Two Neighborhood Semantics for Subintuitionistic Logics

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Abstract. This investigation is concerned with weak subintuitionistic logics with neighbourhood models introduced by the authors in 2016. The two types of neighborhood models introduced in that article are compared and the relationship between the two types of models is clarified. Thereby modal companions for various logics are recognized. Many of the extensions of the basic logics are discussed and characterized.

Keywords: intuitionistic logic, subintuitionistic logic, modal companions, neighborhood models.

1 Introduction

In our [7] two types of neighborhood models were introduced. In the NB-neighborhood frames the neighborhoods are pairs (X, Y) of sets of worlds, in N-neighborhoods the neighborhoods are sets of worlds of the form $\overline{X} \cup Y$. The NB-neighborhoods were best suited to study the basic logic WF, the N-neighborhoods were more suitable for obtaining modal companions with respect to the Gödel type translation discovered for subintuitionistic logics with Kripke models by G. Corsi in [2]. The exact relationship between the two types of models remained unclear in that article. In particular it was not sure whether the two types of models define the same set of valid formulas.

In [4] we gave a partial solution of this problem without complete proofs, the main emphasis of that article was on conservativity results for sets of implications with regard to intuitionistic logic IPC. But we did introduce a rule N in it which, added to WF, gives the system $\mathsf{WF_N}$ that axiomatizes the validities of N-models. In the present article we give complete proofs of this result, and make a finer analysis of the relationship between the two types of models. It turns out that N-neighborhood frames can be seen as a special type of NB-neighborhood frames.

Furthermore, we characterize the properties of many axiom schemata extending WF in both types of models and prove their completeness. In particular a new rule $\mathsf{N_2}$ is introduced related to N which axiomatizes a stronger logic $\mathsf{WF_{N_2}}$ over WF. The modal companion of $\mathsf{WF_{N_2}}$ is the basic monotonic modal logic M. Finding such a logic was of course very desirable, but had escaped us so far.

In Section 2 we introduce the logic WF and its NB-neighborhood semantics. In Section 3 we introduce the logic $\mathsf{WF_N}$ and its N-neighborhood semantics. In Section 4 we discuss some formulas that highlight the difference between the two types of semantics, and in Section 5 some logics are discussed that extend WF. In section 6 we extend Corsi's Gödel type translations of logics above F into modal logic to the logics above WF and provide modal companions to some of them. We also introduced a new logic $\mathsf{WF_{N_2}}$ with monotonic modal logic M as its modal companion. It is based on a rule $\mathsf{N_2}$ which strengthens the rule N.

2 NB-Neighborhood Semantics

We first recall the most general NB-neighborhood frames introduced in [7] and further studied in [4]. Here we choose general frames as the basic notion, restricting the valuations to a subset of the powerset of the set of worlds. We feel that in the case of neighborhood semantics this is the natural choice. Moreover, in the definition we do not assume that the frames have an omniscient element; we show that those can be added without changing validity. We do not give all the proofs in this section. The missing ones can be found in [7], and, anyhow, very similar ones can be found in the next section on N-neighborhoods.

Definition 1. *$\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ is an **NB-Neighborhood Frame** if W is a non-empty set and $\mathcal{X}$ is a non-empty collection of subsets of W such that $\emptyset$ and W belong to $\mathcal{X}$, and $\mathcal{X}$ is closed under $\cup$, $\cap$ and $\to$ defined by*

$$U \to V := \{w \in W \mid (U, V) \in NB(w)\},$$

and NB is a function from W into $\mathcal{P}((\mathcal{X})^2)$ such that

$$\forall w \in W,\ \forall X, Y \in \mathcal{X},\ (X \subseteq Y \ \Rightarrow\ (X, Y) \in NB(w)).$$

*In an **NB-Neighborhood Model** $\mathfrak{M} = \langle W, NB, \mathcal{X}, V\rangle$, $V \colon At \to \mathcal{X}$ is a valuation function on the set of propositional variables At.*

In an NB-neighborhood frame, if there exists an element $g \in W$ such that

$$NB(g) = \big\{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\big\},$$

then g is called **omniscient** and we call such frames **rooted NB-neighborhood frames**.

Definition 2. (*Truth*) *Let $\mathfrak{M} = \langle W, NB, \mathcal{X}, V\rangle$ be an NB-neighborhood model and $w \in W$. Truth of a propositional formula in a world w is defined inductively as follows.*

1. $\mathfrak{M}, w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$;
2. $\mathfrak{M}, w \Vdash A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ and $\mathfrak{M}, w \Vdash B$;
3. $\mathfrak{M}, w \Vdash A \vee B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ or $\mathfrak{M}, w \Vdash B$;
4. $\mathfrak{M}, w \Vdash A \to B \quad \Leftrightarrow \quad \big(A^{\mathfrak{M}}, B^{\mathfrak{M}}\big) \in NB(w)$;

5. $\mathfrak{M}, w \nVdash \bot$,

where $A^{\mathfrak{M}} := \{w \in W \mid \mathfrak{M}, w \Vdash A\}$*. Sets* $X \subseteq W$ *such that* $X = A^{\mathfrak{M}}$ *are called* ***definable****;* A *is* ***valid*** *in* $\mathfrak{M}$*,* $\mathfrak{M} \Vdash A$*, if for all* $w \in W$*,* $\mathfrak{M}, w \Vdash A$*. We write* $\Vdash A$ *if* $\mathfrak{M} \Vdash A$ *for all* $\mathfrak{M}$*. Also we define* $\Gamma \Vdash A$ *iff for all* $\mathfrak{M}$*,* $w \in \mathfrak{M}$*, if* $\mathfrak{M}, w \Vdash \Gamma$ *then* $\mathfrak{M}, w \Vdash A$*.*

Proposition 1. *If* $\mathfrak{M} = \langle W, NB, \mathcal{X}, V\rangle$ *is an NB-neighborhood model then* $\mathfrak{M}$ *can be extended by adding an omniscient world to obtain a rooted NB-neighborhood model* $\mathfrak{M}'$ *such that for all formulas* A *and for all* $w \in W$*,*

$$\mathfrak{M}, w \Vdash A \quad \text{iff} \quad \mathfrak{M}', w \Vdash A.$$

Proof. We add a world g to W and make a new model $\mathfrak{M}' = \langle W', NB', \mathcal{X}', V'\rangle$, with $W' = W \cup \{g\}$, $\mathcal{X}' = \{X, X \cup g \mid X \in \mathcal{X}\}$, for all propositional letters p, $(p)^{\mathfrak{M}'} = (p)^{\mathfrak{M}}$, and for all $w \in W$ and g:

$$NB'(g) = \left\{(X, Y) \in \mathcal{X}'^2 \mid X \subseteq Y\right\},$$

$$\begin{aligned} NB'(w) = NB(w) &\cup \{(X, Y \cup \{g\}) \mid (X, Y) \in NB(w)\} \\ &\cup \{(X \cup \{g\}, Y) \mid (X, Y) \in NB(w)\} \\ &\cup \{(X \cup \{g\}, Y \cup \{g\}) \mid (X, Y) \in NB(w)\}. \end{aligned}$$

The proof is by induction on A. The case where A is a proposition letter follows by definition. Conjunction and disjunction are easy. Now assume $\mathfrak{M}, w \Vdash B \to C$, then $(B^{\mathfrak{M}}, C^{\mathfrak{M}}) \in NB(w)$. By induction hypothesis we have $B^{\mathfrak{M}} = B^{\mathfrak{M}'} \cap W$ and $C^{\mathfrak{M}} = C^{\mathfrak{M}'} \cap W$. So by definition of NB':

$$(B^{\mathfrak{M}'}, C^{\mathfrak{M}'}) \in NB'(w).$$

That is $\mathfrak{M}', w \Vdash B \to C$.

Next assume $\mathfrak{M}', w \Vdash B \to C$. Then $(B^{\mathfrak{M}'}, C^{\mathfrak{M}'}) \in NB'(w)$. So by induction hypothesis and definition of NB', $(B^{\mathfrak{M}}, C^{\mathfrak{M}}) \in NB(w)$. That is $\mathfrak{M}, w \Vdash B \to C$. □

Corollary 1. *Validity in NB-neighborhood frames and rooted NB-neighborhood frames is the same.*

Theorem 1. *Let* $\mathfrak{M}$ *be a rooted NB-neighborhood model, then:*

1. $\mathfrak{M} \Vdash A \to B$ *iff* $A^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$.
2. $\Vdash A \to B$ *iff for all models* $\mathfrak{M}$, $A^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$.

Proof. The easy proof can be found in [7]. □

Definition 3. WF *is the logic given by the following axioms and rules,*

1. $A \to A \vee B$	2. $B \to A \vee B$	3. $A \to A$
4. $A \wedge B \to A$	5. $A \wedge B \to B$	6. $\frac{A \quad A \to B}{B}$

7. $\frac{A \to B \quad A \to C}{A \to B \wedge C}$ 8. $\frac{A \to C \quad B \to C}{A \vee B \to C}$ 9. $\frac{A \to B \quad B \to C}{A \to C}$
10. $\frac{A}{B \to A}$ 11. $\frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \to C) \leftrightarrow (B \to D)}$ 12. $\frac{A \quad B}{A \wedge B}$
13. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$ 14. $\bot \to A$

In this paper we are not concerned with negation. The results are independent of the inclusion of Axiom 14. In this section $\vdash$ will mean $\vdash_{\mathsf{WF}}$.

