Studies in Minimal Mathematics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is a study of minimal mathematics, i.e., mathematics on the basis of minimal logic. We will explore different methods of working in mathematical systems that are based on minimal logic. Special emphasis will be given to finding out which results of and about intuitionistic mathematics still hold in the context of minimal logic, and where the differences lie compared to intuitionistic mathematics.

Contents

Chapter 1

Introduction

Intuitionistic mathematics, as proposed by L.E.J. Brouwer, has been studied formally on the basis of intuitionistic logic, that was developed in 1930 by Brouwer's student Arend Heyting. His logic results from removing the law of excluded middle from classical logic. However, intuitionistic logic still contains the ex falso principle (also called principle of explosion in the paraconsistent tradition) stating that any statement follows from a contradiction. Starting with Brouwer himself and Kolmogorov,[1] objections have been raised that this principle is unintuitive, or even non-constructive. By removing the ex falso principle from intuitionistic logic, one obtains the system of minimal logic as introduced by Ingebrigt Johansson in 1937.

In this thesis, we are going to study minimal mathematics, i.e., mathematics on the basis of minimal logic. Other people have gone in a different direction and suggest studying relevance logic. Mark van Atten remarks that relevance logic may be closest to Brouwer's attitude (see [Att09]), but we like to stay close to the established formal systems. Another direction to consider is Griss' negationless mathematics (see [Gri44]), since without negation one has of course no contradictions. However, Griss rejects hypothetical reasoning which we do not want to do. Moreover, we think—in line with Brouwer (see [Bro48])—that negations give rise to intuitionistically interesting distinctions.

With these thoughts in mind, we choose to consider well-known formal systems but weaken the underlying logic from intuitionistic to minimal logic. A further difficulty of studying minimal mathematics is that there is no doctrine like the one of intuitionistic mathematics that we can follow. This leaves us with different options when deciding how to approach this subject. We will compare the following three approaches in this thesis:

First of all, we should point out that the falsum f of minimal logic is—compared to the falsum $\perp$ of intuitionistic logic—meaningless, i.e., f behaves like an arbitrary propositional variable, whereas $\perp$ implies every formula due to the ex falso principle. This gives rise to a very radical approach to minimal mathematics, in the sense that we subscribe to minimal logic with a meaningless falsum as a basis for our investigations.

[1] In [Dal04], Dirk van Dalen explains that Kolmogorov rejected the ex falso principle as he argued that it "does not have and cannot have any intuitive foundation since it asserts something about the consequence of something impossible". In this same article, van Dalen explains that Brouwer had objections similar to Kolmogorov's. See also Mark van Atten's discussion in [Att09].

We will see that the deductive structure of minimal logic then automatically gives a light, concrete meaning to falsum: $\neg A \wedge \neg\neg A$ for an arbitrary A.

The second, and again somewhat radical approach is trying to find an absurdity that naturally gives rise to the ex falso principle, i.e., providing a sentence that implies every formula over minimal logic. By proving that a certain formula has this property over minimal logic, we show that it is a concrete absurdity satisfying the ex falso principle. Interpreting falsum as this sentence justifies intuitionistic logic, and we can reason as we are used to in intuitionistic mathematics. We are interested in the circumstances for the existence of such a sentence. A perfect candidate for such a sentence should express mathematical content in an attractive manner as, for example, the sentence $0 = 1$ does in the context of Heyting arithmetic HA. This example also illustrates why this option is considered appealing to mathematicians working in intuitionistic formal systems.

A third and less radical approach is to interpret the falsum f of minimal logic by a sentence that we construe as a natural absurdity, i.e., a statement that we claim to be naturally false in the context of the theory at hand. This sentence then conveys more information than the meaningless falsum we use in the first approach, and therefore, possibly allows us to draw more conclusions. As there may be different choices for the absurdity, there may also be different minimal systems, each arising from one of these absurdities.

Let us now discuss how negation is interpreted by intuitionistic mathematicians, and how the ex falso principle is treated. Of course, everything starts with Brouwer, who says:

> The falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely... ([Bro12, pp. 85–86]).

We can draw two insights from Brouwer's thought. Firstly, one may conclude that mathematics can work with unending totalities, such as the natural numbers. More pertinently for our purposes, however, is the conclusion that the numbers 1 and 2 are created by two moments falling apart into *qualitatively different* parts.[2] Therefore, we may construe $1 = 2$ as a fundamental absurdity claiming the equality of two different parts.

Indeed, Heyting interpreted contradictions in the following way:

> I think that contradiction must be taken as a primitive notion. It seems very difficult to reduce it to simpler notions, and it is always easy to recognize a contradiction as such. In practically all cases it can be brought into the form $1 = 2$. ([Hey56, p. 102])

[2]The number 0 was only added later.

We think that absurdity is a better expression here than contradiction. A contradiction seems to refer to two sentences where often only one is needed. This yields the following infinite regress: If proving a negation always involves proving a contradiction, and a contradiction consists of a sentence A and a sentence $\neg A$, then it always involves proving a negation. So ultimately, this has to end in obviously false statements: absurdities.

In a similar manner, Michael Dummett remarks in his "Elements of Intuitionism" that "[u]nsurprisingly, negation is definable in intuitionistic arithmetic by $\neg A \leftrightarrow (A \to 0 = 1)$" ([DM77, p. 35]). Similarly, Anne Troelstra defines negation by stating that "$\neg A$ is proved by giving a proof of something like $A \to 1 = 0$" ([Tro69, p. 5]). In particular this latter interpretation of negation can be seen as similar to how we address this issue in this thesis: we consider different negations that are "something like" the usual absurdity $0 = 1$.

Later in this thesis, we will observe that in arithmetic and analysis, $0 = 1$ proves all formulas in a minimal context (see chapter 5). Let us note here that this is a purely technical result, it does not give $0 = 1$ a special philosophical status. The fact that we can derive all formulas from $0 = 1$ in arithmetic and analysis, does not make $0 = 1$ sacred. Possibly, there are many other formulas from which everything else is derivable, of course always depending on the system we work in (see also chapter 7). Note that, whenever $0 = 1$ is used in this thesis, it can always equivalently be replaced by $1 = 2$ and conceptually this may be the right choice.

Roy T. Cook and Jon Cogburn criticise taking $0 = 1$ for the definition of negation on the basis of a philosophical argument.

> [D]efining negation in terms of any false arithmetical formula results in the most vicious sort of circularity—the sort that immediately destroys the epistemic and/or logical clarity and security associated with intuitionism by its defender. ([CC00, p. 11])

They arrive at this conclusion via a model-theoretic argument where they show that defining $\perp$ as $0 = 1$ yields a one-point model of arithmetic. Moreover, they claim, the absurdity should be a statement that is in principle unprovable and not only, such as $0 = 1$, in the context of an axiom system. However, as van Atten also remarks in [Att09, p. 134], the fact that defining negation in terms of $0 = 1$ in, for instance, HA, enables us to formally derive ex falso, does not depend on whether or not there exists a proof of $0 = 1$.

We would like to point out both worries of Cook and Cogburn are not relevant in the case of minimal mathematics. It will be quite natural to work with models in which f is at least partially true, i.e., models that are certainly not of the intended form.[3] Moreover, the minimal falsum f does not at all possess the strong proof-theoretic properties that the intuitionistic falsum $\perp$ has. Therefore, their proof-theoretic worries do not transfer to the minimal case.

Let us now take a brief look at the intuitionist's justification of the ex falso principle. That one can derive everything from an absurdity, was criticised by Brouwer and only added later by Heyting for practical reasons. Heyting's justification of the ex falso rule shows similarities with the justification of material implication:

[3]By this we mean, for example, models of minimal arithmetic that have finite domains. In these cases, of course, falsum f has to be forced highlighting the absurdity of our situation.

> Axiom X [i.e., ex falso quodlibet in the form $\neg p \to (p \to q)$] may not seem intuitively clear. As a matter of fact, it adds to the precision of the definition of implication. You remember that $p \to q$ can be asserted if and only if we possess a construction which, joined to the construction p, would prove q. Now suppose that $\vdash \neg p$, that is, we have deduced a contradiction from the supposition that p were carried out. Then, in a sense, this can be considered as a construction, which, joined to a proof of p (which cannot exist) leads to a proof of q. I shall interpret the implication in this wider sense [compared to the narrower sense in Johansson's minimal logic]. ([Hey56, p. 106])

Van Atten (see [Att09, pp. 132–133]) critically remarks that this justification neither fits in Heyting's own nor in Kolmogorov's interpretation of intuitionistic logic along constructions.

Of course, our discussion is also connected to relevance logic (also called relevant logic), but distinct from it. Relevance logics were developed to avoid, among other implicational paradoxes, the ex falso principle. However, many relevantists still accept the law of double negation elimination (intuitionistically equivalent to the law of excluded middle) to be valid. Furthermore, negative ex falso (i.e., the principle if 'A' and 'not A', then 'not B') is valid in minimal logic but not in relevance logic. Neil Tennant, as a relevantist, developed his so-called core logic (see [Ten17]), by liberalising some of the deduction rules of intuitionistic logic. He concludes that all intuitionistic mathematics can be done based on this system, thus, avoiding ex falso. This work is distinct from our investigation, as we simply omit ex falso as a rule and investigate different interpretations of negation, without altering the other intuitionistic rules of inference.

Outline

In chapter 2, we will introduce the technical details needed for our analyses. Moreover, we will show a first example of the shortcomings of minimal logic in intuitionistic analysis.

Chapter 3 deals with the differences between minimal and intuitionistic propositional logic. We will restrict our language to different fragments and classify several superminimal-subintuitionistic logics, i.e., logics of strength strictly between minimal and intuitionistic propositional logic.

The preparation of a framework for our analysis is the central topic of chapter 4. First of all, we will show how to interpret a theory in minimal and intuitionistic contexts with different interpretations of negation. Moreover, we will observe several properties of our general definitions.

In chapter 5, we will consider first-order arithmetic in a minimal context. In the context of first-order arithmetic, we will mainly follow the first approach mentioned above and explore minimal arithmetic with an uninterpreted falsum. This system of minimal arithmetic is shown to be weaker than one would hope and to miss certain essential properties. The second approach, interpreting f as $1 = 0$, results in a system which is essentially HA again. For the third approach, we will consider the consequences of interpreting falsum differently, e.g., as $0 = 3$.

Chapter 6 deals with second-order arithmetic. Here, we will follow the third approach mentioned above and do an analysis with an interpretation of falsum as $0 = 1$, noting that not all formulas are derivable from $0 = 1$ in this system and that in this case it is not the strongest possibility of interpreting falsum.

We will treat theories of equality and apartness in chapter 7. Despite our result that in these theories there exists an absurdity that naturally satisfies the ex falso principle, we will work here with an uninterpreted falsum because the system that arises turns out to behave perfectly well. In particular, we will exhibit several conservativity results for the minimal case that have been obtained for the intuitionistic case by van Dalen, Statman, Smoryński and others, and add one of our own.

Because of our finding that the obvious approach to minimal arithmetic leads to an unpleasantly weak system, we have hardly forayed into analysis except for noting that at least for the parts formalised by Kleene, the same holds as for HA: interpreting f as $0 = 1$ leads in essence to the intuitionistic system.

We will close this thesis with a conclusion and directions for further research.

Note that the third chapter is not needed for reading chapters 4, 5, 6 and 7. Chapters 5, 6 and 7 can be read independently of each other.

Chapter 2

Minimal Logic

In this chapter, we will give a technical introduction into minimal logic. We will start with the syntax of minimal logic and give a proof calculus. Afterwards, Kripke semantics for minimal logic will be introduced. Along the way, we will also introduce the syntax and semantics of intuitionistic logic. We conclude this chapter with a discussion of the ex falso principle in intuitionistic analysis.

2.1 Syntax and Derivation System

The language $\mathscr{L}(\mathsf{MPC})$ of minimal propositional logic consists of the logical connectives $\wedge, \vee$ and $\rightarrow$. Additionally, it has a countable set of propositional variables P and an extra propositional variable f. The formulas of $\mathscr{L}(\mathsf{MPC})$ are built recursively in the usual way, where $\neg p$ abbreviates $p \rightarrow f$ and where $p \leftrightarrow q$ is short for $(p \rightarrow q) \wedge (q \rightarrow p)$.

The language $\mathscr{L}(\mathsf{MQC})$ of minimal predicate logic consists of the logical connectives $\wedge, \vee, \rightarrow, \exists$ and $\forall$. Additionally, it has a countable set Q of n-ary predicate symbols and n-ary function symbols for every n, together with an extra nullary predicate symbol f, and individual constants. The formulas of $\mathscr{L}(\mathsf{MPC})$ are built recursively in the usual way, where, again, $\neg A$ abbreviates $A \rightarrow f$ and $A \leftrightarrow B$ is short for $(A \rightarrow B) \wedge (B \rightarrow A)$. An atomic formula is a formula without any logical connectives. With this definition, f is an atomic formula. As usual, we refer to formulas without free variables as sentences.

The languages $\mathscr{L}(\mathsf{IPC})$ for intuitionistic propositional logic and $\mathscr{L}(\mathsf{IQC})$ for intuitionistic predicate logic, are obtained from the languages $\mathscr{L}(\mathsf{MPC})$ and $\mathscr{L}(\mathsf{MQC})$ by replacing f with the symbol $\perp$, denoting the intuitionistic falsum. Hence, in these languages, $\neg A$ is an abbreviation for $A \rightarrow \perp$. Note that we choose the two different symbols f and $\perp$ to emphasise their difference in meaning.

Given a language $\mathscr{L}$, we call any set of $\mathscr{L}$-sentences an *$\mathscr{L}$-theory.* When the language is clear, we will just say *theory.*

We will now introduce minimal and intuitionistic logic by their Prawitz style natural deduction system, following [TS00, Definition 2.1.1]. Minimal propositional logic, MPC, is the theory generated by the following natural deduction system, i.e., by the following introduction and elimination rules:

$$\frac{A \qquad B}{A \wedge B}\,\wedge I \qquad\qquad \frac{A \to B \qquad A}{B}\,{\to}E$$

$$\frac{A \wedge B}{A}\,\wedge E_R \qquad\qquad \frac{A \wedge B}{B}\,\wedge E_L$$

$$\frac{A}{A \vee B}\,\vee I_R \qquad\qquad \frac{B}{A \vee B}\,\vee I_L$$

$$\frac{A \vee B \qquad \begin{array}{c}[A]^n\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]^n\\ \vdots\\ C\end{array}}{C}\,\vee E, n \qquad\qquad \frac{\begin{array}{c}[A]^n\\ \vdots\\ B\end{array}}{A \to B}\,{\to}I, n$$

Minimal predicate logic, MQC, is obtained by adding to the above rules the following inference rules:

$$\frac{A[x/y]}{\forall x A}\,\forall I \qquad\qquad \frac{\forall x A}{A[x/t]}\,\forall E$$

$$\frac{A[x/t]}{\exists x A}\,\exists I \qquad\qquad \frac{\exists x A \qquad \begin{array}{c}[A[x/y]]^n\\ \vdots\\ C\end{array}}{C}\,\exists E, n$$

Note that in $\forall I$, y cannot be free in A nor in any open assumption. In $\exists E$, y can neither be free in A, C or in any open assumption except $[A[x/y]]$.

From the natural deduction systems of MPC and MQC we obtain the systems for intuitionistic propositional logic, IPC, and intuitionistic predicate logic, IQC, respectively, by adding the ex falso rule:

$$\frac{\bot}{A}$$

Note that the following is a derivation in minimal logic, where we use the $\to I$-rule without any assumptions:

$$\frac{f}{A \to f}$$

Since we construe $\neg A$ as an abbreviation for $A \to f$, this observation shows that the ex falso rule is valid in minimal logic for negated formulas. With this in mind, we can easily derive the following equivalence in our derivation systems for minimal propositional and minimal predicate logic:

$$f \leftrightarrow \neg A \wedge \neg\neg A.$$

Therefore, we can interpret falsum f as this particular kind of contradiction. This contradiction is stable in the sense that the derivation holds for any formula A.

Definition 2.1.1. For a logic $\mathsf{S} \in \{\mathsf{MPC}, \mathsf{IPC}, \mathsf{MQC}, \mathsf{IQC}\}$ we define $\Gamma \vdash_{\mathsf{S}} A$ if and only if the formula A is derivable from the set of assumptions Γ in the natural deduction system for S. We write $\vdash_{\mathsf{S}} A$ if A is derivable in the natural deduction system for S from an empty set of assumptions. In that case, we call A a *theorem* of S.

An example of a helpful derivation in MQC, which will also be useful later on, is proven in the following lemma.

Lemma 2.1.2. $\vdash_{\mathsf{MQC}} (\exists x A(x) \to B) \leftrightarrow \forall x(A(x) \to B)$.

Proof. We will prove this by giving derivations of the two implications in MQC:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A(t)]^1}{\exists x A(x)} \qquad [\exists x A(x) \to B]^2}{B}}{A(t) \to B}\;{\scriptstyle 1}}{\forall x(A(x) \to B)}}{(\exists x A(x) \to B) \to \forall x(A(x) \to B)}\;{\scriptstyle 2}$$

The second implication can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A(x) \to B)]^3}{A(t) \to B} \qquad [A(t)]^1}{B} \qquad [\exists x A(x)]^2}{B}\;{\scriptstyle 1}}{\exists x A(x) \to B}\;{\scriptstyle 2}}{\forall x(A(x) \to B) \to (\exists x A(x) \to B)}\;{\scriptstyle 3}$$

This finishes the proof of the lemma. □

So far, we have only defined and discussed pure logical systems. Later, we will extend these definitions by adding non-logical axioms and rules and may then refer to these extensions as *formal systems*, or just *systems*. For such an enriched system S, we denote its language by $\mathscr{L}(\mathsf{S})$.

A desirable property of a system S is the *disjunction property*.

Definition 2.1.3. A system S has the *disjunction property* if whenever $\vdash_{\mathsf{S}} A \vee B$ for some formulas A and B, then also $\vdash_{\mathsf{S}} A$ or $\vdash_{\mathsf{S}} B$.

The disjunction property is a very distinctive property of intuitionistic logic. Classically, $p \vee \neg p$ is an immediate counterexample. That also minimal propositional logic has the disjunction property, was already proven by Johansson in [Joh37].

Moreover, Johansson already gave a translation of intuitionistic logic into minimal logic, as was found in the Johansson-Heyting correspondence (see [Mol16] for an analysis of this correspondence). This translation is defined by induction on the structure of the formula A as follows:

$$\begin{aligned} p^{hj} &:= p \\ \bot^{hj} &:= f \\ (A \wedge B)^{hj} &:= A^{hj} \wedge B^{hj} \\ (A \vee B)^{hj} &:= A^{hj} \vee B^{hj} \\ (A \to B)^{hj} &:= A^{hj} \to (B^{hj} \vee f) \\ (\exists x A)^{hj} &:= \exists x A^{hj} \\ (\forall x A)^{hj} &:= \forall x (A^{hj} \vee f) \end{aligned}$$

The crucial property of this translation is proved in the following proposition.

Proposition 2.1.4. $\vdash_{\mathsf{IQC}} A \;\Leftrightarrow \vdash_{\mathsf{MQC}} A^{hj}$

Proof. The direction from right to left is clear, since $\vdash_{\mathsf{IQC}} (A \vee \bot) \leftrightarrow A$ for all formulas A. In order to prove the other direction, we have to check all the derivation rules. The only interesting rules are $\to E$ and $\forall E$, for which we use that minimal logic has the disjunction property to conclude from $\vdash_{\mathsf{MQC}} B^{hj} \vee f$ and $\vdash_{\mathsf{MQC}} A^{hj}[x/t] \vee f$ that $\vdash_{\mathsf{MQC}} B^{hj}$ and $\vdash_{\mathsf{MQC}} A^{hj}[x/t]$, because $\nvdash_{\mathsf{MQC}} f$. □

2.2 Kripke Models and Completeness

In this section, we will introduce Kripke semantics for minimal logic. Let us start with the semantics for propositional logic.

Definition 2.2.1 (Kripke Frames for MPC)**.** A *Kripke frame for* MPC is a triple $\mathcal{F} = (W, \leq, F)$, where W is a non-empty set of nodes, $\leq$ a partial order on W and F an upwards closed subset of W.

A *Kripke model for* MPC is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a Kripke frame for MPC and V a *valuation*, mapping the set of propositional variables to the set of upwards closed subsets of W.

The Kripke models for minimal predicate logic are defined as follows.

Definition 2.2.2 (Kripke Frames for MQC)**.** A *Kripke frame for* MQC is a quadruple $\mathcal{F} = (W, \leq, F, D)$, where W is a non-empty set of nodes, $\leq$ a partial order on W, F an upwards closed subset of W and $D = \{D_w \mid w \in W\}$ a non-empty set of domains D_w for every node $w \in W$ such that $D_w \subseteq D_v$ whenever $w \leq v$.

A *Kripke model for* MQC is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a Kripke frame for MQC and V a *valuation*, mapping the set of atomic sentences to the set of upward closed subsets of W, such that for any atomic sentence $P(d_1, \ldots, d_n)$, we have $V(P(d_1, \ldots, d_n)) \subseteq \{w \in W \mid d_1, \ldots, d_n \in D_w\}$.

We can now define the forcing relation on our Kripke models.

Definition 2.2.3 (Forcing on Kripke Models)**.** Given an MPC-Kripke model $\mathcal{M} = (W, \leq, F, V)$, a node $w \in W$ and some formula A, we define the forcing relation $w \Vdash A$ inductively as follows:

$$
\begin{aligned}
&w \Vdash f && \Leftrightarrow\ w \in F,\\
&w \Vdash A \wedge B && \Leftrightarrow\ w \Vdash A \text{ and } w \Vdash B,\\
&w \Vdash A \vee B && \Leftrightarrow\ w \Vdash A \text{ or } w \Vdash B,\\
&w \Vdash A \to B && \Leftrightarrow\ \text{for all } v \geq w:\ \text{if } v \Vdash A,\ \text{then } v \Vdash B,\\
&w \Vdash \neg A && \Leftrightarrow\ \text{for all } v \geq w:\ \text{if } v \Vdash A,\ \text{then } v \Vdash f.
\end{aligned}
$$

For an MQC-Kripke model $\mathcal{M} = (W, \leq, F, D, V)$ we add the following clauses:

$$
\begin{aligned}
&w \Vdash P(d_1, \ldots, d_n) && \Leftrightarrow\ w \in V(P(d_1, \ldots, d_n)),\\
&w \Vdash \exists x A(x) && \Leftrightarrow\ w \Vdash A(d) \text{ for some } d \in D_w,\\
&w \Vdash \forall x A(x) && \Leftrightarrow\ \text{for all } v \geq w : v \Vdash A(d) \text{ for all } d \in D_v.
\end{aligned}
$$

The well-known Kripke semantics for intuitionistic logic can be obtained from ours by replacing f with $\perp$ and setting $F = \emptyset$, i.e., the definition of the forcing relation is modified in the sense that $\perp$ is never forced at any node of any Kripke model. Due to this observation, every Kripke model for intuitionistic logic is also a Kripke model for minimal logic. The following soundness and completeness results hold.

Theorem 2.2.4 (see e.g. [Col16])**.** MPC *is sound and complete with respect to the class of finite, rooted Kripke models for* MPC*.*

Theorem 2.2.5. MQC *is sound and complete with respect to the class of rooted Kripke models for* MQC*.*

By a *positive formula*, we denote a formula that does not contain negation, $\perp$ or f. By the *positive fragment* of a logic or system, we mean all positive formulas. MPC and MQC can always be equated to the positive fragments of IPC and IQC, respectively. That this is so, is clear from the fact that in minimal propositional and in minimal predicate logic, falsum, f, behaves as an ordinary propositional variable or nullary predicate.

Lemma 2.2.6 (see e.g. [JZ15])**.** *For any formula A in the positive fragment of* IQC *we have:*

$$\vdash_{\mathsf{MQC}} A \ \Leftrightarrow \vdash_{\mathsf{IQC}} A$$

Finally, we say that a theory is a *consistent theory* if it has a model. Note that the single-noded model forcing all propositional variables, including f, is a model of MPC. Similarly, the single-noded model forcing all atomic formulas, including f, is a model of MQC. Therefore, every theory over MPC or MQC will be consistent.

2.3 An Example of Ex Falso in Intuitionistic Analysis

In Kleene's intuitionistic formal system of analysis, **I**, Church's thesis that every effectively computable number-theoretic function is general recursive, **CT**, can be given by the following schema:

$$\exists \alpha A(\alpha) \to \exists \alpha (GR(\alpha) \wedge A(\alpha)),$$

in which $GR(\alpha)$ is a predicate expressing that α is general recursive, and $A(\alpha)$ contains only α as a free function variable. We denote the instance of **CT** for the predicate A by $\mathbf{CT}_A$. In [Mos71], Joan Moschovakis proves the consistency of **CT** with a certain extension of **I**, under the assumption that $\exists\alpha A(\alpha)$ is closed. Within this argument we find a proof of the statement that the schema $\neg\neg\mathbf{CT}$ is equivalent in proof strength in **I** to the statement $\forall\alpha\neg\neg GR(\alpha)$.

