Loyalty and Faithfulness of Model Constructions for Constructive Set Theory

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written by

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Abstract

Model constructions for intuitionistic and constructive set theories, such as CZF or IZF, are commonly built upon Kripke- or Heyting-semantics for intuitionistic logic. We discuss how much of the logical structure of the underlying Kripke frame or Heyting algebra can be recovered from within the set-theoretical model construction.

We will use Heyting structures (introduced by Fourman and Scott in the 1970s) to develop a framework that allows us to compare the propositional logics of classes of models with the propositional logics of the classes of their underlying Heyting algebras (resp., Kripke frames): A class of models will be called *faithful* if any valuation on an underlying Heyting algebra can be imitated by a collection of sentences in a model of that class. We will call it *loyal* if it has the same logic as the underlying class of Heyting algebras and connect these two notions to the de Jongh property.

The main part of the thesis deals with an analysis of different model constructions by Iemhoff, Lubarsky, and the well-known Heyting-valued and Boolean-valued models for set theory. It turns out that the class of Iemhoff models is loyal and faithful to a very high degree. The class of full Lubarsky models is not faithful to its underlying Kripke frames, and the class of Lubarsky models based on a finite Kripke frame is not loyal as its propositional logic contains the principle of weak excluded middle despite the fact that this principle is not valid in the class of finite Kripke frames. A representation theorem allows us to transfer this result to the class of Heyting-valued models based on a finite Heyting algebra. We will conclude with an analysis of the propositional logics of Boolean-valued and Heyting-valued models for set theory.

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Chapter 1

Introduction

Models of set theory are a crucial tool for establishing independence results. Their impressive success in exhibiting different set-theoretic possibilities has recently led to the development of the *multiverse view*, a philosophical position of higher-order platonism stating the existence of many different set-theoretic worlds (cf., [14]). In particular, this led to a growing branch of multiverse-inspired mathematics that analyses the connections of different models of set theory, and the multiverses they generate. An example of research in this area is the work about the modal logic of forcing (cf., [15]), or set-theoretic geology (cf., [13]), which is a study of the structure of ground models.

The majority of the work that has been done so far in this area of research focuses on *classical* set theory, i.e., Zermelo-Fraenkel set theory, usually with the axiom of choice. The analysis of the modal logic of symmetric extensions (cf., [6, Section 7.4]) is an example of work that takes place in Zermelo-Fraenkel set theory without choice. A great variety of model constructions for constructive and intuitionistic set theories have so far not been analysed in this framework:

Heyting-valued models—a generalisation of the forcing technique that was introduced by Paul Cohen in the 1960s to prove the independence of the continuum hypothesis from Zermelo-Fraenkel set theory—provide a powerful tool for establishing independence results in intuitionistic set theory (cf., [2]). Other model structures for providing independence results in constructive or intuitionistic set theory have been introduced by Robert Lubarsky and Rosalie Iemhoff (cf., [27] and [21]). The latter model structures are built upon Kripke frames for intuitionistic logic, whereas the former ones are based on Heyting and Boolean algebras.

This thesis aims at providing a starting point for a multiverse-inspired analysis of models of set theory that incorporates also these non-classical approaches. After introducing the very general notion of Heyting structure (due to [10]), which covers all of the before-mentioned model constructions, we will focus on the following question:

> How much does the propositional logic of our different model constructions for constructive, intuitionistic or classical set theory re-

flect the logic of their underlying Heyting algebras or Kripke frames?

A first step towards an analysis of this question is to introduce the notions of *loyalty* and—even stronger—*faithfulness* of classes of models to their underlying Kripke frames or Heyting algebras: A class of models that has the same propositional logic as the underlying class of Heyting algebras or Kripke frames will be called loyal to that class of Kripke frames (see Definition 2.39). Even stronger, if it is possible for a class of models to imitate a valuation into a Kripke frame or Heyting algebra for propositional logic by a collection of set-theoretical sentences, we will call this class faithful (see Definition 2.40). We will also see applications of these notions to the *de Jongh property*.

We start off our investigation with the models introduced by Iemhoff. These models satisfy the set theory CZF^*, a weak version of constructive set theory CZF with only weak versions of the collection axioms (see section 3.1 for a discussion of this). The class of Iemhoff models satisfies both of our criteria.

Main Theorem 1 (Theorem 3.19, Corollary 3.20)**.** *The class of Iemhoff models is faithful, and therefore loyal.*

Together with our general considerations about the connections of loyal classes of models and the de Jongh property (cf., Proposition 2.56), this implies the following result for CZF^*.

Main Theorem 2 (Corollary 3.21)**.** *The theory* CZF^* *has the de Jongh property with respect to every logic characterised by a class of Kripke frames.*

Due to the following result, we cannot strengthen this result to obtain the de Jongh property for full CZF.

Main Theorem 3 (Corollary 3.9)**.** *Iemhoff models that involve forcing non-trivially do not satisfy the axiom of exponentiation.*

In contrast to this, faithfulness fails for Lubarsky models in a very strong way. Furthermore, we can derive a result about the loyalty of the Lubarsky models based on a finite Kripke frame.

Main Theorem 4 (Corollary 4.14)**.** *The class of full Lubarsky models is not faithful.*

Main Theorem 5 (Corollary 4.16)**.** *The propositional logic of the class of finite Lubarsky models contains* **KC**. *Therefore, the class of finite Lubarsky models cannot be loyal.*

The question of the general loyalty of the class of all Lubarsky models is open. It is open as well for Heyting-valued models. However, we can again determine it for a certain subclass. To do so, we first prove the following transfer theorem.

Main Theorem 6 (Theorem 5.8, Corollary 5.10)**.** *For every Heyting-valued model based on a Heyting algebra consisting of the upsets of a Kripke frame, there is a full Lubarsky model that proves exactly the same sentences. Indeed, the propositional logics of the two models agree.*

Then, the following result concerning loyalty follows from the analogous result for full Lubarsky models.

Main Theorem 7 (Corollary 5.15)**.** *The class of Heyting-valued models that are based on a finite Heyting algebra is not loyal. Indeed, the propositional logic of this class contains* **KC**.

Concerning faithfulness, we have the following result.

Main Theorem 8 (Corollary 5.13)**.** *The class of Heyting-valued models is not faithful. In particular, it is not faithful to any Heyting algebra with a non-trivial automorphism.*

The case of classical models of set theory is covered by the Boolean-valued models. This class of models is not faithful, but its loyalty follows trivially from the maximality of classical propositional logic.

Main Theorem 9 (Corollary 5.16, Theorem 5.17)**.** *The class of Boolean-valued models is loyal, but not faithful.*

Chapter 2

Semantics for Intuitionistic Logic

This thesis deals with models for intuitionistic and constructive set theories that are inspired by and built upon well-known semantics for intuitionistic logic, namely, Kripke frames and Heyting-valued semantics. In this chapter, we will first briefly introduce these well-known semantics for both intuitionistic propositional logic and intuitionistic predicate logic, before providing a general framework for analysing the propositional logic of (classes of) model structures.

2.1 Intuitionistic Logic

Intuitionistic logic was introduced by Arend Heyting (see, e.g., [19]) to formalise the principles of reasoning behind Brouwer's intuitionism. Compared proof-theoretically to classical logic, we remove the law of excluded middle, $p \vee \neg p$, and the (equivalent) law of double negation elimination, $\neg\neg p \to p$.

We denote intuitionistic propositional logic by **IPC**, intuitionistic predicate logic by **IQC**, classical propositional logic by **CPC** and classical predicate logic by **CQC**. For convenience, we will identify propositional logics with the set of formulas that they derive. An *intermediate logic* is a propositional logic **J** closed under modus ponens and uniform substitution such that $\mathbf{IPC} \subseteq \mathbf{J} \subseteq \mathbf{CPC}$. Jankov's logic **KC** is an example of such a logic and obtained by adding the principle of weak excluded middle, $\neg p \vee \neg\neg p$, to intuitionistic logic **IPC**. For more details on logics, we refer the reader to a short exposition in Appendix A.

A theory T is a set of sentences formulated in a first-order language $\mathcal{L}$ (i.e., a collection of logical and non-logical symbols, cf. Appendix A). We usually assume that a theory T is closed under a logic such as **IQC** or **CQC**. In this situation, we might say that T is based on that logic. Given a propositional logic **J**, we will refer to the closure of a theory T under the inference rules and axioms of **J** as $\mathsf{T}(\mathbf{J})$.

The set theories that we are concerned with in this thesis are all based on predicate logics with equality and formulated in the language $\mathcal{L}_\in$ of set theory that has only one binary relation symbol $\in$ for set-membership additionally to the logical vocabulary. We will consider classical Zermelo-Fraenkel set theory with choice ZFC (ZF without choice), intuitionistic set theory IZF and constructive set theory CZF. See Appendix B for the axiomatisations that we will use.

The *meta-theory* of this thesis is ZFC, additionally assuming the existence of a transitive set model of ZFC. This is mostly a matter of convenience as most of the results of this thesis could be rephrased and proved for a context assuming ZFC alone.

De Jongh's theorem and the de Jongh property

Given a theory based on intuitionistic logic, we may consider its propositional logic, that is, the set of propositional formulas that are derivable after substituting the propositional letters by arbitrary sentences in the language of the theory.

Definition 2.1. Let $\mathcal{L}$ be a language, φ be a propositional formula and let $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ an assignment of propositional variables to $\mathcal{L}$-sentences. By φ^σ we denote the $\mathcal{L}$-sentence obtained from φ by replacing each propositional variable p with the sentence $\sigma(p)$.

Definition 2.2. Let T be a theory in intuitionistic predicate logic, formulated in a language $\mathcal{L}$. A propositional formula φ will be called T-*valid* if and only if $\mathsf{T} \vdash \varphi^\sigma$ for all $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$. The *propositional logic* $\mathbf{L}^{\mathsf{Prop}}(\mathsf{T})$ is the set of all T-valid formulas.

We will later see that the propositional logic of certain model constructions for set theory connects to a proof-theoretic notion: the *de Jongh property*, which is a generalisation of the following classical result that concerns Heyting arithmetic HA.

Theorem 2.3 (de Jongh, [9])**.** *Let φ be a formula of propositional logic. Then $\mathsf{HA} \vdash \varphi^\sigma$ for all $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}_{\mathsf{HA}}$ if and only if $\mathbf{IPC} \vdash \varphi$.*

Definition 2.4. The *de Jongh property for a theory* T is the statement that $\mathbf{L}^{\mathsf{Prop}}(\mathsf{T}) = \mathbf{IPC}$. The *de Jongh property for a theory* T *with respect to an intermediate logic* $\mathbf{J}$ is the statement $\mathbf{L}^{\mathsf{Prop}}(\mathsf{T}(\mathbf{J})) = \mathbf{J}$.

De Jongh's theorem is equivalent to the assertion that Heyting arithmetic has the de Jongh property.

2.2 Kripke Semantics

The goal of this section is to briefly introduce Kripke frames for propositional and predicate logic. We will mainly follow the lecture notes [5] of Bezhanishvili and de Jongh.

Definition 2.5. A *Kripke frame for* **IPC** $(K, \leq)$ consists of a set K equipped with a preorder $\leq$. A *Kripke model for* **IPC** $(K, \leq, V)$ is a Kripke frame with a valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ that is persistent, i.e., if $w \in V(p)$ and $w \leq v$, then $v \in V(p)$.

We will usually identify a Kripke frame $(K, \leq)$ with its underlying set K. As we did in the previous sentence already, we will refer to a *Kripke frame for* **IPC** simply as a *Kripke frame* if it is clear from the context which kind of Kripke frame it is that we are talking about (and similar for Kripke models). We will adopt this policy for the other kinds of Kripke models that we will introduce in this thesis as well. The interpretation of propositional formulas of the language $\mathcal{L}_{\mathsf{Prop}}$ is defined as follows.

Definition 2.6. Let $(K, \leq, V)$ be a Kripke model for **IPC**. We define, by induction on propositional formulas, the forcing relation at every node v of the Kripke frame in the following way:

$$\begin{aligned}
K, V, v \Vdash p &\text{ if and only if } v \in V(p),\\
K, V, v \Vdash \varphi \wedge \psi &\text{ if and only if } K, V, v \Vdash \varphi \text{ and } K, V, v \Vdash \psi,\\
K, V, v \Vdash \varphi \vee \psi &\text{ if and only if } K, V, v \Vdash \varphi \text{ or } K, V, v \Vdash \psi,\\
K, V, v \Vdash \varphi \to \psi &\text{ if and only if for all } w \geq v,\\
&\qquad\qquad K, V, w \Vdash \varphi \text{ implies } K, V, w \Vdash \psi, \text{ and,}\\
K, V, v \Vdash \bot &\text{ never holds.}
\end{aligned}$$

We will write $v \Vdash \varphi$ instead of $K, V, v \Vdash \varphi$, if the Kripke frame or the valuation are clear from the context. Sometimes we will write $K, V \Vdash \varphi$ if $K, V, v \Vdash \varphi$ holds for all $v \in K$. A formula φ is *valid in* K if $K, V, v \Vdash \varphi$ holds for all valuations V on K and $v \in K$, and φ is *valid* if it is valid in every Kripke frame K.

This semantics allows us to define the logic of a Kripke frame and of a class of Kripke frames.

Definition 2.7. If $(K, \leq)$ is a Kripke frame for **IPC**, we define the *propositional logic* $\mathbf{L}^{\mathsf{Prop}}(K)$ to be the set of all propositional formulas that are valid in K. For a class $\mathcal{K}$ of Kripke frames, we define the *propositional logic* $\mathbf{L}^{\mathsf{Prop}}(\mathcal{K})$ to be the set of all propositional formulas that are valid in all Kripke frames $(K, \leq)$ in $\mathcal{K}$.

The persistency of the propositional variables transfers to all formulas, which is proved by induction on the complexity of the formulas.

Proposition 2.8. *Let* $(K, \leq, V)$ *be a Kripke model for* **IPC**, $v \in K$ *and* φ *be a propositional formula such that* $K, v \Vdash \varphi$ *holds. Then* $K, w \Vdash \varphi$ *holds for all* $w \geq v$. □

By extending the Kripke models introduced above, we can obtain models for intuitionistic predicate logic (see also [20]). We will work with a slightly generalised notion that uses *transition functions* instead of inclusions as this will be useful later on.

Definition 2.9. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-Kripke model* $(K, \leq, D, I, f)$ *for* **IQC** is a Kripke frame $(K, \leq)$ for **IPC** with a collection of domains $(D_v)_{v \in K}$, a collection of interpretation function $(I_v)_{v \in K}$, and a collection of transition functions $(f_{vw})_{v \leq w \in K}$, such that the following hold:

(i) $I_v(R)$ is an n-ary relation on D_v for all n-ary relation symbols R of $\mathcal{L}$, $I_v(F)$ is an n-ary function on D_w for any n-ary function symbol F of $\mathcal{L}$, and we have that $I_v(a) = a$ for all $a \in D_v$,[1]

(ii) the interpretation function is extended to all terms t in the usual way, i.e., by inductively defining $I_v(F(t_1, \dots, t_n)) = I_v(F)(I_v(t_1), \dots, I_v(t_n))$,

(iii) equality is interpreted as a congruence relation on each domain,

(iv) the transition functions $f_{vw} : D_v \to D_w$ are injective $\mathcal{L}$-homomorphisms, and,

(v) the system of transition function coheres, i.e, $f_{vv} = \mathrm{id}_v$ for all $v \in K$ and $f_{vw} \circ f_{uv} = f_{uw}$ for all $u \leq v \leq w \in K$.

Indeed, the following persistency requirements are satisfied by all terms $\bar{t}$ and predicates P (as the f_{vw} are embeddings):

(i) if $v \leq w$, then $I_v(P)(I_v(\bar{t}))$ implies $I_w(P)(I_w(\bar{t}))$, and,

(ii) if $v \leq w$, then $I_w(\bar{t}) = f_{vw}(I_v(\bar{t}))$.

We can now extend the forcing relation to Kripke models for **IQC**.

Definition 2.10. Let $(K, \leq, D, I, f)$ be an $\mathcal{L}$-Kripke model for **IQC**. We define, by induction on $\mathcal{L}$-formulas, the forcing relation at every node of a Kripke frame in the following way, where φ and ψ are formulas with all free variables shown, and $\bar{t} = t_0, \dots, t_{n-1}$ are assumed to be terms at the node v considered on the left side.

$$
\begin{aligned}
K, D, I, f, v \Vdash R(t_0, \dots, t_n) &\text{ if and only if } (I_v(t_0), \dots, I_v(t_n)) \in I_v(R), \\
K, D, I, f, v \Vdash \exists x \varphi(x, \bar{t}) &\text{ if and only if there is some } a \in D_v \\
&\qquad \text{with } K, D, I, f, v \Vdash \varphi(a, \bar{t}), \\
K, D, I, f, v \Vdash \forall x \varphi(x, \bar{t}) &\text{ if and only if for all } w \geq v \text{ and } a \in D_w \\
&\qquad \text{we have } K, D, I, f, w \Vdash \varphi(a, f_{vw}(I_v(\bar{t}))), \\
K, D, I, f, v \Vdash \varphi(\bar{t}) \to \psi(\bar{t}) &\text{ if and only if for all } w \geq v, \\
&\qquad K, D, I, f, w \Vdash \varphi(f_{vw}(I_v(\bar{t}))) \\
&\qquad \text{implies } K, D, I, f, w \Vdash \psi(f_{vw}(I_v(\bar{t}))).
\end{aligned}
$$

The cases for $\wedge$, $\vee$ and $\perp$ are analogous to the ones in the above definition of the forcing relation for Kripke models for **IPC**. We will write $v \Vdash \varphi$ (or

[1] We tacitly extend the language by a constant symbol a for every $v \in K$ and $a \in D_v$, not distinguishing between the constant symbol and the actual element of the domain in any way.

$K, v \Vdash \varphi$) instead of $K, D, I, f, v \Vdash \varphi$ if the Kripke model is clear from the context. An $\mathcal{L}$-formula φ is *valid in* K if $v \Vdash \varphi$ holds for all $v \in K$, and φ is *valid* if it is valid in every Kripke frame K.

Intuitionistic logic is sound and complete with respect to these semantics. We note that the result for **IPC** even holds with respect to finite frames.

Theorem 2.11. *A propositional formula φ is derivable in* **IPC** *if and only if it is valid in all Kripke models for* **IPC**. *Similarly, a formula φ of predicate logic is derivable in* **IQC** *if and only if it is valid in all Kripke models for* **IQC**.

A detailed proof of this theorem can be found in [33, Theorem 6.6].

2.3 Heyting-Valued Semantics

We will first introduce Heyting algebras as a tool for semantical analysis of propositional intuitionistic logic. Then, we continue by introducing a very general notion of Heyting-valued model that will be of central importance in this thesis.

Heyting Algebras

For the basic notions in this section and the connections between Heyting algebras and Kripke frames, we will follow [4, chapter 2].

Definition 2.12. We call a partially ordered set $(A, \leq)$ a *lattice* if every two elements $a, b \in A$ have a supremum, denoted by $a \vee b$, and an infimum, $a \wedge b$. We say that $(A, \leq)$ is a *bounded lattice* if it has a greatest element 0_A and a least element 1_A. A lattice $(A, \leq)$ is *complete* if the supremum $\bigvee X$ and the infimum $\bigwedge X$ exist for every $X \subseteq A$. A lattice is called *distributive*, if it is bounded and satisfies the distributivity laws $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$.

Definition 2.13. A bounded distributive lattice $(A, \leq)$ is called a *Heyting algebra* if for every $a, b \in A$, there exists an element $a \to b \in A$ such that

$$a \wedge c \leq b \text{ if and only if } c \leq a \to b$$

holds for all $c \in A$. We then define the pseudo-complement $\neg a = a \to 0$ for every $a \in A$. A Heyting algebra is *complete* if it is so as a lattice.

Heyting algebras can also be defined purely equationally.

Theorem 2.14. *A structure $(A, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, where A is non-empty, $\wedge$,$\vee$ and $\to$ are binary operations on A, and $0, 1 \in A$, if and only if for every $a, b, c \in A$ the following hold:*

(i) $a \vee a = a$, $a \wedge a = a$,
(ii) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

(iv) $a \vee 0 = a$, $a \wedge 1 = a$,

(v) $a \vee (b \wedge a) = a$, $a \wedge (b \vee a) = a$,

(vi) $a \to a = 1$,

(vii) $a \wedge (a \to b) = a \wedge b$,

(viii) $b \wedge (a \to b) = b$,

(ix) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Boolean algebras are a special case of Heyting algebras.

Definition 2.15. A Heyting algebra A satisfying $a \to b = \neg a \vee b$ for every $a \in A$ is called a *Boolean algebra*. An element a is an *atom* of the Boolean algebra A if it is minimal among the non-zero elements of A. We call a Boolean algebra *atomic* if for every $b \in A$, there is a set $C \subseteq A$ of atoms of A such that $b = \bigvee C$.

Proposition 2.16. *The following are equivalent for every Heyting algebra* $(A, \wedge, \vee, \to, 0, 1)$*:*

(i) *A is a Boolean algebra,*

(ii) *$a \vee \neg a = 1$ holds for every $a \in A$,*

(iii) *$\neg\neg a = a$ holds for every $a \in A$.*

Let us close this section by stating the definition of a homomorphism of Heyting algebras.

Definition 2.17. Let $(A, \wedge_A, \vee_A, \to_A, 0_A, 1_A)$ and $(B, \wedge_B, \vee_B, \to_B, 0_B, 1_B)$ be Heyting algebras. A map $h : A \to B$ will be called a *homomorphism of Heyting algebras* if the following conditions are satisfied for all $a_0, a_1 \in A$:

(i) $h(a_0 \wedge_A a_1) = h(a_0) \wedge_B h(a_1)$,

(ii) $h(a_0 \vee_A a_1) = h(a_0) \vee_B h(a_1)$,

(iii) $h(a_0 \to_A a_1) = h(a_0) \to_B h(a_1)$,

(iv) $h(0_A) = 0_B$, and $h(1_A) = 1_B$.