Definition 4. *We define $\Gamma \vdash A$ if there is a derivation of A from Γ and theorems of* WF *using the rules $\frac{A \quad B}{A \wedge B}$ and $\frac{A \quad A \to B}{B}$ where in the latter case the restriction is that $A \to B$ has to be provable in* WF*.*

Theorem 2. (*Weak Deduction Theorem*) *$A \vdash B$ iff $\vdash A \to B$.*

Proof. The proof can be found in [7]. □

Theorem 3. *The logic* WF *is sound with respect to the class of rooted NB-neighborhood frames.*

Proof. The proof uses Theorem 1 and can be found in [7]. □

Theorem 4. *The logic* WF *is sound with respect to the class of NB-neighborhood frames.*

Proof. Assume $\Gamma \vdash A$. We want to show that $\Gamma \Vdash A$. Let $\mathfrak{M}$ be an *NB*-neighborhood model such that $\mathfrak{M}, w \Vdash \Gamma$. Then by Proposition 1, there exists rooted *NB*-neighborhood model $\mathfrak{M}'$ such that $\mathfrak{M}', w \Vdash \Gamma$. So, by Theorem 3, $\mathfrak{M}', w \Vdash A$ and then by Proposition 1, we can conclude that $\mathfrak{M}, w \Vdash A$. That is $\Gamma \Vdash A$. □

Theorem 5. *The logic* WF *is strongly complete with respect to the class of NB-neighborhood frames.*

Proof. The proof can be found in [7]. □

3 N-Neighborhood Semantics

In this section we recall the N-neighborhoods, also introduced in [7]. In [4] the relationship with NB-neighborhoods was clarified to a certain extent, but here with give a much fuller explantion of the connections and differences, and we give complete proofs.

Definition 5. *$\mathfrak{F} = \langle W, N, \mathcal{X} \rangle$ is an **N-Neighborhood Frame** if W is a non-empty set, $\mathcal{X}$ is a non-empty collection of subsets of W such that $\emptyset$ and W belong to $\mathcal{X}$, and $\mathcal{X}$ is closed under $\cup$, $\cap$ and $\to$ defined by*

$$U \to V := \left\{ w \in W \mid \overline{U} \cup V \in N(w) \right\},$$

N is a function from W into $\mathcal{P}(\mathcal{X})$, and for each $w \in W$, $W \in N(w)$.

Valuation $V : At \to \mathcal{X}$ makes $\mathfrak{M} = \langle W, N, \mathcal{X}, V\rangle$ an ***N-Neighborhood Model*** *with the clause:*

$$\mathfrak{M}, w \Vdash A \to B \;\Leftrightarrow\; \{v \mid v \Vdash A \;\Rightarrow\; v \Vdash B\} = \overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \in N(w),$$

and A is ***valid*** *in $\mathfrak{M}$, $\mathfrak{M} \Vdash A$, if for all $w \in W$, $\mathfrak{M}, w \Vdash A$. We write $\Vdash A$ if $\mathfrak{M} \Vdash A$ for all $\mathfrak{M}$. Also we define $\Gamma \Vdash A$ iff for all $\mathfrak{M}$, $w \in \mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash A$.*

In N-neighborhood frames, if there exists element $g \in W$, such that $N(g) = \{W\}$, then g is called **omniscient** and we call such frames, **rooted N-neighborhood frames**.

Proposition 2. *If $\mathfrak{M} = \langle W, N, \mathcal{X}, V\rangle$ is an N-neighborhood model then $\mathfrak{M}$ can be extended by adding an omniscient world to obtain a rooted N-neighborhood model $\mathfrak{M}'$ such that for all formulas A and for all $w \in W$,*

$$\mathfrak{M}, w \Vdash A \;\text{ iff }\; \mathfrak{M}', w \Vdash A.$$

Proof. The proof can be found in [7]. □

Corollary 2. *Validity in N-neighborhood frames and rooted N-neighborhood frames is the same.*

Theorem 6. *Let $\mathfrak{M}$ is a rooted N-neighborhood model, then:*

1. *$\mathfrak{M} \Vdash A \to B$ iff $A^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$.*
2. *$\Vdash A \to B$ iff for all models $\mathfrak{M}$, $A^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$.*

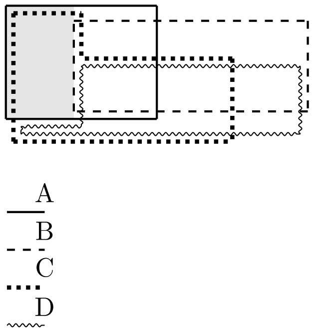
Proof. (1) Assume $\mathfrak{M} \Vdash A \to B$. Then, for all $w \in \mathfrak{M}$, $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \in N(w)$. Therefore $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} = W$, since $N(g) = \{W\}$. So $A^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$. For the other direction, by assumption we have $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} = W$, so, for all $w \in \mathfrak{M}$, $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \in N(w)$, i.e. $\mathfrak{M} \Vdash A \to B$.

(2) Follows immediately from (1). □

In [7] the question whether validity in NB-neighborhoods and N-neighborhoods is the same was left unsolved. In [4] it was discovered that the difference resides in the rule N. To the system WF we add this rule to obtain the logic $\mathsf{WF_N}$:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \quad (\mathsf{N})$$

As usual a rule like N is considered to be valid on a frame $\mathfrak{F}$ if, on each $\mathfrak{M}$ on $\mathfrak{F}$ on which the premises of the rule are valid, the conclusion is valid as well. In the picture below the idea behind the rule is exhibited. The conclusion $(A \to B) \leftrightarrow (C \to D)$ of the rule is valid if $\overline{A} \cup B = \overline{C} \cup D$ or equivalently if $A \cap \overline{B} = C \cap \overline{D}$ or if $A \backslash B = C \backslash D$. In the picture the latter is the grey part. The four assumptions of the rule force the picture to be essentially as given (e.g. $A \to B \vee C$ means $A \subseteq B \cup C$) and make sure that indeed $A \backslash B = C \backslash D$.



Definition 6. *We define* $\Gamma \vdash_{\mathsf{WF_N}} A$ *if there is a derivation of* A *from* Γ *and theorems of* $\mathsf{WF_N}$ *using the rules* $\frac{A \quad B}{A \wedge B}$ *and* $\frac{A \quad A \to B}{B}$, *where in the latter case the restriction is that* $A \to B$ *has to be provable in* $\mathsf{WF_N}$.

Theorem 7. (*Weak Deduction Theorem*) $A \vdash_{\mathsf{WF_N}} B$ *iff* $\vdash_{\mathsf{WF_N}} A \to B$.

Proof. $\Rightarrow$: By induction on the length of the proof.
If B is an axiom. Then $\vdash B$, so by rule 10, $\vdash A \to B$.
If $A \vdash A$ is covered by $\vdash A \to A$.
If $A \vdash B$ and $A \vdash C$, then by the induction hypothesis $\vdash A \to B$ and $\vdash A \to C$, so by rule 7 we obtain $\vdash A \to B \wedge C$.
If $A \vdash B$ and $\vdash B \to C$, then by the induction hypothesis $\vdash A \to B$, so by rule 9 we obtain $\vdash A \to C$.
$\Leftarrow$: This is the use of modus ponens that is allowed. ☐

Theorem 8. *The logic* $\mathsf{WF_N}$ *is sound with respect to the class of rooted* N*-neighborhood frames.*

Proof. Recall that, by Theorem 6(1), for all E, F, $\mathfrak{M} \Vdash E \to F$ iff $E^{\mathfrak{M}} \subseteq F^{\mathfrak{M}}$. We only check rule (N). Assume,

1. $\mathfrak{M} \Vdash A \to B \vee C$, i.e. $A^{\mathfrak{M}} \subseteq B^{\mathfrak{M}} \cup C^{\mathfrak{M}}$,
2. $\mathfrak{M} \Vdash C \to A \vee D$, i.e. $C^{\mathfrak{M}} \subseteq A^{\mathfrak{M}} \cup D^{\mathfrak{M}}$,
3. $\mathfrak{M} \Vdash A \wedge C \wedge D \to B$, i.e. $A^{\mathfrak{M}} \cap C^{\mathfrak{M}} \cap D^{\mathfrak{M}} \subseteq B^{\mathfrak{M}}$,
4. $\mathfrak{M} \Vdash A \wedge C \wedge B \to D$, i.e. $A^{\mathfrak{M}} \cap C^{\mathfrak{M}} \cap B^{\mathfrak{M}} \subseteq D^{\mathfrak{M}}$.

To get the conclusion $(A \to B) \leftrightarrow (C \to D)$ it is sufficent to prove that $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} = \overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}$ because then $w \Vdash A \to B$ iff $w \Vdash C \to D$ and hence $(A \to B)^{\mathfrak{M}} = (C \to D)^{\mathfrak{M}}$. By symmetry, it will suffice to show that $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \subseteq \overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}$. Let $w \in \overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}$. Then $w \in \overline{A^{\mathfrak{M}}}$ or ($w \in A^{\mathfrak{M}}$ and $w \in B^{\mathfrak{M}}$). If $w \in \overline{A^{\mathfrak{M}}}$, we distinguish the cases $w \in D^{\mathfrak{M}}$ and $w \in \overline{D^{\mathfrak{M}}}$. In the first case we are done directly. In the second case, we can conclude from (2) that $w \in \overline{C^{\mathfrak{M}}}$ and we are done as well. If $w \in A^{\mathfrak{M}}$ and $w \in B^{\mathfrak{M}}$, we distinguish the cases $w \in \overline{C^{\mathfrak{M}}}$ and $w \in C^{\mathfrak{M}}$. In the first case we are done directly. In the second case, we can conclude from (4) that $w \in D^{\mathfrak{M}}$ and we are done as well. ☐

Theorem 9. *The logic* $\mathsf{WF_N}$ *is sound with respect to the class of N-neighborhood frames.*

Proof. By Proposition 2 and Theorem 8. □

Remark 1. The rule 11 follows from rule N, by:

1. $\vdash A \leftrightarrow C$	Assumption
2. $\vdash B \leftrightarrow D$	Assumption
3. $\vdash C \to A \vee D$	By 1, axiom 1 and rule 9
4. $\vdash A \wedge C \wedge B \to B$	By 2 and rule 9
5. $\vdash A \wedge C \wedge B \to D$	By 2 and 4
6. $\vdash A \to B \vee C$	By 1, axiom 2 and rule 9
7. $\vdash A \wedge C \wedge D \to D$	
8. $\vdash A \wedge C \wedge D \to B$	By 7, 2
9. $\vdash (A \to B) \leftrightarrow (C \to D)$	From 3,5,6,8 using rule N

Although this is not strictly necessary for completeness because we do not require our models to be rooted we will prove the disjunction property for $\mathsf{WF_N}$. As in our previous papers this is simple by using Kleene's $|$.

