Comparing inductive and circular definitions: parameters, complexity and games

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Abstract. Gupta-Belnap-style circular definitions use all real numbers as possible starting points of revision sequences. In that sense they are boldface definitions. We discuss lightface versions of circular definitions and boldface versions of inductive definitions.

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1. Introduction

It has been suggested that circular definitions are the "next step beyond inductive definitions". On a somewhat superficial level, this seems to be corroborated by the fact that the inductively definable sets are exactly the Π^1_1 sets while the circularly definable sets are exactly the Π^1_2 sets.

In this note, we shall discuss the underlying methodology of inductive definitions and circular definitions, and see that they are quite different: while inductive definability is a lightface (parameter-free) concept, circular definitions as presented in (Gupta & Belnap, 1993) are essentially boldface (using arbitrary parameters in the process).

Therefore, comparing inductive definitions and circular definitions is comparing apples and oranges.

To make these notions comparable, we shall define both a lightface version of circular definability (Section 2) and a boldface version of inductive definability (Section 5), and comment on their relationship. A lightface version of circular definitions has been investigated in (Burgess, 1986) under the name of **arithmetic quasi-inductive definitions**, and it gives an interesting counterpoint to Welch's solution of the limit rule problem of revision theory: for boldface circular definitions, the limit rules don't matter for the complexity of the definition concept, whereas for lightface circular definitions, the limit rules play an important rôle (Section 4). Then, in Section 5, we show that we can define a Π^1_2-complete set with our boldface inductive definitions.

These results highlight the point that complexity issues involving non-monotonicity are much more subtle than the superficial Π^1_1 *vs* Π^1_2 dichotomy mentioned above. Still, there could be some way in which inductive and circular definitions form different levels of a hierarchy. We discuss this in terms of game representations in Section 6.

Prerequisites. This paper presupposes some knowledge of mathematical logic, in particular basic notions of model theory and the set theory of the constructible hierarchy. The books (Moschovakis, 1974), (Barwise, 1975), (Moschovakis, 1980) and (Devlin, 1984) can serve as references. In particular, $\mathbf{L}_\alpha$ denotes the αth level of the constructible hierarchy. We use the symbols Even and Odd to denote the sets of even and odd natural numbers, respectively.

We assume familiarity with the motivations and the background of revision theory of truth, but introduce all necessary definitions in Section 2.

2. Circular Definitions

In the Revision Theory of Truth we have a base language $\mathcal{L}$ which we shall fix for reasons of simplicity as the language of first-order arithmetic for this paper. As usual, the goal is to define the extension of an additional predicate $\dot{x}$ by semantic rules. For reasons of simplicity of the exposition, we shall restrict ourselves to work over the ground model $\mathbb{N}$ (the standard model of arithmetic). We shall call a set $h \subseteq \mathbb{N}$ a **real** or **real number** (because of the well-known embedding of the powerset of the natural numbers into the real numbers). Sometimes, we shall also call it a **hypothesis** (on the denotation level, the terms "real" and "hypothesis" will be synonymous in this paper). For the following definitions, let us fix a set $\mathcal{S} \subseteq \wp(\mathbb{N})$ of hypotheses and a hyperarithmetic operator δ on reals, called the **revision operator**.

In the following we shall consider sequences of reals $\vec{s} = \langle s_\alpha \; ; \; \alpha \in \eta \rangle$ where η is either a limit ordinal or the class of all ordinals.

DEFINITION 2.1. *Let $\vec{s}$ be a sequence of reals of length η (η might be the class of all ordinals), and let d be a natural number. We shall say that "$d \in \dot{x}$"* **is $\vec{s}$-stably true** *if there is a β such that for all $\alpha \geq \beta$ we have $d \in s_\alpha$. The set of all d such that "$d \in \dot{x}$" is $\vec{s}$-stably true will be denoted by* $\mathrm{stab}^+(\vec{s})$.

Likewise, we shall say that "$d \in \dot{x}$" **is $\vec{s}$–stably false** *if there is a β such that for all $\alpha \geq \beta$ we have $d \notin s_\alpha$, and denote this set by* $\mathrm{stab}^-(\vec{s})$.

DEFINITION 2.2. *Let $\vec{s}$ be a sequence of reals. Then a real h is said to be $\vec{s}$–***coherent** *(in symbols:* $\mathrm{Coh}(h, \vec{s})$*) if* $\mathrm{stab}^+(\vec{s}) \subseteq h$ *and* $\mathrm{stab}^-(\vec{s}) \subseteq \mathbb{N} \setminus h$.

Note that, since $\mathrm{stab}^+(\vec{s})$ and $\mathrm{stab}^-(\vec{s})$ are disjoint, $\mathrm{stab}^+(\vec{s})$ is always $\vec{s}$-coherent. It is the minimal $\vec{s}$-coherent real.

DEFINITION 2.3. *We shall call a function γ assigning to a sequence of reals $\vec{s}$ of limit length λ a real $\gamma(\vec{s})$ that is $\vec{s}$–coherent a* **bootstrapping policy**.[1] *Classes of bootstrapping policies Γ will be called* **limit rules**.

Let h be a fixed real. Then γ_h defined by

$$\gamma_h(\vec{s}) := \mathrm{stab}^+(\vec{s}) \cup (h \setminus \mathrm{stab}^-(\vec{s}))$$

is an example of a bootstrapping policy. An interesting special case is when $h = \varnothing$, for which $\gamma_\varnothing$ is the liminf function. We let

$$\gamma_{\mathrm{Gupta}}(\vec{s}) := \gamma_{s_0}(\vec{s}).$$

[1] This is Belnap's terminology.

DEFINITION 2.4. *If Γ is a limit rule, then we shall say that $\vec{s}$ is an* **(ordinary) $\langle\delta,\Gamma,\mathcal{S}\rangle$–revision sequence** *if*

(i) $s_0 \in \mathcal{S}$,

(ii) for all ordinals α, we have $s_{\alpha+1} = \delta(s_\alpha)$, and

(iii) there is a $\gamma \in \Gamma$ such that for each limit ordinal λ we have $s_\lambda = \gamma(\vec{s}\upharpoonright\lambda)$.