A bijective homomorphism of Heyting algebras will be called *isomorphism*, and an isomorphism of a Heyting algebra onto itself is an *automorphism*.

Heyting Algebras for Propositional Logic

Definition 2.18. Let $(A, \wedge, \vee, \to, 0, 1)$ be a Heyting algebra. We call a map $v : \mathsf{Prop} \to A$ a *valuation* into A, and recursively extend it to an interpretation of all propositional formulas by:

(i) $[\![p]\!]^A_v = v(p)$,

(ii) $[\![\varphi \wedge \psi]\!]^A_v = [\![\varphi]\!]^A_v \wedge [\![\psi]\!]^A_v$,

(iii) $[\![\varphi \vee \psi]\!]_v^A = [\![\varphi]\!]_v^A \vee [\![\psi]\!]_v^A$,

(iv) $[\![\varphi \to \psi]\!]_v^A = [\![\varphi]\!]_v^A \to [\![\psi]\!]_v^A$, and,

(v) $[\![\bot]\!]_v^A = 0$.

We then call a propositional formula φ *true in* A *under* v if $[\![\varphi]\!]_v^A = 1$. We call φ *valid in* A if it is true under every valuation into A.

We define the logic $\mathbf{L}^{\mathsf{Prop}}(A)$ of a Heyting algebra A to be the set of formulas φ that are valid in A. If $\mathcal{H}$ is a class of Heyting algebras, then the logic $\mathbf{L}^{\mathsf{Prop}}(\mathcal{H})$ of $\mathcal{H}$ is the set of all formulas valid in every $A \in \mathcal{H}$. The following completeness results hold.

Theorem 2.19. *A formula φ of propositional logic is valid in every Heyting algebra if and only if* $\mathbf{IPC} \vdash \varphi$.

Theorem 2.20. *A formula φ of propositional logic is valid in every Boolean algebra if and only if* $\mathbf{CPC} \vdash \varphi$.

Let us now briefly introduce a correspondence between Kripke frames for intuitionistic logic and Heyting algebras. Given a Kripke frame $(K, \leq)$, consider the set $\mathrm{Up}(K)$ of all upsets of K: An upset of K is a subset $U \subseteq K$ such that for all $x, y \in K$ it holds that if $x \in U$ and $x \leq y$, then $y \in U$.

Theorem 2.21 ([4, Section 2.2.3])**.** *Let K be a Kripke frame for* $\mathbf{IPC}$*. Then the structure* $(\mathrm{Up}(K), \cap, \cup, \to, \emptyset, K)$*, where*

$$U_0 \to U_1 = \{v \in K \mid \forall w \geq v \ (w \in U_0 \to w \in U_1)\},$$

is a complete Heyting algebra. In particular, $\mathbf{L}^{\mathsf{Prop}}(\mathrm{Up}(K)) = \mathbf{L}^{\mathsf{Prop}}(K)$.

This correspondence is a bijection for finite Kripke frames and finite Heyting algebras.

Theorem 2.22 ([4, Theorem 2.2.21])**.** *For every finite Heyting algebra A, there exists a finite Kripke frame $(K, \leq)$ with $\leq$ a partial order such that A is isomorphic to* $\mathrm{Up}(K)$.

To see that this correspondence does not generalise to the infinite case, we need to introduce some definitions.

Definition 2.23. Let A be a Heyting algebra and $a \in A$. We say that a is *completely join prime* if $a \leq \bigvee S$ for some $S \subseteq A$ implies that $a \leq s$ for some $s \in S$. We will say that A is *completely join prime generated* if for each $b \in A \setminus \{0_A\}$, there is a completely join prime $a \in A$ with $a \leq b$.

We can now state the following characterisation of Heyting algebras of the form $\mathrm{Up}(K)$.

Theorem 2.24 ([3, Theorem 4.4])**.** *A Heyting algebra A is isomorphic to the Heyting algebra* $\mathrm{Up}(K)$ *for some Kripke frame K if and only if A is complete and completely join prime generated.*

Proposition 2.25. *There is a Heyting algebra A that is not isomorphic to the Heyting algebra* $\mathrm{Up}(K)$ *for any Kripke frame K.*

Proof. Consider the Heyting algebra $A = \{x \subseteq \omega \mid x = 0 \vee |\omega \setminus x| < \omega\}$, ordered by inclusion. Let a be any non-zero element of A, and take $a = \{a_i \mid i < \omega\}$ to be a bijective enumeration of a. Define $b = a \setminus \{a_0\}$ and $c = a \setminus \{a_1\}$. Clearly, $b, c \in A$, and $a = b \vee c$, but it does not hold that $a \leq b$ or $a \leq c$. This shows that A does not have any (completely) join prime elements and therefore, by Theorem 2.24, A cannot be isomorphic to a Heyting algebra of the form $\mathrm{Up}(K)$. □

Heyting Structures

In this section we will present the notion of H-structure, for a given Heyting algebra H, as defined by Fourman and Scott in [10] (see also [31], and the discussion in the appendix of [2]). We will restrict our attention to the one-sorted case.

Definition 2.26. Let H be a complete Heyting algebra. An *H-set* A is a set equipped with an H-valued equality $e : A \times A \to H$ that is symmetric and transitive, i.e., for all $a, b, c \in A$ it holds that $e(a, b) = e(b, a)$ and $e(a, b) \wedge e(b, c) \leq e(a, c)$.

We further define the *extent* $E : A \to H$ by setting $E(a) = e(a, a)$ and let the *equivalence* $\tilde{e} : A \times A \to H$ be the map with $\tilde{e}(a, b) = E(a) \vee E(b) \to e(a, b)$.

Given the notion of H-set, we can now define what it means to be a Heyting-valued operation or a relation.

Definition 2.27. An n-ary H-function F on an H-set A is defined to be a map $F : A^n \to A$ which respects equivalence in the sense that:

$$\bigwedge_{i<n} \tilde{e}(a_i, b_i) \leq \tilde{e}(F(a_0, \dots, a_{n-1}), F(b_0, \dots, b_{n-1})),$$

for all $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in A$. Similarly, an n-ary H-relation R on A is defined to be a map $R : A^n \to H$ which respects equivalence in the sense that:

$$\bigwedge_{i<n} \tilde{e}(a_i, b_i) \wedge R(a_0, \dots, a_{n-1}) \leq R(b_0, \dots, b_{n-1}),$$

for all $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in A$.

If the Heyting algebra H is clear from the context, we will usually drop the H in front of function and relation. Given an H-set A and a language $\mathcal{L}$, we let $\mathcal{L}(A)$ denote the language that is obtained from $\mathcal{L}$ by adding a constant symbol for each element of A. As usual, we will abuse notation in the sense that we will not distinguish between an element of A and its constant symbol.

Definition 2.28. An *H-structure* (A, F_i, R_i) *for a language* $\mathcal{L}$ consists of an H-set A together with H-relations and H-functions for every relation symbol and function symbol in $\mathcal{L}$, respectively.

The interpretation $[\![\cdot]\!]^A$ of terms and sentences in the language $\mathcal{L}(A)$ is defined as follows:

$$[\![a]\!]^A = a \qquad \text{(for every } a \in A\text{)}$$

$$[\![F(t_0, \dots, t_{n-1})]\!]^A = F([\![t_0]\!]^A, \dots, [\![t_{n-1}]\!]^A) \qquad \text{(for terms } t_0, \dots, t_{n-1} \text{ and } n\text{-ary functions } F\text{)}$$

$$[\![t_0 = t_1]\!]^A = e([\![t_0]\!]^A, [\![t_1]\!]^A) \qquad \text{(for terms } t_0 \text{ and } t_1\text{)}$$

$$[\![R(t_0, \dots, t_{n-1})]\!]^A = R([\![t_0]\!]^A, \dots, [\![t_{n-1}]\!]^A) \qquad \text{(for terms } t_0, \dots, t_{n-1} \text{ and } n\text{-ary relations } R\text{)}$$

$$\begin{aligned}
[\![\bot]\!]^A &= \bot \\
[\![\varphi \wedge \psi]\!]^A &= [\![\varphi]\!]^A \wedge [\![\psi]\!]^A \\
[\![\varphi \vee \psi]\!]^A &= [\![\varphi]\!]^A \vee [\![\psi]\!]^A \\
[\![\varphi \to \psi]\!]^A &= [\![\varphi]\!]^A \to [\![\psi]\!]^A \\
[\![\forall x\ \varphi(x)]\!]^A &= \bigwedge_{a \in A} E(a) \to [\![\varphi(a)]\!]^A \\
[\![\exists x\ \varphi(x)]\!]^A &= \bigvee_{a \in A} E(a) \wedge [\![\varphi(a)]\!]^A
\end{aligned}$$

We will say that a formula $\varphi(x_0, \dots, x_{n-1})$ is valid in A if and only if for all $a_0, \dots, a_{n-1} \in A$ we have that $[\![\varphi(a_0, \dots, a_{n-1})]\!]^A = 1$. If the structure is clear from the context, we will sometimes drop the superscript A and write $[\![\varphi]\!]$ instead of $[\![\varphi]\!]^A$. We will write $A \vDash \varphi$ whenever $[\![\varphi]\!]^A = 1$.

We will more broadly refer to H-structures as *Heyting structures* when the specific Heyting algebra does not matter. Sometimes we say that H is the underlying algebra of A if A is an H-structure.

Moreover, one could add relation symbols E and $\equiv$ to the language $\mathcal{L}$ to interpret them as the extent and equivalence, respectively. It follows then directly from these definitions that $[\![Et]\!] = [\![t = t]\!]$ and $[\![t_0 \equiv t_1]\!] = [\![(t_0 = t_0 \wedge t_1 = t_1) \to t_0 = t_1]\!]$.

An important observation is the following soundness result for Heyting structures.

Theorem 2.29 (Fourman and Scott, [10, Theorem 5.14])**.** *Intuitionistic logic is valid in any Heyting structure.*

2.4 Kripke Models as Heyting Structures

We will show in this section that Kripke models for **IQC** with equality may be interpreted as H-structures for a certain Heyting algebra H.

For the sake of this section, fix a language $\mathcal{L}$, and an $\mathcal{L}$-Kripke model $(K, \leq, D, I, f)$ for **IQC** with a collection of domains $(D_v)_{v\in K}$, a collection of interpretation function $(I_v)_{v\in K}$, and a collection of transition functions $(f_{vw})_{v\leq w\in K}$. We assume that our language $\mathcal{L}$ contains an equality symbol $=$ that is interpreted as a congruence relation on every domain (of course, obeying persistency as any relation in a Kripke model). The set-theoretic models based on Kripke frames that we will consider later on will all interpret $=$ as actual equality on the domains.

We will transform this Kripke model into an H-structure, where H is the Heyting algebra $\mathrm{Up}(K)$ consisting of the upsets of K. Note that Heyting algebras of this form are always complete. Let us first construct the underlying H-set. Given $x \in D_v$, for $v \in K$, let $d^v_x : K \to \bigcup_{v\in K} D_v$ be the minimal partial function that satisfies the following properties:

(i) $d^v_x(v) = x$,

(ii) if $w \in K$ such that $w \geq v$, then $d^v_x(w) = f_{vw}(x)$.

Furthermore, it holds that $d^w_{d^v_x(w)} \subseteq d^v_x$ for all $w \in \mathrm{dom}(d^v_x)$. Now let $A = \bigcup_{v\in K}\{d^v_x \mid x \in D_v\}$ and define the equality $e : A \times A \to H$ as follows:

$$e(a, b) = \{v \in K \mid a(v) \text{ and } b(v) \text{ are defined and } v \Vdash a(v) = b(v)\}.$$

We require the above functions only to be forward closed, i.e., such a function always starts in one node of the Kripke frames and continues in the direction of the relation, but need not be closed under predecessors.

Proposition 2.30. *The set A with the equality e constitutes an H-set.*

Proof. First of all, let us observe that e is well-defined: Persistency in the Kripke model and the definition of the elements of A show that $e(a, b)$ is an upset of the Kripke frame for any $a, b \in A$. Further, symmetry $e(a, b) = e(b, a)$ for all $a, b \in A$ follows from symmetry of equality in the Kripke model. The same holds for transitivity. $\square$

We can now determine the extent E and equivalence $\tilde{e}$ of the H-set A.

Proposition 2.31. *Given $a \in A$, it holds that $E(a) = \mathrm{dom}(a)$. Further, for any $a, b \in A$, it holds that*

$$\begin{aligned}\tilde{e}(a, b) = \{w \in K \mid \forall v \geq w\ ((v \notin \mathrm{dom}(a) \wedge v \notin \mathrm{dom}(b))\\ \vee (v \in \mathrm{dom}(a) \wedge v \in \mathrm{dom}(b) \wedge v \Vdash a(v) = b(v)))\}.\end{aligned}$$

Proof. For the first part of the claim, observe that:

$$\begin{aligned}E(a) &= e(a, a)\\ &= \{v \in K \mid a(v) \text{ and } a(v) \text{ are defined and } v \Vdash a(v) = a(v)\}\\ &= \{v \in K \mid a(v) \text{ is defined}\}\\ &= \mathrm{dom}(a).\end{aligned}$$

The claim about the equivalence $\tilde{e}$ follows similarly, where we are using the definitions of the connectives in Heyting algebras of upsets:

$$\begin{aligned}\tilde{e}(a,b) &= (E(a) \vee E(b)) \to e(a,b) \\ &= (\mathrm{dom}(a) \cup \mathrm{dom}(b)) \to \{v \in K \mid a(v), b(v) \text{ are defined} \\ &\qquad\qquad \text{and } v \Vdash a(v) = b(v)\} \\ &= \{w \in K \mid \forall v \geq w (v \in \mathrm{dom}(a) \cup \mathrm{dom}(b) \\ &\qquad\qquad \to (v \in \mathrm{dom}(a) \cup \mathrm{dom}(b) \text{ and } v \Vdash a(v) = b(v)))\} \\ &= \{w \in K \mid \forall v \geq w (v \notin \mathrm{dom}(a) \cup \mathrm{dom}(b) \\ &\qquad\qquad \vee (v \in \mathrm{dom}(a) \cap \mathrm{dom}(b) \text{ and } v \Vdash a(v) = b(v)))\}.\end{aligned}$$

This finishes the proof of the proposition. □

The next step is to complete the interpretation of all symbols of our language $\mathcal{L}$, i.e., function symbols and relation symbols. Every constant c is just interpreted as $x_c : K \to \bigcup_{v \in K} D_v$ with $x_c(v) = I_v(c)$. Given any n-ary function symbol F, define $\bar{F} : A^n \to A$ by $\bar{F}(a_0, \ldots, a_{n-1}) = b$ where b is defined as follows:

$$\begin{aligned}&v \in \mathrm{dom}(b) \text{ if and only if } v \in \mathrm{dom}(a_i) \text{ for all } i < n, \\ &\qquad\qquad \text{and } b(v) = I_v(F)(a_0(v), \ldots, a_{n-1}(v)).\end{aligned}$$

Proposition 2.32. *The map $\bar{F}$ is an n-ary function on the H-set A.*

Proof. It is clear that $\bar{F}$ is well-defined by the way we defined the elements of A. Further, we need to verify that

$$\bigwedge_{i<n} \tilde{e}(a_i, b_i) \leq \tilde{e}(\bar{F}(a_0, \ldots, a_{n-1}), \bar{F}(b_0, \ldots, b_{n-1})),$$

holds for all $a_i, b_i \in A$. To do so, we need to prove the inclusion from the left hand side to the right hand side. So assume that $w \in \bigwedge_{i<n} \tilde{e}(a_i, b_i)$. Then we know, by Proposition 2.31, that for all $i < n$ and $v \geq w$ we have $v \notin \mathrm{dom}(a_i)$ and $v \notin \mathrm{dom}(b_i)$, or $v \in \mathrm{dom}(a_i)$, $v \in \mathrm{dom}(b_i)$ and $v \Vdash a(v) = b(v)$.

If there is $i < n$ such that for all $v \geq w$ it holds that $v \notin \mathrm{dom}(a_i)$ and $v \notin \mathrm{dom}(b_i)$, then by definition of $\bar{F}$ we will have that both $\bar{F}(a_0, \ldots, a_{n-1})(v)$ and $\bar{F}(b_0, \ldots, b_{n-1})(v)$ are undefined. Therefore, we can conclude that $w \in \tilde{e}(\bar{F}(a_0, \ldots, a_{n-1}), \bar{F}(b_0, \ldots, b_{n-1}))$.

Otherwise, it will be the case that for all $v \geq w$ we have that $v \in \mathrm{dom}(a_i)$, $v \in \mathrm{dom}(b_i)$ and $v \Vdash a_i(v) = b_i(v)$ for all $i < n$. Hence, it follows that $v \Vdash F(a_0(v), \ldots, a_{n-1}(v)) = F(b_0(v), \ldots, b_{n-1}(v))$, i.e, $v \Vdash \bar{F}(\bar{a})(v) = \bar{F}(\bar{b})(v)$. Using Proposition 2.31, this concludes our proof. □

Similarly, if R is a relation (resp. predicate) symbol of our language $\mathcal{L}$, we define the n-ary relation $\bar{R} : A^n \to H$ by

$$\begin{aligned}\bar{R}(a_0, \ldots, a_{n-1}) = \{v \in K \mid v \in \mathrm{dom}(a_i) &\text{ for all } i < n \\ &\text{and } v \Vdash R(a_0, \ldots, a_{n-1})\}.\end{aligned}$$

Proposition 2.33. *The map $\bar{R}$ is an n-ary relation on the H-set A.*

Proof. We need to show that $\bar{R}$ respects equivalence in the sense that

$$\bigwedge_{i<n} \tilde{e}(a_i, b_i) \wedge R(a_0, \ldots, a_{n-1}) \leq R(b_1, \ldots, b_{n-1}).$$

This follows very similarly to the previous proposition using the characterisation of $\tilde{e}$ from Proposition 2.31. □

We have now made all basic definitions and are ready to prove our main theorem that allows us to interpret any Kripke model for **IQC** as a Heyting-structure.

Theorem 2.34. *If $\varphi(x_0, \ldots, x_{n-1})$ is an $\mathcal{L}$-formula with all free variables shown, and $a_0, \ldots, a_{n-1} \in A$, then*

$$\begin{aligned}&[\![\varphi(a_0, \ldots, a_{n-1})]\!] \cap \operatorname{dom}(a_0, \ldots, a_{n-1}) \\ &\qquad = \{v \in K \mid v \in \operatorname{dom}(a_0, \ldots, a_{n-1}) \wedge v \Vdash \varphi(a_0(v), \ldots, a_{n-1}(v))\},\end{aligned}$$

where $\operatorname{dom}(a_0, \ldots, a_{n-1}) = \bigcap_{i<n} \operatorname{dom}(a_i)$.

Proof. We will prove this theorem by induction on the complexity of formulas. For the atomic cases, i.e., equality and relations, the statement of the theorem coincides with the definitions. The case of falsum is trivial. So let us consider the remaining cases one-by-one. We will usually abbreviate $a_0, \ldots, a_{n-1}$ by $\bar{a}$.

