DIRECTIONS IN GENERALIZED QUANTIFIER THEORY *

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1. Introduction

The study of generalized quantifiers is by now an old and respectable field of logic. With the pioneering work of Mostowski and Lindström in the fifties and sixties, quantifiers became a major tool in the model theory for logics extending first-order logic – many of these being representable as first-order logic with added quantifiers. Apart from general structure theorems on how various general properties are distributed in this class of logics (most famous of these is still Lindström's theorem on the properties which characterize first-order logic), particular logics were examined in detail w.r.t. their model theoretic properties and their comparative expressive power, as well as the behaviour of theories expressed within these logics. Though some of the extensions transcend first-order models (e.g. logics with measure-theoretic or probabilistic quantifiers), this work, which reached its peak in the late seventies and early eighties – witness the book *Model-Theoretic Logics* edited by Barwise and Feferman – is squarely situated within classical model theory, with mathematics as its main source of inspiration and set theory as its basic framework.

In the beginning eighties the study of quantifiers received an impetus from a quite different direction, when it was realized (by Barwise and Cooper, Keenan and Stavi, and others) that determiners and noun phrases, which abound in most natural languages, were interpreted in Montague style semantics by means of generalized quantifiers. This brought parts of the established model theory of quantifiers to bear on linguistics, but it also brought new logical questions about quantifiers, motivated by the linguistic perspective and by particular constraints inherent in natural languages (such as conservativity, or the use of finite or at least 'small' models).

Research on quantifiers stemming directly from the original waves of inspiration (Lindström's theorem and Montague semantics, respectively) has perhaps had its heyday, but the field does not show signs of exhaustion. On the contrary, a lot of work on quantifiers is going on, addressing not only 'classical' issues, but also extending them in new directions, charting new territories and establishing sometimes surprising connections with other fields. One such connection is with finite model theory as used in descriptive complexity theory in computer science. Another is with recent developments in modal logic. Both will be elaborated on below.

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* This paper was inspired by the symposium on Generalized Quantifiers held at the 5th European Summer School in Logic, Language and Information in Lisbon, August 1993. We feel that the work presented there motivates a survey of recent research areas and research problems in the field of generalized quantifiers. The speakers at the symposium, Natasha Alechina, Jaap van der Does, Lauri Hella, Michal Krynicki, Michiel van Lambalgen, Kerkko Luosto, Marcin Mostowski, and Jouko Väänänen, have cooperated and made (oral and/or written) contributions and comments to this research survey which we gratefully acknowledge, and without which it would not have been written. But it is easier to produce a paper with two authors than with ten, and so the present two authors take full responsibility for the final formulation of the paper. In addition, we acknowledge comments received from some further colleagues, in particular, Dorit Ben-Shalom, Makoto Kanazawa, Victor Sanchez and Yde Venema.

Our purpose here is to indicate the direction of some of this recent research. We shall sketch a few major research areas and research problems. Such a condensed survey may be useful both for the practitioner in the field and for the interested logician, and also for the logic student who is looking around for something to set his or her teeth in.[1] At least, that is our intention. Moreover, through this unified presentation, we hope to illustrate, and to encourage the current confluence and interaction of more mathematical and more linguistic research lines in this area. After some background, the material is presented under ten distinct headings. This is for ease of exposition, but it will become clear that much of the work is interconnected and some of it belongs under more than one heading.

