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TIME, LOGIC AND COMPUTATION

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ABSTRACT. This paper starts with a survey of temporal logic in its original guise, pointing at its connections with philosophy and linguistics. More specifically, a technical exposition is provided of the basic 'tense-logical' system, based on points or 'moments' of time, with the research program in model theory and proof theory which has grown around it. After that, a more recent stream of 'period' and 'event' based approaches to time is discussed, again with some of the new logical themes engendered by it. Finally, a review is given of some recent computational research in temporal logic. Here, a clear continuity of logical concerns emerges between philosophy, linguistics and computer science. But, the latter adds several new themes and perspectives which might well give it a significant impact on the earlier standard enterprise.

Keywords. Completeness, computational semantics, correspondence, event structure, first-order definability, period structure, point structure, temporal operator, tense logic.

CONTENTS

1. Introduction

The formal properties of Time have attracted the attention of philosophers and mathematicians ever since Antiquity (cf. the anthology Smart 1964). Moreover, abstract temporal structure finds its reflection in our linguistic habits of temporal reasoning, and hence logicians have entered this field too, creating a discipline of 'temporal logic' (Reichenbach 1947, Prior 1957, Prior 1967). But in fact, time is a phenomenon cutting across many academic boundaries: from physics (cf. Reichenbach 1957) to psychology (cf. Michon & Jackson 1985). [A reference spanning virtually this whole spectrum is Whitrow 1980.] Within this wider area, temporal logic finds itself in particularly close contact with philosophy, linguistics (cf. Dowty 1979) and increasingly also with computer science. For, these disciplines share an interest in creating exact systems of temporal representation, coming with calculi for reasoning about change or persistence over time.

The purpose of this paper is twofold. On the one hand, a discursive survey will be given of temporal logic as it has developed in this century - while on the other, there is a review of recent contacts with computer science; whose variety, after a mere decade of research, is already impressive. No complete coverage has been attempted, however, in either respect. For further details on temporal logic, there is an array of informative texts, such as Gabbay 1976, van Benthem 1983, Burgess 1984 and Goldblatt 1987a. Moreover, the present volume itself may be viewed as an anthology of computational research into temporal logic. Therefore, we shall feel free to choose our own path through the area, with an emphasis on general research lines, and an occasional new question or observation.

2. Points

2.1 Basic Framework

2.1.1 An instructive, and much-studied 'minimal' system of temporal logic was developed by Arthur Prior in the fifties and sixties. This so-called 'tense logic' has propositional operators P for *past* and F for *future*. The original motivation for concentrating on these was two-sided. Philosophically, these are the basic operators which assign changing temporal 'perspective' to events. I.e., they express the dynamic 'A-series' of McTaggart 1908, who contrasted this with the static 'B-series' of an immutable order of events. In Prior's system, B-series models provide the semantics for A-series languages, as we shall see. But also, linguistically, the operators P, F correspond to the most basic 'tenses' of natural language:

q	:	Mary cries	(present)
Pq	:	Mary cried	(past)
Fq	:	Mary will cry	(future).

Further iterations then reflect compound tenses:

PPq	:	Mary had cried	(past perfect)
FPq	:	Mary will have cried	(future perfect),

etcetera. The analogy is by no means without its problems: but it has had undeniable heuristic virtues.

Remark. The temporal system of expression in natural languages is much richer than these examples might suggest. In addition to tenses, there are many types of *temporal adverbs* ("now", "then", "yesterday", "for an hour",...), including an elaborate quantificational vocabulary ("always", "mostly", "often", "usually",...). Then, there are *temporal connectives* relating two events ("since", "until", "before", "after",...). And finally, there are so-called *aspects,* recording various kinds of temporal texture which an event can have: finished or unfinished, iterative or unique, and so on. Not all of these have yet received the systematic logical attention which they deserve. (But see Galton 1984 on the logic of aspect treated in a Priorean fashion, as well as Kamp & Rohrer 1988 on more general linguistic semantics of temporal expressions.) □

2.1.2 Semantic structures for the basic system are sets T of *points in time* equipped with a binary order of *precedence* : so-called *frames* $\mathbb{T}$ =

$(T, <)$.

These form the temporal pattern on which histories can take place, assigning to each $t \in T$ some interpretation of the original descriptive language on which the temporal operators have been superimposed. Our paradigm will be a temporal *propositional logic,* having a vocabulary of

proposition letters	:	p,q,r,...
Boolean connectives	:	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
temporal operators	:	F, P

This allows us to define further temporal operators:

$G\phi \quad = \quad \neg F \neg \phi$ (always in the future)

$H\phi \quad = \quad \neg P \neg \phi$ (always in the past)

More generally, one can formulate and compare complex temporal statements.
Here is an illustration, involving relative scope of operators:

$p \rightarrow Gq$: if you sin, you will be punished ever after,

$G(p \rightarrow q)$: you will always be punished if you sin.

For the language, temporal *models* are structures $\mathbb{M}$ =

$(T, <, V)$,

where V is a *valuation* assigning to each atomic proposition p the set V(p) of times when it holds.
A standard inductive truth definition then lifts this to complex formulas ϕ .

$\mathbb{M} \models \phi\ [t]$ (ϕ *is true in model* $\mathbb{M}$ *at point* t):

$\mathbb{M} \models p\ [t]$ iff $t \in V(p)$

$\mathbb{M} \models \neg\phi\ [t]$ iff *not* $\mathbb{M} \models \phi\ [t]$

$\mathbb{M} \models \phi\wedge\psi\ [t]$ iff $\mathbb{M} \models \phi\ [t]$ *and* $\mathbb{M} \models \psi\ [t]$

and likewise for the other Boolean connectives;

$\mathbb{M} \models P\phi\ [t]$ iff $\mathbb{M} \models \phi\ [t']$ *for some* $t'<t$

$\mathbb{M} \models F\phi\ [t]$ iff $\mathbb{M} \models \phi\ [t']$ *for some* $t'>t$

But, one can also superimpose the temporal operators on an underlying *predicate logic*, allowing an interplay between temporal operators and quantifiers:

$F\exists x(Cx\wedge Mx)$: there will be a child (then) landing on Mars

$\exists x(Cx\wedge FMx)$: a child (now) will land (then) on Mars.

In this case, each point in time must carry a complete structure of individuals with relations and operations, in order to interpret the new vocabulary.

Remark. There are in fact some *options* here. For instance, should an atomic statement Px be false at point t if x does not exist at t, or should it be merely undefined? Likewise, should a formula GQx be true at t if Qx is true at all later points, or if it is merely true at all later points where x exists? (When we say that Napoleon was 'always' on the offensive, we obviously mean something like 'at all times of his existence'.) As we are not going to pursue temporal predicate logic here, these issues will not be explored. It should be noted, however, that the technical theory of this system can be rather complex, due to the interaction between quantification over points in time and quantification over individuals. (Cf. Garson 1984, Hughes & Creswell 1984.) □

2.1.3 The above frames only become more genuinely 'temporal structures' when further *constraints* are imposed. What are the basic properties of the temporal order? This is a question of what may be called 'mathematical metaphysics'.

Proposed principles here are of various logical kinds. First, there are simple *first-order* conditions which seem uncontroversial:

$\forall xyz: (x<y \wedge y<z) \rightarrow x<z$ (transitivity)

$\forall x: \neg x<x$ (irreflexivity)

Together, these imply asymmetry $\forall xy: x<y \rightarrow \neg y<x$. Note that all these conditions are *universal Horn* sentences. The first option encountered involves a non-Horn condition, viz.

$\forall xy: x<y \vee y<x \vee x=y$ (linearity)

A weaker form which has also been put forward is

$\forall xyzu: x<y \rightarrow (x<z \vee z<y)$ (almost-connectedness)

But both would exclude genuinely branching time with disjoint paths into the future.

Another option arises with *existence* principles, such as

$\forall x: \exists y\ x<y$ (succession)

An alternative would be to postulate an end to time. Likewise, there is a choice between

$\forall xy: x<y \rightarrow \exists z(x<z \wedge \neg\exists y(x<y \wedge y<z))$ (discreteness)

or

$\forall xy: x<y \rightarrow \exists z(x<z \wedge z<y)$ (density) .

And of course, these principles have leftward past variants too, with similar options.

On top of these first-order conditions, one can formulate *higher-order* postulates on the structure of time. A well-known mathematical example is *Dedekind Continuity*. But also, many philosophers and physicists would subscribe to a principle of *Homogeneity*: 'Time is structurally similar throughout' . Mathematically,

For any two points $t,t' \in T$, there exists an *order-automorphism* of $(T, <)$ mapping t to t'.

For instance, **R**, **Q**, **Z** are homogeneous, whereas **N** is not.

Note that, e.g., a homogeneous structure satisfies all formulas of the form

$\forall x\phi(x) \vee \forall x\neg\phi(x)$ *

In particular, taking $\phi(x) = \exists y(x<y \wedge \neg\exists z(x<z \wedge z<y))$, homogeneous structures must be either discrete or dense.

Remark. No stronger first-order condition than * can be derived from Homogeneity: see van Benthem 1984b. □

Another higher-order principle is *Isotropy*: 'There is no formally preferred direction of Time'. Mathematically,

Every frame $(T, <)$ is isomorphic to its converse $(T, >)$.

This principle enforces the same conditions for $<$ and $>$. [For a more global analogy, one might compare 'duality principles' in Graph Theory.] Actually, some of the earlier principles for $<$ automatically imply their $>$ form already (e.g., transitivity, irreflexivity and linearity). But this is not a general phenomenon, not even for Horn clause conditions. For instance, $\forall xyz(x<y \rightarrow x<z)$ does not imply $\forall xyz(x>y \rightarrow x>z)$.

Digression. Here is an illustration of the utility of classical model-theoretic methods in this area. The complete first-order theory of Isotropy is in fact given by the above conversion schema

$\phi(<) \leftrightarrow \phi(>)$, for all first-order sentences ϕ IS

For, suppose that some first-order sentence α is not a consequence of this schema. Then, by the Completeness and Loewenheim-Skolem theorems, there exists some countable frame $\mathbb{T}$ verifying the schema IS while falsifying α. Now, take any ω-saturated elementary extension $\mathbb{T}^+$ of $\mathbb{T}$ (cf. Chang & Keisler 1973): IS will remain true in $\mathbb{T}^+$, and α false. Let a_1 be any object in $\mathbb{T}^+$. The type of formulas $\{\psi(x) \mid \mathbb{T}^+ \models \psi[a_1]\}$, where $\psi = \psi(<)$, has an obvious transpose with $<$ replaced by $>$, which can be finitely satisfied in $\mathbb{T}^+$, thanks to the validity of IS. By saturation then, the whole transpose is realized by some object b_1 in $\mathbb{T}^+$. Continuing the resulting correspondence a_1, b_1 by the usual zigzag argument, we obtain two enumerations $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ of $\mathbb{T}^+$ such that $(\{a_1, a_2, \ldots\}, <)$ is elementarily equivalent to $(\{b_1, b_2, \ldots\}, <)$. Hence, the map $a_i \mapsto b_i$ $(i \in N)$ is an isomorphism between $\mathbb{T}^+$ and its converse. So, full second-order Isotropy holds for $\mathbb{T}^+$, and hence, α is not among its consequences either. ■

When viewed as a constraint on frames, however, the first-order schema IS will not be equivalent to Isotropy. For instance, the ordinal sum of $(\mathbf{Q}, <)$ and $(\mathbf{R}, <)$, in that order, validates IS, even though this frame is not isomorphic to its converse. But, the idea of the preceding proof is that, on suitably saturated models, first-order schemata approximate their second-order parents (cf. Barwise 1976). □

Evidently, the above considerations do not enforce one single formal picture of Time. Indeed, various mathematical structures still pass most tests. Examples are the classical reals, rationals or integers - but also a relativistic Minkowski Space $\mathbf{M}$ with 'causal precedence' between a point and its successors inside its future light cone. As a concrete, simplified example of the latter kind of structure, take $\mathbb{T} = (T, <)$ with

$T = \mathbf{R} \times \mathbf{R}$

$(x,y) < (u,v)$ if $x<u$ and $y<v$.

This is a branching temporal structure, which still allows 'confluence':

$\forall xy: \exists z(x<z \wedge y<z)$

$\forall xy: \exists z(z<x \wedge z<y)$

(See Goldblatt 1980, Shehtman 1983 for matching relativistic tense logics.)

There is no difficulty in this diversity. Our intuitions about Time, like those concerning other fundamental categories of thought, are rich enough to support different paradigms - and for the purpose of modelling certain phenomena, retaining an open mind toward such options is actually valuable. Moreover, the division into a small set of core conditions on temporal structures and further extras is also useful, in that a given application may need only a weak temporal calculus, rather than the full theory of some particular frame.

Example. Comparing Major Temporal Structures.

- **R**, **Q** have the same *first-order* theory, which differs from that of **M** or **Z** .
- **R**, **Q** have different *second-order* theories: the former, but not the latter, being Dedekind continuous.
- **R**, **Q**, **Z** all have the same *universal* first-order theory, which differs from that of **M** . (The latter is actually contained in the former, as **R** can be embedded as a substructure of **M** .)
- **R**, **Q**, **Z**, **M** all have the same universal *Horn* theory. (For, **M** is a *direct product* of **R** : and this frame operation preserves truth of Horn sentences.) □

Remark. If one takes this diversity of temporal structures quite seriously, then a reasonable next step would be to develop separate kinds of *topology* and *analysis* on them. (E.g., analysis on the rationals rather than the reals has a quite different flavour: cf. van Benthem 1983.) □

2.1.4 The preceding Section already introduced a typical logical theme, namely the interplay between properties of structures in general and what can be expressed about them in specific formal languages set up for the purpose of some genre of reasoning. In particular, one can study *correspondences* between structural properties of the temporal order and principles expressible in the earlier tense logical language. To see this, we must remove any dependence on the 'accidental facts' spread out over a temporal frame:

$(T, <) \models \phi\, [t]$ if $(T, <, V) \models \phi\, [t]$ *for all valuations* V on $(T, <)$

$(T, <) \models \phi$ if $(T, <) \models \phi\, [t]$ *for all points* $t \in T$.

Then, we get equivalences such as the following:

$(T, <) \models FFp \rightarrow Fp$ iff $<$ is *transitive*

$(T, <) \models Fp \rightarrow FFp$ iff $<$ is *dense* .

Often, however, these matches are not quite perfect, due to certain peculiarities of the tense-logical language. E.g., linearity is not definable as it stands, but

$(T, <) \models (Fp \wedge Fq) \rightarrow (F(p \wedge q) \vee F(p \wedge Fq) \vee F(q \wedge Fp))$ iff

$\forall x\colon \forall yz((x<y \wedge x<z) \rightarrow (y<z \vee z<y \vee y=z))$;

and analogously for a leftward variant. Likewise, discreteness can only be approximated:

$p \rightarrow FH(p \vee Fp)$ corresponds to $\forall x\colon \exists y(x<y \wedge \forall z<y(z=x \vee z<x))$.

In general, when viewed in this perspective, Prior's tense-logical formalism expresses *second-order* conditions on the temporal order (because of the above quantification over valuations). And indeed, some tense-logical principles do not correspond to first-order frame conditions on $<$ at all. An example is 'Loeb's Axiom':

$(T, <) \models H(Hp \rightarrow p) \rightarrow Hp$ iff $<$ is transitive and *well-founded.*

On the other hand, certain simple first-order principles are not tense-logically expressible. A well-known example is *irreflexivity* of $<$. Proofs of this and other assertions in this Section may be found in van Benthem 1983. What they presuppose is a deeper analysis of the semantic behaviour of this language. (Section 2.3.1 below has further information on this.)

