusive normal form $\mathcal{G}_{ex}$ of $\mathcal{G}$, or via subsumption-based derivation from the inclusive normal form $(\mathcal{G}_{ex})_{in}$ of $\mathcal{G}_{ex}$.

We thus obtain an effective procedure for transforming priority graphs into semantically equivalent ones—its normal forms. This analysis clarifies various points in the recent literature, especially, the use of inclusion-nested graphs of propositions as in [6] versus other formats, where the latter need additional priority order, whereas the former does not. The syntactic aspects of our analysis also fit quite well with the modal graph logic of [3] and the "internal graph algebra" of [5].

We see various further questions opening up at this point, among which: the study of a richer set of dynamic graph transformations than has been considered so far; the study of priority graphs where properties are described in terms of modal languages, and not just propositional ones.

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