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COMPLETENESS OF THE LAMBEK CALCULUS WITH RESPECT TO RELATIVIZED RELATIONAL SEMANTICS

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COMPLETENESS OF THE LAMBEK CALCULUS WITH RESPECT TO RELATIVIZED RELATIONAL SEMANTICS

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Completeness of the Lambek Calculus with respect to relativized Relational Semantics

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Abstract

Recently M.Szabolcs [8] has shown that many substructural logics including Lambek Calculus L are complete with respect to relativized Relational Semantics. The current paper proves that it is sufficient for L to consider a relativization to the relation "x divides y" in some fixed semigroup G.

1 Introduction

J. Lambek in [5] introduced a formal system L for deriving reduction laws for syntactic types. This calculus was investigated from different semantical points of view (see [1], [2], [3], [4], [6], [7]). In particular, the notion of relational model was introduced in [6].

For a nonempty set D, a function f evaluates types of Lambek Calculus in the set of relations on D so that:

$$f(A \cdot B) = \{(a,b) : \exists c((a,c) \in f(A) \ \& \ (c,b) \in f(B))\},$$

$$f(A \backslash B) = \{(a,b) : \forall c((c,a) \in f(A) \rightarrow (c,b) \in f(B))\},$$

$$f(B/A) = \{(a,b) : \forall c((b,c) \in f(A) \rightarrow (a,c) \in f(B))\},$$

A sequent $X \Rightarrow x$ is true on this model iff $f(X) \subseteq f(x)$.

Lambek Calculus is correct with respect to this semantics ([1], [2]), but is not complete (cf. [4]). There are examples of formulas which are not derivable in L, but are true under any evaluation f. Indeed, for any valuation f we have:

$$f(A) \subseteq f(A \cdot (B \backslash B)) \quad \text{and} \quad f((B \backslash B) \backslash A) \subseteq f(A)$$

whereas neither $A \Rightarrow A \cdot (B \backslash B)$ nor $(B \backslash B) \backslash A \Rightarrow A$ is provable in L.

In [8] the notion of Representable Relational Structure was introduced which is relativization of the notion of relational model to some relation W. In our terms, for a nonempty set D and some relation $W \subseteq D \mathrm{x} D$, a function f is defined such that:

$$\begin{aligned} f(A \cdot B) &= \{(a,b) \in W : \exists c((a,c) \in f(A) \ \& \ (c,b) \in f(B))\}, \\ f(A \backslash B) &= \{(a,b) \in W : \forall c((c,a) \in f(A) \rightarrow (c,b) \in f(B))\}, \\ f(B/A) &= \{(a,b) \in W : \forall c((b,c) \in f(A) \rightarrow (a,c) \in f(B))\}. \end{aligned}$$

And, as it follows from the results of [8], any residuated semigroup (cf.[3]) can be isomorphically embedded into the corresponding Representable Relational Structure with appropriate relation W and thus L is complete with respect to some special W.

The goal of the present paper is to find natural algebraic relation W corresponding to the Lambek Calculus. We prove completeness of the Lambek Calculus with respect to the class of relational models relativized to the following relation W:

For a given semigroup $(G, \cdot)$ and any $x, y \in G$ we define $xWy = (\exists z \in G(x \cdot z = y))$. i.e. W is a divisibility relation in some semigroup G.

Such a choice of W looks very natural from the algebraic viewpoint and has attractive philosophical interpretation. The elements of the semigroup $(G, \cdot)$ may be thought of as information states and the relation W links by arrows any information state x with all information states y such that $y = x \cdot z$ for some $z \in G$, i.e. with all states which are more informative.

The main result of this paper is the following.

There exists semigroup $(G, \cdot)$ such that Lambek Calculus is complete with respect to relational model based on G and relativized to the relation W described above.

Our semigroup $(G, \cdot)$ is obtained by modification from the semigroup of types introduced in [3].

2 Lambek Calculus

Let us describe the formalism of the Lambek Calculus L.