2. Background

We assume familiarity with standard notions and terminology from generalized quantifier theory. In particular, the initial concept of a (generalized) *quantifier* is that of a class of structures of a given similarity type, or, equivalently and more informatively (when the similarity type is finite and involves only relations), a functional relation Q associating with each universe M a quantifier Q_M *on* M, i.e., a relation between relations on M, of that type. The type can then be identified with a finite sequence of natural numbers $<n_1,\ldots,n_k>$, and $(M,R_1,\ldots,R_k) \in Q$ can be written

$$Q_M R_1 \ldots R_k \qquad\qquad \text{where } R_i \subseteq M^{n_i}.$$

Q is usually assumed to be closed under isomorphic structures (ISOM); we will note explicitly when this is not required. The *arity* of Q is $max(n_1,\ldots,n_k)$. Let $\mathbf{Q}_n$ be the class of all n-ary quantifiers. $\mathbf{Q}_1$ is the class of *monadic* quantifiers, i.e., quantifiers of type $<1,1,\ldots,1>$; the others are called *polyadic*.[2] A quantifier Q of type $<n_1,\ldots,n_k>$ comes with a variable-binding operator binding n_i distinct variables in formulas ϕ_i, $i = 1,\ldots,k$, respectively, and when a corresponding formation rule and the obvious truth condition is added to first-order logic we obtain the logic $L_{\omega\omega}(Q)$. Similarly for $L_{\omega\omega}(Q_1,\ldots,Q_m)$, or $L_{\omega\omega}(\mathbf{Q})$ where $\mathbf{Q}$ is a class of quantifiers, and also for $L(\mathbf{Q})$ where L is some other familiar given logic.[3] This is a standard concept of generalized quantifier ('Lindström quantifiers'). Various extended or otherwise different concepts will appear below.

3. Logical definability and expressive power[4]

The first thing you want to know about a quantifier is its expressive power, which in model theory is measured in terms of what you can *say* with the corresponding sentences. $L \leq L'$ iff for every L-sentence there is an equivalent L'-sentence (one with the same models), and $L \equiv L'$ iff $L \leq L'$ and $L' \leq L$. In particular, $L(Q) \leq L(Q')$ iff Q is *definable* in $L(Q')$, i.e., there is an $L(Q')$-sentence ϕ with non-logical symbols matching the

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[1]For further reading in the same vein, cf. Krynicki et al. 1993, which contains both surveys and specimens of recent research on quantifiers.

[2]Note that there are other notions of arity for quantifiers in the literature. For example, a quantifier of type $<n_1,\ldots,n_k>$ is sometimes said to be k-ary.

[3]The notations 'EL(Q)' and 'FO(Q)' for $L_{\omega\omega}(Q)$ are also common. Often 'L(Q)' is also used, but here we let L be any logic which uses the same models as first-order logic, and which allows addition of generalized quantifiers in a similarly straightforward way.

[4]Thanks to Lauri Hella for help with this section!

type of Q such that $Q_M R_1 \ldots R_k \Leftrightarrow (M, R_1, \ldots, R_k) \models \phi$. There are thus innumerable (uncountably many!) definability issues for quantifiers. We may classify them as follows:

(1a) Given two quantifiers Q and Q', when is Q definable in $L_{\omega\omega}(Q')$?
(1b) Given a quantifier Q and a class of quantifiers $\mathbf{Q}$, when is Q definable in $L_{\omega\omega}(\mathbf{Q})$?
(1c) When is Q definable in $L(\mathbf{Q})$ for some other logic L?

To prove definability of a particular quantifier Q you provide a definition. To prove undefinability you either proceed indirectly, using some known property of the target logic L which would fail if Q were definable in it, or directly by providing for each L-sentence ϕ a model over which Q and ϕ disagree. Usually, the latter is done for each quantifier depth d: you find two models which are equivalent for L-sentences of quantifier depth at most d but which differ over Q, where L-equivalence up to d is established by means of an Ehrenfeucht-Fraïssé game for the logic L. The general theme in the background here is the characterization of appropriate *semantic invariances* for quantifier languages, either via comparison games or via some structural connection like 'partial isomorphism' or 'bisimulation' (see van Benthem & Bergstra 1993, De Rijke 1993 on this general theme for families of modal logics and process theories).