Definition 7. [5] We define $|A$ by induction on A, as follows

$\|p$	iff	$\vdash p$,
$\|A \wedge B$	iff	$\|A$ and $\|B$,
$\|A \vee B$	iff	$\|A$ or $\|B$,
$\|A \to B$	iff	$\vdash A \to B$ and (if $\|A$ then $\|B$).

Theorem 10. $|A \ \Leftrightarrow\ \vdash_{\mathsf{WF_N}} A$.

Proof. The proof is by induction on A and a trivial modification of the standard one for IPC. We only check the direction from right to left for the rule N. Let $|A \to B \vee C$, $|C \to A \vee D$, $|A \wedge C \wedge D \to B$ and $|A \wedge C \wedge B \to D$, we want to show $|(A \to B) \leftrightarrow (C \to D)$. By induction hypothsis and rule N we conclude $\vdash_{\mathsf{WF_N}} (A \to B) \leftrightarrow (C \to D)$. Now let $|A \to B$, we will show that $|C \to D$. By induction hypothesis, rule N and MP, it is clear that $\vdash_{\mathsf{WF_N}} C \to D$. So, assume $|C$. Then by $|C \to A \vee D$ and the definition of $|$, $|A$ or $|D$. In the $|D$ case we are done directly. In the $|A$ case, by $|A \to B$, we have $|B$ and so by $|A \wedge C \wedge B \to D$, we have $|D$. The other direction is as usual. □

Theorem 11. *If* $\vdash_{\mathsf{WF_N}} A \vee B$ *then* $\vdash_{\mathsf{WF_N}} A$ *or* $\vdash_{\mathsf{WF_N}} B$.

Proof. Assume $\vdash_{\mathsf{WF_N}} A \vee B$, by Theorem 10, $|A \vee B$. So $|A$ or $|B$. Again Theorem 10 shows that $\vdash_{\mathsf{WF_N}} A$ or $\vdash_{\mathsf{WF_N}} B$. □

Definition 8. *A set of sentences* Δ *is a* ***prime theory*** *if and only if*

1. $A, B \in \Delta \ \Rightarrow\ A \wedge B \in \Delta$,
2. $\vdash A \to B$ and $A \in \Delta \ \Rightarrow\ B \in \Delta$,
3. $\vdash A \ \Rightarrow\ A \in \Delta$,

4. $A \vee B \in \Delta \;\Rightarrow\; A \in \Delta$ or $B \in \Delta$.

Lemma 1. *Δ is a theory $\Longleftrightarrow$ $\Delta \vdash_{\mathsf{WF_N}} A$ if and only if $A \in \Delta$.*

Proof. The proof is similar to the WF case and can be found in [7]. □

Theorem 12. *If $\Sigma \nvdash_{\mathsf{WF_N}} D$ then there is a prime theory Δ such that $\Delta \supseteq \Sigma$ and $D \notin \Delta$.*

Proof. The proof is similar to the WF case and can be found in [7]. □

Definition 9. *Let $W_{\mathsf{WF_N}}$ be the set of all consistent prime theories of $\mathsf{WF_N}$. Given a formula A, we define $[\![A]\!] = \{\Delta \mid \Delta \in W_{\mathsf{WF_N}},\ A \in \Delta\}$. The **N-Canonical model** $\mathfrak{M}_{\mathsf{WF_N}} = \langle W, N, \mathcal{X}, V\rangle$ is defined by:*

1. *$W = W_{\mathsf{WF_N}}$,*
2. *For each $\Gamma \in W$, $N(\Gamma) = \{\overline{[\![A]\!]} \cup [\![B]\!] \mid A \to B \in \Gamma\}$,*
3. *$\mathcal{X}$ is the set of all $[\![A]\!]$,*
4. *If $p \in At$, then $V(p) = [\![p]\!] = \{\Gamma \mid \Gamma \in W \text{ and } p \in \Gamma\}$.*

Lemma 2. *Let C and D are formulas. Then*

$$[\![C]\!] \subseteq [\![D]\!] \quad \textit{iff} \quad \vdash_{\mathsf{WF_N}} C \to D.$$

Proof. Let $\nvdash_{\mathsf{WF_N}} C \to D$. Then by the Weak Deduction Theorem $C \nvdash D$. Let $\Sigma = \{C\}$, then by Theorem 12, there exists a prime theory Γ such that, $C \in \Gamma$ and $D \notin \Gamma$. That is $[\![C]\!] \nsubseteq [\![D]\!]$.

Now let $\Gamma \in W_{\mathsf{WF_N}}$, $C \in \Gamma$ and $\vdash_{\mathsf{WF_N}} C \to D$. Then by definition of theory $D \in \Gamma$. □

Lemma 3. *Let $[\![A]\!] = [\![C]\!]$, $[\![B]\!] = [\![D]\!]$ and $\overline{[\![A]\!]} \cup [\![B]\!] \in N(\Gamma)$, then $C \to D \in \Gamma$.*

Proof. By Lemma 2, we have $\vdash_{\mathsf{WF_N}} A \leftrightarrow C$ and $\vdash_{\mathsf{WF_N}} B \leftrightarrow D$. Then by Remark 1 we will have $\vdash_{\mathsf{WF_N}} (A \to B) \leftrightarrow (C \to D)$. By definition of neighborhood function in N-canonical model, $A \to B \in \Gamma$. Hence by the definition of prime theory we conclude that, $C \to D \in \Gamma$. □

Lemma 4. *Let $\overline{[\![A]\!]} \cup [\![B]\!] = \overline{[\![C]\!]} \cup [\![D]\!]$, then $\mathsf{WF_N} \vdash (A \to B) \leftrightarrow (C \to D)$.*

Proof. Assume $\overline{[\![A]\!]} \cup [\![B]\!] = \overline{[\![C]\!]} \cup [\![D]\!]$. By rule N it is suffices to show

1. $\mathsf{WF_N} \vdash A \to B \vee C$,
2. $\mathsf{WF_N} \vdash A \wedge C \wedge D \to B$,
3. $\mathsf{WF_N} \vdash C \to A \vee D$,
4. $\mathsf{WF_N} \vdash A \wedge C \wedge B \to D$.

We will show 1 and 2; 3 and 4 are analogous.

1. From $\overline{[\![A]\!]} \cup [\![B]\!] = \overline{[\![C]\!]} \cup [\![D]\!]$ we get $[\![A]\!] \cap \overline{[\![B]\!]} = [\![C]\!] \cap \overline{[\![D]\!]}$. We have $[\![A]\!] \subseteq [\![B]\!] \cup [\![A]\!]$, so also, $[\![A]\!] \subseteq [\![B]\!] \cup ([\![A]\!] \cap \overline{[\![B]\!]})$, This means that $[\![A]\!] \subseteq [\![B]\!] \cup ([\![C]\!] \cap \overline{[\![D]\!]})$, so $[\![A]\!] \subseteq [\![B]\!] \cup [\![C]\!]$. By Lemma 2, this implies that $\mathsf{WF_N} \vdash A \to B \vee C$.

2. Again using $[\![A]\!] \cap \overline{[\![B]\!]} = [\![C]\!] \cap \overline{[\![D]\!]}$, we get $[\![A]\!] \cap [\![C]\!] \cap [\![D]\!] \cap \overline{[\![B]\!]} = [\![A]\!] \cap \overline{[\![B]\!]} \cap [\![C]\!] \cap [\![D]\!] = [\![C]\!] \cap \overline{[\![D]\!]} \cap [\![C]\!] \cap [\![D]\!] = \emptyset$. So, $[\![A]\!] \cap [\![C]\!] \cap [\![D]\!] \subseteq [\![B]\!]$, and, reasoning as above, $\mathsf{WF_N} \vdash A \wedge C \wedge D \to B$. □

Lemma 5. *(Truth Lemma) In the N-canonical model* $\mathfrak{M}_{\mathsf{WF_N}}$*,*

$$E \in \Gamma \quad \text{iff} \quad \Gamma \Vdash E.$$

Proof. By induction on E. The atomic case holds by definition of N-canonical model.

($E := A \wedge B$) Let $\Gamma \in W_{\mathsf{WF_N}}$ and $\Gamma \Vdash A \wedge B$ then $\Gamma \Vdash A$ and $\Gamma \Vdash B$. By the induction hypothesis $A \in \Gamma$ and $B \in \Gamma$. Γ is a theory so $A \wedge B \in \Gamma$.

Now let $A \wedge B \in \Gamma$. We have $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, hence by definition of theory we conclude that $A \in \Gamma$ and $B \in \Gamma$. By induction hypothesis $\Gamma \Vdash A$ and $\Gamma \Vdash B$ so $\Gamma \Vdash A \wedge B$.

($E := A \vee B$) Let $\Gamma \in W_{\mathsf{WF_N}}$ and $\Gamma \Vdash A \vee B$ then $\Gamma \Vdash A$ or $\Gamma \Vdash B$. By the induction hypothesis $A \in \Gamma$ or $B \in \Gamma$. We have $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$ so by definition of theory we conclude that $A \vee B \in \Gamma$.

Now let $A \vee B \in \Gamma$. Γ is a prime so $A \in \Gamma$ or $B \in \Gamma$. By induction hypothesis we conclude that $\Gamma \Vdash A$ or $\Gamma \Vdash B$. That is $\Gamma \Vdash A \vee B$.