Moschovakis suggested to us that there is an essential use of ex falso in this proof. We will reconstruct the proof in a more perspicuous way to uncover the minimal invalidity of this statement.

Proposition 2.3.1. *The following hold in the system* **I***:*

(i) $\neg\neg\mathbf{CT} \vdash_{\mathbf{I}} \forall\alpha\neg\neg GR(\alpha)$*;*

(ii) $\forall\alpha\neg\neg GR(\alpha) \vdash_{\mathbf{I}} \neg\neg\mathbf{CT}_A$ *for all predicates* A*.*

Proof.

(i) From $\neg\neg\mathbf{CT}$, we derive $\neg\neg(\exists\alpha\neg GR(\alpha) \to \exists\alpha(GR(\alpha)\wedge\neg GR(\alpha)))$. In **I** we have:

$$\begin{aligned}\neg\neg(\exists\alpha\neg GR(\alpha) \to \exists\alpha(GR(\alpha) \wedge \neg GR(\alpha))) &\to \neg\neg\neg\exists\alpha\neg GR(\alpha)\\ &\to \neg\exists\alpha\neg GR(\alpha)\\ &\to \forall\alpha\neg\neg GR(\alpha)\end{aligned}$$

These steps are all minimally valid as well.

(ii) Let us suppose $\forall\alpha\neg\neg GR(\alpha)$, then we have in **I**:

$$\begin{aligned}\exists\alpha A(\alpha) &\to \exists\alpha(\neg\neg GR(\alpha) \wedge A(\alpha))\\ &\to \exists\alpha(\neg\neg GR(\alpha) \wedge \neg\neg A(\alpha))\\ &\to \exists\alpha\neg\neg(GR(\alpha) \wedge A(\alpha))\\ &\to \neg\neg\exists\alpha(GR(\alpha) \wedge A(\alpha))\end{aligned}$$

Again, these steps are all minimally valid as well.

In intuitionistic logic, we have $\vdash_{\mathsf{IPC}} (p \to \neg\neg q) \to \neg\neg(p \to q)$, so we obtain from our result above, $\exists\alpha A(\alpha) \to \neg\neg\exists\alpha(GR(\alpha) \wedge A(\alpha))$, the desired result: $\neg\neg(\exists\alpha A(\alpha) \to \exists\alpha(GR(\alpha) \wedge A(\alpha))$.[1] □

Note that the proof of (i) works for minimal logic. Regarding (ii), as we can see in the model given below, the intuitionistically valid implication used to derive the final result in the proof above is not minimally valid:

$$\nvdash_{\mathsf{MPC}} (p \to \neg\neg q) \to \neg\neg(p \to q)$$

[1] We note that Moschovakis seemingly made use of another intuitionistically valid propositional implication to derive the final result, namely $\neg(p \to q) \to (\neg\neg p \wedge \neg q)$. However, this formula is minimally equivalent to our $(p \to \neg\neg q) \to \neg\neg(p \to q)$.

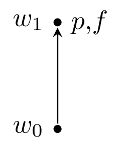


Figure 2.1

Let us now confirm our intuition that the use of minimal logic is not sufficient for the proof of the statement

$$\neg\neg(\exists\alpha A(\alpha) \to \exists\alpha(GR(\alpha) \wedge A(\alpha))) \leftrightarrow \forall\alpha\neg\neg GR(\alpha).$$

Consider the following countermodel with $D_{w_0} = \emptyset$ and $D_{w_1} = D_{w_2} = \{t\}$, then w_0 forces $\forall\alpha\neg\neg GR(\alpha)$ but it does not force the left-hand-side of the equivalence.

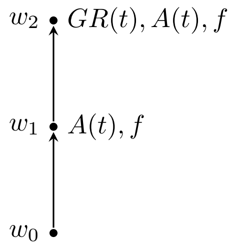


Figure 2.2

Remark 2.3.2. This argument shows that (ii) cannot be proved in minimal logic, but this does not mean that the full power of the ex falso principle is needed: In Remark 3.3.8 in chapter 3, we will see that the principle $(p \to \neg\neg q) \to \neg\neg(p \to q)$ is a weakened form of ex falso. We have seen that this weakened version is sufficient to prove Proposition 2.3.1. Moreover, we will show in chapter 5 that for some parts of intuitionistic analysis the use of intuitionistic logic can be justified even on a minimal base, namely, by interpreting falsum by $1 = 0$.

Let us conclude this discussion by noting that this is an example of a situation where it seems that the ex falso principle is needed, but actually, a weaker principle is sufficient. Adding the principle $(p \to \neg\neg q) \to \neg\neg(p \to q)$ to minimal logic results in the logic $\mathsf{SM1}$ that we are going to discuss in chapter 3.

Chapter 3

Differences between Minimal and Intuitionistic Logic

In this chapter, we will investigate the purely logical differences between the propositional logics MPC and IPC. After discussing the fundamental differences, we explain on which fragments the two logics either coincide or differ and we examine the latter. Finally, we analyse the consequences of adding a minimally invalid formula as an axiom to minimal logic, obtaining *superminimal logics*.

3.1 Fundamental Differences

The difference in the axiomatisations of minimal and intuitionistic propositional logic is of course the ex falso principle, which can be stated as:

$$p \to (\neg p \to q).$$

Over MPC, the ex falso principle is equivalent to the following principle, called *disjunctive syllogism*:

$$((p \vee q) \wedge \neg p) \to q.$$

Hence, the disjunctive syllogism, an important tool of deduction in intuitionistic logic, does not hold in minimal logic. We can see this in following simple countermodel:

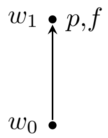


Figure 3.1

We have $w_1 \Vdash (p \vee q) \wedge (p \to f)$, but $w_1 \nVdash q$. Hence $w_0 \nVdash ((p \vee q) \wedge \neg p) \to q$. An even simpler countermodel would be the single-noded model on which p and f are forced.

We now turn to the following important observation:

Remark 3.1.1. In minimal logic, the principle of *negative ex falso* is valid:[1]

$$(p \wedge \neg p) \to \neg q.$$

This means that minimal logic still allows for some kind of 'explosion', as falsum implies all negated formulas. The negative ex falso principle is equivalent to the following special instance of the disjunctive syllogism:

$$((p \vee \neg q) \wedge \neg p) \to \neg q.$$

The following is a derivation of the above principle in minimal logic:

$$
\begin{prooftree}
\AxiomC{$[p \vee \neg q]^3$}
\AxiomC{$[p]^1$}
\AxiomC{$[\neg p]^2$}
\BinaryInfC{f}
\UnaryInfC{$\neg q$}
\AxiomC{$[\neg q]^1$}
\RightLabel{1}
\TrinaryInfC{$\neg q$}
\RightLabel{2}
\UnaryInfC{$\neg p \to \neg q$}
\RightLabel{3}
\UnaryInfC{$(p \vee \neg q) \to (\neg p \to \neg q)$}
\UnaryInfC{$((p \vee \neg q) \wedge \neg p) \to \neg q$}
\end{prooftree}
$$

In the remainder of this section, we will focus on finding intuitionistically valid propositional formulas that are not minimally valid. We do so systematically by examining different fragments of IPC and MPC.

Definition 3.1.2. Let L be either IPC or MPC, and let X be a subset of the logical connectives of L. Then the $[X]$*-fragment of* L consists of all L-formulas φ such that all logical connectives that appear in φ are among X. Given a natural number n, let the $[X]^n$-fragment of L consist of all formulas φ in the $[X]$-fragment such that the variables that appear in φ are among $\{p_1, \dots, p_n\}$. The *full fragment* of L consists of all L-formulas.

We call an $[X]^n$-fragment of L *locally finite* if it contains only finitely many formulas up to equivalence over L. We call an $[X]$-fragment *locally finite* if $[X]^n$ is locally finite for every natural number n.

All fragments of IPC without disjunction are locally finite. This was proven first for the $[\to]$-fragment by Diego in [Die65].

Proposition 3.1.3. *For every formula A in the $[\wedge, \neg]$-fragment of IPC we have, if $\vdash_{\mathsf{IPC}} A$, then $\vdash_{\mathsf{MPC}} A$.*

Proof. Let A be any formula in the $[\wedge, \neg]$-fragment of IPC and let A^* denote the formula obtained from A by replacing all instances of $\bot$ by f. We will first prove by induction on the structure of A that $A^{hj} \leftrightarrow A^*$, where $A \mapsto A^{hj}$ is the Johansson translation defined above Proposition 2.1.4. For the base cases we have $p^{hj} = p$ and

[1] Also the law of non-contradiction, $\neg(p \wedge \neg p)$, is minimally valid, which is often thought of as the law stating that something cannot be both true and false. Nonetheless, minimal theories for sure have models in which a statement is both true and false.

$\perp^{hj} = f$. The induction step for conjunction follows trivially from the induction hypothesis, so the only case left to prove is the case for negation:

$$\begin{aligned}
(\neg A)^{hj} &\leftrightarrow (A \to \perp)^{hj} \\
&\leftrightarrow A^{hj} \to (\perp^{hj} \vee f) \\
&\leftrightarrow A^{hj} \to (f \vee f) \\
&\leftrightarrow A^{hj} \to f \\
&\leftrightarrow A^{*} \to f && \text{(by IH)} \\
&\leftrightarrow \neg A^{*}
\end{aligned}$$

Now, if $\vdash_{\mathsf{IPC}} A$, then by Proposition 2.1.4 we know that $\vdash_{\mathsf{MPC}} A^{hj}$ and thus, by our previous conclusion, $\vdash_{\mathsf{MPC}} A^{*}$. Since A^{*} is simply obtained from A by replacing $\perp$ by f, this means that $\mathsf{MPC} \vdash A$, which finishes the proof. □

Lemma 3.1.4. *Each formula in the $[\wedge, \vee, \neg]^n$-fragment of* MPC *is equivalent to a disjunction of formulas in the $[\wedge, \neg]^n$-fragment of* MPC.

Proof. By induction on the structure of the formula. The base cases are disjunction-free and therefore trivial. We have two induction steps.

Let $A = B \wedge C$, where B and C are equivalent to disjunctions of formulas, $\bigvee_i B_i$ and $\bigvee_j C_j$, in the $[\wedge, \neg]^n$-fragment of MPC. Then A is equivalent to $\bigvee_i B_i \wedge \bigvee_j C_j$, and over MPC we have:

$$\begin{aligned}
\bigvee_i B_i \wedge \bigvee_j C_j &\leftrightarrow (B_1 \wedge \bigvee_j C_j) \vee \ldots \vee (B_n \wedge \bigvee_j C_j) \\
&\leftrightarrow ((B_1 \wedge C_1) \vee \ldots \vee (B_1 \wedge C_m)) \vee \ldots \vee ((B_n \wedge C_1) \vee \ldots \vee (B_n \wedge C_m)) \\
&\leftrightarrow (B_1 \wedge C_1) \vee \ldots \vee (B_1 \wedge C_m) \vee \ldots \vee (B_n \wedge C_1) \vee \ldots \vee (B_n \wedge C_m)
\end{aligned}$$

Hence, A is over MPC equivalent to a disjunction of formulas in the $[\wedge, \neg]^n$-fragment.

Let $A = \neg B$, where B is equivalent to a disjunction of formulas, $\bigvee_i B_i$, in the $[\wedge, \neg]^n$-fragment of MPC. Then A is equivalent to $\neg \bigvee_i B_i$, and over MPC we have:

$$\begin{aligned}
\neg \bigvee_i B_i &\leftrightarrow \neg(B_1 \vee \ldots \vee B_n) \\
&\leftrightarrow \neg B_1 \wedge \ldots \wedge \neg B_n
\end{aligned}$$

And again, A is over MPC equivalent to a disjunction of formulas in the $[\wedge, \neg]^n$-fragment. □

Proposition 3.1.5. *For every formula A in the $[\wedge, \vee, \neg]$-fragment of* IPC *we have, if $\vdash_{\mathsf{IPC}} A$, then $\vdash_{\mathsf{MPC}} A$.*

Proof. Let A be any formula in the $[\wedge, \vee, \neg]$-fragment of IPC and suppose $\vdash_{\mathsf{IPC}} A$. Then, by Lemma 3.1.4, we know that A is minimally equivalent to a disjunction of formulas in the $[\wedge, \neg]$-fragment, say $\vdash_{\mathsf{MPC}} A \leftrightarrow A_1 \vee \ldots \vee A_n$. Then also $\vdash_{\mathsf{IPC}} A \leftrightarrow A_1 \vee \ldots \vee A_n$. Since IPC has the disjunction property, we know that one of the disjuncts, say A_i, is derivable in IPC. Now, since A_i is a formula in the $[\wedge, \neg]$-fragment of IPC, we conclude using Proposition 3.1.3 that $\vdash_{\mathsf{MPC}} A_i$, and thus, using $\vee I$, we conclude $\vdash_{\mathsf{MPC}} A$. □

So, all intuitionistically provable formulas in the fragments without implication are also minimally provable. Together with Lemma 2.2.6 this yields that all intuitionistically provable formulas in the fragments without either implication or negation, are also minimally provable. In other words, minimal and intuitionistic logic are the same on the fragments $[\wedge, \vee, \rightarrow]$ and $[\wedge, \vee, \neg]$. Note, however, that this does not mean that all fragments of minimal logic without implication are isomorphic to some positive fragment of intuitionistic logic. The minimal fragment $[\wedge, \neg]^2$, for instance, has 26 classes compared to the 23 classes of the intuitionistic fragment $[\wedge, \neg]^2$. This difference between the fragments is due to the formulas: $p \wedge \bot$, $q \wedge \bot$ and $p \wedge q \wedge \bot$, which are intuitionistically all equivalent to $\bot$. In the next section, we will go into the details of those fragments in which minimal and intuitionistic logic differ.

3.2 Different Fragments of Minimal Logic

Since falsum behaves in MPC like any other propositional variable, we can easily derive the following conclusion.

Proposition 3.2.1. *For $X \subseteq \{\wedge, \vee\}$, the $[X, \rightarrow, \neg]^n$-fragment of minimal propositional logic is isomorphic the $[X, \rightarrow]^{n+1}$-fragment of intuitionistic propositional logic.*

Proof. We can rewrite the $[X, \rightarrow, \neg]^n$-fragment of MPC as $[X, \rightarrow, f]^n$. □

Before we go into the details of our investigation, we should make a note on the methodology used. We work with computational methods and programs developed for this purpose by Lex Hendriks and will restrict our attention to at most two propositional variables. Hence, only an examination of the four fragments presented in Table 3.1 below, will be of interest for our investigation. The table presents the number of equivalence classes in the different fragments of the two logics.

Fragment	Logic	$n = 1$	$n = 2$
$[\rightarrow, \neg]$	IPC	6	518
	MPC	14	25165802
$[\wedge, \rightarrow, \neg]$	IPC	6	2134
	MPC	18	623662965552330
$[\vee, \rightarrow, \neg]$	IPC	∞	∞
	MPC	∞	∞
$[\wedge, \vee, \rightarrow, \neg]$	IPC	∞	∞
	MPC	∞	∞

Table 3.1: Number of equivalence classes.

Besides the limit on the number of propositional variables, we confine our search by only considering formulas of the form $A \rightarrow B$. This, first of all, for the following reason: If $\nvdash_{\mathsf{MPC}} A \wedge B$, then either $\nvdash_{\mathsf{MPC}} A$ or $\nvdash_{\mathsf{MPC}} B$. If $\nvdash_{\mathsf{MPC}} A \vee B$, then both $\nvdash_{\mathsf{MPC}} A$ and $\nvdash_{\mathsf{MPC}} B$. Hence, if we find a formula of one of these forms that is minimally not derivable, then already one of its subformulas is not. Therefore, implication seems to be the most interesting building block when searching for minimally underivable formulas. Another reason for this restriction is that it simplifies our search.

The $[\wedge, \to, \neg]^1$-fragment of MPC

We will first define what diagrams and exact Kripke models of fragments of propositional logics are (for more details, see [Hen96]). The Lindenbaum-Tarski algebra, or Lindenbaum algebra, of a fragment of a propositional logic is the algebra of equivalence classes of all its formulas, ordered by inclusion. We will also refer to this algebra as *the diagram* of a fragment. The *truth set* of a formula A in a model $\mathcal{M}$ is the set of all nodes in $\mathcal{M}$ that force A.

Definition 3.2.2 (Exact Kripke model)**.** An exact Kripke model[2] for a fragment of propositional logic is a Kripke model $\mathcal{M}$ with the following two properties: Firstly, for formulas A and B in the fragment, if $A \vdash B$, then the truth set of A is contained in the truth set of B. Secondly, for any upwards closed set U of nodes in $\mathcal{M}$ there exists a formula A in the fragment such that U is the truth set of A.

Only when the diagram of a fragment is a lattice, the fragment has an exact Kripke model. In that case, the upwards closed sets of nodes in the exact Kripke model of the fragment correspond to the nodes in the diagram. Below are given both the exact Kripke models and the diagrams of the $[\wedge, \to, \neg]^1$-fragments of the two logics. Note, again, that the $[\wedge, \to, \neg]^1$-fragment of MPC is isomorphic to the $[\wedge, \to]^2$-fragment of IPC.

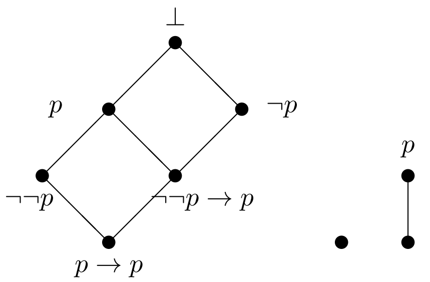


Figure 3.2: Diagram and exact Kripke model of $[\wedge, \to, \neg]^1$-fragment of IPC.

[2] A notion first introduced by de Bruijn in [Bru75].

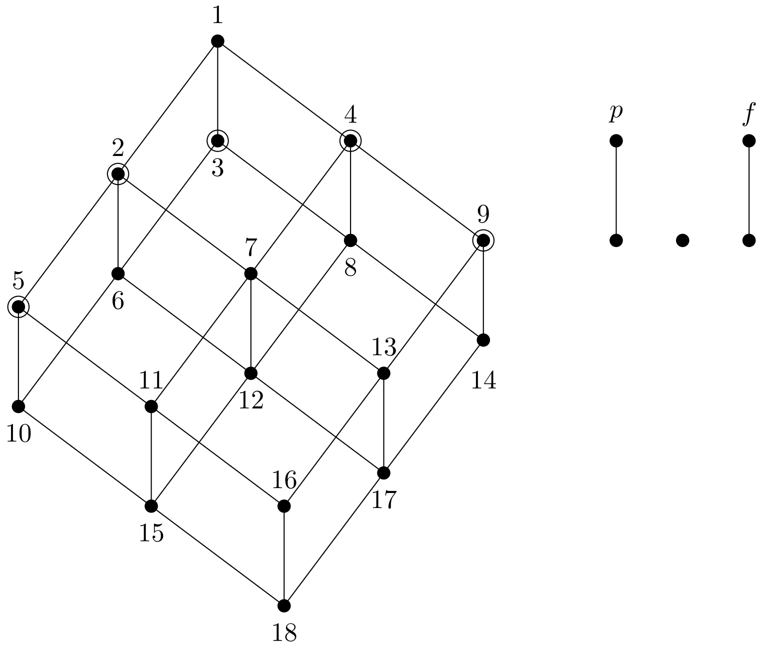


Figure 3.3: Diagram and exact Kripke model of $[\wedge, \rightarrow, \neg]^1$-fragment of MPC.

The formulas in the diagram of the $[\wedge, \rightarrow, \neg]^1$-fragment of MPC:

1. $p \wedge f$	10. $f \rightarrow p$
2. p	11. $\neg\neg p$
3. $(\neg p) \wedge (f \rightarrow p)$	12. $(((\neg p) \rightarrow p) \rightarrow p) \wedge \neg\neg(f \rightarrow p)$
4. f	13. $(f \rightarrow p) \rightarrow p$
5. $\neg p \rightarrow p$	14. $\neg p$
6. $\neg\neg p \rightarrow p$	15. $\neg\neg(f \rightarrow p)$
7. $\neg\neg p \wedge ((f \rightarrow p) \rightarrow p)$	16. $(\neg p \wedge (f \rightarrow p)) \rightarrow p$
8. $\neg((f \rightarrow p) \rightarrow p)$	17. $(\neg p \rightarrow p) \rightarrow p$
9. $\neg(f \rightarrow p)$	18. $p \rightarrow p$

Note that in both IPC and MPC, falsum is equivalent to the formula $\neg(p \rightarrow p)$. Using the diagrams in Figure 3.2 and Figure 3.3, we can now determine all the formulas of the form $A \rightarrow B$ in the $[\wedge, \rightarrow, \neg]^1$-fragment of IPC that are valid in IPC, yet not in MPC. By construction of the diagrams, we know that $A \rightarrow B$ is valid if the node that denotes the equivalence class of A is connected to the node that denotes the equivalence class of B, via a path that only goes downwards. When we do this for the nodes in the diagram of the fragment for IPC we obtain the following three implications that are not valid in MPC:

$\neg(p \to p) \to p$	4 is not above 2
$\neg(p \to p) \to (\neg\neg p \to p)$	4 is not above 6
$\neg p \to (\neg\neg p \to p)$	14 is not above 6

These three formulas are minimally equivalent. Hence, we conclude that in the $[\wedge, \to, \neg]^1$-fragment of IPC there is only one formula of the form $A \to B$ that is intuitionistically but not minimally valid.[3] Note that this formula is $\wedge$-free, and hence is already in the $[\to, \neg]^1$-fragment.

The $[\wedge, \to, \neg]^2$-fragment of MPC

The exact Kripke model of the $[\wedge, \to, \neg]^2$-fragment of IPC is given below.

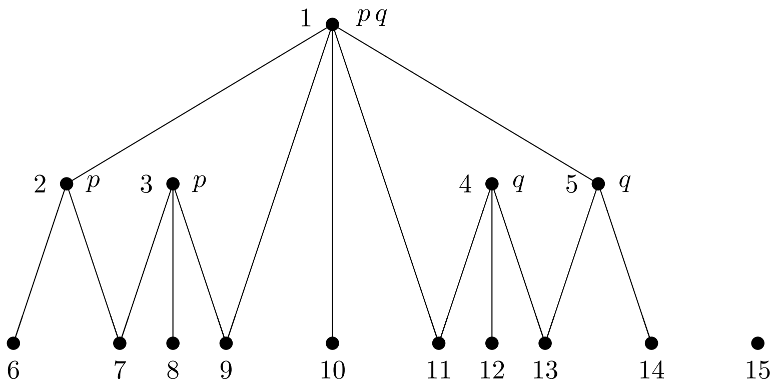


Figure 3.4: The exact Kripke model of the $[\wedge, \to, \neg]^2$-fragment of IPC.

This fragment contains 2134 equivalence classes and is therefore not as manageable as the previous fragment. One solution for making our search more feasible would be to only consider those intuitionistically valid formulas $A \to B$ for which the truth sets of A and B in the exact model differ by one element. We can obtain these formulas by taking a truth set corresponding to some formula A and adding an extra node to it. In order for the formula to be minimally valid, it should be globally true in the exact Kripke model of the $[\wedge, \to, \neg]^2$-fragment of MPC, given below. This fragment contains more than 600 trillion equivalence classes.[4]

[3] We will say more about this formula towards the end of this section, see Table 3.2.

[4] This result was computed first by de Bruijn, who discovered the exact model of the $[\wedge, \to]^3$-fragment of IPC, with 61 nodes.

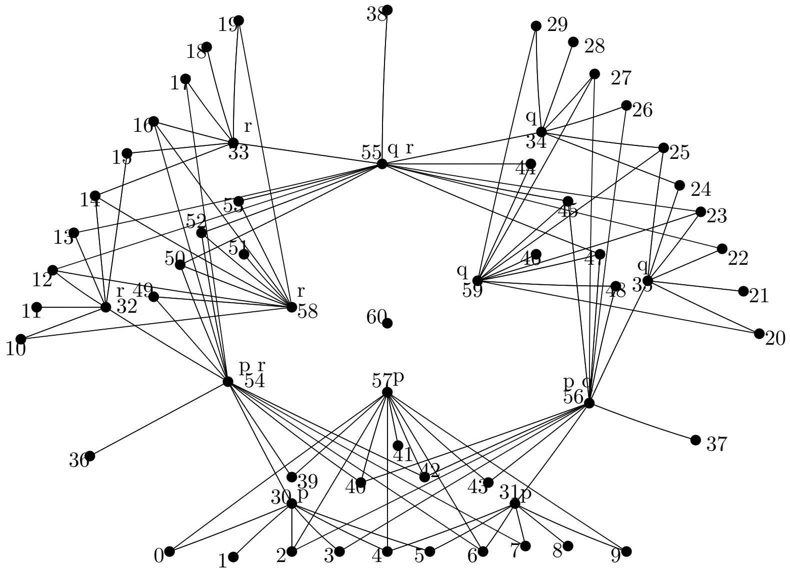


Figure 3.5: The exact Kripke model of the $[\wedge, \to, \neg]^2$-fragment of MPC.