Undefinability proofs range from straightforward to impossibly difficult. Lots of particular results occur in the literature, but systematic attacks on definability questions are only fairly recent. For a start, Corredor 1986 gave a complete characterization for the mutual definability, relative to first-order logic, of two universe-independent type <1> quantifiers on finite structures; the result involves simple arithmetical properties of such quantifiers.[5] Recently Väänänen proved (Väänänen 1994) that the question of definability between any two monadic quantifiers can be reduced to a relationship between certain boolean algebras associated with them. Thus, problem (1a) has been solved in the case of monadic quantifiers. The next natural step would be to consider quantifiers of type <2>. There seem to be no characterization results for mutual definability here, and such results look difficult to obtain for any extensive class of binary quantifiers. Still, it could be worthwhile to find general criteria for definability and undefinability also in the case of polyadic quantifiers. Undefinability results for particular polyadic quantifiers are multiplying (the next section has some examples), and probably some of the constructions behind these could be turned into such general criteria for undefinability.

Questions (1b) and (1c) have attracted attention especially in the case where $\mathbf{Q} = \mathbf{Q}_n$. Already Lindström 1966 proved that the well ordering quantifier W is not definable in $L_{\omega\omega}(\mathbf{Q}_1)$. Krynicki, Lachlan and Väänänen 1984 gave an example of a ternary quantifier not definable in $L_{\infty\omega}(\mathbf{Q}_2)$. Then, Väänänen 1986 established the existence of arity-based hierarchies of quantifiers: for each n there are $(n+1)$-ary quantifiers not definable in $L_{\omega\omega}(\mathbf{Q}_n)$. Using this result as a starting point, Hella 1989 developed a fairly general method for proving that a given quantifier is not definable in $L_{\infty\omega}(\mathbf{Q}_n)$. The paper Hella and Luosto 1992 contains an up to date survey of results obtained by this method. The existence of natural *hierarchies of quantifiers* is another important general theme, which fits in well with developments elsewhere in linguistic semantics (witness various proposed hierarchies of expressive power in categorial, modal or dynamic logics: cf. van

[5]Here universe-independence means that if $A \subseteq M, M'$ then $Q_M A \Leftrightarrow Q_{M'} A$. This property, often called *extension* (EXT) in the literature, applies straightforwardly to quantifiers of other types as well.

Benthem 1991). The notion of arity-based quantifier hierarchies is tightly connected with another definability issue, namely, the *finite generation problem*:

(2a) Given a logic L, does there exist a finite set **Q** of quantifiers such that L $\equiv$ $L_{\omega\omega}(\mathbf{Q})$? (Equivalently, is there a single quantifier Q such that L $\equiv L_{\omega\omega}(Q)$?)

(2b) Is there a finite set **Q** of quantifiers such that L $\equiv$ L'(**Q**) for some other logic L'?

If a logic L is capable of defining a sequence $Q_1, Q_2, \ldots$ of quantifiers such that for each n, Q_{n+1} is not definable in $L_{\omega\omega}(\mathbf{Q}_n)$, then L cannot be finitely generated (or even be a sublogic of a finitely generated logic). Hence, the known arity hierarchies of quantifiers have led to negative answers to the finite generation problem for many extensions of first-order logic familiar from the literature (cf. Hella and Luosto 1992). The finite generation problem was first raised by Makowsky, Shelah and Stavi 1976 for the Δ-closure of the cardinality logic $L_{\omega\omega}(Q_1)$, where Q_1 is now the quantifier "there exist uncountably many". The answer for this special case is still open. The problem is of particular interest in cases, like $\Delta(L_{\omega\omega}(Q_1))$, where the syntax or semantics of the logic under consideration is given in an indirect way: a representation of the form $L_{\omega\omega}(\mathbf{Q})$ with some finite set **Q** of quantifiers would give the logic a simple finitary syntax and a nice semantics.

The study of definability with generalized quantifiers has received new impetus recently through contacts with computer science. Two influential themes here are *finite model theory* and *complexity of queries*, as we shall demonstrate by a few examples.