With richer tense-logical formalisms, the correspondence situation changes. For instance, let us add an operator D ('difference'), whose interpretation is as follows:

$\mathbb{M} \models D\phi\,[t]$ if $\mathbb{M} \models \phi\,[t']$ *for some* $t' \neq t$.

(This proposal is due to several people independently, including Ron Koymans, Patrick Blackburn and Gargov, Passy & Tinchev 1987.) Then, irreflexivity becomes definable after all, by

$Fp \rightarrow Dp$;

and ordinary linearity is expresed by

$Dp \rightarrow Fp \vee Pp$.

Remark. In fact, all universal first-order conditions on $<$ are definable in the F, P, D formalism, as well as various other useful types of formula. □

Of course, one can also consider correspondence on *models* (T, <, V) rather than frames. We shall determine precisely which first-order formulas are tense-logically definable in Section 2.3, using their semantic invariance behaviour.

Remark. Correspondence also makes sense for other base logics than purely propositional ones. For instance, a quantifier / operator interchange principle such as

$\exists xGQx \rightarrow G\exists xQx$

expresses a condition of 'cumulation' on individual domains along successive points of time:

$\forall xy\colon x<y \rightarrow \forall u(u \in D_x \rightarrow u \in D_y)$. □

2.2 Axiomatics

2.2.1 In addition to the semantic aspect of temporal logic, there is also its proof-theoretic side. There, the focus is on syntactic calculi of deduction for temporal expressions. Many such calculi live in the literature, mainly stemming from two sources. On the one hand, philosophers have proposed certain forms of temporal argument, which can be systematized into logical systems. In this case, providing a proper semantics is only a second step. But more often, one starts from some preferred temporal picture, asking for a faithful and complete record of the argument patterns supported by it.

Here are some examples of tense-logical systems. The so-called *minimal tense logic* K_t has the following principles.

Axioms:	(1)	all propositional tautologies	
	(2)	$G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$	(Temporal
		$H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)$	Distribution)
	(3)	$\phi \rightarrow HF\phi$	
		$\phi \rightarrow GP\phi$	
Definition:	(4)	$F\phi \leftrightarrow \neg G \neg \phi$	
		$P\phi \leftrightarrow \neg H \neg \phi$	
Rules of Inference:			
	(5)	$\phi,\ \phi \rightarrow \psi\ /\ \psi$	(Modus Ponens)
	(6)	if ϕ is a theorem, then so are $G\phi$, $H\phi$	(Generalization)

On top of this, one can put any of a great number of possible axioms, reflecting special assumptions on the temporal order, such as the earlier-mentioned principles for transitivity, density or linearity. Here is an illustrative table (precise formulations of the relational conditions are left to the reader):

$FF\phi \rightarrow F\phi$	transitivity
$F\phi \rightarrow FF\phi$	density
$(F\phi \wedge F\psi) \rightarrow F(\phi \wedge F\psi) \vee F(\phi \wedge \psi) \vee F(\psi \wedge F\phi)$	right-linearity
$(P\phi \wedge P\psi) \rightarrow P(\phi \wedge P\psi) \vee P(\phi \wedge \psi) \vee P(\psi \wedge P\phi)$	left-linearity
$F\mathit{true}$	right succession
$P\mathit{true}$	left succession
$\phi \rightarrow FH(\phi \vee F\phi)$	right discreteness
$\phi \rightarrow PG(\phi \vee P\phi)$	left discreteness
$FG\phi \rightarrow GF\phi$	right confluence
$PH\phi \rightarrow HP\phi$	left confluence
$(F\phi \wedge FG\neg\phi) \rightarrow F(HF\phi \wedge G\neg\phi)$	Dedekind continuity

Different choices lead to different 'tense logics', which inhabit the *Lattice of Tense Logics*, lying in between the minimal system and, at the upper end, trivial systems in which (too) many temporal distinctions have collapsed, with principles such as

$G\mathit{false}$ irreflexive isolated points

or

$G\phi \leftrightarrow \phi$ reflexive isolated points.

Thus, as in Section 2.1.3, the issue is not to locate one single preferred system, but rather to understand the varieties of temporal reasoning, by studying the properties of such logics and their inter-relationships.

Example. Relative Interpretation.

The lattice of tense logics cannot be taken at face value; since often, tense logics can be translated into each other. E.g., the transition

$$\tau(G\phi) \mapsto H\tau(\phi), \qquad \tau(H\phi) \mapsto G\tau(\phi)$$

will map an 'isotropic' logic like K_t into itself, whilst switching theorems. On the other hand, a translation like

$$\sigma(G\phi) \mapsto G\sigma(\phi) \wedge \sigma(\phi), \qquad \sigma(H\phi) \mapsto H\sigma(\phi) \wedge \sigma(\phi)$$

will embed K_t into the modal logic T, being K_t with the additional axiom

$G\phi \rightarrow \phi$, corresponding to *reflexivity*.

The latter illustrates the possible location of modalities inside temporal logics (cf. Smirnov 1982). Likewise, the definition of an operator 'somewhere'

$$\Diamond\phi := P\phi \vee \phi \vee F\phi$$

in the tense logic of linear orders will produce the modal logic S5. □

Remark. From certain points of view, K_t would not be a truly minimal system. For instance, it is also possible to treat it as a *bi-modal* system, having separate relations R_G, R_H for interpreting the two modalities G, H. Then, axiom (3) will correspond to an optional interaction condition on the two, namely

$$R_G^{\cup} \supseteq R_H, \qquad R_H \supseteq R_G^{\cup}.$$

Moreover, it is also possible to study weaker logics, lacking the Distribution axioms. Cf. Chellas 1980 on the 'neighbourhood semantics' appropriate for this purpose. (Even weaker basic calculi may also be obtained in the spirit of Lambek 1958, van Benthem 1986a and Girard 1987.) □

Actually, the definition of what constitutes a *tense logic* requires some care. The usual approach is 'extensional': one defines a tense logic by its theorems only, namely, as a set of formulas containing all axioms of K_t and being closed under the action of Modus Ponens, Generalization and *Substitution* for proposition letters. Sometimes, however, a more 'intensional' approach is preferable, defining tense logics by varying some prescribed format of deduction. Then, the choice of 'hard-wired' inference rules can be crucial.

Example. Promoting Derived Rules of Inference.
The following is a derived inference rule of K_t:

if $G\phi$ is a theorem, then so is ϕ itself.

But, if one were to require all tense logics to admit this rule, then certain candidates on the extensional account will be ruled out (such as the logic of 'two step time' axiomatized by GG*false*, F*true*$\vee P$*true*). □

2.2.2 Axiomatic tense logics can be related to classes of temporal frames via (soundness and) completeness results. First, the minimal tense logic K_t indeed captures precisely those truths which are valid at all points in all temporal models, without any constraints on the precedence relation:

Theorem. A tense-logical formula is provable in K_t if and only if it is universally valid.

The standard proof of this result yields even a little more.

Definition. Let Σ be a set of tense-logical formulas. Then ϕ is a *semantic consequence* of Σ if, for all models $\mathbb{M} = (T, <, V)$ and all points $t \in T$, $\mathbb{M} \models \Sigma\,[t]$ (i.e., $\mathbb{M} \models \sigma\,[t]$ for each $\sigma \in \Sigma$) only if $\mathbb{M} \models \phi\,[t]$. Notation: $\Sigma \models \phi$.

What we have in fact is the following *general completeness result:*

A formula ϕ is a semantic consequence of Σ if and only if
it is derivable from Σ using only principles from K_t .

Next, there are *special completeness theorems* of the form

' ϕ is a theorem of logic L if and only if ϕ is universally valid in all frames of class K '.

Such results may arise from two directions. Sometimes, a particular tense logic L is given axiomatically, and one asks for some frame class K modelling it. Or conversely, some frame or class of frames is given, and one looks for some (effective) complete axiomatization of its tense-logical validities.

Example. The full tense logic of the earlier transitive irreflexive frames is axiomatized by K_t4 , being K_t plus the Transitivity axiom. □

Example. The full tense logic of the single frame **R** is axiomatized by K_t4 plus the earlier-mentioned axioms for density, succession and Dedekind continuity. Complete axiomatizations for e.g., **Q**, **Z** or the earlier **M** , may be found in the standard literature. □

As a matter of fact, the usual tense-logical calculi proposed in the literature have all found natural semantic modellings; while, vice versa, all natural temporal structures have turned out to possess elegantly axiomatized theories. These phenomena do not rest on any obvious a priori basis. For, if only for reasons of cardinality, there must be many temporal frames lacking effectively axiomatized tense logics. (There are only countably many of the latter, but uncountably many of the former.) Moreover, as we shall see more clearly in Section 2.3, the complete tense logic of a temporal frame actually comprises part of its *second-order* theory: a system which need not be axiomatizable at all, in view of the known incompleteness of second-order logic (cf. Doets & van Benthem 1983). And also in the other direction, at least as specimina of pathology, there exist many tense logics lacking adequate temporal modelling. The latter phenomenon is known as tense-logical *incompleteness*, first recorded in Thomason 1972.

Example. An Incomplete Tense Logic.

Consider the temporal structure $(\mathbf{Z}, <, \mathbb{W})$ which consists of the integers, provided with all *finite unions of convex subsets* of $\mathbf{Z}$. Call a tense-logical formula *valid* in this structure if it holds everywhere under all valuations taking their values in $\mathbb{W}$ only. The set of valid formulas forms a tense logic L, whose axioms include those for transitivity, left- and right-linearity as well as two restricted versions of Loeb's Axiom:

$$G(G\phi\to\phi)\to(FG\phi\to G\phi), \qquad H(H\phi\to\phi)\to(PH\phi\to H\phi).$$

All these are valid because of general properties of the underlying frame $(\mathbf{Z}, <)$. Not valid in general, but true here thanks to the restriction to valuations in $\mathbb{W}$, are two so-called 'McKinsey Axioms', forbidding infinite alternation in the truth value history of propositions:

$$GF\phi\to FG\phi, \qquad HP\phi\to PH\phi.$$

There can be no adequate frame class K whose logic this is; because of the following observation.

Claim. Any frame validating all axioms of L must consist of isolated reflexive points.

Proof. Consider any frame $\mathbb{T} = (T, <)$ validating L. By the basic axioms of this logic, $<$ is transitive and left- and right-linear. Moroever, on such a frame, the modified Loeb axioms enforce the property that no infinite upward sequence $t_1<t_2<\ldots$ is strictly bounded from above (and likewise for descending sequences). In particular, then, no reflexive point can occur strictly below or above another point. I.e., if a point is reflexive, it must be isolated. Finally, on full transitive frames, the McKinsey axioms correspond to the relational condition that every point has a reflexive final successor and predecessor. In combination with the preceding observation, this implies the claim.

Now, on frames of this kind, the collapsing principle $p\to Gp$ will be universally valid. But, the latter is not a theorem of L, as it is not valid on $(\mathbf{Z}, <, \mathbb{W})$: for a counter-example, take $V(p) = \{0\}$. □

2.2.3 Individual tense-logical completeness theorems abound in the literature. This phenomenon has led to a search for *general methods* establishing completeness for wider ranges of cases at once. A related general question is, of course, to find some principled explanation of why the search for individual completeness theorems has been so succesful, by and large.

As to the second question, some general answers are available. For instance, because of the existence of a *translation* into standard logic (see section 2.3 below), the effective axiomatizability of general tense-logical consequence is derivable from that of ordinary first-order logic. Also, the fact that quite a few natural temporal frames possess effectively axiomatizable (and even decidable) tense logics follows from an observation made in Gabbay 1976: often, they are effectively embeddable into the *monadic second-order logic* of the structure of finite sequences of natural numbers with the relation of 'initial segment' and unary successor functions. By *Rabin's Theorem*, the latter logic is decidable: and hence, so is that of temporal frames such as $\mathbf{N}, \mathbf{Z}$ or $\mathbf{Q}$. An uncountable frame like $\mathbf{R}$ is beyond the scope of this method, however. (Incidentally, the axiomatizability of the tense logic of $\mathbf{Q}$ is also predictable on general logical grounds.)

Remark. Also relevant here is the analysis found in Doets 1987. Although the full second-order theory of structures such as $(\mathbf{N}, <)$ or $(\mathbf{R}, <)$ is highly complex, their monadic second-order fragment turns out to be axiomatizable in a natural way. For instance, for $(\mathbf{N}, <)$, an adequate axiomatization consists of the complete first-order theory of left-bounded discrete linear orders plus Dedekind's Induction Axiom. (Thus, in a sense, Dedekind's well-known abstract proof of categoricity using the latter does have a constructive content after all.) But for $\mathbf{R}$, one needs not just the obvious principles: being the complete first-order theory of unbounded dense linear orders plus Dedekind continuity, but also a monadic variant of the so-called Suslin Property (forbidding uncountable antichains of intervals). □

A penetrating general analysis of monadic second-order theories on linear frames may be found in Burgess & Gurevich 1985, who even prove *decidability* for such cases as: all elementary classes of linear frames, all continuous linear frames.

Remark. The above analysis is restricted to the case of propositional tense logics. For, the tensed *predicate* logic of even a structure like $\mathbf{Z}$ or $\mathbf{R}$ is not arithmetically definable, let alone effectively axiomatizable: a result due to Scott and Lindstroem independently (cf. Garson 1984). The reason is that, using the underlying predicate logic, one can encode enough of arithmetic into the tense logic to fall prey to Tarski's Theorem. □

In the reverse direction, from given tense logics to adequate frame classes, general results have been obtained by means of analysis of the *proof techniques* employed: usually, variants of Henkin set constructions or semantic Beth tableaus. (Cf. Segerberg 1970, Goldblatt 1987, Fitting 1983.) A central early result is *Bull's Theorem* (Bull 1966) concerning modal logics, or equivalently, pure future tense logics involving F,G only:

Theorem. All modal extensions of S4.3 have the finite model property.

What this means is the following. A logic L has the *finite model property* if every one of its non-theorems ϕ can be refuted on some finite model for L . From this, by 'modal collapsing', one can extract a finite *frame* for L which refutes ϕ . So, logics with the finite model property are also *complete* in the above sense. Moreover, provided that L be effectively axiomatizable (and all extensions of S4.3 are), such logics must also be decidable. In all this, S4.3 is the modal logic of the *reflexive linear orders*, whose characteristic axioms, on top of the minimal logic, express transitivity, right-linearity and reflexivity.

This result has been extended considerably in Fine 1974, 1985, where an intensive investigation is made of all modal logics containing the transitivity axiom plus at least one axiom 'Alt_n' restricting the number of incomparable <-successors for any point to at most n . (Thus, linearity is Alt_1 .) Translating to tense logic, we may conclude at least that all pure future (or pure past) tense logics on strict linear time are complete. And this result may be extended to cases of branching with a fixed finite upper limit.