An example of a formula that we have found by the method described above is:

$$((\neg p \to q) \to p) \to ((\neg q \to p) \to p).$$

A derivation of this formula in IPC is given below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg q \to p]^4}{\neg p \to \neg\neg q} \quad [\neg p]^3}{\neg\neg q} \quad \dfrac{[\neg p]^3 \quad \dfrac{[q]^2 \quad \dfrac{\dfrac{\dfrac{[q]^1}{\neg p \to q} \quad [(\neg p \to q) \to p]^5}{p}}{q \to p}\,{}^1}{p}}{\neg q}\,{}^2}{\bot}}{q}}{\neg p \to q}\,{}^3 \quad [(\neg p \to q) \to p]^5}{p}}{(\neg q \to p) \to p}\,{}^4}{((\neg p \to q) \to p) \to ((\neg q \to p) \to p)}\,{}^5$$

It is difficult to get an intuition of what this formula conveys and an examination of the derivation above does not make the meaning of the formula much clearer. We have found several other formulas of the form $A \to B$ for which the truth sets of A and B differ by one node. Yet, often, they were even more complex in meaning and structure. Moreover, there are certainly interesting differences between MPC and IPC that include disjunctions. Considering these reasons, we broaden our research to the full fragment $[\wedge, \vee, \to, \neg]$ and restrict the length of the formulas so that we can avoid high complexity in managing the infinite fragment.

The $[\wedge, \vee, \to, \neg]^2$-fragment of MPC

The $[\wedge, \vee, \to, \neg]^2$-fragment of MPC is the full fragment in two propositional variables. Lex Hendriks designed programs for both generating all intuitionistic equivalence classes of formulas up to a maximal length and for checking whether, for formulas A and B of two different classes, the implication $A \to B$ is intuitionistically, yet not minimally, valid.

The length $len(A)$ of a formula is calculated in the following way:

$$\begin{aligned} len(p) &= 1 \\ len(\neg A) &= len(A) + 1 \\ len(A \circ B) &= len(A) + len(B) + 1 \qquad\qquad \text{where } \circ \in \{\wedge, \vee, \to, \leftrightarrow\} \end{aligned}$$

Note that brackets do not increase the length of a formula. Minimally non-equivalent formulas of the form $A \to B$ with at most length 9 that are intuitionistic validities but not minimal validities are given in Table 3.2 below. Of course, there may be formulas in the same equivalence class of greater length, but we chose representatives with a minimal length.

	Formula	Length
1.	$p \to (\neg p \to q)$	6
2.	$q \to (\neg q \to p)$	6
3.	$\neg(p \to p) \to p$	6
4.	$\neg(p \to p) \to q$	6
5.	$\neg\neg(\neg\neg p \to p)$	7
6.	$\neg\neg(\neg\neg q \to q)$	7
7.	$\neg(p \to p) \to (p \wedge q)$	8
8.	$\neg(p \to p) \to (p \vee q)$	8
9.	$\neg(p \to p) \to (q \leftrightarrow p)$	8
10.	$\neg(p \to q) \to \neg\neg p$	8
11.	$\neg(q \to p) \to \neg\neg q$	8
12.	$\neg(\neg p \to q) \to \neg p$	8
13.	$\neg(\neg q \to p) \to \neg q$	8
14.	$\neg p \to (p \vee (p \to q))$	8
15.	$\neg q \to (q \vee (q \to p))$	8

	Formula	Length
16.	$\neg(p \to q) \to (\neg p \to p)$	9
17.	$\neg(q \to p) \to (\neg q \to q)$	9
18.	$\neg(\neg p \to (p \to q)) \to p$	9
19.	$\neg(\neg p \to (p \to q)) \to p$	9
20.	$((p \to q) \to p) \to \neg\neg p$	9
21.	$((q \to p) \to q) \to \neg\neg q$	9
22.	$(\neg\neg(p \to q) \leftrightarrow p) \to p$	9
23.	$(\neg\neg(q \to p) \leftrightarrow q) \to p$	9
24.	$p \to (\neg p \vee (\neg p \to q))$	9
25.	$q \to (\neg q \vee (\neg q \to p))$	9
26.	$p \to (\neg q \vee (\neg p \to q))$	9
27.	$q \to (\neg p \vee (\neg q \to p))$	9
28.	$\neg p \to (\neg q \vee (p \to q))$	9
29.	$\neg q \to (\neg p \vee (q \to p))$	9

Table 3.2: Representatives of equivalence classes of minimal invalidities.

An interesting question that comes up is whether all formulas we have found will give us intuitionistic logic when adding them as an axiom to minimal logic. By adding a formula of the form $A \to B$ to minimal logic, we mean adding the rule:

$$\frac{A}{B}$$

to the natural deduction system of MPC. As can be seen in the table above, the first two formulas both represent some form of ex falso, despite the fact that they are minimally not equivalent. Hence, when we add one of these formulas as an axiom to MPC, the resulting logic is precisely IPC.

3.3 Superminimal Logics

The results in this section are related to the work on extensions of minimal logic by Krister Segerberg in [Seg68] and by Sergei Odintsov in [Odi08, Chapter 5 and 6]. In fact, we will see that the logics we discover here already appeared in [Seg68]. We, however, have used computational methods to obtain our extensions and we have systematically considered the different fragments of minimal logic with restrictions on the length of the formula. Moreover, in comparison to the paraconsistent logics studied by Segerberg and Odintshov, we are merely interested in logics strictly between MPC and IPC.

The logics we obtain by adding an intuitionistic validity to the axioms of minimal logic, will always be sublogics of IPC. Before we discuss the sublogics we have found, we give two examples of logics obtained from MPC by adding an extra axiom, that are incomparable to IPC.

Take the formula $p \vee \neg p$. If we add this formula as an axiom to MPC, the resulting logic is clearly not a sub-logic of IPC, because the law of excluded middle is not intuitionistically valid. If we add this formula as an axiom to IPC, the resulting logic is classical propositional logic, CPC, but, if we add it to MPC, the resulting logic is not CPC. To see this, just consider the model consisting of a single node forcing f. Then for every formula A, $A \vee \neg A$ is forced on this model. However, $\neg\neg p \to p$ is not. Therefore, when we only add $p \vee \neg p$ as an axiom to MPC, the formula $\neg\neg p \to p$ is not derivable in the newly obtained logic. Hence, this new logic is an extension of minimal logic that is incomparable to intuitionistic logic.

For the other example, take the formula $\neg p \vee \neg\neg p$. This is not an intuitionistically valid formula, adding this formula to IPC results in the intermediate logic KC. All instances of this formula are again valid on the single node that only forces f. However, not all instances of ex falso are valid in this model, take for instance $\neg(p \to p) \to p$. Therefore, adding $\neg p \vee \neg\neg p$ as an axiom to MPC results in a logic incomparable to IPC.

Let us call a logic that extends minimal propositional logic a *superminimal logic*. Since our investigation solely leads to extensions of minimal propositional logic that are contained in intuitionistic propositional logic, the superminimal logics we find will all be subintuitionistic logics. From the list of 29 formulas in Table 3.2, there are 11 formulas that give rise to intuitionistic logic when adding them as an axiom to minimal logic. The remaining 18 formulas are given in Table 3.3 below and give rise to the following four different superminimal logics.

	Formula	Length	Logic
5.	$\neg\neg(\neg\neg p \to p)$	7	SM1
6.	$\neg\neg(\neg\neg q \to q)$	7	SM1
10.	$\neg(p \to q) \to \neg\neg p$	8	SM1
11.	$\neg(q \to p) \to \neg\neg q$	8	SM1
12.	$\neg(\neg p \to q) \to \neg p$	8	SM1
13.	$\neg(\neg q \to p) \to \neg q$	8	SM1
20.	$((p \to q) \to p) \to \neg\neg p$	9	SM1
21.	$((q \to p) \to q) \to \neg\neg q$	9	SM1
22.	$(\neg\neg(p \to q) \leftrightarrow p) \to p$	9	SM1
23.	$(\neg\neg(q \to p) \leftrightarrow q) \to q$	9	SM1
14.	$\neg p \to (p \vee (p \to q))$	8	SM2
15.	$\neg q \to (q \vee (q \to p))$	8	SM2
24.	$p \to (\neg p \vee (\neg p \to q))$	9	SM3
25.	$q \to (\neg q \vee (\neg q \to p))$	9	SM3
26.	$p \to (\neg q \vee (\neg p \to q))$	9	SM4
27.	$q \to (\neg p \vee (\neg q \to p))$	9	SM4
28.	$\neg p \to (\neg q \vee (p \to q))$	9	SM4
29.	$\neg q \to (\neg p \vee (q \to p))$	9	SM4

Table 3.3: Minimal invalidities and superminimal logics.

Note that formulas 5 and 6 are also of the form $A \to B$, where B is f or, for instance, $\neg(p \to p)$. Let us first prove that the formulas that are grouped together in this table, indeed give rise to the same logic. We call a formula A the *p-variant* of formula B, if A is obtained from B by switching p and q, and if in A the most left propositional variable is p. Analogously, we can say A is the q-variant of B. Let us note that adding the p-variant or the q-variant of a formula as an axiom to MPC, gives rise to the same logic. Therefore, it is only left to show that formulas 5, 10, 12, 20 and 22 give rise to the same logic (SM1), and that formulas 26 and 28 give rise to the same logic (SM4).

Let us denote the logic obtained by adding formula n to MPC by Ln.

Proposition 3.3.1. *L*5 = *L*10 = *L*12 = *L*20 = *L*22

Proof. L5 is the logic obtained by adding $\neg\neg(\neg\neg p \to p)$ as an axiom to MPC. Over MPC, $\neg\neg(\neg\neg p \to p)$ implies $\neg\neg(q \vee (q \to p))$, which is equivalent to $\neg(q \to p) \to \neg\neg q$, the q-variant of formula 10. Hence, L10 $\subseteq$ L5. On the other hand, $\neg(p \to q) \to \neg\neg p$ is minimally equivalent to $\neg\neg(p \vee (p \to q))$. If we substitute $\neg\neg p$ for p and p for q, we obtain the formula $\neg\neg(\neg\neg p \vee (\neg\neg p \to p))$, which is minimally equivalent to $\neg\neg(\neg\neg p \to p)$. Hence, $\neg\neg(\neg\neg p \to p)$ is derivable in L10 and thus L5 $\subseteq$ L10. If we substitute $\neg p$ for p in formula 10, we obtain formula 12. Hence, formula 12 is derivable in $\mathsf{L10}$ and thus $\mathsf{L12} \subseteq \mathsf{L10}$. To show the other inclusion, we observe:

$$\begin{aligned} \vdash_{\mathsf{MPC}} p \to \neg\neg p \ &\Rightarrow \vdash_{\mathsf{MPC}} (\neg\neg p \to q) \to (p \to q) \\ &\Rightarrow \vdash_{\mathsf{MPC}} \neg(p \to q) \to \neg(\neg\neg p \to q) \\ &\Rightarrow \vdash_{\mathsf{MPC}} (\neg(\neg\neg p \to q) \to \neg\neg p) \to (\neg(p \to q) \to \neg\neg p) \end{aligned}$$

By substituting $\neg p$ for p in formula 12, we conclude that $\neg(\neg\neg p \to q) \to \neg\neg p$ is derivable in $\mathsf{L12}$. So, by the above reasoning, also $\neg(p \to q) \to \neg\neg p$ is derivable in $\mathsf{L12}$, which is formula 10. We conclude that $\mathsf{L12} = \mathsf{L10}$. Over MPC, formula 10 implies formula 20, so $\mathsf{L20} \subseteq \mathsf{L10}$. If we substitute $\neg p$ for p in formula 20, we get a formula that is minimally equivalent to formula 12. Hence, $\mathsf{L12} \subseteq \mathsf{L20}$ and thus $\mathsf{L5} = \mathsf{L10} = \mathsf{L12} = \mathsf{L20}$. Logic $\mathsf{L22}$ is obtained by adding $(\neg\neg(p \to q) \leftrightarrow p) \to p$ as an axiom. We observe:

$$\begin{aligned} \vdash_{\mathsf{L22}} (\neg\neg(p \to q) \leftrightarrow p) \to p \ &\Rightarrow \vdash_{\mathsf{L22}} \neg p \to \neg(\neg\neg(p \to q) \leftrightarrow p) \\ &\Rightarrow \vdash_{\mathsf{L22}} \neg\neg(\neg\neg(p \to q) \leftrightarrow p) \to \neg\neg p \\ &\Rightarrow \vdash_{\mathsf{L22}} \neg\neg(\neg\neg(\neg p \to q) \leftrightarrow \neg p) \to \neg p \end{aligned}$$

This last formula is over MPC equivalent to $\neg\neg(\neg\neg(\neg p \to q) \to \neg p) \to \neg p$. Moreover, over MPC, $\neg(\neg p \to q)$ implies $\neg\neg(\neg\neg(\neg p \to q) \to \neg p)$. Hence, $\neg(\neg p \to q) \to \neg p$ is derivable in $\mathsf{L22}$ and thus $\mathsf{L12} \subseteq \mathsf{L22}$. Finally, over MPC, formula 10 implies formula 22, so $\mathsf{L22} \subseteq \mathsf{L10}$. And thus $\mathsf{L5} = \mathsf{L10} = \mathsf{L12} = \mathsf{L20} = \mathsf{L22}$. ☐

Proposition 3.3.2. *$\mathsf{L28} = \mathsf{L26}$*

Proof. Clearly $\mathsf{L28} \subseteq \mathsf{L26}$, because formula 28 is obtained from formula 26 by substituting $\neg p$ for p. On the other hand, if we substitute $\neg p$ for p in formula 28, we obtain the formula $\neg\neg p \to (\neg q \vee (\neg p \to q))$. Now, using $\vdash_{\mathsf{MPC}} p \to \neg\neg p$, we conclude that we can derive formula 26, $p \to (\neg q \vee (\neg p \to q))$, in $\mathsf{L28}$. Hence, $\mathsf{L28} = \mathsf{L26}$. ☐

We completed showing that the formulas that are grouped together in Table 3.3, indeed give rise to the same logic. In other words, all these formulas are representatives for the same superminimal logic. In the subsequent propositions, we will prove the strict inclusions between the four superminimal logics as shown in the figure below.

$$\begin{tikzcd} & \mathsf{SM1} \arrow[dr, phantom, sloped, "\subsetneq"] & & & \\ \mathsf{MPC} \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathsf{SM3} \arrow[r, phantom, "\subsetneq"] & \mathsf{SM2} \arrow[r, phantom, "\subsetneq"] & \mathsf{IPC} \\ & \mathsf{SM4} \arrow[ur, phantom, sloped, "\subsetneq"] & & & \end{tikzcd}$$

Figure 3.6: Proper inclusions of superminimal logics.

The two most left strict inclusions are clear, since $\mathsf{SM1}$ and $\mathsf{SM4}$ contain formulas that are not minimally valid. Moreover, $\mathsf{SM1}$ and $\mathsf{SM4}$ are incomparable, in the sense that $\mathsf{SM1} \not\subseteq \mathsf{SM4}$ and $\mathsf{SM4} \not\subseteq \mathsf{SM1}$, which will be shown in Proposition 3.3.7.

Let us describe the way we prove that for two superminimal logics SM_A and SM_B, both obtained by adding a single formula A or B to minimal logic, we have $\mathsf{SM}_A \subsetneq \mathsf{SM}_B$. First, we can show that $\mathsf{SM}_A \subseteq \mathsf{SM}_B$ by proving that A can be derived in SM_B, i.e., that A can be derived over MPC from substitution instances of B. Then, we can show that $\mathsf{SM}_A \subsetneq \mathsf{SM}_B$ by finding a model of MPC on which every substitution instance of A is valid, yet on which some substitution instance of B is not. Since this will be a model of SM_A, this means that B is not in the logic SM_A and thus $\mathsf{SM}_B \not\subseteq \mathsf{SM}_A$. Hence, $\mathsf{SM}_A \subsetneq \mathsf{SM}_B$.

Proposition 3.3.3. $\mathsf{SM2} \subsetneq \mathsf{IPC}$

Proof. We already know that $\mathsf{SM2} \subseteq \mathsf{IPC}$. So, we can prove $\mathsf{SM2} \subsetneq \mathsf{IPC}$ by finding a formula in IPC that is not derivable in $\mathsf{SM2}$. Take the formula $\neg p \to (p \to q)$ and consider the single-noded, minimal Kripke model $\mathcal{M}_1$ below.

$$\bullet\ f, p$$

Figure 3.7: Model $\mathcal{M}_1$

Clearly, $\neg p \to (p \to q)$ is not valid on $\mathcal{M}_1$. But, we will show that every substitution instance of the formula $\neg p \to (p \vee (p \to q))$, formula 14, is valid on this model. Let A and B be arbitrary formulas. Clearly, $\mathcal{M}_1 \Vdash \neg A$. So, we will have to show that $\mathcal{M}_1 \Vdash A \vee (A \to B)$. If $\mathcal{M}_1 \Vdash A$, we are done. If not, then $\mathcal{M} \nVdash A$ and thus $\mathcal{M}_1 \Vdash A \to B$. We can therefore conclude $\mathcal{M}_1 \Vdash \neg A \to (A \vee (A \to B))$. Hence, $\mathcal{M}_1$ is a model of $\mathsf{SM2}$ but not of IPC. Therefore $\mathsf{IPC} \not\subseteq \mathsf{SM2}$ and this finishes our proof. $\square$

Proposition 3.3.4. $\mathsf{SM3} \subsetneq \mathsf{SM2}$

Proof. Logic $\mathsf{SM2}$ is axiomatised by formula 14, $\neg p \to (p \vee (p \to q))$. Hence, $\mathsf{SM2}$ also contains the formula $\neg\neg p \to (\neg p \vee (\neg p \to q))$. Then, using that $\vdash_{\mathsf{MPC}} p \to \neg\neg p$, we conclude that also $p \to (\neg p \vee (\neg p \to q))$ is derivable in $\mathsf{SM2}$. This is precisely formula 24, the formula that axiomatises logic $\mathsf{SM3}$. Hence, we conclude that $\mathsf{SM3} \subseteq \mathsf{SM2}$. Now, consider the minimal Kripke model below.

$$\begin{tikzcd} v \bullet\ f, p \\ w \bullet\ f \arrow[u] \end{tikzcd}$$

Figure 3.8: Model $\mathcal{M}_2$

Clearly, $w \nVdash \neg p \to (p \vee (p \to q))$, so formula 14 is not valid on $\mathcal{M}_2$. On the other hand, we will show that for all formulas A and B we have $w \Vdash A \to (\neg A \vee (\neg A \to B))$, which are all substitution instances of formula 24. This is simple: $w \Vdash \neg A$, since f is forced globally. Hence, $\mathcal{M}_2$ is a model of $\mathsf{SM3}$ but not of $\mathsf{SM2}$. We conclude $\mathsf{SM2} \not\subseteq \mathsf{SM3}$ and thus $\mathsf{SM3} \subsetneq \mathsf{SM2}$. $\square$

Proposition 3.3.5. $\mathsf{SM4} \subsetneq \mathsf{SM3}$

Proof. The logic $\mathsf{SM3}$ is axiomatised by formula 24, $p \to (\neg p \vee (\neg p \to q))$ and the logic $\mathsf{SM4}$ is axiomatised by formula 26, $p \to (\neg q \vee (\neg p \to q))$. Formula 24 is minimally equivalent to $p \to (f \vee (\neg p \to q))$. Hence, using $\vdash_{\mathsf{MPC}} f \to (q \to f)$, we can derive $p \to (\neg q \vee (\neg p \to q))$ in $\mathsf{SM3}$. We conclude that $\mathsf{SM4} \subseteq \mathsf{SM3}$. Consider the minimal Kripke model below.

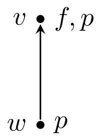


Figure 3.9: Model $\mathcal{M}_3$

It is clear that $w \nVdash p \to (\neg p \vee (\neg p \to q))$. We prove that $w \Vdash A \to (\neg B \vee (\neg A \to B))$ for all formulas A and B, i.e., for all substitution instances of formula 26. Suppose $w \nVdash A$, then we are done, because $v \Vdash \neg B$. Suppose $w \Vdash A$. If $w \Vdash \neg B$, then we are done. If $w \nVdash \neg B$, then necessarily $w \Vdash B$. But then of course $w \Vdash \neg A \to B$ by persistency, so we are also done. We conclude that $w \Vdash A \to (\neg B \vee (\neg A \to B))$ for all A and B. Hence, $\mathcal{M}_3$ is a model of $\mathsf{SM4}$, but not a model of $\mathsf{SM3}$. Therefore $\mathsf{SM3} \not\subseteq \mathsf{SM4}$ and thus $\mathsf{SM4} \subsetneq \mathsf{SM3}$. □

Proposition 3.3.6. $\mathsf{SM1} \subsetneq \mathsf{SM3}$

Proof. Formula 12, $\neg(\neg p \to q) \to \neg p$, gives rise to logic $\mathsf{SM1}$ and formula 24, $p \to (\neg p \vee (\neg p \to q))$, gives rise to $\mathsf{SM3}$. Since, over MPC, 24 implies 12, we conclude that $\mathsf{SM1} \subseteq \mathsf{SM3}$. Consider the minimal Kripke model below.

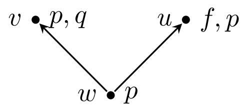


Figure 3.10: Model $\mathcal{M}_4$

We can see that $p \to (\neg p \vee (\neg p \to q))$ is not valid on this model, because $w \Vdash p$ but $w \nVdash \neg p$ and $w \nVdash \neg p \to q$. Now, as in the above propositions, we prove that $\neg(\neg A \to B) \to \neg A$ is valid on $\mathcal{M}_4$ for every formula A and B. Suppose $v \nVdash A$, then $w \Vdash \neg A$ and we are done. So, suppose $v \Vdash A$. Then $v \nVdash \neg A$, so $v \Vdash \neg A \to B$ and thus $w \nVdash \neg(\neg A \to B)$, because $v \nVdash f$. Also $v \nVdash \neg(\neg A \to B)$. Therefore, we conclude that $w \Vdash \neg(\neg A \to B) \to \neg A$. Hence, $\mathcal{M}_4$ is a model of $\mathsf{SM1}$ but not of $\mathsf{SM3}$. We conclude that $\mathsf{SM3} \not\subseteq \mathsf{SM1}$ and thus we obtain the strict inclusion $\mathsf{SM1} \subsetneq \mathsf{SM3}$. □

Proposition 3.3.7. $\mathsf{SM1} \not\subseteq \mathsf{SM4}$ *and* $\mathsf{SM4} \not\subseteq \mathsf{SM1}$

Proof. We first prove that $\mathsf{SM4} \not\subseteq \mathsf{SM1}$. Consider model $\mathcal{M}_4$ in Figure 3.10. We have already shown in the proof of the previous proposition that this is a model of

SM1. However, $p \to (\neg q \vee (\neg p \to q))$ is not valid on this model, because $w \Vdash p$ but $w \nVdash \neg q$ and $w \nVdash \neg p \to q$. Hence, this is not a model of SM4 and therefore SM4 $\nsubseteq$ SM1. For proving SM1 $\nsubseteq$ SM4, consider model $\mathcal{M}_3$ in Figure 3.9. In the proof of proposition Proposition 3.3.5 we have shown that this is a model of SM4. Recall that $\neg(\neg p \to q) \to \neg p$ is a representative for SM1. We will show that this formula is not valid on $\mathcal{M}_3$. We can see that $w \nVdash \neg p \to q$, so we know that $w \Vdash \neg(\neg p \to q)$. However, $w \nVdash \neg p$. So we conclude that $w \nVdash \neg(\neg p \to q) \to \neg p$ and thus $\mathcal{M}_3$ is not a model of SM1. Hence, SM1 $\nsubseteq$ SM4. $\square$

Remark 3.3.8. The formula $\neg(\neg p \to q) \to \neg p$, which gives rise to superminimal logic SM1, is minimally equivalent to $p \to \neg\neg(\neg p \to q)$. This formula can be constructed from the formula $p \to (\neg p \to q)$, a form of ex falso, by weakening the consequent. Therefore, we could see logic SM1 as the result of adding a weakened version of ex falso to MPC. Another way we could weaken ex falso, is by taking the double negation of the whole formula, i.e., $\neg\neg(p \to (\neg p \to q))$. This formula, in fact, gives rise to the same logic, SM1.