Problems similar to those mentioned for the languages $L_{\omega\omega}(\mathbf{Q}_n)$ have recently been studied in the context of finite model theory. Kolaitis and Väänänen 1992 proved that the Härtig quantifier I is not definable in $L^{\omega}_{\infty\omega}(\mathbf{Q})$ for any finite set **Q** of type <1> quantifiers.[6] The main result of Cai, Fürer and Immermann 1989 implies the existence of a binary PTIME computable quantifier (cf. section 6) not definable in $L^{\omega}_{\infty\omega}(\mathbf{Q}_1)$. Extending this result, Hella 1992 proved that for each n there is a PTIME computable (n+1)-ary quantifier which is not definable in $L^{\omega}_{\infty\omega}(\mathbf{Q}_n)$. The result by Kolaitis and Väänänen also points towards other quantifier hierarchies than purely arity-based ones. For example, with monadic quantifiers one may count the number of 1:s in their type. Let $\mathbf{Q}_{(n)}$ be the class of monadic quantifiers with at most n 1:s. By the Kolaitis-Väänänen result, I is not definable in $L^{\omega}_{\infty\omega}(\mathbf{Q}_{(1)})$. Lindström 1992 used a counting argument to show that the classes $\mathbf{Q}_{(n)}$ form a strict hierarchy over finite structures relative to $L_{\omega\omega}$. Extending this, Luosto, Hella, and Väänänen 1994 prove a general hierarchy theorem for a much more refined order between types than by arity (where, for example, (2, 2, 3) > (2, 1, 3)): each type contains a quantifier Q not definable in first-order logic over finite models from any finite number of quantifiers of lower type. Moreover, Q can be made to have various properties, like being monotone or computable in polynomial time. The theorem also yields a *resumption hierarchy*: A type <1> quantifier Q yields a sequence of resumptions $Q^{(n)}$ of type <n>, n = 1,2,…, where $Q^{(n)}$ says of an n-ary relation R what Q says of the

[6] $I_M AB \Leftrightarrow |A| = |B|$. $L^{\omega}_{\infty\omega} = \cup_n L^{n}_{\infty\omega}$, where $L^{n}_{\infty\omega}$ is $L_{\infty\omega}$ except that there are only n variables. Note that $L_{\omega\omega} \leq L^{\omega}_{\infty\omega}$.

set of n-tuples of R $(Q^{(n)}{}_M R \Leftrightarrow Q_{M^n} R)$, and they prove that there exists a Q such that for each n, $Q^{(n+1)}$ is not definable in $L_{\omega\omega}(Q^{(n)})$ over finite structures.[7]

A much studied problem in finite model theory concerns connections with natural complexity classes whose original definition was algorithmic. In particular, there has been a very interesting search for a logical characterization for polynomial-time computability (PTIME). If we consider models with a given ordering then *fixed point logic*, FP, provides such a characterization: a property P of finite ordered structures is computable in polynomial time if and only if P is definable in FP (Immermann 1986, Vardi 1982). However, in the general case where the existence of a linear order is not assumed, this characterization fails badly. Indeed the above mentioned quantifier hierarchy result of Hella 1992 implies that there exists no finite set $\mathbf{Q}$ of quantifiers such that $FP(\mathbf{Q})$ would characterize all PTIME computable properties of finite structures. Thus, *PTIME*, as a logic on finite structures, is not finitely generated even over FP.

This negative result does not rule out the possibility of characterizing PTIME by a uniform sequence of quantifiers: there might exist a single (PTIME computable) quantifier Q such that PTIME $\equiv L_{\omega\omega}(\mathbf{Q})$, where $\mathbf{Q}$ is the set of all relativized resumptions of Q.[8] Note that this is actually a finite generation problem in disguise: if L* is like $L_{\omega\omega}$ except that it has explicit formation rules for relativizing and for quantifying of tuples of variables, then $L_{\omega\omega}(\mathbf{Q}) \equiv L^*(Q)$. Dawar 1993 proves a result that emphasizes the significance of this variation of the finite generation problem for PTIME. Namely, there is a reasonable logic capturing PTIME iff PTIME is finitely generated over L*. In particular, a negative answer to this finite generation problem would immediately yield the separation of PTIME from NPTIME. Regardless of whether the answer in this specific case is negative or positive, it would be desirable to find general tools for proving that a given logic is not finitely generated over L*.