Let $\varphi(v_0, \ldots, v_{n-1})$ and $\psi(v_0, \ldots, v_{m-1})$ and arbitrary elements a_i, $i < n$ and b_i, $i < m$, be given. The case for conjunction follows in a straightforward way.

$$\begin{aligned}&[\![\varphi(\bar{a}) \wedge \psi(\bar{b})]\!] \cap \operatorname{dom}(\bar{a}, \bar{b}) \\ &\qquad = ([\![\varphi(\bar{a})]\!] \wedge [\![\psi(\bar{b})]\!]) \cap \operatorname{dom}(\bar{a}, \bar{b}) \\ &\qquad = \{v \in K \mid v \in \operatorname{dom}(\bar{a}, \bar{b}) \wedge v \Vdash \varphi(a) \wedge v \Vdash \psi(\bar{b})\} && \text{(by I.H.)} \\ &\qquad = \{v \in K \mid v \in \operatorname{dom}(\bar{a}, \bar{b}) \wedge v \Vdash \varphi(a) \wedge \psi(b)\}.\end{aligned}$$

Similarly, we can prove the claim for disjunction.

$$\begin{aligned}&[\![\varphi(\bar{a}) \vee \psi(\bar{b})]\!] \cap \operatorname{dom}(\bar{a}, \bar{b}) \\ &\qquad = ([\![\varphi(\bar{a})]\!] \cup [\![\psi(\bar{b})]\!]) \cap \operatorname{dom}(\bar{a}, \bar{b}) \\ &\qquad = \{v \in K \mid v \in \operatorname{dom}(\bar{a}, \bar{b}) \wedge ((v \in \operatorname{dom}(\bar{a}) \wedge v \Vdash \varphi(a)) \\ &\qquad\qquad \vee (v \in \operatorname{dom}(\bar{b}) \wedge v \Vdash \psi(\bar{b})))\} && \text{(by I.H.)} \\ &\qquad = \{v \in K \mid v \in \operatorname{dom}(\bar{a}, \bar{b}) \wedge v \Vdash \varphi(a) \vee \psi(b)\}.\end{aligned}$$

Let us now focus on the implication, where we use that, by our definitions, the domains of the elements of A are upwards closed.

$$\begin{aligned}
&[\![\varphi(\bar{a}) \to \psi(\bar{b})]\!] \cap \mathrm{dom}(\bar{a}, \bar{b}) \\
&\quad = ([\![\varphi(\bar{a})]\!] \to [\![\psi(\bar{b})]\!]) \cap \mathrm{dom}(\bar{a}, \bar{b}) \\
&\quad = \{v \in K \mid \forall w \geq v(w \in [\![\varphi(\bar{a})]\!] \to w \in [\![\psi(\bar{b})]\!])\} \cap \mathrm{dom}(\bar{a}, \bar{b}) \\
&\quad = \{v \in K \mid \forall w \geq v((w \in \mathrm{dom}(\bar{a}) \wedge w \Vdash \varphi(\bar{a})) \\
&\qquad\qquad \to (w \in \mathrm{dom}(\bar{b}) \wedge w \Vdash \psi(\bar{b})))\} \cap \mathrm{dom}(\bar{a}, \bar{b}) \qquad \text{(I.H.)} \\
&\quad = \{v \in K \mid \forall w \geq v(w \in \mathrm{dom}(\bar{a}, \bar{b}) \wedge ((w \in \mathrm{dom}(\bar{a}) \wedge w \Vdash \varphi(\bar{a})) \\
&\qquad\qquad \to (w \in \mathrm{dom}(\bar{b}) \wedge w \Vdash \psi(\bar{b}))))\} \\
&\quad = \{v \in K \mid \forall w \geq v(w \in \mathrm{dom}(\bar{a}, \bar{b}) \wedge (w \Vdash \varphi(\bar{a}) \to w \Vdash \psi(\bar{b})))\} \\
&\quad = \{v \in K \mid v \in \mathrm{dom}(\bar{a}, \bar{b}) \wedge \forall w \geq v(w \Vdash \varphi(\bar{a}) \to w \Vdash \psi(\bar{b}))\} \\
&\quad = \{v \in K \mid v \in \mathrm{dom}(\bar{a}, \bar{b}) \wedge v \Vdash \varphi(\bar{a}) \to \psi(\bar{b})\}.
\end{aligned}$$

For the quantifier cases, recall that by Proposition 2.31, it holds that $E(x) = \mathrm{dom}(x)$ for all $x \in A$.

$$\begin{aligned}
&[\![\forall x \varphi(x, \bar{a})]\!] \cap \mathrm{dom}(\bar{a}) \\
&\quad = \bigwedge_{x \in A} (E(x) \to [\![\varphi(x, \bar{a})]\!]) \cap \mathrm{dom}(\bar{a}) \qquad \text{(by I.H.)} \\
&\quad = \bigcap_{x \in A} \{v \in K \mid \forall w \geq v(w \in \mathrm{dom}(x) \to w \Vdash \varphi(x(w), \bar{a}))\} \cap \mathrm{dom}(\bar{a}) \\
&\quad = \{v \in K \mid \forall x \in A \ \forall w \geq v(w \in \mathrm{dom}(x) \to w \Vdash \varphi(x(w), \bar{a}))\} \cap \mathrm{dom}(\bar{a}) \\
&\quad = \{v \in K \mid \forall w \geq v \ \forall x \in A(w \in \mathrm{dom}(x) \to w \Vdash \varphi(x(w), \bar{a}))\} \cap \mathrm{dom}(\bar{a}) \\
&\quad = \{v \in K \mid \forall w \geq v \ \forall y \in D_w \ w \Vdash \varphi(d^w_y(w), \bar{a})\} \cap \mathrm{dom}(\bar{a}) \qquad (1) \\
&\quad = \{v \in K \mid \forall w \geq v \ \forall y \in D_w \ w \Vdash \varphi(y, \bar{a})\} \cap \mathrm{dom}(\bar{a}) \\
&\quad = \{v \in K \mid v \Vdash \forall x \varphi(x, \bar{a})\} \cap \mathrm{dom}(\bar{a}) \\
&\quad = \{v \in K \mid v \in \mathrm{dom}(\bar{a}) \wedge v \Vdash \forall x \varphi(x, \bar{a})\}.
\end{aligned}$$

Note that all operations above are well-defined by the fact that we are working in $\mathrm{dom}(\bar{a})$. Equality (1) is justified by recalling that all elements of A are of the form d^v_x and those containing w in their domain are supersets of those starting at w, i.e., $d^w_y \subseteq d^v_x$ for any d^v_x with $w \in \mathrm{dom}(d^v_x)$. As Kripke models only care about what happens at future nodes, it suffices to consider the elements of A that start at w. The final case is existential quantification, where we will make use of the same trick for equality (2).

$$
\begin{aligned}
&[\![\exists x\varphi(x)]\!] \cap \operatorname{dom}(\bar{a}) \\
&\quad = \bigvee_{x\in A} (E(x) \wedge [\![\varphi(x,\bar{a})]\!]) \cap \operatorname{dom}(\bar{a}) \\
&\quad = \bigvee_{x\in A} \{v \in K \mid v \in \operatorname{dom}(x) \wedge v \Vdash \varphi(x(v),\bar{a})\} \wedge \operatorname{dom}(\bar{a}) \\
&\quad = \{v \in K \mid \exists x \in A(v \in \operatorname{dom}(x) \wedge v \Vdash \varphi(x(v),\bar{a}))\} \wedge \operatorname{dom}(\bar{a}) \\
&\quad = \{v \in K \mid \exists y \in D_v\ v \Vdash \varphi(d^v_y(v),\bar{a})\} \cap \operatorname{dom}(\bar{a}) \qquad (2) \\
&\quad = \{v \in K \mid v \in \operatorname{dom}(\bar{a}) \wedge v \Vdash \exists x\varphi(x,\bar{a})\}.
\end{aligned}
$$

This finishes the induction and the proof of the theorem. □

The following corollary is a direct consequence of the theorem.

Corollary 2.35. *If φ is a sentence, then $[\![\varphi]\!]^A = \{v \in K \mid v \Vdash \varphi\}$. Indeed, $A \vDash \varphi$ if and only if $K \vDash \varphi$.* □

2.5 The Propositional Logic of Heyting Structures

Let us first introduce some notation, before we give several central definitions. Given a class $\mathcal{C}$ of Heyting structures for some language $\mathcal{L}$, let $\mathcal{H}_\mathcal{C}$ be the class of all underlying Heyting algebras of $\mathcal{C}$. Then, if $H \in \mathcal{H}_\mathcal{C}$, we let $\mathcal{C}_H \subseteq \mathcal{C}$ consist of all H-structures in $\mathcal{C}$. We should take note of the fact that if H is the underlying Heyting algebra of a Heyting structure, then H must be complete by the definition of H-structures.

Definition 2.36. Let $\mathcal{C}$ be a class of Heyting structures for a language $\mathcal{L}$. We will call $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$ the propositional logic of $\mathcal{C}$, that is, the set of all propositional formulas φ such that for all $C \in \mathcal{C}$ and all substitutions $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ we have that $C \vDash \varphi^\sigma$.

Proposition 2.37. *The propositional logic $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$ is an intermediate logic for any class of Heyting structures $\mathcal{C}$.*

Proof. The proposition follows if we can show that for every Heyting structure A, $\mathbf{L}^{\mathsf{Prop}}(A) = \mathbf{L}^{\mathsf{Prop}}(\{A\})$ is an intermediate logic, i.e., it is closed under modus ponens and uniform substitution such that $\mathbf{IPC} \subseteq \mathbf{L}^{\mathsf{Prop}}(A) \subseteq \mathbf{CPC}$. The first inclusion is a direct consequence of Theorem 2.29. Further observe that $\mathbf{L}^{\mathsf{Prop}}(A)$ is consistent and that it is closed under uniform substitution by definition. In this situation, we know by [7, Theorem 4.1] that $\mathbf{L}^{\mathsf{Prop}}(A)$ is an intermediate logic. □

Before defining the notions of loyalty and faithfulness, which are our main concepts in the analysis of propositional logics of Heyting structures, let us observe the following connection between the logic of a Heyting algebra and the propositional logic of a class of structures based on it.

Proposition 2.38. *Let $\mathcal{C}$ be a class of Heyting structures for a language $\mathcal{L}$. Then $\mathbf{L}^{\mathsf{Prop}}(H) \subseteq \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H)$.*

Proof. Let $\varphi \in \mathbf{L}^{\mathsf{Prop}}(H)$, then φ is true under all valuations $v : \mathsf{Prop} \to H$. To show that $\varphi \in \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H)$, we need to show that for all $A \in \mathcal{C}_H$, and all assignments $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$, we have that $[\![\varphi^\sigma]\!]^A = 1_H$.

By our assumption we have that $[\![\varphi]\!]^H_v = 1_H$, where we define the valuation v such that $v(p) = [\![\sigma(p)]\!]^A$. By a straightforward induction on the complexity of propositional formulas we can now show that $[\![\psi^\sigma]\!]^A = [\![\psi]\!]^H_v$ holds for all propositional formulas ψ. Hence, it follows that $[\![\varphi^\sigma]\!]^A = [\![\psi]\!]^H_v = 1_H$ and this concludes the proof of the proposition. ☐

The notion of loyalty is obtained by strengthening this inclusion to an equality.

Definition 2.39. Let $\mathcal{C}$ be a class of Heyting structures for a language $\mathcal{L}$. We will say that $\mathcal{C}$ *is loyal to* $H \in \mathcal{H}_\mathcal{C}$ if $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H) = \mathbf{L}^{\mathsf{Prop}}(H)$. We call $\mathcal{C}$ *loyal to* $\mathcal{H} \subseteq \mathcal{H}_\mathcal{C}$ if $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_\mathcal{H}) = \mathbf{L}^{\mathsf{Prop}}(\mathcal{H})$, and *loyal* if it is loyal to $\mathcal{H}_\mathcal{C}$.

Definition 2.40. Let $\mathcal{C}$ be a class of Heyting structures for a language $\mathcal{L}$, and further, let $\kappa > 0$ be a (possibly finite) cardinal.

(i) We will say that $\mathcal{C}$ *is κ-faithful to* $H \in \mathcal{H}_\mathcal{C}$ if for every collection $\{h_i \in H \mid i < \kappa\}$ of elements of H, there are some $C \in \mathcal{C}$ and a collection $\{\varphi_i \in \mathcal{L}^{\mathsf{sent}} \mid i < \kappa\}$ of $\mathcal{L}$-sentences such that $[\![\varphi_i]\!]^C = h_i$ for all $i < \kappa$. We call $\mathcal{C}$ *κ-faithful to* $\mathcal{H} \subseteq \mathcal{H}_\mathcal{C}$ if it is κ-faithful to all $H \in \mathcal{H}$, and *κ-faithful* if it is κ-faithful to $\mathcal{H}_\mathcal{C}$.

(ii) We will say that $\mathcal{C}$ *is $<\kappa$-faithful to* $H \in \mathcal{H}_\mathcal{C}$ if it is λ-faithful to H for every cardinal λ with $0 < \lambda < \kappa$. We call $\mathcal{C}$ *$<\kappa$-faithful to* $\mathcal{H} \subseteq \mathcal{H}_\mathcal{C}$ if it is $<\kappa$-faithful to all $H \in \mathcal{H}$, and *$<\kappa$-faithful* if it is $<\kappa$-faithful to $\mathcal{H}_\mathcal{C}$.

(iii) We will say that $\mathcal{C}$ *is faithful to* $H \in \mathcal{H}_\mathcal{C}$ if it is $<\omega$-faithful to H. We call $\mathcal{C}$ *faithful to* $\mathcal{H} \subseteq \mathcal{H}_\mathcal{C}$ if it is $<\omega$-faithful to all $H \in \mathcal{H}$, and *faithful* if it is $<\omega$-faithful to $\mathcal{H}_\mathcal{C}$.

The degree of faithfulness of a class of models describes how strongly the class reflects the structure of the underlying Heyting algebras. The notions of n-faithfulness for a natural number n might be interesting for analysing models of fragments of propositional logic that are restricted to a finite number of propositional variables. This has been done, e.g., in [17] for computational considerations. The notions of κ-faithfulness for infinite κ might be interesting to consider for infinitary logics $\mathcal{L}_{\kappa\lambda}$ in connection with large cardinals (in particular, compact cardinals, cf. [25, Section 4] for characterisations of compact cardinals in terms of compactness for infinitary logics).

In this thesis, however, we restrict our attention to models of set theory based on ordinary propositional logics, i.e., logics with finite formulas and countably many propositional variables. Hence, we will restrict our attention to $<\omega$-faithfulness (which we defined as faithfulness for the sake of simplicity).

This notion seems best suited for an analysis of the logical structures of these models: The behaviour of finite formulas is determined by the valuation of a finite number of propositional variables.

Before continuing with the analysis of the notions we just defined, let us translate these properties for Heyting structures into the world of Kripke models. In particular, classes of Kripke models can now be seen as classes of Heyting structures in which all underlying Heyting algebras are consisting of the upsets of some Kripke frame. We will thus say that a Kripke frame K is the underlying frame of some H-structure A if $H = \mathrm{Up}(K)$. From now on, we will assume that Kripke models are Heyting structures, whenever this is convenient. The following propositions will verify that the notions defined for Heyting structures translate to the Kripke models as expected.

Proposition 2.41. *Assume that K is a Kripke model and A is the associated $\mathrm{Up}(K)$-structure. Then the propositional logic of K and A coincide, i.e., $\mathbf{L}^{\mathsf{Prop}}(K) = \mathbf{L}^{\mathsf{Prop}}(A)$.*

Proof. It holds that $\varphi \in \mathbf{L}^{\mathsf{Prop}}(K)$ if and only if for all σ we have that $K \vDash \varphi^\sigma$. This is equivalent to $A \vDash \varphi^\sigma$ for all σ, i.e., $\varphi \in \mathbf{L}^{\mathsf{Prop}}(A)$. □

The following propositions are direct consequences of our definitions and Corollary 2.35.

Proposition 2.42. *Let $\mathcal{C}$ be a collection of Kripke models and $(K, \leq)$ be the underlying Kripke frame of some of these models. Then $\mathcal{C}$ is κ-faithful to $\mathrm{Up}(K, \leq)$ if and only if for all valuations $V : \mathsf{Prop} \to \mathcal{P}(K)$ and collections $\{p_i \mid i < \kappa\}$, of propositional letters, there is a collection of $\mathcal{L}$-sentences $\{\varphi_i \mid i < \kappa\}$ and a Kripke model $(K, \leq, D, I, f) \in \mathcal{C}$ such that $\{v \in K \mid K, v \Vdash \varphi_i\} = V(p_i)$ for all $i < \kappa$.*

Proof. The crucial observation for this fact is that, by persistency, any valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ can be seen as a valuation for the Heyting-algebra $\mathrm{Up}(K)$ and vice-versa. □

Proposition 2.43. *Let $\mathcal{C}$ be a collection of Kripke models. Then $\mathcal{C}$ is loyal to $\mathrm{Up}(K) \in \mathcal{H}_{\mathcal{C}}$ if and only if $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_{\mathrm{Up}(K)}) = \mathbf{L}^{\mathsf{Prop}}(K)$.*

Proof. This follows from the fact that $\mathbf{L}^{\mathsf{Prop}}(K) = \mathbf{L}^{\mathsf{Prop}}(\mathrm{Up}(K))$. □

These propositions justify the following definition.

Definition 2.44. We will say that a class of Kripke models $\mathcal{C}$ is *loyal to a Kripke frame K* if it is loyal to $\mathrm{Up}(K)$. Similarly, we say that $\mathcal{C}$ is *κ-faithful to K* if it is faithful to $\mathrm{Up}(K)$, and similarly for *$<\kappa$-faithful* and *faithful*.

Let us now observe some connections between the different notions that we have introduced. The first two propositions follow directly from the definitions.

Proposition 2.45. *If a class of Heyting structures $\mathcal{C}$ is κ-faithful to $H \in \mathcal{H}_{\mathcal{C}}$ for a cardinal $\kappa > 0$, then it is $<\kappa$-faithful to H.* □

Proposition 2.46. *If a class of Heyting structures $\mathcal{C}$ is $|H|$-faithful to $H \in \mathcal{H}_{\mathcal{C}}$, then it is κ-faithful to H for every cardinal κ.* □

Proposition 2.47. *Let $\mathcal{C}$ be a class of Heyting structures in a language $\mathcal{L}$, and let κ be the cardinality of the set of $\mathcal{L}$-sentences. For any Heyting algebra $H \in \mathcal{H}_{\mathcal{C}}$ with $|H| > \kappa$, we have that H is not $|H|$-faithful.*

Proof. Suppose that $\mathcal{C}$ is $|H|$-faithful to H. Using the definition faithfulness, we can find a surjection from the set of $\mathcal{L}$-sentences onto H. Hence, $|H| \leq \kappa$, a contradiction. □

The following propositions show that the different notions of κ-faithfulness do not collapse.

Proposition 2.48. *For every cardinal $\kappa \geq \omega$, there is a class $\mathcal{C}$ of Heyting structures for some language $\mathcal{L}$ such that $\mathcal{C}$ is κ-faithful but not κ^+-faithful.*

Proof. Let $\mathcal{L}$ be the language with κ-many nullary relation symbols $\{R_i \mid i < \kappa\}$. Using Theorem 2.34 and Proposition 2.42, it suffices to construct a class $\mathcal{C}$ of Kripke models for $\mathcal{L}$ that is κ-faithful but not κ^+-faithful. So let $\mathcal{C}$ be the class of Kripke models for $\mathcal{L}$ based on the Kripke frame $K = (\kappa^+, \emptyset)$.

Firstly, to show that $\mathcal{C}$ is κ-faithful, we have to show that for every collection $\{U_j \subseteq K \mid j < \kappa\}$ of upsets of K (i.e., subsets of K because the relation is empty), there is a model $C \in \mathcal{C}$ and a collection of sentences $\{\varphi_j \mid j < \kappa\}$ such that $\{v \in K \mid C, v \Vdash \varphi_j\} = U_j$. So let C be the Kripke model on $(\kappa^+, \emptyset)$ with $C, i \Vdash R_j$ if and only if $i \in U_j$. Take $\varphi_j = R_j$ for every $j < \kappa$, and the desired condition holds.

Secondly, to show that $\mathcal{C}$ is not κ^+-faithful, consider the upsets U_j of $(\kappa^+, \emptyset)$ defined by $U_j = j$ for $j \in \kappa^+$. If we wanted to prove κ^+-faithfulness, we would have to provide a collection of sentences whose evaluation matches the U_j's. However, $|\mathcal{L}^{\mathsf{sent}}| \leq \kappa^{<\omega} = \kappa < \kappa^+$. So it is impossible to provide κ^+-many different sentences. Hence, $\mathcal{C}$ is not κ^+-faithful. □

Proposition 2.49. *For every natural number $n > 0$, there is a class $\mathcal{C}$ of Heyting structures for some language $\mathcal{L}$ such that $\mathcal{C}$ is n-faithful but not 2^n-faithful.*

Proof. We will use a very similar technique as in the previous proposition. Let $\mathcal{L}$ be the language with n nullary relation symbols $\{R_i \mid i < n\}$. Consider the class $\mathcal{C}$ of Kripke models C for $\mathcal{L}$ based on the Kripke frame $K = (2^n + 1, \emptyset)$.

The n-faithfulness of $\mathcal{C}$ follows as in the previous proposition by choosing one relation symbol for every subset of the n-many subsets of K. The fact that $\mathcal{C}$ is not 2^n-faithful follows from the observation that there are at most 2^n different combinations of forcing the R_i's at each node. Therefore, every model $C \in \mathcal{C}$ has at least two isomorphic nodes. So we cannot find a collection of sentences representing the $2^n + 1$-many different sets $U_i = \{j \in \omega \mid j \leq i\}$ for $i \leq 2^n$ as this collection requires $2^n + 1$ non-isomorphic nodes. □

We will continue by connecting faithfulness and loyalty.

Proposition 2.50. *If a class of Heyting structures $\mathcal{C}$ is faithful to $H \in \mathcal{H}_{\mathcal{C}}$, then it is loyal to H.*

Proof. We have to show that $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_{\mathcal{H}}) = \mathbf{L}^{\mathsf{Prop}}(\mathcal{H})$. The inclusion from right to left holds by Proposition 2.38 without making use of the assumption of faithfulness. The converse direction follows by a very similar argument where we use the assumption that $\mathcal{C}$ is faithful to H to generate an assignment of propositional variables to sentences that coincides with the given valuation. Indeed, given a propositional formula $\varphi \in \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H)$ and a valuation $v : \mathsf{Prop} \to H$, we need to verify that $[\![\varphi]\!]^H_v = 1_H$. Using that $\mathcal{C}$ is faithful to H, we can obtain an assignment $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ with $[\![\sigma(p)]\!]^A = v(p)$ for all propositional letters p appearing in φ. Then, by a similar induction as in the previous proposition, we can show that $[\![\psi^\sigma]\!]^A = [\![\psi]\!]^H_v$ holds for all propositional formulas ψ with propositional letters among the propositional letters of φ. Hence, it follows that $[\![\varphi]\!]^H_v = [\![\varphi^\sigma]\!]^A = 1_H$ by the assumption that $\varphi \in \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H)$. □

We will later see that the converse direction does not hold. For example, the class of Boolean-valued models of set theory is loyal (as it satisfies classical logic **CPC**), but not faithful to any finite Boolean algebra but the two-point algebra (cf., Corollary 5.16 and Theorem 5.17).

Proposition 2.51. *There is a class of Heyting structures that is loyal but not faithful.* □

Proposition 2.52. *Let $\mathcal{H}$ be a class of Heyting algebras. If a class $\mathcal{C}$ of Heyting structures is loyal to H for all $H \in \mathcal{H}$, then it is loyal to $\mathcal{H}$.*

Proof. Let us assume that $\mathcal{C}$ is loyal to all $H \in \mathcal{H}$. By definition, this means that for all $H \in \mathcal{H}$ we have that $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H) = \mathbf{L}^{\mathsf{Prop}}(H)$. Hence,

$$\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_{\mathcal{H}}) = \bigcap_{H \in \mathcal{H}} \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_H) = \bigcap_{H \in \mathcal{H}} \mathbf{L}^{\mathsf{Prop}}(H) = \mathbf{L}^{\mathsf{Prop}}(\mathcal{H}). \qquad \square$$

The converse direction does not hold in general. For the proof of the following proposition, we need to refer to results that will be proven in later sections (for an introduction of Iemhoff models see chapter 3).

Proposition 2.53. *There is a class of Heyting structures $\mathcal{C}$ that is loyal, but not loyal to every $H \in \mathcal{H}_{\mathcal{C}}$.*

Proof. Let us call an Iemhoff model $K(\mathcal{M})$ *simple* if it associates the same model to every node of the Kripke frame. Clearly, for every simple Iemhoff model $K(\mathcal{M})$ it holds that $\mathbf{L}^{\mathsf{Prop}}(K(\mathcal{M})) = \mathbf{CPC}$. Now, define a class $\mathcal{C}$ of Iemhoff models as the union of all Iemhoff models based on a finite tree and all simple Iemhoff models based on a finite Kripke frame. By the fact that **IPC** is sound and complete with respect to all finitely rooted trees ([33, Theorem 6.12]) and by Theorem 3.19 in combination with Proposition 2.50, we know that $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}) = \mathbf{IPC}$.