Another representative of SM1 is, for instance:

$$(p \to \neg\neg q) \to \neg\neg(p \to q)$$

We will prove this. Let us denote the logic obtained from adding $(p \to \neg\neg q) \to \neg\neg(p \to q)$ as an axiom to MPC by L^*. As we have just mentioned, $p \to \neg\neg(\neg p \to q)$ can be taken as a representative for SM1. We observe:

$$\begin{aligned} \vdash_{\mathsf{SM1}} p \to \neg\neg(\neg p \to q) &\Rightarrow \vdash_{\mathsf{SM1}} \neg p \to \neg\neg(\neg\neg p \to q) && \text{(substitution)} \\ &\Rightarrow \vdash_{\mathsf{SM1}} \neg p \to \neg\neg(p \to q) && (\mathsf{MPC}\text{-valid}) \\ &\Rightarrow \vdash_{\mathsf{SM1}} (p \to \neg\neg q) \to \neg\neg(p \to q) && (\mathsf{MPC}\text{-valid}) \end{aligned}$$

Hence, $\mathsf{L}^* \subseteq$ SM1. For the other inclusion, we observe:

$$\begin{aligned} \vdash_{\mathsf{L}^*} (p \to \neg\neg q) \to \neg\neg(p \to q) &\Rightarrow \vdash_{\mathsf{L}^*} (\neg\neg p \to \neg\neg p) \to \neg\neg(\neg\neg p \to p) && \text{(substitution)} \\ &\Rightarrow \vdash_{\mathsf{L}^*} \neg\neg(\neg\neg p \to p) && (\mathsf{MPC}\text{-valid}) \end{aligned}$$

We note that $\neg\neg(\neg\neg p \to p)$ is a representative for SM1 and conclude that SM1 $\subseteq \mathsf{L}^*$ and thus $\mathsf{L}^* =$ SM1.

Recall from Remark 2.3.2 that the statement $\neg\neg\mathbf{CT} \leftrightarrow \forall\alpha\neg\neg GR(\alpha)$ is not provable using only minimal logic, because, in MPC, $(p \to \neg\neg q) \to \neg\neg(p \to q)$ does not hold. However, we have just proven that there exists a superminimal-subintuitionistic logic in which $(p \to \neg\neg q) \to \neg\neg(p \to q)$ is derivable. Hence, Proposition 2.3.1 can be proved in SM1.

As we have mentioned at the beginning of this section, our work on superminimal logics is related to the work of Segerberg ([Seg68]) and Odintsov ([Odi08]). First of all, our logic SM1 was baptised *Glivenko's Logic* by Odintsov, for the reason that, as Segerberg already remarks, it is the weakest logic in which $\neg\neg A$ is derivable if and only if A is classically derivable.

In Segerberg, the representative for SM1 is written as $\neg\neg(f \to p)$, which is an intuitionistically valid instance of the intuitionistically invalid Peirce's law, because $\neg\neg(f \to p) = ((f \to p) \to f) \to f$.

Moreover, we have found out that our other logics also appear among *Segerberg's Logics* (how Odintsov denotes them, see [Odi08, Chapter 5]), in the form of disjunctions:

$\mathsf{SM2}$ as axiomatised by: $(\neg(p \to p) \to p) \vee (\neg(p \to p) \to (p \to q))$

$\mathsf{SM3}$ as axiomatised by: $\neg(p \to p) \vee (\neg(p \to p) \to p)$

$\mathsf{SM4}$ as axiomatised by: $\neg p \vee (\neg(p \to p) \to p)$

The representatives we have found in our research as axioms for these logics, do not appear in [Seg68] nor [Odi08]. For us, however, they are interesting as theorems, or rules, of intuitionistic logic that are minimally invalid, but that are weaker than ex falso. Finally, our research can easily be extended by systematically computing minimally invalid formulas of the form $A \to B$ of a maximal length greater than 9.

Chapter 4

Obtaining Minimal Theories from Intuitionistic Theories

In this chapter, we will develop a general framework for considering a theory based on different logical systems. Our aim is to investigate the consequences of considering certain theories in the context of minimal logic. We attempt two different approaches, which will be formalised in the remainder of this section: In the first approach, we add an unspecified falsum to the system and the meaning of a negated formula $\neg A$ becomes $A \to f$. In the second approach, we interpret falsum by a formula in the system. There may be several possible candidates, which will become clear when we discuss the conditions such an interpretation has to satisfy. In some cases, there exists a really attractive sentence from which all formulas can be derived. We will investigate this case as well.

4.1 From Axiomatisations to Theories

From here on, we will not consider the connective $\neg$ as an abbreviation anymore, in the sense that $\neg A$ does not abbreviate $A \to \bot$, but we will consider $\neg$ a primitive symbol of our language. We need this syntactical distinction between $\neg A$, $A \to \bot$ and $A \to f$, in order to interpret negation differently depending on context.

Definition 4.1.1. A set of sentences $\mathcal{A}$ is an *axiomatisation* of a theory T with underlying logic L if $\mathsf{T} = \{A \mid \mathcal{A} \vdash_{\mathsf{L}} A\}$. We call $\mathcal{A}$ a *clean axiomatisation* if it is formulated in the fragment $[\wedge, \vee, \to, \neg, \forall, \exists]$ (i.e., without $\bot$ or f).

We will from now assume a theory to have a clean axiomatisation. Only after a certain *translation* of the axioms, which we will now define, negation is given a particular meaning.

Definition 4.1.2. Let T be a theory and ψ a sentence formulated in $\mathscr{L}(\mathsf{T}) \setminus \{\neg\}$.

We define the translation τ_ψ by induction on formulas as follows:

$$\begin{aligned} \tau_\psi(A) &:= A && \text{for } A \text{ propositional or atomic} \\ \tau_\psi(A \circ B) &:= \tau_\psi(A) \circ \tau_\psi(B) && \text{with } \circ \in \{\wedge, \vee, \to\} \\ \tau_\psi(\neg A) &:= \tau_\psi(A) \to \psi \\ \tau_\psi(\forall x A) &:= \forall x \tau_\psi(A) \\ \tau_\psi(\exists x A) &:= \exists x \tau_\psi(A) \end{aligned}$$

Note that, since we require ψ to be negationless, we have $\tau_\psi(\psi) = \psi$. Now, given any clean axiomatisation $\mathcal{A}$ for a theory T, let $\mathcal{A}^\psi := \{\tau_\psi(A) \mid A \in \mathcal{A}\}$. We can then define:

$$\mathsf{T}_\psi := \{B \mid \mathcal{A}^\psi \vdash_{\mathsf{IQC}} \tau_\psi(B)\},$$
$$\mathsf{MT}_\psi := \{B \mid \mathcal{A}^\psi \vdash_{\mathsf{MQC}} \tau_\psi(B)\}.$$

We will say that a formula B is *derivable in* T_ψ, $\mathsf{T}_\psi \vdash B$, whenever $B \in \mathsf{T}_\psi$. And, similarly, $\mathsf{MT}_\psi \vdash B$ whenever $B \in \mathsf{MT}_\psi$.

We require an interpretation ψ of falsum to be negationless because we want to avoid circularity, as we do not want to interpret negation in terms of negation.

The following proposition follows immediately from the above definitions.

Proposition 4.1.3. *Let* T *be a theory and* ψ *a sentence formulated in* $\mathscr{L}(\mathsf{T}) \setminus \{\neg\}$. *Then* T_ψ *is closed under the natural deduction rules of* IQC*, and* MT_ψ *is closed under the natural deduction rules of* MQC. □

The following lemma shows that negation behaves as intended in the systems T_ψ and MT_ψ.

Lemma 4.1.4. *We have* $\mathsf{T}_\psi \vdash \neg A \leftrightarrow (A \to \psi)$ *and* $\mathsf{MT}_\psi \vdash \neg A \leftrightarrow (A \to \psi)$.

Proof. By the definition of the translation τ_ψ we have:

$$\begin{aligned} \tau_\psi(\neg A \leftrightarrow (A \to \psi)) &= \tau_\psi(\neg A) \leftrightarrow \tau_\psi(A \to \psi) \\ &= (\tau_\psi(A) \to \psi) \leftrightarrow (\tau_\psi(A) \to \tau_\psi(\psi)) \\ &= (\tau_\psi(A) \to \psi) \leftrightarrow (\tau_\psi(A) \to \psi) \end{aligned}$$

Hence, using $\vdash_{\mathsf{MPC}} p \to p$, we can conclude that $\mathcal{A}^\psi \vdash_{\mathsf{MQC}} \tau_\psi(\neg A \leftrightarrow (A \to \psi))$, which means that both $\mathsf{MT}_\psi \vdash \neg A \leftrightarrow (A \to \psi)$ and $\mathsf{T}_\psi \vdash \neg A \leftrightarrow (A \to \psi)$ hold. □

We will sometimes say that we add a certain logic to a theory. By adding intuitionistic logic to a theory T, we mean to obtain the theory $\mathsf{T}_\bot$, and by adding minimal logic to a theory T, we mean to obtain the theory MT_f. If the original theory T is an intuitionistic theory, i.e., a theory over IQC such that $\bot \in \mathscr{L}(\mathsf{T})$, then the resulting theory $\mathsf{T}_\bot$ is precisely the original theory T. Take, for instance, Heyting arithmetic, then $\mathsf{HA}_\bot = \mathsf{HA}$. Similarly, if the original theory T is a minimal theory, i.e., a theory over MQC such that $f \in \mathscr{L}(\mathsf{T})$, then the resulting theory MT_f is precisely the original theory T. Therefore, we may assume that every intuitionistic theory is of the form $\mathsf{T}_\bot$ and every minimal theory is of the form MT_f.

Having given the necessary definitions, we can now observe some general properties of the different theories.

A formula A is an *intuitionistic theorem* when $\vdash_{\mathsf{IQC}} A$, i.e., when A is derivable in the natural deduction system for IQC from an empty set of assumptions.

Proposition 4.1.5. *Let* T *be an intuitionistic theory. Then any formula formulated in* $\mathscr{L}(\mathsf{T})$ *is either an intuitionistic theorem, equivalent to* $\perp$*, or, equivalent to a* $\perp$*-free formula.*

Proof. Let T be any intuitionistic theory. We will show by induction on formulas that for any well-formed formula A in $\mathscr{L}(T)$, the above claim holds.

The cases for $A = \perp$ and A atomic are trivial.

Suppose $A = B \vee C$. If B or C is an intuitionistic theorem, then A is too. If B is equivalent to $\perp$ or C is equivalent to $\perp$, then A is equivalent to C or A is equivalent to B, respectively. Finally, if B and C are both equivalent to a $\perp$-free formula, then A as well.

Suppose $A = B \wedge C$. If B or C is equivalent to $\perp$, then A is too. If B is an intuitionistic theorem, then A is equivalent to C and if C is an intuitionistic theorem, then A is equivalent to B. Finally, if B and C are both equivalent to a $\perp$-free formula, then so is A.

Suppose $A = B \to C$. If B is equivalent to $\perp$, then A is an intuitionistic theorem. If C is equivalent to $\perp$, then we can rewrite A as $\neg B$. Then, in case B is an intuitionistic theorem, A is equivalent to $\perp$ and in case B is $\perp$-free, then A is too. If B is an intuitionistic theorem, then A is equivalent to C. If C is an intuitionistic theorem, then A is too. Finally, if both B and C are equivalent to a $\perp$-free formula, then A as well.

Suppose $A = \exists x B$. If B is $\perp$-free, then A is too. If B is equivalent to $\perp$, then A as well. And, if B is an intuitionistic theorem, then so is A.

The case for $A = \forall x B$ works analogously to the previous case.

This finishes our proof by induction.[1] □

An intuitionistic theory T is consistent if $\mathsf{T} \nvdash \perp$. An immediate consequence of the previous proposition is the following corollary.

Corollary 4.1.6. *If* $\mathcal{A}$ *is an axiomatisation of a consistent intuitionistic theory* T*, then there exists a clean axiomatisation* $\mathcal{A}^*$ *of* T*.*

Proof. By Proposition 4.1.5 we know that all formulas in $\mathcal{A}$ are, over T, equivalent to $\perp$-free formulas or intuitionistic theorems. Hence, we can construct an equivalent axiomatisation $\mathcal{A}^*$ of T that is clean. □

[1] Note that we do not have to prove the case for negation, since T is the same theory as $\mathsf{T}_\perp$ and in the latter $\neg A$ is equivalent to $A \to \perp$.

Note that the standard axiomatisations of the intuitionistic theory of arithmetic and the intuitionistic theory of apartness are clean axiomatisations. The following corollary demonstrates that when we add minimal logic to an intuitionistic theory, we may assume that the resulting theory is $\bot$-free.

Corollary 4.1.7. *For an intuitionistic theory* T*, we may assume that all sentences derivable in the theory* MT_ψ *are* $\bot$*-free whenever* ψ *is* $\bot$*-free.*

Proof. Let $\mathcal{A}$ be an axiomatisation of some intuitionistic theory T. We may assume, by Corollary 4.1.6, that $\mathcal{A}$ is clean. Hence, by definition of MT_ψ, $\bot$ will not occur in any sentence derivable in this theory. □

If, in one of the systems T_ψ or MT_ψ, the negation of a formula A is not equivalent to $A \to \bot$, we will call it a *pseudo-negation.*

Example 4.1.8. In chapter 5, we will encounter the theories $\mathsf{MHA}_{k=0}$ for some natural number k. These theories are axiomatised by the axioms of HA with underlying logic MQC, where $\neg A$ is equivalent to $A \to (\overline{k} = 0)$.

4.2 Properties of T_ψ and MT_ψ

As we have mentioned before, we want to examine the consequences of *interpreting* falsum when we add minimal logic to a certain theory. MT_ψ is the theory we obtain when we add minimal logic to the theory T, in which we interpret falsum by some sentence ψ. By definition of MT_ψ, the sentence ψ is negationless. Besides that, we want to add the following requirement for ψ: we require that the negation of ψ is derivable in MT_f, i.e., that $\mathsf{MT}_f \vdash \psi \to f$. This requirement is necessary to preserve the intended meaning of negation: we want ψ to be an absurdity, hence we need ψ to lead to a 'contradiction' in the original minimal system MT_f.[2]

For some theories T, we can find a sentence that naturally satisfies ex falso, i.e., such that from this sentence all formulas are derivable. In general, for any theory in a language containing only finitely many relation symbols, there always exists such a sentence.

Proposition 4.2.1. *In any theory that contains only finitely many relation symbols, there exists a sentence from which all formulas can be derived.*

Proof. Let T be a theory containing the relation symbols $R_1, \ldots, R_n$ with arities $m_1 \ldots, m_n$. We then consider the conjunction of the universal quantifications over all relations between all elements and over the negations of all relations over all elements:

$$\bigwedge_{1 \leq i \leq n} \forall x_1 \ldots \forall x_{m_i} R_i x_1 \ldots x_{m_i} \wedge \bigwedge_{1 \leq i \leq n} \forall x_1 \ldots \forall x_{m_i} \neg R_i x_1 \ldots x_{m_i}.$$

By construction of this sentence it is clear that it implies f, or $\bot$, depending on the logic of the theory, and that it implies all atomic formulas. By an easy induction on formulas we can then prove that from this sentence all formulas can be derived. □

[2]Note that for a complete theory this requirement is directly satisfied if ψ is not derivable.

Note, however, that the sentence in the proof above contains negations and is therefore not of the form we desire for an interpretation of falsum. Still, it might be possible that it is minimally equivalent to a negationless sentence (this is the case, e.g., in the minimal theory of equality and apartness, see Proposition 7.1.2). Moreover, this sentence is a rather obvious candidate for an interpretation of falsum if the goal is to satisfy the ex falso principle. But in most cases, this sentence is not so short and attractive. We are mostly interested in finding sentences that in fact are short and from which perhaps 'surprisingly' all formulas can be derived.

Remark 4.2.2. Let us consider only the positive part of the above sentence, that is:

$$\psi := \bigwedge_{1 \leq i \leq n} \forall x_1 \ldots \forall x_{m_i} R_i x_1 \ldots x_{m_i}.$$

It is clear that we can derive all positive formulas from this sentence. Moreover, if there exists a positive formula A whose negation $\neg A$ is derivable in the theory $(\mathsf{M})\mathsf{T}_\psi$, then we can derive all formulas from ψ in this theory. In this case, we have found a negationless sentence that satisfies the ex falso principle.

Proposition 4.2.3. *If in some formal system $(\mathsf{M})\mathsf{T}_\psi$, there exists a sentence A such that $\neg A$ implies all formulas, then $\neg A$ is equivalent to ψ in $(\mathsf{M})\mathsf{T}_\psi$.*

Proof. Suppose $(\mathsf{M})\mathsf{T}_\psi \vdash \neg A \to B$ for all formulas B, then we also have $(\mathsf{M})\mathsf{T}_\psi \vdash \neg A \to \psi$. Using Lemma 4.1.4 and by the fact that in minimal logic $\psi \to (A \to \psi)$ is provable, we also have $(\mathsf{M})\mathsf{T}_\psi \vdash \psi \to \neg A$. Hence, $(\mathsf{M})\mathsf{T}_\psi \vdash \neg A \leftrightarrow \psi$. □

Recall that every theory over minimal logic is of the form MT_f. The above proposition immediately gives us the following useful insight.

Theorem 4.2.4. *Let T be a minimal theory and A a formula such that in T all formulas are derivable from $\neg A$. Then, f naturally satisfies the ex falso principle.* □

A direct consequence of Proposition 4.2.3 is the following corollary. Recall that a formula A is *stable* in a formal system S if $\mathsf{S} \vdash \neg\neg A \to A$.

Corollary 4.2.5. *If in some formal system $(\mathsf{M})\mathsf{T}_\psi$, there exists a stable sentence A from which all sentences are derivable, then $(\mathsf{M})\mathsf{T}_\psi \vdash A \leftrightarrow \psi$.* □

In particular, if in MT_f there exists a stable sentence A from which all sentences are derivable, then $\mathsf{MT}_f \vdash A \leftrightarrow f$. In that case, all possible interpretations for falsum are equivalent. Unfortunately, we have not been able to find a convincing concrete example to apply this corollary.

4.3 Comparison of Intuitionistic and Minimal Theories: Conservativity Results.

For the comparison of adding either minimal or intuitionistic logic to a theory, we distinguish different theories on the basis of their axiomatisation in the following way: Firstly, the axiomatisation of a theory can be *positively given*, or *positive*, by which we mean that no axiom contains a negation. Secondly, we call a theory *simple* if all axioms in the axiomatisation that contain negation, are of the form $\neg A$ with A

positive. We call a formula of this form, $\neg A$ with A positive, a *simple negation*. Finally, a theory can be neither positive nor simple, i.e., the axiomatisation of the theory contains more complex axioms that contain negation. For a simple system, it seems a reasonable choice to interpret falsum as A where $\neg A$ is a simple axiom. An example of a simple theory is Robinson arithmetic, Q. Heyting arithmetic, HA, is rather close to being such a theory, a fact to which we will return in chapter 5. An example of a theory that is neither positive nor simple is the theory of apartness, AP, which we will discuss in chapter 7.

We will now observe some general features of the different kinds of theories.

Definition 4.3.1 (Conservative extension)**.** A theory T' is a *conservative extension* of, or *conservative over*, a theory T if $\mathsf{T} \subseteq \mathsf{T}'$ and if for every sentence A formulated in the language of T it holds that $\mathsf{T}' \vdash A$ if and only if $\mathsf{T} \vdash A$.

The following proposition shows that any intuitionistic theory proves the same positive sentences as the corresponding minimal theory.

Proposition 4.3.2. *Let* T *be a positive theory. Then* $\mathsf{T}_\bot$ *is conservative over* MT_f *with respect to positive sentences.*

Proof. We will only sketch the proof. Let A be a sentence formulated in $\mathscr{L}(\mathsf{MT}_f)$. We then need to show that $\mathsf{T}_\bot \vdash_{\mathsf{IQC}} \tau_\bot(A)$ if and only if $\mathsf{T}_f \vdash_{\mathsf{MQC}} \tau_f(A)$. But, since T and A are negation-free, this comes down to showing that for any positive sentence A we have $\vdash_{\mathsf{IQC}} A$ if and only if $\vdash_{\mathsf{MQC}} A$. In order to show this, we will temporarily switch from the natural deduction systems for IQC and MQC to one of Gentzen's sequent calculus systems for IQC and MQC, e.g., $\mathsf{G3i}$ and $\mathsf{G3m}$ as described in [TS00, Definition 3.5.1].[3]

Now, if $\vdash_{\mathsf{IQC}} A$, there exists a derivation of A in $\mathsf{G3i}$. The system $\mathsf{G3i}$ has cut elimination and thus, as a consequence of the subformula property, we know that there exists a deduction of A, with an empty set of open assumptions, in which only logical rules and axioms for the logical operators occurring in A appear. Hence, there is a derivation of A in which $\bot$ does not occur. This means that there exists a derivation of A in the weaker minimal sequent calculus $\mathsf{G3m}$, namely, the same derivation. This derivation translates to a derivation of A in the natural deduction system for MQC, i.e., $\vdash_{\mathsf{MQC}} A$. □

An immediate consequence, already stated in Lemma 2.2.6, of the above proposition is that IQC and MQC prove the same positive formulas (i.e., IQC is conservative over MQC with respect to positive formulas).

Let us now give a sequence of helpful results concerning positive theories.

Proposition 4.3.3. *Let* T^+ *be a positive theory. If* $\mathsf{T}^+ \vdash_{\mathsf{MQC}} \neg A \to \neg B$ *for positive sentences* A *and* B*, then* $\mathsf{T}^+ \vdash_{\mathsf{MQC}} B \to A$.

[3] We will not go into the details of these systems here, but refer the reader to the reference mentioned.

Proof. Let T^+ be a positive theory, then:

$$\begin{aligned} \mathsf{T}^+ \vdash \neg A \to \neg B \;&\Rightarrow\; \mathsf{T}^+ \vdash (A \to f) \to (B \to f) \\ &\Rightarrow^{(*)}\; \mathsf{T}^+ \vdash (A \to A) \to (B \to A) \\ &\Rightarrow\; \mathsf{T}^+ \vdash B \to A \end{aligned}$$

$(*)$: This substitution is valid since f does not occur in T^+, A or B. □

Remark 4.3.4. The previous proposition does not hold for IQC. To see this, we can use the fact that $\mathsf{IPC} \vdash \neg p \to \neg((p \to q) \to p)$, but $\mathsf{IPC} \nvdash ((p \to q) \to p) \to p$. In Remark 5.5.1 we will make this counterexample concrete, using a positive formulation of Heyting arithmetic.

Proposition 4.3.5. *Let T^+ be a positive theory. If $\mathsf{T}^+ \vdash_{\mathsf{MQC}} \neg A \to (\neg B \to \neg C)$ for positive sentences A, B and C, then $\mathsf{T}^+ \vdash_{\mathsf{MQC}} C \to A \vee B$.*

Proof. Let T^+ be a positive theory, then:

$$\begin{aligned} \mathsf{T}^+ \vdash \neg A \to (\neg B \to \neg C) \;&\Rightarrow\; \mathsf{T}^+ \vdash (\neg A \wedge \neg B) \to \neg C \\ &\Rightarrow\; \mathsf{T}^+ \vdash \neg(A \vee B) \to \neg C \\ &\Rightarrow\; \mathsf{T}^+ \vdash C \to A \vee B \qquad\qquad \text{(by Proposition 4.3.3)} \end{aligned}$$

□

Proposition 4.3.6. *Let $\mathsf{T} = \mathsf{T}^+ \cup \{\neg A_0\}$ be a simple theory, where T^+ is a positive theory and A_0 a positive sentence. Then $\mathsf{T}^+ \cup \{\neg A_0\}$ is conservative over T^+ with respect to $\mathscr{L}(\mathsf{T}^+)$ in MQC.*

Proof. Let T be a positive theory and A a positive sentence, then:

$$\begin{aligned} \mathsf{T}^+ \cup \{\neg A_0\} \vdash A \;&\Rightarrow\; \mathsf{T}^+ \vdash \neg A_0 \to A \\ &\Rightarrow\; \mathsf{T}^+ \vdash (A_0 \to f) \to A \\ &\Rightarrow^{(*)}\; \mathsf{T}^+ \vdash (A_0 \to A_0) \to A \\ &\Rightarrow\; \mathsf{T}^+ \vdash A \end{aligned}$$

$(*)$: This substitution is valid since f does not occur in T^+ or A. □

Remark 4.3.7. If T is a simple theory of the form $\mathsf{T} = \mathsf{T}^+ \cup \{\neg A_0\}$, then A_0 is a possible interpretation for falsum and, moreover, every other possible interpretation for falsum will be at least as strong as A_0. We can see this as follows. Suppose $\mathsf{T} \vdash \neg A$, with A positive. Then $\mathsf{T}^+ \vdash \neg A_0 \to \neg A$, i.e., $\mathsf{T}^+ \vdash (A_0 \to f) \to (A \to f)$. We can substitute A_0 for f, since f is just a propositional variable not occurring in T^+, A or A_0. We then obtain $\mathsf{T}^+ \vdash A \to A_0$. So, if A_0 proves all sentences, or a subclass of them, then so does A.