With this we hope to have shown that logical definability of generalized quantifiers is an active research area, with a large supply of particular problems, but also with some more general structure theorems, which relate in interesting ways to problems of computational complexity. In the next section they also turn out relate to linguistic issues.

We end this section with a suggestion (as opposed to a conjecture or a well-defined problem) pointing in a different direction. Some undefinability results for generalized quantifiers on finite structures seem to require sophisticated combinatorial methods. For example, the proof of the result about the Härtig quantifier mentioned above starts with type <1> quantifiers $Q_1,\ldots,Q_m$ and a number k, and constructs two models $\mathbf{M}$ and $\mathbf{M'}$, each with two disjoint unary predicates P and R, so that $\mathbf{M}$ and $\mathbf{M'}$ are equivalent relative to the k move Ehrenfeucht-Fraïssé game for $Q_1,\ldots,Q_m$, and P and R have the same cardinality in $\mathbf{M}$ but not in $\mathbf{M'}$. One way of doing this uses van der Waerden's theorem. It also seems that the construction of the models is impossible without some appeal to Ramsey theory, although some work would be needed to make this statement exact. This leads to the question whether every proof of the undefinability result requires Ramsey theory. Or is there perhaps another proof which avoids the construction of such models?[9]

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[7]In fact, it is not definable in $L_{\omega\omega}(\mathbf{Q})$, where $\mathbf{Q}$ is any finite set of quantifiers of the form $Q_1{}^{(m)}$, where Q_1 is of type <1> and $m \leq n$.

[8]I.e., quantifiers $Q^{(n)\mathrm{rel}}$ of type <1,n> defined by $(Q^{(n)\mathrm{rel}})_M AR \Leftrightarrow Q^{(n)}{}_A\, A^n \cap R$.

[9]It may be noted that the corresponding result for the stronger quantifier *more* ($more_M AB \Leftrightarrow |A| > |B|$), also proved in Kolaitis and Väänänen 1992, only requires a simple use of the pigeon hole principle.

Given the very general nature of generalized quantifiers it may be worth-while to do some 'reverse mathematics' in the field of finite combinatorics and definability questions, and thus to assess the *combinatorial content* of certain results about generalized quantifiers. And corresponding results may be obtainable also for infinite models.[10]

4. Polyadic quantifiers and linguistic definability

The quantifiers appearing as denotations of determiners in natural languages are normally monadic – usually of type <1,1> where the first argument belongs to the noun and the second to the verb phrase, though noun phrases with more than one nouns and hence quantifiers of type <1,…,1> occur as well. A typical example is $most_M AB \Leftrightarrow |A \cap B| > |A - B|$. But sentences often combine such monadic quantifiers into polyadic ones. The canonical example is a sentence with quantified subject and object and a transitive verb, like *most students criticized at least two teachers*. With type <1,1> determiner denotations Q_1 and Q_2 of the subject and object, respectively, this construction results in the type <1,1,2> *iteration* $Q_1 Q_2$, defined by

$$(Q_1 Q_2)_M ABR \quad \Leftrightarrow \quad (Q_1)_M A\{a \in M\colon (Q_2)_M BR_a\} \qquad \text{where } R_a = \{b \in M\colon Rab\}.$$

However, linguists have noted that natural language sometimes operates on monadic quantifiers in other ways. Among the examples cited are *branching* (e.g. for two quantifiers which are upward monotone in their right argument, $Br(Q_1, Q_2)ABR \Leftrightarrow \exists X \subseteq A \exists Y \subseteq B[Q_1 X \ \&\ Q_2 Y \ \&\ X \times Y \subseteq R]$), the *resumptions* mentioned above, and *Ramsey quantifiers* ($Ram^k(Q_1)AR \Leftrightarrow \exists X \subseteq A[Q_1 X \ \&\ X^k - Id \subseteq R]$). Initial papers in this move are Keenan 1987,1992, van Benthem 1989, Sher 1990, while Keenan and Westerståhl 1994 is the most up-to-date survey.