Now, let $(K, \leq)$ be the Kripke frame with domain $\{0, 1, 1', 2\}$ such that $\leq$ is the reflexive closure of the relation with $0 \leq 1 \leq 2$, $0 \leq 1' \leq 2$, $1 \leq 1'$ and $1' \leq 1$. Clearly, $(K, \leq)$ is not a finite tree, and $\mathbf{L}^{\mathsf{Prop}}(K)$ does not contain the law of excluded middle. However, $\mathcal{C}$ is not loyal to K as $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}_K) = \mathbf{CPC}$ by our observation it holds that $\mathbf{L}^{\mathsf{Prop}}(K(\mathcal{M})) = \mathbf{CPC}$ for all Iemhoff models based on K. □

If one works with Kripke frames that are based on partial orders instead of preorders, the proof of the previous proposition can be done by considering a union of partial orders that is not rooted. We can combine the previous Proposition 2.50 and Proposition 2.52 to obtain the following one.

Proposition 2.54. *Any faithful class of Heyting structures is loyal.* □

Let us now observe how our model-theoretic notions connect to the more proof-theoretic notion of the de Jongh property.

Proposition 2.55. *Let $\mathcal{C}$ be a class of Heyting structures and T a theory. Suppose that $\mathbf{J} = \mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$ and that $A \vDash \mathsf{T}$ holds for all $A \in \mathcal{C}$. Then T has the de Jongh property with respect to the intermediate logic $\mathbf{J}$.*

Proof. We need to show that $\mathbf{L}^{\mathsf{Prop}}(\mathsf{T}(\mathbf{J})) = \mathbf{J}$, and the direction from right to left is clear. To prove the other direction, assume that $\mathbf{J} \nvdash \varphi$, then $\varphi \notin \mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$ by our assumption. So there is some $A \in \mathcal{C}$ and $\sigma : \mathsf{Prop} \to \mathcal{L}^{\mathsf{sent}}$ such that $A \nvDash \varphi^\sigma$. Hence, $\mathsf{T} \nvdash \varphi^\sigma$, i.e., $\varphi \notin \mathbf{L}^{\mathsf{Prop}}(\mathsf{T}(\mathbf{J}))$. □

The following proposition is a direct consequence.

Proposition 2.56. *If $\mathcal{C}$ is loyal and $\mathcal{C} \vDash \mathsf{T}$, then T has the de Jongh property with respect to $\mathbf{L}^{\mathsf{Prop}}(\mathcal{H}_{\mathcal{C}})$.* □

We summarise the definitions and results of this section in Figure 2.1.

Remark 2.57. In this thesis, we will exclusively consider models of different set theories. Of course, it is possible to use this framework to analyse model constructions for all kinds of theories. For example, [23, Theorem 4.1] shows that Heyting arithmetic HA has the de Jongh property with respect to all logics characterised by a class of finite frames. Interpreted in the framework developed here, the proof of this result shows that the class of Kripke models for HA that are based on a finite frame is faithful.

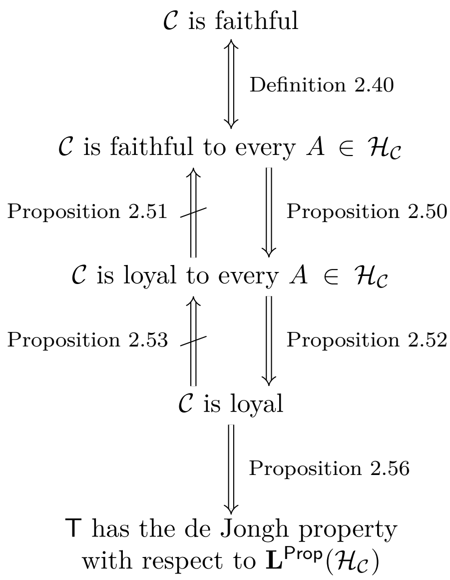

Figure 2.1: Implications between different notions of loyalty and faithfulness, where $\mathcal{C} \vDash \mathsf{T}$.

Chapter 3

Iemhoff's Models for Subtheories of Constructive Set Theory CZF

In this chapter, we will present and analyse a construction that was developed by Iemhoff in [21]. Iemhoff's basic idea is to take a Kripke frame for intuitionistic logic and equip each node with a model of (not necessarily classical) set theory to obtain a new model of a constructive set theory.

In the first section, we will give an overview of Iemhoff's ideas. Afterwards, we will present a limitation of this method—namely, under mild conditions, Iemhoff's models will never satisfy the axiom of exponentiation. An analysis of the underlying propositional logic of the class of Iemhoff models will be the purpose of the third section: We will see that this class is faithful. In conclusion, Iemhoff's method does not so much provide a method for constructing independence results in constructive set theory, but rather for showing the compatibility of constructive set theory with many different underlying logics.

3.1 Kripke Models of Constructive Set Theory

The idea is to obtain models of set theory by putting classical models of set theory at every node of a Kripke model for intuitionistic predicate logic.

Note that our presentation differs from Iemhoff's in that we sacrifice some generality towards a slightly easier presentation and to fit her models better in the context of this thesis. In particular, we will consider models that assign to every node a classical model of ZF set theory. We will start by giving a condition for when an assignment of models to nodes is suitable for our purposes.

Definition 3.1. Let $(K, \leq)$ be a Kripke frame. An assignment $\mathcal{M}$ of nodes to transitive models of ZF set theory is called *sound for K* if for all nodes $i, j \in K$ with $i \leq j$ we have that $\mathcal{M}(i) \subseteq \mathcal{M}(j)$, and the inclusion map is a homomorphism of models of set theory (i.e., it preserves $\in$ and $=$).

This could be readily generalised to homomorphisms of models of set theory that are not necessarily inclusions.

Definition 3.2. An *Iemhoff model* $K(\mathcal{M})$ consists of a Kripke frame $(K, \leq)$ and a sound assignment $\mathcal{M} : K \to V$ of nodes to transitive models of ZF set theory.

We will usually write $\mathcal{M}_v$ for $\mathcal{M}(v)$. The forcing relation is defined as usual for Kripke models of intuitionistic predicate logic (see also section 2.2), with the $\in$-predicate being interpreted as follows:

$$v \Vdash x \in y \text{ if and only if } \mathcal{M}(v) \vDash x \in y.$$

With these definitions, Iemhoff models are just an instance of a Kripke model for **IQC**. So it is clear that persistency holds.

Proposition 3.3. *If $K(\mathcal{M})$ is an Iemhoff model with nodes v and w such that vRw, then for all formulas φ, $v \Vdash \varphi$ implies $w \Vdash \varphi$.* □

Our next goal is to show that these models satisfy some set theory.

Proposition 3.4. *Let $K(\mathcal{M})$ be an Iemhoff model where $\mathcal{M}$ is a transitive sound assignment, $\varphi(x)$ be a Δ_0-formula. Let $a_0, \ldots, a_{n-1} \in \mathcal{M}_v$ and $v \in K$. Then $K(\mathcal{M}), v \Vdash \varphi(a_0, \ldots, a_{n-1})$ if and only if $\mathcal{M}_v \vDash \varphi(a_0, \ldots, a_{n-1})$.*

Proof. We will prove a stronger statement by induction on Δ_0-formulas, simultaneously for all $v \in K$. Namely, we will show that for all $w \geq v$ it holds that $w \Vdash \varphi(a_0, \ldots, a_n)$ if and only if $\mathcal{M}_v \vDash \varphi(a_0, \ldots, a_n)$. Note that it is crucial in for this proof that the quantifier is outside in the sense that:

$$\forall w \geq v(w \Vdash \varphi(a_0, \ldots, a_n) \iff \mathcal{M}_v \vDash \varphi(a_0, \ldots, a_n)).$$

The cases for $\bot$, $=$, $\in$, $\wedge$ and $\vee$ then follow directly from the definitions using the induction hypothesis for the non-atomic cases.

For the first direction of the implication case, let us assume that $w \geq v$ and that $w \Vdash \varphi(\bar{a}) \to \psi(\bar{b})$, where $\bar{a}$ and $\bar{b}$ stand for $a_0, \ldots, a_n \in \mathcal{M}_v$ and $b_0, \ldots, b_n \in \mathcal{M}_v$, respectively. By the semantics of implication, it is the case that $w \Vdash \varphi(\bar{a})$ implies $w \Vdash \psi(\bar{b})$. Therefore, the induction hypothesis yields that $\mathcal{M}_v \vDash \varphi(\bar{a})$ implies $\mathcal{M}_v \vDash \psi(\bar{b})$. But this is just $\mathcal{M}_v \vDash \varphi(\bar{a}) \to \psi(\bar{b})$ by the semantics of classical models.

Conversely, assume that $\mathcal{M}_v \vDash \varphi(\bar{a}) \to \psi(\bar{b})$. This means that $\mathcal{M}_v \vDash \varphi(\bar{a})$ implies $\mathcal{M}_v \vDash \psi(\bar{b})$. In this situation, it holds by our induction hypothesis that for all $w \geq v$ that $w \Vdash \varphi(\bar{a})$ implies $w \Vdash \psi(\bar{b})$. This is $v \Vdash \varphi(\bar{a}) \to \psi(\bar{b})$. By persistency it holds that for all $w \geq v$ we have $w \Vdash \varphi(\bar{a}) \to \psi(\bar{b})$.

The next case is the existential quantifier. As we are concerned with Δ_0-formulas, our assumption for the first direction will be $w \Vdash \exists x(x \in c \wedge \varphi(x, \bar{a}))$ for some $w \geq v$ with $c, \bar{a} \in \mathcal{M}_v$. By the definition of the semantics in the Iemhoff model, it holds that $w \Vdash b \in c \wedge \varphi(b, \bar{a})$ for some $b \in \mathcal{M}_w$. Then $b \in c$ and, by transitivity, it holds that $b \in \mathcal{M}_v$. We can therefore apply our induction hypothesis to $w \Vdash \varphi(b, \bar{a})$ to obtain that $\mathcal{M}_v \vDash \varphi(b, \bar{a})$. Then $\mathcal{M}_v \vDash b \in c \wedge \varphi(b, \bar{a})$ and so $\mathcal{M}_v \vDash \exists x(x \in c \wedge \varphi(b, \bar{a}))$.

For the converse direction assume that $\mathcal{M}_v \vDash \exists x(x \in c \wedge \varphi(x, \bar{a}))$. Then, by transitivity, there is some $b \in c$ such that $\mathcal{M}_v \vDash \varphi(b, \bar{a})$. We can now apply the induction hypothesis to obtain that $w \Vdash \varphi(b, \bar{a})$ for all $w \geq v$. As $b \in c$ and $\mathcal{M}$ is a sound and transitive assignment, it holds that $w \Vdash \exists x(x \in c \wedge \varphi(x, \bar{a}))$.

The last case is the bounded universal quantifier. For the first direction, let us assume that $w \Vdash \forall x(x \in c \to \varphi(x, \bar{a}))$ for some $c, \bar{a} \in \mathcal{M}_v$. It then holds for all $b \in \mathcal{M}_w$ with $b \in c$ that $w \Vdash \varphi(x, \bar{a})$. By transitivity of our models and soundness of the assignment, it follows that the $b \in \mathcal{M}_w$ with $b \in c$ are exactly the $b \in \mathcal{M}_v$ with $b \in c$. So we can apply our induction hypothesis to derive that $\mathcal{M}_v \vDash \varphi(b, \bar{a})$ for all $b \in \mathcal{M}_v$ with $b \in c$. But this is $\mathcal{M}_v \vDash \forall x(x \in c \to \varphi(x, \bar{a}))$.

The converse direction follows similarly: Assume that $\mathcal{M}_v \vDash \forall x(x \in c \to \varphi(x, \bar{a}))$ for some $c, \bar{a} \in \mathcal{M}_v$. By the semantics and our assumptions, we have that $\mathcal{M}_v \vDash \varphi(b, \bar{a})$ for all $b \in c$. In this situation, the induction hypothesis implies for all $w \geq v$ that $w \Vdash \varphi(b, \bar{a})$ for all $b \in c$. As observed above, the $b \in \mathcal{M}_w$ with $b \in c$ are exactly the $b \in \mathcal{M}_v$ with $b \in c$. Therefore, we are allowed to conclude that $w \Vdash \forall x(x \in c \to \varphi(x, \bar{a}))$ for all $w \geq v$. □

We will now see that Iemhoff models indeed satisfy a certain constructive set theory. Note that we will have to deal with fewer technicalities than in Iemhoff's original paper because our setting is slightly less general (in particular, we are only dealing with models of ZF). We will refer to the theory $\mathsf{CZF}^{-\mathsf{c}}$ + Bounded Strong Collection + Set-bounded Subset Collection as CZF^*.

Theorem 3.5 (Iemhoff, [21, Corollary 4])**.** *Let $K(\mathcal{M})$ be an Iemhoff model. Then it holds that $K(\mathcal{M}) \Vdash \mathsf{CZF}^*$.*

Proof. Recall that $\mathsf{CZF}^{-\mathsf{c}}$ is CZF without the collection axioms. We will have to verify all axioms at every node of K, so let $v \in K$.

Extensionality Given $a, b \in \mathcal{M}_v$, we need to show that:

$$v \Vdash a = b \leftrightarrow \forall x(x \in a \leftrightarrow x \in b).$$

Assume that $v \Vdash a = b$. Then $a = b$ and this directly implies that for all $w \geq v$, we will have that $\forall x(x \in a \leftrightarrow x \in b)$. Conversely, if $v \Vdash \forall x(x \in a \leftrightarrow x \in b)$, then by transitivity we know for all $x \in \mathcal{M}_v$ that $x \in a \leftrightarrow x \in b$. This implies $a = b$ by extensionality in our meta-theory.

Empty Set We need to show that $v \Vdash \exists x \forall y(y \notin x)$. Choose the witness for x to be $\emptyset$ (which is a member of $\mathcal{M}_v$ by transitivity). Then the axiom is clearly satisfied.

Pairing Given $a, b \in \mathcal{M}_v$, we need to check that:

$$v \Vdash \exists c \forall x(x \in c \leftrightarrow (x = a \vee x = b)).$$

By Pairing in $\mathcal{M}_v$, take $c = \{a, b\}$. Then the axiom follows directly from the transitivity of the models.

Union Let $a \in \mathcal{M}_v$. We need to verify that:

$$v \Vdash \exists b \forall x(x \in b \leftrightarrow \exists y \in a(x \in y)).$$

Let $b = \bigcup a \in \mathcal{M}_v$ by Union in $\mathcal{M}_v$. The direction from left to right follows directly from this definition. The other direction follows by transitivity of the successor models $\mathcal{M}_w$ of $\mathcal{M}_v$: Every element that $\mathcal{M}_w$ thinks to be a member of b must already have been a member of b in $\mathcal{M}_v$.

Bounded Separation Again, let $a \in \mathcal{M}_v$ and let $\varphi(x)$ be a Δ_0-formula. We need to show that:

$$\exists b \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x)).$$

Let us use Bounded Separation in $\mathcal{M}_v$ to obtain the subset b of a containing exactly of those $x \in a$ with $\varphi(x)$. By transitivity, we have for all $x \in a$ that $x \in \mathcal{M}_v$, and by Proposition 3.4, we know that for those $x \in \mathcal{M}_v$ it holds that $v \Vdash \varphi(x)$ if and only if $\mathcal{M}_v \vDash \varphi(x)$. Hence, our set b satisfies the axiom (using the transitivity of all successor models of $\mathcal{M}_v$).

Strong Infinity As in the previous cases we need to find a witness for the given axiom. Here, we have to show that

$$v \Vdash \exists a \forall x(x \in a \leftrightarrow x = \emptyset \vee \exists y \in a(x = y \cup \{y\})).$$

Again, the canonical candidate $a = \omega^{\mathcal{M}_v}$ will satisfy the axiom.

Set Induction We need to show that

$$v \Vdash \forall x(\forall y \in x\ \varphi(y) \to \varphi(x)) \to \forall x \varphi(x).$$

So assume that $v \Vdash \forall x(\forall y \in x\ \varphi(y) \to \varphi(x))$. It is sufficient to show that $v \Vdash \forall x\ \varphi(x)$. To do so, we will use Set Induction on the meta-level. Let $\psi(x)$ be the formula

$$\forall w \geq v(x \in \mathcal{M}_w \to w \Vdash \varphi(x)).$$

We will show that $\forall x(\forall y \in x\ \psi(y) \to \psi(x))$. Let x be given and assume that $x \in \mathcal{M}_w$ and $\forall y \in x\ \psi(y)$, i.e., we have for all $y \in x$ that $\forall w \geq v(y \in \mathcal{M}_w \to w \Vdash \varphi(x))$. As $x \in \mathcal{M}_w$, we know by transitivity that $y \in \mathcal{M}_w$ for all $y \in x$. Hence, for all $y \in x$ we have that $w \Vdash \varphi(y)$. We know from our assumption that $w \Vdash \forall y \in x\ \varphi(y) \to \varphi(x)$, and can therefore conclude $w \Vdash \varphi(x)$. Hence, $\forall w \geq v\ \varphi(x)$, i.e., $\psi(x)$ holds.

We can now conclude that for all x, we have $\psi(x)$. This means that $\forall w \geq v \forall x \in \mathcal{M}_w(w \Vdash \varphi(x))$. By the semantics of the Iemhoff model, we have $v \Vdash \forall x \varphi(x)$, and this concludes the proof of Set Induction.

Bounded Strong Collection Let $a \in \mathcal{M}_v$. The axiom we have to verify reads as follows, where φ is a Δ_0-formula,

$$(\forall x \in a \exists y\ \varphi(x, y)) \to \exists b(\forall x \in a \exists y \in b\ \varphi(x, y) \wedge \forall y \in b \exists x \in a\ \varphi(x, y)).$$

From Proposition 3.4 we know that Δ_0-formulas are decided locally, i.e., we know that $v \Vdash \varphi(x, y)$ if and only if $\mathcal{M}_v \vDash \varphi(x, y)$ for all $x, y \in \mathcal{M}_v$. Now, as $a \in \mathcal{M}_v$, let $b \in \mathcal{M}_v$ be the set obtained from Bounded Strong Collection in $\mathcal{M}_v$. As all formulas in the right hand side are bounded with objects in $\mathcal{M}_v$, it is decided within $\mathcal{M}_v$. By persistency, the axiom is proven.

Set-bounded Subset Collection Let $a \in \mathcal{M}_v$. We have to verify that

$$\begin{aligned}&\exists c \forall z(\forall x \in a \exists y \in b\ \varphi(x, y, z) \to \\ &\quad \exists d \in c(\forall x \in a \exists y \in d\ \varphi(x, y, z) \wedge \forall y \in d \exists x \in a\ \varphi(x, y, z))),\end{aligned}$$

where φ is a Δ_0-formula such that z is set-bounded in φ, i.e., it is possible to derive $z \in t$ for some term t that appears in φ in intuitionistic logic from $\varphi(x, y, z)$. As the only terms in the language of set theory are variables, it must be derivable that $z \in u$ for some u. As φ is Δ_0 and the only free variables are x, y and z, the transitivity of $\mathcal{M}_w$ implies that whenever $w \Vdash z \in u$, we have that $w \Vdash z \in x$, $w \Vdash z \in y$ or $w \Vdash z \in z$. By the semantics $\in$ and transitivity of $\mathcal{M}_v$, it follows that $z \in \mathcal{M}_v$.

Note that $\forall x \in a \exists y \in b\ \varphi(x, y, z)$ is a Δ_0-formula, i.e., we have by Proposition 3.4 that

$$\begin{aligned}&v \Vdash \forall x \in a \exists y \in b\ \varphi(x, y, z) \\ &\qquad\qquad \text{if and only if } \mathcal{M}_v \vDash \forall x \in a \exists y \in b\ \varphi(x, y, z).\end{aligned}$$

Let $c \in \mathcal{M}_v$ be the element of which Set-bounded Subset Collection of φ, a and b holds in $\mathcal{M}_v$. It suffices now to show that for every $w \geq v$ and $z \in \mathcal{M}_w$, $w \Vdash \forall x \in a \exists y \in b\ \varphi(x, y, z)$ implies

$$w \Vdash \exists d \in c(\forall x \in a \exists y \in d\ \varphi(x, y, z) \wedge \forall y \in d \exists x \in a\ \varphi(x, y, z)).$$

From the assumption $w \Vdash \forall x \in a \exists y \in b\ \varphi(x, y, z)$ we can derive that either $w \Vdash \neg \exists x \in a$, or $z \in \mathcal{M}_v$, since z is set-bounded, and $v \Vdash \forall x \in a \exists y \in b\ \varphi(x, y, z)$.

In the first case, we also must have $v \Vdash \neg \exists x \in a$, and so will get in $\mathcal{M}_v$ that $\neg \exists x \in d$ for all d as in the axiom. This implies $v \Vdash \neg \exists x \in d$ and, by persistency, $w \Vdash \neg \exists x \in d$. But then the axiom clearly holds in w.

In the second case we know that $\mathcal{M}_v \vDash \forall x \in a \exists y \in b\ \varphi(x, y, z)$ and can thus use our assumptions on b and persistency to derive the axiom in w. This is what we needed to show to verify the axiom in v. □

Let us end this section with the following observation.