($E := A \to B$) Let $\Gamma \in W_{\mathsf{WF_N}}$, then

$$\begin{aligned} \Gamma \Vdash A \to B &\iff \overline{A^{\mathfrak{M}_{\mathsf{WF_N}}}} \cup B^{\mathfrak{M}_{\mathsf{WF_N}}} \in N(\Gamma) \\ \text{(by induction hypothesis)} &\iff \overline{[\![A]\!]} \cup [\![B]\!] \in N(\Gamma) \\ \text{(by definition, Lemma 3 and 4)} &\iff A \to B \in \Gamma. \end{aligned}$$

□

Theorem 13. *(Completeness of* $\mathsf{WF_N}$*) The logic* $\mathsf{WF_N}$ *is sound and strongly complete with respect to the class of N-neighbourhood frames.*

Proof. Soundness already shown in earlier lemmas.

Let $\Sigma \nvdash A$, then by Theorem 12, there is a prime theory $\Delta \supseteq \Sigma$ such that $A \notin \Delta$. So in the N-canonical model $\mathfrak{M}_{\mathsf{WF_N}}$ we will have $\mathfrak{M}_{\mathsf{WF_N}}, \Delta \Vdash \Sigma$ and $\mathfrak{M}_{\mathsf{WF_N}}, \Delta \nVdash A$. That is $\Sigma \nVdash_{\mathsf{WF_N}} A$. □

Definition 10. ***(Equivalence of NB-neighborhood and N-neighborhood frames)*** *Let* $\langle W, NB, \mathcal{X}\rangle$ *be an NB-neighborhood frame and* $\langle W, N, \mathcal{X}\rangle$ *be an N-neighborhood frame. We say that* $\langle W, NB, \mathcal{X}\rangle$ *and* $\langle W, N, \mathcal{X}\rangle$ *are* ***equivalent*** *if for all* $X, Y \in \mathcal{X}$*:*

$$(X, Y) \in NB(w) \quad \text{iff} \quad \overline{X} \cup Y \in N(w).$$

Definition 11. *In NB-neighborhood frames we define the* ***equivalence relation*** *as follows:*

$$(X, Y) \equiv (X', Y') \quad \text{iff} \quad \overline{X} \cup Y = \overline{X'} \cup Y'.$$

Lemma 6. *Let* $\langle W, N, \mathcal{X}\rangle$ *be an N-neighborhood frame. Then there is an equivalent NB-neighborhood frame* $\langle W, NB, \mathcal{X}\rangle$ *closed under equivalence relation.*

Proof. The proof is straightforward by considering, for each $w \in W$,
$NB(w) = \left\{(X, Y) \mid \overline{X} \cup Y \in N(w)\right\}$. □

Theorem 14. *The logic* $\mathsf{WF_N}$ *is sound and strongly complete with respect to the class of NB-neighbourhood frames that are closed under equivalence relation.*

Proof. By Theorem 13 and Lemma 6. □

Lemma 7. *Let $\langle W, NB, \mathcal{X}\rangle$ be an NB-neighborhood frame closed under equivalence relation. Then there is an equivalent N-neighborhood frame $\langle W, N, \mathcal{X}\rangle$.*

Proof. The proof is straightforward by considering, for each $w \in W$,
$N(w) = \left\{\overline{X} \cup Y \mid (X, Y) \in NB(w)\right\}$. □

We list some relevant properties of N-neighborhood frames:

Definition 12. *1. $\mathfrak{F}$ is closed under **N-intersection** if and only if for all $w \in W$, if $\overline{X} \cup Y \in N(w)$ and $\overline{X} \cup Z \in N(w)$ then $\overline{X} \cup (Y \cap Z) \in N(w)$.*
*2. $\mathfrak{F}$ is closed under **N-union** if and only if for all $w \in W$, if $\overline{X} \cup Z \in N(w)$ and $\overline{Y} \cup Z \in N(w)$ then $\overline{X \cup Y} \cup Z \in N(w)$.*
*3. $\mathfrak{F}$ is an **N-transitive** frame if and only if for all $w \in W$, if $\overline{X} \cup Y \in N(w)$ and $\overline{Y} \cup Z \in N(w)$ then $\overline{X} \cup Z \in N(w)$.*
*4. $\mathfrak{F}$ is closed under **N-intersection superset** if and only if for all $w \in W$, if $\overline{X} \cup (Y \cap Z) \in N(w)$ and $Y, Z \in \mathcal{X}$, then $\overline{X} \cup Y \in N(w)$ and $\overline{X} \cup Z \in N(w)$.*
*5. $\mathfrak{F}$ is closed under **N-union superset** if and only if for all $w \in W$, if $\overline{X \cup Y} \cup Z \in N(w)$ and $X, Y \in \mathcal{X}$, then $\overline{X} \cup Z \in N(w)$ and $\overline{Y} \cup Z \in N(w)$.*

Lemma 8. *(a) The formula $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ characterizes the class of rooted N-neighborhood frames $\mathfrak{F}$ satisfying closure under N-intersection.*
(b) The formula $(p \to q) \wedge (r \to q) \to (p \vee r \to q)$ characterizes the class of rooted N-neighborhood frames $\mathfrak{F}$ satisfying closure under N-union.
(c) The formula $(p \to q) \wedge (q \to r) \to (p \to r)$ characterizes the class of rooted N-transitive N-nieghborhood frames $\mathfrak{F}$.
(d) The formula $(p \to q \wedge r) \to (p \to q) \wedge (p \to r)$ characterizes the class of rooted N-neighborhood frames $\mathfrak{F}$ satisfying closure under N-intersection superset.
(e) The formula $(p \vee r \to q) \to (p \to q) \wedge (r \to q)$ characterizes the class of rooted N-neighborhood frames $\mathfrak{F}$ satisfying closure under N-union superset.

Proof. (a) Let $\mathfrak{F} = \langle W, N, \mathcal{X}\rangle$ be closed under N-intersection and $\mathfrak{M}$ be any model based on $\mathfrak{F}$. We have to prove for all $w \in W$,

$$\{v \mid v \Vdash (p \to q) \wedge (p \to r) \Rightarrow v \Vdash p \to q \wedge r\} \in N(w).$$

For this purpose it is sufficient to show that

$$K = \{v \mid v \Vdash (p \to q) \wedge (p \to r) \Rightarrow v \Vdash p \to q \wedge r\} = W.$$

Let $w \in W$, $w \Vdash p \to q$ and $w \Vdash p \to r$ then

$$\overline{p^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \in N(w)$$

$$\overline{p^{\mathfrak{M}}} \cup r^{\mathfrak{M}} \in N(w).$$

The frame is closed under intersection so, $\overline{p^{\mathfrak{M}}} \cup (q \wedge r)^{\mathfrak{M}} \in N(w)$, that is $w \Vdash p \to q \wedge r$. Hence $W = K$ and so for all $w \in W$, $K \in N(w)$, since by the definition of N-neighborhood frames for all $w \in W$, $W \in N(w)$.

For the other direction, we use contraposition. Suppose that the class is not closed under N-intersection. That is there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $\overline{X} \cup Y \in N(w)$ and $\overline{X} \cup Z \in N(w)$, but $\overline{X} \cup (Y \cap Z) \notin N(w)$. To falsify $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ in the frame $\mathfrak{F}$ we should find $u \in W$ such that $u \nVdash (p \to q) \wedge (p \to r) \to (p \to q \wedge r)$.

For this purpose consider the valuation such that $p^{\mathfrak{M}} = X$, $r^{\mathfrak{M}} = Z$ and $q^{\mathfrak{M}} = Y$. Then we will have

$$\overline{p^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \in N(w)$$

$$\overline{p^{\mathfrak{M}}} \cup r^{\mathfrak{M}} \in N(w)$$

$$\overline{p^{\mathfrak{M}}} \cup (q \wedge r)^{\mathfrak{M}} \notin N(w).$$

So, $w \Vdash (p \to q) \wedge (p \to r)$ and $w \nVdash p \to q \wedge r$. That is $w \in (p \to q)^{\mathfrak{M}}$, $w \in (p \to r)^{\mathfrak{M}}$ and $w \notin (p \to q \wedge r)$. Therefore

$$w \notin \overline{(p \to q)^{\mathfrak{M}}} \cup \overline{(p \to r)^{\mathfrak{M}}} \cup (p \to q \wedge r)^{\mathfrak{M}}.$$

So we have

$$\overline{(p \to q)^{\mathfrak{M}}} \cup \overline{(p \to r)^{\mathfrak{M}}} \cup (p \to q \wedge r)^{\mathfrak{M}} \neq W.$$

Hence,

$$\mathfrak{M}, g \nVdash (p \to q) \wedge (p \to r) \to (p \to q \wedge r),$$

and $\mathfrak{F} \nVdash (p \to q) \wedge (p \to r) \to (p \to q \wedge r)$.

(b) Right to left is similar to (a).

$$\overline{p^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \in N(w)$$

$$\overline{r^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \in N(w)$$

$$\overline{(p \vee r)^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \notin N(w)$$

and we proceed as under (a).

(c) Right to left is similar to (a). For the other direction we use contraposition. Suppose that the class is not N-transitive. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $\overline{X} \cup Y \in N(w)$ and $\overline{Y} \cup Z \in N(w)$, but $\overline{X} \cup Z \notin N(w)$. Consider the valuation such that $p^{\mathfrak{M}} = X$, $r^{\mathfrak{M}} = Z$ and $q^{\mathfrak{M}} = Y$. Then we will have

$$\overline{p^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \in N(w)$$

$$\overline{q^{\mathfrak{M}}} \cup r^{\mathfrak{M}} \in N(w)$$

$$\overline{p^{\mathfrak{M}}} \cup r^{\mathfrak{M}} \notin N(w)$$

and we proceed as under (a).

