Hunting for Tractable Languages for Judgment Aggregation

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Abstract

Judgment aggregation is a general framework for collective decision making that can be used to model many different settings. Due to its general nature, the worst case complexity of essentially all relevant problems in this framework is very high. However, these intractability results are mainly due to the fact that the language to represent the aggregation domain is overly expressive. We initiate an investigation of representation languages for judgment aggregation that strike a balance between (1) being limited enough to yield computational tractability results and (2) being expressive enough to model relevant applications. In particular, we consider the languages of Krom formulas, (definite) Horn formulas, and Boolean circuits in decomposable negation normal form (DNNF). We illustrate the use of the positive complexity results that we obtain for these languages with a concrete application: voting on how to spend a budget (i.e., participatory budgeting).

Introduction

Judgment aggregation is a general framework to study methods for collective opinion forming, that has been investigated in the area of computational social choice (see, e.g., Endriss 2016, Grossi and Pigozzi 2014). The framework is set up in such a general way that it can be used to model an extremely wide range of scenarios—including, e.g., the setting of voting (Dietrich and List 2007). On the one hand, this generality is an advantage: methods studied in judgment aggregation can be employed in all these scenarios. On the other hand, however, this generality severely hinders the use of judgment aggregation methods in applications. Because there are no restrictions on the type of aggregation settings that are modeled, relevant computational tasks across the board are computationally intractable in the worst case. In other words, no performance guarantees are available that warrant the efficient use of judgment aggregation methods for applications—not even for simple settings. For example, computing the outcome of a judgment aggregation scenario is NP-hard for all aggregation procedures studied in the literature that satisfy the rudimentary quality condition of *consistency* (Endriss and De Haan 2015; De Haan and Slavkovik 2017; Lang and Slavkovik 2014).

These negative computational complexity results are in many cases due purely to the expressivity of the language used to represent aggregation scenarios (full propositional logic, or CNF formulas)—not to the structure of the scenario being modeled. In other words, the known negative complexity results draw an overly negative picture

To correct this gloomy and misleading image, a more detailed and more fine-grained perspective is needed on the way that application settings are modeled in the general framework of judgment aggregation. In this paper, we take a first look at the complexity of judgment aggregation scenarios using this more sensitive point of view. That is, we initiate an investigation of representation languages for judgment aggregation that (1) are modest enough to yield positive complexity results for relevant computational tasks, yet (2) are general enough to model interesting and relevant applications.

Concretely, we look at several restricted propositional languages that strike a balance between expressivity and tractability in other settings, and we study to what extent such a balance is attained in the setting of judgment aggregation. In particular, we look at Krom (2CNF), Horn and definite Horn formulas, and we consider the class of Boolean circuits in decomposable negation normal form (DNNF). We study the impact of these restricted languages on the complexity of computing outcomes for a number of judgment aggregation procedures studied in the literature. We obtain a wide range of (positive and negative) results. Most of the results we obtain are summarized in Tables 3, 4 and 5, located in later sections.

In particular, we obtain several interesting positive complexity results for the case where the domain is represented using a Boolean circuit in DNNF. Additionally, we illustrate how this representation language of Boolean circuits in DNNF—that combines expressivity and tractability—can be used to get tractability results for a specific application: voting on how to spend a budget. This application setting can be seen as an instantiation of the setting of *Participatory Budgeting* (see, e.g., Benade et al. 2017).

Related Work Judgment aggregation has been studied in the field of computational social choice from (a.o.) a philosophy, economics and computer science perspective (see, e.g., Dietrich 2007, Endriss 2016, Grossi and Pigozzi 2014, Lang et al. 2017, List and Pettit 2002, Rothe 2016). The complexity of computing outcomes for judgment aggregation procedures has been studied by, a.o., Endriss, Grandi, and Porello (2012), Endriss et al. (2016), Endriss and De Haan (2015), De Haan and Slavkovik (2017) and Lang and Slavkovik (2014). See Table 2 for complexity results that are relevant for this paper.

Roadmap We begin by explaining the framework of judgment aggregation. We then study to what extent the known languages of Krom and (definite) Horn formulas lead to suitable results for judgment aggregation. We continue with looking at the class of DNNF circuits—studied in the field of knowledge compilation—and we illustrate how results for this class of circuits can be used for a concrete application of judgment aggregation (that of voting on how to allocate a budget). We conclude with outlining some promising ways in which the research path that we set out can be followed.

An overview of notions from propositional logic and computational complexity theory that we use can be found in the appendix. The proofs of some results are omitted from the main paper and are located in the additional material at the end—these results are marked with a star ($\star$).

Judgment Aggregation

We begin by introducing the setting of Judgment Aggregation (Dietrich 2007; Endriss 2016; Grossi and Pigozzi 2014; List and Pettit 2002). In this paper, we will use a variant of the framework that has been studied by, e.g., Grandi (2012), Grandi and Endriss (2013) and Endriss et al. (2016).[1]

Let $\mathcal{I} = \{x_1, \dots, x_n\}$ be a finite set of *issues*, in the form of propositional variables. Intuitively, these issues are the topics about which the individuals want to combine their judgments. A truth assignment $\alpha : \mathcal{I} \to \{0, 1\}$ is called a *ballot*, and represents an opinion that individuals and the group can have. We will also denote ballots α by a binary vector $(b_1, \dots, b_n) \in \{0, 1\}^n$, where $b_i = \alpha(x_i)$ for each $i \in [n]$—we use $[n]$ to denote $\{1, \dots, n\}$ for each $n \in \mathbb{N}$. Moreover, we say that $(p_1, \dots, p_n) \in \{0, 1, \star\}^n$ is a *partial ballot*, and that $(p_1, \dots, p_n)$ *agrees with* a ballot $(b_1, \dots, b_n)$ if $p_i = b_i$ whenever $p_i \neq \star$, for all $i \in [n]$. We use an integrity constraint Γ to restrict the set of feasible opinions (for both the individuals and the group). The integrity constraint Γ is a propositional formula (or more generally, a single-output Boolean circuit), whose variables can include $x_1, \dots, x_n$. We define the set $\mathcal{R}(\mathcal{I}, \Gamma)$ of *rational ballots* to be the ballots (for $\mathcal{I}$) that are consistent with the integrity constraint Γ. We say that finite sequences $\boldsymbol{r} \in \mathcal{R}(\mathcal{I}, \Gamma)^+$ of rational ballots are *profiles*. A profile contains a ballot for each individual participating in the judgment aggregation scenario. Where convenient we equate a profile $\boldsymbol{r} = (r_1, \dots, r_p)$ with the multiset containing $r_1, \dots, r_p$.

A *judgment aggregation procedure* (or *rule*), for the set $\mathcal{I}$ of issues and the integrity constraint Γ, is a function F that takes as input a profile $\boldsymbol{r} \in \mathcal{R}(\mathcal{I}, \Gamma)^+$, and that produces a non-empty set of ballots. A procedure F is called *consistent* if for all $\mathcal{I}$, Γ and $\boldsymbol{r}$ it holds that each $r^* \in F(\boldsymbol{r})$ is consistent with Γ. Consistency is a central requirement for judgment aggregation procedures, and all rules that we consider in this paper are consistent.

An example of a simple judgment aggregation procedure is the *majority rule* (defined for profiles with an odd number of ballots). We let the majority outcome $m_{\boldsymbol{r}}$ be the partial ballot such that for each $x \in \mathcal{I}$, $m_{\boldsymbol{r}}(x) = 1$ if a strict majority of ballots $r_i \in \boldsymbol{r}$ satisfy $r_i(x) = 1$, $m_{\boldsymbol{r}}(x) = 0$ if a strict majority of ballots $r_i \in \boldsymbol{r}$ satisfy $r_i(x) = 0$, and $m_{\boldsymbol{r}}(x) = \star$ otherwise. The majority rule returns the majority outcome $m_{\boldsymbol{r}}$. The majority rule is efficient to compute, but is not consistent (as shown in Example 1).

Example 1. *Consider the judgment aggregation scenario. where* $\mathcal{I} = \{x_1, x_2, x_3\}$, $\Gamma = (\neg x_1 \vee \neg x_2 \vee \neg x_3)$, *and the profile* $\boldsymbol{r} = (r_1, r_2, r_3)$ *is as shown in Table 1. The majority outcome* $\text{MAJ}(\boldsymbol{r})$ *is inconsistent with* Γ.

$\boldsymbol{r}$	x_1	x_2	x_3
r_1	1	1	0
r_2	1	0	1
r_3	0	1	1
$\text{MAJ}(\boldsymbol{r})$	1	1	1

Table 1: Example of a judgment aggregation scenario.

Judgment Aggregation Procedures

Next, we introduce the judgment aggregation rules that we use in this paper. These procedures are all consistent and are many of the ones that have been studied in the literature (for an overview see, e.g., Lang et al. 2017).

Several procedures that we consider can be seen as instantiations of a general template: *scoring procedures*. Let $\mathcal{I}$ be a set of issues and Γ be an integrity constraint. Moreover, let $s : \mathcal{R}(\mathcal{I}, \Gamma) \times \text{Lit}(\mathcal{I}) \to \mathbb{N}$ be a *scoring function* that assigns a value to each literal $l \in \text{Lit}(\mathcal{I})$ with respect to a ballot $r \in \mathcal{R}(\mathcal{I}, \Gamma)$. The scoring judgment aggregation procedure F_s that corresponds to s is defined as follows:

$$F_s(\boldsymbol{r}) = \operatorname*{arg\,max}_{r \in \mathcal{R}(\mathcal{I},\Gamma)} \sum_{r_i \in \boldsymbol{r}} \sum_{\substack{l \in \text{Lit}(\mathcal{I}) \\ r(l)=1}} s(r_i, l).$$

That is, F_s selects the rational ballots $r \in \mathcal{R}(\mathcal{I}, \Gamma)$ that maximize the cumulative score for all literals agreeing with r with respect to all ballots $r_i \in \boldsymbol{r}$.

The *median (or Kemeny) procedure* MED is based on the scoring function and is defined by letting $s_{\text{K}}(r, l) = r(l)$ for each $r \in \mathcal{R}(\mathcal{I}, \Gamma)$ and each $l \in \text{Lit}(\mathcal{I})$. Alternatively, the MED procedure can be defined as the rule that selects the ballots $r^* \in \mathcal{R}(\mathcal{I}, \Gamma)$ that minimize the cumulative Hamming distance to the profile $\boldsymbol{r}$. The *Hamming distance* between two ballots r, r' is $d_{\text{H}}(r, r') = |\{\, x \in \mathcal{I} : r(x) \neq r'(x) \,\}|$.

The *reversal scoring procedure* REV is based on the scoring function $s_{\text{R}}(r, l)$ such that $s_{\text{R}}(r, l) = \min_{r' \in \mathcal{R}(\mathcal{I},\Gamma), r'(l)=0} d_{\text{H}}(r, r')$ for each $r \in \mathcal{R}(\mathcal{I}, \Gamma)$ and each $l \in \text{Lit}(\mathcal{I})$. That is, the score $s_{\text{R}}(r, l)$ of l w.r.t. r is

[1] This framework is also known under the name of *binary aggregation with integrity constraints*, and can be used interchangeably with other Judgment Aggregation frameworks from the literature —as shown by Endriss et al. (2016).

the minimal number of issues whose truth value needs to be flipped to get a rational ballot r' that sets l to false.

The *max-card Condorcet (or Slater) procedure* MCC is also based on the Hamming distance. Let $\boldsymbol{r}$ be a profile. The MCC procedure is defined by letting $\text{MCC}(\boldsymbol{r}) = \arg\min_{r^* \in \mathcal{R}(\mathcal{I},\Gamma)} d_{\text{H}}(r^*, m_{\boldsymbol{r}})$. That is, the MCC procedure selects the rational ballots that minimize the Hamming distance to the majority outcome $m_{\boldsymbol{r}}$.

The *Young procedure* YOUNG selects those ballots that can be obtained as a rational majority outcome by deleting a minimal number of ballots from the profile. Let $\boldsymbol{r}$ be a profile, and let d denote the smallest number such that deleting d individual ballots from $\boldsymbol{r}$ results in a profile $\boldsymbol{r}'$ such that $m_{\boldsymbol{r}'}$ is a complete and rational ballot. We let the outcome $\text{YOUNG}(\boldsymbol{r})$ of the Young procedure be the set of rational ballots r^* such that deleting d individual from $\boldsymbol{r}$ results in a profile $\boldsymbol{r}'$ with $m_{\boldsymbol{r}'} = r^*$.