Proposition 3.6. *If $K(\mathcal{M})$ is an Iemhoff model such that every $\mathcal{M}_v$ is a model of the axiom of choice, then the axiom of choice holds in $K(\mathcal{M})$.*

Proof. Recall that the axiom of choice is the following statement:

$$\forall a((\forall x \in a \forall y \in a\ (x \neq y \to x \cap y = \emptyset)) \to \exists b \forall x \in a \exists! z \in b\ z \in x). \qquad (\mathsf{AC})$$

Let $v \in K$ and $a \in \mathcal{M}_v$ such that $v \Vdash \forall x \in a \forall y \in a\ (x \neq y \to x \cap y = \emptyset)$. This is a Δ_0-formula, so we can apply Proposition 3.4 to derive that $\mathcal{M}_v \vDash \forall x \in a \forall y \in a\ (x \neq y \to x \cap y = \emptyset)$. As $\mathcal{M}_v \vDash \mathsf{AC}$, there is some $b \in \mathcal{M}_v$ such that $\mathcal{M}_v \vDash \forall x \in a \exists! z \in b\ z \in x$. Again, this is a Δ_0-formulas, so it holds that $v \Vdash \forall x \in a \exists! z \in b\ z \in x$. As $b \in \mathcal{M}_v$, we have $v \Vdash \exists b \forall x \in a \exists! z \in b\ z \in x$. But this shows that $v \Vdash \mathsf{AC}$. □

3.2 A Failure of Exponentiation

In this section, we will investigate the limits of Iemhoff models from the set-theoretical point of view. We will exhibit a failure of the axiom of exponentiation:

$$\forall x\ \forall y\ \exists z\ \forall f(f \in z\ \leftrightarrow\ f : x \to y) \qquad \text{(Exponentiation, } \mathsf{Exp})$$

Note that $f : x \to y$ is an abbreviation for the Δ_0-formula $\varphi(f, x, y)$ stating that f is a function from x to y. Recall that this axiom is an intuitionistic consequence of the axiom of subset collection (cf., Proposition B.1). Therefore, a failure of exponentiation implies a failure of subset collection.

Proposition 3.7. *Let $K(\mathcal{M})$ be an Iemhoff model such that there are $v, w \in K$ with $v < w$. If $a, b \in \mathcal{M}_v$ and $g : a \to b$ is a function contained in $\mathcal{M}_w$ but not in $\mathcal{M}_v$, then $K(\mathcal{M}) \nVdash \mathsf{Exp}$.*

Proof. Informally, the argument boils down to the fact that the semantics allow us to require the existence of the set of functions from a to b within $\mathcal{M}_v$. But there, of course, it cannot capture g.

Assume, for contradiction, that $K(\mathcal{M}) \Vdash \mathsf{Exp}$. Further, assume that $a, b \in \mathcal{M}_v$ and $g : a \to b$ is a function contained in $\mathcal{M}_w$ but not in $\mathcal{M}_v$. Then,

$$K(\mathcal{M}), v \Vdash \forall x\ \forall y\ \exists z\ \forall f(f \in z\ \leftrightarrow\ f : x \to y),$$

and by the definition of our semantics, we can deduce

$$K(\mathcal{M}), v \Vdash \exists z\ \forall f(f \in z\ \leftrightarrow\ f : a \to b),$$

but this just means that there is some $c \in \mathcal{M}_v$ such that

$$K(\mathcal{M}), v \Vdash \forall f(f \in c\ \leftrightarrow\ f : a \to b).$$

By the semantics of universal quantification, this means that

$$K(\mathcal{M}), w \Vdash g \in c\ \leftrightarrow\ g : a \to b.$$

As $g \in c \leftrightarrow g : a \to b$ is just an abbreviation for $(g \in c \to (g : a \to b)) \wedge ((g : a \to b) \to g \in c)$, we may deduce that

$$K(\mathcal{M}), w \Vdash (g : a \to b) \to g \in c,$$

which yields, as g is indeed a function from $a \to b$, that

$$K(\mathcal{M}), w \Vdash g \in c.$$

As c is assumed to be a member of $\mathcal{M}_v$, we have $g \in c \in \mathcal{M}_v$. Therefore, by transitivity, $g \in \mathcal{M}_v$. This is a contradiction to our assumption that g is not contained in $\mathcal{M}_v$. □

To see the full impact of this observation on Iemhoff models, we need to remind ourselves of the following side-effect of set-theoretical forcing.

Proposition 3.8. *If M is a model of classical* ZFC *set theory, $\mathbb{P} \in M$ a non-trivial forcing notion and G a $\mathbb{P}$-generic filter over M, then the generic extension $M[G]$ contains a function that is not contained in the ground model.*

Proof. As $G \notin M$, it follows that $\chi_G : \mathbb{P} \to 2$, the characteristic function of G in $\mathbb{P}$, is not contained in M. □

Let us now say that an Iemhoff model *involves forcing non-trivially* if there are nodes $v < w \in K$ such that $\mathcal{M}_w$ is a non-trivial generic extension of $\mathcal{M}_v$ (i.e., $\mathcal{M}_w = \mathcal{M}_v[G]$ for some generic $G \notin \mathcal{M}_v$).

Corollary 3.9. *If the Iemhoff model $K(\mathcal{M})$ involves forcing non-trivially, then it is not a model of* CZF. □

Of course, the concepts of "not forcing an axiom" and "forcing the negation of an axiom" do not coincide. The above results show that the former is true for many Iemhoff models, now we will provide an example of the latter for the exponentiation axiom.

Proposition 3.10. *There is an Iemhoff model $K(\mathcal{M})$ that forces the negation of the exponentiation axiom, i.e., $K(\mathcal{M}) \Vdash \neg\mathsf{Exp}$.*

Proof. Consider the Kripke frame $K = (\omega, <)$ where $<$ is the standard ordering of the natural numbers. Construct the assignment $\mathcal{M}$ as follows: Choose $\mathcal{M}_0$ to be any countable and transitive model of ZFC. If $\mathcal{M}_i$ is constructed, let $\mathcal{M}_{i+1} = \mathcal{M}_i[G_i]$ where G_i is generic for Cohen forcing over $\mathcal{M}_i$. Clearly, $\mathcal{M}$ is a sound and transitive assignment of models of set theory.

We want to show that for every $i \in \omega$ we have that $i \Vdash \neg\mathsf{Exp}$, i.e., for all $j \geq i$ we need to show that $j \Vdash \mathsf{Exp}$ implies $j \Vdash \bot$. This, however, works just as in the proof of Proposition 3.7. Note that the witnesses are the characteristic functions χ_{G_i} as provided by Proposition 3.8. □

3.3 The Propositional Logic of Iemhoff Models

Our next goal is to show that Iemhoff models are faithful, and therefore loyal. This terminology is justified as we can interpret Iemhoff models as Heyting structures via Theorem 2.34.

The basic idea of proving faithfulness of Iemhoff models is the following: Given a valuation for some Kripke frame, we need to construct a sound assignment and a collection of sentences such that we can imitate the valuation of the standard Kripke model within an Iemhoff model. The rest of this section will provide the technical results needed to do so.

Technical preliminaries

We define the usual relativisation $\varphi \mapsto \varphi^{\mathrm{L}}$ of a formula of set theory to the constructible universe L. Note, however, that in our setting the evaluation of universal quantifiers and implications is not locally. We should, therefore, recall some important facts about the constructible universe to make sure that everything works as expected.

Proposition 3.11. *There is a Σ_1-formula $\varphi(x)$ such that in any model $M \vDash$ ZFC, we have $M \vDash \varphi(x) \leftrightarrow x \in \mathrm{L}$.*

Proof. By [22, Lemma 13.14], the assignment $\alpha \mapsto \mathrm{L}_\alpha$ is Δ_1, so we can find a Σ_1-formula $\exists z \psi$, where ψ is Δ_0, such that $y = \mathrm{L}_\alpha \leftrightarrow \exists z \psi(y, \alpha)$ holds. Note that the statement 'α is an ordinal' is Δ_0. Hence, the following is a Σ_1-formula:

$$\varphi(x) = \exists \alpha (\text{“}\alpha \text{ is an ordinal”} \wedge \exists y (x \in y \wedge \exists z \psi(y, \alpha))).$$

This finishes the proof of the proposition. ☐

From now on, let $x \in \mathrm{L}$ be an abbreviation for $\varphi(x)$, where φ is as in Proposition 3.11.

Proposition 3.12. *Let K be a Kripke frame and $\mathcal{M}$ a sound assignment of nodes to transitive models of* ZFC*. Then $K(\mathcal{M}), v \Vdash x \in \mathrm{L}$ if and only if $\mathcal{M}_v \vDash x \in \mathrm{L}$.*

Proof. This follows directly from the fact that existential quantifiers are evaluated locally and Δ_0-formulas are decided locally by Proposition 3.4. ☐

The crucial detail of the following technical lemma is the fact that the constructible universe is absolute between inner models of set theory. We will need a strengthened notion of a sound assignment.

Definition 3.13. Let K be a Kripke frame. We call an assignment $\mathcal{M} : K \to V$ *meticulous* if it is an assignment of nodes to transitive models of ZFC such that for all $v, w \in K$ we have that $\mathrm{L}^{\mathcal{M}_v} = \mathrm{L}^{\mathcal{M}_w}$ and, moreover, $v \leq w$ implies that $\mathcal{M}_v$ is an inner model of $\mathcal{M}_w$ (i.e., $\mathcal{M}_v$ is a transitive class in $\mathcal{M}_w$ that contains all the ordinals and satisfies the axioms of ZFC).

Lemma 3.14. *Let K be a Kripke frame and $\mathcal{M}$ a meticulous assignment. Let* L *be the constructible universe from the point of view of all models in $\mathcal{M}$. Then the following are equivalent for any formula $\varphi(x)$ in the language of set theory, and all parameters $a_0, \dots, a_{n-1} \in \mathrm{L}$:*

(i) $K(\mathcal{M}) \Vdash (\varphi(a_0, \dots, a_{n-1}))^{\mathrm{L}}$,

(ii) for all $v \in K$ we have $\mathcal{M}_v \vDash (\varphi(a_0, \dots, a_{n-1}))^{\mathrm{L}}$,

(iii) there is a $v \in K$ such that $\mathcal{M}_v \vDash (\varphi(a_0, \dots, a_{n-1}))^{\mathrm{L}}$*, and,*

(iv) $\mathrm{L} \vDash \varphi(a_0, \dots, a_{n-1})$.

Proof. Note that the $a_0, \dots, a_{n-1}$ can be accessed from every node as they are assumed to be members of the constructible universe. The equivalence of (ii), (iii) and (iv) follows directly from the fact L is absolute between inner models of ZFC. For the equivalence of (i) and (ii) observe that a relativised formula φ^{L} behaves like a Δ_0-formula because all nodes in $\mathcal{M}$ agree on L. It is then possible to prove the equivalence as in the proof of Proposition 3.4. □

The family of sentences that we are going to use to imitate the behaviour of propositional variables in a valuation of a Kripke frame has been presented by Friedman, Fuchino and Sakai [11]. Consider the following statements:

$$\text{There is an injection from } \aleph_{n+2}^{\mathrm{L}} \text{ to } \mathcal{P}(\aleph_n^{\mathrm{L}}).$$

There are different ways to formalise these statements that are classically equivalent. In our situation, we choose to define the sentence $b(n)$ like this:

$$\begin{aligned}\exists x \exists y \exists g ((x = \aleph_{n+2})^{\mathrm{L}} \wedge (y = \aleph_n)^{\mathrm{L}} & \\ & \wedge g \text{ “is an injective function”} \\ & \wedge \mathrm{dom}(g) = x \\ & \wedge \forall \alpha \in x \forall z \in g(\alpha)\ z \in y)\end{aligned}$$

The main reason for this choice of representation is that the existential quantifiers are evaluated locally, which will be handy in the proof of the following crucial observation.

Proposition 3.15. *Let K be a Kripke frame and $\mathcal{M}$ a meticulous assignment. Then $K(\mathcal{M}), v \Vdash b(n)$ if and only if $\mathcal{M}_v \vDash b(n)$.*

Proof. This follows from Lemma 3.14, Proposition 3.4 and the fact that the existential quantifier is evaluated locally, i.e., the injection g of the above statement must (or may not) be found within $\mathcal{M}_v$.

Assume that $K(\mathcal{M}), v \Vdash b(n)$, i.e., there are $x, y, g \in \mathcal{M}_v$ such that:

$$\begin{aligned}K(\mathcal{M}), v \Vdash (x = \aleph_{n+2})^{\mathrm{L}} \wedge (y = \aleph_n)^{\mathrm{L}} \wedge g \text{ “is an injective function”} & \\ \wedge \mathrm{dom}(g) = x & \\ \wedge \forall \alpha \in x \forall z \in g(\alpha)\ z \in y. &\end{aligned}$$

To show that this conjunction is evaluated locally, it suffices to argue that every conjunct is. For the first two conjuncts of the form φ^{L} this holds by Lemma 3.14. Moreover, note that the final three conjuncts are Δ_0-formulas. So we can apply Proposition 3.4 and the desired result follows. □

The construction of the assignment

We will now construct a collection of models of set theory that will later be used to construct a meticulous assignment. The forcings that we are going to use have been constructed in [11].

So let us begin by setting up the forcing construction. We start from some countable transitive constructible universe L (that is, a countable transitive model of set theory satisfying the axiom $V = \mathrm{L}$). Let $\mathbb{Q}_n$ be the forcing $\mathrm{Fn}(\aleph_{n+2}^{\mathrm{L}}, 2, \aleph_n^{\mathrm{L}})$, defined within L. Given $A \subseteq \omega$, let us define the forcings:

$$\mathbb{P}_n^A = \begin{cases} \mathbb{Q}_n, & \text{if } n \in A, \\ \mathbb{1}, & \text{otherwise.} \end{cases}$$

Then let $\mathbb{P}^A = \prod_{n<\omega} \mathbb{P}_n^A$ be the full support product of the forcing notions $\mathbb{P}_n^A$. Recall that the ordering $<$ on $\mathbb{P}^A$ is defined by $(a_i)_{i\in\omega} < (b_i)_{i\in\omega}$ if and only if $a_i <_i b_i$ for all $i \in \omega$. Now, let G be $\mathbb{P}^\omega$-generic over L, and let $G_n = \pi_n[G]$ be the n-th projection of G. Let H be the trivial generic filter on the trivial forcing $\mathbb{1}$. Now, for $A \subseteq \omega$ and $n \in \omega$ define the collection of filters

$$G_n^A = \begin{cases} G_n, & \text{if } n \in A, \\ H, & \text{otherwise.} \end{cases}$$

and let $G^A = \prod_{n<\omega} G_n^A$.

Proposition 3.16. *The filter G^A is $\mathbb{P}^A$-generic over* L.

Proof. We know that every G_n^A is a generic filter on $\mathbb{P}_n^A$ (cf. [22, Proposition 15.10]). So let us check that G^A is a generic filter. The fact that G^A is non-empty, upwards-closed and downwards-directed follows directly from the definition of the ordering on $\mathbb{P}^A$ and the fact that G^A has these properties component-wise.

So let $D \subseteq \mathbb{P}^A$ be dense. It follows that the projection $D_i = \pi_i[D]$ is dense for each $i < \omega$. By assumption, we can pick $p_i \in D_i \cap G_i^A$ for each $i < \omega$, and then, by definition, $(p_i)_{i<\omega} \in D \cap G^A$. □

Proposition 3.17. *If $A \subseteq B \subseteq \omega$ and $A \in \mathrm{L}[G^B]$, then $\mathrm{L}[G^A] \subseteq \mathrm{L}[G^B]$. Indeed, $\mathrm{L}[G^A]$ is an inner model of $\mathrm{L}[G^B]$.*

Proof. Work in $\mathrm{L}[G^B]$. We can define the collection of sets $(X_i)_{i<\omega}$ as follows, where H is the generic for $\mathbb{1}$,

$$X_i = \begin{cases} \pi_i[G^B], \text{if } i \in A, \\ H, \text{otherwise.} \end{cases}$$

This allows us to define $X = \prod_{i<\omega} X_i$.

Work in V. With transitivity, it is clear that $X = G^A$, and so it follows that $G^A \in \mathrm{L}[G^B]$. By the minimality of forcing extensions (cf. [22, Lemma 14.31]), it follows that $\mathrm{L}[G^A] \subseteq \mathrm{L}[G^B]$ is an inner model. □

The additional assumption $A \in \mathrm{L}[G^B]$ is necessary as otherwise (as we automatically add a real), we could have incomparable universes: Take a real c coding that L is countable. If we take a splitting c_0, c_1 of c, then $c_0, c_1 \subseteq \omega$ but not both $\mathrm{L}[G^{c_i}] \subseteq \mathrm{L}[G^\omega]$ can hold because otherwise $c \in \mathrm{L}[G^\omega]$. For a discussion of forcing extensions that cannot be amalgamated, see [13, Observation 35].

The following proposition, for whose proof we refer to the literature, shows that our construction works.

Proposition 3.18 (Friedman, Fuchino and Sakai, [11, Proposition 5.1])**.** *It holds that* $\mathrm{L}[G^A] \vDash b(n)$ *for all* $n \in A$ *and* $\mathrm{L}[G^A] \vDash \neg b(n)$ *for all* $n \in \omega \setminus A$.

The construction of the Iemhoff model

Theorem 3.19. *The Iemhoff models are faithful.*

Proof. Let K be a Kripke frame. By Proposition 2.42, we need to show that for any valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ on K, and every finite collection p_i, $i < n$ of propositional letters, there is a collection of set-theoretical sentences φ_i and an Iemhoff model $K(\mathcal{M})$ such that $\{v \in K \mid K(\mathcal{M}), v \Vdash \varphi_i\} = V(p_i)$ for all $i < n$.

Now, let $\bar{V}$ be the valuation with $\bar{V}(p_i) = V(p_i)$ for each p_i, $i < n$, and $\bar{V}(p) = \emptyset$ otherwise. Observe that $\bar{V}^{-1}(v)$ is finite for every $v \in K$ and we can define $A_v = \{i < \omega \mid v \in \bar{V}(p_i)\}$ for any $v \in K$. It holds that $A_v \in \mathrm{L}$ as it is a finite subset of ω. Note that $v \leq w \in K$ implies that $A_v \subseteq A_w$ by monotonicity of the original valuation V. Therefore, we will have by Proposition 3.17 that $\mathrm{L}[G^{A_v}]$ is an inner model of $\mathrm{L}[G^{A_w}]$ for all $v \leq w \in K$. Hence, the assignment $\mathcal{M} : K \to V$ with $\mathcal{M}_v = \mathrm{L}[G^{A_v}]$ is a meticulous assignment. This yields the Iemhoff model $K(\mathcal{M})$. Choose $\varphi_i = b(i)$ for all $i < n$. It then holds that:

$$
\begin{aligned}
K(\mathcal{M}), v \Vdash \varphi_i &\iff K(\mathcal{M}), v \Vdash b(i) && \text{(by the definition of } \sigma) \\
&\iff \mathcal{M}_v \vDash b(i) && \text{(by Proposition 3.15)} \\
&\iff \mathrm{L}[G^{A_v}] \vDash b(i) && \text{(by definition of } \mathcal{M}) \\
&\iff i \in A_v && \text{(by Proposition 3.18)} \\
&\iff p_i \in \bar{V}^{-1}(v) && \text{(by definition of } A_v) \\
&\iff K, \bar{V}, v \Vdash p_i.
\end{aligned}
$$

Therefore, it holds that $\{v \in K \mid K(\mathcal{M}), v \Vdash \varphi_i\} = V(p_i)$ for all $i < n$, and this shows that the Iemhoff models are faithful to K. Since K was chosen arbitrary, it follows that the Iemhoff models are faithful. □

In this situation, Proposition 2.54 implies the following result.

Corollary 3.20. *The class of Iemhoff models is loyal to any class of Kripke frames.* □

The corollary shows that, for example, the propositional logic of the linear Iemhoff models is Dummett's logic **LC**. The logic of the weak excluded middle **KC** is the underlying propositional logic of the finite Iemhoff models with a largest element (cf. [5] for details on logics that are characterised by a class of Kripke frames, also see Appendix A). The next corollary can be derived with Proposition 2.56.

Corollary 3.21. *The theory* CZF^* *has the de Jongh property with respect to every logic that is characterised by a class of Kripke frames.* □

Our limiting result Corollary 3.9 shows that we cannot easily push this method to derive the de Jongh property for stronger set theories than CZF^*, such as CZF or even IZF. Results connected to the de Jongh property for IZF can be found in [12].

Chapter 4

Lubarsky's Kripke Models for Intuitionistic and Constructive Set Theory

In this chapter, we will analyse another model-construction for intuitionistic set theory, mainly developed by Robert Lubarsky based on Kripke frames. These models have first been introduced in [27] to prove the independence of the power set axiom from the subset collection axiom over constructive set theory CZF. Later on, similar models have been constructed in [28] and [29] to discuss the constructive Cauchy and Dedekind reals. The models in [18] are used to separate *omniscience principles*, such as different weakenings of the law of excluded middle, in constructive mathematics.

4.1 Kripke Models for IZF and CZF

There are several different definitions of these Kripke models around; we follow the quite general one of [18].

Definition 4.1. Let $(K, \leq)$ be a Kripke frame. We will call a system $(M_v)_{v \in K}$ of models and elementary embeddings $(f_{vw})_{v \leq w \in K}$ *an elementary system of models of* ZFC *for* K, if it holds that $M_v \models \mathsf{ZFC}$ for every $v \in K$, $f_{vv} = \mathrm{id}_{M_v}$ for all $v \in K$ and $f_{wu} \circ f_{vw} = f_{vu}$ for all $v \leq w \leq u$.

Definition 4.2. Let K be a Kripke frame and $(M_v)_{v \in K}, (f_{vw})_{v \leq w \in K}$ be an elementary system of models of ZFC. Assume that K, in particular $K^{\geq v} = \{w \in K \mid w \geq v\}$, and $(M_w)_{w \geq v}$ are definable in each M_v.

At each node $v \in K$, we simultaneously define the domains D^v by induction on $\alpha \in \mathrm{Ord}^{M_v}$ and transition functions $g_{vw} : D^v \to D^w$ as follows. An object x of D^v_α is a function such that for all $w \geq v$,

(i) $\mathrm{dom}(x) = K^{\geq v}$,

(ii) $x \restriction K^{\geq w} \in M_w$,

(iii) $x(w) \subseteq \bigcup_{\beta < f_{vw}(\alpha)} D^w_\beta$,

(iv) if $h \in x(w)$ and $w \leq u$, then $g_{wu}(h) \in x(u)$.

