The Lambek Calculus: Relational Semantics and the Method of Labelling

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Relational Semantics as it is introduced in [vB88], [vB91], [Bus.86] and [Or89] provides a *dynamic* interpretation for the Lambek Calculus: valuation of formulae is carried out on pairs of points, which can be treated as informational states. Propositions can be understood as programs of transformation of informational states. In [M92] Szabolcs Mikulás presents an algebraic proof of the corresponding Completeness Theorem.
The purpose of this paper is to propose a new, rather simple completeness proof by using the method of labelling. The general theory of labelled deductive systems is proposed by D.Gabbay. In the labelled version of the Lambek Calculus a pair of labels is attached to every formula and the way of labelling reflects corresponding semantical truth conditions. Straighforward parallelism between syntax and semantics enables one to obtain the completeness result.
Below LC is an abbreviation for the Lambek Calculus, and LLC - for the labelled version of the Lambek Calculus.

§1. The Lambek Calculus, Relational Semantics and the method of labelling: the general idea behind the Completeness proof.

The language of the LC is a propositional language with countable set of propositional letters and the binary connectives $\bullet$, $\rightarrow$, $\leftarrow$. We use A,B,C,...,A_1,B_1 etc.

as schematic letters for formulae. Expressions of the form $A_1,...A_n \Rightarrow B$ will be called sequents.

The Lambek Calculus

LC can be presented by the following axiom (0) and inference rules (1)-(6) where A,B,C stand for formulae and X,Y,Z stand for finite sequences of formulae including the empty sequence.

(O) $A \Rightarrow A$

(1) $\dfrac{X \Rightarrow A \quad Y,B,Z \Rightarrow C}{Y,X,A \rightarrow B,Z \Rightarrow C}$ (2) $\dfrac{A,X \Rightarrow B}{X \Rightarrow A \rightarrow B}$

(3) $\dfrac{X \Rightarrow A \quad Y,B,Z \Rightarrow C}{Y,B \leftarrow A,X,Z \Rightarrow C}$ (4) $\dfrac{X,A \Rightarrow B}{X \Rightarrow B \leftarrow A}$

(5) $\dfrac{X,A,B,Y \Rightarrow C}{X,A \bullet B,Y \Rightarrow C}$ (6) $\dfrac{X \Rightarrow A \quad Y \Rightarrow B}{X,Y \Rightarrow A \bullet B}$

The usual notion of derivable sequent is going to be used.
Adding the Cut Rule is conservative. Cut Rule:

$\dfrac{X \Rightarrow A \quad Y,A,Z \Rightarrow C}{Y,X,Z \Rightarrow C}$ _____

Relational Semantics

A Relational Frame is a pair < K,D > where K is a non-empty set, D=KxK
A Relational Model adds a valuation function ψ which can be extended as follows:
let a,b,c, be elements of K
$ab \models p$ iff $ab \in \psi(p)$
$ab \models A \rightarrow B$ iff $\forall c(ca \models A \Rightarrow cb \models B)$
$ab \models B \leftarrow A$ iff $\forall c(bc \models A \Rightarrow ac \models B)$
$ab \models A \bullet B$ iff $\exists c(ac \models A \ \& \ cb \models B)$
Notions of true sequent and of valid sequent can be defined as follows:
A sequent $A_1,...A_n \Rightarrow A$ (where n>0) is true in a Relational Model M iff,
for any $a,b \in K$, if $ab \models A_1 \bullet ... \bullet A_n$, then $ab \models A$
A sequent $\Rightarrow A$ is true in a Relational Model M iff, for any $a \in K$, $aa \models A$
A sequent ϕ is valid in Relational Semantics iff it is true in all Relational models.

If LC is considered just as a sequent calculus then the Completeness Theorem says that every semantically valid sequent ϕ is derivable.
We will call this theorem the Weak Completeness Theorem.