We hereby conclude our general observations and we will turn to concrete examples in the following three chapters. There are several formal theories in which there exists a short and clear sentence from which all formulas are derivable without using ex falso. Hence, when we add minimal logic to such a theory and interpret falsum as such a particular sentence, the resulting minimal theory naturally satisfies the ex falso principle. We will see in the subsequent chapter that the sentence $0 = 1$, or for instance $1 = 2$, is an example of such short and clear, but powerful sentence in the context of several theories.

Chapter 5

First-Order Minimal Heyting Arithmetic

In this chapter, we will consider first-order minimal Heyting arithmetic. After introducing Heyting arithmetic in section 5.1, we will investigate the system MHA_f, in sections 5.2–5.4, with an uninterpreted falsum. This system has the disjunction and existence property, and satisfies the propositional variant of de Jongh's theorem. However, we will see that MHA_f is rather weak: Equality is not stable and not all primitive recursive functions are representable. To prove these results we will introduce several non-standard models for minimal arithmetic.

Finally, in section 5.5, we will briefly investigate the system $\mathsf{MHA}_{0=1}$, where falsum is interpreted as $0 = 1$, and prove that it is equivalent to the intuitionistic system $\mathsf{HA}_{0=1}$. The latter system has the same proof-strength as HA. Moreover, $0 = 1$ is the strongest possible interpretation for falsum. We will see that there are many other candidates for falsum that are weaker than $0 = 1$. The reason for this is that even though it is provable in MHA_f that 0 is not a successor, MHA_f does have models in which 0 is a successor.

5.1 Heyting Arithmetic

Let us first take a look at Heyting arithmetic, HA, the formal system of intuitionistic first-order arithmetic (see e.g. [TD88, Chapter 3.3.1]). It has the same language and non-logical axioms as Peano arithmetic, but uses first-order intuitionistic predicate logic as its underlying logic.

(1) $\forall x(x = x)$

(2) $\forall x\forall y(x = y \to y = x)$

(3) $\forall x\forall y\forall z(x = y \to (y = z \to x = z))$

(4) $\forall x\forall y(x = y \to Sx = Sy)$

(5) $\forall x\forall y(Sx = Sy \to x = y)$

(6) $\forall x\neg(Sx = 0)$

(7) $\forall x(x + 0 = x)$

(8) $\forall x \forall y(x + Sy = S(x + y))$

(9) $\forall x(x \times 0 = 0)$

(10) $\forall x \forall y(x \times Sy = x \times y + x)$

Induction schema, for every formula A:

(I) $(A(0) \wedge \forall x(A(x) \to A(Sx))) \to \forall x A(x)$

Let us denote the axioms and the induction schema above by $\mathcal{A}_{\mathsf{HA}}$. We note that HA has the disjunction and existence property and satisfies de Jongh's theorem. De Jongh's theorem is the result that intuitionistic propositional logic is precisely the logic of Heyting arithmetic.

Theorem 5.1.1 (de Jongh, [Jon70])**.** *Let A be any propositional formula, then $\mathsf{IPC} \vdash A(p_1, \ldots, p_n)$ if and only if for all arithmetical sentences $\alpha_1, \ldots, \alpha_n$ we have $\mathsf{HA} \vdash A(\alpha_1, \ldots, \alpha_n)$.*

A Kripke model for HA is simply a Kripke model K for intuitionistic predicate logic for the language of HA, such that $K \Vdash \mathsf{HA}$. Equality is interpreted at each node as a congruence relation.

The following result conveys the idea that in Heyting arithmetic, $0 = 1$ is a natural candidate for falsum.

Proposition 5.1.2. *In Heyting arithmetic, we can derive all formulas from $0 = S0$, without using ex falso.*

Proof. First, note that $\perp$ is derivable from $0 = S0$, since $0 = S0$ implies $\exists x(Sx = 0)$. Now, we will show that $\vdash_{\mathsf{HA}} 0 = S0 \to \forall x(x = 0)$ by proving $\vdash_{\mathsf{HA}} \forall x(0 = S0 \to x = 0)$ by the following easy induction:

$$\begin{aligned}
x = 0 &: \text{Since } \vdash_{\mathsf{HA}} 0 = 0, \text{ then } \vdash_{\mathsf{HA}} 0 = S0 \to 0 = 0; \\
x \to Sx &: \text{If we have } \vdash_{\mathsf{HA}} 0 = S0 \to x = 0, \text{ then } \vdash_{\mathsf{HA}} 0 = S0 \to Sx = S0 \\
&\quad \text{and thus } \vdash_{\mathsf{HA}} 0 = S0 \to Sx = 0.
\end{aligned}$$

This yields that for all x and y we have $0 = S0 \to x = 0$ and $0 = S0 \to y = 0$, and thus $0 = S0 \to (x = 0 \wedge y = 0) \to x = y$. Therefore, we conclude $\vdash_{\mathsf{HA}} \forall xy(0 = S0 \to x = y)$, i.e., $\vdash_{\mathsf{HA}} 0 = S0 \to \forall xy(x = y)$. The atomic formulas of HA, except $\perp$, are of the form $s = t$, where s and t are terms. Since $\vdash_{\mathsf{HA}} 0 = S0 \to \forall xy(x = y)$, then $\vdash_{\mathsf{HA}} 0 = S0 \to s = t$ for all terms s, t. Therefore, we conclude that $\vdash_{\mathsf{HA}} 0 = S0 \to P$ for all atomic formulas P. By induction on formulas we establish that $\vdash_{\mathsf{HA}} S = S0 \to A$ for all formulas A:

$A = B \wedge C$:

$$\dfrac{\dfrac{\dfrac{[0 = S0]^1}{B}\,\text{IH} \qquad \dfrac{[0 = S0]^1}{C}\,\text{IH}}{B \wedge C}}{0 = S0 \to B \wedge C}\,1$$

$A = B \vee C$:

$$\dfrac{\dfrac{\dfrac{[0 = S0]^1}{B}\,\text{IH}}{B \vee C}}{0 = S0 \to B \vee C}\,1$$

$A = B \to C$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[0 = S0]^2 \qquad [B]^1}{0 = S0 \wedge B}}{0 = S0}\,\text{IH}}{C}}{B \to C}\,1}{0 = S0 \to (B \to C)}\,2$$

$A = \exists x B(x)$:

$$\dfrac{\dfrac{\dfrac{[0 = S0]^1}{B(x)}\,\text{IH}}{\exists x B(x)}\,\exists I \text{ (}x\text{ free for }x\text{ in }B(x)\text{)}}{0 = S0 \to \exists x B(x)}\,1$$

$A = \forall x B(x)$:

$$\dfrac{\dfrac{\dfrac{[0 = S0]^1}{B(x)}\,\text{IH}}{\forall x B(x)}\,\forall I \text{ (}x\text{ not free in `}0{=}S0\text{')}}{0 = S0 \to \forall x B(x)}\,1$$

This finishes the proof of the theorem. □

Let us note that this result extends to Kleene's system I and, for instance, to the system of intuitionistic analysis with variables for sequences, EL. Moreover, it extends to any system where function symbols are used exclusively via their values. Even equality between functions can be added as long as this equality is extensional.

5.2 Metamathematical Properties of Minimal Arithmetic

We can add minimal logic to the theory HA in the way described in Definition 4.1.2. We then obtain the theory $\mathsf{MHA}_f = \{A \mid \mathcal{A}^f_{\mathsf{HA}} \vdash_{\mathsf{MQC}} \tau_f(A)\}$, which is the deductive closure under minimal predicate logic of $\mathcal{A}_{\mathsf{HA}}$, where $\neg A$ is defined as $A \to f$ and where f can be used in instances of the induction schema. We baptise the theory MHA_f as the formal system of 'minimal first-order arithmetic'.

Besides instances of the induction schema, the only axiom containing negation in HA and in MHA_f is $\forall x \neg(Sx = 0)$. Note that $\mathsf{MHA}_f \vdash \forall x \neg(Sx = 0) \leftrightarrow \forall x(Sx = 0 \to f)$, by Lemma 4.1.4. Hence, using Lemma 2.1.2, we know that in MHA_f, and thus in HA, $\forall x \neg(Sx = 0)$ is equivalent to $\neg \exists x(Sx = 0)$. For this reason HA is close to being a simple theory, but not quite, because there can be negations in instances of the induction schema.

Similar to the case of HA, a Kripke model for MHA_f is simply a Kripke model K for minimal predicate logic for the language of MHA_f, as defined in Definition 2.2.2, such that $K \Vdash \mathsf{MHA}_f$. Equality is interpreted at each node as a congruence relation.

We will first show that MHA_f with regards to metamathematical properties is much like HA. It has the *disjunction property* and the *existence property* and we will prove *de Jongh's theorem* for MHA_f with respect to MPC. The latter makes clear that MHA_f really behaves as a 'minimal' theory.

Remark 5.2.1. In HA, and in various other intuitionistic systems, the disjunction property is reducible to the existence property by the following observation:

$$\mathsf{HA} \vdash (A \vee B) \leftrightarrow \exists x((x = 0 \to A) \wedge (\neg(x = 0) \to B)).$$

However, this is not provable anymore when we consider minimal arithmetic:

$$\mathsf{MHA}_f \nvdash (A \vee B) \to \exists x((x = 0 \to A) \wedge (\neg(x = 0) \to B)).$$

Let us prove this. Suppose $\mathsf{MHA}_f \vdash (A \vee B) \to \exists x((x = 0 \to A) \wedge (\neg(x = 0) \to B))$ and instantiate B with $1 = 0$. Then:

$$\begin{aligned}
& \mathsf{MHA}_f \vdash (A \vee 1 = 0) \to \exists x((x = 0 \to A) \wedge (\neg(x = 0) \to 1 = 0)) \\
\Rightarrow\ & \mathsf{MHA}_f \vdash (f \to (A \vee 1 = 0) \to \exists x((x = 0 \to A) \wedge (\neg(x = 0) \to 1 = 0))) \\
\Rightarrow\ & \mathsf{MHA}_f \vdash (f \to (A \vee 1 = 0) \to \exists x((x = 0 \to A) \wedge (1 = 0))) \\
\Rightarrow\ & \mathsf{MHA}_f \vdash (f \to (A \vee 1 = 0) \to 1 = 0 \wedge \exists x((x = 0 \to A))) \\
\Rightarrow\ & \mathsf{MHA}_f \vdash f \to (A \to 1 = 0)
\end{aligned}$$

Let A be f. We observe that the above derivation cannot be the case, because, we would have to conclude that $\mathsf{MHA}_f \vdash f \to (f \to 1 = 0)$, i.e., $\mathsf{MHA}_f \vdash f \to 1 = 0$. That this fails, will become clear in Lemma 5.3.2, where we encounter models of MHA_f in which f is forced, but $0 = 1$ is not. This means that for MHA_f the disjunction property needs to be proven separately from the existence property.

The disjunction property of MHA_f can be proved similarly to the disjunction property for HA, see e.g. [JVV11, Lemma 5.1].

Theorem 5.2.2. MHA_f *has the disjunction property.*

Proof. To prove this, we use what is known as Smoryński's trick: Given any set of MHA_f-models, we can take their disjoint union and add the standard model $\mathbb{N}$ below as the new root α_0. A quick check of the axioms allows us to conclude that this is again a model of MHA_f.
Now, if $\mathsf{MHA}_f \vdash A \vee B$, but $\mathsf{MHA}_f \nvdash A$ and $\mathsf{MHA}_f \nvdash B$. We know, by completeness of MQC, that there exist MHA_f-countermodels for A and B. Apply Smoryński's trick and conclude that $\alpha_0 \nVdash A \vee B$, which is a contradiction. Hence, MHA has the disjunction property. $\square$

Theorem 5.2.3. MHA_f *has the existence property.*

Proof. Works analogously to the proof for HA, using again Smoryński's trick. $\square$

Let us now prove de Jongh's theorem for MHA_f with respect to MPC.

Theorem 5.2.4. *Let A be a propositional formula, then $\mathsf{MPC} \vdash A(p_1, \ldots, p_n, f)$ if and only if for all arithmetical sentences $\alpha_1, \ldots, \alpha_n$ we have $\mathsf{MHA}_f \vdash A(\alpha_1, \ldots, \alpha_n, f)$.*

Proof. The direction from left to right is trivial, since the underlying logic of MHA_f is MQC. We prove the other direction by contraposition. If $\mathsf{MPC} \nvdash A(p_1, \ldots, p_n, f)$, then $\mathsf{IPC} \nvdash A(p_1, \ldots, p_n, f)$, because MPC corresponds to the positive fragment of IPC and $A(p_1, \ldots, p_n, f)$ is positive in IPC. Then we know, by de Jongh's theorem, that there exists a model of HA with root w_0 and arithmetical sentences $\alpha_1, \ldots, \alpha_n$ and β such that $w_0 \nVdash A(\alpha_1, \ldots, \alpha_n, \beta)$. Let us give f the valuation of β. Then $w_0 \nVdash A(\alpha_1, \ldots, \alpha_n, f)$. We show that $\mathcal{M}$ is a model of MHA_f, where we take the valuation for f to be the one for β. To do so, we have to show that all axioms of MHA_f hold in α_0. There are only two non-trivial cases to check. Firstly, the axiom $\exists x(Sx = 0) \to f$. This axiom trivially holds in w_0 since $\exists x(Sx = 0)$ is forced nowhere, because $\mathcal{M}$ is a model of HA. Secondly, we have to make sure that the induction axioms go through. But since f is interpreted as β and the induction axioms hold with instances of β, they also hold with instances of f. Hence, $\mathcal{M}$ is a model of MHA_f and therefore we conclude $\mathsf{MHA}_f \nvdash A(p_1, \ldots, p_n, f)$. $\square$

We will now prove that in minimal arithmetic, like in HA, all atomic formulas are *decidable*.

Lemma 5.2.5. $\mathsf{MHA}_f \vdash \forall xy(x = y \vee (x = y \to \exists x(Sx = 0)))$

Proof. By induction on x.

The case $x = 0$ we show by induction on y.

(i) For $y = 0$ we have $0 = 0$, so we are done.

(ii) For $y \to Sy$ we have $0 = Sy \to \exists x(Sx = 0)$ and so we are also done.

For the induction step, we assume $\forall y(x = y \vee (x = y \to \exists x(Sx = 0)))$ and we show $\forall y(Sx = y \vee (Sx = y \to \exists x(Sx = 0)))$ by induction on y.

(i) For $y = 0$ we of course have $Sx = 0 \to \exists x(Sx = 0)$.

(ii) For $y \to Sy$ we assume $x = y \vee (x = y \to \exists x(Sx = 0))$. Clearly, if $x = y$ then $Sx = Sy$, and, if $x = y \to \exists x(Sx = 0)$ then $Sx = Sy \to \exists x(Sx = 0)$, by using that $\forall xy(x = y \leftrightarrow Sx = Sy)$. Hence, we conclude $Sx = Sy \vee (Sx = Sy \to \exists x(Sx = 0))$ and this finishes our proof by induction. $\square$

Corollary 5.2.6. *In MHA_f, all atomic formulas are decidable.*

Proof. As a consequence of the above lemma, since $\mathsf{MHA}_f \vdash \exists x(Sx = 0) \to f$, we conclude $\mathsf{MHA}_f \vdash \forall xy(x = y \vee \neg(x = y))$. $\square$

In the next section, we will see that, contrary to decidability, stability fails for atomic formulas in MHA_f.

5.3 Models of Minimal Arithmetic

In this section, we will study the structure of non-standard models of minimal arithmetic. As we have mentioned before, a model of minimal arithmetic is a Kripke model for MQC on which the axioms of MHA_f are valid. The model needs, at each node w, an interpretation of the constant 0 and an interpretation of Sx for each element x in the domain of w. Recall that $D_w \subseteq D_v$ whenever $w \leq v$ and note that $S^k x$ is an abbreviation for applying the successor function k times to the element x.

Definition 5.3.1 (false k-circle)**.** For any natural number k, the *false k-circle* is the single-noded Kripke model for MQC, $\mathcal{M} = \{*\}$, with domain $D_* = \{0, \ldots, k-1\}$, such that $\{*\} \Vdash f$, where we interpret the function symbols and constant as follows:

$$0 := 0,\ a + b := (a+b)_{mod\ k},\ a \cdot b := (a \cdot b)_{mod\ k} \text{ and } Sa := (a + S0)_{mod\ k}.$$

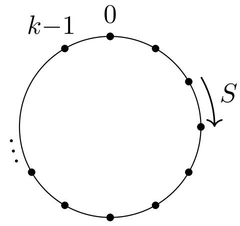


Figure 5.1

Lemma 5.3.2. *The false k-circle is a model of* MHA_f*.*

Proof. Let k be given and let $\mathcal{M}$ be the false k-circle. We have to check axioms (1-10) and the induction schema (I). Almost all axioms are clear to hold from the way we defined the operations, as the modulo operations 'inherit' these properties from the usual operations $\cdot$ and $+$. We will check axiom (8) and (I):

$$\begin{aligned} a + Sb &= (a + Sb)_{mod\ k} \\ &= (a + (b+1)_{mod\ k})_{mod\ k} \\ &= (a + (b+1))_{mod\ k} \\ &= ((a+b)+1)_{mod\ k} \\ &= S(a+b) \end{aligned}$$

Suppose, for some formula A, that $A(\overline{0})$ and $\forall x(A(x) \to A(Sx))$ are forced at the only node in $\mathcal{M}$. Then, by applying $\forall x(A(x) \to A(Sx))$ k times, we know that $A(\overline{0}), A(\overline{1}), \ldots, A(\overline{k})$ are forced. So, for all x in the domain of the singleton node in the model, $A(\overline{x})$ is forced. Hence, $\forall x A(x)$ is forced and thus the induction scheme is valid. $\square$

Abusing terminology, we will also speak of a false k-circle as part of a Kripke model with several nodes, so, when at some node the interpretations of 0 and S on its domain generate a natural number k such that $S^k 0 = 0$. We can also imagine domains with circles in which 0 does not occur, but for which there is an element x such that $S^k x = x$ for some natural number k. Such a circle we will denote with a *zeroless false k-circle.*

Lemma 5.3.3. $\mathsf{MHA}_f \vdash \exists x(S^k x = x) \leftrightarrow S^k 0 = 0$.

Proof. The direction from right to left is clear. For the other direction, we know by Lemma 2.1.2 that $\exists x(S^k x = x) \to S^k 0 = 0$ is over MQC equivalent to $\forall x(S^k x = x \to S^k 0 = 0)$. We prove the latter by induction on x. The case for $x = 0$ is clear. Now suppose $S^k x = x \to S^k 0 = 0$, then:

$$\begin{aligned} S^k Sx = Sx &\to S^{k+1} x = Sx \\ &\to SS^k x = Sx \\ &\to S^k x = x && \text{(by axiom (5))} \\ &\to S^k 0 = 0 \end{aligned}$$ □

A consequence of this lemma is that whenever a node in a model of MHA_f contains a zeroless false l-circle, the same node also contains a false k-circle for some divisor k of l, because the induction schema fails in models with nodes where we have a false k-circle together with a zeroless false l-circle where k is not a divisor of l. In particular, if a node has a domain with a zeroless false l-circle for some l, the domain also has a false k-circle, for some k.

We will now prove a much stronger result. Namely, a model of MHA_f cannot have a node with a domain containing a zeroless false l-circle for any natural number l. In order to show this, we will first prove the following lemma.

Lemma 5.3.4. *No domain of any node in any model of* MHA_f *contains both a false k-circle and a zeroless false l-circle, for some natural numbers k and l.*

Proof. Suppose, towards a contradiction, there exists a model $\mathcal{M}$ of MHA_f with node w of which the domain D_w contains both a false k-circle and a zeroless false l-circle, for some natural numbers k and l. This means that D_w is the set $\{0, 1, \dots, k-1, 0', 1', \dots, (l-1)'\}$, where the successor function is defined in the following way:

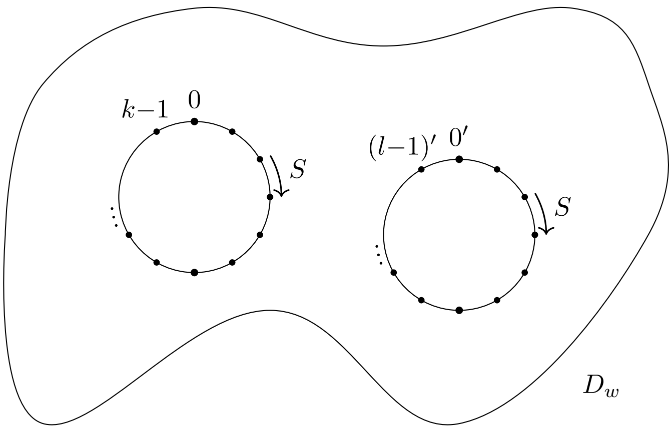


Figure 5.2

Let $A(x)$ be the formula $x = 0 \vee Sx = 0 \vee \ldots \vee S^{k-1}x = 0$. We will prove that $w \Vdash A(0) \wedge \forall x(A(x) \to A(Sx))$. We have $w \Vdash A(0)$, because $w \Vdash 0 = 0$. It is left to show that $w \Vdash \forall x(A(x) \to A(Sx))$, i.e., that for every $v \geq w$, every $d \in D_v$ and every $u \geq v$ we have, if $u \Vdash A(d)$, then $u \Vdash A(Sd)$. Suppose $u \geq v \geq w$ and $u \Vdash A(d)$ for some $d \in D_v$. Then $u \Vdash d = 0 \vee Sd = 0 \vee \ldots \vee S^{k-1}d = 0$. We need to show:

$$u \Vdash Sd = 0 \vee SSd = 0 \vee \ldots \vee S^k d = 0$$

From $u \Vdash d = 0 \vee Sd = 0 \vee \ldots \vee S^{k-1}d = 0$, we know that $u \Vdash S^l d = 0$ for some $l \in \{0, \ldots, k-1\}$. In case of $u \Vdash Sd = 0$ or $\ldots$ or $u \Vdash S^{k-1}d = 0$ we can conclude what we needed to show. Now suppose that $u \Vdash d = 0$. Then by applying axiom 4 k times we know that $u \Vdash S^k d = S^k 0$. Since $w \Vdash S^k 0 = 0$, then by persistency $u \Vdash S^k 0 = 0$, hence $u \Vdash S^k d = 0$. We conclude $u \Vdash Sd = 0 \vee SSd = 0 \vee \ldots \vee S^k d = 0$. Now, by the induction schema we should be able to conclude that $w \Vdash \forall x Ax$. However, we can see that for instance $w \nVdash A(0')$, because $w \nVdash 0' = 0 \vee S0' = 0 \vee \ldots \vee S^{k-1}0' = 0$. This is a contradiction to $\mathcal{M} \Vdash \mathsf{MHA}_f$, which proves the lemma. □

From here on, we will sometimes leave out k when denoting a false k-circle, and simply say false circle, when there is no need to specify k. Analogously, we will sometimes say zeroless false circle, or even, zeroless circle.

Using the previous two lemmas, we can now prove the subsequent proposition.

Proposition 5.3.5. *No domain of any node in any model of* MHA_f *contains a zeroless false circle, i.e., a circle in which* 0 *does not occur.*

Proof. Suppose there exists a model of MHA_f with a node of which the domain contains a zeroless false circle. As a consequence of Lemma 5.3.3 we know that this domain then also contains a false circle. We now apply Lemma 5.3.4 to conclude that this cannot be the case. □

Lemma 5.3.6. $\mathsf{MHA}_f \vdash \forall x(S^k x = x) \leftrightarrow S^k 0 = 0$.

Proof. From left to right is clear. For the other direction, suppose $S^k 0 = 0$ is forced at some node w in some model of MHA_f. We prove by an easy induction on x, similar to the induction in the proof of Lemma 5.3.3, that then $\forall x(S^k x = x)$ must be forced at every successor v of w. The base case is clear. Suppose $v \Vdash S^k a = a$ for some $a \in D_v$. Then $v \Vdash S(S^k a) = Sa$, hence $v \Vdash S^k(Sa) = Sa$ and thus $v \Vdash \forall x(S^k x = x)$. □

Let $\mathcal{M}$ be a model of MHA_f with node w. We say that $a, b \in D_w$ are *not equivalent* if $w \nVdash a = b$. By the previous lemma we know that whenever a domain contains a false circle, then all elements in the domain occur in a circle. Therefore, a direct consequence of combining Lemma 5.3.6 with Proposition 5.3.5 is the following corollary.