Extend g_{wu} by $g_{wu}(x) = x \restriction K^{\geq u}$ for each $u \geq w$. Finally, let $D^v = \bigcup_{\alpha \in \mathrm{Ord}^{M_v}} D^v_\alpha$. Now, we can define the forcing relation in the Kripke model as follows:

(i) $v \Vdash x \in y$ if and only if $M_v \vDash x \in y(v)$,

(ii) $v \Vdash x = y$ if and only if $M_v \vDash x = y$,

and the other definitions being standard as in Kripke semantics for intuitionistic predicate logic. This constitutes the *full Lubarsky model* $K^{\mathrm{L}}(M)$.

Theorem 4.3 (Hendtlass-Lubarsky, [18, Theorem 3.1])**.** *The full Lubarsky model $K^{\mathrm{L}}(M)$ satisfies* IZF*.*

Proof. We will verify the axioms of IZF one-by-one. As we have to verify the axioms at all nodes of K, let $v \in K$.

Extensionality We need to show that $v \Vdash \forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \leftrightarrow a = b)$. To do so, it suffices to show that for all $a, b \in D^v$ we have that $v \Vdash \forall x (x \in a \leftrightarrow x \in b) \leftrightarrow a = b$. Assuming that $v \Vdash a = b$, we know that $a = b$ holds and therefore the right hand side follows directly. Conversely, if $v \Vdash \forall x (x \in a \leftrightarrow x \in b)$ this means that for all $w \geq v$ we have that $x \in a(w)$ if and only if $x \in b(w)$, i.e., $a = a \restriction K^{\geq v} = b \restriction K^{\geq v} = b$ (using Extensionality on the meta-level) and so $v \Vdash a = b$.

Emptyset To verify that $v \Vdash \exists a \; \forall x \in a \; \bot$ it suffices to construct a witness. So let $a \in D^v$ be the function with $a(w) = \emptyset$ for all $w \geq v$. Then $v \Vdash \forall x (x \in a \to \bot)$ which is what we needed to show.

Pairing It suffices to show that for all $a, b \in D^v$, there is a $y \in D^v$ such that $v \Vdash \forall x (x \in y \leftrightarrow (x = a \vee x = b))$. Let $y \in D^v$ be the function with $y(w) = \{g_{vw}(a), g_{vw}(b)\}$ for all $w \geq v$. Now, let $x \in D^w$ for some $w \geq v$. From these definitions it follows that $x \in g_{vw}(y)(w) = y(w)$ if and only if $x = g_{vw}(a)$ or $x = g_{vw}(b)$, and this is exactly what we had to prove.

Union Given $a \in D^v$, we will show that $v \Vdash \exists y \forall x (x \in y \leftrightarrow \exists u (u \in a \wedge x \in u))$. Again, this is done by providing a witness y. So let $y \in D^v$ be such that $y(w) = \bigcup_{u \in a(w)} u(w)$. Now let $w \geq v$ and $x \in D^w$. Then if $x \in y(w)$, it follows that $x \in u(w)$ for some $u \in a(w)$, i.e., $w \Vdash \exists u (u \in g_{vw}(a) \wedge x \in u)$. Conversely, if there is some $u \in D^w$ such that $w \Vdash u \in g_{vw}(a)$ and $w \Vdash x \in u$, then $u \in a(w)$ and $x \in u(w)$, so $x \in y(w)$. Hence, $w \Vdash x \in g_{vw}(y)$. This shows that Union holds.

Power set Given $a \in D^v$, we need to construct its power set $b \in D^v$ and verify that $v \Vdash \forall x (x \in b \leftrightarrow x \subseteq a)$. Let us first observe that by the definition of the semantics in the Lubarsky model, the assertion $w \Vdash x \subseteq a$ for some $w \geq v$ is equivalent to the following statement:

$$\text{at every node } u \geq w \text{ we have that } x(u) \subseteq a(u). \qquad (\mathrm{Sub}^a_w(x))$$

Note that we can extend any object $x \in D^w$, $w \geq v$, to an object $\hat{x} \in D^v$ by stipulating that $\hat{x}(u) = x(u)$ if $u \geq v$, and $\hat{x}(u) = \emptyset$ otherwise. Now, we can define the power set b of a to be the function with $b(v) = \{\hat{x} \mid \mathrm{Sub}^a_w(x)\}$ and $b(w) = g_{vw}[b(v)]$ for $w \geq v$. This function is well-defined by the fact that our Kripke frame K is a set, and therefore, the collection $b(v)$ is a set as well.

Using the above equivalence, it is now straightforward to see that b satisfies the definition of the power set: If $w \geq v$ and $w \Vdash x \in b$, then $w \Vdash x \subseteq a$ by definition of b. Conversely, if $w \Vdash x \subseteq a$, then we have $\mathrm{Sub}^a_w(x)$ and therefore, $x = g_{vw}(\hat{x}) \in b(w)$, i.e., $w \Vdash x \in b$.

Infinity We need to verify that $v \Vdash \exists a(\exists x\ x \in a \wedge \forall x \in a \exists y \in a\ x \in y)$. To do so, we will construct simultaneously at every node $v \in K$ objects n_v for every $n \in \omega$ and finally ω_v. Start with $0_v = \emptyset_v$, where $\emptyset_v$ is as in the above proof of the empty set axiom. Assume that m_v has already been constructed for all $w \geq v$ and $m < n$. Let $n_v(w) = \{m_w \mid m < n\}$ for $w \geq v$, and let $n_w = n_v \restriction K^{\geq w}$ for all $w \geq v$. Finally, set $\omega_v(w) = \{n_w \mid n < \omega\}$ for $w \geq v$.

Now, clearly $v \Vdash \exists x\ x \in \omega_v$. Further, observe that for all $n \in \omega$ we have that $v \Vdash n_v \in (n+1)_v$ holds by definition. Therefore,

$$v \Vdash \forall x \in \omega_v \exists y \in \omega_v\ x \in y$$

holds and this concludes the proof of Infinity.

Set Induction Let us assume, towards a contradiction, that Set Induction does not hold. That is, there is a node $v \in K$ such that:

$$v \Vdash (\forall x(\forall y \in x\ \varphi(y) \to \varphi(x))), \text{ but } v \nVdash \forall x \varphi(x)$$

for some formula $\varphi(x)$. So there is $w \geq v$ and $a \in D^w$ such that $v \nVdash \varphi(a)$. Then, by the hypothesis, we have that $w \nVdash \forall y \in a\ \varphi(y)$. So there is $u \geq w$ with $b \in D^w$ such that $w \Vdash b \in a$, but $w \nVdash \varphi(b)$.

Now, the statement that Set Induction fails at v is, in fact, a statement internal to M_v as we have (by our requirements on definability) that:

$$M_v \models \text{“There is a Lubarsky model with bottom element } \bot$$
$$\text{such that there is a counterexample to set-induction in } M_\bot\text{”}$$

As $M_v \models \mathsf{ZFC}$, we can take a rank-minimal counterexample a to this. Then, by our observation above, there is $w \geq v$ and $b \in a(w)$ such that b is a counterexample to set induction as well, but within M_w. In M_v, the rank of b is lower than the rank of a. By elementarity, it holds that the rank of $f_{vw}(b)$ is lower than the rank of $f_{vw}(a)$. However, elementarity also implies that the rank of $f_{vw}(a)$ is minimal of the ranks of counterexamples. A contradiction. Hence, Set Induction holds.

Separation The verification of Separation, reduces again to the construction of a witness. So let $a \in D^v$ and $\varphi(x)$ be given. We define $b \in D^v$ to be the function with $b(w) = \{x \in a(w) \mid w \Vdash \varphi(x)\}$. To verify is that $\forall x(x \in b \leftrightarrow (x \in a \wedge \varphi(x)))$ holds, but this follows directly from the definition of b. Note that this definition is only possibly because we are using again the fact that we can internalise the Lubarsky model at every node.

Collection Given a formula $\varphi(x, y)$ and an element $a \in D^v$ such that $v \Vdash \forall x \in a \exists y\, \varphi(x, y)$, we need to find an object $b \in D^v$ with $v \Vdash \forall x \in a \exists y \in b\, \varphi(x, y)$. As Collection is a theorem of ZFC, which holds in M_v, we can apply it to a to obtain a set $b_w \subseteq D^w$ from the formula $\psi_w(x, y)$, defined as "$y \in D^w \wedge w \Vdash \varphi(g_{vw}(x), y)$." Note that this set is non-empty whenever $a(w)$ is non-empty by our hypothesis. We can then define $b \in D^v$ to be the function with $b(w) = b_w$ for all $w \geq v$. This will indeed be an element of D^v (using monotonicity). It follows now directly from this definition that $v \Vdash \forall x \in a \exists y \in b\, \varphi(x, y)$.

This finishes the verification of all the axioms of IZF and concludes the proof of the theorem. □

Not only the Collection scheme but also the Reflection scheme holds in the above model (see [18]). Let us now give an example of such a model before moving on to the construction of a class of models of CZF.

Example 4.4. Let M be a model of ZFC, and take $K = \{v, w\}$ to be the Kripke frame ordered by the reflexive closure $\leq$ of $\{(v, w)\}$. Take $M_v = M_w = M$ and the identity for the embedding. By the previous theorem, the full Lubarsky model $K^{\mathrm{L}}(M)$ is a model of IZF, and there are three subsets of $1 = \{\emptyset\}$ at the bottom node v: The element $0 \in D^v$ with $0(v) = 0(w) = \emptyset$, the element $1 \in D^0$ with $1(v) = 1(w) = \{\emptyset\}$, and the element $\frac{1}{2} \in D^0$ with $\frac{1}{2}(v) = \emptyset$ and $\frac{1}{2}(w) = \{\emptyset\}$. It then holds that $v \Vdash 0 \subseteq 1 \wedge 1 \subseteq 1 \wedge \frac{1}{2} \subseteq 1$. However, $v \nVdash 0 = \frac{1}{2}$ and $v \nVdash \frac{1}{2} = 1$.

In the remainder of this section, we will exhibit a theory of Kripke models for CZF. Building on the models for IZF defined above, we want to give an alternative proof for a result presented in [27]. Our proof will avoid the class constructions done by Lubarsky.[1]

[1] Lubarsky already mentioned in [27, p. 3] that such a construction would be possible, "Alternatively, [the partial order] could be taken to be ω and the Kripke sets to be eventually hereditarily constant. In fact, this latter approach could be read off from the former by taking a cofinal ω-sequence through [the ordinals] (from outside of V, naturally) and cutting the full model down to those nodes." Our construction will not strictly follow this suggestion, but rather bound the size of the elements at each domain.

Definition 4.5. Let K be a Kripke frame with a root 0 and let $(M_v)_{v\in K}$, $(f_{vw})_{v\leq w\in K}$ be an elementary system of models of ZFC such that $\kappa \in M_0$ with $M_0 \models$ "κ is an inaccessible cardinal." Assume that K, in particular $K^{\geq v} = \{w \in K \mid w \geq v\}$, and $(M_w)_{w\geq v}$ are definable in M_v.

At each node $v \in K$ we simultaneously define the domains D^v by induction on $\alpha \in \mathrm{Ord}^{M_v}$ and transition functions $g_{vw} : D^v \to D^w$ as follows. An object x of D^v_α is obtained as in the definition of the full Lubarsky model with the additional requirement that $M_w \models |x(w)| < f_{0w}(\kappa)$ for all $w \geq v$. The forcing relation is defined as in the full Lubarsky model. This constitutes the *bounded Lubarsky model* $K^L_\kappa(M)$.

Note that this is, by definition, a submodel of the full Lubarsky model (cf. Definition 4.2). To show that this class of models indeed satisfies CZF follows exactly along the lines of Theorem 4.3 observing that the size-restriction does not impact the axioms of CZF. In fact, every axiom except for Powerset goes through as in the proof of that theorem.

Theorem 4.6. *The bounded Lubarsky model satisfies* CZF. □

There is a bounded Lubarsky model in which the negation of the power set axiom holds true.

Theorem 4.7. *Let M be a transitive model of* ZFC *set theory and $\kappa \in M$ such that $M \models$ "κ is an inaccessible cardinal." Let K be the Kripke frame $(\kappa, <)$, and consider the system where $M_v = M$ for all $v \in K$. Then $K^L_\kappa(M) \models$* CZF + ¬Powerset.

Proof. The previous theorem shows that $K^L_\kappa(M)$ is a model of CZF. Given $v \in \kappa$, let us consider the element $h_v \in D^0$ defined by

$$h_v(w) = \begin{cases} \emptyset, & \text{if } w < v, \\ \{\emptyset\}, & \text{if } w \geq v. \end{cases}$$

It is clear that $0 \Vdash h_v \subseteq \{\emptyset\}$ for all $v \in K$ (where $\{\emptyset\} = h_0$). Now, assume towards a contradiction that x is the power set of $\{\emptyset\}$, then $0 \Vdash h_v \in x$ for all v. By definition of our semantics this implies that $x \supseteq \{h_v \mid v \in K\}$, but as $h_v \neq h_w$ for $v \neq w$, it holds that the cardinality of x is at least the cardinality of κ. A contradiction to $K^L_\kappa(M)$ being a bounded model. □

Corollary 4.8 (Lubarsky, [27, Theorem 2.0.3])**.** *The power set axiom is independent of* CZF. □

4.2 The Propositional Logic of Lubarsky Models for IZF

Our next objective is an analysis of the propositional logic of the class of full Lubarsky models. To do so, we interpret Lubarsky models as Heyting structures along the lines of Theorem 2.34.

Technical Preliminaries

We would like to first exhibit a technical result that will greatly simplify our analysis of the underlying logic of Lubarsky models.

Let us first observe that, given a Kripke frame K, we can define a Kripke frame $K/\sim$ as the quotient of K under the equivalence relation $\sim$, defined by

$$v \sim w \text{ if and only if } v \leq w \text{ and } w \leq w.$$

The Kripke frame $K/\sim$ is then partially ordered.

Now, let $K^{\mathrm{L}}(M)$ be a Lubarsky model based on any Kripke frame K and an elementary system $(M_v)_{v\in K}$, $(f_{vw})_{v\leq w\in K}$. We will construct a Lubarsky model based on $K/\sim$.

Note that if $v \sim w$ holds, then there are elementary embeddings $f : M_v \to M_w$ and $g : M_w \to M_v$ such that $f \circ g = \mathrm{id}_{M_w}$ and $g \circ f = \mathrm{id}_{M_v}$, i.e., M_v and M_w are isomorphic. We will therefore assume without loss of generality that $M_v = M_w$ for all $v \sim w$. Then the elementary system $\tilde{M}$ with $\tilde{M}_{[v]} = M_v$ is well-defined. We can therefore consider the Lubarsky model $(K/\sim)^L(\tilde{M})$ with domains $D^{[v]}$.

Now, we are going to observe what happens to the elements of our domains when taking the quotient. As the domains of the elements are upwards-closed it follows that if $v \sim w$ and $v \in \mathrm{dom}(a)$ for some $a \in D^u$, then $w \in \mathrm{dom}(a)$. Hence, $v \in \mathrm{dom}(a)$ if and only if $w \in \mathrm{dom}(a)$.

Proposition 4.9. *If $v, w \in \mathrm{dom}(a)$ for some $a \in D^u$ and $v \sim w$, then $a(v) = a(w)$.*

Proof. We assumed without loss of generality that $M_v = M_w$, and therefore, the transition functions are the identity. By property (iv) in Definition 4.2, it then follows from $v \sim w$ that $a(v) \subseteq a(w)$ and that $a(w) \subseteq a(v)$. □

Recursively on $\alpha \in \mathrm{Ord}^{M_u}$ for $a \in D^u_\alpha$, we can define the map $f_u : D^u \to D^{[u]}$ by $(f_u(a))([v]) = f_u[a(v)]$. It is well-defined by the previous proposition. We will write $\tilde{a}$ for $f_u(a)$ whenever u is clear from the context. Note that f_u is a bijection as the converse map $f_u^{-1} : D^{[u]} \to D^u$ is given by $f_u^{-1}(b)(v) = f_u^{-1}[b([v])]$. Moreover, the maps f_u commute with the transition functions in the sense that $f_u \circ f_{wu} = f_{[w][u]} \circ f_w$.

Theorem 4.10. *Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula in the language of set theory. It is the case that*

$$K^{\mathrm{L}}(M), v \Vdash \varphi(a_0, \ldots, a_{n-1}) \text{ if and only if } (K/\sim)^L(\tilde{M}), [v] \Vdash \varphi(\tilde{a}_0, \ldots, \tilde{a}_{n-1}).$$

Proof. We prove the theorem by induction on the complexity of the formula φ, simultaneously for all $v \in K$.

The case for $\bot$ follows trivially. The cases for equality and the $\in$-relation follow easily from the definitions made above:

$$\begin{aligned} v \Vdash a \in b &\iff a \in b(v) \\ &\iff \tilde{a} \in \tilde{b}([v]) && \text{(by definition of } f_v\text{)} \\ &\iff [v] \Vdash \tilde{a} \in \tilde{b} \end{aligned}$$

$$\begin{aligned} v \Vdash a = b &\iff a = b \\ &\iff \tilde{a} = \tilde{b} && (f_v \text{ bijective}) \\ &\iff [v] \Vdash \tilde{a} = \tilde{b}. \end{aligned}$$

The statement follows for $\wedge$ and $\vee$ in the same way, so let us only consider one of the two cases:

$$\begin{aligned} v \Vdash &\varphi(a_0, \dots, a_{n-1}) \wedge \psi(b_0, \dots, b_{m-1}) \\ &\iff v \Vdash \varphi(a_0, \dots, a_{n-1}) \text{ and } v \Vdash \psi(b_0, \dots, b_{m-1}) \\ &\iff [v] \Vdash \varphi(\tilde{a}_0, \dots, \tilde{a}_{n-1}) \text{ and } [v] \Vdash \psi(\tilde{b}_0, \dots, \tilde{b}_{m-1}) && \text{(by I.H.)} \\ &\iff [v] \Vdash \varphi(\tilde{a}_0, \dots, \tilde{a}_{n-1}) \wedge \psi(\tilde{b}_0, \dots, \tilde{b}_{m-1}). \end{aligned}$$

For the case of the implication, we will use the commutativity of the transition functions and the bijections $f_v : D^v \to D^{[v]}$.

$$\begin{aligned} v \Vdash &\varphi(a_0, \dots, a_{n-1}) \to \psi(b_0, \dots, b_{m-1}) \\ &\iff \forall w \geq v (w \Vdash \varphi(f_{vw}(a_0), \dots, f_{vw}(a_{n-1})) \\ &\qquad\qquad \text{implies } w \Vdash \psi(f_{vw}(b_0), \dots, f_{vw}(b_{m-1}))) \\ &\iff \forall w \geq v ([w] \Vdash \varphi(f_w(f_{vw}(a_0)), \dots, f_w(f_{vw}(a_{n-1}))) \\ &\qquad\qquad \text{implies } [w] \Vdash \psi(f_w(f_{vw}(b_0)), \dots, f_w(f_{vw}(b_{m-1})))) \\ &\iff \forall w \geq v ([w] \Vdash \varphi(f_{[v][w]}(f_v(a_0)), \dots, f_{[v][w]}(f_v(a_{n-1}))) \\ &\qquad\qquad \text{implies } [w] \Vdash \psi(f_{[v][w]}(f_v(b_0)), \dots, f_{[v][w]}(f_v(b_{m-1})))) \\ &\iff \forall [w] \geq [v] ([w] \Vdash \varphi(f_{[v][w]}(f_v(a_0)), \dots, f_{[v][w]}(f_v(a_{n-1}))) \\ &\qquad\qquad \text{implies } [w] \Vdash \psi(f_{[v][w]}(f_v(b_0)), \dots, f_{[v][w]}(f_v(b_{m-1})))) \\ &\iff [v] \Vdash \varphi(\tilde{a}_0, \dots, \tilde{a}_{n-1}) \to \psi(\tilde{b}_0, \dots, \tilde{b}_{m-1}). \end{aligned}$$

The final two cases are the existential and the universal quantifier. Let us begin with the existential quantifier:

$$\begin{aligned} v \Vdash &\exists x\ \varphi(x, a_0, \dots, a_{n-1}) \\ &\iff \text{there exists } x \in D^v \text{ such that } v \Vdash \varphi(x, a_0, \dots, a_{n-1}) \\ &\iff \text{there exists } x \in D^v \text{ such that } [v] \Vdash \varphi(\tilde{x}, \tilde{a}_0, \dots, \tilde{a}_{n-1}) \\ &\iff \text{there exists } x \in D^{[v]} \text{ such that } [v] \Vdash \varphi(x, \tilde{a}_0, \dots, \tilde{a}_{n-1}) && (1) \\ &\iff [v] \Vdash \exists x\ \varphi(x, \tilde{a}_0, \dots, \tilde{a}_{n-1}). \end{aligned}$$

Note that we are relying on the fact that f_v is a bijection when we change the domain of quantification in equivalence (1). The same argument will be applied in the final case, the universal quantifier for equivalence (2):

$$\begin{aligned} v \Vdash & \forall x\ \varphi(x, a_0, \ldots, a_{n-1}) \\ & \iff \text{for all } w \geq v \text{ and for all } x \in D^w\ w \Vdash \varphi(x, a_0, \ldots, a_{n-1}) \\ & \iff \text{for all } w \geq v \text{ and for all } x \in D^w\ [w] \Vdash \varphi(\tilde{x}, \tilde{a}_0, \ldots, \tilde{a}_{n-1}) \\ & \iff \text{for all } [w] \geq [v] \text{ and for all } x \in D^{[w]}\ [w] \Vdash \varphi(x, \tilde{a}_0, \ldots, \tilde{a}_{n-1}) \quad (2) \\ & \iff [v] \Vdash \forall x \varphi(x, \tilde{a}_0, \ldots, \tilde{a}_{n-1}). \end{aligned}$$

This finishes the proof of the theorem. □

The discussion in this section and the theorem above allow us to directly deduce the following corollary.