We fix a denumerable set Pr, of constants, called primitive types. The set Tp, of types, is the smallest one, satisfying:

(i) $Pr \subseteq Tp$,

(ii) if $x, y \in Tp$ then $(x \cdot y), (x \backslash y), (y/x) \in Tp$.

The variables x, y, z (resp. p, q, r; resp. X, Y, Z) with or without indices, will range over types (resp. primitive types; resp. finite sequences of types). Any sequent $X \Rightarrow x \quad (X \neq \emptyset)$ will be called a formula.

The system L is given by the following axiom schema and rules:

axiom schema: $x \Rightarrow x$,

rules (all sequents are formulas):

$$(\backslash 2)\ \frac{X \Rightarrow a \qquad Y,\, b,\, Z \Rightarrow c}{Y,\, X,\, (a\backslash b),\, Z \Rightarrow c} \qquad (/2)\ \frac{X \Rightarrow a \qquad Y,\, b,\, Z \Rightarrow c}{Y,\, (b/a),\, X,\, Z \Rightarrow c}$$

$$(\backslash 1)\ \frac{a,\, X \Rightarrow b}{X \Rightarrow a\backslash b} \qquad (/1)\ \frac{X,\, a \Rightarrow b}{X \Rightarrow b/a}$$

$$(\cdot 1)\ \frac{X,\, a,\, b,\, Y \Rightarrow c}{X,\, a \cdot b,,\, Y \Rightarrow c} \qquad (\cdot 2)\ \frac{X \Rightarrow a \qquad Y \Rightarrow b}{X\, Y \Rightarrow a \cdot b}$$

For any sequent by $L \vdash X \Rightarrow x$ we denote derivability of $X \Rightarrow x$ in L.

3 Semigroup $(G, \cdot)$

We turn now to the description of the semigroup $(G, \cdot)$ which plays essential role in our investigation. As the set of elements of $(G, \cdot)$ we take the set of irreducible terms,defined below, which is modification of the corresponding notion from [3].

Definition 1 *The set of terms denoted by Tr is defined by induction:*

1. *each type is a term,*
2. *if t is a term and $x, y \in Tp$, then $(t, x \cdot y, 1)$ and $(t, x \cdot y, 2)$ are terms,*
3. *if t is a term and $x \in Tp$, then $[t, x]$ is a term,*
4. *if t, u are terms then tu is a term.*

As a measure of complexity of term t we choose the number $c(t)$ of all occurrences of primitive types in t.

Below it will be convenient to consider another measure of complexity $m(\cdot)$. The corresponding notion is introduced by the Definition 4.

Definition 2 *By redex we shall call any term of the form*

$$(t, x \cdot y, 1)(t, x \cdot y, 2) \quad or \quad [t, x]x.$$

The term t will be called the reduct of those redexes. We say that t directly reduces to u (and write $t \mapsto u$) if u arises from t by replacing a single occurrence of a redex by the reduct of this redex. We say that t reduces to u (and write $t \mapsto^* u$) if there exist $t_1 \ldots t_n$ such that $t_1 = t$, $t_n = u$ and $t_i \mapsto t_{i+1}$, for all $1 \leq i < n$. A term is said to be irreducible if it contains no redex.

In the similar way as in [3], we can prove that our notion of the reduction of terms satisfies the Church-Rosser condition.

Lemma 1 *If* $t \mapsto^* u_1$ *and* $t \mapsto^* u_2$*, then* $u_1 \mapsto^* w$ *and* $u_2 \mapsto^* w$ *for some* w*.*

Proof. First of all we prove the following claim:
if $t \mapsto u_1$ and $t \mapsto u_2$, then $u_1 \mapsto^* w$ and $u_2 \mapsto^* w$ for some w.

We proceed by induction on the complexity of the term t. According to our definition of the set Tr we have four possibilities:

- $t \in Tp$,
- $t = (t_1, x \cdot y, i)$ provided $i = 1$ or 2,
- $t = [t_1, x]$,
- $t = t't''$.

As the first three cases are easy we only deal with the last one. There are also three possibilities of different occurrences of redexes in the term $t = t't''$:

1. Both the redexes are subterms of t' or t'', or one of them is a subterm of t' while another is a subterm of t''.
2. One of the redexes is a subterm of t' or t'' while another is a subterm of neither t', nor t''.
3. Both the redexes are subterms of neither t', nor t''.