Corollary 5.3.7. *If a domain of some node in some model of* MHA_f *contains a false* k*-circle, then the domain contains precisely* k *non-equivalent elements.* □

Hence, we know that whenever a domain contains a false circle, it contains no other elements outside of this circle. However, it remains a question which other non-standard models MHA_f has, as we have not fully determined what kind of models can find a place in nodes of models of MHA_f.

Using the false circles we can make the following important observation. For any natural number k greater than 1 we have:

$$\mathsf{MHA}_f \nvdash k = 0 \to 1 = 0.$$

This is witnessed by the false k-circle. Hence, we conclude:

$$\mathsf{MHA}_f \nvdash \exists x(Sx = 0) \to 1 = 0.$$

And, therefore:

$$\mathsf{MHA}_f \nvdash f \to 1 = 0.$$

An immediate consequence of the last observation is stated in the following corollary.

Corollary 5.3.8. $\mathsf{MHA}_{0=1}$ *is not a subtheory of* MHA_f. □

Another observation, following from the existence of the false k-circles as models, is that we have the following two underivable fundamental formulas of HA:

$$\mathsf{MHA}_f \nvdash x + y = 0 \to (x = 0 \wedge y = 0)$$

$$\mathsf{MHA}_f \nvdash x \cdot y = 0 \to (x = 0 \vee y = 0)$$

Take for instance the false 2-circle as a countermodel for the first one and the false 6-circle as a countermodel for the latter one.

The following model is drawn to give an example of a non-standard model of MHA_f:

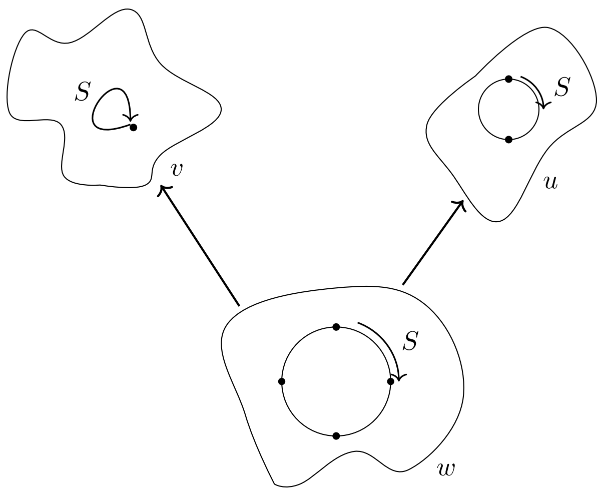


Figure 5.3

Remark 5.3.9. Recall from Corollary 5.2.6 that in MHA_f all atomic formulas are decidable, i.e., $\mathsf{MHA}_f \vdash \forall xy(x = y \vee \neg(x = y))$. Stability of the equality relation, i.e., in our case, stability of all atomic sentences, means that $\forall xy(\neg\neg(x = y) \to x = y)$. The model above shows us that not all atomic formulas are stable, because $w \Vdash \neg\neg(1 = 0)$, but $w \nVdash 1 = 0$. Note that it is also possible to see this in a single-noded model, such as the standard model where f is forced, or, the false 2-circle.

Note that the following is derivable:

$$\mathsf{MHA}_f \vdash \forall xy(\neg\neg(x = y) \leftrightarrow (x = y \vee f)).$$

From right to left is clear. The other direction is proven as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall xy(x = y \vee \neg(x = y))}{u = v \vee \neg(u = v)} \qquad \dfrac{[u = v]^1}{u = v \vee f} \qquad \dfrac{\dfrac{[\neg(u = v)]^1 \qquad [\neg\neg(u = v)]^2}{f}}{u = v \vee f}}{u = v \vee f}\,{}^{1}}{\neg\neg(u = v) \to (u = v \vee f)}\,{}^{2}}{\forall xy(\neg\neg(x = y) \to (x = y \vee f))}$$

In this respect, MHA_f is very different from HA, because we know that $\mathsf{HA} \vdash \forall xy(\neg\neg(x = y) \leftrightarrow x = y)$. Even more surprising is the following result.

Lemma 5.3.10. *In* MHA_f*,* $\exists x(Sx = 0)$ *has the disjunction property.*

Proof. Suppose for a contradiction that there exist formulas A and B such that $\mathsf{MHA}_f \vdash \exists x(Sx = 0) \to A \vee B$, but $\mathsf{MHA}_f \nvdash \exists x(Sx = 0) \to A$ and $\mathsf{MHA}_f \nvdash \exists x(Sx = 0) \to B$. By completeness, we then know there exist models $\mathcal{M}_A$ and $\mathcal{M}_B$ with roots r_A and r_B such that $r_A \Vdash \exists x(Sx = 0)$ and $r_B \Vdash \exists x(Sx = 0)$, but $r_A \nVdash A$ and $r_B \nVdash B$. This means that the domains of r_A and r_B consist of false circles. Let us say that D_{r_A} contains the false k-circle and D_{r_B} contains the false l-circle for some natural numbers k and l. Now, let m be the least common multiple of k and l. We then construct a new model $\mathcal{M}$ by placing a new root r below the models $\mathcal{M}_A$ and $\mathcal{M}_B$, with domain the false m-circle. This is again a model of MHA_f (for an example of such a model, see Figure 5.3). Hence, $r \Vdash \exists x(Sx = 0) \to A \vee B$. Since $r \Vdash \exists x(Sx = 0)$, then $r \Vdash A$ or $r \Vdash B$. This is a contradiction and thus we conclude that $\exists x(Sx = 0)$ has the disjunction property. $\square$

5.4 Representability

In this section, we discuss some results concerning representability of functions in minimal arithmetic. In a particular formal system S, the numeral $\overline{n}$ is the symbol that stands for the n-th successor of the interpretation of the symbol 0 in that system. Note that $\exists! yAy$ is an abbreviation for $\exists yAy \wedge \forall z(Az \to y = z)$.

Definition 5.4.1 (Numeralwise Representability, [Kle52, p. 200])**.** We call a function $F(x_1, \ldots, x_k)$ *numeralwise representable* in a formal system S if there exists a formula $\varphi(x_1, \ldots, x_k, y)$ such that for all natural numbers $n_1, \ldots, n_k$ and n we have:

(i) if $F(n_1, \dots, n_k) = n$, then $\vdash_{\mathsf{S}} \varphi(\overline{n_1}, \dots, \overline{n_k}, \overline{n})$;

(ii) $\vdash_{\mathsf{S}} \exists! y \varphi(\overline{n_1}, \dots, \overline{n_k}, y)$.

In MHA_f, in contrast to HA, not every primitive recursive function is numeralwise representable (for a definition of primitive recursive functions see e.g. [Kle52, p. 219]).

Proposition 5.4.2. *The primitive recursive function $F : \mathbb{N} \to \mathbb{N}$ defined by:*

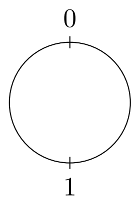
$$n \mapsto \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{otherwise} \end{cases}$$

is not numeralwise representable in MHA_f.

Proof. Suppose F is numeralwise representable, i.e., suppose there exists a formula $\varphi(x, y)$ such that for every natural number n:

(i) if $F(n){=}m$, then $\vdash_{\mathsf{MHA}_f} \varphi(\overline{n}, \overline{m})$;

(ii) $\vdash_{\mathsf{MHA}_f} \exists! y \varphi(\overline{n}, y)$.



Now, consider the false 2-circle $\mathcal{M}$:

By Lemma 5.3.2 we know that $\mathcal{M}$ is a model of MHA_f. Using (i), we have $\mathcal{M} \Vdash \varphi(\overline{2}, \overline{1})$. We know that $\mathcal{M} \Vdash \overline{2} = \overline{0}$, hence we obtain $\mathcal{M} \Vdash \varphi(\overline{0}, \overline{1})$. However, by (i) we also have $\mathcal{M} \Vdash \varphi(\overline{0}, \overline{0})$. Now we use (ii), which is an abbreviation for:

$$\vdash_{\mathsf{MHA}_f} \exists y (\varphi(\overline{x}, y) \wedge \forall z (\varphi(\overline{x}, z) \to y = z)).$$

Thus, we derive $\mathcal{M} \Vdash \overline{0} = \overline{1}$, but $\mathcal{M} \nVdash \overline{0} = \overline{1}$, a contradiction. □

Let us call a function *ill-defined modulo k*, if there exist m and n such that in the full k-circle we have $m = n \ (mod\ k)$, but $F(m) \neq F(n) \ (mod\ k)$.

Proposition 5.4.3. *If F is a function that is ill-defined modulo k, then F is not numeralwise representable in* MHA_f.

Proof. This works analogously to the proof of the previous proposition, by simply taking the model $\mathcal{M}$ to be the false k-circle. □

The above proposition shows that many primitive recursive functions will not be numeralwise representable in MHA_f. In Heyting arithmetic, the fact that every primitive recursive predicate is numeralwise expressible is a consequence of the fact that every primitive recursive function is numeralwise representable (see e.g. [Kle52, p. 244]). Hence, the next natural question we can ask is whether there are primitive recursive predicates that are not numeralwise expressible in MHA_f. We have not answered this question yet.

Remark 5.4.4. The addition of a predecessor function, recursively defined by $pd(0) = 0$ and $pd(Sx) = x$, to MHA_f, would ensure that all possible candidates for interpretations of falsum are equivalent. Because, if we add a predecessor function, then, using Lemma 5.5.4, we conclude that $\mathsf{MHA}_f \vdash A \to 1 = 0$ for A positive such that $\mathsf{MHA}_f \vdash \neg A$. An interesting question is whether, when adding a predecessor function, there would still be a difference between minimal arithmetic and Heyting arithmetic.

Let us conclude this section with the following rather negative observation. If we practice minimal arithmetic with an uninterpreted falsum, i.e., if we work in MHA_f, then many of the valuable properties that Heyting arithmetic has, will no longer hold. It might therefore be more interesting to examine systems MHA_β that are stronger than MHA_f. A consequence of the existence of MHA_f-models with circles in their domains, is that there are several non-equivalent possible candidates β for interpretations of falsum, giving rise to different systems MHA_β. We investigate this in the following section.

5.5 Interpreting Falsum in Minimal Arithmetic

Let β be a sentence formulated in $\mathscr{L}(\mathsf{HA}) \setminus \{\neg\}$. As defined in Definition 4.1.2, we can then investigate the following theories:

$$\mathsf{HA}_\beta := \{A \mid \mathcal{A}^\beta_{\mathsf{HA}} \vdash_{\mathsf{IQC}} \tau_\beta(A)\}$$

$$\mathsf{MHA}_\beta := \{A \mid \mathcal{A}^\beta_{\mathsf{HA}} \vdash_{\mathsf{MQC}} \tau_\beta(A)\}$$

We first examine the systems $\mathsf{HA}_{0=1}$ and $\mathsf{MHA}_{0=1}$. We observe that $\mathsf{HA}_{0=1}$ is a subtheory of HA, simply because $\mathsf{HA} \vdash \exists x(Sx = 0) \to 0 = 1$ and because $\mathsf{HA} \vdash \bot \leftrightarrow 0 = 1$. This is not the case for the corresponding minimal systems, because $\mathsf{MHA}_{0=1}$ is not a subtheory of MHA_f, see Corollary 5.3.8.

Note that $\mathcal{A}^{0=1}_{\mathsf{HA}}$ is a set of positive axioms, since the only axiom containg a negation in fact only contains a pseudo-negation, namely, $\exists x(Sx = 0) \to 1 = 0$. Hence, $\mathcal{A}^{0=1}_{\mathsf{HA}}$ is a positive theory. We can now return to Remark 4.3.4 and provide a concrete counterexample for Proposition 4.3.3 in the following remark, when we replace minimal logic by intuitionistic logic.

Remark 5.5.1. We show that there exist positive formulas A and B such that $\mathcal{A}^{0=1}_{\mathsf{HA}} \vdash_{\mathsf{IQC}} \neg A \to \neg B$, but $\mathcal{A}^{0=1}_{\mathsf{HA}} \nvdash_{\mathsf{IQC}} B \to A$. We know that Peirce's law is not intuitionistically valid, i.e., $\mathsf{IPC} \nvdash ((p \to q) \to p) \to p$. Hence, by de Jongh's theorem we know that there exist arithmetical sentences α and β such that $\mathsf{HA} \nvdash ((\alpha \to \beta) \to \alpha) \to \alpha$. Now, let $\alpha' := \tau_{0=1}(\alpha)$ and $\beta' := \tau_{0=1}(\beta)$ (recall Definition 4.1.2). We obtain $\mathcal{A}^{0=1}_{\mathsf{HA}} \nvdash_{\mathsf{IQC}} ((\alpha' \to \beta') \to \alpha') \to \alpha'$. For suppose $\mathcal{A}^{0=1}_{\mathsf{HA}} \vdash_{\mathsf{IQC}} ((\alpha' \to \beta') \to \alpha') \to \alpha'$. Then by definition $\mathsf{HA}_{0=1} \vdash ((\alpha \to \beta) \to \alpha) \to \alpha$, and thus $\mathsf{HA} \vdash ((\alpha \to \beta) \to \alpha) \to \alpha$, because $\mathsf{HA}_{0=1}$ is a subtheory of HA. Hence, we conclude that $\mathcal{A}^{0=1}_{\mathsf{HA}} \nvdash_{\mathsf{IQC}} ((\alpha' \to \beta') \to \alpha') \to \alpha'$. On the other hand, we know that $\mathsf{IPC} \vdash \neg p \to \neg((p \to q) \to p)$, hence $\mathcal{A}^{0=1}_{\mathsf{HA}} \vdash_{\mathsf{IQC}} \neg\alpha' \to \neg((\alpha' \to \beta') \to \alpha')$ follows immediately.

Recall, from Proposition 5.1.2, that in HA, all formulas are derivable from $0 = 1$ without using ex falso. It is important to note, however, that the intuitionistic system $\mathsf{HA}_{0=1}$ is not the same as HA. The false 1-circle is a model of $\mathsf{HA}_{0=1}$, but not of HA.

Closely examining the proof of Proposition 5.1.2, we see that it can also be carried out in any of the systems HA_β and MHA_β, where only for MHA_f it also needs to be proved that $0 = 1$ implies f, which follows from axiom (6).

Lemma 5.5.2. *For positive formulas A, we have $\mathsf{MHA}_{1=0} \vdash A$ if and only if $\mathsf{HA} \vdash A$.*

Proof. Let A be a positive formula, that is, A does not contain $\neg$, $\bot$ nor f. We show: $\mathsf{MHA}_{1=0} \vdash A \;\Leftrightarrow\; \mathsf{HA} \vdash A$.

$$\begin{aligned}
\mathsf{MHA}_{1=0} \vdash A \;&\Leftrightarrow\; \mathcal{A}_{\mathsf{HA}}^{1=0} \vdash_{\mathsf{MQC}} \tau_{1=0}(A) && \text{(by definition)}\\
&\Leftrightarrow\; \mathcal{A}_{\mathsf{HA}}^{1=0} \vdash_{\mathsf{MQC}} A && \text{(A is positive)}\\
&\Leftrightarrow\; \mathcal{A}_{\mathsf{HA}}^{1=0} \vdash_{\mathsf{IQC}} A && \text{(by Lemma 2.2.6)}\\
&\Leftrightarrow\; \mathcal{A}_{\mathsf{HA}} \vdash_{\mathsf{IQC}} A && (\mathsf{HA} \vdash \bot \leftrightarrow 1 = 0)\\
&\Leftrightarrow\; \mathsf{HA} \vdash A && \text{(by definition)}
\end{aligned}$$
□

Since $\mathsf{MHA}_{1=0}$ is a positive theory, the above lemma immediately gives rise to the following corollary.

Corollary 5.5.3. *$\mathsf{MHA}_{1=0}$ is precisely the positive fragment of HA.* □

Besides $0 = 1$, there are various possibilities for the interpretation of falsum in MHA_f. The only condition a candidate A has to satisfy is that $\mathsf{MHA}_f \vdash \neg A$, i.e., $\mathsf{MHA}_f \vdash A \to f$. With the subsequent lemma, we show that $\exists x(Sx = 0)$ is the weakest possible candidate for falsum in MHA_f.

Lemma 5.5.4. *If $\mathsf{MHA}_f \vdash \neg A$ and A positive, then $\mathsf{MHA}_f \vdash A \to \exists x(Sx = 0)$.*

Proof. As we have mentioned before, MHA_f is not precisely a simple theory, because there can be negations in instances of the induction schema. However, let MHA_0 be the theory axiomatised by MQC together with $\mathcal{A}_{\mathsf{HA}} \setminus \{\neg\exists x(Sx = 0)\}$. Suppose $\mathsf{MHA}_f \vdash \neg A$ for some positive formula A. Then $\mathsf{MHA}_0 \vdash \neg\exists x(Sx = 0) \to \neg A$, i.e., $\mathsf{MHA}_0 \vdash (\exists x(Sx = 0) \to f) \to (A \to f)$. Since we know nothing in particular about f in MHA_0, f behaves just like a propositional variable. So, we can in the deduction of $(\exists x(Sx = 0) \to f) \to (A \to f)$ replace all instances of f by $\exists x(Sx = 0)$, including the possible instances of f in the induction schema. We then obtain $\mathsf{MHA}_0 \vdash A \to \exists x(Sx = 0)$. □

Hence, for positive A such that $\mathsf{MHA}_f \vdash \neg A$, we know that $\exists x(Sx = 0)$ is either weaker than A or equivalent to A and thus we conclude that $\exists x(Sx = 0)$ is the weakest possible candidate for falsum in MHA_f.

Lemma 5.5.5. *In $\mathsf{MHA}_{1=0}$, all possible candidates for an interpretation of falsum are equivalent to 1=0.*

Proof. Suppose $\mathsf{MHA}_{1=0} \vdash \neg A$. Then, by Lemma 4.1.4, we know that $\mathsf{MHA}_{1=0} \vdash A \to 1{=}0$. Moreover, we know that $\mathsf{MHA}_{1=0} \vdash 1 = 0 \to A$ by Corollary 5.5.3. Hence, $\mathsf{MHA}_{1=0} \vdash A \leftrightarrow 0 = 1$. □

We have seen that in MHA_f, the weakest possible candidate for interpreting falsum is $\exists x(Sx = 0)$ and the strongest one is $1 = 0$. Other possible interpretations are for instance $2 = 0$, $3 = 0$, $4 = 0$, and so on. Or, for instance, a disjunction of those atomic sentences, like $2 = 0 \vee 3 = 0$. The conjunction $S^k 0 = 0 \wedge S^l 0 = 0$ of two atomic formulas is equivalent to $S^m 0 = 0$, where m is the greatest common divisor of k and l. Note that for any k and l we have $\mathsf{MHA}_f \nvdash (S^k 0 = 0 \to S^l 0 = 0) \to f$, since the standard model always forces $S^k 0 = 0 \to S^l 0 = 0$. Hence, these implications do not satisfy the requirements of an interpretation for falsum. Finally, for every k, we know by Lemma 5.3.3 that the candidate $\exists x(S^k x = 0)$ is equivalent to $S^k 0 = 0$, hence, also atomic. Unfortunately, we do not have a complete description of the hierarchy of candidates for falsum.

We finish this chapter by remarking that the system MHA_f is clearly in various respects not as one would wish. We can ask ourselves what can be done about this. As we have seen in this section, the system $\mathsf{MHA}_{0=1}$ has the same proof-theoretic strength as HA. Another suggestion would be to add $\exists x(Sx = 0) \to 0 = 1$ as an axiom to MHA_f, to collapse all possible candidates for falsum. This comes close to the system $\mathsf{MHA}_{0=1}$, in which the same is true. However, one has to defend extending MHA_f in this way, from a minimal point of view.

Chapter 6

Second-Order Minimal Heyting Arithmetic

In this chapter, we study second-order intuitionistic arithmetic, i.e., second order Heyting arithmetic (see e.g. [Tro73]). We concentrate on [JS76], in which Smoryński and de Jongh introduce Kripke models for second-order Heyting arithmetic, HAS. There, the consistency of the principles UP!, UP^C, MP and IP_0 with HAS was shown by giving models for them. We will introduce these principles along the way and we will replicate some, but not all of these results, in a minimal context, where in the case of UP^C we have to use much more complicated models than in [JS76]. Furthermore, in [JS76], models in which non-intuitionistic logical principles are contradictory were given. We will do the same here for non-minimal logical principles and even prove a stronger form of de Jongh's theorem for second-order minimal arithmetic.

6.1 Models of Second-Order Minimal Arithmetic

By second-order Heyting arithmetic, HAS, we understand the intuitionistic theory of species, given by HA together with unary species variables $X_0, X_1, \dots$ and the following two axioms:

> For any formula $A(x)$ the comprehension axiom, where X a possible free set variable in $A(x)$:
> $$\exists X \forall x (x \in X \leftrightarrow A(x)).$$
>
> For any second order formula A, where m and X are possibly free numerical and set variables in Ax, respectively, the second order induction scheme is given by:
> $$\forall m \forall X (A(0) \wedge \forall n (A(n) \to A(Sn))) \to \forall n A n,$$
> where an essential instance is the second order induction axiom:
> $$\forall X ((0 \in X \wedge \forall n (n \in X \to Sn \in X)) \to \forall n (n \in X)).$$

Let second-order minimal arithmetic, a "minimal theory of species", be the theory MHAS_f as defined in Definition 4.1.2. This is the theory given by MHA_f together with unary species variables $X_0, X_1, \dots$ and the above axioms.

Definition 6.1.1 (MHAS_f frame)**.** An MHAS_f frame is of the form $(K, \leq, F, D_1, D_2)$ where $(K, \leq)$ is a partially ordered set of nodes, containing a least node α_0, and $F \subseteq K$ is upwards closed with respect to $\leq$. D_1 and D_2 are domain functions. For $\alpha \in K$ we have $D_1\alpha = \omega = \{0, 1, \ldots\}$ and $D_2\alpha$ is the family of all *systems of sets*, or *species*. A species is a class of subsets of the natural numbers indexed by the nodes of K, $T = \{T_\alpha \mid \alpha \in K\}$, such that:

$$\alpha \leq \beta \Rightarrow T_\alpha \subseteq T_\beta$$

Definition 6.1.2 (MHAS_f model)**.** An MHAS_f model $\mathcal{M}$ for minimal logic is of the form $(K, \leq, F, D_1, D_2, \Vdash)$ where $(K, \leq, F, D_1, D_2)$ an MHAS_f frame for minimal logic and $\Vdash$ a forcing relation defined by:

$$\begin{array}{lll}
\alpha \Vdash A & \Leftrightarrow & \omega \Vdash A \textit{ for } A \textit{ atomic in } \mathscr{L}(\mathsf{HA}) \\
\alpha \Vdash f & \Leftrightarrow & \alpha \in F \\
\alpha \Vdash t \in T & \Leftrightarrow & t \in T_\alpha \textit{ where } t \in \omega \textit{ and } T = \{T_\alpha \ : \ \alpha \in K\} \\
\alpha \Vdash A \wedge B & \Leftrightarrow & \alpha \Vdash A \textit{ and } \alpha \Vdash B \\
\alpha \Vdash A \vee B & \Leftrightarrow & \alpha \Vdash A \textit{ or } \alpha \Vdash B \\
\alpha \Vdash A \to B & \Leftrightarrow & \textit{for all } \beta \geq \alpha\textit{: if } \beta \Vdash A\textit{, then } \beta \Vdash B \\
\alpha \Vdash \exists x A(x) & \Leftrightarrow & \textit{for some } t \in D_1\alpha : \ \alpha \Vdash A(t) \\
\alpha \Vdash \exists X A(X) & \Leftrightarrow & \textit{for some } T \in D_2\alpha : \ \alpha \Vdash A(T) \\
\alpha \Vdash \forall x A(x) & \Leftrightarrow & \textit{for all } \beta \geq \alpha \textit{ and all } t \in D_1\beta : \ \beta \Vdash A(t) \\
\alpha \Vdash \forall X A(X) & \Leftrightarrow & \textit{for all } \beta \geq \alpha \textit{ and all } T \in D_2\beta : \ \beta \Vdash A(T)
\end{array}$$

Note that the upwards closed set F is part of the frame. This means that the set of nodes where falsum is forced is independent of the valuation we choose. From here on, we will use Xx as an abbreviation for $x \in X$.