Remark. For a more negative result, see Thomason 1982, which shows that the general question whether a given modal formula axiomatizes a frame-complete logic is *undecidable* . □

Remark. There is also a curious *limitation* to current proof techniques - in that they do not explicate why not just the propositional tense logic of temporal frames, but often also their complete monadic second-order theory is effectively axiomatizable. □

Finally, we mention another possible type of general result, this time involving tense logics of temporal frames. Given that we know complete tense logics of certain frames, can we also describe the tense logic of other frames formed out of these by natural mathematical operations? For instance, our conjecture would be that the *ordinal sum* of two frames having effectively axiomatizable theories itself has such a theory, effectively obtainable from the two 'components'. And a similar issue may be raised for *direct products*: e.g., can the full tense logic of the earlier Minkowski frame $\mathbf{M} = \mathbf{R} \times \mathbf{R}$ be obtained effectively from that of $\mathbf{R}$ itself?

2.2.4 Much of the general (completeness) theory for intensional systems has been developed for *modal* logic, rather than for tense logic. One prevalent assumption is that generalization to the latter case is straightforward, since we are merely duplicating the modal operator into a bi-modal system. While this is true for many topics, there still remains a non-trivial problem of *transfer*:

Which techniques and results from modal logic generalize to tense logic?

First, here is a simple case of non-transfer.

Example. Disjunction Property.

The minimal modal logic has the following property:

If $G\phi \vee G\psi$ is provable, then so is either $G\phi$ or $G\psi$.

The proof of this uses a common model-theoretic technique called 'rooting'. Suppose that neither $G\phi$ nor $G\psi$ are theorems. Then there exist models $\mathbb{M}_1, \mathbb{M}_2$ refuting ϕ, ψ respectively, at some points t_1, t_2 . Now, take the union of two disjoint copies of these models, held together by one new point, $<$-joined to only t_1, t_2 . Thus, a new model is obtained in which $G\phi \vee G\psi$ fails at the new root: whence the latter formula is not a theorem either.

This technique is no longer available in tense logic. The reason is as follows. Evaluation of pure future formulas in the separate models is not affected in their rooted join. But, for past formulas, changes may occur: e.g., rooting of two linear frames will produce leftward branching. And in fact, K_t does not have the disjunction property. For instance, $GP\phi \vee GP\neg\psi$ is among its theorems, whereas neither $GP\phi$ nor $GP\neg\psi$ is. □

Remark. The disjunction property and semantic rooting are also behind the notion of 'honesty' as developed in Halpern & Moses 1985. □

Next, here is a more serious example of non-transfer. *Bull's Theorem* no longer holds for tense logic. A counter-example is provided by the earlier case of an incomplete tense logic on discrete integer time (cf. Section 2.2.2). So, can anything be saved here? At least, we have a

Conjecture. Bull's Theorem holds for all tense logics on *dense* linear time.

2.3 Model Theory

2.3.1 The general theory of intensional logic has produced some useful model-theoretic tools. In particular, there are certain operations on frames and models which leave truth of modal or tense-logical formulas *invariant*. First, there is the following central notion and result.

- $\mathbb{M}_1 = (T_1, <_1, V_1)$ is a *generated submodel* of $\mathbb{M}_2 = (T_2, <_2, V_2)$ if

(1) $(T_1, <_1)$ is a subframe of $(T_2, <_2)$ which is closed under $<_2$- successors and $<_2$- predecessors (i.e., $\mathbb{T}_1$ is a 'generated subframe' of $\mathbb{T}_2$)

(2) for each proposition letter p , $V_1(p) = V_2(p) \cap T_1$.

- Then, for all tense-logical formulas ϕ , and all $t \in T_1$,

$\mathbb{M}_1 \models \phi[t]$ iff $\mathbb{M}_2 \models \phi[t]$.

It follows that tense-logical formulas valid in a frame are also valid in all its *generated subframes*, and also that formulas valid in a family of frames remain valid in their *disjoint union*.

Another invariance employs a notion of back-and-forth connection which is quite similar to the *bisimulations* of current process algebra (cf. Klop 1988):

- A binary relation C between T_1 and T_2 is a *zigzag relation* between two models $\mathbb{M}_1, \mathbb{M}_2$ if

(1) the domain of C equals T_1 and its range T_2

(2) for all $x \in T_1, y \in T_2$ with xCy , and each $z \in T_1$ such that $x <_1 z$ $(z <_1 x)$, there exists some $u \in T_2$ such that zCu and $y <_2 u$ $(u <_2 y)$; and vice versa

(3) for each proposition letter p , if xCy , then $x \in V_1(p)$ iff $y \in V_2(p)$.

- Then, for all tense-logical formulas ϕ , and all $x \in T_1, y \in T_2$ with xCy ,
  $\mathbb{M}_1 \models \phi[x]$ iff $\mathbb{M}_2 \models \phi[y]$.

As a consequence, tense-logical formulas valid in a frame are also valid in its *p-morphic images*; where a 'p-morphism' is a function preserving < , and obeying the above back-and-forth condition in the inverse direction.

<u>Application</u>. The following conditions on temporal precedence are not tense-logically definable:

$\exists x\ x<x$	(not preserved under generated subframes)
$\forall x \forall y\ (x<y \vee y<x \vee x=y)$	(not preserved under disjoint unions)
$\forall x\ \neg x<x$	(not preserved under p-morphic images) □

There are several further important operations on models, such as *filtration* ('modal collapsing') or *unraveling*: for which we refer to the literature (van Benthem 1983, Goldblatt 1987a).

Semantic invariances describe the characteristic behaviour of a certain (fragment of a) formal language. As such, they have a dual aspect. On the one hand, they are a sign of weakness: the language cannot detect genuine differences in temporal structure; on the other hand, they are useful, in that they allow transfer of truth from one temporal situation to another.

2.3.2 As has been observed already, tense logic can be related systematically to more standard systems of logic on binary orders. First, there is a *standard translation* from tense-logical formulas into formulas of a first-order language L_0 having one binary predicate letter < as well as unary predicate letters P corresponding to proposition letters p . The latter have one free variable t_0 , reflecting the 'current point of evaluation':

$$(p)^* = Pt_0$$
$$(\neg\phi)^* = \neg(\phi)^*$$
$$(\phi\#\psi)^* = (\phi)^*\#(\psi)^* \text{ , for all Boolean connectives } \#$$
$$(F\phi)^* = \exists t(t_0<t \wedge [t/t_0](\phi)^*) \text{ , where } t \text{ is some}$$
$$(P\phi)^* = \exists t(t<t_0 \wedge [t/t_0](\phi)^*) \text{ fresh variable}$$

Models (T, <, V) may be naturally regarded as semantic structures for the first-order language L_0 too - and then we have, properly viewed,

$\mathbb{M} \models \phi[t]$ iff $\mathbb{M} \models (\phi)^*[t]$.

Thus, many facts about standard first-order logic become available for tense logic too. For instance, tense-logical truth on models is insensitive to semantic operations which preserve first-order truth: such as the formation of elementary submodels or ultraproducts. In particular, one obtains *Loewenheim-Skolem* and *Compactness* theorems. Moreover, semantic consequence on models ("$\Sigma \models \phi$") reduces to first-order consequence: $(\Sigma)^* \models (\phi)^*$. The exact form of a properly tense-logical axiomatization requires additional scrutiny, of course. And so does the known *decidability* of the minimal tense logic: where a special purpose argument is needed - say, by filtration - to establish the finite model property.

The above translation takes tense-logical formulas into a *fragment* of the full corresponding first-order language L_0 . For instance, the only quantifiers required are *restricted*, being of the forms '$\exists y(x<y\wedge$' or '$\exists y(y<x\wedge$' . This fragment can be determined precisely, both syntactically and semantically (cf. van Benthem 1985).

Theorem. A first-order formula in L_0 with one free variable t_0 is equivalent to the translation of some tense-logical formula if and only if it is invariant for generated submodels and zigzag relations.

Thus, the basic tense logical formalism has an intimate connection with bisimulation: it is the largest description language for models which is invariant for such a structural connection.

With richer formalisms, this invariance disappears.

Example. Progressive Tense versus Bisimulation.

A tense operator not expressible in the basic P, F formalism is the progressive tense ("Mary is crying"):

$$\mathbb{M} \models \Pi\phi\,[t] \quad \text{iff} \quad \exists t_1<t\ \exists t_2>t\ \forall u\,(t_1<u<t_2 \Rightarrow \mathbb{M} \models \phi\,[u])\,.$$

This operator essentially involves *betweenness*, rather than mere succession or precedence. It is not even P, F definable on the rational frame **Q** . For, consider the valuation $V(q) = \bigcup\{(i,i+1] \mid i \text{ is even}\}$. By induction, we have, for basic tense-logical formulas ϕ , and all points $t_1, t_2 \in V(q)$ that $(\mathbf{Q}, <, V) \models \phi\,[t_1]$ iff $(\mathbf{Q}, <, V) \models \phi\,[t_2]$. By contrast, Πq will be true in the topological interior of $V(q)$ only: not in its boundary points.

That the progressive Πq need not survive bisimulation, may be seen as follows. Consider the following bisimulation, where corresponding numbers indicate points to be identified ('fold the left-hand model'):

Set $V(q) = \{4\}$ in both cases. Then, Πq will be true on the left (consider, e.g., some upper 2 and its diagonally opposite 3); but, it fails on the right-hand side. $\square$

Continuing the analysis behind the earlier characterization theorem, we can develop a hierarchy of ever finer notions of 'simulation', preserving ever stronger fragments of the first-order language L_0 .

Closer inspection reveals further syntactic peculiarities of the basic tense-logical fragment of L_0 . All translations can be taken into a fragment employing only *two variables*. The reason is that the 'fresh variables' for operators F, P may be chosen in an alternating manner.

Example. $G(FPq\wedge r)$ may be translated as

$$\forall t(t_0<t \rightarrow (\exists t_0(t<t_0 \wedge \exists t<t_0\ Qt) \wedge Rt)\,. \quad \square$$

This is quite significant, since fragments of predicate logic with a restriction to some fixed finite number of bound variables always allow a *functionally complete* variable-free *operator* formulation. (See Section 2.4.2

for more details.) Thus, tense logic is one of a larger family of operator formalisms approximating predicate logic.

Remark. In some respects, the basic tense logic is not yet well-chosen. For, its formulas do not exhaust the two-variable fragment of L_0 : as has been observed by various authors. For instance, none of its formulas expresses that $\exists y(t_0<y \wedge y<y)$ or $\exists y(t_0<y \wedge y<t_0 \wedge Py)$. Thus, there have been several attempts at increasing its coverage slightly, without losing pleasant axiomatizations and decidability. A systematic program to this effect is reported in Gargov, Passy & Tinchev 1987, who add various operations, such as 'window':

$$M \models W\phi\,[t] \quad \text{iff} \quad \forall t'(\, M \models \phi\,[t'] \Rightarrow t<t')\,.$$

What we need in general for two-variable L_0 are operators of the form

$\exists x($ 'some conjunction of atoms $\{x<y,\ y<x,\ x<x,\ y<y\}$ and their negations' $\wedge\ \phi(x)$).

One elegant way of using an intensional logic framework here after all employs a small fragment of *Dynamic Logic* (cf. Harel 1984). Ordinary modal logic may be viewed as a poor dynamic logic of one binary relation (or atomic program) R , without any operations on the latter. Tense logic may be said to add the operation of *converse* here. But, we can also add the *Boolean* operations on programs: $-$, $\cap$ and $\cup$. The above first-order language is then captured by this dynamic logic, provided that one further propositional constant *loop* is added, true at only those points t where t<t . The resulting logic has been axiomatized in Gargov & Passy 1988. Its model theory still awaits further exploration (but cf. Goranko 1987). □

2.3.3 One strand in the above discussion is the more fundamental issue of *natural primitives* in setting up temporal logic. This may be approached in a more classical mathematical spirit as follows (cf. van Benthem 1986c).

Temporal propositional operators such as tenses may be viewed semantically as *operations* on sets of points in time: the latter being the correlates of propositions. Then, the question is which of these operations have a reasonable temporal motivation. For this purpose, fix some temporal frame $\mathbb{T} = (T, <)$. Because temporal operations are only concerned with temporal 'perspective', as encoded in the precedence order $<$, they should be insensitive to those transformations of the underlying points which preserve this precedence order. Thus (recall also the Homogeneity of Section 2.1.3),

$f: P(T) \rightarrow P(T)$ should be *invariant for automorphisms* of $(T, <)$, in the sense that,

for all subsets X of T and all <-automorphisms π , $\pi[f(X)] = f(\pi[X])$.

This commutation with automorphisms can also be formulated as follows:

$$t \in f(X) \quad \text{iff} \quad \pi(t) \in f(\pi[X]), \qquad \text{for all } t \in T\,.$$

Whether this is a restrictive requirement depends on the number of automorphisms in $\mathbb{T}$. E.g., $(\mathbf{N}, <)$ has only one (namely, the identity map), whereas $(\mathbf{R}, <)$ admits many translations and contractions. In the latter frame, then, automorphism invariance imposes a certain uniformity on temporal operations f , in that

(1) for any singleton $\{t\}$, $f(\{t\})$ is a union of certain of the three ranges
$\{u \in T \mid u<t\}$, $\{t\}$ and $\{u \in T \mid t<u\}$,

(2) the choice under (1) is made uniformly .

Still, all temporal operations definable in standard logical languages, whether first-order or higher-order, pass this test.

Another kind of restriction which might be imposed concerns 'local computability' of temporal operations. For instance, here is a common condition of 'pointwise computability' or *Continuity*:

$$f(\bigcup\{X_i \mid i\in I\}) = \bigcup\{f(X_i) \mid i\in I\} ,\text{ for all families of point sets } \{X_i \mid i\in I\} \text{ on } \mathbb{T} .$$

In particular, this implies that $f(X) = \bigcup\{f(\{x\}) \mid x\in X\}$, for any set X .

Proposition. On **R** , the only automorphism-invariant and continuous temporal operations are F, P and identity, plus their disjunctions.

Thus, the Priorean basic framework captures the best-behaved operations on real time.

On other frames, however, this outcome may differ. For instance, on a discrete structure like **Z** , one would also have to allow "to-morrow", "yesterday" and their iterations.

This type of analysis can be extended, to create a structural hierarchy of temporal operations. For instance, instead of pointwise computability, one may require only computability by means of 'small episodes':

$$f(X) = \bigcup\{f(Y) \mid Y \text{ is a } convex \text{ subset of } X\}.$$

Unlike Continuity, this would allow the earlier *progressive* tense, as well as various operations selecting *beginnings* or *endings* of convex intervals within propositions.

Moreover, a similar analysis can be given for *polyadic* temporal operations.

The attraction of this perspective is its language independence, giving us a separate stance from which to judge the design of temporal calculi.

2.3.4 Although the initial standard translation took tense-logical formulas to first-order ones, in L_0 , there is also a natural sense in which it makes them rather *second-order*. For, the earlier notion of truth in frames induces a correspondence with formulas in a *monadic* second-order logic over L_0 :

for formulas $\phi=\phi(p_1,\ldots,p_n)$,

$$(T, <) \models \phi\,[t] \quad \text{iff} \quad (T, <) \models \forall P_1 \ldots \forall P_n(\phi)^*\,[t] .$$

More precisely, the latter formulas have a shape which is $\Pi^1{}_1$: all their second-order quantifiers occur in one universal block in front. Thus, on frames, tense logic behaves like a fragment of a higher-order logic; be it one whose logical properties are sometimes better than those of the full system. (Cf. van Benthem 1985 for some further technical background in higher-order logic.)

