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A Remark on the Maximal Extensions of the Relevant Logic R

Kazimierz Świrydowicz

1.**Preliminaries.C_R-matrices.** Let a set of propositional variables $p, q, r, \ldots$ be given and let F be the set of propositional formulae built up from propositional variables by means of the connectives: $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The Anderson and Belnap logic R with relevant implication (cf. [75]) is defined as the subset of propositional formulae of F which are provable from the set of axiom schemas indicated below, by application of the rule of Modus Ponens (MP; $A, A \rightarrow B/B$) and the Rule of Adjunction ($A, B/A \wedge B$):

$A1.$ $A \rightarrow A$
$A2.$ $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
$A3.$ $A \rightarrow ((A \rightarrow B) \rightarrow B)$
$A4.$ $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
$A5.$ $A \wedge B \rightarrow A$
$A6.$ $A \wedge B \rightarrow B$
$A7.$ $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
$A8.$ $A \rightarrow A \vee B$
$A9.$ $B \rightarrow A \vee B$
$A10.$ $(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$
$A11.$ $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$
$A12.$ $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
$A13.$ $\neg\neg A \rightarrow A$

A *matrix* is a pair $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ where $\mathbf{A}$ is an algebra while $\nabla_{\mathbf{A}}$ is a subset of the domain of $\mathbf{A}$. To the logic R and its extensions we can associate a set of so-called C_R-matrices (cf .W.Dziobiak [83]), their characterization is given by the following

Theorem 1 (W.Dziobiak (83),L.Maximowa (73)) *Let* $\mathbf{A} = \langle A, \rightarrow, \wedge, \vee, \neg \rangle$ *be an algebrą similar to* F *and let* $\nabla_{\mathbf{A}}$ *be a subset of* A. *then the following conditions are equivalent:*
(i) $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ *is a* C_R*-matrix,*
(ii) $\langle A, \wedge, \vee \rangle$ *is a distributive lattice with* $\wedge$ *and* $\vee$ *as its meet and join, respectively and* $\nabla_{\mathbf{A}}$ *is a filter on* A *with the property: for all* $a, b \in A, a \wedge b = a$ *iff* $a \rightarrow b \in \nabla_{\mathbf{A}}$*; and moreover, the following conditions are satisfied for all x,y,z*

of A,

(c1) $(x \to y) \leq (y \to z) \to (x \to z)$,

(c2) $x \leq (x \to y) \to y$,

(c3) $x \to (x \to y) \leq x \to y$,

(c4) $(x \to y) \wedge (x \wedge z) \leq x \to (y \wedge z))$,

(c5) $(x \to y) \wedge (x \wedge z) \leq (x \vee y) \to z$,

(c6) $x \to \neg y \leq y \to \neg x$,

(c7) $\neg\neg x = x$,

here $\leq$ is ordering of the lattice $\langle A, \wedge, \vee \rangle$.

Let us add some additional properties of C_R-matrices:

Lemma 2 (L.Maximowa (73)) *Let $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix and let the relation $\leq$ be defined as follows:*
$x \leq y$ *iff* $x \to y \in \nabla_{\mathbf{A}}$.
Then the relation $\leq$ satisfes the following implications and inequalities:

(i) *if* $x \in \nabla_{\mathbf{A}}$ *then* $x \to y \leq y$

(ii) *if* $x \leq y$ *then* $y \to z \leq x \to z$

(iii) $x \to \neg x \leq \neg x$.

Let us quote moreover a lemma and two propositions proved by W.Dziobiak in [83] which are important for our further investigations. Let $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix and let $X \subseteq A$. By $[X)$ we shall denote the least filter on $\mathbf{A}$ containing X. Moreover, each filter ∇ on $\mathbf{A}$ will be called *normal* iff $\nabla_{\mathbf{A}} \subseteq \nabla$. We have

Lemma 3 (W.Dziobiak (83)) *Let $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix. Then*
(i) $\nabla_{\mathbf{A}} = [\{a \to a : a \in A\})$,
(ii) If $\mathbf{A}$ is generated by elements $a_1, \ldots, a_{n-1}$ then

$$\nabla_{\mathbf{A}} = [\bigwedge_{i<n} (a_i \to a_i)).$$

Theorem 4 (W.Dziobiak (83)) *Let $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix and let $NF(\mathbf{A})$ be the set of normal filters on $\mathbf{A}$. Then the lattices: $\langle NF(\mathbf{A}), \subseteq \rangle$ and $\langle Con(\mathbf{A}), \subseteq \rangle$ are isomorphic.*