For the remaining part of this chapter, we interpret f by 0=1, i.e, we investigate the system $\mathsf{MHAS}_{0=1}$. We will first show that interpreting falsum by $1 = 0$ falls under the third approach we discussed in chapter 1. For simplicity, we will denote the system $\mathsf{MHAS}_{0=1}$ by MHAS. We obtain for every node α in every MHAS model:

$$\alpha \Vdash f \to \forall X(\exists x Xx \to \forall x Xx).$$

Because, if $\alpha \Vdash 0{=}1$, then $\alpha \Vdash \forall xy(x = y)$. Hence, for every $T \in D_2\alpha$, either $T_\alpha = \emptyset$ or $T_\alpha = \omega$, where we note that $=$ is not interpreted simply as identity. However, we do not obtain $\alpha \Vdash f \to \forall XY(X = Y)$. In fact, recall Proposition 4.2.1 (where we now have to take two kinds of variables into account), from $\forall xy(x = y) \wedge \forall XY(X = Y)$, all formulas can be proven. Taking this as an interpretation for falsum feels a little forced and unnatural, in contrast to taking $0 = 1$. We therefore find interpreting negation in terms of $0 = 1$ more appealing, yet it does not naturally give rise to the ex falso principle. So, although second order arithmetic is not like first order arithmetic the same in minimal and intuitionistic logic if we interpret f by 0=1, the influence of this is still heavily felt. In particular, $\mathsf{MHAS} \vdash 0{=}1 \to A$ still holds for all purely arithmetical statements A.

We observe that the model defined in Definition 6.1.2, where we interpret f by 0=1, indeed satisfies MHAS. The proof of this statement proceeds like the proof of the analogous statement for HAS in [JS76, Theorem 1.1].

In [JS76], all results were obtained by using models on the full binary or full $\aleph_0$-ary tree. Here, we will need to use a model on a more complicated tree to show consistency of the parameter-free from of the Uniformity Principle and to prove de Jongh's theorem in stronger form.

6.2 Principles of Second-Order Minimal Arithmetic

Let us first prove the consistency of the weak Uniformity Principle with MHAS. Note that this is a weaker result than the consistency of the paramater-free form of the Uniformity Principle, which we will prove later on.

Theorem 6.2.1. *Let $\mathcal{M}$ be an* MHAS *model on the full binary (or $\aleph_0$-ary) tree, then the weak Uniformity Principle:*

$$\forall X \exists ! y A(X, y) \to \exists ! y \forall X A(X, y) \qquad \text{(UP!)}$$

is valid in the model $\mathcal{M}$.

Proof. Suppose we have $\alpha \Vdash \forall X \exists ! y A(X, y)$, $\alpha \Vdash A(S, m)$ and $\alpha \Vdash A(T, n)$. We will show that $m = n$. Note that if $\alpha \Vdash f$, i.e., $\alpha \Vdash 0{=}1$, then we have $\alpha \Vdash \forall xy(x = y)$, hence $\alpha \Vdash \exists ! y \forall X A(X, y)$ directly follows.

Let $U := \{U_\beta \mid \beta \in K\}$ be defined by:

(i) $U_\beta = \emptyset$ for $\beta \not\geq \alpha$;

(ii) $U_\beta = S_\beta$ if $\beta = \alpha * \langle 0 \rangle * \sigma$ for some sequence σ;

(iii) $U_\beta = T_\beta$ if $\beta = \alpha * \langle 1 \rangle * \sigma$ for some sequence σ.

We have:

$$\begin{aligned} &\alpha * \langle 0 \rangle \Vdash U{=}S, \text{ hence } \alpha * \langle 0 \rangle \Vdash A(U, m), \text{ and we have} \\ &\alpha * \langle 1 \rangle \Vdash U{=}T, \text{ hence } \alpha * \langle 1 \rangle \Vdash A(U, n). \end{aligned}$$

Since $\alpha \Vdash \forall X \exists ! y A(X, y)$, then $\alpha \Vdash A(U, p)$ for a unique p. Whence we get $\alpha * \langle 0 \rangle \Vdash A(U, p)$ and $\alpha * \langle 1 \rangle \Vdash A(U, p)$ and thus $m = p = n$. We note that if $\alpha * \langle i \rangle \Vdash f$, this will still hold. Thus there exists precisely one $m \in \omega$ such that $\alpha \Vdash A(X, m)$ for all $X \in D_2\alpha$. Since $D_2\beta = D_2\alpha$ for all $\beta \geq \alpha$, we conclude $\alpha \Vdash \exists ! y \forall X A(X, y)$. □

We will now show that, unlike in the intuitionistic case, Markov's Principle is not valid on all models of MHAS.

Proposition 6.2.2. *Markov's principle* (MP) *is not valid on all* MHAS *models:*

$$\forall X \forall x(((Xx \vee \neg Xx) \wedge \neg\neg \exists x Xx) \to \exists x Xx) \qquad \text{(MP)}$$

Proof. Consider the MHAS model on the full $\aleph_0$-ary tree $(K, \leq)$ in which the root α_0 has some successor β that forces f. Define the species T by $T_\alpha = \emptyset$ for all $\alpha \in K$. Since $T \in D_2\beta$ and $\beta \Vdash \forall x(Tx \vee (Tx \to f)) \wedge ((\exists x Tx \to f) \to f)$ but $\beta \nVdash \exists x Tx$, we conclude $\alpha_0 \nVdash$ MP. □

Note that, if MP is valid on a model, then also $\forall X(f \to \exists x Xx)$ is valid on the model. This is not the case, as we have mentioned before.[1] The Independence of premise principle, on the other hand, is valid on all MHAS models.

Proposition 6.2.3. *The Independence of premise principle* (IP_0) *is valid on all* MHAS *models:*

$$\forall XY((\forall x(Xx \vee \neg Xx) \wedge (\forall x Xx \to \exists y Yy)) \to \exists y(\forall x Xx \to Yy)) \quad (\mathrm{IP}_0)$$

Proof. Let $\mathcal{M}$ be any MHAS model and α any node of $\mathcal{M}$. To show $\alpha \Vdash \mathsf{IP}_0$ we need to show that for any successor β of α and any two species X and Y, if $\beta \Vdash \forall x(Xx \vee \neg Xx) \wedge (\forall x Xx \to \exists y Yy)$, then $\beta \Vdash \exists y(\forall x Xx \to Yy)$. Let β and X, Y be any such successor and species. We know that either $\beta \Vdash \forall x Xx$, or $\beta \nVdash \forall x Xx$. If $\beta \Vdash \forall x Xx$, then by our assumption $\beta \Vdash \exists y Yy$ and thus $\beta \Vdash \exists y(\forall x Xx \to Yy)$. If $\beta \nVdash \forall x Xx$, then for any successor γ of β that forces $\forall x Xx$, we have $\gamma \Vdash \exists y Yy$, because $\beta \Vdash \forall x Xx \to \exists y Yy$. Moreover, for any such successor γ we have $\gamma \Vdash 0 = 1$, because $\beta \Vdash \forall x(Xx \vee \neg Xx)$. Hence, for every successor γ forcing $\forall x Xx$, we know that $\gamma \Vdash Y0$ and thus $\beta \Vdash \forall x Xx \to Y0$, from which we conclude $\beta \Vdash \exists y(\forall x Xx \to Yy)$. $\square$

As we observed in the proof above, an even stronger principle holds in all MHAS models for minimal logic:

$$\forall XY((\forall x(Xx \vee \neg Xx) \wedge (\forall x Xx \to \exists y Yy)) \to \forall \mathbf{y}(\forall x Xx \to Yy))$$

A consequence of Proposition 6.2.3 is that the corresponding rule, IR_0, also holds for MHAS: for any species X and Y, if $\mathsf{MHAS} \vdash \forall x(Xx \vee \neg Xx) \wedge (\forall x Xx \to \exists y Yy)$, then $\mathsf{MHAS} \vdash \exists y(\forall x Xx \to Yy)$.

Theorem 6.2.4. *Let $A(p_1, \ldots, p_n, f)$ be an unprovable formula of* MPC*. Then there exists a model $\mathcal{M}$ on the full $\aleph_0$-ary tree such that $\mathcal{M} \nVdash A(\exists x X_1 x, \ldots, \exists x X_n x, 0{=}1)$ for some species $X_1, \ldots, X_n$.*

Proof. Suppose $A(p_1, \ldots, p_n, f)$ is an unprovable formula of MPC. Since MPC has the finite model property, see [Col16], there exists a finite Kripke countermodel for A. Since the full $\aleph_0$-ary tree $(K, \leq)$ can be projected on this finite model through a p-morphism, we find a countermodel $\mathcal{M}$ on $(K, \leq)$ such that $\mathcal{M} \nVdash A(p_1, \ldots, p_n, f)$. We define for $i \in \{1, \ldots, n\}$ species $X_i = \{X_i, \alpha \mid \alpha \in K\}$, where:

$$X_i, \alpha := \begin{cases} \emptyset & \text{if } \alpha \nVdash p_i \\ \omega & \text{if } \alpha \Vdash p_i \end{cases}$$

By definition of the species we then obtain $\mathcal{M} \nVdash A(\exists x X_1 x, \ldots, \exists x X_n x, 0{=}1)$. $\square$

The conclusion we can draw from Theorem 6.2.1, MHAS+UP! is consistent, is by itself rather weak, as HAS+UP! is also consistent. From Theorem 6.2.1 together with Theorem 6.2.4 we can however obtain the following interesting corollary:

[1] This makes clear that validating Markov's Principle on a model for MHAS is not at all trivial. Markov's Principle does seem to have a rather different status in minimal logic from the one in intuitionistic logic.

Corollary 6.2.5. *For each unprovable formula $A(p_1, \ldots, p_n, f)$ of* MPC *we have:*

$$\mathsf{MHAS}{+}\mathrm{UP!} \nvdash \forall X_1 \ldots X_n A(\exists x X_1 x, \ldots, \exists x X_n x, 0{=}1)$$

□

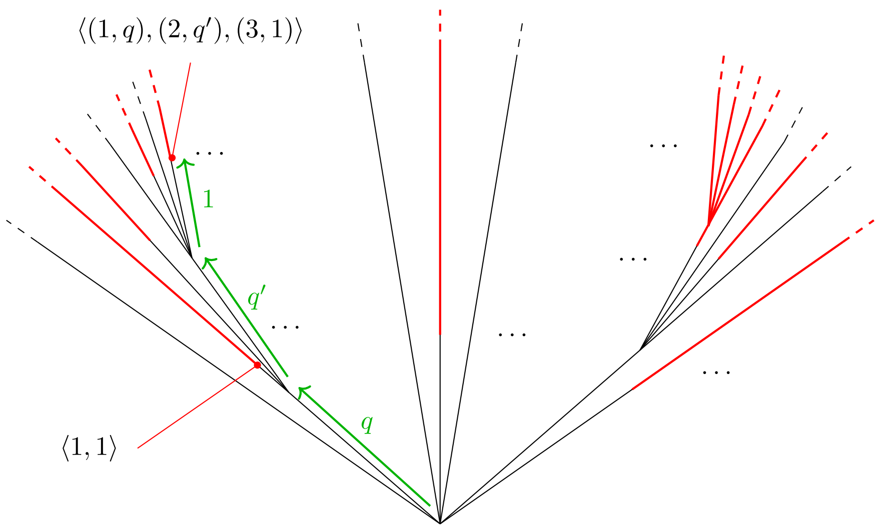
So far, we have been able to use very similar methods as in [JS76] to obtain results for MHAS. The following principle, the parameter-free form of the Uniformity Principle, does not hold in all MHAS models on the $\aleph_0$-ary tree:

$$\forall X \exists y A(X, y) \to \exists y \forall X A(X, y) \qquad (\mathrm{UP}^c)$$

The proof of the consistency of UP^C (in [JS76, Theorem 1.6]) is dependent on an isomorphism between a submodel of an HAS model and the model itself. In minimal logic however, such an isomorphism does not exist for every MHAS model on the full $\aleph_0$-ary tree, when we define f to be $0 = 1$. In order to prove the consistency of UP^C, we therefore introduce a more homogeneous tree-structure with a more complicated partial ordering in which such an isomorphism does exist:

Definition 6.2.6. We define the tree $K^{\aleph_0}_{\mathbb{Q}} := (K, \leq, F)$ of the form $(\mathbb{N} \times \mathbb{Q})^{<\omega}$ as follows. The root of the tree is the empty sequence $\langle\rangle$. For every $\alpha \in K$, also $\alpha * (n, q) \in K$ for all $n \in \mathbb{N}$ and $q \in \mathbb{Q}_{>0}$. We define $\alpha < \beta$ if and only if α is a proper initial segment of β. Furthermore $a * (n, q) * (0, q') := \alpha * (n, q + q')$. Finally, we define $F := \{\alpha \in K \mid \alpha \geq \beta * (2n + 1, 1) \text{ for some } \beta \in K \text{ and } n \in \mathbb{N}\}$.

Note that $\alpha * (n, q')$ is an initial segment of $\alpha * (n, q)$ for $q' < q$ and thus for every $q' < q$ we have $\alpha * (n, q') < \alpha * (n, q)$.



Now, for an MHAS model on $K^{\aleph_0}_{\mathbb{Q}}$, the set F is fixed. So, whenever we take a generated submodel of the MHAS model on $K^{\aleph_0}_{\mathbb{Q}}$, either f holds globally, or, due to the structure of the model, the submodel is isomorphic to the whole model.

Theorem 6.2.7. *The* MHAS *model on the tree* $K_{\mathbb{Q}}^{\aleph_0}$ *satisfies the paramater-free form of the Uniformity Principle* UP^{C}*:*

$$\forall X \exists y A(X, y) \to \exists y \forall X A(X, y)$$

Proof. Let $\mathcal{M}$ be the MHAS model on $K_{\mathbb{Q}}^{\aleph_0}$ and let $A(X, y)$ be parameter-free. Suppose:

$$\alpha \Vdash \forall X \exists y A(X, y)$$

$$\alpha \nVdash \exists y \forall X A(X, y)$$

Note that $\alpha \nVdash 0 = 1$. Also, since D_1 constant and D_2 practically constant we obtain from $\alpha \nVdash \forall X A(X, n)$ that there exists $T \in D_2\alpha$ such that $\alpha \nVdash A(T, n)$.

Let for each n, a species S_n be given such that:

$$\alpha \nVdash A(S_n, n)$$

Define the species S by:

$$\begin{array}{ll} S_\beta = \emptyset & \text{if } \ \beta \not\geq \alpha \\ S_{\alpha*(n,q)*\beta} = \omega & \text{if } \ \alpha * (n, q) * \beta \Vdash 0 = 1 \\ S_{\alpha*(n,q)*\beta} = S_{n,\alpha*\beta} & \text{if } \ \alpha * (n, q) * \beta \nVdash 0 = 1 \end{array}$$

Now, there exists m such that $\alpha \Vdash A(S, m)$. Hence $\alpha * (m, q) \Vdash A(S, m)$ for all $q \in \mathbb{Q}$. Since $\alpha \nVdash 0 = 1$, we know by definition of $K_{\mathbb{Q}}^{\aleph_0}$ that $\alpha * (m, q) \nVdash 0 = 1$ for $q < 1$.

Let $q < 1$. The submodels generated by α and by $\alpha * (m, q)$ are isomorphic, as they are both isomorphic to $\mathcal{M}$, because $\alpha, \alpha * (m, q) \notin F$. Since $A(X, y)$ contains no parameters, we obtain $\alpha \Vdash A(S_m, m)$, because S_m above α looks like S above $\alpha * (m, q)$. This contradicts our assumption, hence UP^C is valid in the model. □

Our next goal is to prove Theorem 6.2.10, a stronger version of de Jongh's theorem. To do so we need the following. For any unprovable formula in MPC, we want to find a Kripke countermodel such that there exists a model on the tree $K_{\mathbb{Q}}^{\aleph_0}$ that can be projected, via a p-morphism, on this finite countermodel. This requires the finite countermodel to only have end nodes where f is forced. That such a model can always be found, was for example shown in [JZ15, Theorem 5]. We will first define the necessary notions and then state the part of the implicated theorem that we need.

Definition 6.2.8 (Top-model)**.**

(i) For any Kripke model $\mathcal{M}$ we define its *top-model* $\mathcal{M}^+$ by adding a node at the top of the model, accessible from all the other nodes and forcing all propositional variables, including f.

(ii) A formula φ has the *top-model property* if for any Kripe model $\mathcal{M}$ and all w we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^+, w \Vdash \varphi$.

Theorem 6.2.9. *Every formula* φ *in the language of* MPC *has the top-model property.*

Proof. By induction on the length of φ. □

Theorem 6.2.10. *Let $A(p_1, \ldots, p_n, f)$ be an unprovable formula of* MPC*. Then there exists a model $\mathcal{M}$ on the tree $K_{\mathbb{Q}}^{\aleph_0}$ such that:*

$$\alpha_0 \Vdash \neg\forall X_1 \ldots X_n A(\exists x X_1 x, \ldots, \exists x X_n x, 0 = 1)$$

Proof. Suppose $A(p_1, \ldots, p_n, f)$ is an unprovable formula of MPC. By the finite model property of MPC we find a finite Kripke countermodel $\mathcal{N}$ for A. By the previous theorem, the top-model $\mathcal{N}^+$ is still a Kripke countermodel for A and we can project the tree $K_{\mathbb{Q}}^{\aleph_0}$ on this finite model $\mathcal{N}^+$ through a p-morphism. Hence, we find a countermodel $\mathcal{M}$ on $K_{\mathbb{Q}}^{\aleph_0}$ such that $\alpha_0 \nVdash A(p_1, ..., p_n, f)$. We can now define a species X_i as done in the proof of Theorem 6.2.4 and obtain $\alpha_0 \nVdash A(\exists x X_1 x, \ldots, \exists x X_n x, 0 = 1)$ and thus $\alpha_0 \nVdash \forall X_1 \ldots \forall X_n A(\exists x X_1 x, \ldots, \exists x X_n x, 0 = 1)$. Since for every $\beta > \alpha_0$ where f is not forced, the submodel generated by β is isomorphic to $\mathcal{M}$, we conclude $\alpha_0 \Vdash \neg\forall X_1 \ldots \forall X_n A(\exists x X_1 x, \ldots, \exists x X_n x, 0 = 1)$. □

The following and final theorem we state without a proof, as it can be proved in precisely the same manner as the analogous theorem for HAS in [JS76, Corollary 2.2].

Theorem 6.2.11. MHAS *has the disjunction property and the numerical existence property* (ED_0)*.*

Chapter 7

A Minimal Theory of Equality and Apartness

In this chapter, we consider minimal theories of equality and apartness. We will see that in the minimal theory of apartness there exists a candidate for interpreting falsum which satisfies ex falso. However, we will only follow our first approach, i.e., we will not interpret falsum as it has natural strength in these systems.

Starting with van Dalen and Statman [DS79], a number of conservativity results were proved for theories of equality and apartness over theories with just equality. In other words, the theories of pure equality induced by the theories with both equality and apartness were determined. We prove a number of these results for the minimal case, starting with an analogue of the main result in [DS79]. Extending this result, we prove the minimal analogue of a result by Smoryński in [Smo77]. Finally, we prove a conservativity result for MQC that concerns the existence of a choice function in the theory of apartness. This result is also new for IQC.

7.1 Preliminaries

We start by introducing the intuitionistic theories EQ and AP (see e.g. [DS79, p. 95]).

Definition 7.1.1. A binary relation # is called an apartness relation if:

(i) $\forall xy(x = y \leftrightarrow \neg(x \# y))$

(ii) $\forall xy(x \# y \leftrightarrow y \# x)$

(iii) $\forall xyz(x \# y \to x \# z \vee y \# z)$

We assume the theory of equality, EQ, to be the first-order intuitionistic theory with the following non-logical axioms: 7

Let us denote axioms (1) – (2) by $\mathcal{A}_{\mathsf{EQ}}$ and axioms (1) – (6) by $\mathcal{A}_{\mathsf{AP}}$. Recall from Definition 4.1.2 that we obtain the corresponding minimal theories in the following way:

$$\mathsf{MEQ}_f = \{A \mid \mathcal{A}_{\mathsf{EQ}}^f \vdash_{\mathsf{MQC}} \tau_f(A)\}$$

$$\mathsf{MAP}_f = \{A \mid \mathcal{A}_{\mathsf{AP}}^f \vdash_{\mathsf{MQC}} \tau_f(A)\}$$

As we will not consider any interpretations of falsum in this chapter, we will write MEQ instead of MEQ_f, and MAP instead of MAP_f. Note that the minimal theory of equality, MEQ, has the same non-logical axioms as EQ, because $\mathcal{A}_{\mathsf{EQ}}$ is positive. Moreover, in both MEQ and MAP, just as in the minimal theories before, we have $\neg A \leftrightarrow (A \to f)$. In particular, in MAP we have $\neg(x \# y) \leftrightarrow (x \# y \to f)$.

Due to the axiomatisation, MAP is much stronger than one might expect from our experiences in arithmetic. Examining axiom (4), we conclude that from falsum we can derive that all elements are equal, shown in the following short derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{[f]^1}{\neg(t \# s)}}{t = s}\,ax(4)}{\forall xy(x = y)}}{f \to \forall xy(x = y)}$$

This does not mean however, that f by itself is strong enough to prove all sentences. In particular, we cannot derive from falsum that all elements are apart. To prove this, consider the single-noded model $\mathcal{M} = \{*\}$, with a singleton as domain $D_* = \{a\}$, in which f is forced: $* \Vdash f$. A quick check of the axioms allows us to conclude that this is a model of MAP. Moreover, $\mathcal{M} \nVdash a \# a$ and thus $\mathcal{M} \nVdash \forall xy(x \# y)$. Hence, $\mathsf{MAP} \nvdash f \to \forall xy(x \# y)$.

We will now exhibit an example of an attractive sentence that does naturally satisfy the ex falso principle.

Proposition 7.1.2. *In* MAP, $\exists x(x \# x)$ *implies all formulas.*

Proof. Let us first give the following helpful derivations.
D:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[t \# t]^1 \qquad \dfrac{\dfrac{\dfrac{}{\forall x(x = x)}\,ax(1)}{t = t}}{\neg(t \# t)}\,ax(4)}{f}}{\neg(s \# s')}}{s = s'}\,ax(4)}{\forall xy(x = y)} \qquad \exists x(x \# x)}{\forall x(x = y)}\,1$$

D':

$$\dfrac{\dfrac{\dfrac{}{\forall xyz(x = y \wedge z \# x \to z \# y)}\,ax(3)}{t = s \wedge t \# t \to t \# s} \qquad t = s \wedge t \# t}{t \# s}$$

D'':

$$\dfrac{\dfrac{\dfrac{}{\forall xyz(x = y \wedge z \# x \to z \# y)}\,ax(3)}{t = s \wedge s \# t \to s \# s} \qquad \dfrac{t = s \wedge t \# s}{t = s \wedge s \# t}}{s \# s}$$

D''' :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall xyz(x \# y \to z \# x \lor z \# y)}\,ax(6)}{\forall xz(x \# x \to z \# x \lor z \# x)}}{\forall xy(x \# x \to y \# x)}}{s \# s \to s' \# s} \qquad s \# s}{s' \# s}$$

Now, the final derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}[\exists x(x \# x)]^1 \\ \diagdown\, D \,\diagup \\ \forall xy(x = y)\end{array}}{t = s} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}[\exists x(x \# x)]^1 \\ \diagdown\, D \,\diagup \\ \forall xy(x = y)\end{array}}{t = s} \qquad t \# t}{t = s \ \land \ t \# t} \\ \diagdown\, D' \,\diagup \\ t \# s\end{array}}{\begin{array}{c} t = s \land t \# s \\ \diagdown\, D'' \,\diagup \\ s \# s \\ \diagdown\, D''' \diagup \\ s' \# s \end{array}}}{\forall xy(x \# y)} \qquad \begin{array}{c}[\exists x(x \# x)]^1 \\ \diagdown\, D \,\diagup \\ \forall xy(x = y)\end{array}}{\dfrac{\forall xy(x \# y \land x = y)}{\exists x(x \# x) \to \forall xy(x \# y \land x = y)}\,1}$$

Since the only relation symbols in MAP are $=$ and $\#$, we can now use Proposition 4.2.1 and axiom (4) to conclude that from $\forall xy(x \# y \land x = y)$, we can derive all other formulas. $\square$

Remark 7.1.3. In intuitionistic logic, we know that if equality is decidable on a set, then inequality is an apartness relation. This holds because, in intuitionistic logic, a decidable equality is a stable equality, which follows from:

$$\vdash_{\mathsf{IPC}} (p \lor \neg p) \to (\neg\neg p \to p)$$

This however, does not hold in minimal logic, as we have have already discussed in Remark 5.3.9. Hence, minimal logic allows for a set with a decidable equality on which inequality is not an apartness relation.

We will now introduce the van Dalen Statman result, immediately in the minimal context.

Let $x \neq y$ be an abbreviation of $\neg(x = y)$. We define the following inequalities, inductively on $n \in \mathbb{N}$:

$$x \neq_0 y := (x \neq y)$$
$$x \neq_{n+1} y := \forall z(z \neq_n x \lor z \neq_n y)$$

As stated in [DS79, p. 96], it is immediate that for $n \geq m$, the following axiom schema is provable in EQ:

$$\forall xy(x \neq_n y \to x \neq_m y) \tag{$I_{n,m}$}$$

This can be shown by proving that $\mathsf{EQ} \vdash \forall xy(x \neq_{n+1} y \to x \neq_n y)$, i.e. $\mathsf{EQ} \vdash \forall xy(\forall z(z \neq_n x \vee z \neq_n y) \to x \neq_n y)$, for all n. By an easy induction we therefore need to show that $\mathsf{EQ} \vdash \forall x \neg(x \neq_n x)$ and the claim then follows by using the disjunctive syllogism:

$$((p \vee q) \wedge \neg p) \to q$$

As we have seen in chapter 3, the disjunctive syllogism is not minimally valid. However, we recall from Remark 3.1.1 that any instance of the disjunctive syllogism where a negated formula is substituted for q, is minimally provable. For the proof of the following proposition only instances of the disjunctive syllogism of this form are needed.