The *Max-Hamming procedure* MAXHAM is also based on the Hamming distance. Let r be a single ballot, and let $\boldsymbol{r} = (r_1, \dots, r_p)$ be a profile. We define the max-Hamming distance between r and $\boldsymbol{r}$ to be $d_{\text{max,H}}(r, \boldsymbol{r}) = \max_{r_i \in \boldsymbol{r}} d_{\text{H}}(r, r_i)$. The Max-Hamming procedure is defined by letting $\text{MAXHAM}(\boldsymbol{r}) = \arg\min_{r^* \in \mathcal{R}(\mathcal{I},\Gamma)} d_{\text{max,H}}(r^*, \boldsymbol{r})$. That is, the Max-Hamming procedure selects the rational ballots that minimize the max-Hamming distance to $\boldsymbol{r}$.

The *ranked agenda (or Tideman) procedure* RA is based on the notion of majority strength.[2] Let $\boldsymbol{r}$ be a profile and let $l \in \text{Lit}(\mathcal{I})$. The majority strength $\text{ms}(\boldsymbol{r}, l)$ of l for $\boldsymbol{r}$ is the number of ballots $r \in \boldsymbol{r}$ such that $r(l) = 1$. Let $<_{\text{tb}}$ be a fixed linear order on $\text{Lit}(\mathcal{I})$ (the tie-breaking order). Based on $<_{\text{tb}}$ and the majority strength, we define the linear order $<_{\boldsymbol{r}}$ on $\text{Lit}(\mathcal{I})$. Let $l_1, l_2 \in \text{Lit}(\mathcal{I})$. Then $l_1 <_{\boldsymbol{r}} l_2$ if either (i) $\text{ms}(\boldsymbol{r}, l_1) > \text{ms}(\boldsymbol{r}, l_2)$ or (ii) $\text{ms}(\boldsymbol{r}, l_1) = \text{ms}(\boldsymbol{r}, l_2)$ and $l_1 <_{\text{tb}} l_2$. Then $\text{RA}(\boldsymbol{r}) = \{r^*\}$ where the ballot r^* is defined inductively as follows. Let $l_1, l_2, \dots, l_{2n}$ be such that for each $i \in [2n-1]$ it holds that $l_i <_{\boldsymbol{r}} l_{i+1}$. Let s_0 be the empty truth assignment. For each $i \in [2n-1]$, check whether both $s_i(l_i) \neq 0$ and s'_i is consistent with Γ, where s'_i is obtained from s_i by setting l_i to true (and keeping the assignments to variables not occurring in l_i unchanged). If both are the case, then let $s_{i+1} = s'_i$. Otherwise, let $s_{i+1} = s_i$. Then $r^* = s_{2n}$. Intuitively, the procedure iterates over the assignments $l_1, l_2, \dots$ in the order specified by $<_{\boldsymbol{r}}$. Each literal l_i is set to true whenever this does not lead to an inconsistency with previously assigned literals.

Outcome Determination

When given a judgment aggregation scenario (i.e., an agenda, an integrity constraint, and a profile of individual opinions), an important computational task is to compute a possible collective opinion, for a fixed judgment aggregation procedure. This task is often referred to as *outcome determination*. Moreover, often it makes sense to seek possible collective opinions that satisfy certain properties (e.g., whether or not a given issue is accepted in the collective opinion).

Essentially, this is a search problem: the task is to find one of (possibly) multiple solutions. However, to make the theoretical complexity analysis easier, we will consider the following decision variant of this problem.

> OUTCOME(F)
> *Instance:* A set $\mathcal{I}$ of issues with an integrity constraint Γ a profile $\boldsymbol{r} \in \mathcal{R}(\mathcal{I},\Gamma)^+$ and a partial ballot s (for $\mathcal{I}$).
> *Question:* Is there a ballot $r^* \in F(\boldsymbol{r})$ such that s agrees with r^*?

An outcome r^* witnessing a yes-answer can be obtained by solving this decision problem a linear number of times. In addition to the basic task of finding one outcome (that agrees with a given partial ballot s), one could consider other computational tasks, e.g., representing the set $F(\boldsymbol{r})$ of outcomes in a succinct way that admits certain queries/operations to be performed efficiently. For example, it might be desirable to enumerate all (possibly exponentially many) outcomes with polynomial delay. It could also be desirable to check whether all outcomes agree with a given partial ballot s (*skeptical reasoning*). For the sake of simplicity, in this paper we will stick to the decision problem described above. All tractability results that we obtain for the decision problem can straightforwardly be extended to tractability results for the above computational tasks.

For the judgment aggregation procedures F that we considered above, OUTCOME(F) is Θ_2^{p}-hard. For an overview, see Table 2.

F	complexity of OUTCOME(F)	
MED	Θ_2^{p}-c	(Lang and Slavkovik 2014)
REV	Θ_2^{p}-c	(De Haan and Slavkovik 2017)
MCC	Θ_2^{p}-c	(Lang and Slavkovik 2014)
YOUNG	Θ_2^{p}-c	(Endriss and De Haan 2015)
MAXHAM	Θ_2^{p}-c	(De Haan and Slavkovik 2017)
RA	Δ_2^{p}-c	(Endriss and De Haan 2015)

Table 2: The computational complexity of outcome determination for various procedures F.

Krom and (Definite) Horn Formulas

In this section, we consider the fragments of Krom (2CNF), Horn and definite Horn formulas—for a formal definition of these fragments, see the appendix. These fragments can be used to express settings where only basic dependencies between issues play a role—see Example 2 for an indication.

Example 2. *Krom (2CNF) formulas can be used to express dependencies of the form "if we decide to use software tool 1 (s_1) or software tool 2 (s_2), then we need to purchase the entire package (p):"* $(s_1 \lor s_2) \rightarrow p \equiv (\neg s_1 \lor p) \land (\neg s_2 \lor p)$.

Definite Horn formulas can be used to express dependencies of the form "if we hire both researcher 1 (r_1) and researcher 2 (r_2), then we need to rent another office o:" $(r_1 \land r_2) \rightarrow o \equiv (\neg r_1 \lor \neg r_2 \lor o)$.

[2] Here, we consider a variant of the ranked agenda procedure that works with a fixed tie-breaking order. Other variants, where all possible tie-breaking orders are considered in parallel, have also been studied in the literature (see, e.g., Lang et al. 2017).

For some judgment aggregation rules these fragments make computing outcomes tractable, and for other judgment aggregation rules they do not. We begin with considering the rules MED and MCC. Computing outcomes for these rules is tractable when restricted to Krom formulas, but not when restricted to (definite) Horn formulas.

Proposition⋆ 1. OUTCOME(MED) *is Θ_2^p-hard even when restricted to the case where $\Gamma \in$ DEFHORN.*

Proposition⋆ 2. OUTCOME(MCC) *is Θ_2^p-hard even when restricted to the case where $\Gamma \in$ DEFHORN.*

The following result refers to the notion of majority consistency (see, e.g., Lang and Slavkovik 2014). A profile $\boldsymbol{r}$ is *majority consistent* (with respect to an integrity constraint Γ) if the majority outcome $m_{\boldsymbol{r}}$ is consistent with Γ. A judgment aggregation procedure is *majority consistent* if for each integrity constraint Γ and each profile $\boldsymbol{r}$ that is majority consistent (w.r.t. Γ), the procedure outputs all and only those complete ballots that agree with the (partial) ballot $m_{\boldsymbol{r}}$.

Theorem 3. *For all judgment aggregation procedures F that are majority consistent, e.g., $F \in \{\text{MED}, \text{MCC}\}$,* OUTCOME-(F) *is polynomial-time solvable when $\Gamma \in$ KROM.*

Proof. The general idea behind this proof is to use the property that when $\Gamma \in$ KROM, the majority outcome $m_{\boldsymbol{r}}$ is always Γ-consistent. Let $(\mathcal{I}, \Gamma, \boldsymbol{r}, s)$ be an instance of OUTCOME(F) with $\Gamma \in$ KROM. Let $\boldsymbol{r} = (r_1, \dots, r_p)$. We consider the majority outcome $r^* = m_{\boldsymbol{r}}$.

We show that the partial ballot r^* is consistent with Γ. Suppose, to derive a contradiction, that r^* is inconsistent with Γ. Then there must be some clause $(l_1 \vee l_2)$ of size 2 such that $\Gamma \models (l_1 \vee l_2)$ and r^* sets both l_1 and l_2 to false. By definition of r^*, then a strict majority of the ballots in $\boldsymbol{r}$ set l_1 to false, and a strict majority of the ballots in $\boldsymbol{r}$ set l_2 to false. By the pigeonhole principle then there must be some ballot r_i in $\boldsymbol{r}$ that sets both l_1 and l_2 to false. However, since $\Gamma \models (l_1 \vee l_2)$, we get that r_i does not satisfy Γ, which is a contradiction with our assumption that all ballots in the profile satisfy Γ. Thus, we can conclude that r^* is consistent with Γ.

Since F is majority consistent, we know that $F(\boldsymbol{r})$ contains all ballots that are consistent with both r^* and Γ. Since $\Gamma \in$ KROM, deciding if $F(\boldsymbol{r})$ contains a ballot that is consistent with s can be done in polynomial time. $\square$

We continue with the MAXHAM procedure for which computing outcomes is not tractable when restricted to Krom formulas nor when restricted to definite Horn formulas.

Proposition⋆ 4. OUTCOME(MAXHAM) *is Θ_2^p-hard even when restricted to the case where $\Gamma = \top$.*

OUTCOME(MAXHAM) restricted to the case where $\Gamma = \top$ coincides with a problem known as CLOSEST STRING for binary alphabets (see, e.g., Li, Ma, and Wang 2002). To the best of our knowledge, this is the first time that the exact complexity of (this variant of) this problem has been identified. OUTCOME(MAXHAM) is also very similar to the problem of computing outcomes for the minimax rule in approval voting (Brams, Kilgour, and Sanver 2004).

Corollary 5. OUTCOME(MAXHAM) *is Θ_2^p-hard even when restricted to the case where $\Gamma \in$ DEFHORN $\cap$ KROM.*

Finally, we consider the procedure RA, for which computing outcomes is tractable for both Krom and Horn formulas.

Theorem 6. *Let $\mathcal{C}$ be a class of propositional formulas (or Boolean circuits) with the following two properties:*

- *$\mathcal{C}$ is closed under instantiation, i.e., for any $\Gamma \in \mathcal{C}$ and any partial truth assignment $\alpha : \text{Var}(\Gamma) \to \{0, 1\}$ it holds that $\Gamma[\alpha] \in \mathcal{C}$; and*
- *satisfiability of formulas in $\mathcal{C}$ is polynomial-time solvable.*

Then OUTCOME(RA) *is polynomial-time solvable when restricted to the case where $\Gamma \in \mathcal{C}$.*

Proof (sketch). Let $\mathcal{C}$ be a class of propositional formulas that satisfies the conditions stated above, and let $\Gamma \in \mathcal{C}$. We can then compute OUTCOME(RA) $= \{r^*\}$ by directly using the iterative definition of r^* given in the description of the ranked agenda procedure. This definition iteratively constructs partial ballots $s_0, \dots, s_{2n}$. Ballot s_0 is the empty ballot, and for each $i > 0$, ballot s_i is constructed from s_{i-1} by using only the operations of instantiating the integrity constraint and checking satisfiability of the resulting formula. Due to the properties of $\mathcal{C}$, these operations are all polynomial-time solvable. Thus, constructing $r^* = s_{2n}$ can be done in polynomial time. $\square$

Corollary 7. *For each $\mathcal{C} \in \{\text{KROM}, \text{HORN}\}$,* OUTCOME-(RA) *is polynomial-time solvable when restricted to the case where $\Gamma \in \mathcal{C}$.*

An overview of the complexity results that we established in this section can be found in Table 3.

F	complexity of OUTCOME(F) restricted to HORN / DEFHORN	
MED	Θ_2^p-c	(Proposition 1)
MCC	Θ_2^p-c	(Proposition 2)
MAXHAM	Θ_2^p-c	(Corollary 5)
RA	in P	(Corollary 7)
F	complexity of OUTCOME(F) restricted to KROM	
MED	in P	(Theorem 3)
MCC	in P	(Theorem 3)
MAXHAM	Θ_2^p-c	(Corollary 5)
RA	in P	(Corollary 7)

Table 3: The computational complexity of outcome determination for several procedures F restricted to the case where $\Gamma \in$ KROM, the case where $\Gamma \in$ HORN, and the case where $\Gamma \in$ DEFHORN.