Theorem 5 (W.Dziobiak (83)) *The class of all C_R-matrices form a variety.*

2.**RRPg-spaces.** Let $\mathbf{S} = \langle S, RP, g \rangle$ be an ordered 4-tuple where S is a nonempty set, R is a ternary relation on S, P is a nonempty subset of S and $g : S \longrightarrow S$ a function. Then $\mathbf{S}$ is said to be a *$RRPg$-space* i.e RPg-space for the logic R (por.W.Dziobiak [83], L.Maximowa [73],R.Routley,R.K.Meyer[73]) iff for all $x, y, v, z \in S$ the following conditions are satisfied:

(s1) $\exists y \in P : R(y, x, x)$,
(s2) $\exists y \in P : R(x, y, z)$,
(s3) if $R(x, y, z)$ and $R(z, v, t)$ and $x \in P$ then $R(y, v, t)$,
(s4) if $R(x, y, z)$ and $R(u, z, v)$ and $x \in P$ then $R(u, y, v)$,
(s5) if $R(x, y, z)$ and $R(u, z, v)$ and $u \in P$, then $R(x, y, v)$,
(s6) $g(g(x)) = x$,
(s7) if $R(x, y, z)$ then $R(x, g(z), g(y))$,
(s8) $R(x, g(x), x)$,
(s9) $R(x, y, z)$ implies $R(x, y, t)$ and $R(t, y, z)$ for some t,
(s10) $R(x, y, z)$ and $R(z, v, w)$ imply $R(x, v, s)$ and $R(y, s, w)$ for some s ,
(s11) $R(x, y, z)$ and $R(z, v, w)$ imply $R(x, v, s)$ and $R(s, y, w)$ for some s.

Let us define a binary relation $\leq_{\mathbf{S}}$ as follows: $x \leq_{\mathbf{S}} y$ iff $R(s, x, y)$ for some $s \in P$. By (s1) and (s4), the relation $\leq_{\mathbf{S}}$ is both reflexive and transitive. Now let $A(\mathbf{S})$ denote the family of all subsets of S which are closed under $\leq_{\mathbf{S}}$. Put (cf.Maximowa [73]) for all $X, Y \in A(\mathbf{S}) : X \wedge Y = X \cap Y, X \vee Y = X \cup Y, X \to Y = \{s \in S : \forall y, z \in S$ (if $R(s, y, z)$ and $y \in X$ then $z \in Y)\}$ and $\neg X = g^{-1}(S \setminus X)$. Setting $\nabla(\mathbf{S}) = \{X \in A(\mathbf{S}) : P \subseteq X\}$ we have the following

Lemma 6 (L.Maximowa (73)) *If* $\mathbf{S} = \langle S, R, P, g\rangle$ *is a RRPg-space then* $\langle\langle A(\mathbf{S}), \wedge, \vee, \to, \neg\rangle, \nabla(\mathbf{S})\rangle$ *is a* C_R*-matrix.*

It is known (cf. L.Maximowa [73], Routley and Meyer [74]) that from each C_R-matrix we can get a $RRPg$-space; the construction is based on prime filters.

3.**Maximal extensions of the logic** R. Let us start with the following lemma [1]

Lemma 7 *Let* $\mathbf{S} = \langle S, R, P, g\rangle$ *be RRPg-space and let the set P have the least element with respect to the relation* $\leq_{\mathbf{S}}$. *Then* $(S \to X) = \emptyset$ *for each proper subset X of the set S, which is* $\leq_{\mathbf{S}}$*-hereditary.*

Proof: (a) We show first that if P have the least element with respect to the relation $\leq_{\mathbf{S}}$ (we denote this element by 0) then it is true that (*) $\forall x \forall y \exists z : R(x, z, y)$. By (s8) we have: $R(0, g(O), 0)$, i.e. $g(0) \leq_{\mathbf{S}} 0$. By (s2) we get: $(\exists y \in P)\, R(g(x), y, g(x))$, but since $0 \leq_{\mathbf{S}} y$, $R(g(x), 0, g(x))$ (by (s4)). Since $R(0, y, y)$ and $g(0) \leq_{\mathbf{S}} 0$, $R(g(0), y, y)$ (by (s3)). By (s7) and (s8) $R(g(x), 0, g(x))$ implies $R(g(x), x, g(0))$. At last, $R(g(x), x, g(0))$ and $R(g(0), y, y)$ imply (by (s10)) that $\exists t \in S) : R(x, t, y)$.
(b) Now, let $(S \to X) \neq \emptyset$ for some $X \subseteq S$, i.e. let there exists an x_0 such that $x_0 \in (S \to X)$. By the definition of the function $\to$ (see above) the following implication holds for each $y, z \in S$:

[1] The Lemmas 7 and 8 are proved by dr. W.Dziobiak.

if $R(x_0, y, z)$ then $z \in X$.