The first case is obvious.

As it follows from our definition of the set Tr, a term $t't''$ contains a redex which is a subterm of neither t', nor t'' if and only if

$$\begin{array}{lllll} t' & = & v_1(v, x \cdot y, 1), & t'' & = (v, x \cdot y, 2)v_2 \\ \textit{or} & & & & \\ t' & = & v_1[v, x], & t'' & = xv_2. \end{array}$$

Let $t' = v_1(v, x \cdot y, 1)$; $t'' = (v, x \cdot y, 2)v_2$.

- If the second redex occurs in v_1 then, moving the first redex, we get v_1vv_2 and, moving the second, we get $v_1'(v, x \cdot y, 1)(v, x \cdot y, 2)v_2$. Moving the redex in v_1, we get $v_1vv_2 \mapsto v_1'vv_2$ and, moving the redex $(v, x \cdot y, 1)(v, x \cdot y, 2)$ in $v_1'(v, x \cdot y, 1)(v, x \cdot y, 2)v_2$, we obtain the same term $v_1'vv_2$.
- If the second redex occurs in v_2 then we proceed in the similar way as in the previous case.
- If the second redex occurs in v then, moving the first redex, we get v_1vv_2 and, moving the second, we get $v_1(v', x \cdot y, 1)(v', x \cdot y, 2)v_2$. Then

$$\begin{array}{rcll} v_1vv_2 & \mapsto & v_1v'v_2 & \text{by moving the redex in } v \text{ and} \\ v_1(v', x \cdot y, 1)(v', x \cdot y, 2)v_2 & \mapsto & v_1v'v_2. & \end{array}$$

Let $t' = v_1[v, x]$; $t'' = xv_2$.

- If the second redex occurs in v_1 then, moving the first redex, we get v_1vv_2 and, moving the second, we get $v_1'[v,x]xv_2$. Then we have

$$\begin{array}{rcll} v_1vv_2 & \mapsto & v_1'vv_2 & \text{by moving the redex in } v_1, \\ v_1'[v,x]xv_2 & \mapsto & v_1'vv_2 & \text{by moving the redex } [v,x]x. \end{array}$$

- If the second redex occurs in v_2 then we proceed in the similar way.
- If the second redex occurs in v then, moving the first redex, we get v_1vv_2 and, moving the second, we get $v_1[v',x]xv_2$. Then we have

$$\begin{array}{rcll} v_1vv_2 & \mapsto & v_1v'v_2 & \text{by moving the redex in } v, \\ v_1[v',x]xv_2 & \mapsto & v_1v'v_2 & \text{by moving the redex } [v',x]x. \end{array}$$

According to our definition of the set Tr, both the redexes are subterms of neither t', nor t'' iff both redexes are equal.This case is obvious.

To complete the proof of the lemma we proceed by standard induction on $c(t)$.

The basis of induction is trivial.The following is induction step.

If $t \mapsto^* u_1$ and $t \mapsto^* u_2$ then there are terms t_1 and t_2 such that $t \mapsto t_1$, $t \mapsto t_2$ and $t_1 \mapsto^* u_1$, $t_2 \mapsto^* u_2$. (We omit the trivial case $t = u_1$ or $t = u_2$, which is straightforward.)

By the first part of the proof we can find v such that $t_1 \mapsto^* v$ and $t_2 \mapsto^* v$. Since $c(t_1) < c(t)$ and $c(t_2) < c(t)$, by induction hypothesis, there are w_1 and w_2 such that

$$\begin{array}{llllll} u_1 & \mapsto^* & w_1, & v & \mapsto^* & w_1 \quad \text{and} \\ u_2 & \mapsto^* & w_2 & v & \mapsto^* & w_2. \end{array}$$

Since $c(v) < c(t)$, we can apply the induction hypothesis again. We get a term w such that

$$w_1 \mapsto^* w \text{ and } w_2 \mapsto^* w.$$

Since $u_1 \mapsto^* w_1$ and $u_2 \mapsto^* w_2$, we have also $u_1 \mapsto^* w$ and $u_2 \mapsto^* w$. □

Corollary 1 *Each term t has a unique irreducible term $ir(t)$ such that $t \mapsto^* ir(t)$.*

It easily follows from Lemma 1 if we notice that each application of reduction decreases the complexity of term.