The results that we obtained for Horn formulas can all be straightforwardly extended to the fragment of renamable Horn formulas—e.g., the fragment of renamable Horn formulas satisfies the requirements of Theorem 6. A propositional formula φ is *renamable Horn* if there is a set $R \subseteq \text{Var}(\varphi)$ of variables such that φ becomes Horn when all literals over R are complemented.

Boolean Circuits in DNNF

Next, we consider the case where the integrity constraints are restricted to Boolean circuits in Decomposable Negation Normal Form (DNNF). This is a class of Boolean circuits studied in the area of knowledge compilation. We illustrate how this class of circuits is useful for judgment aggregation.

Knowledge Compilation

Knowledge compilation (see, e.g., Darwiche and Marquis 2002, Darwiche 2014, Marquis 2015) refers to a collection of approaches for solving reasoning problems in the area of artificial intelligence and knowledge representation and reasoning that are computationally intractable in the worst-case asymptotic sense. These reasoning problems typically involve knowledge in the form of a Boolean function—often represented as a propositional formula. The general idea behind these approaches is to split the reasoning process into two phases: (1) compiling the knowledge into a different format that allows the reasoning problem to be solved efficiently, and (2) solving the reasoning problem using the compiled knowledge. Since the entire reasoning problem is computationally intractable, at least one of these two phases must be intractable. Indeed, typically the first phase does not enjoy performance guarantees on the running time—upper bounds on the size of the compiled knowledge are often desired instead. One of the advantages of this methodology is that one can reuse the compiled knowledge for many instances, which could lead to a smaller overall running time.

A prototypical example of a problem studied in the setting of knowledge compilation is that of *clause entailment* (see, e.g., Darwiche and Marquis 2002, Cadoli et al. 2002). In this problem, one is given a knowledge base, say in the form of a propositional formula φ in CNF, together with a clause δ. The question is to decide whether $\varphi \models \delta$. This problem is co-NP-complete in general. The knowledge compilation approach to solve this problem would be to firstly compile the CNF formula φ into an equivalent expression in a different format. For example, one could consider the formalism of Boolean circuits in *Decomposable Negation Normal Form (DNNF)* (or *DNNF circuits*, for short).

DNNF circuits are a particular class of Boolean circuits in *Negation Normal Form (NNF)*. A Boolean circuit C in NNF is a direct acyclic graph with a single root (a node with no ingoing edges) where each leaf is labelled with $\top$, $\bot$, x or $\neg x$ for a propositional variable x, and where each internal node is labelled with $\wedge$ or $\vee$. (An arc in the graph from N_1 to N_2 indicates that N_2 is a child node of N_1.) The set of propositional variables occurring in C is denoted by $\text{Var}(C)$. For any truth assignment $\alpha : \text{Var}(C) \to \{0, 1\}$, we define the truth value $C[\alpha]$ assigned to C by α in the usual way, i.e., each node is assigned a truth value based on its label and the truth value assigned to its children, and the truth value assigned to C is the truth value assigned to the root of the circuit. DNNF circuits are Boolean circuits in NNF that satisfy the additional property of decomposability. A circuit C is *decomposable* if for each conjunction in the circuit, the conjuncts do not share variables. That is, for each node d in C that is labelled with $\wedge$ and for any two children d_1, d_2 of this node, it holds that $\text{Var}(C_1) \cap \text{Var}(C_2) = \emptyset$, where C_1, C_2 are the subcircuits of C that have d_1, d_2 as root, respectively. An example of a DNNF circuit is given in Figure 1.

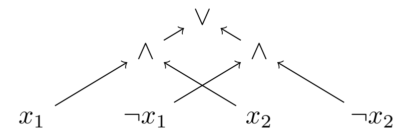


Figure 1: An example of a DNNF circuit.

The problem of clause entailment can be solved in polynomial time when the propositional knowledge is given as a DNNF circuit (Darwiche and Marquis 2002). Moreover, every CNF formula can be translated to an equivalent DNNF circuit—without guarantees on the size of the circuit. Thus, one could solve the problem of clause entailment by first compiling the CNF formula φ into an equivalent DNNF circuit C (without guarantees on the running time or size of the result) and then solving $C \models \delta$ in time polynomial in $|C|$.

Next, we will show how representation languages such as DNNF circuits can be used in the setting of Judgment Aggregation, and we will argue how Judgment Aggregation can benefit from the approach of first compiling knowledge (without performance guarantees) before using the compiled knowledge to solve the initial problem.

Algebraic Model Counting

We will use the technique of algebraic model counting (Kimmig, Van den Broeck, and De Raedt 2017) to execute several judgment aggregation procedures efficiently using the structure of DNNF circuits. Algebraic model counting is a generalization of the problem of counting models of a Boolean function that uses the addition and multiplication operators of a commutative semiring.

Definition 1 (Commutative semiring)**.** *A* semiring *is a structure* $(\mathcal{A}, \oplus, \otimes, e^{\oplus}, e^{\otimes})$*, where:*

- *addition* $\oplus$ *is an associative and commutative binary operation over the set* $\mathcal{A}$*;*
- *multiplication* $\otimes$ *is an associative binary operation over the set* $\mathcal{A}$*;*
- $\otimes$ *distributes over* $\oplus$*;*
- $e^{\oplus} \in \mathcal{A}$ *is the neutral element of* $\oplus$*, i.e., for all* $a \in \mathcal{A}$*,* $a \oplus e^{\oplus} = a$*;*
- $e^{\otimes} \in \mathcal{A}$ *is the neutral element of* $\otimes$*, i.e., for all* $a \in \mathcal{A}$*,* $a \otimes e^{\otimes} = a$*; and*
- $e^{\oplus}$ *is an annihilator for* $\otimes$*, i.e., for all* $a \in \mathcal{A}$*,* $e^{\oplus} \otimes a = a \otimes e^{\oplus} = e^{\oplus}$*.*

When $\otimes$ *is commutative, we say that the semiring is commutative. When* $\oplus$ *is idempotent, we say that the semiring is idempotent.*

Definition 2 (Algebraic model counting)**.** *Given:*

- *a Boolean function* f *over a set* $\mathcal{I}$ *of propositional variables;*
- *a commutative semiring* $(\mathcal{A}, \oplus, \otimes, e^{\oplus}, e^{\otimes})$*, and*

- *a labelling function $\lambda : \text{Lit}(\mathcal{I}) \to \mathcal{A}$ mapping literals over the variables in $\mathcal{I}$ to values in the set $\mathcal{A}$,*

the task of algebraic model counting (AMC) *is to compute:*

$$\boldsymbol{A}(f) = \bigoplus_{\substack{\alpha:\mathcal{I}\to\{0,1\} \\ f(\alpha)=1}} \bigotimes_{\substack{l\in\text{Lit}(\mathcal{I}) \\ \lambda(l)=1}} \lambda(l).$$

We can solve the task of algebraic model counting efficiently for DNNF circuits when the semiring satisfies an additional condition.

Definition 3 (Neutral $(\oplus, \alpha)$)**.** *Let $(\mathcal{A}, \oplus, \otimes, e^{\oplus}, e^{\otimes})$ be a semiring, and let $\lambda : \text{Lit}(\mathcal{I}) \to \mathcal{A}$ be a labelling function for some set $\mathcal{I}$ of propositional variables. A pair $(\oplus, \lambda)$ is* neutral *if for all $x \in \mathcal{I}$ it holds that $\lambda(x) \oplus \lambda(\neg x) = e^{\otimes}$.*

Theorem 8 (Kimmig, Van den Broeck, and De Raedt 2017, Thm 5)**.** *When f is represented as a DNNF circuit, and the semiring $(\mathcal{A}, \oplus, \otimes, e^{\oplus}, e^{\otimes})$ and the labelling function λ have the properties that (i) $\oplus$ is idempotent, and (ii) $(\oplus, \lambda)$ is neutral, then the algebraic model counting problem is polynomial-time solvable—when given f and λ as input, and when the operations of addition ($\oplus$) and multiplication ($\otimes$) over $\mathcal{A}$ can be performed in polynomial time.*

We will use the result of Theorem 8 to show that outcome determination for several judgment aggregation procedures is tractable for the case where Γ is a DNNF circuit. To do so, we will consider the following commutative, idempotent semiring (also known as the *max-plus algebra*). We let $\mathcal{A} = \mathbb{Z} \cup \{-\infty, \infty\}$, we let $\oplus = \max$, $\otimes = +$, $e^{\oplus} = -\infty$, and $e^{\otimes} = 1$. Whenever we have a labelling function α such that $(\oplus, \lambda)$ is neutral—i.e., such that $\max(\lambda(x), \lambda(\neg x)) = 0$ for each $x \in \mathcal{I}$—we satisfy the conditions of Theorem 8.

Theorem 9. OUTCOME(MED) *and* OUTCOME(MCC) *are polynomial-time computable when Γ is a DNNF circuit.*

Proof. We prove the statement for OUTCOME(MED). The case for OUTCOME(MCC) is analogous. Let $(\mathcal{I}, \Gamma, \boldsymbol{r}, s)$ be an instance of OUTCOME(MED). We solve the problem by reducing it to the problem of algebraic model counting. For $(\mathcal{A}, \oplus, \otimes, e^{\oplus}, e^{\otimes})$, we use the max-plus algebra described above. We construct the labelling function λ as follows. For each $x \in \mathcal{I}$, we count the number $n_{x,1}$ of ballots $r \in \boldsymbol{r}$ such that $r(x) = 1$ and we count the number $n_{x,0}$ of ballots $r \in \boldsymbol{r}$ such that $r(x) = 0$. That is, we let $n_{x,0}$ and $n_{x,1}$ be the majority strength of $\neg x$ and x, respectively, in the profile $\boldsymbol{r}$. We pick a constant c_x such that $\max\{n'_{x,0}, n'_{x,1}\} = 0$ where $n'_{x,0} = n_{x,0} + c_x$ and $n'_{x,1} = n_{x,1} + c_x$. We then let $\lambda(x) = n'_{x,1}$ and $\lambda(\neg x) = n'_{x,0}$. This ensures that $(\oplus, \lambda)$ satisfies the condition of neutrality (i.e., that $\lambda(x) \oplus \lambda(\neg x) = e^{\otimes}$ for each $x \in \mathcal{I}$).

This choice of $(\mathcal{A}, \oplus, \otimes, e^{\oplus}, e^{\otimes})$ and λ has the property that the ballots $r^* \in \text{MED}(\boldsymbol{r})$ are exactly those complete ballots r^* that satisfy Γ and for which holds that $\boldsymbol{A}(\Gamma) = \bigotimes_{l\in\text{Lit}(\mathcal{I}), r^*(l)=1} \lambda(l)$. That is, the set $\text{MED}(\boldsymbol{r})$ consists of those rational ballots that achieve the solution of the algebraic model counting problem $\boldsymbol{A}(\Gamma)$. We can solve the instance of decision problem OUTCOME(MED) by solving the algebraic model counting problem twice: once for Γ and once for $\Gamma[s]$. The instance is a yes-instance if and only if $\boldsymbol{A}(\Gamma) = \boldsymbol{A}(\Gamma[s])$. By Theorem 8, this can be done in polynomial time.

To make this algorithm work for the case of OUTCOME(MCC), one only needs to adapt the values of $n_{x,0}$ and $n_{x,1}$. Instead of setting $n_{x,0}$ and $n_{x,1}$ to the majority strength of $\neg x$ and x, respectively, we let $n_{x,0} = 0$ if a strict majority of ballots $r \in \boldsymbol{r}$ have that $r(x) = 1$, and we let $n_{x,0} = 1$ otherwise. Similarly, we let $n_{x,1} = 0$ if a strict majority of ballots $r \in \boldsymbol{r}$ have that $r(x) = 0$, and we let $n_{x,0} = 1$ otherwise. ☐

Representing the integrity constraint as a DNNF circuit makes it possible to perform more tasks efficiently than just the decision problem OUTCOME(F). For example, the algorithms for algebraic model counting can be used to produce a DNNF circuit that represents the set $F(\boldsymbol{r})$ of outcomes, allowing further operations to be carried out efficiently.