Since X is a proper subset of the set S, there exists $z_0 \notin X$. Let y_0 be an arbitrary element of S; then the following implication is satisfied::

if $R(x_0, y_0, z_0)$ then $z_0 \in X$.

But $z_0 \notin X$, thus it is not true that $R(x_0, y_0, z_0)$. However, by (a) for $x_0, z_0 \in S$ there exists an y_0 such that $R(x_0, y_0, z_0)$. Thus the set $S \to X$ must be empty, and it finishes the proof. .

This Lemma enables us to prove the following

Lemma 8 *Let $\langle \mathbf{A}, \nabla\mathbf{A} \rangle$ be a C_R-matrix. Let* **A** *be a subdirectly irreducible algebra and let* **A** *have the least element $0_{\mathbf{A}}$. Then for each $x \neq 1_{\mathbf{A}}$ the algebra* **A** *satisfies the equality: $(1_{\mathbf{A}} \to x) = 0_{\mathbf{A}}$.*

Proof: Let $\mathbf{S}_{\mathbf{A}}$ be an $RRPg$-space constructed of prime filters on **A** and let $\mathcal{A}(\mathbf{S}_{\mathbf{A}})$ be the C_R-matrix build up of the $RRPg$-space $\mathbf{S}_{\mathbf{A}}$. It is obvious that the function $f : \mathbf{A} \longrightarrow \mathcal{A}(\mathbf{S}_{\mathbf{A}})$ defined by equality

$f(a) = \{\nabla \in S_{\mathbf{A}} : a \in \nabla\}$

(where $S_{\mathbf{A}}$ is the set of all prime filters on **A**) is an embedding. Of course

$f(x) = S_{\mathbf{A}}$ iff $x = 1_{\mathbf{A}}$, and

$f(x) = \emptyset$ iff $x = 0_{\mathbf{A}}$.

Let $x \neq 1_{\mathbf{A}}$. Then we have $f(x) \neq S_{\mathbf{A}}$. By the previous lemma we have $f(1_{\mathbf{A}} \to x) = f(1_{\mathbf{A}}) \to f(x) = S_{\mathbf{A}} \to f(x) = \emptyset = f(0_{\mathbf{A}})$, i.e. $f(1_{\mathbf{A}} \to x) = f(0_{\mathbf{A}})$. But since f is an embedding, $(1_{\mathbf{A}} \to x) = 0_{\mathbf{A}}$.

Lemma 9 *Let $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix. If all finitely generated subalgebras of* **A** *are Boolean algebras (in the signature $(\wedge, \vee, \neg)$) i.e.the operation $\neg$ satisfies the equality $x \to y = \neg x \vee y$ then* **A** *is a Boolean algebra and $\nabla_{\mathbf{A}} = \{1_{\mathbf{A}}\}$.*

Proof: We prove that for each $t \in \nabla_{\mathbf{A}}, t = x \vee \neg x$ and that the element $x \vee \neg x$ is the unit of **A**. So, we have $x \leq x \vee \neg y$ (because $x \to (x \vee \neg y) \in \nabla_{\mathbf{A}}$), thus by the assumption $x \leq (y \to x)$. However, since $(x \to (y \to z)) \leq (y \to (x \to z))$, $y \leq (x \to x)$ and in consequence $y \leq (x \vee \neg x)$. Thus $x \vee \neg x$ is the unit of **A** and in particular, for each $t \in \nabla_{\mathbf{A}}$,$t \leq (x \vee \neg x)$. Now, let $t \in \nabla_{\mathbf{A}}$. By $x \leq (y \to x)$ we have $t \leq (x \vee \neg x) \to t$ and in consequence $x \vee \neg x \leq t$.