(d) Right to left is similar to (a). For the other direction we use contraposition. Suppose that the class is not closed under N-intersection superset. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $\overline{X} \cup (Y \cap Z) \in N(w)$ and $Y, Z \in \mathcal{X}$, but $\overline{X} \cup Y \notin N(w)$ or $\overline{X} \cup Z \notin N(w)$. Assume $\overline{X} \cup Y \notin N(w)$ and consider the valuation such that, $p^{\mathfrak{M}} = X$, $r^{\mathfrak{M}} = Z$ and $q^{\mathfrak{M}} = Y$. Then we will have

$$\overline{p^{\mathfrak{M}}} \cup (q \wedge r)^{\mathfrak{M}} \in N(w)$$

$$\overline{p^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \notin N(w)$$

$$\overline{(p)^{\mathfrak{M}}} \cup (r)^{\mathfrak{M}} = V$$

and we proceed as under (a).

(e) Right to left is similar to (a). For the other direction we use contraposition. Suppose that the class is not closed under N-union superset. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $\overline{X \cup Y} \cup Z \in N(w)$ and $X, Y \in \mathcal{X}$, but $\overline{X} \cup Z \notin N(w)$ or $\overline{Y} \cup Z \notin N(w)$. Consider the valuation such that $p^{\mathfrak{M}} = X$, $r^{\mathfrak{M}} = Y$ and $q^{\mathfrak{M}} = Z$. Then we will have

$$\overline{p^{\mathfrak{M}} \cup r^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \in N(w)$$

$$\overline{p^{\mathfrak{M}}} \cup q^{\mathfrak{M}} \notin N(w)$$

$$\overline{r^{\mathfrak{M}}} \cup (q)^{\mathfrak{M}} = V$$

and we proceed as under (a). □

In the remainder of this section we will be interested in the following axiom schemas.

(C) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(D) $(A \to B) \wedge (C \to B) \to (A \vee C \to B)$
(I) $(A \to B) \wedge (B \to C) \to (A \to C)$
($\widehat{\mathsf{C}}$) $(A \to B \wedge C) \to (A \to B) \wedge (A \to C)$
($\widehat{\mathsf{D}}$) $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$

Lemma 9. *(a) If $\mathsf{WF_N C} \subseteq L$, then the N-canonical model of logic L is rooted and closed under N-intersection.*

(b) If $\mathsf{WF_N D} \subseteq L$, then the N-canonical model of logic L is rooted and closed under N-union.

(c) If $\mathsf{WF_N I} \subseteq L$, then the N-canonical model of logic L is rooted and closed under N-transitive.

(d) If $\mathsf{WF_N}\widehat{\mathsf{C}} \subseteq L$, then the N-canonical model of logic L is rooted and closed under N-intersection superset.

(e) If $\mathsf{WF_N}\widehat{\mathsf{D}} \subseteq L$, then the N-canonical model of logic L is rooted and closed under N-union superset.

Proof. We only prove (a). The other cases are similar.

(a) Suppose that in the N-canonical model of logic L, $\overline{X} \cup Y \in N(\Gamma)$ and $\overline{X} \cup Z \in N(\Gamma)$. By definition of N in the N-canonical model there exist formulas A, B and C such that $[\![A]\!] = X$, $[\![B]\!] = Y$ and $[\![C]\!] = Z$, where $A \to B \in \Gamma$ and $A \to C \in \Gamma$. Hence $(A \to B) \wedge (A \to C) \in \Gamma$ and so using (C), $A \to B \wedge C \in \Gamma$. Hence $\overline{[\![A]\!]} \cup [\![B \wedge C]\!] = \overline{X} \cup (Y \cap Z) \in N(\Gamma)$. So N is closed under intersection.

4 Differences between N-validity and NB-validity

In this section we will be interested in the following axiom schemas which make the difference between NB-frames and N-frames more concrete.

($\mathsf{N_a}$) $(A \to B) \leftrightarrow (A \vee B \to B)$
($\mathsf{N_b}$) $(A \to B) \leftrightarrow (A \to A \wedge B)$
($\mathsf{N_c}$) $(A \wedge B \to C) \leftrightarrow (A \wedge B \to A \wedge C)$
($\mathsf{N_d}$) $(A \to B \vee C) \leftrightarrow (B \vee A \to B \vee C)$

Lemma 10. *1.* $\mathsf{WF_N} \vdash \mathsf{N_a}$.
2. $\mathsf{WF_N} \vdash \mathsf{N_b}$.
3. $\mathsf{WF_N} \vdash \mathsf{N_c}$.
4. $\mathsf{WF_N} \vdash \mathsf{N_d}$.

Proof. The proofs are easy. We only prove 1:

1. $\vdash A \vee B \leftrightarrow A \vee B$
2. $\vdash A \wedge (A \vee B) \wedge B \to B$
3. $\vdash A \to B \vee (A \vee B)$
4. $\vdash (A \to B) \leftrightarrow (A \vee B \to B)$ From 1,2 and 3 using rule N

□

Lemma 11. *1.* $\mathsf{WF} \nvdash \mathsf{N_a}$.
2. $\mathsf{WF} \nvdash \mathsf{N_b}$.
3. $\mathsf{WF} \nvdash \mathsf{N_c}$.
4. $\mathsf{WF} \nvdash \mathsf{N_d}$.

Proof. 1. ⇒: Consider the rooted NB-neighborhood frame $\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ with

$$W = \{w, g\},\ NB(w) = \{(\{g\}, \{w\})\} \cup \big\{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\big\}.$$

Also consider the valuation $p^{\mathfrak{M}} = \{g\}$, $q^{\mathfrak{M}} = \{w\}$. With this valuation we can conclude $g \nVdash (p \to q) \to (p \vee q \to q)$.

⇐: Consider the rooted NB-neighborhood frame $\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ with

$$W = \{w, g\},\ NB(w) = \{(\{g, w\}, \{w\})\} \cup \big\{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\big\}.$$

Also consider the valuation $p^{\mathfrak{M}} = \{g\}$, $q^{\mathfrak{M}} = \{w\}$. With this valuation we can conclude $g \nVdash (p \vee q \to q) \to (p \to q)$.

2. ⇒: Consider the rooted NB-neighborhood frame $\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ with

$$W = \{w, g\},\ NB(w) = \{(\{g\}, \{w\})\} \cup \big\{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\big\}.$$

Also consider the valuation $p^{\mathfrak{M}} = \{g\}$ and $q^{\mathfrak{M}} = \{w\}$. With this valuation we can conclude $g \nVdash (p \to q) \to (p \to p \wedge q)$.

⇐: Consider the rooted NB-neighborhood frame $\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ with

$$W = \{w, g\},\ NB(w) = \{(\{g\}, \varnothing)\} \cup \big\{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\big\}.$$

Also consider the valuation $p^{\mathfrak{M}} = \{g\}$ and $q^{\mathfrak{M}} = \{w\}$. With this valuation we can conclude $g \nVdash (p \to p \wedge q) \wedge (p \to q)$.

3. $\Rightarrow$: Consider the rooted NB-neighborhood frame $\mathfrak{F} = \langle W, NB, \mathcal{X} \rangle$ with

$$W = \{w, v, g\}, NB(w) = NB(g)$$

$$NB(v) = \{(\{g\}, \{w, v\})\} \cup \{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\}.$$

Also consider the valuation $p^{\mathfrak{M}} = \{w, g\}$, $q^{\mathfrak{M}} = \{v, g\}$ and $V(c) = \{w, v\}$. With this valuation we can conclude $g \nVdash (p \wedge q \to c) \to (p \wedge q \to p \wedge c)$.

$\Leftarrow$: Consider the NB-neighborhood frame $\mathfrak{F} = \langle W, g, NB, \mathcal{X} \rangle$ with

$$W = \{w, v, g\}, NB(w) = NB(g)$$

$$NB(v) = \{(\{g\}, \{w\})\} \cup \{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\}.$$

Also consider the valuation $p^{\mathfrak{M}} = \{w, g\}$, $q^{\mathfrak{M}} = \{v, g\}$ and $V(c) = \{w, v\}$. With this valuation we can conclude $g \nVdash (p \wedge q \to p \wedge c) \to (p \wedge q \to c)$.

4. The proof of this case is similar to 3. □

5 Completeness for some Logics above WF with NB-neighborhood frames

We list some relevant properties of NB-neighborhood frames:

Definition 13. *1. $\mathfrak{F}$ is closed under the $\mathsf{N_a}$-condition if and only if for all $w \in W$ and $X, Y \in \mathcal{X}$,*

$$(X, Y) \in NB(w) \iff (X \cup Y, Y) \in NB(w).$$

2. $\mathfrak{F}$ is closed under the $\mathsf{N_b}$-condition if and only if for all $w \in W$ and $X, Y \in \mathcal{X}$,

$$(X, Y) \in NB(w) \iff (X, X \cap Y) \in NB(w).$$

3. $\mathfrak{F}$ is closed under the $\mathsf{N_c}$-condition if and only if for all $w \in W$ and $X, Y \in \mathcal{X}$,

$$(X \cap Y, Z) \in NB(w) \iff (X \cap Y, X \cap Z) \in NB(w).$$

4. $\mathfrak{F}$ is closed under the $\mathsf{N_d}$-condition if and only if for all $w \in W$ and $X, Y \in \mathcal{X}$,

$$(X, Y \cup Z) \in NB(w) \iff (Y \cup X, Y \cup Z) \in NB(w).$$

*5. $\mathfrak{F}$ is closed under **weak intersection** if and only if for all $w \in W$ and $X, Y \in \mathcal{X}$, If $(X, Y) \in NB(w)$ then for all $Z \in \mathcal{X}$, $(X \cap Z, Y \cap Z) \in NB(w)$.*

*6. $\mathfrak{F}$ is closed under **weak union** if and only if for all $w \in W$ and $X, Y \in \mathcal{X}$, If $(X, Y) \in NB(w)$ then for all $Z \in \mathcal{X}$, $(X, Y \cup Z) \in NB(w)$.*