Theorem 10. OUTCOME(REV) *is polynomial-time computable when Γ is a DNNF circuit.*

Proof (sketch). The polynomial-time algorithm for OUTCOME(REV) is analogous to the algorithm described for OUTCOME(MED) described in the proof of Theorem 9. The only modification that needs to be made to make this algorithm work for OUTCOME(REV) is to adapt the numbers $n_{x,0}$ and $n_{x,1}$, for each $x \in \mathcal{I}$. Instead of identifying these numbers with the majority strength of $\neg x$ and x, respectively, we identify them with the total reversal score of x and $\neg x$, over the profile $\boldsymbol{r}$. That is, we let $n_{x,0} = \sum_{r\in\boldsymbol{r}} s_{\text{R}}(r, \neg x)$ and we let $n_{x,1} = \sum_{r\in\boldsymbol{r}} s_{\text{R}}(r, x)$. For general propositional formulas Γ, the reversal scoring function s_{R} is NP-hard to compute. However, since Γ is given as a DNNF circuit, we can compute the scoring function s_{R}, and thereby $n_{x,0}$ and $n_{x,1}$, in polynomial time—by using another reduction to the problem of algebraic model counting. We omit the details of this latter reduction. ☐

Intuitively, the results of Theorems 9 and 10 are a consequence of the fact that DNNF circuits allow polynomial-time weighted maximal model computation, and that the judgment aggregation procedures MED, MCC and REV are based on weighted maximal model computation. These results can therefore also straightforwardly be extended to other judgment aggregation procedures that are based on weighted maximal model computation.

Other Results

We can extend some previously established results (Proposition 4 and Theorem 6) to the case of DNNF circuits.

Corollary 11. OUTCOME(RA) *is polynomial-time computable when restricted to the case where Γ is a DNNF circuit.*

Corollary 12. OUTCOME(MAXHAM) *is Θ_2^{p}-complete when restricted to the case where Γ is a DNNF circuit.*

A similar result for YOUNG follows from a result that we will establish in the next section (Proposition 18).

Corollary 13. OUTCOME(YOUNG) *is Θ_2^{p}-complete when restricted to the case where Γ is a DNNF circuit.*

An overview of the results established so far in this section can be found in Table 4.

F	complexity of OUTCOME(F)	
MED	in P	(Theorem 9)
REV	in P	(Theorem 10)
MCC	in P	(Theorem 9)
YOUNG	Θ_2^p-c	(Corollary 13)
MAXHAM	Θ_2^p-c	(Corollary 12)
RA	in P	(Corollary 11)

Table 4: The computational complexity of outcome determination for various procedures F restricted to the case where Γ is a DNNF circuit.

A Compilation Approach

The results of Theorems 9 and 10 and Corollary 11 pave the way for another approach towards finding cases where judgment aggregation procedures can be performed efficiently. The idea behind this approach is to compile the integrity constraint into a DNNF circuit—regardless of whether this compilation process enjoys a polynomial-time worst-case performance guarantee. There are several off-the-shelf tools available that compile CNF formulas into DNNF circuits using optimized methods based on SAT solving algorithms (Darwiche 2004; Muise et al. 2012; Oztok and Darwiche 2014b). Since the class of DNNF circuits is expressively complete—i.e., every Boolean function can be expressed using a DNNF circuit—it is possible to compile any integrity constraint Γ into a DNNF circuit C_Γ.

The downside is that the circuit C_Γ could be of exponential size, or it could take exponential time to compute it. However, once the circuit C_Γ is computed and stored in memory, one can use several judgment aggregation procedures efficiently: MED, MCC, REV and RA.

Thus, this approach restricts the computational bottleneck to the *compilation phase*, before any judgments are solicited from the individuals in the judgment aggregation scenario. Once the compilation phase has been completed, there are polynomial-time guarantees on the *aggregation phase* (polynomial in the size of the compiled DNNF circuit C_Γ).

CNF Formulas of Bounded Treewidth

The tractability results for DNNF circuits can be leveraged to get parameterized tractability results for the case where the integrity constraint is a CNF formula with a 'treelike' structure.

Parameterized Complexity Theory & Treewidth In order to explain the results that follow, we briefly introduce some relevant concepts from the theory of parameterized complexity. For more details, we refer to textbooks on the topic (see, e.g., Cygan et al. 2015, Downey and Fellows 2013). The central notion in parameterized complexity is that of *fixed-parameter tractability*—a notion of computational tractability that is more lenient than the traditional notion of polynomial-time solvability. In parameterized complexity running times are measured in terms of the input size n as well as a problem parameter k. Intuitively, the parameter is used to capture structure that is present in the input and that can be exploited algorithmically. The smaller the value of the problem parameter k, the more structure the input exhibits. Formally, we consider *parameterized problems* that capture the computational task at hand as well as the choice of parameter. A parameterized problem Q is a subset of $\Sigma^* \times \mathbb{N}$ for some fixed alphabet Σ. An instance (x, k) of Q contains the problem input $x \in \Sigma^*$ and the parameter value $k \in \mathbb{N}$. A parameterized problem is *fixed-parameter tractable* there is a deterministic algorithm that for each instance (x, k) decides whether $(x, k) \in Q$ and that runs in time $f(k)|x|^c$, where f is a computable function of k, and c is a fixed constant. Algorithms running within such time bounds are called *fpt-algorithms*. The idea behind these definitions is that fixed-parameter tractable running times are scalable whenever the value of k is small.

A commonly used parameter is that of the treewidth of a graph. Intuitively, the treewidth measures the extent to which a graph is like a tree—trees and forests have treewidth 1, cycles have treewidth 2, and so forth. The notion of treewidth is defined as follows. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\mathcal{T}, (B_t)_{t \in T})$ where $\mathcal{T} = (T, F)$ is a tree and $(B_t)_{t \in T}$ is a family of subsets of V such that:

- for every $v \in V$, the set $B^{-1}(v) = \{ t \in T : v \in B_t \}$ is nonempty and connected in $\mathcal{T}$; and
- for every edge $\{v, w\} \in E$, there is a $t \in T$ such that $v, w \in B_t$.

The *width* of the decomposition $(\mathcal{T}, (B_t)_{t \in T})$ is the number $\max\{ |B_t| : t \in T \} - 1$. The *treewidth* of G is the minimum of the widths of all tree decompositions of G. Let G be a graph and k a nonnegative integer. There is an fpt-algorithm that computes a tree decomposition of G of width k if it exists, and fails otherwise (Bodlaender 1996).

Encoding Results We can then use results from the literature to establish tractability results for computing outcomes of various judgment aggregation procedures for integrity constraints whose variable interactions have a treelike structure. Let $\Gamma = c_1 \wedge \dots \wedge c_m$ be a CNF formula. The *incidence graph of* Γ is the graph (V, E), where $V = \text{Var}(\Gamma) \cup \{c_1, \dots, c_u\}$ and $E = \{ \{c_j, x\} : 1 \leq j \leq m, x \in \text{Var}(\Gamma), x \text{ occurs in the clause } c_j \}$. The *incidence treewidth of* Γ is defined as the treewidth of the incidence graph of Γ.

We can leverage the results of Theorems 9 and 10 and Corollary 11 to get fixed-parameter tractability results for computing outcomes of MED, MCC, REV and RA for integrity constraints with small incidence treewidth.

Proposition 14 (Oztok and Darwiche 2014a, Bova et al. 2015)**.** *Let Γ be a CNF formula of incidence treewidth k. Constructing a DNNF circuit Γ' that is equivalent to Γ can be done in fixed-parameter tractable time.*

Corollary 15. *The problems* OUTCOME(MED)*,* OUTCOME(MCC)*,* OUTCOME(REV) *and* OUTCOME(RA) *are fixed-*

parameter tractable when parameterized by the incidence treewidth of Γ.

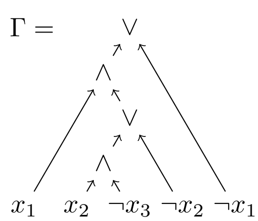
Case Study: Budget Constraints

In this section, we illustrate how the results of the previous section can contribute to providing a computational complexity analysis for an application setting. The setting that we consider as an example is that of budget constraints. This setting is closely related to that of *Participatory Budgeting* (see, e.g., Benade et al. 2017), where citizens propose projects and vote on which projects get funded by public money. In the setting that we consider, each issue $x \in \mathcal{I}$ represents whether or not some measure is implemented. Each such measure has an implementation cost c_x associated with it. Moreover, there is a total budget B that cannot be exceeded—that is, each ballot (individual or collective) can set a set of variables x to true such that the cumulative cost of these variables is at most B (and set the remaining variables to false). The integrity constraint Γ encodes that the total budget B cannot be exceeded by the total cost of the variables that are set to true. (For the sake of simplicity, we assume that all costs and the total budget are all positive integers.)

The concepts and tools from judgment aggregation are useful and relevant in this setting. This is witnessed, for instance, by the fact that simply taking a majority vote will not always lead to a suitable collective outcome. Consider the example where there are three measures that are each associated with cost 1, and where there is a budget of 2. Moreover, suppose that there are three individuals. The first individual votes to implement measures 1 and 2; the second votes for measures 1 and 3, and the third for 2 and 3. Each of the individuals' opinions is consistent with the budget. However, taking a majority measure-by-measure vote results in implementing all three issues, which exceeds the budget. (In other words, the individual opinions r_1, r_2, r_3 are all rational, whereas the collective majority opinion $m_{\boldsymbol{r}}$ is not.) This example is illustrated in Figure 2—in this figure, we encode the budget constraint using a DNNF circuit Γ.

$\boldsymbol{r}$	x_1	x_2	x_3
r_1	1	1	0
r_2	1	0	1
r_3	0	1	1
$\text{MAJ}(\boldsymbol{r})$	1	1	1

(a) The profile $\boldsymbol{r}$



(b) The integrity constraint Γ

Figure 2: Example of an aggregation scenario with a budget constraint (for $B = 2$ and $c_x = 1$ for all $x \in \mathcal{I}$), where the budget constraint is represented as a DNNF circuit Γ.

Encoding into a Polynomial-Size DNNF Circuit

To use the framework of judgment aggregation to model settings with budget constraints, we need to encode budget constraints using integrity constraints Γ. One can do this in several ways. We consider an encoding using DNNF circuits (as in Figure 2b). Let $\mathcal{I}$ be a set of issues, let $\{c_x\}_{x \in \mathcal{I}}$ be a vector of implementation costs, and let $B \in \mathbb{N}$ be a total budget. We say that an integrity constraint Γ encodes the budget constraint for $\{c_x\}_{x \in \mathcal{I}}$ and B if for each complete ballot $r : \mathcal{I} \to \{0, 1\}$ it holds that r satisfies Γ if and only if $\sum_{x \in \mathcal{I}, r(x)=1} c_x \leq B$.

We can encode budget constraints efficiently using DNNF circuits by expressing them as binary decision diagrams. A *binary decision diagram (BDD)* is a particular type of NNF circuit. Let Γ be an NNF circuit. We say that a node N of Γ is a *decision node* if (i) it is a leaf or (ii) it is a disjunction node expressing $(x \wedge \alpha) \vee (\neg x \wedge \beta)$, where $x \in \text{Var}(\Gamma)$ and α and β are decision nodes. A binary decision diagram is an NNF circuit whose root is a decision node. A *free binary decision diagram (FBDD)* is a BDD that satisfies decomposability (see, e.g., Darwiche and Marquis 2002, Gergov and Meinel 1994).

Theorem 16. *For each* $\mathcal{I}$, $\{c_x\}_{x \in \mathcal{I}}$ *and* B, *we can construct a DNNF circuit* Γ *encoding the budget constraint for* $\{c_x\}_{x \in \mathcal{I}}$ *and* B *in time polynomial in* $B + |\mathcal{I}|$.

Proof. We construct an FBDD Γ encoding the budget constraint for $\{c_x\}_{x \in \mathcal{I}}$ and B as follows. Without loss of generality, suppose that $c_x > 0$ for each $x \in \mathcal{I}$. Let $\mathcal{I} = \{x_1, \dots, x_n\}$. We introduce a decision node $N_{i,j}$ for each $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, B\}$. Take arbitrary $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, B\}$. If $i = n$, we let $N_{i,j} = \top$. If $i < n$, we distinguish two cases: either (i) $j' \leq B$ or (ii) $j' > B$, where $j' = j + c_{x_i}$.. In case (i), we let $N_{i,j} = (x_i \wedge N_{i+1,j'}) \vee (\neg x_i \wedge N_{i+1,j})$. In case (ii), we let $N_{i,j} = (x_i \wedge \bot) \vee (\neg x_i \wedge N_{i+1,j})$. We let the root of the FBDD be the node $N_{0,0}$—and we remove all nodes that are not descendants of $N_{0,0}$. Intuitively, the subcircuit rooted at $N_{i,j}$ represents all truth assignments to the variables $x_{i+1}, \dots, x_n$ that fit within a budget of $B - j$. For each node $N_{i,j}$ it holds that the variables in the leaves reachable from $N_{i,j}$ are among $x_{i+1}, \dots, x_n$. Therefore, we constructed an FBDD. Moreover, each complete ballot r satisfies the circuit Γ if and only if $\sum_{x \in \mathcal{I}, r(x)=1} c_x \leq B$. Thus, Γ is a DNNF circuit constructed in time polynomial in $B + |\mathcal{I}|$ encoding the budget constraint for $\{c_x\}_{x \in \mathcal{I}}$ and B. $\square$

An example of a DNNF circuit resulting from the encoding described in the proof of Theorem 16—after some simplifications—can be found in Figure 2b.