This is the proper setting, for instance, for pursuing the 'correspondences' of Section 2.1.4. See van Benthem 1984a for a survey of results, addressing such questions as

- Which model-theoretic behaviour is necessary and sufficient for first-order frame definability of tense-logical axioms?
- And conversely, for tense-logical frame definability of first-order conditions on precedence?
- Which algorithms (if any) produce standard ordering conditions equivalent to given tense-logical axioms?
- How can the account be extended to higher-order logic in general?

For instance, a tense-logical axiom defines a first-order condition on temporal frames if and only if it is preserved under the formation of *ultrapowers* of frames. And, e.g., a first-order sentence in $\{<, =\}$ is tense-logically definable on frames only if it is equivalent to one constructed from atoms $x<y$, $x=y$ and *false* using only one initial unrestricted universal quantifier, and after that only restricted ones $\forall u<v$, $\exists u<v$, $\forall v<u$, $\exists v<u$ as well as $\wedge, \vee$.

Remark. This type of question has a certain computational interest, in that one likes to see just where reductions are possible from second-order formalisms to first-order ones having better automated theorem provers. (For a parallel in the theory of *Circumscription*, cf. Lifschitz 1985, van Benthem 1988c.) □

Here, we shall consider a simpler question, however, related more directly to the expressive power of the basic tense logic on frames. For the purpose of illustration, let us restrict attention to unbounded strict linear orders which are also homogeneous in the sense of Section 2.1.3 . Thus, they are either dense or discrete. In this case, all equivalence classes of frames may be enumerated which have the same tense logic. Moreover, all these tense logics can be axiomatized effectively. The bare enumeration can be obtained by the method of 'filtration and inflation' found in van Benthem 1983 (theorem II.2.1.6), but also by means of a novel technique developed in De Jongh, Veltman & Verbrugge 1988. The latter also yields the axiomatizability of all logics concerned. The outcomes are as follows:

- On *dense* orders, the only types of frame which can be distinguished are those having exactly n exceptions to Dedekind Continuity (where $n= 0,1,2,.., \omega$).

Here, Dedekind Continuity itself is the special case with $n=0$.

Remark. This result shows that, even here, the basic tense logic is weaker in expressive power than the monadic $\Pi^1{}_1$ logic of $<$. For, Doets 1987 has a principle of the latter kind (being the $\Pi^1{}_1$ derivative of the Suslin Property mentioned in Section 2.2.3) which holds in $\mathbf{R}$, but not in some of its Dedekind continuous non-Suslin homogeneous elementary equivalents. □

- On *discrete* orderings, the only types of frame which can be distinguished are those having exactly n consecutive copies of the integers $\mathbf{Z}$ (where $n=1,2,..., \omega$).

Finally, despite its apparent simplicity, propositional tense logic on frames can also be intractable. Thus, Thomason 1974, 1975 establish that the notion of *frame-consequence* (as opposed to the earlier consequence relation on *models*) defined by:

$\Sigma \models^* \phi$ if ϕ is true in every frame where Σ holds ,

is fully as complex as that of second-order logic in general.

2.3.5 *Appendix: Algebraic Semantics*

The basic tense logic can also be studied in an *algebraic* setting, as a formalism defining a Boolean algebra with two added operators p and f , which obey some minimal constraints, such as

$$p(x+y) = p(x)+p(y), \qquad f(x+y) = f(x)+f(y) .$$

Then, formulas become polynomials over algebras of the relevant similarity type: and different axiomatic systems correspond to different equational varieties. Here, general techniques from Universal Algebra may be applied. (Cf. Blok 1976, 1980; e.g., for many fundamental results about the extent of the incompleteness phenomenon among modal logics.)

For certain technical purposes, this algebraic approach seems more convenient than the (purely) model-theoretic one. Nevertheless, the two can be systematically related by introducing a somewhat more general notion of a temporal frame.

Definition. A *general frame* $(T, <, \mathbb{W})$ is a temporal frame $(T, <)$ together with a family $\mathbb{W}$ of subsets of T satisfying the following conditions:

(1) $\mathbb{W}$ is closed under the Boolean operations

(2) $\mathbb{W}$ is closed under the set-theoretic operations f, p defined by

$f(X) = \{t \in T \mid \exists x \in X\ t<x\}$ $\qquad$ $p(X) = \{t \in T \mid \exists x \in X\ x<t\}$.

General frames implement the reasonable idea that the range of 'admissible propositions' on a temporal frame may obey certain restrictions: something which happened already with the structure $(\mathbb{Z}, <, \mathbb{W})$ used in Section 2.2.2, where only finite unions of convex sets were allowed.

Evaluation on general frames takes place as usual, but with only those valuations V whose values on proposition letters fall inside $\mathbb{W}$ .The stated closure conditions then guarantee that the set of valid formulas on a general frame will be a (substitution-closed) tense logic. All the earlier basic invariance relations between models are easily adapted to this new case, with corresponding outcomes for tense-logical formulas.

Remark. Another motivation for general frames stems from second-order logic. As in Henkin 1950, one can plausibly move from full 'standard models' for a higher-order formalism to a richer universe of 'general models', thereby restoring some form of general completeness. □

Now, there is a tight connection between general frames and temporal algebras. For, any general frame induces a temporal algebra of sets, whose universe is $\mathbb{W}$, and whose operations are the set-theoretic Booleans as well as the above-defined f and p . But also conversely, each temporal algebra may be *represented* as such a frame-induced set algebra, via the well-known Stone Ultrafilter Representation, adapted to this purpose (cf. Jonsson & Tarski 1951, Goldblatt 1976, van Benthem 1979). This two-way correspondence may be elaborated into a whole categorial correspondence, as the natural morphisms on either side are related too. The analogy is as follows:

generated subframes	homomorphic images
disjoint unions	direct products
p-morphic images	subalgebras .

The most elegant presentation of this categorial duality to date is Sambin & Vaccaro 1987.

Through this connection, many results have been obtained, such as the following (Goldblatt & Thomason 1975):

Theorem. An elementary class of frames K is tense-logically definable if and only if

(1) K is closed under the formation of p-morphic images, generated subframes and disjoint unions,

(2) both K and its complement are closed under the formation of ultrafilter extensions.

Remark. The algebraic viewpoint too, suggests certain generalizations of the basic tense-logical framework. For instance, it would also be quite natural to consider quasi-varieties, or classes of algebras defined by arbitrary *universal* statements of the algebraic language. Some corresponding changes (for the better) in the framework of modal logic are found in Kapron 1987. □

One reason why this framework has not been more prominent in our presentation is that it does not seem to adapt very easily to the richer formalisms to be considered below; where some form of *cylindric algebra* would be needed.

2.4 Further Developments

The account of temporal logic so far has concentrated on Prior's basic system, as a paradigm for developing a more general technical theory. From this base, one can set out in different directions.

2.4.1 *Locating Special Fragments*

We start with a direction which may not be the most obvious one, but which makes good computational sense, namely, looking for *fragments* of a formalism with better computational behaviour.

One important fragment provides a format for temporal axioms which guarantees two desirable properties for our logics, namely *frame completeness* and *first-orderness*. The relevant shapes are implications

$\phi \to \psi$, where ϕ is constructed using proposition letters and $\wedge, \vee, G, H, P, F$ such that no G or H governs a $\vee, F$ or P , and ψ is any *positive* formula constructed using proposition letters and $\wedge, \vee, G, H, F, P$.

The following result comes from Sahlqvist 1975:

Theorem. Each temporal logic extending K_t whose axioms consist of the above forms defines a first-order frame class, with respect to which it is complete.

In fact, the relevant frame condition on precedence may be obtained effectively from the axioms. Many of the earlier axioms are Sahlqvist forms (or can be transformed into them), witness

$FF\phi \to F\phi$, $\quad F\phi \to FF\phi$, $\quad FG\phi \to GF\phi$, $\quad F\phi \wedge F\psi \to F(\phi \wedge F\psi) \vee F(\phi \wedge \psi) \vee F(\psi \wedge F\phi)$.

A typical non-example is the McKinsey Axiom $GF\phi \to FG\phi$, which is not first-order (cf. van Benthem 1984a).

A general Sahlqvist-type theorem for multi-modal logics occurs in Catach 1988.

These special forms can also be studied for their model-theoretic *transfer* behaviour (cf. Section 2.2.3). For instance, Sahlqvist forms without disjunctions in their antecedents define 'general Horn sentences' in the sense of Chang & Keisler 1973, which are all preserved under *direct products* of frames. (Can the preservation result of Section 2.3.4 be made to fit exactly those first-order conditions corresponding to tense-logical formulas preserved under direct products?) And also, the more general Sahlqvist format, consisting of 'generalized Horn clauses' having positive consequents, might be of interest by itself.

Remark. The search for fragments can also start inside the standard frame formalisms, of course. For instance, even though there is no known necessary *and sufficient* condition for a first-order sentence to be defined by a tense-logical axiom, one can in fact find such a result for all tense-logically definable *Horn clauses*, etcetera. □

Another type of syntactic fragmentation arises in the study of so-called 'normal forms' for intensional languages (cf. Fine 1975a; a topic rediscovered in Fagin & Vardi 1985). For instance, we can characterize the formulas up to nested operator depth n by means of their insensitivity to those parts of models which lie more than n $<$-successor / predecessor steps removed from their point of evaluation.

Even the process of generalization of a language itself always raises an inverse issue of fragmentation, viz. how to recognize the old language within the new. For instance, how can 'pure past' formulas be located within the full tense-logical language? (For a computational motivation of this type of question, see Pnueli's contribution to this volume, on the description of safety and liveness. See also Gabbay 1981b for a connection with functional completeness, via so-called 'separability'.) There is one trivial syntactic answer here: "look at the F, G - free formulas". But, the more interesting version would be a semantic one. Call a formula ϕ *pure past* if always

$\mathbb{M} \models \phi\,[t]$ iff $(\mathbb{M}{\leftarrow}, t) \models \phi\,[t]$, where $(\mathbb{M}\leftarrow, t)$ is the submodel of $\mathbb{M}$ whose domain consists of just t together with all its ancestors in the precedence relation. What is the syntactic counterpart of this notion? In a full predicate logic, the answer is simple. For formulas $\phi=\phi(t_0)$, at least on *transitive* frames, pure past formulas are just those logically equivalent to one having all its quantifiers 'past-restricted' ('$\exists y(y\leq x$', '$\forall y(y\leq x$'). For, the latter formulas are obviously pure past in the semantic sense - and conversely, if ϕ is pure past, then it is logically equivalent to its own relativization to the predicate '$\lambda x.x\leq t_0$'.

For the tense-logical language, however, such a simple argument does not work. (Here is one instance where living inside a fragment has its drawbacks.) But, by more complex reasoning, similar to that proving the earlier invariance characterization of Section 2.3.2, one may show that a tense-logical formula is pure past if and only if it is logically equivalent to some formula without P, H operators.

More computationally, is being pure past a *decidable* notion? At least in general predicate logic, it is not, because of the following effective reduction of an undecidable problem to this one. Let α be an arbitrary predicate-logical sentence, not containing $<$. Let α^* be the syntactic restriction of α to the predicate '$\lambda x.x\leq t_0$'. Finally, set $\beta := \alpha \vee \exists y\, t_0<y$.

Claim. β is pure past if and only if α is universally valid.

Proof. 'If': With α universally valid, β is equivalent to the pure past formula *false*.

'Only if': Suppose that α is not universally valid. Say, $\mathbb{M}$ falsifies it. Then, expand $\mathbb{M}$ by choosing some t_0 and imposing a binary relation $<$ which makes all other points $<$-predecessors of t_0. Finally, add one new point which becomes an $<$-successor of t_0. In the new model $\mathbb{N}$, β holds, since $\exists y\, t_0<y$. But, in $(\mathbb{N}\leftarrow, t_0)$, both disjuncts of β are false: whence it is not pure past. ■

As for the question of decidability of pure pastness within the basic tense logic, we conjecture that the answer is positive.

2.4.2 *Ascending to Stronger Formalisms*

In many applications, the expressive limitations of the basic tense logic become a hindrance. Therefore, various enrichments have been studied. For instance, already in the study of tenses proper, we encountered the *progressive* tense, which is not P, F definable. This is even more striking with certain temporal connectives, such as "since" and "until", whose definitions may be rendered as

$$\mathbb{M} \models S\phi\psi\,[t] \quad \text{iff} \quad \exists u<t\, (\mathbb{M} \models \phi\,[u] \wedge \forall s(u<s<t \Rightarrow \mathbb{M} \models \psi\,[s])$$

$$\mathbb{M} \models U\phi\psi\,[t] \quad \text{iff} \quad \exists u>t\, (\mathbb{M} \models \phi\,[u] \wedge \forall s(t<s<u \Rightarrow \mathbb{M} \models \psi\,[s])$$

Such further operators may be viewed as successive steps towards drawing the full first-order language into the scope of temporal logic. Indeed, Kamp 1966 has the following functional completeness result.

Theorem. On continuous linear orders, S, U are sufficient for defining all first-order formulas in L_0.

This result has been generalized by Stavi, who found two additional operators S', U' such that the resulting set is functionally complete on arbitrary linear orders (cf. Gabbay 1981b).

Remark. Note that this result depends on the particular first-order language under consideration. For instance, if one also allows *binary* predicates over points as denotations of proposition letters (as in the temporal interval logics of Section 3), then no functional completeness exists, not even on **R** (cf. Venema 1988). □

Explicit completeness results for the S, U logic, both in general and on specific frames, may be found in Burgess 1982a, Goldblatt 1987a.

The general issue in the background here is that of the possible approximation of first-order logic by means of *variable-free* operator formalisms (cf. Section 2.3.2). As far as the present case is concerned, the situation was much clarified by Gabbay, Pnueli, Shelah & Stavi 1980 who showed that the following two statements are equivalent for temporal frames:

(1) the first-order predicate logic L_0 can be restricted without loss of expressive power to some fixed finite number of bound variables

(2) this logic has a finite functionally complete operator formalism.

Probably the most elegant presentation to date is to be found in Immerman & Kozen 1987, which characterizes these fixed bound variable fragments by means of the existence of winning strategies in an *Ehrenfeucht-Fraisse game* with pebbling. (Immerman 1982 has a connection with complexity measures for evaluation of first-order sentences.)

Concerning axiomatics, Gabbay 1981a, 1981b give a general (though abstract) method for producing completeness theorems for arbitrary temporal operators on *irreflexive* frames. (The trick here involves direct definition of the additional operators in terms of the old F, P, using proposition letters true at only one point as 'parameters'.) Also, the earlier-mentioned decidability results of Burgess & Gurevich 1985 are relevant here, as they were already about full first-order formalisms, rather than just the basic tense logic. As yet, however, there are few *explicit* completeness results for such natural special cases as the unary tense fragment of the full first-order language on $\mathbf{R}$ or $\mathbf{Z}$.

Remark. Also in this full first-order setting, there is an issue of *incompleteness*, when looking in the opposite direction. Starting from a set of axioms, the following minimal system of $\Pi^1{}_1$ inference seems plausible and tractable: all deduction rules of *first-order* logic plus a *substitution rule* replacing universal second-order quantifiers by their first-order instances. But then, e.g., the earlier incomplete logic of Section 2.2.2 is still incomplete. For, on full frames, it implied $p \rightarrow Gp$ which does not follow from it on the general frame $(\mathbf{Z}, <, \mathbb{W})$. But, the latter range $\mathbb{W}$ is even closed under arbitrary sets which are parametrically first-order definable from it: and hence, it validates all consequences of our tense logic in the minimal $\Pi^1{}_1$ logic too. □

One prominent issue is how the earlier general tense-logical theory fares under successive extensions of the formalism. In fact, many results seem to generalize in one way or another. For instance, the earlier-mentioned Sahlqvist theorem will remain valid without any change in its proof for antecedents of the kind described in Section 2.4.1 with *arbitrary* positive first-order consequents. More technically, van Benthem 1986 has an extension of Goldblatt & Thomason type results (cf. Section 2.3.5) to definability of frame classes in monadic $\Pi^1{}_1$ logic (which corresponds to using full first-order languages on models).