Corollary 4.11. *For every full Lubarsky model there exists a full Lubarsky model whose Kripke frame is a partial order such that the propositional logics of the two models coincide.*

Proof. Let $K^{\mathrm{L}}(M)$ be a full Lubarsky model, and let $(K/{\sim})^{\mathrm{L}}(M)$ be the corresponding model whose Kripke frame is a partial order. Suppose that φ is an $\mathcal{L}_{\in}$-sentence with $K^{\mathrm{L}}(M) \Vdash \varphi$. By Theorem 4.10, this is equivalent to $(K/{\sim})^{\mathrm{L}}(M) \Vdash \varphi$. Hence, $\mathbf{L}^{\mathsf{Prop}}(K^{\mathrm{L}}(M)) = \mathbf{L}^{\mathsf{Prop}}((K/{\sim})^{\mathrm{L}}(M))$. □

Analysis of the Propositional Logic

We shall now show that faithfulness fails in a very strong way for full Lubarsky models. Let us say that two nodes v, w of a Kripke frame $(K, <)$ are in the *same component* if for the transitive reflexive closure R of $<$ it holds that vRw.

Theorem 4.12. *The class of full Lubarsky models is not faithful to any Kripke frame K with two distinct end-points in the same component.*

We will prove Theorem 4.12 with help of the following lemma.

Lemma 4.13. *Let $K^{\mathrm{L}}(M)$ be a full Lubarsky model, and let $v_0, v_1 \in K$ be end-nodes (i.e., nodes without a proper successor) in the same component. Then for any sentence φ in the language of set theory it holds that $K^{\mathrm{L}}(M), v_0 \Vdash \varphi$ if and only if $K^{\mathrm{L}}(M), v_1 \Vdash \varphi$.*

Proof. Let us first observe that $v_i \Vdash \varphi$ is equivalent to $M_{v_i} \vDash \varphi$: As v_i does not have any proper successor, the partial order $K^{\geq v_i}$ consists of only one point. By Definition 4.2, the domain D^{v_i} will be an isomorphic copy of M inside M. Using this and the fact that the definitions of forcing in a one-point Kripke frame collapse to the case of classical models, it can be shown by a short induction on formulas that $v_i \Vdash \psi$ if and only if $M_{v_i} \vDash \psi$. Then, as M_{v_0} and M_{v_1} are elementary equivalent (v_0 and v_1 are in the same component), it holds that $M_{v_0} \vDash \varphi$ if and only if $M_{v_1} \vDash \varphi$ for any sentence φ, and the proposition follows. □

We are now prepared to prove the theorem.

Proof of Theorem 4.12. Let K be a Kripke frame with two distinct end-nodes v_0 and v_1 in the same component of K. By Proposition 2.42 it suffices to show that there is a valuation $V : \mathsf{Prop} \to \mathcal{P}(K)$ and a propositional letter p that we cannot imitate by a sentence φ in the language of set theory. So let V be any valuation with $V(p) = \{v_0\}$, and $K^{\mathrm{L}}(M)$ be any full Lubarsky model based on K. Now, by Lemma 4.13, for any sentence φ with $v_0 \Vdash \varphi$ it holds that $\{v \in K \mid K^{\mathrm{L}}(M), v \Vdash \varphi\} \supseteq \{v_0, v_1\}$. Hence, for all sentences φ we have that $\{v \in K \mid K^{\mathrm{L}}(M), v \Vdash \varphi\} \neq V(p)$. This shows that the Lubarsky models are not faithful to K. □

The following is a direct consequence of the Theorem 4.12.

Corollary 4.14. *The class of full Lubarsky models is not faithful.* □

Regarding the loyalty of Lubarsky models, let us make the following observation.

Theorem 4.15. *Let $\mathcal{C}^{\mathrm{L}}$ be the class of full Lubarsky models. Let K be a Kripke frame in which every node has a successor that is an end-node. Then it holds that $\mathbf{KC} \subseteq \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}^{\mathrm{L}}_K)$.*

Proof. We will show that weak excluded middle holds in $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}^{\mathrm{L}}_K)$. Let φ be a sentence in the language of set theory. By Lemma 4.13, we know that there are two cases. Without loss of generality, we can assume that all nodes in K are in the same component (i.e., K has one component).

In the first case, it holds that $v \Vdash \varphi$ in all end-nodes v. Then, by our assumption, for all $w \in K$ and $u \geq w$, there exists some end-node $v \geq u$ with $v \Vdash \varphi$. Hence, $w \Vdash \neg\neg\varphi$.

In the second case, it holds that $v \nVdash \varphi$ in all end-nodes v. Therefore, $v \Vdash \neg\varphi$ in all end-nodes v. An analogous argument to the first case show that $w \Vdash \neg\neg\neg\varphi$ for all $w \in K$, but this means $w \Vdash \neg\varphi$ for all $w \in K$, by the fact that $\neg\neg\neg\varphi \to \neg\varphi$ is a theorem of **IQC**.

Putting the two cases together, it holds that $w \Vdash \neg\varphi \vee \neg\neg\varphi$ for all $w \in K$ and sentences φ in the language of set theory. Hence, $\neg p \vee \neg\neg p \in \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}^{\mathrm{L}}_K)$, i.e., $\mathbf{KC} = \mathbf{IPC} + \neg p \vee \neg\neg p \subseteq \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}^{\mathrm{L}}_K)$. □

Let us say that a full Lubarsky model is finite if its underlying Kripke frame is finite.

Corollary 4.16. *The class of finite full Lubarsky models is not loyal.*

Proof. Let $K^{\mathrm{L}}(M)$ be a finite full Lubarsky model. Then consider the model $(K/{\sim})^L(\tilde{M})$ that is based on a partial order Kripke frame. By Corollary 4.11, the propositional logic of both models coincide. Since every finite partial order has the property that every node has a successor that is an end-node, the previous Theorem 4.15 implies for the class $\mathcal{C}^{\mathrm{L}}_{\mathsf{fin}}$ of finite Lubarsky models that

$$\mathbf{L}^{\mathsf{Prop}}(\mathcal{H}_{\mathcal{C}^{\mathrm{L}}_{\mathsf{fin}}}) = \mathbf{IPC} \subsetneq \mathbf{KC} \subseteq \bigcap_{C \in \mathcal{C}^{\mathrm{L}}_{\mathsf{fin}}} \mathbf{L}^{\mathsf{Prop}}(C) = \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}^{\mathrm{L}}_{\mathsf{fin}}).$$

However, the logic of the class of finite Kripke frames is **IPC**. □

Chapter 5

Heyting-Valued Models for IZF

This chapter deals with Heyting-valued models for IZF. A special case of these models are the Boolean-valued models that are a means to the fruitful theory of forcing in set theory. After a short introduction to the necessary technical details of the theory of Heyting-valued models, we will analyse their propositional logic and derive corollaries for the Boolean-valued models, as well.

5.1 Heyting-Valued Models

Our exposition follows [2], to which we also refer for a complete exposition of Boolean- and Heyting-valued models of set theory. Here, we will only provide the necessary details for our analysis for the propositional logic without giving a proper introduction into the set-theoretic power of this theory.

Definition 5.1. Let H be a complete Heyting algebra and $M \vDash \mathsf{ZFC}$. We define $M_\alpha^{(H)}$ within M recursively on $\alpha \in \mathrm{Ord}^M$ as follows:

$$M_\alpha^{(H)} = \{x \in M \mid \text{“}x \text{ is a function”} \wedge \mathrm{ran}(x) \subseteq H \wedge \exists \xi < \alpha(\mathrm{dom}(x) \subseteq M_\xi^{(H)})\}$$

Then, let $M^{(H)} = \bigcup_{\alpha \in \mathrm{Ord}^M} M_\alpha^{(H)}$.

To allow us to make statements about $M^{(H)}$, we need to augment our first-order language of set theory $\mathcal{L}_\in$ to $\mathcal{L}_\in^{(H),M}$ additionally containing a constant symbol for every element of $M^{(H)}$ (note that by a standard coding argument, $M^{(H)}$ can be construed as a definable class within M). This augmented language is also called *the forcing language*. With this definition at hand, we may define the forcing relation in $M^{(H)}$.

Definition 5.2. We inductively define a map from the $\mathcal{L}_\in^{(H),M}$-sentences to values in the complete Heyting algebra. First, by simultaneous induction on $M_\alpha^{(H)}$, $\alpha \in \mathrm{Ord}^M$, we define $=$ and $\in$ as follows:

(i)
$$[\![u \in v]\!]^H = \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![u = y]\!]^H),$$

(ii)

$$[\![u = v]\!]^H = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \to [\![x \in v]\!]^H) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(x) \to [\![y \in u]\!]^H).$$

The definitions of the other cases are standard:

(iii) $[\![\bot]\!] = \bot$,

(iv) $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$,

(v) $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$,

(vi) $[\![\varphi \to \psi]\!] = [\![\varphi]\!] \to [\![\psi]\!]$,

(vii) $[\![\exists x \varphi(x)]\!] = \bigvee_{u \in M^{(H)}} [\![\varphi(u)]\!]$,

(viii) $[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in M^{(H)}} [\![\varphi(u)]\!]$.

As usual, $\neg\varphi$ abbreviates $\varphi \to \bot$ and $\varphi \leftrightarrow \psi$ denotes $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Note that the above definition is a definition of truth in $M^{(H)}$ and can therefore only be fully formalised externally, as we did here. The definitions of the quantifiers are well-defined as H is assumed to be a complete Heyting algebra. For the remainder of this section, let M be a model of set theory, H be a Heyting algebra and $M^{(H)}$ be the Heyting-valued model. The following is an analogue of [2, Theorem 1.17] for Heyting algebras, see also [2, chapter 8].

Theorem 5.3 ([2, Theorem 1.17 and p. 165])**.** *All axioms of first-order intuitionistic logic are true in $M^{(H)}$, and all its rules of interference are valid. Further, the following hold:*

(i) $[\![u = u]\!] = 1$,

(ii) $u(x) \leq [\![x \in u]\!]$ *for all* $x \in \mathrm{dom}(u)$,

(iii) $[\![u = v]\!] = [\![v = u]\!]$,

(iv) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$,

(v) $[\![u = v]\!] \wedge [\![v \in w]\!] \leq [\![u \in w]\!]$,

(vi) $[\![v = w]\!] \wedge [\![u \in v]\!] \leq [\![u \in w]\!]$, *and,*

(vii) $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]$.

It is now clear that Heyting-valued models are an instance of Heyting structures, where the underlying H-set is $M^{(H)}$ with equality and set membership defined as above. The definitions for the existential and universal quantifier in the definition of forcing for $M^{(H)}$ are exactly what is required in the definition of an H-structure, using that $E(a) = [\![a = a]\!] = 1$ for all $a \in M^{(H)}$. Moreover, $\tilde{e}(a, b) = [\![a = b]\!]$ by (i); and so (v) shows that $\in$ respects equivalence.

We only mention the following results without proof.

Theorem 5.4 ([2, p. 166])**.** *The law of excluded middle holds in $M^{(H)}$ if and only if H is a Boolean algebra.*

Theorem 5.5 ([2, Theorem 1.33 and p. 165])**.** *If H is a complete Heyting algebra and M a model of* ZFC*, then all the axioms of* IZF *are true in $M^{(H)}$.*

Theorem 5.6 ([2, Theorem 1.33])**.** *If B is a complete Boolean algebra and M a model of* ZFC*, then all the axioms of* ZFC *are true in $M^{(B)}$.*

5.2 Heyting-Valued Models as Full Lubarsky Models

In this section, we will express a certain class of Heyting-valued models as full Lubarsky models. This will be possible for those Heyting-valued models whose underlying Heyting algebra is one that consists of the upsets of a Kripke frame. The full Lubarsky model will then be based on that Kripke frame.

Let H be a complete Heyting algebra such that $H = \mathrm{Up}(K)$ for some Kripke frame $(K, \leq)$, M be a model of ZFC and $M^{(H)}$ the resulting Heyting-valued model. Further, let $K^{\mathrm{L}}(M)$ be the full Lubarsky model where the elementary directed system is the one with $M_v = M$ for all $v \in K$, and all transition functions are the identity on M. Note that the transition functions $f_{vw} : D^v \to D^w$ are the restriction maps $x \mapsto x{\restriction}K^{\geq w}$.

We begin by constructing a collection of surjections $f_v : M^{(H)} \to D^v$ for every $v \in K$ simultaneously. We will do so recursively on $\alpha \in \mathrm{Ord}^M$, but externally to M (note that Ord^M is an ordinal in V). Let $x \in M^{(H)}_\alpha$. By our recursive assumption, $f_v(y)$ is defined for every $v \in K$ and $y \in M^{(H)}_\beta$ for some $\beta < \alpha$. Let us define $f_v(x)$ as follows:

$$\begin{aligned} &f_v(x) : K^{\geq v} \to V, \\ &f_v(x)(w) = f_w[\{y \in M^{(H)} \mid w \in x(y)\}]. \end{aligned}$$

By the recursive assumption, every $f_w(y)$ is a well-defined element of D^w and therefore, the definition of $f_v(x)$ is well-defined as an element of D^v. We can now prove the following essential lemma.

Lemma 5.7. *For every $v \in K$, the map f_v is a surjection.*

Proof. We will provide the witnesses by induction on the complexity of $x \in D^v_\alpha$. So let us assume that for every $a \in D^w_\beta$, $\beta < \alpha$, we already constructed an element $b \in M^{(H)}$ such that $f_v(b) = a$. Now define $y \in M^{(H)}$ as follows:

$$\begin{aligned} &y : M^{(H)} \to H = \mathrm{Up}(K), \text{ with} \\ &\mathrm{dom}(y) = \{b \in M^{(H)} \mid \exists w \geq v \ \exists a \in x(w) \ f_w(b) = a\}, \\ &y(b) = \{w \in K \mid w \geq v \wedge \exists a \in x(w) \text{ such that } f_w(b) = a\}. \end{aligned}$$

We need to show that $f_v(y) = x$. So let $w \geq v$.

$$\begin{aligned} f_v(y)(w) &= f_w[\{b \in M^{(H)} \mid w \in y(b)\}] \\ &= f_w[\{b \in M^{(H)} \mid w \in \{w \in K \mid w \geq v \wedge \exists a \in x(w)\ f_w(b) = a\}\}] \\ &= f_w[\{b \in M^{(H)} \mid \exists a \in x(w)\ f_w(b) = a\}] \qquad (\star) \\ &\subseteq x(w). \end{aligned}$$

For the converse direction of the inequality, we use our induction hypothesis: f_w is a surjection onto $\bigcup_{\beta<\alpha} D^w_\beta$. So let $a \in x(w)$. Then, in particular, $a \in \bigcup_{\beta<\alpha} D^w_\beta$. Hence, there must be some $b \in M^{(H)}$ such that $f_w(b) = a$. Clearly, b is a member of the set in $(\star)$, and so we are allowed to conclude that $f_v(y)(w) = x(w)$.

As the derivation holds for all $w \geq v$, we have shown that $f_v(y) = x$. This finishes the proof of the lemma. □

We can now prove the main theorem of this section.

Theorem 5.8. *For every formula $\varphi(x_0, \ldots, x_{n-1})$ and all $a_0, \ldots, a_{n-1} \in M^{(H)}$ it is the case that:*

$$[\![\varphi(a_0, \ldots, a_{n-1})]\!]^H = \{v \in K \mid K^{\mathrm{L}}(M), v \Vdash \varphi(f_v(a_0), \ldots, f_v(a_{n-1}))\}.$$

Proof. Before proving the theorem by induction on the complexity of the formulas, let us make some general observations. By the definitions made above, it holds that:

$$f_v(y) \in f_v(b)(v) \text{ if and only if } v \in b(y), \qquad \text{(M)}$$

for any $y \in \mathrm{dom}(b)$ and $b \in M^{(H)}$. Another useful observation is the fact that our maps $f_v : M^{(H)} \to D^v$ commute with the transition functions $f_{vw} : D^v \to D^w$ in the sense that:

$$f_{vw} \circ f_v = f_w \qquad \text{(C)}$$

holds for all $v \leq w \in K$. This follows from the fact that the transition functions are truncations of the form $x \mapsto x \restriction K^{\geq w}$.

We can now begin with the actual proof. The cases for equality and set-membership follow via a simultaneous induction. Let us begin with the case for $\in$:

$$\begin{aligned} [\![a \in b]\!]^H &= \bigvee_{y \in \mathrm{dom}(b)} (b(y) \wedge [\![a = b]\!]^H) \\ &= \{v \in K \mid \exists y \in \mathrm{dom}(b)(v \in b(y) \wedge [\![a = y]\!]^H)\} \\ &= \{v \in K \mid \exists y \in \mathrm{dom}(b)(f_v(y) \in f_v(b)(v) \wedge v \Vdash f_v(a) = f_v(y))\} \\ &\qquad \text{(by (M) and I.H.)} \\ &= \{v \in K \mid \exists y \in \mathrm{dom}(b)(f_v(y) \in f_v(b)(v) \wedge f_v(a) = f_v(y))\} \\ &\qquad \text{(Definition of = in } K^{\mathrm{L}}(M)) \\ &= \{v \in K \mid f_v(a) \in f_v(b)(v)\} \qquad \text{(using (M))} \\ &= \{v \in K \mid v \Vdash f_v(a) \in f_v(b)\}. \qquad \text{(Definition of } \in \text{ in } K^{\mathrm{L}}(M)) \end{aligned}$$

Similarly, we can prove the case for equality:

$$
\begin{aligned}
[\![a = b]\!]^H &= \bigwedge_{x \in \mathrm{dom}(a)} (a(x) \to [\![x \in b]\!]^H) \wedge \bigwedge_{y \in \mathrm{dom}(b)} (b(y) \to [\![y \in a]\!]^H) \\
&= \{v \in K \mid \forall x \in \mathrm{dom}(a) \forall w \geq v (w \in a(x) \to w \in [\![x \in b]\!]^H) \\
&\qquad \wedge \forall y \in \mathrm{dom}(b) \forall w \geq v (w \in b(y) \to w \in [\![y \in a]\!]^H)\} \\
&= \{v \in K \mid \forall x \in \mathrm{dom}(a) \forall w \geq v \\
&\qquad\qquad (f_w(x) \in f_w(a)(w) \to w \Vdash f_w(x) \in f_w(b)) \\
&\qquad \wedge \forall y \in \mathrm{dom}(b) \forall w \geq v \\
&\qquad\qquad (f_w(y) \in f_w(b)(w) \to w \Vdash f_w(y) \in f_w(a))\} \\
&\qquad\qquad\qquad \text{(by (M) and I.H.)} \\
&= \{v \in K \mid \forall x \in \mathrm{dom}(a) \forall w \geq v \\
&\qquad\qquad (f_w(x) \in f_w(a)(w) \to f_w(x) \in f_w(b)(w)) \\
&\qquad \wedge \forall y \in \mathrm{dom}(b) \forall w \geq v \\
&\qquad\qquad (f_w(y) \in f_w(b)(w) \to f_w(y) \in f_w(a)(w))\} \\
&= \{v \in K \mid \forall w \geq v (f_w(a)(w) = f_w(b)(w))\} && \text{(by (M))} \\
&= \{v \in K \mid \forall w \geq v (f_{vw}(f_v(a))(w) = f_{vw}(f_v(b))(w))\} && \text{(by (C))} \\
&= \{v \in K \mid \forall w \geq v (f_v(a)(w) = f_v(b)(w))\} \\
&\qquad\qquad\qquad \text{(transition functions are truncations)} \\
&= \{v \in K \mid f_v(a) = f_v(b)\} \\
&= \{v \in K \mid v \Vdash f_v(a) = f_v(b)\}. && \text{(Definition of = in } K^{\mathrm{L}}(M))
\end{aligned}
$$

This finishes the simultaneous induction for equality and set-membership.

We will now prove the other cases one-by-one. Let us start with $\wedge$ and $\vee$, and $a_0, \ldots, a_{n-1} \in A$ be given. To improve readability, we will write $\bar{a}$ for $a_0, \ldots, a_{n-1}$ and $f_v(\bar{a})$ for $f_v(a_0), \ldots, f_v(a_{n-1})$.

$$
\begin{aligned}
[\![\varphi(\bar{a}) \wedge \psi(\bar{b})]\!] &= [\![\varphi(\bar{a})]\!] \wedge [\![\psi(\bar{b})]\!] \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar{a}))\} \cap \{v \in K \mid v \Vdash \psi(f_v(\bar{b}))\} && \text{(by I.H.)} \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar{a})) \text{ and } v \Vdash \psi(f_v(\bar{b}))\} \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar{a})) \wedge \psi(f_v(\bar{b}))\}.
\end{aligned}
$$

And completely analogous, for $\vee$ it holds that:

$$
\begin{aligned}
[\![\varphi(\bar{a}) \vee \psi(\bar{b})]\!] &= [\![\varphi(\bar{a})]\!] \vee [\![\psi(\bar{b})]\!] \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar{a}))\} \cup \{v \in K \mid v \Vdash \psi(f_v(\bar{b}))\} && \text{(by I.H.)} \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar{a})) \text{ or } v \Vdash \psi(f_v(\bar{b}))\} \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar{a})) \vee \psi(f_v(\bar{b}))\}.
\end{aligned}
$$

The implication-case follows similarly, with an essential use of (C) for equation (1):

$$\begin{aligned}
[\![\varphi(\bar a) \to \psi(\bar b)]\!]^H &= [\![\varphi(\bar a)]\!]^H \to [\![\psi(\bar b)]\!]^H \\
&= \{v \in K \mid \forall w \geq v (w \Vdash \varphi(f_w(\bar a)) \to w \Vdash \psi(f_w(\bar b)))\} \quad \text{(I.H.)} \\
&= \{v \in K \mid \forall w \geq v (w \Vdash \varphi(f_{vw}(f_v(\bar a))) \\
&\qquad \to w \Vdash \psi(f_{vw}(f_v(\bar b))))\} \quad (1) \\
&= \{v \in K \mid v \Vdash \varphi(f_v(\bar a)) \to \psi(f_v(\bar b))\}.
\end{aligned}$$

The only remaining cases are the quantifiers. Here, the surjectivity of the f_v (as proved in Lemma 5.7) will be important.

$$\begin{aligned}
[\![\exists x\ \varphi(x, \bar a)]\!]^H &= \bigvee_{x \in M^{(H)}} [\![\varphi(x, \bar a)]\!]^H \\
&= \{v \in K \mid \exists x \in M^{(H)}\ v \Vdash \varphi(f_v(x), f_v(\bar a))\} \quad \text{(by I.H.)} \\
&= \{v \in K \mid \exists x \in D^v\ v \Vdash \varphi(x, f_v(\bar a))\} \quad (2) \\
&= \{v \in K \mid v \Vdash \exists x\ \varphi(x, f_v(\bar a))\}.
\end{aligned}$$

The inclusion from left to right in equality (2) follows by the fact that $f_v(x) \in D^v$, the converse inclusion follows because f_v is surjective. Our final case is the one of the universal quantifier:

$$\begin{aligned}
[\![\forall x\ \varphi(x, \bar a)]\!]^H &= \bigwedge_{x \in M^{(H)}} [\![\varphi(x, \bar a)]\!]^H \\
&= \{v \in K \mid \forall x \in M^{(H)}\ v \Vdash \varphi(f_v(x), f_v(\bar a))\} \quad \text{(by I.H.)} \\
&= \{v \in K \mid \forall w \geq v\ \forall x \in M^{(H)}\ w \Vdash \varphi(f_w(x), f_w(\bar a))\} \quad (3) \\
&= \{v \in K \mid \forall w \geq v\ \forall x \in D^w\ w \Vdash \varphi(x, f_{vw}(f_v(\bar a)))\} \quad (4) \\
&= \{v \in K \mid v \Vdash \forall x\ \varphi(x, f_v(\bar a))\}.
\end{aligned}$$

Note that the inclusion from left to right of equality (3) follows by (C), persistency of the Kripke model, and the fact that every element $x \in D^w$ for $w \geq v$ can be extended to an element of D^v by setting $x(u) = \emptyset$ for all $u \not\geq w$. The converse inclusion is trivial. Furthermore, equality (4) holds because the image of f_v is D^w (by definition and surjectivity). This was the last case of the induction and therefore, we have finished the proof of the theorem. □

The essential corollaries read as follows.