Proposition 7.1.4. $\vdash_{\mathsf{MEQ}} \forall xy(x \neq_{n+1} y \to x \neq_n y)$

Proof. By induction on n. Base case:

$$
\begin{prooftree}
\AxiomC{$[t \neq_1 s]^2$}
\RightLabel{def.}
\UnaryInfC{$\forall z(z \neq t \vee z \neq s)$}
\UnaryInfC{$t \neq t \vee t \neq s$}
\AxiomC{$[t \neq t]^1$}
\AxiomC{$[\forall x(x = x)]^{ax.}$}
\UnaryInfC{$t = t$}
\BinaryInfC{f}
\UnaryInfC{$t \neq s$}
\AxiomC{$[t \neq s]^1$}
\RightLabel{1}
\TrinaryInfC{$t \neq s$}
\RightLabel{def.}
\UnaryInfC{$t \neq_0 s$}
\RightLabel{2}
\UnaryInfC{$t \neq_1 s \to t \neq_0 s$}
\UnaryInfC{$\forall xy(x \neq_1 y \to x \neq_0 y)$}
\end{prooftree}
$$

For the induction step, we assume $\forall xy(x \neq_{n+1} y \to x \neq_n y)$:

$$
\begin{prooftree}
\AxiomC{$[t \neq_{n+2} s]^2$}
\RightLabel{def.}
\UnaryInfC{$\forall z(z \neq_{n+1} t \vee z \neq_{n+1} s)$}
\UnaryInfC{$r \neq_{n+1} t \vee r \neq_{n+1} s$}
\AxiomC{$[r \neq_{n+1} t]^1$}
\RightLabel{IH}
\UnaryInfC{$r \neq_n t$}
\UnaryInfC{$r \neq_n t \vee r \neq_n s$}
\AxiomC{$[r \neq_{n+1} s]^1$}
\RightLabel{IH}
\UnaryInfC{$r \neq_n s$}
\UnaryInfC{$r \neq_n t \vee r \neq_n s$}
\RightLabel{1}
\TrinaryInfC{$r \neq_n t \vee r \neq_n s$}
\UnaryInfC{$\forall z(z \neq_n t \vee z \neq_n s)$}
\RightLabel{def.}
\UnaryInfC{$t \neq_{n+1} s$}
\RightLabel{2}
\UnaryInfC{$t \neq_{n+2} s \to t \neq_{n+1} s$}
\UnaryInfC{$\forall xy(x \neq_{n+2} y \to x \neq_{n+1} y)$}
\end{prooftree}
$$

This finishes the induction. □

We conclude that for $n \geq m$, the axiom schema $(I_{n,m})$ is provable in MEQ.

We define for any n the following generalized stability axiom:

$$\forall xy(\neg(x \neq_n y) \to x = y) \tag{S_n}$$

Proposition 7.1.5.

(i) $\mathsf{MAP} \vdash S_n$ *for all* n*;*

(ii) $\mathsf{MEQ} \vdash S_n \to S_m$ *for* $n \geq m$*.*

Proof. We first note that since the positive parts of MPC and IPC are the same, we have: $\vdash_{\mathsf{MPC}} (p \to q) \to ((q \to r) \to (p \to r))$.

(i) By a straightforward induction on n we can prove $\mathsf{MAP} \vdash x \# y \to x \neq_n y$, and thus $\mathsf{MAP} \vdash \neg(x \neq_n y) \to \neg(x \# y)$, for all n. From there we conclude using axiom (4) that $\mathsf{MAP} \vdash \forall xy(\neg(x \neq_n y) \to x = y)$ for all n.

(ii) Let $n \geq m$, then $\mathsf{MEQ} \vdash x \neq_n y \to x \neq_m y$ by the previous proposition. Hence $\mathsf{MEQ} \vdash \neg(x \neq_m y) \to \neg(x \neq_n y)$, and thus:

$$\mathsf{MEQ} \vdash (\neg(x \neq_n y) \to x = y) \to (\neg(x \neq_m y) \to x = y)$$

Hence $\mathsf{MEQ} \vdash \forall xy(\neg(x \neq_n y) \to x = y) \to \forall xy(\neg(x \neq_m y) \to x = y)$. $\square$

We define for each $n \in \mathbb{N}$:

$$\mathsf{SMEQ}^n := \mathsf{MEQ} + S_n,$$

And:

$$\mathsf{SMEQ}^\omega := \mathsf{MEQ} + \{S_n \mid n \in \omega\}.$$

Henkin models, also called canonical models, of MQC are obtained by a similar construction as for IQC (e.g. [Smo73, Theorem 5.1.5 - 5.1.11], or [TD88, pp. 87 - 89]). We can leave out the condition that the theories need to be consistent, for any theory in MQC will be. We will state the saturation lemma and the truth lemma without proofs, as they are essentially the same as in the case for IQC (see e.g. [TD88, Lemma 6.3 and 6.5]).

Definition 7.1.6 (Saturated theory)**.** Let C be a set of constants. A theory Γ, formulated in some language $\mathscr{L}$, is *C-saturated* if for any sentences A, B formulated in $\mathscr{L}$, and for any formula $C(x)$ formulated in $\mathscr{L}$ with only x free in C, we have:

(i) if $\Gamma \vdash A$, then $A \in \Gamma$;

(ii) if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$;

(iii) if $\Gamma \vdash \exists x C(x)$, then $C(c) \in \Gamma$ for some $c \in C$.

Lemma 7.1.7 (Saturation lemma)**.** *Let Γ be a theory and A a sentence, both formulated in some language $\mathscr{L}$, and let C be a countable set of constants not in $\mathscr{L}$. If $\Gamma \nvdash A$, then there exists a C-saturated theory $\Delta \supseteq \Gamma$ such that $\Delta \nvdash A$.* $\square$

Definition 7.1.8 (Henkin model construction for MQC)**.** Let $C_0, C_1, C_2, \ldots$ be a countable sequence of disjoint countable sets of constants not occurring in some language $\mathscr{L}$, where we define $C_n^* := C_0 \cup C_1 \cup \ldots \cup C_n$. For any C_0-saturated theory Γ_0 we can construct a Henkin model $\mathcal{M} = (\mathcal{W}, F, \subseteq, D, \Vdash)$ with root Γ_0 as follows:

(i) W consists of all C_n^*-saturated theories $\Gamma \supseteq \Gamma_0$ formulated in $\mathscr{L} \cup C_n^*$, for all n;

(ii) F is an upwards closed set of W such that $\Gamma \in F$ if and only if $\Gamma \Vdash f$;

(iii) $D(\Gamma) = C_n^*$, where Γ is C_n^*-saturated and formulated in $\mathscr{L} \cup C_n^*$;

(iv) For atomic sentences A in $\mathscr{L}(D(\Gamma))$, we define $\Gamma \Vdash A$ if and only if $A \in \Gamma$.

Lemma 7.1.9 (Truth lemma)**.** *For each theory Γ in any Henkin model $\mathcal{M}$ and for each sentence A formulated in $\mathscr{L}(D(\Gamma))$, we have:*

$$\Gamma \Vdash A \iff A \in \Gamma \qquad \Box$$

Now, for each Γ in any Henkin model we know that $A \in \Gamma$ if and only if $\Gamma \vdash A$ if and only if $\Gamma \Vdash A$.

7.2 Conservativity Results

Let us now start proving the conservativity results. The proof of the following theorem is an adaptation of Smoryński's proof as sketched in [DS79, p.115]. Note again that any MQC-theory is consistent. In our proof we use the following definition for a set of MQC-sentences Γ:

$$\Gamma \text{ is } f\textit{-consistent} \text{ if and only if } \Gamma \nvdash f.$$

Theorem 7.2.1. MAP *is conservative over* SMEQ^ω*.*

Proof. Let Γ_0 be any saturated extension of SMEQ^ω and let $\mathcal{M}$ be the Henkin model of MQC with root Γ_0. We will prove that $\mathcal{M}$ is a model of MAP. For any Γ in $\mathcal{M}$ and all $a, b \in D(\Gamma)$, we define:

$$\Gamma \Vdash a \# b \;\text{ if and only if for all } n \in \omega \;\; \Gamma \Vdash a \neq_n b.$$

Claim. *The relation $\#$ is an apartness relation on $\mathcal{M}$.*
Proof of the Claim. The only interesting axiom to check is the left to right direction of (4):

$$\forall xy(\neg(x \# y) \to x = y)$$

Suppose for contradiction that $\Gamma_i \Vdash \neg(a \# b)$ and $\Gamma_i \nVdash a = b$ for some theory $\Gamma_i \in \mathcal{W}$. Since $\mathcal{M}$ is a model of SMEQ^ω, $\Gamma_i \Vdash S_n$ for all n. Hence $\Gamma_i \Vdash (a \neq_n b \to f) \to a = b$ for all n. Since we have $\Gamma_i \nVdash a = b$, we know $\Gamma_i \nVdash f$, hence by the truth lemma, Lemma 7.1.9, we conclude that Γ_i is f-consistent. Consider $\Gamma_i \cup \{a \neq_n b \mid n \in \omega\}$. If $\Gamma_i \cup \{a \neq_n b \mid n \in \omega\}$ is not f-consistent, then we have $\Gamma_i \cup \{a \neq_n b \mid n \in \omega\} \vdash f$ and thus it follows that $\Gamma_i \cup \{a \neq_n b \mid n \leq k\} \vdash f$ for some $k \in \omega$. We thus have $\Gamma_i \vdash a \neq_0 b \wedge \ldots \wedge a \neq_k b \to f$. Since $\mathcal{M}$ is a model of SMEQ^ω, by the axiom schema $(I_{n,m})$ we know that $\Gamma_i \vdash a \neq_n b \to a \neq_m b$ for $n \geq m$. We may then conclude that $\Gamma_i \vdash a \neq_k b \to f$. By $\Gamma_i \Vdash S_k$, we have $\Gamma_i \Vdash (a \neq_k b \to f) \to a = b$, hence $\Gamma_i \vdash a = b$, a contradiction. We conclude that $\Gamma_i \cup \{a \neq_n b \mid n \in \omega\}$ is f-consistent. Then, by the saturation lemma, Lemma 7.1.7, there exists a theory $\Delta \in \mathcal{W}$ such that $\Gamma_i \cup \{a \neq_n b \mid n \in \omega\} \subseteq \Delta$ and $f \notin \Delta$. Our assumption $\Gamma_i \Vdash \neg(a \# b)$ implies $\Delta \Vdash \neg(a \# b)$, by the truth lemma, Lemma 7.1.9. On the other hand, $\Gamma_i \cup \{a \neq_n b \mid n \in \omega\} \subseteq \Delta$ implies $a \neq_n b \in \Delta$, and thus $\Delta \Vdash a \neq_n b$, for all n. By our definition above, this gives us $\Delta \Vdash a \# b$. From this we obtain $\Delta \Vdash f$, which contradicts $f \notin \Delta$. We therefore conclude that $\Gamma_i \Vdash a = b$, hence $\Gamma_0 \Vdash \forall xy(\neg(x \# y) \to x = y)$. $\blacksquare$

Suppose $\mathsf{SMEQ}^\omega \nVdash A$, for some sentence A formulated in the language of equality. Then, by the saturation lemma, we know that there exists a saturated extension Γ of SMEQ^ω such that $\Gamma \nVdash A$. Take $\Gamma_0 := \Gamma$. Since $\#$ defines an apartness relation on $\mathcal{M}$, it follows that $\mathcal{M}$ is a model of MAP an thus $\mathsf{MAP} \nVdash A$. This shows that SMEQ^ω is the equality fragment of MAP, i.e. MAP is conservative over SMEQ^ω. □

We will now present an extension of the above theorem, which was proved for intuitionistic logic by Smoryński in [Smo77, Theorem 2]. We mimic the proof for MQC. First we define for $n \geq 2$:

$$\mathsf{MAP}_n := \mathsf{MAP} + \exists x_1 \dots \exists x_n (\bigwedge_{i<j} x_i \# x_j)$$

$$\mathsf{SMEQ}^\omega_n := \mathsf{SMEQ}^\omega + \exists x_1 \dots \exists x_n (\bigwedge_{i<j} x_i \neq_m x_j), \quad \text{for all } m$$

Moreover, we define:

$$\mathsf{MAP}_\omega := \bigcup_n \mathsf{MAP}_n$$

$$\mathsf{SMEQ}^\omega_\omega := \bigcup_n \mathsf{SMEQ}^\omega_n$$

Theorem 7.2.2. *For any $2 \leq n \leq \omega$, MAP_n is conservative over SMEQ^ω_n.*

Proof. It suffices to proof this for finite n, the case for MAP_ω then easily follows. Let n be given and let Γ be any f-consistent extension of SMEQ^ω_n. Let C_0 be a set of constants and $c_1, \dots, c_n \in C_0$ pairwise distinct.

Claim. $\Gamma \cup \{c_i \neq_m c_j \mid i < j \leq n, m \geq 0\}$ *is f-consistent.*

Proof of the Claim. Suppose not, then $\Gamma \cup \{c_i \neq_k c_j \mid i < j \leq n, 0 \leq k \leq m\} \vdash f$ for some finite m. Then $\Gamma \cup \{c_i \neq_m c_j \mid i < j \leq n\} \vdash f$, by the axioms $I_{m,k}$. Then $\Gamma \cup \{\exists x_1 \dots \exists x_n (\bigwedge_{i<j\leq n} x_i \neq_m x_j)\} \vdash f$ and thus $\Gamma \vdash f$, contradicting our assumption. ■

Let $\Gamma' := \Gamma \cup \{c_i \neq_m c_j \mid i < j \leq n, m \geq 0\}$. Now suppose $\mathsf{SMEQ}^\omega_n \nVdash A$. We use the saturation lemma to find a C_0-saturated extension Γ_0 of Γ'. Let $\mathcal{M}$ be the Henkin model for MQC with root Γ_0. Let the relation $\#$ be again defined by $a \# b := \forall n (a \neq_n b)$, then $\#$ again defines an apartness relation. Now $\Gamma_0 \Vdash c_i \# c_j$ for any $i < j \leq n$, hence $\Gamma_0 \Vdash \bigwedge_{i<j\leq n} c_i \# c_j$ and thus $\Gamma_0 \Vdash \exists x_1 \dots \exists x_n (\bigwedge_{i<j\leq n} x_i \# x_j)$. We conclude that the Henkin model $\mathcal{M}$ is a model of MAP_n. Hence $\mathsf{MAP}_n \nVdash A$ and thus MAP_n is conservative over SMEQ^ω_n. □

Remark 7.2.3. An alternative and simplified way to do this proof is by leaving out the condition of f-consistency. We can simply take Γ to be any extension of SMEQ^ω_n and Γ_0 any C_0-saturated extension of $\Gamma \cup \{c_i \neq_m c_j \mid i < j \leq n, m \geq 0\}$. The claim in the above proof is then superfluous. That there always exists a C_0-saturated extension for a certain set C_0 for any theory Γ, can be easily shown by simplifying the proof of the saturation lemma.

The Addition of a Choice Function

Classically, a choice function g for which $\forall x R(x, gx)$, can be conservatively added to the existence axiom $\forall x \exists y Rxy$. In both [Smo77] and [Smo78], Smoryński showed that, intuitionistically, the following axioms are needed to satisfy extensionality of g, a result that was first proved by Mints (unpublished):

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \big(\bigwedge_{i \leq n} Rx_i y_i \wedge \bigwedge_{i \leq j} (x_i = x_j \to y_i = y_j) \big) \quad (*_n)$$

Let us denote the theory in the language of $=, R$ and g with the axioms of MEQ and $\forall x Rxgx$ as T_1 and let T_2 be the theory in the language of $=$ and R with the axioms of MEQ and all the axioms $(*_n)$.

Theorem 7.2.4. *T_1 is conservative over T_2.*

Proof. If $T_2 \nvdash A$ for some sentence A formulated in T_2, we can find a saturated extension Γ of T_2 such that $\Gamma \nvdash A$. Now we can construct a Henkin Model of Γ which is a model of T_2 by mimicking the proof of the analogous theorem for intuitionistic logic found in [Smo77, Theorem 1]. We can do this by merely leaving out 'consistency' (which in fact simplifies the proof) and replacing 'strong C-saturated extension' by 'C-saturated extension'. $\square$

We can strengthen this theorem by adding $\forall xy(gx \# gy \to x \# y)$ to the theory of apartness. Intuitively, this should have no consequences for the theory of equality and indeed it does not.

Theorem 7.2.5. $\mathsf{MAP} \cup T_2 \cup \{\forall xy(gx \# gy \to x \# y)\}$ *is conservative over* $\mathsf{SMEQ}^{\omega} \cup T_1$.

Proof. Let Γ be any f-consistent saturated extension of $\mathsf{SMEQ}^{\omega} \cup T_1$. Then construct the Henkin Model $\mathcal{M}$ with root Γ as in Theorem 7.2.4. By Theorem 7.2.1 and Theorem 7.2.4 we already know that $\mathcal{M}$ is a model of MAP and T_2, respectively. We will show that the choice function g as constructed in Theorem 7.2.4 satisfies $\forall xy(gx \# gy \to x \# y)$, where $\#$ defined as in Theorem 7.2.1. This will complete our proof.

Claim. *In $\mathsf{SMEQ}^{\omega} \cup \mathrm{T}_2$, the following is derivable for every n:*

$$\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \big(\bigwedge_{i} Rx_i y_i \wedge \bigwedge_{i \leq j} (y_i \neq_n y_j \to x_i \neq_n x_j) \big)$$

Proof of the Claim. We prove this claim by induction on n. The case for $n = 0$ follows immediately by using axiom $(*_n)$ and because $(p \to q) \to (\neg q \to \neg p)$ in minimal logic. We assume the case for n and thus it is left to show that for all k:

$$\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \big(\bigwedge_{i} Rx_i y_i \wedge \bigwedge_{i \leq j} (y_i \neq_{n+1} y_j \to x_i \neq_{n+1} x_j) \big). \quad (1)$$

Let k be arbitrary. Using the induction hypothesis for $k + 1$ we know that:

$$\forall x_1 \exists y_1 \ldots \forall x_{k+1} \exists y_{k+1} \big(\bigwedge_{i} Rx_i y_i \wedge \bigwedge_{i \leq j} (y_i \neq_n y_j \to x_i \neq_n x_j) \big).$$

Hence, for arbitrary t_1 there exists s_1 such that for arbitrary t_2 there exists s_2 such that ... such that for arbitrary t_k there exists s_k such that:

$$\forall x_{k+1} \exists y_{k+1} \big(\bigwedge_{i \leq k} R t_i s_i \ \wedge \ R x_{k+1} y_{k+1} \ \wedge \ \bigwedge_{i<j\leq k} (s_i \neq_n s_j \to t_i \neq_n t_j) \ \wedge$$

$$\bigwedge_{i \leq k} (s_i \neq_n y_{k+1} \to t_i \neq_n x_{k+1}) \big). \quad (2)$$

By (2) we already know that $\bigwedge_{i\leq k} R t_i s_i$, hence it finishes the proof if we can show that $\bigwedge_{i<j\leq k} (s_i \neq_{n+1} s_j \to t_i \neq_{n+1} t_j)$, because then we have proved (1). Let i and j be arbitrary such that $i < j \leq k$ and suppose $s_i \neq_{n+1} s_j$, i.e. $\forall z(z \neq_n s_i \vee z \neq_n s_j)$. We need to show that $t_i \neq_{n+1} t_j$, i.e. that $\forall z(z \neq_n t_i \vee z \neq_n t_j)$. Let t_{k+1} be arbitrary. Then, using (2), we can find s_{k+1} such that $R t_{k+1} s_{k+1}$ and $s_i \neq_n s_{k+1} \to t_i \neq_n t_{k+1}$ and $s_j \neq_n s_{k+1} \to t_j \neq_n t_{k+1}$. Since $\forall z(z \neq_n s_i \vee z \neq_n s_j)$, we know that $s_{k+1} \neq_n s_i \vee s_{k+1} \neq_n s_j$. Hence we conclude that $t_{k+1} \neq_n t_i \vee t_{k+1} \neq_n t_j$. Since t_{k+1} was arbitrary we obtain $\forall z(z \neq_n t_i \vee z \neq_n t_j)$, i.e. $t_i \neq_{n+1} t_j$. ■

Now, if $\Gamma_i \Vdash ga \# gb$ for any Γ_i in $\mathcal{M}$, then $\Gamma_i \Vdash ga \neq_n gb$, for all n. Hence, by the above claim, $\Gamma_i \Vdash a \neq_n b$, for all n, and thus $\Gamma_i \Vdash a \# b$. Therefore, we conclude $\Gamma \Vdash \forall xy(gx \# gy \to x \# y)$ and this completes the proof. □

In this chapter, it has been shown that regarding the theories of equality and apartness, minimal and intuitionistic logic differ very little.

Chapter 8

Conclusions and Further Research

We will conclude this thesis with an overview of what we have done and ideas for further research.

In chapter 2, we have seen an example for the use of a minimally invalid law in analysis. However, this problem could be solved in Glivenko's logic, which is strictly weaker than IPC but stronger than MPC. We found new characterisations of this logic in our research on superminimal logics in chapter 3. There we systematically examined the differences between minimal and intuitionistic propositional logic by restricting the length of formulas in the infinite fragment. This work can be taken further by allowing formulas with a greater maximal length. A natural question is whether there are differences between minimal and intuitionistic predicate logic that do not reduce to differences between minimal and intuitionistic propositional logic.

We developed a general framework for studying formal systems on the basis of minimal logic with different interpretations of falsum in chapter 4. Throughout this thesis, we have attempted three different approaches of studying theories on the basis of minimal logic: First, working with an uninterpreted falsum as in pure minimal logic. Second, trying to find a natural sentence satisfying the ex falso principle, and third, considering candidates for falsum of strength strictly between the previous two approaches. Besides these approaches, one can think of working with several interpretations of falsum at once. That is, if we have an axiomatisation with several occurrences of negation, one can choose a different interpretation for falsum for different occurrences.

In chapter 5, we have studied first-order minimal arithmetic and found out that this system is quite weak: Equality is not stable, not all primitive recursive functions are representable and several fundamental principles of arithmetic do not hold (e.g., $x + y = 0$ does not imply that $x = 0$ and $y = 0$). Naturally, the discovery of these weaknesses of minimal arithmetic lets us wonder about ways to strengthen the theory MHA_f. This is possible, for example, by adding an axiom like $\exists x(Sx = 0) \to 1 = 0$. Moreover, it would be interesting to study other intuitionistic systems of arithmetic and analysis, and see whether their minimal versions exhibit similar deficiencies.

To summarise our work on first-order arithmetic, note that minimal mathematics in its purest sense is not suitable for studying arithmetic because of its many weaknesses. However, there are several possible solutions which one can still further investigate: Interpreting falsum as $0 = 1$, adding logical axioms (similar to the case of Glivenko's logic and intuitionistic analysis) or adding non-logical axioms (as we have seen above). Of course, one can think of other solutions as well.

When considering minimal second-order arithmetic in chapter 6, we decided to interpret falsum as $0 = 1$ (following the third approach, as $0 = 1$ does not imply all formulas). Among other things, we have seen that Markov's Principle has a rather different status in minimal than in intuitionistic logic as it is not at all trivial to validate it on models for minimal second-order arithmetic, and its justification seems less clear in the minimal case. This is worth more investigation.

Different theories of equality and apartness are the topic of chapter 7. We proved a number of conservativity results for those theories. We decided to not interpret falsum, for the reason that minimal logic possesses natural strength in these systems. As a consequence, it may be interesting to add apartness to other minimal systems to strengthen minimal logic there. For instance, this can be investigated for minimal arithmetic.

Finally, a general direction of further research is the decidability of minimal theories. Every minimal theory is consistent and, therefore, every extension of every minimal theory will be consistent. Hence, no minimal theory is essentially undecidable, as the maximal extension of all sentences in the language of our theory is trivially decidable. Take, for instance, MHA_f. If we add $0 = 1$ as an axiom, we obtain a consistent extension, since the false 1-circle is a model. This extension is decidable, as every formula is a theorem. Our question boils down to the fact whether we can sensibly redefine essentially undecidability to obtain an interesting notion for minimal theories. A possible reformulation is to say that a minimal theory is essentially undecidable if every f-consistent extension is undecidable.

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