Now we have the fundamental

Proposition 10 *Let $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be an infinite C_R-matrix where* **A** *is not a Boolean algebra. Then the variety $V(\mathcal{A})$ generated by the algebrę* **A** *contains a finitely generated C_R-matrix which is simple and is not a Boolean algebra.*

Proof: Let $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ satisfies the assumptions of Proposition. We will consider finitely generated subalgebras of **A**.

(a) All finitely generated subalgebras of **A** are either Boolean algebras or or infinite algebras without 0 and 1. Then let us consider a finitely generated algebra **B** which is infinite and does not have 0. In such a case in the matrix $\mathcal{B} = \langle \mathbf{B}, \nabla_{\mathbf{B}} \rangle$ we have $\nabla_{\mathbf{B}} = [b)$ for some $b \in B$ (cf. Lemma 3.(ii)). By the Jonsson's theorem (cf.B.Jonsson [72]) there exists a finitely generated simple algebra which is a isomorphic image of **B**. Let us assume that this algebra is the two-valued Boolean algebra **2** (in the signature $(\wedge, \vee, \neg)$). Then by the Rival-Sands theorem (cf.I.Rival, B.Sands [78]) the congruence relation which determines this homomorphic image is compact in the congruence lattice **Con**(**B**) and in consequence the normal filter which is connected with this congruence relation is a principal filter the in Boolean sense (i.e. is of the form $[b_0)_{\mathbf{B}}$ for some $b_0 \in B$). However, since the algebra **B** does not have the least element, there exists an element $b_1 \in B$ such that $b_1 < b_0 < b$, which is impossible because the filter $[b_1)_{\mathbf{B}}$ is a normal filter as well. So we conclude that this simple algebra which is a homomorphic image of the algebra **B** cannot be a Boolean algebra. Moreover, since each homomorphic image of a finitely generated algebra is finitely generated as well, this simple algebra whose existence follows from Jonsson's theorem have 0 and 1 .

(b) All finitely generated subalgebras of the algebra **A** are either Boolean algebras or infinite algebras with 1 and 0. So let us consider an infinite, finitely generated subalgebra **B** of the algebra **A** and let **B** has 1 and 0. By $V(\mathbf{B})$ we denote the variety generated by the algebra **B**. It is obvious that $V(\mathbf{B})$ contains a subdirectly irreducible algebra **C** such that **C** is not a Boolean algebra and that **C** has unit and zero (we denote these elements by $1_{\mathbf{C}}$ and $0_{\mathbf{C}}$, respectively). By Lemma 8, $1_{\mathbf{C}} \rightarrow x = 0_{\mathbf{C}}$ for each $x \neq 1_{\mathbf{C}}$, thus the two-valued Boolean algebra **2** cannot be a homomorphic image of **C**. Let us consider a finitely generated subalgebra **D** of the algebra **C** (assume that $1_{\mathbf{C}}, 0_{\mathbf{C}}$ are between generators of **D** ; now we denote them by $1_{\mathbf{D}}$ and $0_{\mathbf{D}}$, respectively. It is clear that the algebra **D** satisfies the equality $1_{\mathbf{D}} \rightarrow x = 0_{\mathbf{D}}$ for each $x \neq 1_{\mathbf{D}}$, thus the two-valued Boolean algebra cannot be a homomorphic image of the algebra **D** and by the Jonsson's theorem (cf.Jonsson [72]) there exists a simple algebra which is a homomorphic image of **D**.

(c) If all finitely generated subalgebras of the algebra **A** are either Boolean algebras or finite (proper) C_R-algebras then the proof of the existence of a simple algebra in the variety $V(\mathbf{A})$ can be obtained from (b).

The fundamental result of this note follows from the following

Proposition 11 *Let* $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ *be a* C_R*-matrix such that* **A** *is not a Boolean algebra and has the elements 1 and 0. Moreover let* $\nabla_{\mathbf{A}} = [a)$ *where* $a \neq 1$ *and* a *be an atom in the algebra* **A**. *Then* **A** *has a finite subalgebra different from the two-valued Boolean algebra* **2**.

Proof: Let us consider a subalgebra of the algebra **A**, generated by elements

a and 0. It is clear that $a \neq 0$. Thus the elements $\neg a$ and 1 belong to this subalgebra. We show now that the set $\{0, a, \neg a, 1\}$ is closed under operations $\wedge, \vee, \rightarrow, \neg$; this implies that this subalgebra consists only of these four elements .

(a) Let us observe first that $\neg a \neq 1, \neg a \neq 0$, because if $\neg a = 0$ then $\neg\neg a = a = 1$, and if $\neg a = 1$ then $a = 0$.