The most interesting cases of limit rules are the **Belnap rule** Γ_∞ of all bootstrapping policies, the **Gupta rule** $\Gamma_{\text{Gupta}} = \{\gamma_{\text{Gupta}}\}$, and the **fixed Gupta rules** $\Gamma_h := \{\gamma_h\}$. An important special case is the **Herzberger rule**, $\Gamma_\varnothing$.

DEFINITION 2.5. *A real h is called $\langle\delta,\Gamma,\mathcal{S}\rangle$–**recurring** if h occurs cofinally often in some $\langle\delta,\Gamma,\mathcal{S}\rangle$–revision sequence $\vec{s}$ of length* Ord. *The set of $\langle\delta,\Gamma,\mathcal{S}\rangle$–recurring reals will be denoted by* $\mathrm{Rec}_{\delta,\Gamma,\mathcal{S}}$.

PROPOSITION 2.6. *Let $\vec{s} = \langle s_\alpha\,;\,\alpha \in \mathrm{Ord}\rangle$ be a $\langle\delta,\Gamma,\mathcal{S}\rangle$-revision sequence, then there is an $\alpha \in \mathrm{Ord}$ such that for all $\beta \geq \alpha$, s_β is $\langle\delta,\Gamma,\mathcal{S}\rangle$-recurring.*

Proof: Each non-recurring real h has some largest ordinal at which it occurs, say α_h. Since there is only a set of reals, $\alpha := \sup\{\alpha_h\,;\,h$ is a real$\}$ is an ordinal and obviously has the property claimed. q.e.d.

Though almost obvious, Proposition 2.6 gives the crucial motivation for the definition of the semantic relation of the Gupta-Belnap systems: If we revise long enough, there will be only recurring reals left, and so the recurring reals are exactly the ones that survive revision.

With this motivation in mind, we can now define the semantic relation for the Gupta–Belnap systems $\mathbf{S}^*$:

$$\mathbb{N} \models^{\mathbf{S}^*}_{\delta,\Gamma,\mathcal{S}} \varphi(\dot{x}) \iff \forall h \in \mathrm{Rec}_{\delta,\Gamma,\mathcal{S}}(\langle\mathbb{N},h\rangle \models \varphi(\dot{x})).$$

DEFINITION 2.7. *We shall say that a real z is $\langle\Gamma,\mathcal{S}\rangle$**-revision theoretically definable** (in symbols: $z \in \mathrm{RTD}_{\Gamma,\mathcal{S}}$) if there is a hyperarithmetic operator δ such that for all natural numbers n the following holds:*

$$n \in z \iff \mathbb{N} \models^{\mathbf{S}^*}_{\delta,\Gamma,\mathcal{S}} \text{“}n \in \dot{x}\text{”}.$$

Note that the parameter $\mathcal{S}$ in this definition is a new feature in this paper. The original Gupta-Belnap approach of (Gupta & Belnap, 1993) uses the special case $\mathcal{S} = \wp(\mathbb{N})$ and $\Gamma = \Gamma_\infty$.[2] At the other end of the spectrum are Burgess' **arithmetical quasi-inductive definitions** (Burgess, 1986, § 13) where $\mathcal{S} = \{\varnothing\}$.[3] Our parametrized definition allows to explore the area between these two extremes. We call the definability concept connected to $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$ **boldface circular definitions** and the definability concept connected to $\mathrm{RTD}_{\Gamma,\Delta^1_1}$ **lightface circular definitions**. Before discussing complexity issues further, let us briefly discuss a possible philosophical methodology for the lightface notion in the next section.

3. A brief argument for lightface circular definitions

In (Gupta & Belnap, 1993), Gupta and Belnap argue for the philosophical soundness of their circular definitions, corresponding to our $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$. They argue that a good notion of circular definition

> "must satisfy two competing desiderata. On the one hand, it should attribute to the defienda a rich *content*. On the other, the content attributed should not be so rich as to violate the *conservativeness* of definitions. (Gupta & Belnap, 1993, p. 146)"

In traditional revision theory of truth, the intersection over all recurring reals corresponds to a restriction of arbitrariness: the truth-teller stabilizes on true on the starting hypothesis true, and on false on the starting hypothesis false, but neither of these stabilizing patterns alone is enough to evaluate the truth-teller properly. Only the view at all (in this case, two) possible revision sequences gives the proper analysis.

In order to ascertain that the restricted concepts $\mathrm{RTD}_{\Gamma,\mathcal{S}}$ are still reasonable candidates for truth definitions, we have to ensure that by restricting the set of starting hypotheses, we do not lose the power to weed out arbitrariness. We shall not give a full philosophical argument for that, but we want to point out some methodological problems with $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$ and a point of view that is more favourable of the restricted notions of circular definability. (For reasons of simplicity, let's fix $\Gamma = \Gamma_\varnothing$ for this section.)

Impredicativity of the boldface concept of circular definitions. One of the troubles of $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$ is that the underlying concept

[2] We write $\mathrm{RTD}_\Gamma := \mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$.

[3] Properly speaking, this would be "hyperarithmetical quasi-inductive definitions". Burgess allows only arithmetical operators δ where we allow hyperarithmetical operators, yet there is no relevant effect on complexity issues.

of definability is **impredicative** or **self-referential** in an informal sense[4]: If $x \in \mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$, and you want to know whether $n \in x$, then you have to check all revision sequences, including the one starting with x. In other words, the starting hypotheses should come from some set of things defined by a definability notion strictly simpler than the circular definitions we are trying to implement here. One option (the one we chose) is to restrict ourselves to hyperarithmetic starting hypotheses, so we stay within the realm of inductively definable sets.