Another direction of generalization in the literature has been toward 'multi-dimensional' tense logics, allowing evaluation of formulas at *sequences* of points. For instance, interpretation of sentences with temporal adverbs like "now", "then" will require a record of not just the current point of evaluation, but also certain auxiliary points. Thus, not just formulas $\phi(t_0)$, but also formulas with an arbitrary number of free temporal variables become relevant. See Segerberg 1973, Gabbay 1976, Burgess 1984a for some results in this area.

Remark. With arbitrary finite numbers of argument places, the issue of an adequate variable-free operator notation for first-order predicate logic starts shifting. For, in this area, there are several general candidates which do that job uniformly: witness Quine 1966 with a well-known proposal, and Bacon 1985 with a complete axiomatization of the latter. Van Benthem 1977 discusses the propriety of the latter kind of move for a genuine temporal logic. But also, one could use a formalism like Combinatory Logic (cf. Hindley & Seldin 1986) for the purpose.

There is a subtlety here, in that the earlier-mentioned (im)possibility results for temporal operator languages presuppose a narrower conception of what constitutes an 'admissible' operator; which would rule out the Quine apparatus, strictly speaking. It is perhaps too early to adjudicate the issue. □

Remark. There has been a lively debate on the utility, and indeed morality, of temporal operator formalisms versus predicate-logical ones employing explicit quantification over points in time (cf. Massey 1969, Needham 1975). There appears to be no uniform answer here. For certain purposes, restricted operator formalisms continue proving their value, in terms of perspicuity and simple computation. (A recent example of the virtues of 'operationalizing' may be found in so-called arithmetical 'provability logic': cf. Smorynski 1984.) On the other hand, the bounds of any particular operator formalism can sometimes become artificial - and then, displaying explicit variables can be definitely superior. □

As was already observed in Section 2.4.1, the very tendency toward increasing expressive power induces a general question as to what are *natural fragments* of first-order predicate logic. This issue can be approached in many different ways, depending on the intended application. For instance, in the semantics of natural language, a natural principle of division proceeds by various restrictions on the *binding patterns* for quantifiers: cf. van Benthem 1986a, 1988b. One possible route for temporal logic is to extend the analysis of invariant operations initiated in Section 2.3.3. For instance, we may generalize the result given there as follows:

Any automorphism-invariant continuous n-ary operation on propositions in $\mathbf{R}$ may be defined using P, F, $\wedge$, $\vee$.

Not surprisingly, in view of Kamp 1966, a central role is played by Since and Until when it comes to enumerating all polyadic automorphism-invariant operations satisfying not continuity, but the earlier computability by convex subintervals. Thus, we can give a more principled underpinning of specific operators extending the basic framework.

2.4.3 *Further Directions*

The above survey of variants on the basic tense logic is by no means complete. Of the many other directions of research , we mention merely three.

- Linear versus branching time logics.

This topic is illustrated by various contributions in this volume. Basic publications in the logical tradition are Burgess 1980, Thomason 1984 and Gurevich & Shelah 1985. (See also de Jongh & Bowen 1986.)

The systems presented up till now allow branching models just as well as linear ones: witness the earlier Minkowski spaces. So, what is genuinely 'branching' temporal logic? One answer is that this is just a restriction to special branching frames; which then suggest a somewhat different set of basic temporal operators (often with a second-order flavour, as in 'eventual truth along all future branches'). Another, perhaps more principled answer is that branching indicates a two-sorted semantic perspective: of *points* in time and *paths* through them, requiring evaluation at point-on-path pairs. Philosophically, the question here is if this is really a temporal field, or whether we have crossed the border toward the land of *modality*.

Thus, on the previous pattern of this Section, the technical task would be to sort out possible choices of temporal structures, fundamental temporal operators and temporal axioms. For instance, should the set of paths always be 'full' in the set-theoretic sense, or could it be varied? (Such decisions affect the model theory considerably, inducing varying correspondences between axioms and structural conditions: cf. the appendix to Rodenburg 1986.) Then, what are appropriate basic temporal operators in such a setting? Probably the most plausible choice is the one found in much of the computational literature, namely to have an existential quantifier over paths diverging from the current one only at the time of evaluation, in addition to the ordinary future operator along the current path. Such systems have a nice axiomatic completeness theory (witness Emerson 1988, Stirling 1988), but they remain to be studied in the model-theoretic detail already obtained for the basic tense-logical framework.

- Introducing metric structure.

In basic research on Measurement (cf. Krantz et al. 1971), studying temporal order is just a prelude towards taking a second step, imposing *metric* structure for measuring duration. Little attention has been paid to this theme within temporal logic (see Burgess 1984a on what little there is). Nevertheless, there is no natural frontier here, since numerical representation may arise out of qualitative relational structure: witness the paradigm case of Geometry (Tarski 1969, Goldblatt 1987b) and Physics (Field 1980). For instance, what would be the proper temporal logic of the reals with precedence and *equidistance*? Probably, as in other cases of richer 'numerical' systems, the pure temporal logic component will be only a modest one, with much of the mathematical action occurring in the superstructure.

- Combinations of time and other phenomena.

It is often difficult to maintain a rigid separation between purely temporal phenomena and space, modality or causation. There is a good deal of philosophical and logical literature on such combinations, witness Chellas 1980, van Eck 1981, Thomason and Gupta 1981, van Benthem 1983, Thomason 1984 and van Benthem 1988a. Perhaps, in the final philosophical analysis, there is no such thing as a separate phenomenon of 'time' at all. Even so, to justify the enterprise of temporal logic, one may paraphrase an aphorism of the Dutch scholar Brandt Corstius as follows:

> "By 'science' we understand that immensely succesful human activity which proceeds by resolutely ignoring the fact that everything depends on everything else."

3. Intervals and Events

Over the past decade, a reappraisal has taken place of the choice of basic temporal entities. After all, the idea that these should be durationless *points* or moments in time, although prominent in modern science, reflects only one intuitive view of Time. But, even within mathematics, there has also been another broad intuition, viewing Time more 'continuously' as consisting of primitive pieces which always have duration: say, *periods*.

To be sure, the two views are not mutually exclusive. A 'point theorist' can understand periods as *intervals*, i.e., sets of points satisfying certain (convexity) constraints. And also conversely, a 'period theorist' can admit points as the result of a *limit* process extrapolating from ever smaller periods. In any case, by now, there has been a large number of attempts at (re)constructing a viable temporal period paradigm, of which we shall survey a few salient traits, in order to illustrate the workings of this new perspective in temporal logic.

3.1 Basic Period Ontology

The move from points to periods has had various motivations stemming from the original areas of application for temporal logic, being philosophy and linguistics. For instance, philosophers have become increasingly interested in so-called *part-whole* relations in addition to *element-set* relations, which leads to more 'continuous' approaches (cf. Smith 1982). But also in linguistic semantics, many temporal expressions in natural language seem to involve extended periods rather than points in time as their natural indices of evaluation. For instance, on dense structures like $\mathbf{Q}$ or $\mathbf{R}$, the earlier account of the *progressive* tense (Sections 2.3.2, 2.4.2) would imply that, if I am suffering now, I have been suffering already (in some open interval far enough to the left in the ongoing interval of suffering). This inference seems unwarranted: and the problem highlights the fact that "be suffering" is more naturally predicated of extended intervals in the first place, without any direct reduction to points within these (if any). Further examples of mismatches may be found in Dowty 1979, Creswell 1985, van Benthem 1988a. Finally, there has been a number of proposals coming from Computer Science and Artificial Intelligence advocating this same move, for instance, by suggesting that in temporal data bases, periods rather than points are the most efficient carriers of temporal information (cf. Allen 1983).

Despite all these pointers to periods, no single ontological paradigm has emerged yet. As it turns out, there are a good many options as to the proper mathematical modelling. Clearly, there should be some form of *precedence* ; and also, at least one relation displaying the extended nature of periods, such as *inclusion* or *overlap* . But, many further candidates are around in the literature.

There are various possibilities here for a more principled selection. E.g., Ladkin & Maddux 1987 have a systematization in terms of *relational algebra.* And one could also extend the earlier *invariance* analysis of Sections 2.3.3, 2.4.2 on intervals of linear orders, in order to enumerate basic options. [A conceptual disadvantage of the latter procedure is that development of the period paradigm is made to depend essentially on pictures supplied by its point-based rival.]

<u>Example</u>. With two convex intervals on the reals $\mathbf{R}$, there are only thirteen possible relative positions which are invariant under automorphisms: cf. Allen 1983, van Benthem 1983. For finite unions of convex intervals, however, the situation is less clear (cf. Ladkin 1987b), calling for further constraints. □

Incidentally, a period theorist is not committed to a sparse unitary ontology. For instance, another option might be to allow *co-existence* of different sorts of primitive temporal objects: periods, points and perhaps others as well.

<u>Remark</u>. A rather original perspective on the choice of primitive relations is found in Thomason 1987, who shows that a proper categorial duality between point structures and period structures may be set up only if we assume the following items:

'total precedence', 'beginning before', 'ending before' and 'abutment'. □

In addition to the choice of primitive relations, there is the issue of choosing appropriate *axioms*. Here, the same classification may be used as in Section 2.1.3. That is, there may be a core set of uncontroversial (Horn clause) first-order principles, surrounded by progressively more debatable further assumptions. Here is a reasonable system for precedence and overlap, due to Russell 1926:

(1) $\forall x: \neg x<x$

(2) $\forall x: xOx$

(3) $\forall xy: xOy \rightarrow yOx$

(4) $\forall xy: xOy \rightarrow \neg x<y$

(5) $\forall xyz: x<yOz<u \rightarrow x<u$

These principles imply transitivity for $<$.

For inclusion, one may require partial ordering:

(6) $\forall x: x \subseteq x$

(7) $\forall xyz: x \subseteq y \subseteq z \rightarrow x \subseteq z$

(8) $\forall xy: x \subseteq y \subseteq x \rightarrow x=y$

as well as *monotonicity*:

(9) $\forall xyzu: x \subseteq y<z \supseteq u \rightarrow x<u$

(10) $\forall xyzu: x \supseteq yOz \subseteq u \rightarrow xOu$.

Beyond these plausible properties of and connections between the primitive relations, there are also some less evident universal options, such as

$\forall xy: x<y \vee y<x \vee xOy$ (linearity)

$\forall xyzu: u \supseteq x<y<z \subseteq u \rightarrow y \subseteq u$ (convexity) .

On top of this, then, first-order principles with existential import may be formulated, expressing the existence of neighbours, subintervals, etcetera. Some full first-order theories of intervals may be found in van Benthem 1983, Allen & Hayes 1985, Ladkin 1987a, Ladkin & Maddux 1987. For instance, the complete first-order theory of $<, \subseteq$ on convex intervals of rational numbers is effectively axiomatizable (and indeed decidable), via a Cantor-style zigzag argument establishing countable categoricity.

Remark. Subtleties of formulation may matter here. For instance, some first-order interval theories will be decidable, whereas other related variants are not (cf. Schulz 1987). □

Then also, *higher-order* intuitions on period structures may be formulated. One example with a metaphysical pedigree is the idea of *Reflection*: 'The structure of the whole universe must be reflected in its smallest parts'. Technically, let $\mathbb{I} = (I, <, \subseteq, O)$ be some period structure, with $i \in I$. The substructure $\mathbb{I}_i$ has a universe $\{j \in I \mid j \subseteq i\}$ with the obvious restrictions of $<, \subseteq, O$. Then, one might require that:

For each $i \in I$, $\mathbb{I}$ is isomorphic to $\mathbb{I}_i$.

Note that this also makes all $\mathbb{I}_i$ isomorphic among themselves: Reflection involves a form of *Homogeneity*.

Remark. The pattern of the exposition here has followed that for point structures rather closely. Thus, whatever ontological rivalry may exist, the temporal point paradigm can serve as a model for its period alternative, at least in a methodological sense. □

3.2 Temporal Logic

First-order theories of intervals are one, direct way of describing period structures. But, it is also possible to proceed as before in the Priorean fashion, using period frames for the evaluation of suitable propositional languages. Again, there is a variety of possible 'temporal logics' here, differing in their choice of operators.

One relatively standard system of this kind is developed in van Benthem 1983. It has the earlier temporal operators P, F referring to total precedence, as well as one operator typically exploiting the additional inclusion structure, namely

$\mathbb{M} = (I, <, \subseteq, V) \models \Box\phi\,[i]$ iff $\forall j \subseteq i$: $\mathbb{M} \models \phi\,[j]$.

To preserve the symmetry here, we might add an upward dual of $\Box$.

This tense logic can be treated by the methods already developed in Section 2. For instance, various conditions on the period relations may correspond to principles expresssible in the temporal logic.

Example. The axioms $\Box\phi \to \Box\Box\phi$ and $\Box\Diamond\phi \to \Diamond\Box\phi$ together will now express that $\subseteq$ is an atomic transitive order. (Compare the McKinsey Axioms of Sections 2.2.2, 2.4.1.) $\Box$

Example. The axiom $F\phi \to \Box F\phi$ corresponds to leftward monotonicity, being $\forall xyz: x \subseteq y < z \to x < z$; and likewise, rightward monotonicity corresponds to $P\phi \to \Box P\phi$. $\Box$

Similarly, the basic model theory for this system can be developed using suitable notions of generated submodel and zigzag morphism.

Finally, as for completeness results, the following logic has been proposed as a 'minimal extended tense logic', and proved frame-complete in van Benthem 1983:

(1) a complete propositional base
(2) the modal logic S4 for $\Box$
(3) the minimal tense logic K_t for F, P
(4) the above two monotonicity axioms for $\Box$, F, P .

At the back of these results lies again a *translation* into a suitable first-order logic (on models), or a monadic second-order logic (on frames), as before.

Nevertheless, there is one important special case to be noted here. Quantification in the latter logics will be over individuals which are *periods* (and subsets of these, in the second-order case). But, when one is dealing with *intervals* which have been derived from some underlying point frame, quantification over periods itself may be viewed as being already second-order, vis-a-vis the underlying points. In many cases, however, the latter complexity can be reduced. Intervals are often given by their *boundary points*: and then, quantification over them reduces to mere quantification over ordered pairs of individual points, and hence to ordinary quantification over points.

Even thus, there remains a subtlety. Frame validity of an interval axiom will then involve quantification over all predicates of ordered pairs of points, that is, essentially, over all *binary relations* among points. Therefore, when measured in terms of the underlying points, frame validity of temporal axioms on interval structures becomes more complex than in Section 2 above, involving *dyadic* instead of *monadic* second-order logic. And, unlike the monadic theory of, say, the integers $\mathbb{Z}$, its dyadic second-order theory is non-effectively axiomatizable. (Cf. van Benthem 1983, Halpern & Shoham 1986.)