Complexity Results

Using the encoding result of Theorem 16, we can establish polynomial-time solvability results for computing outcomes for several judgment aggregation procedures in the setting of budget constraints.

Corollary 17. OUTCOME(MED), OUTCOME(MCC), OUTCOME(REV), *and* OUTCOME(RA) *are polynomial-time computable when restricted to the case where* Γ *expresses a budget constraint.*

Proof. The result follows from Theorems 9, 10 and 16, and Corollary 11. $\square$

For the YOUNG and MAXHAM procedures, we obtain intractability results for the case of budget constraints—for both procedures computing outcomes is Θ_2^{p}-hard.

Proposition* 18. OUTCOME(YOUNG) *is Θ_2^{p}-hard when restricted to the case where Γ expresses a budget constraint.*

Corollary 19. OUTCOME(MAXHAM) *is Θ_2^{p}-hard when restricted to the case where Γ expresses a budget constraint.*

Proof. The result follows directly from Proposition 4. □

An overview of the complexity results that we established in this section can be found in Table 5.

F	complexity of OUTCOME(F)	
MED	in P	(Corollary 17)
REV	in P	(Corollary 17)
MCC	in P	(Corollary 17)
YOUNG	Θ_2^{p}-c	(Proposition 18)
MAXHAM	Θ_2^{p}-c	(Corollary 19)
RA	in P	(Corollary 17)

Table 5: The computational complexity of outcome determination for various procedures F restricted to the case where Γ is a budget constraint.

Directions for Future Research

In this paper, we provided a set of initial results for restricted languages for judgment aggregation, but these results are only the tip of the iceberg that is to be explored. We outline some directions for interesting future work on this topic.

One first direction is to establish the complexity of OUTCOME(F) for cases that are left open in this paper—for example, for YOUNG and REV for the case of Krom and (definite) Horn formulas. Another direction is to pinpoint the complexity of OUTCOME(F) for the languages that we considered for other judgment aggregation rules studied in the literature (see, e.g., Lang et al. 2017).

Yet another direction is to extend tractability results obtained in this paper—e.g., for Krom and Horn formulas—to formulas that are 'close' to Krom or Horn formulas. One could use the notion of backdoors for this (see, e.g., Gaspers and Szeider 2012).

Finally, further restricted languages of propositional formulas or Boolean circuits need to be studied, to get a more complete picture of where the boundaries of the expressivity-tractability balance lie in the setting of judgment aggregation. A good source for additional languages is the field of knowledge compilation (see, e.g., Darwiche and Marquis 2002, Darwiche 2014, Marquis 2015), where many restricted languages have been studied with respect to their expressivity and support for performing various operations tractably.

Conclusion

In this paper, we initiated the hunt for representation languages for the setting of judgment aggregation that strike a balance between (1) allowing relevant computational tasks to be performed efficiently and (2) being expressive enough to model interesting and relevant application settings. Concretely, we considered Krom and (definite) Horn formulas, and we studied the class of Boolean circuits in DNNF. We studied the impact of these languages on the complexity of computing outcomes for a number of judgment aggregation procedures studied in the literature. Additionally, we illustrated the use of these languages for a specific application setting: voting on how to spend a budget.

Appendix: Preliminaries

We give an overview of some notions from propositional logic and computational complexity that we use in the paper.

Propositional Logic

Propositional formulas are constructed from propositional variables using the Boolean operators $\wedge, \vee, \rightarrow$, and $\neg$. A *literal* is a propositional variable x (a *positive literal*) or a negated variable $\neg x$ (a *negative literal*). A *clause* is a finite set of literals, not containing a complementary pair $x, \neg x$, and is interpreted as the disjunction of these literals. A formula in *conjunctive normal form (CNF)* is a finite set of clauses, interpreted as the conjunction of these clauses. For each $r \geq 1$, an r*-clause* is a clause that contains at most r literals, and rCNF denotes the class of all CNF formulas consisting only of r-clauses. 2CNF is also denoted by KROM, and 2CNF formulas are also known as *Krom formulas*. A *Horn clause* is a clause that contains at most one positive literal. A *definite Horn clause* is a clause that contains exactly one positive literal. We let HORN denote the class of all CNF formulas that contain only Horn clauses (*Horn formulas*), and we let DEFHORN denote the class of all CNF formulas that contain only definite Horn clauses (*definite Horn formulas*).

For a propositional formula φ, Var(φ) denotes the set of all variables occurring in φ. Moreover, for a set X of variables, Lit(X) denotes the set of all literals over variables in X, i.e., Lit(X) $= \{ x, \neg x : x \in X \}$. We use the standard notion of *(truth) assignments* α : Var(φ) $\rightarrow \{0, 1\}$ for Boolean formulas and *truth* of a formula under such an assignment. For any formula φ and any truth assignment α, we let $\varphi[\alpha]$ denote the formula obtained from φ by instantiating variables s in the domain of α with $\alpha(x)$ and simplifying the formula accordingly. By a slight abuse of notation, if α is defined on all Var(φ), we let $\varphi[\alpha]$ denote the truth value of φ under α.

Computational Complexity Theory

We assume the reader to be familiar with the complexity classes P and NP, and with basic notions such as polynomial-time reductions. For more details, we refer to textbooks on computational complexity theory (see, e.g., Arora and Barak 2009).

In this paper, we also refer to the complexity classes Θ_2^{p} and Δ_2^{p} that consist of all decision problems that can be solved by a polynomial-time algorithm that queries an NP oracle $O(\log n)$ or $n^{O(1)}$ times, respectively. Formally, algorithms with access to an oracle are defined as follows. Let O be a decision problem. A Turing machine $\mathbb{M}$ with access to an O *oracle* is a Turing machine with a dedicated *oracle tape* and dedicated states q_{query}, q_{yes} and q_{no}. Whenever $\mathbb{M}$ is in the state q_{query}, it does not proceed according to the transition relation, but instead it transitions into the state q_{yes} if the oracle tape contains a string x that is a yes-instance for the problem O, i.e., if $x \in O$, and it transitions into the state q_{no} if $x \notin O$. Intuitively, the oracle solves arbitrary instances of O in a single time step. The class Θ_2^{p} (resp. Δ_2^{p}) consists of all decision problems Q for which there exists a deterministic Turing machine that decides for each instance x of size n whether $x \in Q$ in time polynomial in n by querying some oracle $O \in \text{NP}$ at most $O(\log n)$ (resp. $n^{O(1)}$) times.

Let $\mathcal{C}$ be a class of propositional formulas. The following problem is complete for the class Θ_2^{p} under polynomial-time reductions when $\mathcal{C}$ is the class of all propositional formulas (Chen and Toda 1995; Krentel 1988; Wagner 1990).

MAX-MODEL($\mathcal{C}$)
Instance: A satisfiable propositional formula $\varphi \in \mathcal{C}$, and a variable $z \in \text{Var}(\varphi)$.
Question: Is there a model of φ that sets a maximal number of variables in $\text{Var}(\varphi)$ to true (among all models of φ) and that sets z to true?

For any class $\mathcal{C}$ of propositional formulas, we let MAX-MODEL($\mathcal{C}$) denote the problem MAX-MODEL restricted to formulas $\varphi \in \mathcal{C}$.

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Additional Material: Lemmas and Proofs

As additional material, we provide proofs for all statements in the main paper marked with a star ($\star$), as well as additional lemmas used for these proofs.

Lemma 20. MAX-MODEL(3CNF) *is* Θ_2^{p}*-complete.*

Proof. We sketch a reduction from MAX-MODEL for arbitrary propositional formulas. Let (φ, z) be an instance of MAX-MODEL. By using the standard Tseitin transformation, we can transform φ into a 3CNF formula φ' with $\text{Var}(\varphi') = \text{Var}(\varphi) \cup Z$ for some set Z of new variables, such that for each truth assignment $\alpha : \text{Var}(\varphi) \to \{0,1\}$ it holds that $\varphi[\alpha]$ is true if and only if there exists a truth assignment $\beta : Z \to \{0,1\}$ such that $\varphi'[\alpha \cup \beta]$ is true.

We then transform φ' into a 3CNF formula φ'' with $\text{Var}(\varphi'') = \text{Var}(\varphi') \cup Z'$, for the set $Z' = \{ z' : z \in Z \}$ of fresh variables, such that the maximal models of φ'' correspond exactly to the maximal models of φ. We define φ'' as follows:

$$\varphi'' = \varphi' \wedge \bigwedge_{z \in Z} ((\neg z \vee \neg z') \wedge (z \vee z')).$$

Each model of φ'' then must set the same number of variables in $Z \cup Z'$ to true—namely $|Z|$ of them. $\square$

Lemma 21. MAX-MODEL(HORN $\cap$ KROM) *is* Θ_2^{p}*-complete.*

Proof. We give a reduction from MAX-MODEL(3CNF). Let (φ, z) be an instance of MAX-MODEL(3CNF), where $\text{Var}(\varphi) = X = \{x_1, \dots, x_n\}$ and where φ consists of the clauses $c_1, \dots, c_m$. Without loss of generality, we may assume that each clause c_j is of size exactly 3. Also, without loss of generality, we may assume that φ is satisfied by the "all zeroes" assignment, that is, by the assignment α_0 such that $\alpha_0(x_i) = 0$ for all $i \in [n]$. Moreover, we may assume without loss of generality that $m \geq n$. We construct an instance (φ', z') of MAX-MODEL(HORN $\cap$ KROM) as follows.

For each clause c_j, we introduce fresh variables y_j^u and $y_{j,\ell}^u$, for $u \in [3]$ and $\ell \in [n]$. Moreover, for each x_i, we introduce fresh variables x_i^1, x_i^0, $z_{i,\ell}^1$ for $\ell \in [m+1]$ and $z_{i,\ell}^0$ for $\ell \in [m]$. We then let φ' consist of the following clauses. For each $j \in [m]$, we add the clauses:

$$(\neg y_j^1 \vee \neg y_j^2), (\neg y_j^1 \vee \neg y_j^3), (\neg y_j^2 \vee \neg y_j^3),$$

ensuring that at most one variable among y_j^1, y_j^2, y_j^3 can be true. Moreover, for each $j \in [m]$ and each $u \in [3]$, we add the clauses:

$$(y_j^u \to y_{j,1}^u), (y_{j,1}^u \to y_{j,2}^u), \dots, (y_{j,n-1}^u \to y_{j,n}^u), (y_{j,n}^u \to y_j^u),$$

ensuring that the variables y_j^u and $y_{j,\ell}^u$ get the same truth value, for each $u \in [3]$ and each $j \in [m]$.

Then, for each $i \in [n]$, we add the clause $(\neg x_i^1 \vee \neg x_i^0)$, ensuring that at most one variable among x_i^1, x_i^0 is true. Moreover for each $i \in [n]$ we add the clauses:

$$(x_i^1 \to z_{i,1}^1), (z_{i,1}^1 \to z_{i,2}^1), \dots, (z_{i,m}^1 \to z_{i,m+1}^1), (z_{i,m+1}^1 \to x_i^1),$$

and:

$$(x_i^0 \to z_{i,1}^0), (z_{i,1}^0 \to z_{i,2}^0), \dots, (z_{i,m-1}^0 \to z_{i,m}^0), (z_{i,m}^0 \to x_i^0),$$

ensuring that the variables x_i^u and $z_{i,\ell}^u$ get the same truth value, for each $u \in \{0,1\}$ and each $i \in [n]$.

Finally, we add the following clauses to φ', for each clause c_j of φ. Let c_j be a clause of φ, and let $l_{j,u}$ be the u-th literal in c_j, for $u \in [3]$. If $l_{j,u} = x_i$ for some $i \in [n]$, we add the clause $(y_j^u \to x_i^1)$, and if $l_{j,u} = \neg x_i$ for some $i \in [n]$, we add the clause $(y_j^u \to x_i^0)$.

To finish our construction, we let $z' = x_i^1$, for the unique i such that $z = x_i$.