Plausibility arguments for the lightface concept. Now let us give two scenarios that show how one could argue for the restricted versions of circular definitions of truth.

Scenario I. Imagine (in an approximation of Hintikka-style game theoretic semantics) the circular definition as an interaction of two players, the **revisionist** and the **spoiler**. The players are given a real x and should decide whether it is circularly definable. Between them is a machine that can perform the revision process: given a starting hypothesis h and a revision rule δ, it displays the entire revision sequence to both players.[5] The revisionist claims that x is circularly definable and provides a revision rule δ. The spoiler now has to present a starting hypothesis h, the players feed h and δ in the machine, stare at the display where the machine calculates $\vec{s}$ with $s_0 = h$, and the revisionist wins if there is some α such that $n \in s_\beta$ for all $\beta > \alpha$.

If you take the two players as entities with bounded resources (human beings, or computers), then the sentence

> "the spoiler now has to present a starting hypothesis h"

implies that h must be given in some definable way, for otherwise, the spoiler could not present it.

Scenario II. Going back to the original application of circular definitions, the theory of truth, what is the standard philosophical technique to test a theory of truth? We normally test it against our intuitions. When given a set of sentences that is not too complicated, we have intuitions about what the proper truth values should be, and we can compare them to the outcome of the technical theory of truth that is being tested. A large part of the monograph (Gupta & Belnap, 1993) deals with this technique: different approaches are tested against standard examples. If taken seriously, this yields a natural sciences methodology to philosophical concept analysis that takes empirical inductive

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[4] Of course, in the formal sense of proof theory, even inductive definitions are impredicative. For a discussion of the borders of predicativity, *cf.* (Simpson, 2002).

[5] Note that by Burgess' converse of Theorem 4.3, the machine is essentially performing a task which is Σ_2 over $\mathbf{L}_{\varrho_0}$, so it is not a machine whose action can be defined inductively.

corroboration as measure of quality of a concept analysis.[6] Here, the empirical data are our intuitions of truth formed by experience.

Whether we have any deep intuitions about very complicated sets can be arguably denied. Therefore, if a circular concept of truth is to be tested and rejected, the counterexample (*i.e.*, the example of sentences such that the intuition gives a different analysis than the formal circular definition) must be simple enough to be in the realm of our intuitions. Of course, it depends on your foundational convictions what you consider to be accessible by intuitions.[7]

As mentioned, **Scenario I** and **Scenario II** serve as arguments against $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$ and for some $\mathrm{RTD}_{\Gamma,\mathcal{S}}$ where $\mathcal{S}$ consists only of definable reals of the right kind. They cannot provide more detailed insight into what could be the most fitting class of circular definitions. This would require more detailed work in both analyzing the concepts of iterating definability notions and the mathematical structure of revision.

4. The complexity of circular definitions

4.1. Arbitrary starting hypotheses.

The **Limit Rule Problem of Revision Theory** asked whether the parameter Γ is relevant for the notion of revision theoretic definability.[8] It was solved in the negative in the case $\mathcal{S} = \wp(\mathbb{N})$:

THEOREM 4.1 (Welch 1999). *Let Γ be any limit rule. Then every Π^1_2 subset of* Even *is in $\mathrm{RTD}_\Gamma = \mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$. In particular this is true for Π^1_2-complete subsets of* Even*, so there are examples of extremely complicated sets in RTD_Γ.*

Let us give a brief sketch of the relevant part of the argument and explain why the assumption $\mathcal{S} = \wp(\mathbb{N})$ is necessary here:

We say that a level $\mathbf{L}_\alpha$ of the constructible hierarchy has the Σ_1 **hull property** if each element $x \in \mathbf{L}_\alpha$ is definable by a Σ_1-Skolem

[6] Surely, not all readers will agree with this reduction of philosophical logic to something called **mathematical modelling** in the applied mathematics community.

[7] Note that this is also the methodology underlying the definition of realistically varied revision sequences to deal with the Π^1_3 complexity of the fully varied revision sequences in (Welch, 2001, p. 355).

[8] For a more detailed discussion, *cf.* (Welch, 2001, Theorem 2.1) and (Löwe & Welch, 2001, § 4); for a full proof of Theorem 4.1, *cf.* (Welch, 2003, Theorem 2.1).

term inside $\mathbf{L}_\alpha$. We say that a real $h \subseteq \mathbb{N}$ is a **good code** if $h \cap \text{Even}$ is a code for the Π_1 theory of a level of the constructible hierarchy with the Σ_1 hull property. It is easy to see[9] that a set $p \subseteq \text{Even}$ is Π^1_2 if and only if there is a total recursive function $f_p : \text{Even} \to \text{Even}$ such that

$$x \in p \iff f_p(x) \text{ is an element of every good code.}$$

For each Π^1_2 set $p \subseteq \text{Even}$, we can now construct a hyperarithmetic revision operator δ_p such that the following two conditions hold (regardless of the choice of Γ):

(i) If you start a $\langle \delta_p, \Gamma\rangle$-revision sequence $\vec{s}$ with a hypothesis $s_0 = h$ such that
$$(h \cap \text{Odd}) \cup \{f_p(x)\,;\, x \in h \cap \text{Even}\}$$
is not a good code, then there is some β such that for all $\eta > \beta$, $s_\eta = \mathbb{N}$.

(ii) If you start a $\langle \delta_p, \Gamma\rangle$-revision sequence $\vec{s}$ with a hypothesis $s_0 = h$ such that
$$(h \cap \text{Odd}) \cup \{f_p(x)\,;\, x \in h \cap \text{Even}\}$$
is a good code, then $h \cap \text{Even}$ is recurring in $\vec{s}$ and all recurring reals in $\vec{s}$ are of the form $X \cup (h \cap \text{Even})$ where $X \subseteq \text{Odd}$.

COROLLARY 4.2. *The intersection p^* of all $\langle \delta_p, \Gamma, \wp(\mathbb{N})\rangle$-recurring reals is p.*

Proof. Let $h \subseteq \text{Even}$ be a good code. By (ii), h is recurring in the revision sequence starting with h, so in particular, p^* is a subset of Even.