A systematic study of such *polyadic patterns* would thus be quite interesting. First, what are most general schemas of definition for polyadic quantification (cf. Sher 1991, Spaan 1993)? More systematically, one may inquire how much of the existing monadic theory (cf. van Benthem 1986, Westerståhl 1989) can be lifted to the polyadic case. Second, what are typical properties of the lifts, and can some lifts be interestingly characterized in terms of such properties? Third, can we get an illuminating overview of all the lifts that occur in natural languages? Fourth, there are obvious questions of definability in this connection: how far (and in what sense) do these lifts increase expressive power? And finally, what are the prospects for axiomatizability here?

An indication of recent research on points one to four can be found in the survey paper mentioned above – the subject is far from exhausted. Here, we shall just emphasize some general themes. First, the general linguistic challenge in this area involves the extent and precise nature of the principle of semantic *compositionality* (cf. Janssen 1994). Where lies the 'Frege Boundary' of standard iteration of (quantified) components of expressions, and where do we need additional forms of 'logical glue' to construct the sentence meanings that we use? In the limit, one might use full lambda calculus or type theory for this purpose (cf. van Benthem 1991), but intuitively there are strong constraints on what would be admissible in empirical 'linguistic definability' (which will be weaker

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[10]Barwise 1972a and Friedman 1974 studied how much set theory is needed to prove the existence of the Hanf number of second order logic. How much set theory is required to prove that the game quantifier is not definable in $L_{\infty\omega}$?

than logical definability tout court). These general concerns may actually be translated into a variety of specific technical questions of definability in the earlier sense. Here is a quite recent illustration.

There are two natural definability questions for a polyadic lift F, namely, (1) is $F(Q_1,\ldots,Q_k)$ definable in terms of $Q_1,\ldots,Q_k$, and (2) is $F(Q_1,\ldots,Q_k)$ definable in terms of any monadic quantifiers? As logicians we thus ask if $F(Q_1,\ldots,Q_k)$ is definable in $L_{\omega\omega}(Q_1,\ldots,Q_k)$, or in $L_{\omega\omega}(\mathbf{Q}_1)$. (A case might also be made for considering other basis logics than $L_{\omega\omega}$ here.) But it is not clear that logical definability is really what the linguist wants: one might reasonably restrict attention to definitions that are somehow easily expressed in natural languages. Definability as an iteration, or as a boolean combination of iterations, are obvious candidates that have been studied, but common constructions in natural languages would yield richer notions of linguistic definability, tending towards full lambda calculus and type theory in the limit.

This area remains largely unexplored, but note that in so far as linguistic definitions are expressible in the logical language, a logical *un*definability result yields linguistic undefinability as well. In this connection, Hella, Väänänen and Westerståhl 1994 obtain a characterization of precisely when $Br(Q_1,Q_2)$ is definable in $L_{\omega\omega}(Q_1,Q_2)$, and in $L_{\omega\omega}(\mathbf{Q}_1)$, on finite structures, and likewise for $Ram^k(Q_1)$. In general, e.g. when $Q_1 = Q_2$ = *most* or some other 'proportional' quantifier, the branching (and the corresponding Ramsey quantifier) is not definable in $L_{\omega\omega}(\mathbf{Q}_1)$. The case of resumption seems harder, but recently Luosto (1994) succeeded in proving (using van der Waerden's theorem) the conjecture that $most^{(2)}$ is not definable in $L_{\omega\omega}(\mathbf{Q}_1)$ on finite structures.