Before we show correctness of this reduction, we establish several other properties of the formula φ'. Any maximal model of φ' sets at least $n(m+1) + m(n+1) = 2nm + n + m$ variables to true. Since the "all zeroes" assignment α_0 satisfies φ, we can satisfy φ' by setting all variables $x_i^0, z_{i,\ell}^0$ to true, setting all variables $x_i^1, z_{i,\ell}^1$ to false, and for each $j \in [m]$ setting all variables $y_j^u, y_{j,\ell}^u$ to true for some $u \in [3]$, and setting all variables $y_j^{u'}, y_{j,\ell}^{u'}$ to false for the other $u' \in [3]$. This model of φ' sets $2nm + n + m$ variables to true.

Moreover, by construction of φ', we know that each model of φ' sets at most $n(m+2) + m(n+1) = 2nm + 2n + m$ variables to true.

By construction of φ', we know that any model of φ' sets variables $x_i^u, z_{i,\ell}^u$ to true for **at most** one $u \in \{0,1\}$ for each $i \in [n]$, and that it sets variables $y_j^u, y_{j,\ell}^u$ to true for **at most** one $u \in [3]$ for each $j \in [m]$. We argue that any maximal model of φ' must set variables $x_i^u, z_{i,\ell}^u$ to true for **exactly** one $u \in \{0,1\}$ for each $i \in [n]$, and must set variables $y_j^u, y_{j,\ell}^u$ to true for **exactly** one $u \in [3]$ for each $j \in [m]$. Suppose that there is some maximal model of φ' that sets all variables $x_i^1, x_i^0, z_{i,\ell}^1, z_{i,\ell}^0$ to false, for some $i \in [n]$. Then we know that this model can set at most $2nm + 2n - 2$ variables to true. Since $m \geq n$, we know that this model cannot be maximal, since there is a model that sets $2nm + n + m > 2nm + 2n - 2$ variables to true. From this we can conclude that each maximal model of φ' must set variables $x_i^u, z_{i,\ell}^u$ to true for exactly one $u \in \{0,1\}$ for each $i \in [n]$. An entirely similar argument can be used to show that each maximal model of φ' must set variables $y_j^u, y_{j,\ell}^u$ to true for exactly one $u \in [3]$ for each $j \in [m]$.

Then, for each maximal model α' of φ', we can construct a truth assignment $\alpha : X \to \{0,1\}$ as follows. For each $x_i \in X$, we let $\alpha(x_i) = 1$ if and only if α' sets x_i^1 to true, and we let $\alpha(x_i) = 0$ if and only if α' sets x_i^0 to true. Moreover, this truth assignment α satisfies φ. To derive a contradiction, suppose that α does not satisfy φ, that is, that there is some clause c_j of φ that α does not satisfy. Then there must be a clause of the form $(y_j^u \to x_i^{u'})$ in φ', for some $u \in [3]$ and some $u' \in \{0,1\}$, that is not satisfied by α'. This is a contradiction with our assumption that α' satisfies φ'. Therefore, we can conclude that α satisfies φ.

Conversely, for any model α of φ we can construct a model α' of φ' as follows. For each $i \in [n]$ and each $u \in \{0, 1\}$, α' sets the variables $x_i^u, z_{i,\ell}^u$ to true if and only if $\alpha(x_i) = u$. Moreover, since α satisfies φ, we know that for each $j \in [m]$ there is some $u_j \in [3]$ such that α satisfies the u_j-th literal in clause c_j. Then, for each $j \in [m]$ and each $u \in [3]$, α' sets the variables $y_j^u, y_{j,\ell}^u$ to true if and only if $u = u_j$. It is straightforward to verify that α' satisfies φ'.

We will now argue that there is a maximal model of φ that sets z to true if and only if there is a maximal model of φ' that sets z' to true.

($\Rightarrow$) Suppose that there is a maximal model α of φ that sets z to true. We can then construct a model α' of φ', as described above. It is easy to verify that α' sets z' to true. We argue that α' is a maximal model of φ'. Suppose, to derive a contradiction, that α' is not a maximal model of φ'—that is, there is some model β' of φ' that sets more variables to true than α'. Then, as described above, we can construct a model β of φ from β'. It is straightforward to verify that β sets more variables in X to true than α. This is a contradiction with our assumption that α is a maximal model of φ. Therefore, we can conclude that α' is a maximal model of φ'.

($\Leftarrow$) Conversely, suppose that there is a maximal model α' of φ' that sets z' to true. We can then construct a model α of φ', as described above. It is easy to verify that α sets z to true. We argue that α is a maximal model of φ. Suppose, to derive a contradiction, that α is not a maximal model of φ—that is, there is some model β of φ that sets more variables to true than α. Then, as described above, we can construct a model β' of φ' from β. It is straightforward to verify that β' sets more variables in Var(φ') to true than α'. This is a contradiction with our assumption that α' is a maximal model of φ'. Therefore, we can conclude that α is a maximal model of φ. □

Lemma 22. OUTCOME(MED) *is Θ_2^{p}-hard even when restricted to the case where* $\Gamma \in$ HORN.

Proof. We give a reduction from MAX-MODEL(HORN). Let (φ, z) be an instance of MAX-MODEL(HORN), where Var(φ) $= X = \{x_1, \dots, x_n\}$. We may assume without loss of generality that the "all zeroes" assignment $\alpha_0 : X \to \{0, 1\}$, for which $\alpha_0(x_i) = 0$ for all $i \in [n]$, satisfies φ. We construct an instance $(\mathcal{I}, \Gamma, \boldsymbol{r}, s)$ of OUTCOME(MED), with $\Gamma \in$ HORN, as follows.

We let $\mathcal{I} = X \cup \{\, y_{i,j}, y'_{i,j} : i \in [n], j \in [3] \,\}$. We define Γ as follows: $\Gamma = \varphi \wedge \bigwedge_{i \in [n]} ((y_{i,1} \wedge y_{i,2} \wedge y_{i,3} \to x_i) \wedge (y'_{i,1} \wedge y'_{i,2} \wedge y'_{i,3} \to x_i))$. We define the profile $\boldsymbol{r} = (r_1, r_2, r_3)$ as shown in Figure 3. Finally, we let s be the partial ballot that only sets z to 1.

Clearly, each rational ballot $r^* \in \mathcal{R}(\mathcal{I}, \Gamma)$ must satisfy φ, since $\Gamma \models \varphi$. Moreover, to satisfy Γ, each rational ballot r^* must—for each $i \in [n]$—either (i) set x_i to 1 or (ii) set at least one variable among $y_{i,1}, y_{i,2}, y_{i,3}$ and at least one variable among $y'_{i,1}, y'_{i,2}, y'_{i,3}$ to 0. In case (i), the total Hamming distance to the profile $\boldsymbol{r}$ increases with 3, and in case (ii), the total Hamming distance to the profile $\boldsymbol{r}$ increases with at least 4. Therefore, the rational ballots r^* with minimal cumulative Hamming distance to the profile $\boldsymbol{r}$ correspond exactly to the models of φ that set a maximal number of variables $x \in X$ to true. From this it immediately follows that there exists some $r^* \in$ MED($\boldsymbol{r}$) that agrees with s if and only if there is a maximal model of φ that sets z to true. □

Proof of Proposition 1 (sketch). We give a reduction from OUTCOME(MED) restricted to the case where $\Gamma \in$ HORN. Let $(\mathcal{I}, \Gamma, \boldsymbol{r}, s)$ be an instance of OUTCOME(MED) with $\Gamma \in$ HORN. Let $\boldsymbol{r} = (r_1, \dots, r_p)$. Also, let $c_1, \dots, c_m$ denote the clauses of Γ. Moreover, suppose that the clauses $c_1, \dots, c_u$ are non-definite Horn clauses, and that the clauses $c_{u+1}, \dots, c_m$ are definite Horn clauses. We construct an equivalent instance $(\mathcal{I}', \Gamma', \boldsymbol{r}', s)$ of OUTCOME(MED) with $\Gamma' \in$ DEFHORN, as follows.

Firstly, we let $\mathcal{I}' = \mathcal{I} \cup \{\, y_{j,\ell} : j \in [u], \ell \in [n+1] \,\}$. We obtain the definite Horn formula Γ' from Γ as follows. Firstly, we add the clauses $c_{u+1}, \dots, c_m$ to Γ'. Then, for each non-definite Horn clause c_j, with $j \in [u]$, we add a definite Horn clause $(c_j \vee y_{j,\ell})$ to Γ' for each $\ell \in [n+1]$. We obtain the profile $\boldsymbol{r}' = (r'_1, \dots, r'_p)$ from $\boldsymbol{r}$ as follows. For each $i \in [p]$, we let r'_i agree with r_i on the issues in $\mathcal{I}$. Moreover, for each $i \in [p]$ and each $x' \in \mathcal{I}' \backslash \mathcal{I}$, we let $r'_i(x') = 0$. It is straightforward to verify that each r'_i is rational.

We firstly show that for each $r^* \in$ MED($\boldsymbol{r}'$) and for each $j \in [u]$, it holds that r^* sets all variables $y_{j,\ell}$ to 0. We proceed indirectly, and suppose that this is not the case, i.e., that there is some $j \in [u]$ such that r^* does not set all variables $y_{j,\ell}$ to 0. We distinguish two cases: either (i) for all $\ell \in [n+1]$ it holds that r^* sets $y_{j,\ell}$ to 1, or (ii) this is not the case. In case (i), we know that the cumulative Hamming distance from r^* to the profile $\boldsymbol{r}'$ is at least $p(n+1)$. However, the ballot r_0 such that $r_0(x) = 0$ for all $x \in \mathcal{I}'$ is rational and has cumulative distance of at most pn to $\boldsymbol{r}'$. Thus, r^* does not have minimal distance to $\boldsymbol{r}'$, which contradicts our assumption that $r^* \in$ MED($\boldsymbol{r}'$). In case (ii), we know that there exists some $\ell, \ell' \in [n+1]$ such that r^* sets $y_{j,\ell}$ to 1 and $y_{j,\ell'}$ to 0. Then, we know that $r^* \models c_j$, since $r^* \models (c_j \vee y_{j,\ell'})$. However, then modifying r^* by setting $y_{j,\ell}$ to 0 would result in a rational ballot with strictly smaller cumulative distance to the profile $\boldsymbol{r}'$, which is a contradiction with our assumption that $r^* \in$ MED($\boldsymbol{r}'$). Thus, we can conclude that for each $r^* \in$ MED($\boldsymbol{r}'$) and for each $j \in [u]$, it holds that r^* sets all variables $y_{j,\ell}$ to 0.

It is then straightforward to verify that each $r^* \in$ MED($\boldsymbol{r}'$) satisfies Γ, and that there exists a ballot $r^* \in$ MED($\boldsymbol{r}$) such that s agrees with r^* if and only if there exists a ballot $r^* \in$ MED($\boldsymbol{r}'$) such that s agrees with r^*. □

$\boldsymbol{r}$	x_i	$y_{i,1}$	$y_{i,2}$	$y_{i,3}$	$y'_{i,1}$	$y'_{i,2}$	$y'_{i,3}$
r_1	0	1	1	0	1	1	0
r_2	0	1	0	1	1	0	1
r_3	0	0	1	1	0	1	1

Figure 3: The profile $\boldsymbol{r} = (r_1, r_2, r_3)$ in the proof of Lemma 22—here i ranges over $[n]$.

Lemma 23. OUTCOME(MCC) *is* Θ_2^{p}*-hard even when restricted to the case where* $\Gamma \in$ HORN*.*

Proof. The proof of this statement is analogous to the proof of Lemma 22—we use the same reduction from MAX-MODEL(HORN). That is, we construct $\mathcal{I}$, Γ, $\boldsymbol{r}$ and s in exactly the same way. What remains to show is that this reduction is also correct for the problem OUTCOME(MCC).