*7. $\mathfrak{F}$ is closed under the **superset equivalence relation** if and only if for all $w \in W$ and $X, Y, X', Y' \in \mathcal{X}$, if $(X, Y) \in NB(w)$ and $\overline{X} \cup Y \subseteq \overline{X'} \cup Y'$ then $(X', Y') \in NB(w)$.*

Lemma 12. *1.* $(p \to q) \leftrightarrow (p \vee q \to q)$ *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F}$ *closed under the* $\mathsf{N_a}$*-condition.*

2. $(p \to q) \leftrightarrow (p \to p \wedge q)$ *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F}$ *closed under the* $\mathsf{N_b}$*-condition.*

3. $(p \wedge q \to c) \leftrightarrow (p \wedge q \to p \wedge c)$ *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F}$ *closed under the* $\mathsf{N_c}$*-condition.*

4. $(p \to q \vee c) \leftrightarrow (q \vee p \to q \vee c)$ *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F}$ *closed under the* $\mathsf{N_d}$*-condition.*

5. $(p \to q) \to (r \wedge p \to r \wedge q)$ *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F}$ *satisfying closure under weak intersection.*

6. $(p \to q) \to (p \to r \vee q)$ *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F}$ *satisfying closure under weak union.*

Proof. The proofs are easy. We only prove 6:

6. Let $\mathfrak{F} = \langle W, NB, \mathcal{X} \rangle$ be closed under weak union and $\mathfrak{M}$ be any model based on $\mathfrak{F}$. We have to prove for all $w \in W$,

$$((p \to q)^{\mathfrak{M}}, (p \to r \vee q)^{\mathfrak{M}}) \in NB(w).$$

For this purpose it is sufficient to show that, $(p \to q)^{\mathfrak{M}} \subseteq (p \to r \vee q)^{\mathfrak{M}}$. Let $w \in W$, $w \Vdash p \to q$ then, $(p^{\mathfrak{M}}, q^{\mathfrak{M}}) \in NB(w)$. The frame is closed under weak union so, $(p^{\mathfrak{M}}, r^{\mathfrak{M}} \cup q^{\mathfrak{M}}) \in NB(w)$. That is, $w \Vdash p \to r \vee q$. Hence, by definition of neighborhood frames for all $w \in W$, $((p \to q)^{\mathfrak{M}}, (p \to r \vee q)^{\mathfrak{M}}) \in NB(w)$.

For the other direction we use contraposition. Suppose that the class is not closed under weak union. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $(X, Y) \in NB(w)$ and $Z \in \mathcal{X}$, but $(X, Y \cup Z) \notin NB(w)$. Consider the valuation such that, $p^{\mathfrak{M}} = X$, $q^{\mathfrak{M}} = Y$ and $r^{\mathfrak{M}} = Z$. Then we will have

$$(p^{\mathfrak{M}}, q^{\mathfrak{M}}) \in N(w)$$

$$(p^{\mathfrak{M}}, (r \vee q)^{\mathfrak{M}}) \notin N(w).$$

So $(p \to q)^{\mathfrak{M}} \subseteq (p \to r \vee q)^{\mathfrak{M}}$. Then by the definition of neighborhood frames $g \nVdash (p \to q) \to (p \to r \vee q)$. Therefore $\mathfrak{F} \nVdash (p \to q) \to (p \to r \vee q)$. □

In the remainder of this section we will be interested in the following axiom schemas and rule.

($\mathsf{C_W}$) $(A \to B) \to (C \wedge A \to C \wedge B)$
($\mathsf{D_W}$) $(A \to B) \to (A \to C \vee B)$
($\mathsf{N_2}$) $\dfrac{C \to A \vee D \qquad A \wedge C \wedge B \to D}{(A \to B) \to (C \to D)}$

Lemma 13. *(a) If* $\mathsf{WFN_a} \subseteq L$*, then the NB-canonical model of logic* L *is rooted and closed under the* $\mathsf{N_a}$*-condition.*

(b) If $\mathsf{WFN_b} \subseteq L$*, then the NB-canonical model of logic* L *is rooted and closed under the* $\mathsf{N_b}$*-condition.*

(c) If $\mathsf{WFN_c} \subseteq L$*, then the NB-canonical model of logic* L *is rooted and closed under the* $\mathsf{N_c}$*-condition.*

(d) If $\mathsf{WFN_d} \subseteq L$*, then the NB-canonical model of logic* L *is rooted and closed under the* $\mathsf{N_d}$*-condition.*
(f) If $\mathsf{WFC_W} \subseteq L$*, then the NB-canonical model of logic* L *is rooted and closed under weak intersection.*
(g) If $\mathsf{WFD_W} \subseteq L$*, then the NB-canonical model of logic* L *is rooted and closed under weak union.*

Proof. The proofs are easy. □

Lemma 14. *The rule* N *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ *that are closed under equivalence relation.*

Proof. The direction from right to left is immediate from Lemma 7.

For the other direction, we use contraposition. Suppose that the class is not closed under equivalence relation. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $(X, Y) \in NB(w)$ and $U, V \in \mathcal{X}$ and $\overline{X} \cup Y = \overline{U} \cup V$, but $(U, V) \notin NB(w)$. Since $p \vee q \leftrightarrow p \vee q$, $p \wedge (p \vee q) \wedge q \to q$ and $p \to q \vee (p \vee q)$ are provable, it suffices to falsify $(p \to q) \leftrightarrow (p \vee q \to q)$ on the frame. Consider the valuation such that $p^{\mathfrak{M}} = X$, $q^{\mathfrak{M}} = Y$ and assume $U = X \cup Y$ and $V = Y$. It is easy to show that $\overline{X} \cup Y = \overline{U} \cup V$. So $(p^{\mathfrak{M}}, q^{\mathfrak{M}}) \in NB(w)$ and $((p \vee q)^{\mathfrak{M}}, q^{\mathfrak{M}}) \notin NB(w)$, and consequently, $(p \to q)^{\mathfrak{M}} \nsubseteq (p \vee q \to q)^{\mathfrak{M}}$. So, $\mathfrak{M}, g \nVdash (p \to q) \leftrightarrow (p \vee q \to q)$. □

Lemma 15. *The rule* N_2 *characterizes the class of rooted NB-neighborhood frames* $\mathfrak{F} = \langle W, NB, \mathcal{X}\rangle$ *that are closed under the superset equivalence relation.*

Proof. Let for some $\mathfrak{M} = \langle W, NB, \mathcal{X}, V\rangle$ on a frame $\mathfrak{F}$, which is closed under the superset equivalence relation, $\mathfrak{M} \Vdash C \to A \vee D$ and $\mathfrak{M} \Vdash A \wedge C \wedge B \to D$. We have to prove that, $\mathfrak{M} \Vdash (A \to B) \to (C \to D)$. For this purpose we show that $(A \to B)^{\mathfrak{M}} \subseteq (C \to D)^{\mathfrak{M}}$. Let $w \in W$ and $w \Vdash A \to B$. Then $(A^{\mathfrak{M}}, B^{\mathfrak{M}}) \in NB(w)$. Now to show $(C^{\mathfrak{M}}, D^{\mathfrak{M}}) \in NB(w)$, it is sufficient to prove $\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \subseteq \overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}$. By assumption and Theorem 2.13(1) of [7], $C^{\mathfrak{M}} \subseteq A^{\mathfrak{M}} \cup D^{\mathfrak{M}}$ and $A^{\mathfrak{M}} \cap C^{\mathfrak{M}} \cap B^{\mathfrak{M}} \subseteq D^{\mathfrak{M}}$. So $\overline{C^{\mathfrak{M}}} \cup A^{\mathfrak{M}} \cup D^{\mathfrak{M}} = W$ and $\overline{A^{\mathfrak{M}}} \cup \overline{C^{\mathfrak{M}}} \cup \overline{B^{\mathfrak{M}}} \cup D^{\mathfrak{M}} = W$. By these assumption we have:

$$\begin{aligned}
\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} &= (\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}) \cap \overline{A^{\mathfrak{M}}} \cup \overline{C^{\mathfrak{M}}} \cup \overline{B^{\mathfrak{M}}} \cup D^{\mathfrak{M}} \\
&= ((\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}) \cap (\overline{A^{\mathfrak{M}}}) \cup ((\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}) \cap (\overline{C^{\mathfrak{M}}}) \cup \\
&\quad ((\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}) \cap (\overline{B^{\mathfrak{M}}}) \cup ((\overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}}) \cap (\overline{D^{\mathfrak{M}}}) \\
&= (B^{\mathfrak{M}} \cap \overline{A^{\mathfrak{M}}}) \cup (B^{\mathfrak{M}} \cap D^{\mathfrak{M}}) \cup \overline{A^{\mathfrak{M}}} \\
&= (B^{\mathfrak{M}} \cap (\overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}})) \cup \overline{A^{\mathfrak{M}}} \\
&= (B^{\mathfrak{M}} \cup (\overline{A^{\mathfrak{M}}}) \cap ((\overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}) \cup \overline{A^{\mathfrak{M}}}) \\
&\subseteq ((\overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}) \cup \overline{A^{\mathfrak{M}}}) = \overline{C^{\mathfrak{M}}} \cup D^{\mathfrak{M}}.
\end{aligned}$$

For the other direction, we use contraposition. Suppose that the class is not closed under the superset equivalence relation. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $(X, Y) \in NB(w)$ and $U, V \in \mathcal{X}$ and $\overline{X} \cup Y \subseteq \overline{U} \cup V$, but $(U, V) \notin NB(w)$. Since $p \wedge q \to q \vee (p \wedge r)$ and $q \wedge (p \wedge q) \wedge r \to p \wedge r$ are provable it suffices to falsify $(q \to r) \to (p \wedge q \to p \wedge r)$ on the frame. Consider the valuation such that $q^{\mathfrak{M}} = X$, $r^{\mathfrak{M}} = Y$, $(p \wedge q)^{\mathfrak{M}} = U$ and $(p \wedge r)^{\mathfrak{M}} = V$. It is easy to show that $\overline{(p)^{\mathfrak{M}}} \cup \overline{(q)^{\mathfrak{M}}} \cup (r)^{\mathfrak{M}} = \overline{(p \wedge q)^{\mathfrak{M}}} \cup (p \wedge r)^{\mathfrak{M}}$. Hence $\overline{X} \cup Y \subseteq \overline{U} \cup V$. So $(q^{\mathfrak{M}}, r^{\mathfrak{M}}) \in NB(w)$ and $((p \wedge q)^{\mathfrak{M}}, (p \wedge r)^{\mathfrak{M}}) \notin NB(w)$, and consequently, $(q \to r)^{\mathfrak{M}} \nsubseteq (p \wedge q \to p \wedge r)^{\mathfrak{M}}$. So, $\mathfrak{M}, g \nVdash (q \to r) \to (p \wedge q \to p \wedge r)$. □

Notation. The rule N can be derived from $\mathsf{N_2}$, so in the following we will write $\mathsf{WF_{N_2}}$, instead of $\mathsf{WF_N N_2}$.