The observed smooth sailing along the path charted in Section 2 reflects the rather cautious selection of a minimal period tense logic. Many further choices of primitive predicates will lead to a more complex system, closer in spirit to the first-order generalizations of tense logic discussed in Section 2.4.2.

What is worth emphasizing are rather some *new* aspects of working with a period-based temporal logic, which have not emerged before.

One basic issue is this. Several authors have claimed that, on our intuitive understanding of even the standard *logical constants*, these no longer have their usual truth definitions in a period setting. For instance, Humberstone 1979 claims that *negation* now should have the following 'intuitionistic' reading (compare Troelstra & van Dalen 1988):

$\mathbb{M} \models \neg\phi\,[i]$ iff $\forall j \subseteq i$: *not* $\mathbb{M} \models \phi\,[j]$ ('nowhere during i') .

And Creswell 1977 has proposed that our natural temporal reading of conjunction involves a join of subperiods where the conjuncts hold:

$\mathbb{M} \models \phi\wedge\psi\,[i]$ iff $\exists j, k \in I$: $j \cup k = i$ *and* $\mathbb{M} \models \phi\,[j]$ *and* $\mathbb{M} \models \psi\,[k]$.

These views also come with general intuitions concerning the admissible truth value patterns of propositions across nested periods. Thus, Hamblin 1971 has stated that

"there are no indefinitely finely intermingled intervals of both truth and falsity for any statement".

Probably, a better strategy here is to leave the old logical constants undisturbed, while studying some of their compounds or variants. For instance, the above intuitionistic negation is already definable in our language as $\Box\neg\phi$. And, the Creswell conjunction might be added as a new connective, comparable to the computationally well-known chop operator CHOP $\phi\psi$, which says that a period can be divided up into two consecutive subperiods verifying ϕ, ψ in that order.

A more general new topic which arises naturally in this context is the semantic *preservation* behaviour of certain types of formula. For a long time, linguists have observed that temporal (verbal) expressions in natural language may be classified into various kinds, distinguishable by their behaviour under inclusion. For instance, some are downward preserved in passing from a period to its subperiods; others may be upward cumulative, in that they hold of any union of overlapping periods in which they hold. Moreover, there are linguistic constructions taking expressions from one such 'aspectual class' to another (cf. Dowty 1979). E.g., "Mary was writing" is preserved under subperiods, whereas its extended direct object form "Mary was writing a love letter" is not (cf. Krifka 1987). As a formal counterpart, we can study various types of preservation behaviour for syntactic classes of formulas in period tense logic.

<u>Example</u>. Temporal Propagation.

Here is a phenomenon which also makes sense within the framework of Section 2. Which formulas are such that, once true, they will remain true? Examples are all forms $G\phi$, $P\phi$(on transitive frames) as well as anything obtainable from them using $\wedge, \vee, \Box$. In fact, any future-propagated formula is equivalent to a form $G^*\phi$ (standing for: $G\phi\wedge\phi$). In particular, the class of formulas equivalent to some G^* shape is already closed under $\wedge, \vee, \Box$. $\Box$

<u>Example</u>. Temporal Descent.

Which formulas are such that their truth at any period trickles down to all its subperiods? Examples are all forms $P\phi$, $F\phi$, $\Box\phi$ as well as anything obtainable from them using $\wedge, \vee$. And conversely, any downward-preserved formula is equivalent to the form $\Box\phi$. Again, the class of such formulas is closed under $P, F, \Box, \wedge$ and $\vee$. $\Box$

Finally, here is an example of a richer interval logic, taken from Halpern & Shoham 1986. Structures are frames $(T, <)$ with intervals $[t_1,t_2]$ $(t_1 \leq t_2)$. Basic operators include the following:

BEGIN ϕ is true at $[t_1,t_2]$	iff	there exists $t_3<t_2$ such that ϕ is true at $[t_1,t_3]$
START ϕ is true at $[t_1,t_2]$	iff	there exists $t_3>t_2$ such that ϕ is true at $[t_1,t_3]$
BEFORE ϕ is true at $[t_1,t_2]$	iff	there exists $t_3 \leq t_1$ such that ϕ is true at $[t_3,t_1]$,

as well as their obvious duals (such as END opposite to BEGIN).

The expressive power of this formalism can be studied by the earlier techniques, viewing it as a two-dimensional multi-modal logic, whose primitive relations are required to show certain interactions, encodable in suitable axioms. See Venema 1988 for further details. Specifically,

(1) on linear orders, all point-based tense-logical statements are expressible, since the logic has the means of defining the functionally complete Stavi set S, U, S', U' of Section 2.4.2.

(2) as to interval expressions, the earlier CHOP operator is not definable

(3) the formalism is stronger than any point-based temporal logic, in that there exist two countable linear frames having the same monadic second-order theory which can be distinguished in this logic. (Recall the earlier remark about the *dyadic* second-order nature of the present formalism.)

Remark. Using an idea from van Benthem 1983, Venema reinterprets this interval logic as one of points in a direct product of the underlying frame with itself. Thus, the above operators become *topological* ones: and 'logic of time' becomes also 'logic of space'. □

As for completeness, Halpern & Shoham 1986 prove a number of negative results, establishing $\Pi^1{}_1$-hardness for the interval logics of e.g., **N**, **Z** or **R** . For general logical reasons, however, the rationals **Q** must remain an exception (as was observed already in Section 2.2.3) - and indeed, Venema provides a complete axiomatization of their interval logic, using the 'irreflexivity rule' of Gabbay 1981a. At the base of this lies a general interval logic, whose axioms include such typical cases as

$$\text{BEGIN BEGIN } \phi \rightarrow \text{BEGIN } \phi$$

$$\text{END START } \phi \rightarrow \text{START END } \phi$$

All these axioms express *first-order* conditions on the above topological structures, and hence on their underlying point frames. The reason is that all of them are *Sahlqvist* forms (cf. Section 2.4.1).

Remark. The last observation suggests that *fragments* of the full language may be better behaved (as is already pointed out by Halpern & Shoham themselves). In particular, are the Sahlqvist fragments of **N** , **Z** and **R** axiomatizable after all? □

Remark. The above logic is still traditional, in that intervals are convex sets on a *linear* order.
P. Thiagarajan has suggested looking into similar logics of intervals on non-linear frames, such as the Minkowski spaces of section 2.2.1, 2.2.3 , whose convex intervals are geometric rectangles or , in higher dimensions, blocks. □

3.3 Relating Periods and Points

Not just philosophically, but also mathematically, the point-based paradigm of Time can peacefully coexist with the period-based one.

- Every point frame $(T, <)$ induces an interval structure $(I, <, \subseteq, O)$ where
  (1) I consists of all non-empty convex subsets of T
  (2) $i<i'$ if $\forall t\in i, t'\in i'$: $t<t'$
  (3) $i\subseteq i'$ if i is a subset of i'
  (4) iOi' if $i\cap i'$ is non-empty .

Such interval structures will validate almost all of the earlier basic postulates: and truly all, if one starts with an underlying transitive irreflexive frame $(T, <)$. See van Benthem 1983,1984b for details, including a proof that these principles axiomatize the complete first-order theory of convex interval structures on strict partial orders.

- Conversely, period structures may be *represented* as families of intervals on point frames via any one of a number of mathematical constructions, such as (maximal) filters (see van Benthem 1983) or Dedekind cuts (see Burgess 1984b, Thomason 1979). [Whitrow 1980 has an account of historically prior attempts, going back to Russell and Wiener.] A short exposition of the filter method and its mathematical properties is found in van Benthem 1984b. For instance, one interesting issue is the *correspondence* between the earlier basic constraints on the two kinds of temporal structure, when looking along the two representation maps. Likewise, one can try to relate Priorean tense logics interpreted at the two levels.
- This mathematical correspondence between point structures and period structures can be elaborated into a full-fledged *duality between categories* (cf. van Benthem 1983, Thomason 1987). The basic observation here is that natural kinds of morphism turn out to correspond on both sides. For instance, van Benthem 1984b correlates the following two notions (out of many possibilities):

(1) *positive extension* among period structures $\mathbb{I}_1, \mathbb{I}_2$:
 I_1 is contained in I_2, and likewise $<_1$ in $<_2 \restriction I_1$, $\subseteq_1$ in $\subseteq_2 \restriction I_1$.

(2) *anti-morphic surjective functions* between point structures $\mathbb{T}_1, \mathbb{T}_2$:
 i.e., partial maps f from T_2 into T_1 such that $f(x) <_1 f(y) \rightarrow x <_2 y$,
 which also satisfy a suitable continuity condition on distinguished intervals.

Intuitively, (1) describes a period structure $\mathbb{I}_1$ growing into a larger one $\mathbb{I}_2$ having new periods as well as (possibly) more precedence and inclusion facts about old periods. Thus, their induced point frames become richer, with the obvious notion of backward restriction among (suitable) filters satisfying (2). Further technical details may be found in the earlier references. What we have here, in fact, is an instance of the general mathematical duality between more point-based set-theoretic approaches and more interval-based topological ones , which also figures e.g. in the semantics of intuitionistic versus classical logic.

3.4 Extending to Events

In some ways, the move from temporal points to temporal periods seems only a half-way house towards more radical measures. Many recent authors in the philosophical and linguistic tradition have broken with the primacy, or at least independence, of a purely temporal ontology, by making *events* the major furniture of our universe. This is an old, and respectable idea. Already Leibniz tried to derive the structure of Time (and Space) out of the formal pattern of events in this world (cf. Winnie 1977). And also more recently, for quite different reasons, Davidson 1967 has been a very influential plea for making 'event' a basic category, both in the philosophy of language and in practical linguistics.

Remark. More recently, various formal theories of events have appeared, intended as a vehicle for the logical semantics of natural language (cf. Krifka 1987). One issue here is how to develop a good version of the *Lambda Calculus* based on both ordinary individuals and events. □

Events are not purely temporal entities, in that they also have (at least) spatial and causal aspects. Nevertheless, much of their formal theory, as far as temporality is concerned, has been similar to what we have already seen for period structures. For instance, the earlier basic axioms for the latter were originally put forward to capture the behaviour of events (cf. Russell 1926) - and a similar switch may be observed with many modern authors. What is different are rather the types of *question* suggested for logical investigation.

Example. Private and Public Time.

For Russell, one of the interesting phenomena to be clarified by logical analysis was the interplay between 'common sense' and 'scientific' notions of Time. Our private experiences come in the form of (finite) event structures, which may be pooled (via the earlier notion of positive extension) into some large public experience. In parallel, these event structures may be represented as point structures, running from private times to public time. The categorial perspective of Section 3.3 then allows us to make exact comparisons between various routes towards creating the public time of science (cf. Thomason 1979, van Benthem 1984b, 1988a). □

Example. Temporal Representation of Discourse.

In Kamp 1979, the process of interpreting natural language is analyzed as follows. Ongoing text produces 'discourse representations', whose temporal format is that of event structures. These should then be related to actual intervals in physical point-based time, in order to establish objective truth. In this perspective, the representation method of Section 3.3 turns out to explain some of the peculiarities of actual temporal expressions in natural language. □

Digression. Partial Models.

One of the other novelties in the paper just-mentioned (at least, as far as temporal logic is concerned) is the introduction of *partial* relations of precedence and inclusion, having positive parts to their extension, but also negative and undecided ones. The latter reflect temporary lack of information. We shall return to the underlying issue of *partial information* in Section 4 below. □

Partial information is just one of several computational issues which surface in the account so far. For instance, as we shall see, connections between various levels of temporal representation also arise naturally in computer science. And, on the topic of formal theories of events, much of the material in Winskel 1988 seems relevant in a wider philosophical / linguistic setting too.

In reality, the ontological picture behind natural language is more complex than has been suggested so far. We are living in a rich universe of not just individuals and events, but also such temporal entities as 'cases' (Lewis 1975), 'processes', 'states', etcetera. An interesting concrete reflection of this wealth is found in the aspectual calculus of Galton 1984, where states and events appear on a par as the basic temporal entities. The Priorean tenses F, P then become operators from states to states, whereas the progressive (PROG) as well as the perfect (PERF) change events into states. Conversely, there are operators turning states into events, such as INGR ('begin to') and PO ('spend a while'). A fair sample of Galton's principles reads as follows:

$$\mathrm{PERF\ INGR}\ q \leftrightarrow P^*(P\neg q \wedge q) \qquad (\text{where } P^*q := Pq \vee q)$$

$$\mathrm{PROG\ PO}\ q \leftrightarrow P\neg q \wedge q \wedge F\neg q$$

$$\mathrm{PERF}\ q \rightarrow G\ \mathrm{PERF}\ q\ .$$

Thus, tense logic can be pursued in richer temporal ontologies too.

Although these theories have their motivation in linguistics and philosophy, they may be suggestive also as a source for computational accounts of storage of temporal information. Presumably, natural language has good reasons for preferring a rich many-sorted approach over some austere view of temporality.

4. Perspectives from Computer Science

The main topic of this survey has been temporal logic as it has developed from its original sources. But recently, there has been an outburst of application, and indeed independent development, of temporal logic within computer science: as is amply demonstrated by the present volume. Without any pretense at exhaustiveness or authority, the following is an attempt at stating some of the main phenomena which make this computational connection of interest to 'native' temporal logicians. Examples will be taken from both point-based and period or event-based paradigms.

4.1 Application with Innovation

Current applications of temporal logic in computer science look congenial to a temporal logician. This starts already with their general attitude. There seems to be a healthy engineering approach in computational work, stressing the construction of different temporal systems geared towards intended applications and *issues*, rather than the search for any single 'philosophers' stone'. This is quite in line with the tendency towards plurality and generality recorded earlier for classical temporal logic. Moreover, the kinds of technical question being studied are usually quite recognizable logical concerns, in particular, expressive power, axiomatizability and decidability. To an outsider, this might actually come as a surprise. A priori, one might expect computer scientists to arrive at quite different fundamental questions concerning known temporal logics. And perhaps, with time, this will yet come about.

Even so, not *all* logical concerns seem to have found computational application yet. For instance, there seems to be a relative neglect of other possible meta-properties of temporal calculi, such as *interpolation*, and likewise, of other types of model-theoretic question, such as *preservation*. It might be worth the effort to investigate their potential computational relevance too. After all, there are already quite surprising examples of new computational purposes for old logical notions, witness the use of the *finite model property* for deriving machines or protocols producing certain desired behaviour (cf. Clarke & Emerson 1981 and subsequent publications).

Computational research has not just been a customer of existing logic: it also has much of its own to offer which enriches traditional temporal logic in various ways.

For a start, it appears to have introduced one more 'standard concern' for research, in adddition to expressive power and axiomatization, namely that of precise *computational complexity* of proposed systems. Such concerns were largely absent from the classical literature, where 'decidability' was as far as practical concerns went. Moreover, issues of complexity have also started penetrating elsewhere in logic - witness the recent interest in deciding various other syntactic and semantic notions. Perhaps, we shall have a 'computational model theory' one day.

The concern with complexity may also be classified as an instance of another general trend in the computational literature, namely the study of the *fine-structure* of earlier logical systems. [In a sense, the fine-structure below logical Recursion Theory is what computer science is all about.] Often, there is as much to be gained by *restricting* as by *generalizing* from some standard temporal calculus. This is also true for expressive power and syntactic form; witness the emphasis on special syntactic shapes, such as those expressing 'liveness' and 'safety'. (See Pnueli 1988 and Katz 1988 for illustrations.)