If $c \subseteq \text{Even}$ is a good code, we can let revision sequences start with $f_p^{-1}[c]$ and get this set as a recurring real. Thus, by (i) and (ii), we have

$$p^* = \bigcap \{h \subseteq \text{Even}\,;\, \{f_p(x)\,;\, x \in h\} \text{ is a good code}\}$$

and so $p^* = p$. q.e.d.

For the simpler rules Γ (*e.g.*, $\Gamma_\varnothing$, Γ_∞, and Γ_h for $h \in \Delta^1_2$), a simple computation[10] shows that reals in $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$ are Π^1_2, so Theorem 4.1 gives an exact characterization of the definable reals.

[9] *Cf.* (Löwe & Welch, 2001, p. 37).
[10] *Cf.* (Löwe & Welch, 2001, Proposition 4.1).

4.2. Hyperarithmetic starting hypotheses and the Herzberger limit rule.

Clearly, this proof uses the fact that we include also revision sequences with very complicated starting hypotheses in our intersection. Burgess has shown in his (Burgess, 1986) that this cannot be avoided for Herzberger sequences. An ordinal ϱ is called **Σ_2-end extendible** if there is some $\sigma > \varrho$ such that $\mathbf{L}_\varrho \prec_{\Sigma_2} \mathbf{L}_\sigma$. Let ϱ_0 be the least Σ_2-end extendible. It is well-known that sets definable over $\mathbf{L}_{\varrho_0}$ are Δ^1_2.

THEOREM 4.3 (Burgess). *Every set of natural numbers in* $\mathrm{RTD}_{\Gamma_\varnothing,\{\varnothing\}}$ *is* Σ_2 *over* $\mathbf{L}_{\varrho_0}$.

Actually, the two concepts mentioned in Theorem 4.3 are even equivalent. For a proof, *cf.* (Burgess, 1986, Theorem 14.1). Using the fact that ϱ_0 is an admissible limit of admissibles, Burgess' theorem is readily extended to yield:

COROLLARY 4.4. *Every set of natural numbers in* $\mathrm{RTD}_{\Gamma_\varnothing,\Delta^1_1}$ *is* Σ_2 *over* $\mathbf{L}_{\varrho_0}$.

Proof. Let $\langle x_n\,;\, n \in \mathbb{N}\rangle$ be an enumeration of the hyperarithmetic hypotheses. This enumeration can be found in $\mathbf{L}_{\varrho_0}$. Recursively partition the natural numbers into infinitely many infinite sets $\langle N_n\,;\, n \in \mathbb{N}\rangle$ with bijections $b_n : N_n \to \mathbb{N}$. Let $h_* := \bigcup_{n\in\mathbb{N}} b^{-1}_{n+1}[x_n]$. Clearly, h_* is an element of $\mathbf{L}_{\varrho_0}$, so by (the proof of) Theorem 4.3, for each hyperarithmetic revision operator the intersection of all reals recurring in the revision sequence starting with h_* is Σ_2 over $\mathbf{L}_{\varrho_0}$.

Let δ be a hyperarithmetic operator, and let $z \in \mathrm{RTD}_{\Gamma_\varnothing,\Delta^1_1}$ be defined by δ. Define

$$\delta^*(h) := \bigcup_{n\in\mathbb{N}} b^{-1}_{n+1}\left[\delta\left(b_{n+1}\left[h \cap N_{n+1}\right]\right)\right] \cup b^{-1}_0 \left[\bigcap_{n\in\mathbb{N}} \delta\left(b_{n+1}\left[h \cap N_{n+1}\right]\right)\right].$$

In other words, for each hyperarithmetic real x_n, we run the δ-revision sequence on N_n and have on N_0 the intersection of the reals on the N_n (for $n \geq 1$).

Call the resulting sequence $\vec{s}$. Let R be the set of reals recurring in $\vec{s}$, and $x := \bigcap R$. It is easy to see that $x \cap N_0 = z$. But since x is Σ_2 over $\mathbf{L}_{\varrho_0}$, z is also Σ_2 over $\mathbf{L}_{\varrho_0}$. q.e.d.

COROLLARY 4.5. *Every set of natural numbers in* $\mathrm{RTD}_{\Gamma_\varnothing,\Delta^1_1}$ *is* Δ^1_2. *In particular, no* Π^1_2*-complete set is in* $\mathrm{RTD}_{\Gamma_\varnothing,\Delta^1_1}$.

4.3. Hyperarithmetic starting hypotheses and the Belnap limit rule.

From the Herzberger rules, we now move to the Belnap rules. We use the fact that Belnap rules can generate arbitrary complexity out of an infinite set of unstable points at a limit to show that Belnap rules can define complicated sets from simple starting hypotheses:

THEOREM 4.6. *Every* Π^1_2 *subset of* Even *is in* $\mathrm{RTD}_{\Gamma_\infty,\Delta^1_1}$.

Proof. We fix a Π^1_2 set $p \subseteq$ Even and intend to define a revision operator δ^*_p such that the intersection of all $\langle\delta^*_p, \Gamma_\infty, \Delta^1_1\rangle$-recurring reals is p. In this construction, we shall use the operator δ_p (*cf.* Section 4.1).

We split up the natural numbers in three parts: Even, $\mathrm{Odd}_1 = \{n\,;\, n \equiv 1 \pmod 4\}$, and $\mathrm{Odd}_3 = \{n\,;\, n \equiv 3 \pmod 4\}$. We shall define a hyperarithmetic operator δ^*_p that simulates the operator δ_p from Section 4.1 on $\mathrm{Even} \cup \mathrm{Odd}_1$ and uses Odd_3 as counting space.