LC can be also treated as a "sequent-axiomatic" system with a notion of an inference of a sequent ϕ from some sequent premises $\phi_1,...,\phi_n$.. In this case the Completeness Theorem says that, if $\phi_1,...,\phi_n$ semantically entail ϕ, then ϕ is derivable from the premises $\phi_1,...,\phi_n$. We will call this theorem the Strong Completeness Theorem.

To obtain the completeness result we are going to use the *Labelled* version of the Lambek Calculus. By a *labelled* formula we mean an expression of the form "a pair: formula" (ab:A). By a *labelled* sequent we mean an expression of the form $ac_1{:}A_1, c_1c_2{:}A_2, .., c_{n-1}b{:}A_n \Rightarrow ab{:}A$

The Labelled version of the Lambek Calculus (LLC)

Let K' be a set of labels, D'=K'×K'; a,b,c∈K';
Let α be a labelled formula and X,Y,Z finite sequences of labelled formulas
LLC is presented by an axiom scheme (O') and rules (1')-(6')

(0') $ab{:}A \Rightarrow ab{:}A$ (for any a,b∈K)

(1') $\dfrac{X \Rightarrow ca{:}A \quad Y, cb{:}B, Z \Rightarrow \alpha}{Y, X, ab{:}A\rightarrow B, Z \Rightarrow \alpha}$ (2') $\dfrac{ca{:}A, X \Rightarrow cb{:}B \quad (*)}{X \Rightarrow ab{:}A\rightarrow B}$

(3') $\dfrac{X \Rightarrow bc{:}A \quad Y, ac{:}B, Z \Rightarrow \alpha}{Y, X, ab{:}B\leftarrow A, Z \Rightarrow \alpha}$ (4') $\dfrac{X, bc{:}A \Rightarrow ac{:}B \quad (*)}{X \Rightarrow ab{:}A\leftarrow B}$

(5') $\dfrac{X, ac{:}A, cb{:}B, Y \Rightarrow \alpha \quad (*)}{X, ab{:}A\bullet B, Y \Rightarrow \alpha}$ (6') $\dfrac{X \Rightarrow ac{:}A \quad Y \Rightarrow cb{:}B}{X, Y \Rightarrow ab{:}A\bullet B}$

Remark. In the rules marked with * c is fresh.

The labelled version of the Cut rule has a form

$$\frac{X \Rightarrow ab{:}A \quad Y, ab{:}A, Z \Rightarrow C}{Y, X, , Z \Rightarrow C} \qquad \text{_____}$$

Canonical sequences of labels
A sequence of labels $<a, c_1, ..., c_n, b>$ is a canonical one iff
(1) if n>0, then each c_i is different from c_j,a,b, where $1 \le i \ne j \le n$

(2) if n=0, then a=b

Lemma 1 establishes the obvious connection between LC and LLC that we need for the completeness proof.

Lemma 1.
If for any canonical sequence of labels $<a,c_1,...,c_n,b>$ a sequent $ac_1:A_1,c_1c_2:A_2,..\ ,c_{n-1}b:A_n \Rightarrow ab:A$ is derivable in LLC, then $A_1,A_2,..,A_n \Rightarrow A$ is derivable in LC.
Proof.
If there is a derivation of $ac_1:A_1,c_1c_2:A_2,..,c_{n-1}b:A_n \Rightarrow ab:A$ in LLC, then dropping labels we immediately have a derivation of $A_1,A_2,..,A_n \Rightarrow A$ in LC.

The general idea of the completeness proof can be expressed as follows:
(a) Suppose $A_1,A_2,..,A_n \Rightarrow A$ (where $n \geq 0$) is not derivable in LC ;
(b) According to Lemma 1 there is a canonical sequence of labels $<a,c_1,...,c_n,b>$ such that the sequence $ac_1:A_1,c_1c_2:A_2,..,c_{n-1}b:A_n \Rightarrow ab:A$ is not derivable in LLC;
(c) We mark with T all left labelled formulas of the labelled sequent and with F -the right formula. The resulting T-F set is

$\Delta_0=\{Tac_1:A_1, Tc_1c_2:A_2,.., Tc_{n-1}b:A_n, Fab:A\}$

(d) To prove the completeness theorem one has to construct a model such that $ac_1|=A_1$, $c_1c_2|=A_2,..,c_{n-1}b|=A_n$, but not $ab|=A$.