Another interesting linguistic topic is the semantic and inferential behaviour of quantifier combinations, involving the central notion of *scope*. For instance, how strict is the position-dependence of individual quantifiers in a sequence: when can they be interchanged, etcetera? For a technical illustration, once again connecting linguistics with mathematics, let us mention the Prefix Theorem (Keenan 1993, cf. also Westerståhl 1992), which is formulated for the linear prefixes of iteration, but holds in a suitable form for the 'vertical' prefixes of simple branching as well. One version says that if $Q_1,\ldots,Q_k$ and $Q_1',\ldots,Q_k'$ are positive (do not hold of $\varnothing$) non-trivial type <1> quantifiers on M and $Q_1\cdots Q_k = Q_1'\cdots Q_k'$ on M, then $Q_i' = Q_i'$ for $i = 1,\ldots,k$. Compare this with the Linear Prefix Theorem of Keisler and Walkoe 1973 which says that, for $Q_1,\ldots,Q_k,Q_1',\ldots,Q_k' \in \{\forall,\exists\}$, if $(Q_1,\ldots,Q_k)$ and $(Q_1',\ldots,Q_k')$ are distinct prefixes then there is a sentence with the $(Q_1,\ldots,Q_k)$-prefix which is not equivalent to *any* sentence with the $(Q_1',\ldots,Q_k')$-prefix. Keenan's Prefix Theorem generalizes this to arbitrary quantifiers, but the conclusion is weaker, namely, only that the two sentences $Q_1x_1\ldots Q_kx_kRx_1\ldots x_k$ and $Q_1'x_1\ldots Q_k'x_kRx_1\ldots x_k$ are not equivalent. The proof is surprisingly simple, whereas the Keisler-Walkoe theorem uses Ramsey theory. So the obvious question is this: Can the Keisler-Walkoe result be generalized to other quantifiers than $\forall$ and $\exists$? (See the last paragraphs of Keenan 1993 for some caveats.) For truly polyadic constructions, of course, these issues would require more sophisticated formulations. (Some linguists have even claimed that absence of scope is a hall-mark of the latter: cf. de Mey 1990.)

As a final linguistic issue, we mention the pervasive phenomenon of plural predicates and *collective quantification.* So far we have been talking about so-called 'distributive quantification', i.e., quantification over individuals, but an equally common natural language phenomenon, especially in connection with plurals, is collective quantification, which can be construed as quantification over *sets* of individuals. This suggests a second-order version of generalized quantifiers, or, more generally, a higher-order version. Here too there are natural notions of *lifts* from the first-order (monadic) domain to the higher-order one, and all of the issues we mentioned above for the polyadic lifts have their counterparts. These lifts have been investigated in van der Does 1992a,b, but a general study from the perspective suggested here does not yet exist. And in contrast with the polyadics there is not this time an established model theory to fall back on.

5. Weak semantics and axiomatizability

Going higher-order, as in the previous section, carries connotations of a substantial increase in complexity, loss of nice properties, etc. But in fact this need not be so. The familiar technique of *general models* allows great freedom in the choice of sets, while retaining a *many-sorted* first-order framework. Thus, in the analysis of collective quantification, one avoids set theoretic complexity and can be explicit about which sets to invoke for the treatment of plurals. In other words, one may profitably use a 'weak' semantics tailored to one's needs. Likewise, polyadic patterns may actually involve only the existence of certain restricted families of 'choice functions', rather than full quantification over Skolem functions and the like. This perspective on lowering semantic complexity is discussed at length in van Benthem 1993, where a plea is made for reconsidering many received views on semantic complexity in the semantics of natural languages and computation. Of course, the art will be not just to switch to some broad abstract model class, but to find some informative yet more tractable 'intermediate' modelling. (Many successful examples of this kind may be found in the field of algebraic logic, which has to navigate between standard set-theoretic models at one extreme and trivial Lindenbaum algebras at the other.) Again, this general theme has definite technical counterparts.