The rough idea is that δ^*_p will work in three steps: It will count to ω on Odd_3 and at each step toggle all entries on $\mathrm{Even} \cup \mathrm{Odd}_1$; after ω steps, the Belnap rule has complete freedom to choose the entries of $\mathrm{Even} \cup \mathrm{Odd}_1$ (so some such revision sequences will pick preimages of good codes); from that point on, we continue with δ_p restricted to $\mathrm{Even} \cup \mathrm{Odd}_1$, and will end up with either p or $\mathbb{N}$ in the end as in the argument for Corollary 4.2.

Let us call $c \subseteq \mathrm{Odd}_3$ a **countdown** if there is some $n \in \mathbb{N}$ such that $4k+3 \in c$ if and only if $k \geq n$. If $c = \{4k+3\,;\, k \geq n\}$ is a countdown, we define $\mathsf{step}(c) := \{4k+3\,;\, k \geq n+1\}$. In addition to the step function, we define a flip function for the rest of $\mathbb{N}$:

$$n \in \mathsf{flip}(h) \iff n \in (\mathbb{N}\backslash h) \cap (\mathrm{Even} \cup \mathrm{Odd}_1).$$

For $h \subseteq \mathbb{N}$ let $h_0 := h \cap \mathrm{Even}$, $h_1 := h \cap \mathrm{Odd}_1$ and $h_2 := h \cap \mathrm{Odd}_3$. If $h \subseteq \mathbb{N}$, we define $\mathsf{squeeze}(h) := \{4k+1\,;\, 2k+1 \in h\} \cup (h \cap \mathrm{Even})$, and if $h \subseteq \mathrm{Odd}_1$, we let $\mathsf{stretch}(h) := \{2k+1\,;\, 4k+1 \in h\}$. We define $\delta^*_p(h)$ as follows:

Case 1. If h_2 is a countdown, then we let

$$\delta^*_p(h) := \mathsf{flip}(h_0 \cup h_1) \cup \mathsf{step}(h_2).$$

Case 2. If h_2 is not a countdown, then

$$\delta^*_p(h) := \mathsf{squeeze}(\delta_p(h_0 \cup \mathsf{stretch}(h_1))).$$

Let us look at an arbitrary hypothesis $h \subseteq \mathbb{N}$ and the possible Belnap revision sequences that start with h. Let $\vec{s}$ be a $\langle\delta^*_p, \Gamma_\infty\rangle$-revision sequence with $s_0 = h$.

Case A. If h_2 is not a countdown, the operator δ^*_p deletes Odd_3 and essentially behaves as δ_p while ignoring the elements of Odd_3.

Subcase A.1. If $f_p[h_0]$ is not a good code, the sequence defaults to $\mathrm{Even} \cup \mathrm{Odd}_1$ at some point, so these are the only recurring reals.

Subcase A.2. If $f_p[h_0]$ is a good code, the recurring reals are $X \cup Y$ where X is a code of the Π_1 theory of an $\mathbf{L}_\alpha$ with the Σ_1 hull property and Y is a subset of Odd_1.

Case B. If h_2 is a countdown, the operator δ^*_p starts counting on Odd_3 to ω while flipping the rest of the integers. After ω steps following this pattern, all of the elements of Odd_3 are stably out, and all other elements are unstable. Thus s_ω can be an arbitrary subset of $\mathrm{Even} \cup \mathrm{Odd}_1$. In particular, $(s_\omega)_2$ is not a countdown, so we're in **Case A**.

In order to prove the theorem, we need to show for all good codes $g \subseteq \mathrm{Even}$, the sets $f_p^{-1}[g]$ occur as recurring sequences: Start to present a revision sequence $\vec{s}$ with $s_0 = h = \mathrm{Odd}_3$. Clearly, h_2 is a countdown, the sequence starts with **Case B**, and with the Belnap limit rule, we can freely choose a subset of $\mathrm{Even} \cup \mathrm{Odd}_1$ as s_ω.[11] Just choose $f_p^{-1}[g]$. By **Subcase A.2**, $f_p^{-1}[g]$ is recurring in $\vec{s}$. q.e.d.

As promised, Corollary 4.5 and Theorem 4.6 yield a counterpoint to Theorem 4.1: if we restrict the starting hypotheses to hyperarithmetic sets, then the choice of the limit rule does matter for the complexity of the definability notion.

This answers the question

Where does the complexity of revision come from?:

It is not the process of revision alone that makes circular definitions Π^1_2 as can be seen in Corollary 4.5, it is either the intersection over all starting hypotheses ("external universal quantifier") or the choice of a powerful (nondeterministic) limit rule.

4.4. A remark on pointer semantics

We would like to mention that there is a remarkable discrepancy between the complexity of revision theory and the examples of applications of revision theory in philosophy. In most cases, we are interested in either finite or very simple infinite sets of sentences, mostly expressible in some propositional language. This is something that could be easily expressed in a very basic logic (propositional logic with pointers), and doesn't need the coding apparatus of arithmetic. This is the approach

[11] Note that this is the only application of the limit rule Γ_∞ needed. Consequently, the proof works for the limit rule Γ_{exc} consisting of all bootstrapping policies that follow the Herzberger policy $\gamma_\varnothing$ with at most one exception.

of **Gaifman's Pointer Semantics**.[12] While very different from the discussions in this paper, it is an interesting endeavour to look at the complexity of these propositional logics involving pointers without the power of arithmetic.[13]

5. Inductive Definitions with parameters

In this section we present a boldface version of the theory of inductive definitions following the pattern from revision theory.

We call a revision operator $\delta : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ **monotone** if for all $h, h^* \in \wp(\mathbb{N})$ with $h \subseteq h^*$, we have $\delta(h) \subseteq \delta(h^*)$. We call it **increasing** if $h \subseteq \delta(h)$.