Then, as is natural in any applied field, the computational perspective also highlights certain *concrete models* or model classes, whose temporal logic can then be developed in much more depth, using available special purpose tools. This may be observed, for instance, in the use of Automata Theory on discrete temporal frames of suitable order types in Thomas 1986, 1988. [The latter perspective may even be of general semantic significance after all: witness van Benthem 1987 on 'semantic automata' in a computational semantics.]

Another computational restriction is that to *finite models* (advocated in Gurevich 1985). Here, a number of results has been reached already within classical temporal logic. For instance, much of the earlier work on the finite model property falls under this heading. And there are also some results on expressive power of temporal logic on finite frames (cf. van Benthem 1986b). Contrary to prevailing prejudices among logicians, the finiteness restriction does not always make the original theory easier. For, useful classical methods based on Compactness fail, and have to be replaced by laborious combinatorial reasoning.

In addition to the process of specialization and concretization of classical temporal logic in a computational setting, it must be said that the area has also produced at least one important *generalization*. For, *Dynamic Logic* (see Harel 1984) may be viewed as a general form of temporal logic, where our earlier systems have become embedded in a theory of complex actions.

4.2 Semantic Parallels

The above examples were all taken from point-based temporal logic. Next, we review some topics from more recent period or event-based developments, which are sometimes surprisingly similar to those having occurrred in the classical framework. Since there are fewer ready-made tools here (compare Section 3), straightforward 'application' is not our main concern. What we want to demonstrate is rather how this parallel development by itself suggests a natural community of interests between logic, linguistics and computer science.

We have already seen a proposed temporal *interval logic* by Halpern & Shoham 1986, which turned out to be quite close to the standard tense-logical tradition (cf. Section 3.2).

Arguments put forward by computer scientists for making the move towards intervals or periods, rather than points, have been partly similar to classical ones ('greater ease and naturalness of expression': Halpern & Shoham 1986, Kowalski & Sergot 1985), partly new, stressing computational advantages of period-based inferential algorithms (Allen 1983). An interesting general philosophy supporting this ontological shift is found in the research program of *Naive Physics* in Artificial Intelligence (cf. Hayes 1979, Hobbs 1985). The guiding idea here is that efficient reasoning and computation requires knowledge representations which are similar to those used in the common sense world, rather than those found in abstract science. The technical elaboration of this program shows unmistakeable analogies with the research reported in Section 3 above. But of course, the scope of Naive Physics is much wider in principle than mere temporality.

Digression. As a matter of history of ideas, Naive Physics shows how 'lost causes' in science can still be of computational interest. What Hayes and his followers are doing seems close to an attempt at recapturing an Aristotelean, pre-Galilean physics. Thus, contrary to the tenets of modern orthodoxy, maintaining a plurality of scientific frameworks seems useful, even across so-called 'decisive turns' in history. □

There are also interesting analogies between more concrete proposals made in Artificial Intelligence, and Computer Science in general, concerning temporal representation and temporal inference, and the more linguistic/philosophical interests described earlier. For instance, the central issues in such papers as McDermott & Shoham 1985 or Kowalski & Sergot 1985 are quite close to those formulated in Section 3 concerning various forms of temporal persistence along precedence or inclusion orderings.

A final example, which brings together many earlier strands, arises from the much-cited recent work Lamport 1985 on the proper temporal modelling of *parallel computation*. Lamport presents a system of axioms for 'system executions', being sets of 'operation executions' ordered by two relations of total precedence and partial precedence. Moreover, he studies the connection between different 'views' of a system execution, arising from different groupings of individual actions into complex units. Without going into detail, we may note some interesting analogies with the themes developed in Section 3.

First, the choice of primitives is easily understood via Lamport's intuitive explanation in terms of subsets on a strict partial order $(T, <)$:

total precedence	$x<y$	$\forall e\in x\ \forall e'\in y:\ e<e'$
partial precedence	$x\leq y$	$\exists e\in x\ \exists e'\in y:\ e\leq e'$

Slightly more elegantly, we may rewrite the second notion as a disjunction of

overlap	xOy	$\exists e\in x\ \exists e'\in y:\ e=e'$
partial precedence*	$x<<y$	$\exists e\in x\ \exists e'\in y:\ e<e'$

Lamport's axioms express some obvious Horn clause conditions on these relations, which look similar to those encountered in the earlier literature (Kamp 1979, Allen 1983). That they are well-chosen may be seen from various mathematical results proven since (cf. Angers 1986, Ben-David 1987), which show that the models of the Lamport system, slightly extended, are precisely those admitting of a *representation* in terms of subsets on an underlying strict partial order. Such a result was also obtained independently using the methods of van Benthem 1983, with the outcome that the following Lamport-type calculus axiomatizes the complete first-order theory of point sets on strict partial orders:

(1) all Russell axioms on $<$, O listed in Section 3.1 (except for linearity)

(2) the further axioms

$x<y\rightarrow\neg y<<x$, $\quad$ $x<<y<z<<u\rightarrow x<<u$, $\quad$ $x<y<<z<u\rightarrow x<u$,

$xOy<z<<u\rightarrow x<<u$, $\quad$ $x<<y<zOu\rightarrow x<<u$, $\quad$ $xOy<zOu\rightarrow x<<u$

Remark. *Linearity* would be an optional item. Ben-David 1987 has some results on relating system executions to linear 'global time models' which is reminiscent of the embedding of discourse representations into real physical time discussed in Section 3.4. (This author also shows that, for certain types of modal statement, working with underlying partial orders or linear orders makes no difference. At the level of first-order relational conditions, this may also be shown to be true for *Horn clauses*.) □

Remark. In line with Section 3.1, *inclusion* may be added as a primitive relation. Then, the above representation result can be extended using some obvious axioms on $\subseteq$ and its interplay with $<<$, $<$, O .
□

Next, the idea of various related system executions, having different 'grain size', brings in the earlier categorial perspective of a wider world of event structures, with suitable morphisms connecting them. For instance, here are two candidates which may be discerned behind Lamport's presentation:

(1) f is an *embedding* from one system execution into another if it respects $<$ as well as $<<$.

(2) f is a *higher level view* if it is a surjection acting as follows:

$x<y \rightarrow f(x)<<f(y)$, $\quad f(x)<f(y) \rightarrow x<y$.

Explanation. Higher level views may be interpreted as grouping several events into a single new one, and deriving the new relational structure from the old:

$u<v$ if $\quad \forall x \in f^{-1}(u)\ \forall y \in f^{-1}(v)$: $x<y$

$u<<v$ if $\quad \exists x \in f^{-1}(u)\ \exists y \in f^{-1}(v)$: $x<y$. □

Such morphisms can be studied model-theoretically as to their *transfer* behaviour. Which descriptions true in one approximation of a system remain valid in another? Notably, embeddings will preserve all existential positive statements concerning $<$ and $<<$. Higher level views will guarantee (amongst others) the following more complex transfer:

If $\phi = \phi(<)$ is true in the source, then ϕ^* is true in the image, where ϕ^* is obtained from ϕ by rewriting it to a form having negations only in front of atoms, and then replacing positive $x<y$ by $x<<y$ (negative atoms $\neg x<y$ remaining undisturbed).

Example. Succession $\forall x\ \exists y\ x<y$ induces $\forall x\ \exists y\ x<<y$; while discreteness $\forall x\ \exists y(\ x<y \wedge \neg \exists z(\ x<z \wedge z<y\ )\ )$ becomes $\forall x\ \exists y(\ x<<y \wedge \neg \exists z(\ x<z \wedge z<y\ )\ )$. □

In addition, of course, we may also study these morphisms in combination with their categorial counterparts at the level of the underlying point frames, by various representation methods.

Thus, the Lamport perspective too involves the richer temporal picture which was already required by philosophical and linguistic developments in current temporal logic. This final case of convergence concludes our survey of more tentative recent trends in temporal representation.

4.3 Reinterpreting Logic

One intriguing phenomenon to be observed these days is how computationally inspired ideas are making themselves felt even within the heartland of standard logic, leading to a reappraisal of basic semantic frameworks. It is too early yet to judge the intrinsic merits of this development: but it is certainly worth demonstrating in some detail here, for the case of the basic tense logic. We shall consider variations on all of its central aspects: whether its models, notions of inference or even modes of evaluation.

4.3.1 *Time Machines*

The imagery of *machines* is ubiquitous in computational logic, sometimes even occurring at several levels within one type of problem. This may be partly a matter of fashion. John Searle once pointed out in this connection how seventeenth century thinkers saw *clocks* everywhere, and their ancestors in Antiquity again whatever piece of technology was 'en vogue' then. Nevertheless, there is an interesting issue here concerning the nature of time. Traditional temporal logic has looked at time as a receptacle for histories, and accordingly, its models are regarded intuitively as sets of possible *traces* of the evolution of some system over time. What recent computational applications show, however, is that these models may also reflect the systems or *machines themselves* which produce these histories. [And of course, even on the classical view, one tries to see the grand System in or through the observed traces.]

The two perspectives mingle rather nicely in those semantics for branching time where some finite possible worlds frame, viewed as a machine, is considered together with the set of all its possible evolutions, so that statements about its 'states' and its 'paths' can be brought together. This also highlights a difference with traditional temporal logic of course: one is investigating 'machine time', rather than actual time. But on the other hand, this move toward conceptual, rather than actual temporal structure could also be observed already in those classical theories which were designed with an eye toward temporal representation (cf. Section 3).

In fact, classical logical techniques work just as well when dealing with machine-like temporal models. In particular, the semantic invariance relations of Section 2.3.1 make sense for machines too. For instance, given any finite state machine, its complete evolution tree is precisely its *unraveling* in the earlier sense, which is zigzag related to it: and hence the two verify the same tense-logical formulas at corresponding points. Conversely, given any temporal model, we may construct some minimal model verifying the same formulas at corresponding points by means of the earlier-mentioned technique of *filtration*. When the collapsed model is *finite*, we may view it as a machine producing the original 'behaviour'.

Remark. There are two subtleties here. One is that filtration may be *relativized* to work for special classes of formulas only. With a finite class of relevant formulas, the collapse is even guaranteed to be finite too. The other point is that the collapsing function is not necessarily a zigzag relation. For that to occur, one needs additional conditions on the original model, such as 'modal saturation' (cf. Fine 1975b). In particular, if the original model is only *finitely branching*, filtration will always produce p-morphic collapsing maps. □

Of course, the machine perspective suggests various new questions here, which would not be so evident in a more general modal or temporal perspective. One case in point are possible ways of *retrieving* machines from their output behaviour. (Compare the *Nerode representation* for finite automata.)

Also relevant is the earlier remark about bisimulation in process algebra and zigzag relations in temporal logic (cf. Sections 2.3.1, 2.3.2). Without going into details here about the encoding of transitions and states into propositions, we can make such simple observations as the following:

Two connected indeterministic finite automata bisimulate each other if and only if their initial states, when viewed as roots of possible worlds models, verify the same Priorean pure future formulas.

Example. Failure of Distribution.
The well-known example from Process Algebra concerning the difference between $a\cdot(b+c)$ and $(a\cdot b)+(a\cdot c)$ is reflected in the temporal non-equivalence between $F(a\wedge(Fb\wedge Fc))$ and $F(a\wedge b)\wedge F(a\wedge c)$.
□

If one is merely interested in recognized sequences, then only a *fragment* of the temporal language will be relevant, using just syntactic forms $F(a_1\wedge F(a_2\wedge\ldots\wedge F(a_n\wedge \mathit{accept})\ldots))$. Again, there is an analogy here between temporal logic and automata theory. Two deterministic finite automata bisimulate each other if and only if they recognize the same strings. But also, two corresponding 'deterministic' possible worlds models verify the same, possibly 'branching', pure future formulas if and only if they verify the same formulas of the above restricted 'linear' forms.

Thus, even though a precise elaboration is beyond the scope of this survey, predominantly machine-oriented formalisms for temporal behaviour and more traditional trace-oriented ones seem very closely connected - even though they may give rise to different points of emphasis in the agenda for research.

4.3.2 *Partial Information*

Another trend in current logic and semantics whose inspiration has clear computational roots is the interest in *information,* and potential lack of it. In particular, how can one draw inferences with *incomplete* information? *Partiality* is a broad concern by now (cf. Blamey 1986, Urquhart 1986), which again provides a fresh look at established logical systems. (Cf. Fenstad et al. 1987 on partial predicate logic and related theories.)

Within temporal logic, partiality may occur at several levels. Already in Section 3.4, we encountered temporal frames having 'positive', 'negative' as well as 'undecided' / 'unknown' parts to their relations of precedence and inclusion. This move can be made without losing the insights from the earlier 'total' framework.

Example. Partializing Temporal Constraints.
Let Σ be a set of first-order conditions on frames, defining a class K of admissible temporal structures. Let K_{part} be the class of those partial structures which can still be made into classical frames satisfying Σ by addition of individuals and appropriate closure of 'truth value gaps'. (Thus, K_{part} is a natural 'partial hull' for K.) Then, K_{part} is first-order definable too, in the obvious first-order companion language having two predicates $<^+, <^-$ and likewise for other temporal relations. (See Kamp 1979 for a proof.) Moreover, the new partial axioms may be extracted effectively from the original Σ . □

But not just the temporal *ontology* may be partialized. The same holds for the tense logic over it. For instance, even on standard frames, we may have partiality at the level of *interpretation.* Let us assume that propositional valuations V now have positive, negative and undecided parts to their extensions. Following a well-known pattern, one can then set up conditions of 'verification' and 'rejection':

$\mathbb{M} \models^+ p[t]$	iff	$t \in V^+(p)$
$\mathbb{M} \models^- p[t]$	iff	$t \in V^-(p)$
$\mathbb{M} \models^+ \neg\phi[t]$	iff	$\mathbb{M} \models^- \phi[t]$
$\mathbb{M} \models^- \neg\phi[t]$	iff	$\mathbb{M} \models^+ \phi[t]$
$\mathbb{M} \models^+ \phi\wedge\psi[t]$	iff	$\mathbb{M} \models^+ \phi[t]$ *and* $\mathbb{M} \models^+ \psi[t]$
$\mathbb{M} \models^- \phi\wedge\psi[t]$	iff	$\mathbb{M} \models^- \phi[t]$ *or* $\mathbb{M} \models^- \psi[t]$.

For the temporal operators, there seem to be *options*; of which we choose

$\mathbb{M} \models^+ F\phi[t]$	iff	*for some* $t'>t$: $\mathbb{M} \models^+ \phi[t]$
$\mathbb{M} \models^- F\phi\,[t]$	iff	*for each* $t'>t$: $\mathbb{M} \models^- \phi[t]$

An alternative would be to stipulate here: *for no* $t'>t$: $\mathbb{M} \models^+ \phi[t]$.

The option chosen here makes evaluation *persistent*, in that,

If $\mathbb{M} \models^+ \phi[t]$, then $\mathbb{M}' \models^+ \phi\,[t]$, for every model $\mathbb{M}'$ differing from $\mathbb{M}$ only in that V'-values for proposition letters may have become 'more defined'; and likewise for $\mathbb{M} \models^- \phi\,[t]$.

The latter stipulation leads to possible non-persistence: growth of information may lead us to accept an already rejected statement after all. Eventually, having non-persistent operators available seems unavoidable when dealing with partial information - and the most natural candidate is the following two-valued negation:

$\mathbb{M} \models^+ \sim\phi\,[t]$	if	*not* $\mathbb{M} \models^+ \phi[t]$
$\mathbb{M} \models^- \sim\phi\,[t]$	otherwise.	