(b) Of course, $a \wedge \neg a \leq a$. However, a is an atom, thus either $a \wedge \neg a = a$ or $a \wedge \neg a = 0$. This entails that either $a \vee \neg a = \neg a$ or $a \vee \neg a = 1$.

To show that the set $\{0, a, \neg a, 1\}$ is closed under the operation $\rightarrow$ we need some useful inequalities .

(c) $1 \rightarrow y \leq y$. (Of course, if a C_R-matrix $\mathcal{A}$ has a subdirectly irreducible algebra then by Lemma 8 something more is true, but the inequality (c) holds in each case.) To justify it let us observe that $x \leq (x \rightarrow y) \rightarrow y$, thus $1 \leq (1 \rightarrow y) \rightarrow y$ and in consequence $1 \rightarrow y \leq y$.

(d) $0 \rightarrow x = 1$. To prove it we take the inequality $1 \leq (1 \rightarrow y) \rightarrow y$; and by the implication: if $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$ (cf.Lemma 2) and the inequality $0 \leq 1 \rightarrow x$ we get $(1 \rightarrow x) \rightarrow x \leq 0 \rightarrow x$, thus $1 \leq 0 \rightarrow x$.

(e) Besides of joins, meets and "complements" the following elements belong to the C_R-subalgebra of the algebra $\mathbf{A}$ generated by elements $a, \neg a, 1, 0$:
1)$a \rightarrow a$, 2)$\neg a \rightarrow \neg a$, 3) $a \rightarrow \neg a$. 4) $\neg a \rightarrow a$, 5) $0 \rightarrow a$, 6) $0 \rightarrow \neg a$, 7) $a \rightarrow 0$, 8) $\neg a \rightarrow 0$, 9) $a \rightarrow 1$, 10) $\neg a \rightarrow 1$, 11) $1 \rightarrow a$, 12) $1 \rightarrow \neg a$, 13) $0 \rightarrow 1$, 14) $0 \rightarrow 0$, 15) $1 \rightarrow 0$, 16) $1 \rightarrow 1$.
We prove now that each of the elements 1) - 16) is one of the elements $0, 1, a, \neg a$. We have

1) $a \rightarrow a = a$. By Lemma 2, $a \rightarrow a \leq a$. But $[a)$ is the filter of designated elements of the algebra $\mathbf{A}$, thus $a \leq a \rightarrow a$.

2) $\neg a \rightarrow \neg a = a$, because by Theorem 1 ((c6),(c7)) we have : $a \rightarrow a \leq \neg a \rightarrow \neg a \leq a \rightarrow a$

3) $a \rightarrow \neg a = \neg a$. Since $x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z)$, $a \rightarrow (\neg a \rightarrow \neg a) \leq \neg a \rightarrow (a \rightarrow \neg a)$. By 1) and 2), $a \leq \neg a \rightarrow (a \rightarrow \neg a)$, thus $\neg a \leq a \rightarrow \neg a$. For the converse, by Lemma 2 we have $a \rightarrow \neg a \leq \neg a$.

4) Since $\neg a \rightarrow a \leq a$ (cf.Lemma 2), either $\neg a \rightarrow a = a$ or $\neg a \rightarrow a = 0$, because a is an atom.

5) $0 \rightarrow a = 1$ (cf. (d) above).

6) $0 \rightarrow \neg a = 1$, as above. .

7) $a \rightarrow 0 = 0$, for the proof - cf. Lemma 2.

8) $\neg a \rightarrow 0 = 0$. Let us note first that $\neg a \rightarrow 0 \leq 1 \rightarrow a$, and by (c) $\neg a \rightarrow 0 \leq a$, thus either $\neg a \rightarrow 0 = 0$ or $\neg a \rightarrow 0 = a$, because a is an atom. Let us assume that $\neg a \rightarrow 0 = a$. Thus $a \leq 1 \rightarrow a$, i.e. $a \rightarrow (1 \rightarrow a) \in [a)$. However, by

$x \to (y \to z) \leq y \to (x \to z)$ we get $1 \leq (a \to a)$, i.e. $a = 1$. Since it is impossible, $\neg a \to 0 = 0$.

9) $a \to 1 = 1$. To state it observe that $0 \to \neg a \leq a \to 1 \leq 0 \to \neg a$..

10) $\neg a \to 1 = 1$. The proof as for 9)..