Lemma 16. *Let* $\overline{[\![A]\!]} \cup [\![B]\!] \subseteq \overline{[\![C]\!]} \cup [\![D]\!]$, *then* $\mathsf{WF_{N_2}} \vdash (A \to B) \to (C \to D)$.

Proof. The proof is similar to Lemma 4. □

Lemma 17. *If* $\mathsf{WF_{N_2}} \subseteq L$, *then the NB-canonical model of logic L is rooted and closed under the superset equivalence relation.*

Proof. Suppose that in the *NB*-canonical model of logic L, $(X, Y) \in NB(\Gamma)$ and $\overline{X} \cup Y \subseteq \overline{X'} \cup Y'$. By definition of *NB* in the *NB*-canonical model there exist formulas A, B, C and D such that $(X, Y) = ([\![A]\!], [\![B]\!])$ and $(X', Y') = ([\![C]\!], [\![D]\!])$, where $A \to B \in \Gamma$. Using Lemma 16, $\vdash (A \to B) \to (C \to D)$. Hence $C \to D \in \Gamma$ and $(X', Y') \in NB(\Gamma)$. □

Definition 14. *The N-neighborhood frame* $\mathfrak{F} = \langle W, N, \mathcal{X} \rangle$ *is closed under* ***N-superset*** *if and only if for all* $w \in W$, *if* $\overline{X} \cup Y \in N(w)$ *and* $U, V \in \mathcal{X}$ *and* $\overline{X} \cup Y \subseteq \overline{U} \cup V$, *then* $\overline{U} \cup V \in N(w)$.

Lemma 18. *Let* $\langle W, NB, \mathcal{X} \rangle$ *be an NB-neighborhood frame closed under the superset equivalence relation. Then there is an equvalent N-neighborhood frame* $\langle W, N, \mathcal{X} \rangle$, *closed under N-superset.*

Proof. The proof is straightforward by considering, for each $w \in W$,
$N(w) = \left\{ \overline{X} \cup Y \mid (X, Y) \in NB(w) \right\}$. □

Theorem 15. *The logic* $\mathsf{WF_{N_2}}$ *is sound and strongly complete with respect to the class of N-neighbourhood frames that are closed under N-superset.*

Proof. By Lemmas 17 and 18. □

Lemma 19. *1.* $\mathsf{WF_{N_2}} \vdash \widehat{\mathsf{C}}$.
2. $\mathsf{WF_{N_2}} \vdash \widehat{\mathsf{D}}$.
3. $\mathsf{WF_{N_2}} \vdash \mathsf{C_W}$.
4. $\mathsf{WF_{N_2}} \vdash \mathsf{D}_W$.
5. $\mathsf{WF_{N_2}} \vdash (A \to A \wedge B \wedge C) \to (A \to A \wedge B)$.
6. $\mathsf{WF_{N_2}} \vdash (A \to A \wedge B) \to (C \wedge A \to C \wedge A \wedge B)$

Proof. The proofs are easy, we only prove 3.

3. By Lemma 17, $\mathsf{WF_{N_2}}$ is sound and complete with respect to the class of NB-neighborhood frames that are closed under the superset equivalence relation. It will be enough to show that in such frames $(A \to B)^{\mathfrak{M}} \subseteq (C \wedge A \to C \wedge B)^{\mathfrak{M}}$. Assume $\mathfrak{M}$ be an NB-neighborhood model on these frames, $w \in \mathfrak{M}$ and $w \Vdash$

$A \to B$, then $(A^{\mathfrak{M}}, B^{\mathfrak{M}}) \in NB(w)$. On the other hand, $\overline{(C \wedge A)^{\mathfrak{M}}} \cup (C \wedge B)^{\mathfrak{M}} = \overline{(C)^{\mathfrak{M}}} \cup \overline{(A)^{\mathfrak{M}}} \cup (B)^{\mathfrak{M}}$ and so, $\overline{(A)^{\mathfrak{M}}} \cup (B)^{\mathfrak{M}} \subseteq \overline{(C \wedge A)^{\mathfrak{M}}} \cup (C \wedge B)^{\mathfrak{M}}$. Then by the superset equivalence relation condition of these frames $((C \wedge A)^{\mathfrak{M}}, (C \wedge B)^{\mathfrak{M}}) \in NB(w)$, that is $w \Vdash C \wedge A \to C \wedge B$. □

Proposition 3. *1.* $\mathsf{WF}\widehat{\mathsf{C}} \vdash (A \to A \wedge B \wedge C) \to (A \to A \wedge B)$.
2. $\mathsf{WF}\widehat{\mathsf{D}} \vdash (A \vee B \vee C \to A) \to (A \vee B \to A)$.
3. $\mathsf{WFC}_W \vdash (A \to A \wedge B) \to (C \wedge A \to C \wedge A \wedge B)$.

Proof. The proofs are easy. □

The logic $\mathsf{WF}\widehat{\mathsf{C}}\widehat{\mathsf{D}}$ is complete with respect to the class of NB-neighborhood frames that are closed under upset and downset [7], i.e. for all $w \in W$,
(Upset) if $(X, Y) \in NB(w)$ and $Y \subseteq Z$ then $(X, Z) \in NB(w)$,
(Downset) if $(X, Y) \in NB(w)$ and $Z \subseteq X$ then $(Z, Y) \in NB(w)$.

Lemma 20. $\mathsf{WF}\widehat{\mathsf{C}}\widehat{\mathsf{D}} \nvdash (p \to q) \to (r \wedge p \to r \wedge q)$.

Proof. Consider the rooted NB-neighborhood frame $\mathfrak{F} = \langle W, NB, \mathcal{X} \rangle$ with,

$$\begin{aligned} &W = \{w, v, g\},\\ &NB(v) = NB(g),\\ &NB(w) = \{(\{w, v\}, \{v, g\}), (\{w\}, \{v, g\}), (\{w\}, \{v\}), (\{w, v\}, \{v\})\}\\ &\qquad\quad \cup \left\{(X, Y) \in \mathcal{X}^2 \mid X \subseteq Y\right\}. \end{aligned}$$

The frame $\mathfrak{F}$ is closed under upset and downset. Then consider the valuation $(p)^{\mathfrak{M}} = \{w, v\}$, $(q)^{\mathfrak{M}} = \{v\}$ and $(r)^{\mathfrak{M}} = \{w\}$. With this valuation we can conclude, $g \nVdash (p \to q) \to (r \wedge p \to r \wedge q)$, since $(p \to q)^{\mathfrak{M}} \nsubseteq (r \wedge p \to r \wedge q)^{\mathfrak{M}}$. □

Again, we list some relevant properties for NB-neighborhood frames:

Definition 15.
*1. $\mathfrak{F}$ is **quasi-reflexive** iff for all $w \in W$, if $(X, Y) \in NB(w)$ and $w \in X$, then $w \in Y$.*
*2. $\mathfrak{F}$ is **quasi-persistent** iff for all $w \in W$, if $w \in X$ and $X \in \mathcal{X}$, then for all $Y \in \mathcal{X}$, $(Y, X) \in NB(w)$.*
*3. $\mathfrak{F}$ is **quasi-transitive** iff for all $w \in W$, if $(X, Y) \in NB(w)$, then for all $C \in \mathcal{X}$, $(C, \{v \mid (X, Y) \in NB(v)\}) \in NB(w)$.*
*4. $\mathfrak{F}$ is **pseudo-transitive** iff for all $w \in W$, if $(X, Y) \in NB(w)$, then for all $C \in \mathcal{X}$, $(\{v \mid (Y, C) \in NB(v)\}, \{v \mid (X, C) \in NB(v)\}) \in NB(w)$.*

Lemma 21.

(a) *The formula $A \wedge (A \to B) \to B$ characterizes the class of rooted NB-neighborhood quasi-reflexive frames.*
(b) *The formula $A \to (B \to A)$ characterizes the class of rooted NB-neighborhood quasi-persistent frames.*
(c) *The formula $(A \to B) \to (C \to (A \to B))$ characterizes the class of rooted NB-neighborhood quasi-transitive frames.*

(c) The formula $(A \to B) \to ((B \to C) \to (A \to C))$ *characterizes the class of rooted NB-neighborhood pseudo-transitive frames.*

Proof. We only prove (b). The other cases are similar.