Definition 3 *We define the semigroup $(G, \cdot)$ in the following way*

1. *G consists of all irreducible terms.*
2. *For $u, v \in G$ we define $u \cdot v = ir(uv)$.*

Associativity of the operation $\cdot$ follows from the Church-Rosser property for reduction. Indeed,

$$u \cdot (v \cdot w) = ir(u\ ir(vw)) = ir(uvw) = ir(ir(uv)w) = (u \cdot v) \cdot w.$$

Definition 4 *In addition to $c(\cdot)$, we define the measure of complexity $m(\cdot)$ on the set of terms by induction:*

1. $m(x) = 0$ *if* $x \in Tp$,
2. $m(s) = m(t) + 1$ *if* $s = (t, a \cdot b, i)$ $(i = 0, 1)$ *or* $s = [t, a]$,
3. $m(s) = m(u) + m(v)$ *if* $s = uv$.

The following two lemmas express properties of the semigroup $(G, \cdot)$ we shall need in the sequel.

Lemma 2 *Let $s, t, u, v \in G$, term u cannot be divided into two subterms u_1, u_2 such that $u = u_1 u_2$, uv– irreducible term.*
If $st \mapsto^* uv$ then $m(u) \leq m(s)$.

Proof. Since s and t are irreducible terms, the reduction process, which leads from st to uv is deterministic. Therefore, we can use induction on the number n of steps in the reduction process.

If $n = 0$ then $st = uv$ and, taking into account that u is subterm of s, we get $m(u) \leq m(s)$.

Let the assertion of Lemma 2 hold for $n \leq k$ and assume that $st \mapsto^* uv$ and the reduction process takes $k + 1$ steps $(k \geq 1)$.

Since s and t are irreducible terms, we have:

$$\begin{array}{lllllll} s & = & s'(w, a \cdot b, 1), & t & = & (w, a \cdot b, 2)t' & \text{or} \\ s & = & s'[w, a], & t & = & at' & \text{for some terms } s', t', w \text{ and types } a, b. \end{array}$$

We get a chain of reductions:

$$st \mapsto^* ir(s'w)t' \mapsto^* uv.$$

The reduction $ir(s'w)t' \mapsto^* uv$ has a number of steps less than $(k + 1)$.

Therefore,

$$m(u) \leq m(ir(s'w)).$$

On the other hand,

$$m(ir(s'w)) \leq m(s'w),$$

because elimination of redex decreases the measure $m(\cdot)$ of term.

So we have

$$m(u) \leq m(s'w) < m(s).\square$$

Lemma 3 *Let u, v_1, v_2 be irreducible terms. If $ir(uv_1) = ir(uv_2)$ then $v_1 = v_2$.*

Proof. We proceed by induction on parameter $m(u)$.

- Let $m(u) = 0$. It means that u is a type and terms uv_1 and uv_2 do not contain redexes. Therefore, we conclude that $uv_1 = uv_2$ and, obviously, $v_1 = v_2$.
- We assume that the assertion of lemma holds for each term t such that $m(t) \leq n$, and for all terms v_1, v_2.

  Let u be any term with $m(u) = n + 1$.

Let w_1, w_2 be arbitrary terms such that $ir(uw_1) = ir(uw_2)$.