Clearly, each rational ballot $r^* \in \mathcal{R}(\mathcal{I}, \Gamma)$ must satisfy φ, since $\Gamma \models \varphi$. Moreover, to satisfy Γ, each rational ballot r^* must—for each $i \in [n]$—either (i) set x_i to 1 or (ii) set at least one variable among $y_{i,1}, y_{i,2}, y_{i,3}$ and at least one variable among $y'_{i,1}, y'_{i,2}, y'_{i,3}$ to 0. In case (i), the total Hamming distance to the majority outcome $m_{\boldsymbol{r}}$ increases with 1, and in case (ii), the total Hamming distance to the majority outcome $m_{\boldsymbol{r}}$ increases with at least 2. Therefore, the rational ballots r^* with minimal cumulative Hamming distance to the profile $\boldsymbol{r}$ correspond exactly to the models of φ that set a maximal number of variables $x \in X$ to true. □

Proof of Proposition 2 (sketch). The proof of this statement is analogous to the proof of Proposition 1. That is, we take the reduction from OUTCOME(MED) to OUTCOME(MED) from the proof of Proposition 1, and we employ it as a reduction from OUTCOME(MCC) to OUTCOME(MCC). Since this reduction results in an instance where $\Gamma \in$ DEFHORN, this suffices. The argument for correctness of the reduction is entirely analogous. □

Lemma 24. *Let φ be a 3CNF formula with clauses $c_1, \dots, c_m$ (all of size exactly 3) and with n variables such that $\varphi \setminus \{c_1\}$ is 1-in-3-satisfiable. We can then in polynomial time construct a set $\mathcal{I}$ of issues together with a profile $\boldsymbol{r}$ for $\mathcal{I}$ (and for $\Gamma = \top$), and positive integers u_1, u_2, u_3 (with $u_3 < u_1$) that are polynomial in $|\varphi|$, and that depend only on n and m, such that:*

- *if φ is 1-in-3-satisfiable, then the minimum max-Hamming distance from any ballot r^* to $\boldsymbol{r}$ is u, and moreover, there exists some ballot r^* such that the Hamming distance from r^* to each individual ballot in $\boldsymbol{r}$ is exactly u_1;*
- *if φ is not 1-in-3-satisfiable, then the minimum max-Hamming distance from any ballot r^* to $\boldsymbol{r}$ is $u_1 + u_2$, and moreover, for each ballot r^* that achieves his minimum max-Hamming distance to $\boldsymbol{r}$ it holds that the Hamming distance from r^* to each individual ballot in $\boldsymbol{r}$ is exactly $u+w$; and*
- *the Hamming distance from any ballot r^* that achieves the minimum max-Hamming distance to the profile $\boldsymbol{r}$ to the all-zeroes ballot r_0 is exactly u_3.*

Proof. Take an arbitrary 3CNF formula φ with clauses $c_1, \dots, c_m$ and $\text{Var}(\varphi) = \{x_1, \dots, x_n\}$ such that $\varphi \setminus \{c_1\}$ is 1-in-3-satisfiable. Without loss of generality, suppose that n is a power of 2.

We proceed in two steps. In the first step, we will construct a set of issues and a set of ballots such that the minimum max-Hamming distance to this set of ballots is lower than a particular threshold if and only if φ is 1-in-3-satisfiable. Then, in the second step, we will use these issues and ballots to construct another set of issues and another set of ballots that satisfy the conditions specified in the statement of the lemma.

We begin by introducing $2n + 4$ issues $y_1, \dots, y_n, y'_1, \dots, y'_n, z_1, \dots, z_4$, together with a set of $2 \log n + 3m$ ballots on these issues. We define the first $2 \log n$ ballots $r_1, \dots, r_{\log n}$ and $r'_1, \dots, r'_{\log n}$ as follows:

$$r_i(y_j) = r_i(y'_j) = \begin{cases} 1 & \text{if the } i\text{-th bit of } j \text{ is 1,} \\ 0 & \text{otherwise,} \end{cases}$$

and $r'_i(y_j) = r'_i(y'_j) = 1 - r_i(y_j)$.

It is straightforward to verify that any ballot r^* that sets exactly one of y_j and y'_j to true (for each $j \in [n]$) achieves the minimum possible max-Hamming distance to these ballots (namely distance n). Moreover, any ballot that has a higher minimum max-Hamming distance to these ballots has a Hamming distance strictly higher than n to more than one of the ballots. Intuitively, setting y_j to true in a ballot r^* corresponds to setting variable x_j to true, and setting y'_j to true corresponds to setting variable x_j to false.

Next, for each clause c_k of φ, we add a ballot s_k, that is defined as follows. For all variables $x_j \notin \text{Var}(c_k)$, we let $s_k(y_j) = s_k(y'_j) = 0$. Then, for each literal $l \in c_k$, if $l = x_j$, we let $s_k(y_j) = 1$ and $s_k(y'_j) = 0$, and if $l = \neg x_j$, we let $s_k(y_j) = 0$ and $s_k(y'_j) = 1$. Moreover, for each $\ell \in [4]$, $s_k(z_\ell) = 0$.

Then, for each clause c_k of φ, we add ballots s'_k and s''_k, that are defined as follows. For all variables $x_j \notin \text{Var}(c_k)$, we let $s'_k(y_j) = s''_k(y_j) = s'_k(y'_j) = s''_k(y'_j) = 0$. Then, for each literal $l \in c_k$, if $l = x_j$, we let $s'_k(y_j) = s''_k(y_j) = 0$ and $s'_k(y'_j) = s''_k(y'_j) = 1$, and if $l = \neg x_j$, we let $s'_k(y_j) = s''_k(y_j) = 1$ and $s'_k(y'_j) = s''_k(y'_j) = 0$. Moreover, for both $\ell \in [2]$, $s'_k(z_\ell) = 0$ and $s''_k(z_\ell) = 1$, and for both $\ell \in [3, 4]$, $s'_k(z_\ell) = 1$ and $s''_k(z_\ell) = 0$.

It is now routine to verify the following statements. (1) If φ is 1-in-3-satisfiable, then there is a ballot r^* that achieves a minimum max-Hamming distance of $n + 4$ to these ballots—namely by setting the variables z_ℓ to 0, and by setting the variables y_j and y'_j according to the truth assignment witnessing exactly-1-satisfiability. (2a) If φ is not 1-in-3-satisfiable, then the minimum max-Hamming distance of any ballot r^* to these ballots is strictly more than $n + 4$. (2b) If φ is not 1-in-3-satisfiable, then any ballot r^* has a cumulative Hamming distance to these ballots of at least $(2 \log n + 3m)(n + 4) + 2$. (2c) If φ is not 1-in-3-satisfiable, there is a ballot r^* that has Hamming distance $n + 4$ to all ballots except one, to which it has Hamming distance $n + 6$. (3) The Hamming distance from the all-zeroes ballot r_0 to any ballot r^* achieving the minimum max-Hamming distance or the minimum cumulative Hamming distance to these ballots is exactly n.

Next, in the second step, we will use the issues and ballots that we constructed above to construct the set $\mathcal{I}$ of issues and the profile $\boldsymbol{r}$ of ballots as specified in the statement of the lemma. We do this by making $2 \log n + 3m$ copies of each of the issues y_j, y'_j, z_i. Then, we construct the profile $\boldsymbol{r}$ that consists of $2 \log n + 3m$ ballots, each of which consists of a different ballot (among y_j, y'_j, z_i) for each set of copies of the issues. This can be done as follows. Let $t_1, \dots, t_b$ be the ballots that we defined above, where $b = 2\log n + 3m$. Then let $\boldsymbol{r} = \{t'_1, \dots, t'_b\}$. For each $i \in [2\log n + 3m]$, the ballot $t'_i \in \boldsymbol{r}$ agrees with ballot $t_{i+\ell \mod 2\log n+3m}$ on the ℓ-th copies of y_j, y'_j, z_i, for each $\ell \in [2\log n + 3m]$.

It is now straightforward to verify that if φ is 1-in-3-satisfiable, then the minimum max-Hamming distance from any ballot r^* to $\boldsymbol{r}$ is $u_1 = (2\log n + 3m)(n + 4)$, and that there exists some ballot r^* that has Hamming distance u to each ballot in $\boldsymbol{r}$. Also, if φ is not 1-in-3-satisfiable, then the minimum max-Hamming distance from any ballot r^* to $\boldsymbol{r}$ is $u_1 + u_2$, where $u_2 = 2(2\log n + 3m)$, and that any ballot r^* that achieves this minimum has Hamming distance exactly $u_1 + u_2$ to each ballot in $\boldsymbol{r}$.

Moreover, any ballot r^* that achieves the minimum max-Hamming distance to the profile $\boldsymbol{r}$ has Hamming distance exactly $u_3 = n(2\log n + 3m)$ to the all-zeroes ballot r_0. $\square$

Proof of Proposition 4. Membership in Θ_2^{p} (for the general case) has been shown before (de Haan and Slavkovik 2017). We show Θ_2^{p}-hardness for the case where $\Gamma = \top$ by giving a reduction from the Θ_2^{p}-complete problem of deciding whether the maximum number of variables set to true in any satisfying assignment of a (satisfiable) propositional formula φ is odd. Let φ be an arbitrary satisfiable propositional formula with n variables. Suppose without loss of generality that n is even.

For each $i \in [n]$, we construct a 3CNF formula ψ_i that is 1-in-3-satisfiable if and only if there is a truth assignment that satisfies φ and that sets at least i variables among Var(φ) to true (by NP-completeness of 1-in-3SAT, using the standard reduction). We can do this in such a way that all of the formulas ψ_i have the same number of clauses and the same number of variables, and such that for each ψ_i, there is some clause $c \in \psi_i$ such that $\psi_i \setminus \{c\}$ is 1-in-3-satisfiable.

Then, since the formulas ψ_i satisfy the requirements for Lemma 24, we can construct sets $\mathcal{I}_1, \dots, \mathcal{I}_n$ of issues and profiles $\boldsymbol{r}_1, \dots, \boldsymbol{r}_n$ such that for each $i \in [n]$, the issues $\mathcal{I}_i$ and the profile $\boldsymbol{r}_i$ satisfy the conditions mentioned in the statement of Lemma 24. We can do this in such a way that the sets $\mathcal{I}_1, \dots, \mathcal{I}_n$ are disjoint. Moreover, the profiles $\boldsymbol{r}_1, \dots, \boldsymbol{r}_n$ have the same number b of individual ballots. For each $i \in [n]$, let $\boldsymbol{r}_i$ consist of the ballots $r^i_1, \dots, r^i_b$. We then use these sets $\mathcal{I}_1, \dots, \mathcal{I}_n$ and profiles $\boldsymbol{r}_1, \dots, \boldsymbol{r}_n$ to construct a single set $\mathcal{I}$ of issues and a single profile $\boldsymbol{r}$. We let $\mathcal{I} = \bigcup_{i=1}^{n} \mathcal{I}_i \cup \{z\}$, where z is a fresh propositional variable. We let $\boldsymbol{r}$ consist of the ballots $r_1, \dots, r_b, r'_1, \dots, r'_b$, that we will define below.

For each $j \in [b]$, we define r_j as follows. For each odd $i \in [n]$ and each $x \in \mathcal{I}_i$, we let r_j agree with r^i_j, i.e., $r_j(x) = r^i_j(x)$. For each even $i \in [n]$ and each $x \in \mathcal{I}_i$, we let $r_j(x) = 0$. Finally, we let $r_j(z) = 0$.

For each $j \in [b]$, we define r'_j as follows. For each even $i \in [n]$ and each $x \in \mathcal{I}_i$, we let r'_j agree with r^i_j, i.e., $r'_j(x) = r^i_j(x)$. For each odd $i \in [n]$ and each $x \in \mathcal{I}_i$, we let $r'_j(x) = 0$. Finally, we let $r_j(z) = 1$.

Finally, we let s be the partial ballot defined by letting $l(z) = 1$ and $l(x) = \star$ for all $x \in \mathcal{I} \setminus \{z\}$. We show that the maximum number of variables among Var(φ) that are set to true in any satisfying assignment of φ is odd if and only if there is some $r^* \in \text{MAXHAM}(\boldsymbol{r})$ that agrees with s.

($\Rightarrow$) Suppose the maximum number of variables among Var(φ) that are set to true in any satisfying assignment of φ is odd. Then the number of formulas ψ_i that are not 1-in-3-satisfiable is the same as the number of formulas ψ_i that are 1-in-3-satisfiable. As a result, for any ballot r^* over $\mathcal{I} \setminus \{z\}$ that minimizes the max-Hamming distance to $\boldsymbol{r}$ (restricted to $\mathcal{I} \setminus \{z\}$), the Hamming distance to the ballots $r_1, \dots, r_b$ is equal to the Hamming distance to the ballots $r'_1, \dots, r'_b$. As a result, any such ballot r^* over $\mathcal{I} \setminus \{z\}$ minimizing the max-Hamming distance to $\boldsymbol{r}$ (restricted to $\mathcal{I} \setminus \{z\}$) can be extended to a ballot minimizing the max-Hamming distance to $\boldsymbol{r}$ by setting z to 1.

($\Leftarrow$) Suppose the maximum number of variables among Var(φ) that are set to true in any satisfying assignment of φ is even. Then there are more formulas ψ_i that are not 1-in-3-satisfiable than formulas ψ_i that are 1-in-3-satisfiable. As a result, for any ballot r^* over $\mathcal{I} \setminus \{z\}$ that minimizes the max-Hamming distance to $\boldsymbol{r}$ (restricted to $\mathcal{I} \setminus \{z\}$), the Hamming distance to the ballots $r_1, \dots, r_b$ is larger than the Hamming distance to the ballots $r'_1, \dots, r'_b$. As a result, any ballot r^* that minimizes the max-Hamming distance to $\boldsymbol{r}$ must set z to 0. $\square$

MINVERTEXCOVER *Instance:* A graph $G = (V, E)$, and a vertex $v^* \in V$. *Question:* Is there a minimum-size vertex cover $C \subseteq V$ that includes v^*?