§2. *Weak completeness of the Lambek Calculus with respect to Relational Semantics*

The method of construction of the canonical model requires the description of some properties of the T-F set, i.e. a set of labelled formulas marked with T or F. We say that a labelled formula α is a T-member (F-member) of T-F set Δ iff $T\alpha \in \Delta$ ($F\alpha \in \Delta$).

Properties of T-F sets

Definitions.

- The T-F set Δ is consistent iff, for any α, if $T\alpha \in \Delta$, then $F\alpha$ does not belong to Δ.

- TheT-F set is complete iff, for any α, if $T\alpha$ does not belong to Δ, then $F\alpha \in \Delta$.

•The T-F set is saturated (or Henkin complete) iff (i)-(iii) hold:
(i) If Fab:A→B belongs to Δ , then there is c such that Tca:A and Fcb:B belong to Δ.
(ii) If Fab:A← B belongs to Δ, then there is c such that Tbc:A and Fac:B belong to Δ.
(iii) If Tab:A•B belongs to Δ, then there is c such that Tac:A and Tcb:B belong to Δ .

•The T-F set is deductively closed iff if $\gamma_1...\gamma_k$ are T-members of Δ and the sequent $\gamma_1...\gamma_k \Rightarrow \gamma$ is derivable, then Tγ belongs to Δ.

Lemma 2.
If Δ is a consistent, complete, saturated, closed T-F set, then there is a model $M^*=<K^*,D^*,\psi^*>$ such that for any $xy\in D^*$ it the case that

Txy:α∈Δ iff xy|=α

Proof.
Define $M^*=<K^*,D^*,\psi^*>$ as follows:
(a) K^* is a set of labels that occur in Δ
(b) $D^*=K^*\times K^*$
(c) ψ^* is such that xy|=p iff Txy:p∈Δ
The first case: α=A→B
Suppose ab|=A→B but Tab:A→B∉Δ. Since Δ is complete, Fab:A→B∈Δ
and since Δ is saturated, there is c such that TcaA∈Δ but Tcb:B∉Δ. Using the inductive hypothesis we easily get a contradiction.
Suppose Tab:A→B∈Δ and ca|=A. By the inductive hypothesis and the closedness of Δ conclude cb|=B and therefore ab|=A→B.
The second case (α=A←B) is analogous.
The third case: α=A•B.
Suppose ab|=A•B. Then there is c such that ac|=A and cb|=B. By the inductive hypothesis and the closedness of Δ we get Tab:A•B∈Δ.
Suppose Tab:A•B∈Δ. Since Δ is saturated, there is c such that Tac:A∈Δ and Tcb:B∈Δ. By the inductive hypothesis we can easily get ab|=A•B.

Lemmas 1,2 enable one to claim, that if a sequent
$ac_1:A_1,c_1c_2:A_2,..,c_{n-1}b:A_n \Rightarrow ab:A$ is not derivable in LLC and the corresponding T-F set
$\Delta_{0=}\{\cdot T\cdot ac_1:A_1, Tc_1c_2:A_2,.., Tc_{n-1}b:A_n, Fab:A\}$
can be extended to a consistent, complete, saturated, closed T-F set,
then there is a model where $A_1,A_2,..,A_n \Rightarrow A$ is falsified.

To describe how Δ_0 can be extended to a consistent, complete, saturated, closed T-F set we need the following definition:

•The T-F set Δ is <u>deeply consistent</u> (d.c.) iff, if $\gamma_1,...,\gamma_k$ are T-members of Δ and the sequent $\gamma_1,...,\gamma_k$