- If both terms uw_1 and uw_2 do not contain redexes then $w_1 = w_2$ obviously follows from $ir(uw_1) = ir(uw_2)$.
- If both terms uw_1 and uw_2 contain redexes then, taking into account that u, w_1, w_2 are irreducible terms, we have:

$$\begin{aligned} u &= u'(t, a \cdot b, 1), \\ w_1 &= (t, a \cdot b, 2)w_1', \\ w_2 &= (t, a \cdot b, 2)w_2', \end{aligned}$$

  or

$$\begin{aligned} u &= u'[t, a], \\ w_1 &= aw_1', \\ w_2 &= aw_2'. \end{aligned}$$

  Therefore,

$$ir(uw_1) = ir(u'tw_1') = ir(u'tw_2') = ir(uw_2).$$

  We can apply the induction hypothesis to the equality $ir(u'tw_1') = ir(u'tw_2')$ because

$$m(u't) < m(u'(t, a \cdot b, 1)) = m(u).$$

  We deduce $w_1' = w_2'$ and, therefore, $w_1 = w_2$
- If uw_1 contains a redex and uw_2 does not, then we have:

$$\begin{aligned} u &= u'(t, a \cdot b, 1); \quad w_1 = (t, a \cdot b, 2)w_1' \quad \textit{or} \\ u &= u'[t, a]; \qquad\quad\;\; w_1 = aw_1'. \end{aligned}$$

  Therefore,

$$\begin{aligned} ir(uw_1) &= ir(u'tw_1') = ir(u'(t, a \cdot b, 1)w_2) = ir(uw_2) \quad \textit{or} \\ ir(uw_1) &= ir(u'tw_1') = ir(u'[t, a]w_2) = ir(uw_2). \end{aligned}$$

  Applying the induction hypothesis to the equality

$$ir(u'tw_1') = ir(u'(t, a \cdot b, 1)w_2)$$

  or

$$ir(u'tw_1') = ir(u'[t, a]w_2),$$

  we get

$$ir(tw_1') = (t, a \cdot b, 1)w_2$$

  or

$$ir(tw_1') = ir([t, a]w_1').$$

  We can do that because $m(u') < m(u)$.

But, as it follows from Lemma 2, we obtain

$$m((t, a \cdot b, 1)) \leq m(t)$$

or

$$m([t, a]) \leq m(t)$$

in contradiction with our definition of the measure $m(\cdot)$.

So if we are within conditions of Lemma 3 and uw_1 contains redex then uw_2 contains redex too. □

4 System *ND*

We shall use the system *ND*, introduced in [3], which conservatively extends the Lambek Calculus. Let us describe the formalism of *ND*.

Its formulas are to be of the form $t \epsilon x$, where $t \in Tr, \quad x \in Tp$.

The system *ND* is given by the following axiom schema and rules:

axiom schema: $x \epsilon x$ for all $x \in Tp$,

rules:

$$\frac{t \epsilon x/y \qquad u \epsilon y}{tu \epsilon x} \qquad \frac{t \epsilon x \qquad u \epsilon x\backslash y}{tu \epsilon y}$$

$$\frac{ty \epsilon x}{t \epsilon x/y} \qquad \frac{xt \epsilon y}{t \epsilon x\backslash y}$$

$$\frac{t \epsilon x \cdot y}{(t, x \cdot y, 1) \epsilon x} \qquad \frac{t \epsilon x \cdot y}{(t, x \cdot y, 2) \epsilon y}$$

$$\frac{t \epsilon x \qquad u \epsilon y}{tu \epsilon x \cdot y} \qquad \frac{t \epsilon x \qquad t \mapsto u}{u \epsilon x}$$

For justification of this system the reader is referred to [3].

In fact, the notion of term introduced in the present paper is larger than that from [3], including terms of the form $[t, a]$, where t is a term and a is a type. We have also an extra reduction rule: $[t, a]a \mapsto t$. But we can easily conclude by inspecting the inference rules of *ND* that if $ND \vdash t \epsilon x$ then this derivation does not contain any term $[q, b]$. So the system *ND* with extended notion of reduction introduced in this paper coincides with that from [3]. This remark justifies applicability of results from [3] to the system *ND* with our notion of reduction.