Corollary 5.9. *Let M be a model of* ZFC*, $(K, \leq)$ be a Kripke frame and H the Heyting algebra* $\mathrm{Up}(K)$*. For every sentence φ in the language of set theory, it is the case that $M^{(H)} \models \varphi$ if and only if $K^{\mathrm{L}}(M) \models \varphi$.* □

Corollary 5.10. *Let M be a model of* ZFC*, $(K, \leq)$ be a Kripke frame and H the Heyting algebra* $\mathrm{Up}(K)$*. The logics of the Heyting-valued model $M^{(H)}$ and the full Lubarsky model $K^{\mathrm{L}}(M)$ coincide, that is, $\mathbf{L}^{\mathsf{Prop}}(M^{(H)}) = \mathbf{L}^{\mathsf{Prop}}(K^{\mathrm{L}}(M))$.* □

As we have seen in Proposition 2.25, not every Heyting algebra is of the form $\mathrm{Up}(K)$ for some Kripke frame K.

5.3 The Propositional Logic of Heyting-Valued Models

Recall from definition Definition 2.17 that an isomorphism $f : H \to H'$ of Heyting algebras is a bijective map that respects the structure of the algebras.

Given any isomorphism $f : H \to H'$ of Heyting algebras, we can define an isomorphism $\hat{f} : M^{(H)} \to M^{(H')}$ by induction on the ranks of the elements of $M^{(H)}$ as follows:

(i) Let $\hat{f}(\emptyset) = \emptyset$, and,

(ii) given $x : M^{(H)} \to H$, let $\hat{f}(x) : M^{(H')} \to H'$ be given by $\mathrm{dom}(\hat{f}(x)) = \hat{f}[\mathrm{dom}(f(x))]$, and $\hat{f}(x)(\hat{f}(y)) = f(x(y))$.

The following proposition is a generalisation of the Symmetry Lemma for Boolean-valued models (see [22, Lemma 14.37]). The proof is the same.

Proposition 5.11. *If $f : H \to H'$ is an isomorphism of Heyting algebras, then $\hat{f} : M^{(H)} \to M^{(H')}$ is an isomorphism of Heyting-valued models such that $[\![\varphi(\hat{f}(a_0), \dots, \hat{f}(a_{n-1}))]\!]^{H'} = f([\![\varphi(a_0, \dots, a_{n-1})]\!]^{H})$.*

We call an automorphism f of a Heyting algebra H *non-trivial* if $f \neq \mathrm{id}_H$.

Proposition 5.12. *Let H be a Heyting algebra. If the class of Heyting-valued models is 1-faithful to H, then H does not have non-trivial automorphisms.*

Proof. Suppose that the Heyting-valued models are 1-faithful to H, and let f be an automorphism of H. By 1-faithfulness of $M^{(H)}$, we can find a sentence φ_h in the language of set theory for every $h \in H$ such that $[\![\varphi]\!]^H = h$ holds in some $M^{(H)}$. Then Proposition 5.11 implies that $h = [\![\varphi]\!]^H = f([\![\varphi]\!]^H) = f(h)$ for all $h \in H$. Hence, f is the identity on H. □

Let us discuss this result for a bit. There are Heyting algebras without non-trivial automorphisms. For example, the finite element Heyting algebra $\mathbf{3} = (\{a, b, c\}, \leq)$ ordered by the reflexive and transitive closure of $a \leq b \leq c$ does not permit a non-trivial automorphism: Any automorphism needs to be bijective and has to respect $0_{\mathbf{3}} = a$ and $1_{\mathbf{3}} = b$. For an infinite Heyting algebra without non-trivial automorphisms see [24].

Note that every permutation of the atoms (that is, minimal non-zero elements) of a complete atomic Boolean algebra gives rise to an automorphism in the following way. By [8, Theorem 10.24], we know that the complete atomic Boolean algebras are exactly the Boolean algebras of the form $(\mathcal{P}(X), \subseteq)$ for some set X and the atoms are exactly the singletons. Given any bijection $f : X \to X$, the induced map $f[\cdot] : \mathcal{P}(X) \to \mathcal{P}(X)$, $Y \mapsto f[Y]$ is an automorphism of $\mathcal{P}(X)$. By the previous proposition, this means that the class of Heyting-valued models is not faithful to any of these Boolean algebras (except for the trivial two-element Boolean algebra).

An example of a finite Heyting algebra with a non-trivial automorphism can be seen in Figure 5.1. A non-trivial automorphism f is obtained by $f(a) = b$, $f(b) = a$ and $f(x) = x$ for all other elements x of the Heyting algebra. Again,

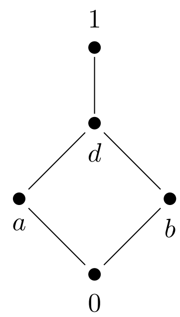

Figure 5.1: A Heyting algebra that admits a non-trivial automorphism.

the above proposition implies that the class of Heyting-valued models is not faithful to this Heyting algebra.

Finally, applications in set-theoretical forcing explicitly use the existence of non-trivial automorphisms, e.g., for proving the consistency of $\neg\mathsf{AC}$ (see [22, chapter 14]).

We summarise this discussion in the following corollary.

Corollary 5.13. *The class of Heyting-valued models is not faithful to any Heyting algebra with non-trivial automorphisms.* □

The question of loyalty of the class of all Heyting-valued models is still open. However, we can discuss it for a subclass of the Heyting-valued models.

Theorem 5.14. *Let $(K, \leq)$ be a Kripke frame in which every node has a successor that is an end-node. Then $\mathbf{KC} \subseteq \mathbf{L}^{\mathsf{Prop}}(M^{(\mathrm{Up}(K))})$.*

Proof. By Corollary 5.10, we know that $\mathbf{L}^{\mathsf{Prop}}(M^{(\mathrm{Up}(K))}) = \mathbf{L}^{\mathsf{Prop}}(K^{\mathrm{L}}(M))$. In this situation, Theorem 4.15 implies that

$$\mathbf{KC} \subseteq \mathbf{L}^{\mathsf{Prop}}(K^{\mathrm{L}}(M)) = \mathbf{L}^{\mathsf{Prop}}(M^{(\mathrm{Up}(K))})$$

and this proves the theorem. □

Corollary 5.15. *The class of Heyting-valued models that are based on a finite Heyting algebra is not loyal. Indeed, the propositional logic of this class contains at least* $\mathbf{KC}$.

Proof. Denote with $\mathcal{C}$ the class of Heyting-valued models based on a finite Heyting algebra. By Theorem 2.22, for every finite Heyting algebra H, there exists a partially ordered Kripke frame $(K, \leq)$ such that $H = \mathrm{Up}(K)$. Note that every successor in such a finite Kripke frame has a successor that is an end-node. Therefore, we can apply the previous Theorem 5.14 to derive that $\mathbf{KC} \subseteq \mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$. However, $\mathbf{L}^{\mathsf{Prop}}(\mathcal{H}_{\mathcal{C}}) = \mathbf{IPC} \subsetneq \mathbf{KC}$. □

The Propositional Logic of Boolean-Valued Models

We want to spend the rest of this chapter analysing the propositional logic of the Boolean-valued models, a subclass of the class of Heyting-valued models. Let us first observe that the result of faithfulness for Heyting-valued models transfers directly to the Boolean-valued case.

Corollary 5.16. *The class of Boolean-valued models is not faithful to any Boolean algebra with non-trivial automorphisms.* □

As discussed above, this includes many examples of Boolean algebras used in the context of the forcing technique but also applies to every complete atomic Boolean algebra.

The law of excluded middle holds in any Boolean-valued model as it does so in any Boolean-algebra. Therefore, it holds that the propositional logic of the Boolean-valued models is **CPC**, and so we can state the following theorem.

Theorem 5.17. *The class of Boolean-valued models is loyal.*

Proof. Let $\mathcal{C}$ be the class of Boolean-valued models. From Proposition 2.37 we know that $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$ is an intermediate logic, i.e., $\mathbf{IPC} \subseteq \mathbf{L}^{\mathsf{Prop}}(\mathcal{C}) \subseteq \mathbf{CPC}$. By Theorem 5.4, we know that the law of excluded middle is contained in $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C})$, but this means that $\mathbf{L}^{\mathsf{Prop}}(\mathcal{C}) = \mathbf{CPC}$. □

Chapter 6

Conclusion and Open Questions

In the first chapter of this thesis, we used the theory of Heyting structures to develop a framework that allows us to define and classify the propositional logics of classes of Heyting structures.

To summarise, we have seen that Iemhoff models are faithful structures that provide many classes of models for CZF^* with different underlying logics. Our results about the limitations of this method indicate that Iemhoff models are a tool rather for showing the compatibility of set theory with many different underlying propositional logics than for independence proofs in constructive set theory.

Lubarsky's models, in contrast, exhibit a strong failure of faithfulness. The class of finite Lubarsky models is not loyal. This seems to be a consequence of requiring *elementary* embeddings in the definition. The loyalty of the class of all Lubarsky models is still open.

Question 6.1. Is the class of all Lubarsky models loyal?

We discovered that the class of Heyting-valued models is not faithful to any Heyting algebra with a non-trivial automorphism. This result transfers directly to the (sub-)class of Boolean-valued models. The class of Boolean-valued models for set theory is loyal.

Question 6.2. Is the class of all Heyting-valued models loyal?

To answer these questions will be future work. Let us conclude by stating some further open questions and ideas for research in this area.

Question 6.3. In this thesis, we mainly consider the notion of $<\omega$-faithfulness as defined in Definition 2.40. What happens for logics of the form $\mathcal{L}_{\kappa\lambda}$? Are there connections to compact cardinals? Is the class of Iemhoff models ω-faithful?

Question 6.4. Is it possible to strengthen Proposition 2.49 and show that n-faithful does not imply $(n+1)$-faithful for natural numbers n? Moreover, are there applications of n-faithfulness? Are there connections to computational aspects of logic?

Question 6.5. Modal algebras provide semantics for propositional modal logic similar to how Heyting algebras do so for intuitionistic logic. Is it possible to extend the theory of Heyting structures to obtain models of modal theories? Can we then similarly analyse the underlying propositional logic of classes of models in terms of loyalty and faithfulness? Can this be applied to modal set theories as presented in [26] or [32]?

Question 6.6. The authors of [16] generalise the set-theoretical ultrapowers to Boolean ultrapowers. Can the class of Boolean ultrapowers be evaluated within our framework? Moreover, can it be generalised to Heyting ultrapowers (of course, based on a Heyting algebra)? Are these structures loyal or even faithful?

Question 6.7. Similar to the previous question, we can ask whether the topological models as defined in [18] can be analysed within our framework? If so, are they loyal or faithful?

Question 6.8. The notion of Heyting structure that we took from [10] is used in that paper in a topos-theoretic context. In how far can our analyses be generalised for this context?

Question 6.9. Is it possible to extended our framework to analyse (set) theories based on weaker logics such as minimal logic?

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Appendix A

Logics

For a general introduction to intuitionistic logic we refer to [30] and [5]. A general overview on constructivism can be found in [33], and a proof-theoretic exposition is [34].

We assume that we are given a first-order language $\mathcal{L}$: The logical primitives are $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\forall$ and $\exists$; the non-logical symbols might include constants, function symbols and relation symbols. The latter are also known as predicate symbols. The inductive definitions of terms and formulas are as usual, where the negation $\neg\varphi$ of a formula is defined as $\varphi \rightarrow \bot$. We will use $x, y, z, \ldots$ to denote variables of the language and $t, s, \ldots$ to denote terms of our language. The set of formulas of the language $\mathcal{L}$ will be denoted by $\mathcal{L}^{\mathsf{form}}$ and the set of sentences, again defined as usual, by $\mathcal{L}^{\mathsf{sent}}$.

Definition A.1. Intuitionistic predicate logic **IQC** can be axiomatised in a Hilbert-style system in the following way. The axioms are all formulas of the following forms:

(i) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(ii) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi))$
(iii) $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$
(iv) $\varphi \wedge \psi \rightarrow \varphi$
(v) $\varphi \wedge \psi \rightarrow \psi$
(vi) $\varphi \rightarrow \varphi \vee \psi$
(vii) $\psi \rightarrow \varphi \vee \psi$
(viii) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi))$
(ix) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$
(x) $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$
(xi) $\forall x\varphi(x) \rightarrow \varphi(t)$
(xii) $\varphi(t) \rightarrow \exists x\varphi(x)$

We will write $\vdash_{\mathbf{IQC}} \varphi$ to denote that φ is deducible in **IQC**. For every φ in the list of axioms, we have that $\vdash_{\mathbf{IQC}} \varphi$. The following three rules of interference hold:

Modus Ponens If $\vdash_{\mathbf{IPC}} \varphi \to \psi$ and $\vdash_{\mathbf{IPC}} \varphi$, then $\vdash_{\mathbf{IPC}} \psi$.
$\forall$-Introduction If $\vdash_{\mathbf{IPC}} \psi \to \varphi(t)$, then $\vdash_{\mathbf{IPC}} \psi \to \forall x\varphi$.
$\exists$-Introduction If $\vdash_{\mathbf{IPC}} \varphi(t) \to \psi$, then $\vdash_{\mathbf{IPC}} \exists x\varphi \to \forall\psi$.

If we want to consider **IQC** with equality, we have to add the following two axiom schemes for all terms t and s:

(i) $t = t$, and,
(ii) $t = s \to (\varphi(t) \leftrightarrow \varphi(s))$ for all formulas $\varphi(x)$ with possibly more free variables.

This finishes the definition of the Hilbert-style system for **IQC**.

We can obtain classical predicate logic **CQC** from **IQC** by adding the axiom scheme $\varphi \vee \neg\varphi$. Given a set of propositional letters Prop (also called propositional variables) we can consider the restricted first-order language $\mathcal{L}_{\mathsf{Prop}}$ where all quantifiers and variables are removed, and Prop is taken as a set of nullary predicates. Intuitionistic propositional logic **IPC** is obtained from **IQC** by restricting to the language $\mathcal{L}_{\mathsf{Prop}}$, restricting to the first 10 axioms, and using modus ponens as the only inference rule. We obtain **CPC** by adding the axioms scheme $p \vee \neg p$ to **IPC** (or by restricting **CQC** in a similar way).

Having fixed a language $\mathcal{L}$, we will sometimes abuse notation and identify our logics with the set of formulas that they derive. We will do so in particular for propositional logics, such as **IPC** and **CPC**. Then we can make the following definition.

Definition A.2. A set of propositional formulas **J** is an *intermediate logic* if it is closed under modus ponens and uniform substitution, and it holds that $\mathbf{IPC} \subseteq \mathbf{J} \subseteq \mathbf{CPC}$.

A set of sentences T of a first-order language $\mathcal{L}$ will be called a theory. We will usually assume that a theory is closed under some deductive system such as **IQC** or **CQC**. Moreover, we will sometimes close a theory T under an intermediate logic **J**.

Definition A.3. Let T be a theory and **J** be an intermediate logic. With $\mathsf{T}(\mathbf{J})$ we will denote the closure of T under substitutions into the formulas of **J** and modus ponens.

Appendix B

Set Theories

The main three axiom systems used in this thesis are classical Zermelo-Fraenkel set theory with choice ZFC (or without choice ZF), intuitionistic set theory IZF and constructive set theory CZF. For the presentation of these axiom systems, we will follow the book [1] of Aczel and Rathjen, to which we also refer for a deeper introduction to intuitionistic and constructive set theory.

Before listing the three axiom systems in detail, we refer to the following table for an overview.

ZFC	IZF	CZF	$\mathsf{CZF}^{-\mathsf{c}}$
Extensionality	Extensionality	Extensionality	Extensionality
Empty set	Empty set	Empty set	Empty set
Pairing	Pairing	Pairing	Pairing
Union	Union	Union	Union
Power set	Power set		
Infinity	Infinity	Strong Infinity	Strong Infinity
Foundation			
	Set Induction	Set Induction	Set Induction
Separation	Separation	Bounded Separation	Bounded Separation
Replacement			
	Collection		
		Strong Collection	
		Subset Collection	
Choice			

Zermelo-Fraenkel Set Theory ZFC

The axiom system ZF of Zermelo-Fraenkel set theory consists of the following axioms.

Extensionality $\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b)$

Empty set $\exists a \ \forall x \in a \ \bot$

Pairing $\forall a \forall b \exists y \forall x (x \in y \leftrightarrow (x = a \vee x = b))$

Union $\forall a \exists y \forall x (x \in y \leftrightarrow \exists u (u \in a \wedge x \in u))$

Power set $\forall a \exists y \forall x (x \in y \leftrightarrow x \subseteq a)$

Infinity $\exists a (\exists x\, x \in a \wedge \forall x \in a \exists y \in a\, x \in y)$

Foundation $\forall a (\exists x (x \in a) \rightarrow \exists x \in a \forall y \in a (y \notin x))$

Separation $\forall a \exists y \forall x (x \in y \leftrightarrow (x \in a \wedge \varphi(x)))$, for all formulas $\varphi(x)$.

Replacement $\forall a \forall x \in a \exists! y\, \varphi(x, y) \rightarrow \exists b \forall y (y \in b \leftrightarrow \exists x \in a\, \varphi(x, y))$, for all formulas $\varphi(x, y)$, where b is not free in $\varphi(x, y)$.

By adding the following axiom of choice AC, we obtain the system ZFC.

Choice $\forall a ((\forall x \in a \forall y \in a\, (x \neq y \rightarrow x \cap y = \emptyset)) \rightarrow \exists b \forall x \in a \exists! z \in b\, z \in x)$

Note that ZF and ZFC use classical logic.

Intuitionistic Zermelo-Fraenkel Set Theory IZF

Intuitionistic Zermelo-Fraenkel set theory IZF consists of the axioms (or axiom schemes) of Extensionality, Empty set, Pairing, Union, Power set, Infinity, Separation together with the following axioms:

Set Induction $(\forall a (\forall x \in a\, \varphi(x) \rightarrow \varphi(a))) \rightarrow \forall a \varphi(a)$, for all formulas $\varphi(x)$.

Collection $\forall a (\forall x \in a \exists y\, \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b\, \varphi(x, y))$, for all formulas $\varphi(x, y)$, where b is not free in $\varphi(x, y)$.

Intuitionistic set theory IZF uses intuitionistic logic instead of classical logic.

Constructive Set Theory CZF

The constructive set theory CZF consists of the axioms of Extensionality, Empty set, Pairing, Union, Set Induction and the following axioms:

Bounded Separation $\forall a \exists y \forall x (x \in y \leftrightarrow x \in a \wedge \varphi(x))$, for all bounded (i.e., Δ_0) formulas $\varphi(x)$, where y does not appear free in $\varphi(x)$.

Strong Infinity $\exists a (\mathrm{Ind}(a) \wedge \forall b (\mathrm{Ind}(b) \rightarrow \forall x \in a (x \in b)))$, where $\mathrm{Ind}(a)$ is the formula denoting that a is an inductive set:

$$\mathrm{Ind}(a) \text{ abbreviates } \emptyset \in a \wedge \forall x \in a \exists y \in a\; y = \{x\}.$$

Strong Collection

$$\forall a (\forall x \in a \exists y\, \varphi(x, y) \rightarrow \exists b (\forall x \in a \exists y \in b\, \varphi(x, y) \wedge \forall y \in b \exists x \in a\, \varphi(x, y))),$$

for all formulas $\varphi(x, y)$ with potentially more free variables.

Subset Collection

$$\forall a \forall b \exists c \forall u (\forall x \in a \exists y \in b\ \psi(x,y,u) \rightarrow$$
$$\exists d \in c (\forall x \in a \exists y \in d\ \psi(x,y,u) \wedge \forall y \in d \exists x \in a\ \psi(x,y,u))),$$

for all formulas $\psi(x, y, u)$ with potentially more free variables.

Note that CZF is based on *intuitionistic logic*. By $\mathsf{CZF}^{-\mathsf{c}}$ we denote CZF without the collection axioms.

Proposition B.1 ([1, Theorem 5.1.2])**.** *The axiom of Subset Collection implies the principle of exponentiation, i.e., the statement that for any two sets a and b, the set of functions from a to b does exist.*

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