11) $1 \to a = 0$. By (c) $1 \to a \leq a$, so either $1 \to a = a$ or $1 \to a = 0$. The case $1 \to a = a$ can be eliminated as in 8).

12) $1 \to \neg a = 0$. We have: $a \to 0 \leq 1 \to \neg a \leq a \to 0$, but by 7) $a \to 0 = 0$.

13) $0 \to 1 = 1$ - by (d).

14) $0 \to 0 = 1$ - by (d).

15) $1 \to 0 = 0$ - by (c).

16) $1 \to 1 = 1$ because $0 \to 0 = 1$.

Thus the set $\{0, a, \neg a, 1\}$ is closed under all basic operations of the algebra **A**, an it finishes the proof.

Let PC denote the set of tautologies of the classical propositional logic. We have now

Theorem 12 *The interwal $[R, PL]$ of the lattice of extensions of the relevant logic R has exactly three co-atoms.*

Proof: Proposition 10 entails that each variety of C_R-matrices which contains a proper C_R-matrix $\mathcal{A}$ (i.e. a matrix, whose algebra **A** is not a Boolean algebra) contains a finitely generated C_R-matrix $\mathcal{B}$, whose algebra **B** is a simple algebra different from the two-valued Boolean algebra **2**, and in consequence whose filter of designated elements is generated by an atom of **B**. By the previous theorem each such a simple algebra **B** has a subalgebra whose uniwerse consists of elements $0, a, \neg a, 1$, where a is the generator of the filter of designated elements of the matrix $\mathcal{B}$.

Let us consider now connections between the element $\neg a$ and the remaining elements. There exist the following three cases :

(a) $\neg a \notin [a)$. It is known that $\neg a \neq 0, \neg a \neq 1, a \wedge \neg a = 0$, $a \vee \neg a = 1$; in consequence the operations $\wedge, \vee, \neg$ in this algebra are defined as in the four-element Boolean algebra. The filter of designated elements consists of the elements $a, 1$. To find the table of values of the function $\to$, we use the proof of the previous proposition. The only doubtful point is the value of $\neg a \to a$ (point (e) 4) of the proof of the previous proposition). It follows from the proof that $\neg a \to a \leq a$, thus either $\neg a \to a = a$ or $\neg a \to a = 0$. If the first possibility holds then we have $a \leq \neg a \to a$, i.e. $a \to (\neg a \to a) \in [a)$. But $x \to (y \to z) \leq y \to (x \to z)$, thus $\neg a \leq (a \to a)$, i.e. $\neg a \leq a$. Since a is an atom and $\neg a \neq 0$, $\neg a = a$. By the assumption it is impossible, thus $\neg a \to a = 0$. In consequence the tables of values for the operation $\to$ in the case in question will have the following form:

$\to$	0	a	$\neg a$	1
0	1	1	1	1
a	0	a	$\neg a$	1
$\neg a$	0	0	a	1
1	0	0	0	1

If $\neg a \in [a)$ then of course we have $a \leq \neg a$, thus $a \wedge \neg a = a, a \vee \neg a = \neg a$. So we need to consider two cases:
(b) $a = \neg a$. Thus the operations $\wedge, \vee$, are defined as in three-element chain and the operations $\neg, \to$ are defined as follows:

x	$\neg x$
a	a
1	0
0	1

$\to$	0	a	1
0	1	1	1
a	0	a	1
1	0	0	1

It is easy to observe, that the algebra of the matrix we characterize now is a Sugihara matrix; this matrix generates the logic, which is the maximal extension of the relevant logic RM (cf. M.Dunn [70]) .

(c) $a \neq \neg a$. Then the C_R algebra in question is defined on the four-element chain where the elements are ordered in the following way: $0 < a < \neg a < 1$; the operations $\wedge, \vee$ are defined as in this chain. Operations $\neg, \to$ are defined as follows:

x	$\neg x$
1	0
a	$\neg a$
$\neg a$	a
0	1

$\to$	0	a	$\neg a$	1
0	1	1	1	1
a	0	a	$\neg a$	1
$\neg a$	0	0	a	1
1	0	0	0	1

The value of $\neg a \to a$ we establish as in the case (a). Of course, in each of these cases the filter of designated elements is the filter $[a)$; in the case (c) consists of three elements.

In this way we have found three C_R matrices which characterize three maximal extensions of the relevant logic R

Let us observe that none of these logics satisfies the relevant principle (cf. N.D.Belnap [60]).

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