Lemma 4 *For any types x and y if $ND \vdash x \epsilon y$ then $L \vdash x \Rightarrow y$.*

Proof. For the proof the reader is referred to [3] (p.21, lemma 8). □

5 Completeness

Definition 5 *For a given semigroup $(S,\cdot)$ we define binary relation W_S on S such that*

$$\forall a,b \in S \quad aW_Sb = (\exists c \in S : a \cdot c = b)$$

Definition 6 *By a relational model relativized to W_S we mean the couple $(P(W_S), f)$ where f is a map from the set of finite sequences of types of the Lambek Calculus into $P(W_S)$ satisfying the following properties. For any types x, y we have:*

$$\begin{aligned} f(A \cdot B) &= \{(a,b) \in W : \exists c((a,c) \in f(A) \ \& \ (c,b) \in f(B))\}, \\ f(A \backslash B) &= \{(a,b) \in W : \forall c((c,a) \in f(A) \rightarrow (c,b) \in f(B))\}, \\ f(B/A) &= \{(a,b) \in W : \forall c((b,c) \in f(A) \rightarrow (a,c) \in f(B))\}. \end{aligned}$$

We extend this map on the set of finite sequences of syntactic types by putting:

$$f(x_1 \dots x_n) = f(x_1 \cdot \dots \cdot x_n)$$

for all $n \in N$.

For a given semigroup $(S,\cdot)$ we shall denote by $RM(S,f)$ the relational model relativized to W_S. We say that a sequent $X \Rightarrow x$ is true in a model $RM(S,f)$ and write $RM(S,f) \models X \Rightarrow x$ if $f(X) \subseteq f(x)$.

By $\mathcal{RM}$ we denote the set of all models $RM(S,f)$. A formula of the Lambek Calculus is said to be valid with respect to the class of models $\mathcal{RM}$ iff it is true in every model $RM(S,f)$.

Theorem 1 (Completeness of Lambek Calculus with respect to $\mathcal{RM}$) *For any sequent $X \Rightarrow x$ of Lambek Calculus*

$$L \vdash X \Rightarrow x \text{ iff } X \Rightarrow x \text{ is valid with respect to } \mathcal{RM}.$$

Proof. Soundness follows from the Soundness Theorem for Lambek Calculus with respect to Relational Semantics (see [2]).

To prove Completeness we construct a universal model from the class $\mathcal{RM}$ where all underivable sequents fail.

As such a model we take $RM(G,f)$, where G is the semigroup of irreducible terms introduced above and f is a valuation such that

$$\text{for any atomic type } p \quad f(p) = \{(u, ir(uv)) : ND \vdash v \,\epsilon\, p\}$$

and it is canonically extended on arbitrary types.

Lemma 5 *For any $x \in Tp$*

$$f(x) = \{(u, ir(uv)) : ND \vdash v \,\epsilon\, x\}.$$

Proof. We proceed by induction on $c(x)$.

- For atomic types the assertion of Lemma 5 follows from the definition of f.
- Let $x = x_1 \cdot x_2$.
  1. If $(u, ir(uv)) \in f(x_1 \cdot x_2)$ then it means that there exist irreducible terms s and w such that

$$(u, ir(us)) \in f(x_1), \tag{1}$$

$$(ir(us), ir(ir(us)w)) \in f(x_2) \tag{2}$$

$$ir(ir(us)w) = ir(uv) \tag{3}$$

By the Church-Rosser property for reduction and Lemma 3 we have from (3):

$$v = ir(sw).$$

By induction hypothesis, the Church-Rosser property and Lemma 3 we have from (2):

$$ND \vdash w \in x_2.$$

We have from (1):

$$ND \vdash s \in x_1.$$

By rules of ND we get

$$ND \vdash sw \in x_1 \cdot x_2.$$

and

$$ND \vdash ir(sw) \in x_1 \cdot x_2.$$

  2. Let v be a term such that $ND \vdash v \in x_1 \cdot x_2$. Then we have:

$$ND \vdash (v, x_1 \cdot x_2, 1) \in x_1,$$

$$ND \vdash (v, x_1 \cdot x_2, 2) \in x_2.$$

By induction hypothesis

$$(u, ir(u(v, x_1 \cdot x_2, 1))) \in f(x_1) \tag{4}$$

$$(ir(u(v, x_1 \cdot x_2, 1)), ir(ir(u(v, x_1 \cdot x_2, 1))(v, x_1 \cdot x_2, 2))) \in f(x_2) \tag{5}$$

From (4) and (5) we obtain $(u, ir(uv)) \in f(x_1 \cdot x_2)$.