LEMMA 5.1. *Let δ be an increasing revision operator, and Γ and Γ^* two different limit rules. If h is an arbitrary real, $\vec{s}$ a $\langle \delta, \Gamma, \{h\}\rangle$-revision sequence, and $\vec{s^*}$ a $\langle \delta, \Gamma^*, \{h\}\rangle$-revision sequence, then $\vec{s} = \vec{s^*}$.*

Moreover, for every bootstrapping policy γ and every revision sequence $\vec{s}$ of limit length λ, we have

$$\gamma(\vec{s}) = \bigcup \{s_\alpha \,;\, \alpha < \lambda\}.$$

Proof. It is enough to show that for any sequence $\vec{t}$ of limit length λ, and any bootstrapping policy γ, $\gamma(\vec{t}) = \gamma_\varnothing(\vec{t})$.

Suppose $n \in \gamma(\vec{t})$, then by the definition of bootstrapping policy there must be some $\alpha < \lambda$ such that $n \in t_\alpha$. But because δ is increasing, the number n can never leave the reals of $\vec{t}$ after α, so $n \in \mathrm{stab}^+(\vec{t})$ and consequently, $n \in \gamma_\varnothing(\vec{t})$. q.e.d.

Thus, if we restrict our attention to increasing revision operators, we can ignore the differences of the limit rules.

COROLLARY 5.2. *Let Γ and Γ^* be two limit rules, δ an increasing and monotone operator, $\mathcal{S}$ a set of hypotheses, and $\vec{s}$ be a sequence of reals. Then $\vec{s}$ is a $\langle \delta, \Gamma, \mathcal{S}\rangle$-revision sequence if and only if it is a $\langle \delta, \Gamma^*, \mathcal{S}\rangle$-revision sequence.*

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[12] *Cf.* (Gaifman, 1988; Gaifman, 1992; Cook, ∞).

[13] In this context, it is interesting to remark that finite model theorists have been looking at propositional versions of inductive definitions for a while, and recently, Kreutzer has (without any previous knowledge of the revision theory literature) proposed an alternative definition for so-called "partial fixed-point logics" on infinite structures that turns out to be arithmetic quasi-inductive definitions (Kreutzer, 2002, Definition 3.3).

An inductive definition can now be seen as a special form of a revision sequence using monotone and increasing operators. The following definition is deliberately kept close to the definitions in Section 2:

DEFINITION 5.3. *Let δ be a monotone and increasing operator and $\mathcal{S}$ be a set of hypotheses. We call a sequence of reals $\vec{s}$ a* **(boldface) $\langle\delta,\mathcal{S}\rangle$–inductive definition** *if it is a $\langle\delta,\Gamma,\mathcal{S}\rangle$-revision sequence for some (and hence by Corollary 5.2 all) limit rules Γ.*

Further we can set

$$\mathbb{N} \models^{\mathrm{IND}}_{\delta,\mathcal{S}} \varphi(\dot{x}) \iff \forall h \in \mathrm{Rec}_{\delta,\mathcal{S}}(\langle\mathbb{N},h\rangle \models \varphi(\dot{x})).$$

Notice that the previous definition as well as the next definition make sense since it does not matter which limit rules are applied.

DEFINITION 5.4. *We shall say that a real z is* ***$\mathcal{S}$-inductively definable*** *(in symbols: $\mathrm{IND}_{\mathcal{S}}$) if there is a monotone hyperarithmetic operator δ and a first order formula $\Phi(x,n)$ such that for all natural numbers n we have*

$$n \in z \iff \mathbb{N} \models^{\mathrm{IND}}_{\delta,\mathcal{S}} \Phi(n,\dot{x}).$$

The case $\mathcal{S} = \{\varnothing\}$ gives us some (slightly more liberal) version of the usual (lightface) inductive definitions: usually (*cf.*, *e.g.*, (Moschovakis, 1974)) we would say that h is inductively definable if there is a real h^* that is $\{\varnothing\}$-inductively definable with the formula $\Phi(n,\dot{x}) \doteq n \in \dot{x}$ and there is a recursive subset $p \subseteq \mathbb{N}$ such that $h = p \cap h^*$. However, $\mathrm{IND}_{\{\varnothing\}}$ gives us exactly the expressive power of Kreisel's theory ID_1 of first-order non-iterated positive inductive definitions.[14]

The next proposition shows that there is a Π^1_2-complete real definable with our boldface inductive definitions. The proof can be easily extended in the spirit of Theorem 4.6 to show that all Π^1_2 reals can be boldface inductively defined. Let $\mathcal{S}_{\mathrm{Odd}}$ be the set of all starting hypotheses vanishing on the even bits.

PROPOSITION 5.5. *There is a Π^1_2-complete set in $\mathrm{IND}_{\mathcal{S}_{\mathrm{Odd}}}$.*

Proof. We shall give a boldface inductive definition of the set of all Π_1 sentences φ that are true in all countable well-founded extensional structures. We call this set p in this proof. It is well-known that p is Π^1_2-complete.

[14] The class $\mathrm{IND}_{\{\varnothing\}}$ is roughly equivalent to the first-order closure of Π^1_1 (*i.e.*, $\Sigma^0_\omega(\Pi^1_1)$.)

We shall define a monotone operator δ that builds the *accessible part* of the binary relation coded by the *complement* of the odd part of the starting hypothesis (using the complement ensures that δ is monotone).

We can interpret h as a binary relation *via* its complement as follows:

$$nE_hm \iff 2 \cdot \ulcorner n, m \urcorner + 1 \notin h.$$

We try to exhaust the structure $\langle \mathbb{N}, E_h \rangle$ by an induction and use Even to list those elements of $\mathbb{N}$ that we already reached.

Formally, we define δ as follows:

$$\delta(h) := h \cup \{2m\,;\, \forall n(nE_hm \to 2n \in h)\}.$$

The operator is clearly hyperarithmetic and increasing. Because of the negation in the definition of E_h, the formula defining δ is positive in h, and therefore δ is monotone.

It is obvious that the recurring reals of the operator δ are exactly the reals h such that $h \cap \mathrm{Even}$ codes the accessible part of the relation coded in the above sense by $h \cap \mathrm{Odd}$.