Then, the second reading of the future operator becomes definable as $\sim\sim F\phi$.

Again, this system can be translated into a classical 'total' variant, by a method of Gillmore (cf. Langholm 1988). In fact, it may be embedded back into the basic Priorean tense logic, so that its *decidability* and other meta-properties follow automatically.

Example. For the sake of illustration, here is the translation which embeds partial temporal logic into its total classical version. Each formula gets a 'positive' and a 'negative' counterpart, via the following mutual recursion:

$$
\begin{array}{llll}
(p)^+ = & p^+ & (p)^- = & p^- \\
(\neg\phi)^+ = & (\phi)^- & (\neg\phi)^- = & (\phi)^+ \\
(\phi\wedge\psi)^+ = & (\phi)^+\wedge(\psi)^+ & (\phi\wedge\psi)^- = & (\phi)^-\vee(\psi)^- \\
(F\phi)^+ = & F(\phi)^+ & (F\phi)^- = & G(\phi)^- \ .
\end{array}
$$

This allows us to reduce various proposed notions of partial consequence to their classical counterparts. □

Despite these reductions, the partial perspective also raises some interesting questions of its own, witness the characterization of persistence in an extended language proved in Langholm 1988. Further logical aspects of partiality are discussed in van Benthem 1988a.

4.3.3 *Circumscription*

One striking phenomenon in the field of automated reasoning is the emergence of computational systems having other modes of inference than just standard logical consequence. As such further inferences are often defeasible, the resulting logics will usually be non-monotonic. One such system, whose logical theory is relatively well-developed is the calculus of *circumscription* (cf. McCarthy 1980). Again, it is of interest to observe how even the basic temporal logic admits of circumscription, as an alternative to its standard notion of consequence.

Definition. Let ϕ be a tense-logical formula, and p some proposition letter. $(\mathbb{M}, t)$ is a p-*minimal* model for ϕ if (1) $\mathbb{M} \models \phi\,[t]$, and (2) for no $\mathbb{M}'$ differing from $\mathbb{M}$ only in that $V'(p)$ is properly contained in $V(p)$, $\mathbb{M}' \models \phi\,[t]$.

There is an immediate generalization to minimizing over several proposition letters simultaneously.

Remark. Actually, this notion is not foreign to standard tense logic. For instance, the proof of Sahlqvists's Theorem (Section 2.4.1) depends on the fact that its antecedent formulas admit of minimal models having first-order definitions for the extensions of the proposition letters involved (cf. van Benthem 1985). □

Now, circumscription gives us additional inferences, through the following notion of consequence, which looks only at special models for premises, where as few p-facts happen as possible:

$$\Sigma \models^{p}_{circ} \phi \quad \text{if} \quad \phi \text{ is true at all } p\text{-minimal models of } \Sigma \ .$$

Evidently, every standard consequence of Σ remains valid in this new sense; but in general, additional ones arise, such as $F(p \wedge q) \models^{p}_{circ} G(p \to q)$.

Many logical questions arise concerning circumscription in the basic tense logic. In particular, it is known that circumscription in *full* predicate logic is a highly complex, non-axiomatizable notion. On the other hand, for *monadic* predicate logic, circumscription is decidable. For the present formalism, intermediate between these two, we only have a

Conjecture. Circumscriptive inference in the minimal tense logic is decidable.

As for reduction to standard logics, the obvious translation matching the above definition takes circumscribed formulas (say) $\mu p{\cdot}\phi(p)$ to *monadic second-order* sentences (cf. van Benthem 1988c). Nevertheless, we may investigate just when tense-logical formulas have first-order circumscriptions, on the pattern of the earlier Correspondence Theory (cf. Lifschitz 1985, van Benthem 1988c).

Example. First-Order Circumscription.

By results of Lifschitz, all formulas of the following forms are first-order:

(1) $\mu p{\cdot}\phi(p)$ with only positive occurrences of p, (2) $\mu p{\cdot}\phi(p)$ with only negative occurrences of p. E.g.,

$\mu p{\cdot}GFp$ is equivalent to $(\forall x{>}t_0\, \exists y{>}x\, Py) \wedge \forall u{:}\ \neg\forall x{>}t_0\, \exists y{>}x\, (Py \wedge \neg y{=}u)$,

$\mu p{\cdot}GF\neg p$ is equivalent to $(\forall x{>}t_0\, \exists y{>}x\, \neg Py) \wedge \neg\exists x Px$.

The following formula is not first-order, however: $\mu p{\cdot}(Fp \wedge G(p{\rightarrow}Fp) \wedge G(p{\rightarrow}Hp))$. For, it holds at 0 in the ordinal sum $(\mathbf{N}, <) + (\mathbf{Z}, <)$ when $V(p) = \mathbf{N}$. But, it fails in the many elementary equivalents of the latter structure having $V(p)$ true beyond the initial copy of $\mathbf{N}$. Therefore, no first-order definition can be adequate. □

4.3.4 *Dynamic Interpretation*

Logical formalisms are often contrasted with programming languages, as being *declarative* rather than *procedural* or 'imperative'. But recently, various authors have become interested in procedural aspects of logical interpretation, to the extent of giving procedural semantics for standard logical formalisms too. (Cf. Groenendijk & Stokhof 1987, taking up an earlier idea of Barwise; and van Benthem 1988c for further logical background.) When applied to the basic tense logic, the following mechanism of interpretation arises.

Intuitively, evaluation of tense-logical formulas in some model is a process taking us along various shifting points of evaluation. Thus, one might assign some set of succesful 'verification traces' to each formula. [This would be an alternative to the 'multi-dimensional' tense logics of Section 2.4.2.] But, as in Hoare-style operational semantics for programs, a convenient level of abstraction is provided by the mere *transition relation* associated with each formula, viewed as if it were an instruction for its own evaluation. Here is a formal implementation, restricted (for convenience only) to pure future tense logic.

Let $\mathbb{M}$ be a model $(T, <, V)$. We define a binary relation $[[\phi]]$ for each tense-logical formula ϕ through the following recursion:

$$[[q]] \quad = \quad \{ (t, t) \mid t \in V(p) \}$$

i.e., atomic propositions function as tests,

$$[[\phi \wedge \psi]] \quad = \quad [[\phi]] \circ [[\psi]]$$

i.e., conjunction denotes sequential composition,

$$[[F\phi]] \quad = \quad \{ (t, t') \mid \exists t''{:}\ t < t'' \textit{ and } (t'', t') \in [[\phi]] \}$$

i.e., future truth is established by making some succesful jump along $<$,

$$[[\neg\phi]] \quad = \quad \{ (t, t) \mid \forall t'{:}\ (t, t') \notin [[\phi]] \}$$

i.e., negation is a test again, of 'strong failure'.

Remark. The third clause may also be rephrased as a composition of $[[\phi]]$ and $[[F]] = <$. □

On this account, interesting procedural differences emerge between formulas which used to be equivalent. For instance, $Fp \wedge Fq$ does not denote the same relation as $Fq \wedge Fp$. The reason behind this failure is in fact the *validity* of the following non-theorem of the standard system: formulas $Fp \wedge Fq$ denote the same relation as $F(p \wedge Fq)$. More generally, the effect of the new interpretation procedure is to widen scopes of F-operators toward the right. There are limits here, though, set by negation: $\neg Fp \wedge Fq$ is not equivalent to $\neg F(p \wedge Fq)$. Thus, *inference* in the new system will be startlingly unlike before.

In fact, there are several options for defining semantic consequence. One measures inclusion of interpretative transition relations:

$$\phi_1,\ldots,\phi_n \models^1 \psi \quad \text{if} \quad \text{in all models,} \quad [[\phi_1 \wedge\ldots\wedge \phi_n]] \subseteq [[\psi]]\,.$$

Another notion would look at possible continuations for evaluating the conclusion after the premises have been processed:

$$\phi_1,\ldots,\phi_n \models^2 \psi \quad \text{if} \quad \text{in all models, } (t_1, t_2) \in [[\phi_1 \wedge\ldots\wedge \phi_n]] \text{ only if there exists some } t_3 \text{ such that } (t_2, t_3) \in [[\psi]]\,.$$

The reason why many properties of classical logic are lost here is that various 'side-effects' of interpretation become important to consequence. Nevertheless, one can still embed classical tense logic into the new one, for instance, by translating formulas so that every F-operator is packed in a double negation. (See the final example below, however, for a different principle.)

The purpose of this passage is merely to point out how even the well-known basic tense logic becomes challenging again under this oblique procedural perspective. For instance, one question now becomes whether the original set of *logical operators* is adequate to the new situation. The answer seems to be negative: on binary relations, many other operations become important (cf. van Benthem 1988c). For instance, it seems reasonable at least to have genuine *Booleans* too. This calls for the following alternatives for negation and conjunction:

$$[[\sim\phi]] \quad = \quad T\times T - [[\phi]] \qquad (\text{'absence'})$$
$$[[\phi\cap\psi]] \quad = \quad [[\phi]] \cap [[\psi]] \qquad (\text{'parallel success'})$$

Moreover, it seems reasonable to add a *modality* over extensions

$$[[\Box\phi]] \quad = \quad \{ (t, t) \mid \forall t' \in T: (t, t') \in [[\phi]] \}\,.$$

This makes the earlier negation $\neg$ definable, as well as other useful modalities of interpretation. Thus, the system with operators $\sim$, $\cap$, $\wedge$, $\Box$ and F seems a reasonable candidate for further investigation.

As for its complexity, we conjecture that the resulting system is still *decidable*. [But of course, in line with earlier remarks, one would like to know its *precise* complexity vis-a-vis K_t.]

What can be established a priori is the effective axiomatizability of these dynamic systems. The reason is that they admit of a*translation* into standard formalisms, be it a different one from that studied in Section 2.3.2. Tense logic now becomes a vehicle for defining binary *relations* rather than subsets over points in time:

$$\tau(p) \quad = \quad t_1=t_0 \wedge Pt_1$$
$$\tau(\sim\phi) \quad = \quad \neg\tau(\phi)$$
$$\tau(\phi\cap\psi) \quad = \quad \tau(\phi) \wedge \tau(\psi)$$
$$\tau(\phi\wedge\psi) \quad = \quad \exists x([x/t_1]\tau(\phi) \wedge [x/t_0]\tau(\psi))$$
$$\tau(F\phi) \quad = \quad \exists x(t_0<x \wedge [x/t_0]\tau(\phi))$$
$$\tau(\Box\phi) \quad = \quad t_1=t_0 \wedge \forall x([x/t_1]\,\tau(\phi))$$

Again, we can study the model theory of this system. For instance, which classes of relations can be tense-logically defined? Thus, the whole theory of Section 2 might be re-examined, under such a new embedding of tense logic as a fragment of first-order logic.

Remark. By exercising a little more care, the translation may even be taken into the *three-variable* fragment of L_0 with identity. The following instructions use only t_0, t_1, x to send every tense-logical ϕ to a first-order relation $\tau^*(\phi)$ with free variables t_0, t_1 :

$$\begin{array}{lcl}
\tau^*(p) & = & t_1=t_0 \wedge Pt_1 \\
\tau^*(\sim\phi) & = & \neg\tau^*(\phi) \\
\tau^*(\phi\cap\psi) & = & \tau^*(\phi) \wedge \tau^*(\psi) \\
\tau^*(\phi\wedge\psi) & = & \exists x(\exists t_1(\tau^*(\phi) \wedge t_1=x) \wedge \exists t_0(\tau^*(\psi) \wedge t_0=x)) \\
\tau^*(F\phi) & = & \exists x(t_0<x \wedge \exists t_0(\tau^*(\phi) \wedge t_0=x)) \\
\tau^*(\Box\phi) & = & t_1=t_0 \wedge \forall t_1\tau^*(\phi) \ . \quad \Box
\end{array}$$

Remark. A more radical move would be to drop the clauses '$t_1=t_0$' in the above translation, and making evaluation of proposition letters a non-instantaneous action. (Thus, the formalism would become even more analogous to Dynamic Logic.) □

Digression. The above translation also suggests that dynamic point-based tense logic is closer to *interval* tense logic. Perhaps a better general perspective is this. We have a structure $\mathbb{T} = (T, <)$ modelling 'real time'. But, it also generates a notion of 'evaluation time', reflected in the structure $\mathbb{T}^*$ of all *finite sequences* of points from T ; which may be provided with various 'modalities' describing properties of the process of evaluation. Some of the Halpern & Shoham operators (see Section 3.2) would be appropriate here, but so would CHOP (in expressing that some sequence has recognized a conjunction $\phi\wedge\psi$ sequentially). □

Finally, we point out how, at least for the original system with operators $\wedge$, F, $\neg$, there is a classical road towards decidability by embedding the present dynamic system into classical tense logic.

Here is a sketch of a procedure for converting consequence problems in the dynamic system to those in classical tense logic. First, every formula ϕ can be transformed into an equivalent ϕ^*, working inside out, by widening scopes of F-operators as far as possible toward the right (until some negation boundary is hit). For formulas ϕ^* in this special form, it may be shown that their succesful transitions (x,y) are those having some finite sequence $x=t_1<t_2<...<t_n=y$, each of whose points corresponds to some outermost F operator (outside of the scope of any negation), while each t_i reflects some test (t_i, t_i) for a subformula p or $\neg\alpha$, which is satisfied if and only if the latter formulas are classically true at t_i . Using this perspective, various notions of dynamic consequence can be reduced. As a concrete illustration, one can check that

$$F\neg(Fp \wedge q) \wedge r \wedge Fp \models_2 Fr \wedge p \qquad \text{iff}$$
$$F(\neg F(p \wedge q) \wedge r \wedge F(p \wedge a)) \models FF(a \wedge F(r \wedge p)) \quad \text{classically.}$$

Explanation. The reason for this reduction lies in the following translation σ , where the *new* proposition letter a is used to mimick the second argument of the transition relation:

$$\begin{array}{lcl}
\sigma(p) & = & p \wedge a \\
\sigma(F\phi) & = & F\sigma(\phi) \\
\sigma(\phi\wedge\psi) & = & [\sigma(\psi)/a]\sigma(\phi) \\
\sigma(\neg\phi) & = & \neg[\mathit{true}/a]\sigma(\phi) \wedge a
\end{array}$$

By a straightforward induction, then, the following facts may be proven:

(1) $\mathbb{M} \models \sigma(\phi)\,[t]$ iff there exists some $t' \in V(a)$ such that $\mathbb{M}(t') \models \sigma(\phi)\,[t]$;
where $\mathbb{M}(t')$ is the model which differs from $\mathbb{M}$ only in that $V(a)$ is the singleton $\{t'\}$.
(2) $(t, t') \in [[\phi]]$ iff $\mathbb{M}(t') \models \sigma(\phi)\,[t]$.
As a corollary, e.g., the following reduction holds:

$$\phi \models^2 \psi \quad \text{iff} \quad \sigma(\phi) \models F^k(\, a \wedge [\mathit{true}/a]\sigma(\phi)\,)\ ;$$

where k is the number of outermost F-operators in ϕ (not governed by any negation). □

Here, our travels on the border-line between temporal logic and computer science come to an end. We have noticed short-term benefits of this contact, such as useful additions to the traditional logical fund of tools and applications. But, there may also be a long-term effect, as we have seen, in that logic itself might become transformed in a computational mode.

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