- Let $x = x_1 \backslash x_2$.

1. If $(u, ir(uv)) \in f(x_1 \backslash x_2)$ then it means that for any w such that $(w, u) \in f(x_1)$ we have

$$(w, ir(uv)) \in f(x_2).$$

As a term w we take $[u, x_1]$. Then we get by induction hypothesis

$$([u, x_1], ir([u, x_1]x_1)) \in f(x_1). \tag{6}$$

Therefore, there exists a term r such that

$$([u, x_1], ir([u, x_1]r)) \in f(x_2) \tag{7}$$

$$ir([u, x_1]r) = ir(uv) \tag{8}$$

and

$$ND \vdash r \,\epsilon\, x_2 \tag{9}$$

We can rewrite (8) as

$$ir([u, x_1]r) = ir([u, x_1]x_1 v). \tag{10}$$

By Lemma 3 from (10) we get $r = ir(x_1 v) = x_1 v$ (because x_1 is a type and v is irreducible term).
So we have from (9)

$$ND \vdash x_1 v \,\epsilon\, x_2$$

and, therefore,

$$ND \vdash v \,\epsilon\, x_1 \backslash x_2.$$

2. Let v be a term such that $ND \vdash v \,\epsilon\, x_1 \backslash x_2$. Take any term w such that $(w, u) \in f(x_1)$. Then by induction hypothesis there exists a term r such that:

$$u = ir(wr) \tag{11}$$

$$ND \vdash r \,\epsilon\, x_1 \tag{12}$$

By (11) we get $ir(uv) = ir(ir(wr)v) = ir(wrv)$.
Taking into account (12) and $ND \vdash v \,\epsilon\, x_1 \backslash x_2$ we have

$$ND \vdash rv \,\epsilon\, x_2$$

and, therefore,

$$ND \vdash ir(rv) \,\epsilon\, x_2.$$

By induction hypothesis,

$$(w, ir(wrv)) \in f(x_2)$$

and

$$ir(wrv) = ir(uv).$$

So $(w, ir(uv)) \in f(x_2)$ and, therefore, $(u, ir(uv)) \in f(x_1 \backslash x_2)$.

- Let $x = x_2/x_1$.
  1. If $(u, ir(uv)) \in f(x_2/x_1)$ then it means that for any term w such that $(ir(uv), w) \in f(x_1)$ we have
  $$(uw) \in f(x_2). \tag{13}$$
  As a term w we take $ir(uv)x_1$. Then we get by induction hypothesis
  $$(ir(uv), ir(uv)x_1) \in f(x_1). \tag{14}$$
  By (13) we have
  $$ir(uw) = ir(uvx_1) \tag{15}$$
  and
  $$ND \vdash vx_1 \epsilon x_2.$$
  Therefore, $ND \vdash v \epsilon x_2/x_1$.
  2. Let v be a term such that $ND \vdash v \epsilon x_2/x_1$. Take any term w such that
  $$(ir(uv), w) \in f(x_1).$$
  Then by induction hypothesis there exists a term r such that
  $$w = ir(ir(uv)r), \tag{16}$$
  $$ND \vdash r \epsilon x_1. \tag{17}$$
  By the Church-Rosser property we have
  $$w = ir(ir(uv)r) = ir(u\, ir(vr)).$$
  By (17) we have also
  $$ND \vdash vr \epsilon x_2$$
  and
  $$ND \vdash ir(vr) \epsilon x_2$$
  Therefore, by induction hypothesis $(uw) \in f(x_2)$

□

To conclude the proof of Theorem 1 we notice that if we take a sequent $x \Rightarrow y$ such that $L \nvdash x \Rightarrow y$ then by Lemma 4 $ND \nvdash x \epsilon y$. Then by Lemma 5 we have

$$(x, ir(xx)) \in f(x)$$

but

$$(x, ir(xx)) \notin f(y). \square$$

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