Now let us define a formula Φ as follows. We call a real h **extensional** if E_h is an extensional relation, *i.e.*, if

$$\mathrm{ext}(h) \;\doteqdot\; \forall n, m\,((\forall k(kE_hn \;\leftrightarrow\; kE_hm)) \to n = m)$$

holds. Note that a relation E_h is extensional and well-founded if and only if revision by δ eventually fills in all of the elements of Even. Fix a recursive list $\{\varphi_n\,;\, n \in \mathbb{N}\}$ of all Π_1 sentences of the language of set theory. There is a first order formula $\mathrm{sat}(n, h)$ such that

$$\mathrm{sat}(n, h) \iff \langle \mathbb{N}, E_h \rangle \models \phi_n.$$

Now let

$$\Phi(n, h) \;\;\doteqdot\;\; (\mathrm{ext}(h)\ \&\ \mathrm{Even} \subseteq h) \;\to\; \mathrm{sat}(n, h).$$

By the above remarks, δ and Φ provide a boldface inductive definition of p by

$$n \in p \iff \mathbb{N} \models^{\mathrm{IND}}_{\delta, \mathcal{S}_{\mathrm{Odd}}} \Phi(n, \dot{x}).$$

q.e.d.

6. Game representations

The key concept to connect definability classes to games is the notion of a **game quantifier**. Game quantifiers can be interpreted as generalized quantifiers or as (finite or infinite) strings of the usual quantifiers $\forall$ and $\exists$. For example, the quantifier string $\forall\exists\forall\exists$ can be interpreted as a winning strategy for player II in a two-player game as follows:

Suppose we have a formula $\varphi(x_0, x_1, x_2, x_3, x_4)$ and want to describe the set

$$Q := \{x\,;\ \forall x_0 \exists x_1 \forall x_2 \exists x_3 \varphi(x_0, x_1, x_2, x_3, x)\}$$

in terms of games. Consider the set

$$P_x := \{\langle x_0, x_1, x_2, x_3, x\rangle\,;\ \neg\varphi(x_0, x_1, x_2, x_3, x)\}$$

as the payoff set for a game of two rounds in which player I plays x_0 first, then player II plays x_1, then player I plays x_2, and finally player II answers with x_3. Let's say that player I wins, if the sequence played by players I and II together with x lies in P_x. Then it's obvious that x lies in Q if and only if player II has a winning strategy in the game with payoff P_x.

We shall be looking at game representations through infinite games, and we shall define a game quantifier for our purpose as follows (this is the standard game quantifier from set theory):

First of all, we define a standard infinite game for sets T of reals which we call **the game on** T.

$$\begin{array}{lccccccc} \text{Player I} & f(0) & & f(2) & & f(4) & & \ldots \\ \text{Player II} & & f(1) & & f(3) & & f(5) & \ldots \end{array}$$

Figure 1. The game on T

In this game, player I controls the even numbers and player II controls the odd numbers. One by one, the players successively communicate their decisions about the numbers under their control by playing $f(n) \in \{\mathsf{in}, \mathsf{out}\}$. After infinitely many steps as pictured in Figure 6, the players have generated a function f which can be interpreted as a real h_f by $n \in h_f :\Longleftrightarrow f(n) = \mathsf{in}$. We say that player I wins a run of this game if $h_f \in T$, otherwise we say player II wins.

If $R \subseteq \mathbb{N} \times \wp(\mathbb{N})$, we write

$$R_n := \{h\,;\ \langle n, h\rangle \in R\}$$

for every natural number $n \in \mathbb{N}$. This is a set of reals, so we can play games on R_n. Now we can define $\Game R \in \wp(\mathbb{N})$ to be the real

$$\Game R := \{n\,;\ \text{Player I has a winning strategy in the game on } R_n\}.$$

If Ξ is a complexity class (*e.g.*, Σ^0_1, or Σ^1_1), we can define $\Game\Xi$ by setting

$$\Game\Xi = \{h\,;\, \exists R \in \wp(\mathbb{N} \times \wp(\mathbb{N})) \cap \Xi\, (h = \Game R)\,\}.$$

We call a set $D \subseteq \wp(\mathbb{N})$ of reals a **class of definitions** if there is some concept of definition such that the elements of D are exactly those reals which are definable according to that concept. (Of course, this is an informal definition building on the undefined notion of "concept of definition".) We say that a class of definitions D has a **game representation** by a complexity class Ξ if $D = \Game\Xi$. In words, this means that to each definable real d we can associate a family of games in Ξ such that $n \in d$ if and only if player I wins the nth game.

One of the most famous game examples is the already mentioned representation of inductive sets by open games due to Moschovakis (Moschovakis, 1972):

THEOREM 6.1. *The class* Ind *has a complete game representation by the class* Σ^0_1 *of (lightface) open relations, i.e.,*

$$\mathrm{Ind} = \Game\Sigma^0_1.$$

From Theorem 4.1 we can now derive a game representation for $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})}$ for simple limit rules Γ as an immediate corollary.[15] Since the class Π^1_2 has a complete game representation under reasonable set theoretic assumptions (*viz.* that of Σ^1_1 determinacy)[16], this representation transfers to the class of revision-theoretically definable reals as well.

THEOREM 6.2 ($\mathsf{ZFC} + \mathsf{Det}(\Sigma^1_1)$). *For simple limit rules* Γ *(see footnote 15), we have* $\mathrm{RTD}_{\Gamma,\wp(\mathbb{N})} = \Game\Sigma^1_1$.