Lemma⋆ 25. MINVERTEXCOVER *is* Θ_2^{p}*-complete.*

Proof. Membership in Θ_2^{p} can be shown routinely. We show Θ_2^{p}-hardness by reducing from the problem of deciding whether the maximum number of variables satisfied by any model for a given (satisfiable) propositional formula is odd. Let φ be an arbitrary satisfiable propositional formula, and let $n = |\text{Var}(\varphi)|$. Since the propositional satisfiability problem is NP-complete, we can construct propositional formulas $\psi_1, \dots, \psi_n$ such that for each $i \in [n]$, ψ_i is satisfiable if and only if φ can be satisfied by setting at least i variables among Var(φ) to true. Then, by NP-completeness of the problem of deciding whether a graph has a clique of size at least m, we can transform these formulas ψ_i into graphs $G_1, \dots, G_n$ together with positive integers $m_1, \dots, m_n$ such that for each $i \in [n]$ it holds that G_i has a clique of size at least m_i. Moreover, we can ensure that no G_i has a clique of size $m_i + 1$. Let $m_{\max} = \max\{m_1, \dots, m_n\}$. Assume without loss of generality that $m_{\max}$ is even. Then we can straightforwardly transform the graphs $G_1, \dots, G_n$ and the integers $m_1, \dots, m_n$ into graphs $G'_1, \dots, G'_n$ and integers $m'_1, \dots, m'_n$ such that (1) for each $i \in [n]$ it holds that ψ_i is true if and only if G'_i has a clique of size m'_i, (2) for each $i \in [n]$, G'_i has no clique of size $m'_i + 1$, and (3) $m'_1 < m'_2 < \dots < m'_n$. Assume without loss of generality that $G'_1, \dots, G'_n$ are pairwise disjoint. Then construct the graph G'' by putting together $G'_1, \dots, G'_n$, adding two additional vertices v^*_1, v^*_2, connecting v^*_1 to all vertices in G'_i for odd i, and connecting v^*_2 to all vertices in G'_i for even i. It is straightforward to verify that every clique of G'' of maximum size does not contain v^*_2 if and only if the maximum number of variables satisfied by any model of φ is odd. Then the instance (G''', v_2^*) of MINVERTEXCOVER—where G''' is the complement of G''—is a yes-instance of MINVERTEXCOVER if and only if the maximum number of variables satisfied by any model of φ is odd. This completes our proof of Θ_2^{p}-hardness. □

Proof of Proposition 18. We show Θ_2^{p}-hardness by reducing from MINVERTEXCOVER. Let (G, v^*) be an instance of MINVERTEXCOVER, where $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. Without loss of generality, assume that $2n+1$ is a multiple of 3. Moreover, without loss of generality, assume that $v^* = v_1$.

For each $i \in [n]$, let $d_i = |\{\, e_j : j \in [m], v_i \in e_j \,\}|$ denote the degree of vertex v_i. For each $j \in [m]$, let $d'_j = |\{\, v_i : i \in [n], v_i \in e_j \,\}|$ denote the degree of edge e_j. Moreover, for each $i \in [n], j \in [m]$, let $a_{i,j} = 1$ if and only if $v_i \in e_j$, and $a_{i,j} = 0$ otherwise. That is, $a_{i,j}$ encodes whether v_i is incident to edge e_j. Also, for each $i \in [n], j \in [m]$, let $t_{i,j} = 1$ if and only if $i \leq n+1-d'_j$, and let $t_{i,j} = 0$ otherwise.

We construct an instance $(\mathcal{I}, \Gamma, \boldsymbol{r}, s)$ of OUTCOME-(YOUNG) as follows. We let $\mathcal{I} = \{\, x_j : j \in [m] \,\} \cup \{y, z\} \cup \{\, w_i, w'_i : i \in [e] \,\}$. Then, we let Γ be a budgetary constraint that assigns cost 2 to each variable in $\{\, x_j : j \in [m] \,\}$, cost 2 to z, cost 1 to y, cost $2m$ to each variable in $\{\, w_i, w'_i : i \in [e] \,\}$, and that assigns a total budget of $12m + 1$. Then, we let $\boldsymbol{r}$ be the profile as depicted in Figure 4. It is straightforward to verify that each ballot in the profile satisfies the budgetary constraint Γ. Finally, we let s be the partial ballot defined by $l(y) = 0$ and $l(v) = \star$ for all $v \in \mathcal{I} \setminus \{y\}$.

Clearly, the majority outcome $m_{\boldsymbol{r}}$ does not satisfy the budgetary constraint Γ, as all variables in $\mathcal{I} \setminus \{z\}$ enjoy majority support, and the total cost of these variables is $14m + 1 > 12m + 1$. There are two ways of saving a total cost of at least $2m$ by deleting individual ballots: either (1) delete a set of ballots such that some w_i or w'_i is not supported by a majority anymore, or (2) delete a set of ballots such that all variables in $\{\, x_j : j \in [m] \,\} \cup \{z\}$ are not supported by a majority. Option (1) requires deleting more than $1/3(2n + 1)$ individual ballots, as each w_i and w'_i enjoys a two-thirds majority support. Without loss of generality, we may assume that the smallest vertex cover of G is of size less than $1/6(2n + 1)$—if this is not the case, we can simply add unconnected vertices to increase n. We show that option (2) requires less than $1/3(2n + 1)$ individual ballots. Let $C \subseteq V$ denote some vertex cover of G. Now remove from $\boldsymbol{r}$ those individual ballots r_i and the $|C|$ individual ballots $r_{n+1}, \dots, r_{n+|C|}$. Without loss of generality, we may assume that these ballots $r_{n+1}, \dots, r_{n+|C|}$ support all variables x_j—again, if this were not the case, we could increase n by adding unconnected vertices. It is straightforward to verify that removing these ballots results in a profile where the variables in $\{\, x_j : j \in [m] \,\} \cup \{z\}$ do not enjoy majority support. Moreover, since $|C| < 1/6(2n + 1)$, we deleted less than $1/3(2n + 1)$ individual ballots. Thus, we can restrict our attention to deleting individual ballots that ensure that the variables in $\{\, x_j : j \in [m] \,\} \cup \{z\}$ do not enjoy majority support.

Let $I \subseteq [2n + 1]$ be a set of indices (of size smaller than $1/3(2n + 1)$) such that if we delete the individual ballots r_i for all $i \in I$, then the variables in $\{\, x_j : j \in [m] \,\} \cup \{z\}$ do not enjoy majority support. By the way the Young judgment aggregation procedure is defined, it suffices to look at sets I of even size. Without loss of generality, we can assume that $2n + 1 \notin I$—if this were not the case, one could replace $2n + 1$ by any other index. Moreover, without loss of generality, we can assume that for each $i \in I \cap [n + 1, 2n]$ it holds that r_i accepts all variables in $\{x_j : j \in [m]\,\}$—if this were not the case, we could replace such an i by another $i' \in [n+1, 2n]$ for which this is the case; as mentioned above, since we can arbitrarily increase n by adding variables, we may assume without loss of generality that enough such indices i' exist. Now, let $I_1 = I \cap [n]$ and let $I_2 = I \cap [n+1, 2n]$. If $|I_2| > |I_1|$, we know that in the resulting profile (after deleting the individual ballots according to I), the variable z has majority support. This contradicts our assumption, and thus we can conclude that $|I_1| \geq |I_2|$. Then, if $|I_1| > |I_2|$, we could replace some indices in I_1 by other indices in $[n + 1, 2n] \setminus I_2$, and we would end up with another suitable set I of indices. Therefore, we can restrict our attention to the case where $|I_1| = |I_2|$.

Each such set I corresponds to a vertex cover of G in the following way. Let $C_I \subseteq V$ be defined as $C_I = \{\, v_i : i \in I \cap [n] \,\}$. Suppose, to derive a contradiction, that C_I is not a vertex cover, i.e., that there is some $e_j \in E$ such that $C_I \cap e_j = \emptyset$. Then in the profile resulting from deleting the individual ballots with indices in I, the variable x_j enjoys majority support. This is a contradiction with our assumption that deleting the ballots corresponding to I results in a profile where all variables in $\{\, x_j : j \in [m] \,\} \cup \{z\}$ do not enjoy majority support. Thus, we can conclude that C_I is a vertex cover of G.

We will now show that there is a minimum-size vertex cover $C \subseteq V$ of G that includes v^* if and only if there is some $r^* \in \text{YOUNG}(\boldsymbol{r})$ that agrees with s.

($\Rightarrow$) Take a minimum-size vertex cover $C \subseteq V$ of G that includes v^*. We show how to construct a minimum size set of individual ballots to delete to result in a majority outcome r^* that satisfies Γ. Moreover, we show that deleting this set of ballots results in an outcome r^* that agrees with s. Define the set I of indices of ballots to delete as follows. Let $I = \{\, i \in [n] : v_i \in C \,\} \cup \{v_{n+1}, \dots, v_{n+|C|}\}$. It is straightforward to verify, since C is a vertex cover of G, that deleting individual ballots according to I results in a consistent majority outcome that does not include y (and thus that agrees with s). We show that I is of minimum size (among all such I that lead to a consistent majority outcome). Suppose, to derive a contradiction, that this is not the case, i.e., that there is some suitable I' that is smaller than I. Then, as described above, we can construct a vertex cover $C_{I'}$ of G that is smaller than C, which is a contradiction. Therefure, I is of minimum size.

($\Leftarrow$) Conversely, suppose that there is some $r^* \in \text{YOUNG}(\boldsymbol{r})$ that agrees with s, i.e., such that $r^*(y) = 0$. Then r^* results as the majority outcome of the profile after deleting individual ballots according to some (minimum size) set $I \subseteq [2n + 1]$. As described above, we can construct a vertex cover C_I of G. Since $r^*(y) = 0$, it is straight-

$\boldsymbol{r}$	r_1	r_2	r_3	$\cdots$	r_n	r_{n+1}	r_{n+2}	$\cdots$	r_{2n}	r_{2n+1}	$m_{\boldsymbol{r}}$	cost
x_1	$a_{1,1}$	$a_{2,1}$	$a_{3,1}$	$\cdots$	$a_{n,1}$	$t_{1,1}$	$t_{2,1}$	$\cdots$	$t_{n,1}$	0	1	**2**
x_2	$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	$\cdots$	$a_{n,2}$	$t_{1,2}$	$t_{2,2}$	$\cdots$	$t_{n,2}$	0	1	**2**
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	$a_{1,m}$	$a_{2,m}$	$a_{3,m}$	$\cdots$	$a_{n,m}$	$t_{1,m}$	$t_{2,m}$	$\cdots$	$t_{n,m}$	0	1	**2**
y	1	0	0	$\cdots$	0	1	1	$\cdots$	1	0	1	**1**
z	1	1	1	$\cdots$	1	0	0	$\cdots$	0	0	0	**2**
w_1	1	1	0	1	1	0	$\cdots$	1	1	0	1	$\boldsymbol{2m}$
w_2	1	0	1	1	0	1	$\cdots$	1	0	1	1	$\boldsymbol{2m}$
w_3	0	1	1	0	1	1	$\cdots$	0	1	1	1	$\boldsymbol{2m}$
w'_1	1	1	0	1	1	0	$\cdots$	1	1	0	1	$\boldsymbol{2m}$
w'_2	1	0	1	1	0	1	$\cdots$	1	0	1	1	$\boldsymbol{2m}$
w'_3	0	1	1	0	1	1	$\cdots$	0	1	1	1	$\boldsymbol{2m}$

Figure 4: Construction of the profile $\boldsymbol{r}$ in the proof of Proposition 18.

forward to verify that $v^* \in C$. We show that C is a minimum size vertex cover. Suppose, to derive a contradiction, that there exists a smaller vertex cover C' of G. Then define the set I' of indices of ballots to delete as follows. Let $I' = \{ i \in [n] : v_i \in C' \} \cup \{v_{n+1}, \dots, v_{n+|C'|}\}$. It is straightforward to verify, since C' is a vertex cover of G, that deleting individual ballots according to I' results in a consistent majority outcome. Moreover, since C' is smaller than C, we get that I' is smaller than I. This is a contradiction with our assumption that I is of minimum size. Thus, we can conclude that C is a minimum size vertex cover of G. $\square$