(b) Let $\mathfrak{F}$ be quasi-persistent and $\mathfrak{M} = \langle W, NB, \mathcal{X}, V \rangle$ be based on $\mathfrak{F}$. We have to prove for all $w \in W$, $(A^{\mathfrak{M}}, (B \to A)^{\mathfrak{M}}) \in NB(w)$. For this purpose it is sufficient to show that $A^{\mathfrak{M}} \subseteq (B \to A)^{\mathfrak{M}}$. Let $w \in W$ and $w \in A^{\mathfrak{M}}$. The frame is quasi-persistent, so by definition for all $Y \in \mathcal{X}$, $(Y, A^{\mathfrak{M}}) \in NB(w)$ and then $(B^{\mathfrak{M}}, A^{\mathfrak{M}}) \in NB(w)$. That is, $w \Vdash B \to A$. Hence, for all $w \in W$, $(A^{\mathfrak{M}}, (B \to A)^{\mathfrak{M}}) \in NB(w)$.

For the other direction, suppose that $\mathfrak{F} = \langle W, NB, \mathcal{X} \rangle$ is not quasi-persistent. Then there are $X, Y \in \mathcal{X}$ and $w \in X$ such that $(Y, X) \notin N(w)$. Consider the valuation such that, $A^{\mathfrak{M}} = X$ and $B^{\mathfrak{M}} = Y$. Then $w \Vdash A$ and $w \nVdash B \to A$, because $(B^{\mathfrak{M}}, A^{\mathfrak{M}}) \notin NB(w)$. So, we have $A^{\mathfrak{M}} \nsubseteq (B \to A)^{\mathfrak{M}}$. Then $g \nVdash A \to (B \to A)$ and hence $\mathfrak{F} \nVdash A \to (B \to A)$. □

In the remainder of this section we will be interested in the following axiom schemas known as extensions of the basic logic F of Kripke frames [2].

(R) $A \wedge (A \to B) \to B$
($\mathsf{P_T}$) $A \to (B \to A)$
($\mathsf{T_1}$) $(A \to B) \to (C \to (A \to B))$
($\mathsf{T_2}$) $(A \to B) \to ((B \to C) \to (A \to C))$

Lemma 22. *(a) If* $\mathsf{WFR} \subseteq L$*, then the NB-canonical model of logic L is rooted and quasi-reflexive.*
(b) If $\mathsf{WFP_T} \subseteq L$*, then the NB-canonical model of logic L is rooted and quasi-persistent.*
(c) If $\mathsf{WFT_1} \subseteq L$*, then the NB-canonical model of logic L is rooted and quasi-transitive.*
(d) If $\mathsf{WFT_2} \subseteq L$*, then the NB-canonical model of logic L is rooted and pseudo-transitive.*

Proof. We only prove (a). The other cases are similar.

(a) Suppose that in the *NB*-canonical model of logic L, $(X, Y) \in NB(\Gamma)$ and $\Gamma \in X$. By definition of *NB* in the *NB*-canonical model there exist formulas A, B such that $(X, Y) = ([\![A]\!], [\![B]\!])$, where $A \to B \in \Gamma$ and $A \in \Gamma$. Hence $A \wedge (A \to B) \in \Gamma$ and so, using (R), $B \in \Gamma$. Hence $\Gamma \in [\![B]\!] = Y$. □

6 Modal Companions

In this section we clarify the connection with modal logic, specifically with classical modal logic and monotone modal logic. We consider the translation $\Box$ from L, the language of propositional logic, to $L_{\Box}$, the language of modal propositional logic [2]. It is given by:

1. $p^{\Box} = p$;

2. $(A \wedge B)^\Box = A^\Box \wedge B^\Box$;
3. $(A \vee B)^\Box = A^\Box \vee B^\Box$;
4. $(A \to B)^\Box = \Box(A^\Box \to B^\Box)$.

Definition 16. *A system of modal logic is* ***classical*** *iff it is closed under RE* ($\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$) [1].

E is the smallest classical modal logic. The logic EN extends E by adding the axiom scheme $\Box\top$. Completeness holds for EN with respect neighborhood frames that contain the unit, i.e. for all $w \in W$, $W \in N(w)$ [1]. The clause for $\Box A$ in the neighborhood models is: $w \Vdash \Box A$ iff $A^{\mathfrak{M}} \in N(w)$.

Definition 17. *A system of modal logic is* ***monotone*** *iff it is closed under RM* ($\frac{A \to B}{\Box A \to \Box B}$) [3].

EM (M) is the smallest monotonic modal logic. Completeness holds for M with respect monotonic neighborhood frames, i.e. in $\mathfrak{F} = \langle W, N\rangle$, N is closed under superset [3].

We can interpret the neighborhood models in two ways. As an N-neighborhood model with $\Vdash_{\mathsf{WF_{N_2}}}$ and as a modal model with $\Vdash_{\mathsf{M}}$.

The N-neighborhood model $\langle W, N, V\rangle$ is closed under N-superset if and only if the N-neighborhood frame $\langle W, N\rangle$ is closed under N-superset. The logic $\mathsf{WF_{N_2}}$ is sound and strongly complete with respect to the class of N-neighbourhood frames that are closed under N-superset (Theorem 15).

Lemma 23. *Let $\mathfrak{M} = \langle W, N, V\rangle$ be an N-neighborhood model closed under N-superset. Then for all $w \in W$,*

$$\mathfrak{M}, w \Vdash_{\mathsf{WF_{N_2}}} A \quad \text{iff} \quad \mathfrak{M}, w \Vdash_{\mathsf{M}} A^\Box.$$

Proof. The proof is by induction on A. The atomic case holds by induction and the conjunction and disjunction cases are easy. We only check the implication case. So let $A = C \to D$, then

$$\begin{aligned}
\mathfrak{M}, w \Vdash_{\mathsf{WF_{N_2}}} C \to D &\iff \big\{v \mid v \nVdash_{\mathsf{WF_{N_2}}} C\big\} \cup \big\{v \mid v \Vdash_{\mathsf{WF_{N_2}}} D\big\} \in N(w) \\
\text{(by induction hypothesis)} &\iff \big\{v \mid v \nVdash_{\mathsf{M}} C^\Box\big\} \cup \big\{v \mid v \Vdash_{\mathsf{M}} D^\Box\big\} \in N(w) \\
&\iff \big\{v \mid v \Vdash_{\mathsf{M}} \neg C^\Box\big\} \cup \big\{v \mid v \Vdash_{\mathsf{M}} D^\Box\big\} \in N(w) \\
&\iff \big\{v \mid v \Vdash_{\mathsf{M}} \neg C^\Box \vee D^\Box\big\} \in N(w) \\
&\iff \mathfrak{M}, w \Vdash_{\mathsf{M}} \Box(\neg C^\Box \vee D^\Box) \\
&\iff \mathfrak{M}, w \Vdash_{\mathsf{M}} (C \to D)^\Box.
\end{aligned}$$

□

Theorem 16. *For all formulas A,*

$$\vdash_{\mathsf{WF_{N_2}}} A \quad \text{iff} \quad \vdash_{\mathsf{M}} A^\Box.$$

Proof. By Lemma 23. □

The classical modal logic EC extends E by adding the axiom scheme $(\Box A \wedge \Box B) \to \Box(A \wedge B)$. Completeness holds for EC with respect to the class of neighborhood frames that are closed under intersection.

Definition 18. *The NB-neighborhood frame* $\mathfrak{F}= \langle W, NB, \mathcal{X}\rangle$ *is closed under NB-**intersection** if and only if for all* $w \in W$*, if* $(X,Y) \in N(w)$ *and* $(X,Z) \in NB(w)$ *then* $(X, Y \cap Z) \in NB(w)$*.*

Theorem 17. *The logic* WFC *is sound and strongly complete with respect to the class of NB-neighborhood frames that are closed under NB-intersection.*

Proof. The proof can be found in [7]. □

The next result readily follows.

Theorem 18. *The logic* $\mathsf{WF_N C}$ *is sound and strongly complete with respect to the class of NB-neighborhood frames that are closed under NB-intersection and equivalence relation.*

Lemma 24. *Let* $\langle W, NB, \mathcal{X}\rangle$ *be an NB-neighborhood frame closed under equivalence relation and NB-intersection. Then there is an equivalent N-neighborhood frame* $\langle W, N, \mathcal{X}\rangle$*, closed under N-intersection.*

Proof. The proof is straightforward by considering, for each $w \in W$,
$N(w) = \left\{\overline{X} \cup Y \mid (X,Y) \in NB(w)\right\}$. □

Theorem 19. *The logic* $\mathsf{WF_N C}$ *is sound and strongly complete with respect to the class of N-neighborhood frames that are closed under N-intersection.*

Proof. By Theorem 18 and Lemma 24. □

Theorem 20. *For all formulas A,*

$$\vdash_{\mathsf{WF_N C}} A \quad \text{iff} \quad \vdash_{\mathsf{ENC}} A^{\Box}.$$

Proof. By Theorem 19 and Lemma 23. □

The logic K is the smallest normal, or Kripkean, modal logic. Corsi in [2] showed that the modal companion of subintuitionistic logic F is the logic K.

Lemma 25. *The logic* MCN *equals the logic* K [6]*.*

Completeness holds for K with respect to the class of augmented neighborhood frames, i.e. closed under superset and containes its core provided for all $w \in W$, $\bigcap_{X \in N(w)} X \in N(w)$.

Theorem 21. *The logic* $\mathsf{WF_{N_2} C}$ *is sound and strongly complete with respect to the class of N-neighborhood frames that are closed under N-superset and N-intersection.*

Proof. By Theorems 9 and 15. □

Theorem 22. *For all formulas A,*

$$\vdash_{\mathsf{WF_{N_2} C}} A \quad \text{iff} \quad \vdash_{\mathsf{K}} A^{\Box}.$$

Proof. By Theorem 21 and Lemma 23. □

Corollary 3. *The logic* $\mathsf{WF_{N_2} C}$ *equals the logic* F*.*

7 Conclusion

In the above the similarities and differences between N-frames and NB-frames were fully clarified. An interesting new logic $\mathsf{WF}_{\mathsf{N}_2}$ was axiomatized and it was established that the monotone logic M is a modal companion of this logic. It will be worthwhile to further study this logic, which takes a central place amid subintuitionistic logics.

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