Proof. By Theorem 4.1, we have to show that $\Pi^1_2 = \Game\Sigma^1_1$.[17]

"$\supseteq$": Let $R \subseteq \mathbb{N} \times \wp(\mathbb{N})$ be a Σ^1_1 relation. We want to show that the set $\Game R = \{n\,;$ Player I has a winning strategy in the game on $C_n\}$ is Π^1_2. At first glance, the set is Σ^1_3, since the definition gives us

$$n \in \Game R \iff \exists\sigma\forall\tau\,(\langle n, \sigma * \tau\rangle \in R).^{18}$$

[15] Again, as in Section 4.1, $\Gamma = \Gamma_\varnothing$, $\Gamma = \Gamma_\infty$, or $\Gamma = \Gamma_h$ for some $h \in \Delta^1_2$.

[16] We say that Σ^1_1 **determinacy** holds (in symbols: $\mathsf{Det}(\Sigma^1_1)$) if $h \subseteq \mathfrak{P}(\mathbb{N})$ is Σ^1_1, then the game on h has the property that either player I or player II has a winning strategy. For details, *cf.* (Moschovakis, 1980, Section 6A).

[17] In the proof, we shall ignore trivial technicalities: Of course, strategies are not subsets of $\mathbb{N}$ but functions from finite sequences of natural numbers to natural numbers, and they have to be coded in order to be in the scope of a genuine second-order quantifier. Since this is all standard technical work in definability theory, we omit the details here.

[18] Here, $\sigma * \tau$ denotes the unique set that is the outcome of the game if we let the strategy σ play against the strategy τ.

But since R_n is Σ^1_1 and we're assuming determinacy for these games, the fact that I has a winning strategy is equivalent to II having no winning strategy, *i.e.*,

$$n \in \Game R \iff \forall\tau\exists\sigma\,(\langle n, \sigma * \tau\rangle \in R)$$

which gives a Π^1_2 description of $\Game R$.

"$\subseteq$": Fix a Π^1_2 real p. By definition, there is a Σ^1_1 relation $R \subseteq \mathbb{N} \times \wp(\mathbb{N})$ such that $n \in p \iff \forall h\,(\langle n, h\rangle \in R)$. Define a new set R^* by $\langle n, h\rangle \in R^* :\iff \langle n, h \cap \mathrm{Odd}\rangle \in R$. Note that R^* is still Σ^1_1 and that $p = \Game R^*$ since the game on R^*_n doesn't depend on the moves of player I. q.e.d.

Let us discuss the "reasonable set theoretic assumptions" for a moment. The theory $\mathsf{ZFC} + \mathsf{Det}(\Sigma^1_1)$ is stronger than ZFC in several respects: $\mathsf{Det}(\Sigma^1_1)$ cannot be proved in ZFC, and even worse, the consistency of $\mathsf{Det}(\Sigma^1_1)$ cannot be shown either. Determinacy axioms are equivalent to the so-called "Large Cardinal Axioms" or "Strong Axioms of Infinity", this particular determinacy axiom is equivalent to "$0^\#$ exists".[19]

Nonetheless, its assumption in this context is natural. As pointed out by Welch in (Welch, 2001, Remark 4) and discussed further by Löwe and Welch in (Löwe & Welch, 2001, Section 6), the high descriptive complexity of revision-theoretic definitions yields certain dependencies between Revision Theory and aspects of the surrounding set theoretic universe. The answers to some questions about revision-theoretic objects depend strongly on set theory, they have different answers, *e.g.*, in the axiom systems $\mathsf{ZFC} + \mathbf{V}{=}\mathbf{L}$ and $\mathsf{ZFC} + \mathsf{Det}(\Sigma^1_1)$. That means that in order to make definite claims about Revision Theory, we need to make a choice what sort of set theory we want to work in. Since we are interested in game-theoretic characterizations, choosing the game-theoretically smooth $\mathsf{ZFC} + \mathsf{Det}(\Sigma^1_1)$ over the theory $\mathsf{ZFC} + \mathbf{V}{=}\mathbf{L}$ that produces odd Σ^1_1 games seems to be the natural choice here.

The question about the existence of a game representation for lightface circular definitions (or arithmetical quasi-inductive definitions) is interesting for more than just technical reasons:

> Can you find a complexity class Ξ such that $\Game\Xi = \mathrm{RTD}_{\Gamma_\varnothing,\Delta^1_1}$? Furthermore, can you provide a proof that gives informative insight in the structure of revision?

[19] For details on $0^\#$ and its connection to determinacy, *cf.* (Kanamori, 2003, § 9 & § 31).

We believe that this question is the correct question for a game representation of revision theoretic definitions. When we look at the hierarchy of $\langle \Game\Sigma^0_n\,;\, n \in \mathbb{N}\rangle$, the inductive definitions $\mathrm{Ind} = \Game\Sigma^0_1$ form one of the most basic game-theoretically defined definability notions. At the next level, Σ^0_2 games are connected to Σ^1_1-monotone inductive definitions via a theorem of Tanaka's (Tanaka, 1991, Theorems 3.1 & 4.2).

It would be interesting to see where in this hierarchy the revision-theoretic definitions $\mathrm{RTD}_{\Gamma,\Delta^1_1}$ come up, and whether they would have a prominent and interesting place in this hierarchy.

Recent results of Welch indicate that if there is a pointclass connected to circular definitions, it should be a proper subclass of Σ^0_3: Welch proves that the proof-theoretic system Δ^1_3-CA_0+"every arithmetic quasi-inductive definition converges" is strictly weaker than the system Δ^1_3-$\mathsf{CA}_0 + \mathsf{Det}(\Sigma^0_3)$ (Welch, ∞). As arithmetic quasi-inductive definitions are connected to the first Σ_2-end extendible ordinal by Theorem 4.3, the strategies for Σ^0_3 games lie between the first Σ_2-end extendible ordinal and the first ordinal initiating a 2-chain of Σ_2-substructures $\mathbf{L}_{\alpha_0} \prec_{\Sigma_2} \mathbf{L}_{\alpha_1} \prec_{\Sigma_2} \mathbf{L}_{\alpha_2}$. It is open whether there is a natural class $\Sigma^0_2 \subsetneqq \Xi \subsetneqq \Sigma^0_3$ corresponding to circular definitions via game representations.

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