Few logics with generalized quantifiers are axiomatizable in the sense of having a recursively enumerable set of standard validities. For example, in $L_{\omega\omega}(most)$ one can characterize the order of the natural numbers, so there is no axiomatization (by Tarski's Theorem). But here again there are moves to 'weak' semantics which sometimes restore axiomatizability. This was originally used by Keisler and others as a technical step in proofs of ordinary completeness and omitting types results for certain quantifiers, but we have already hinted that 'weak' semantics has an independent motivation.

A *weak model* has the form $(\mathbf{M},q)$, where $\mathbf{M}$ is an ordinary model and q is a quantifier on M – in this context ISOM is *not* assumed. If we now take a (generalized) quantifier of type τ to be a class Q of weak models instead, where the right elements are of type τ, satisfaction of the usual formulas of $L_{\omega\omega}(Q)$ in weak models is defined as expected, and the earlier notion of a quantifier is essentially the special case of a class of models of the form $(\mathbf{M},Q_M)$.[11] Such quantifiers are called *ambiguous* in Krynicki and Mostowski 1993, the idea being that the variety of local instances of Q on a given universe may reflect an ambiguity of meaning.[12] An example would be *most*, which on infinite models might need some form of measure to give a reasonable interpretation.

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[11]Equivalently, we can view Q as a functional relation which with each $\mathbf{M}$ associates a class $Q_{\mathbf{M}}$ of quantifiers of type τ on M.

[12]This is then an extended concept of a generalized quantifier. What constraints does it obey? Some proposals are discussed in Krynicki and Mostowski 1993.

Continuing with *most* as our example, there are now at least two ways in which axiomatizability could be obtained. First, we may consider a type <1,1> ambiguous quantifier which allows all local quantifiers with some typical *properties* of *most*, like monotonicity, conservativity, existential import, etc. Now valid reasoning with *most* which only depends on these properties can be axiomatized – Doets 1991 in fact shows (roughly) that universal properties like these are always axiomatizable.

Another route to axiomatization goes via the observation that (the ordinary quantifier) *most* is definable in second-order logic, which already has a familiar complete semantics in terms of general models, i.e., models of the form (**M**, K), where K is a class of relations on M over which the second-order variables vary. M. Mostowski 1993b provides proof systems for second-order definable quantifiers, which are complete with respect to any class of general models (**M**, K) such that K is closed under definability over (**M**, K) by $L_{\omega\omega}(Q)$-formulas. This again turns *most* into an ambiguous quantifier, whose ambiguity now resides in the choice of the class K. Other cases are the Henkin quantifiers (quantifiers with partially ordered prefixes), but of course there is a vast supply of further examples. For example, if Q_1 and Q_2 are second-order definable, so are $Br(Q_1,Q_2)$ and $Ram^k(Q_1)$, and hence they can be axiomatized by the same methods.[13]

Yet another class of ambiguous quantifiers are the so-called *relational quantifiers*; cf. Krynicki (1994a,b). Using models of the form (**M**, R), where R is a binary relation on M, let $Qx\phi(x)$ mean $\exists a\in M\forall b\in M(aRb \rightarrow \phi(b))$. Here the ambiguity lies in the choice of R, and Krynicki gives completeness results for various classes of models of this form. The motives for studying these quantifiers have been mostly technical, but there also seem to exist affinities with modal logic – cf. section 8 for a general exploration of the analogies between quantifiers and modal operators.

A final proposal for capturing *most* (due to Krynicki and Mostowski) is via using a *measure* in the sense of a function μ from M to [0,1] which is finitely additive and homogeneous in that $\mu(\{a\}) = \mu(\{b\})$ for $a,b\in M$. Then on each (**M**, μ) one interprets *most* as usual but using the measure instead of cardinality. Does this give an axiomatizable logic?[14]

6. Computational semantics

Intuitively, quantifiers may be viewed in two different ways. On the one hand, they express static quantitative relationships that may hold between predicates of individuals. But on the other hand, we can also think of them through their associated semantic procedures. This theme has already emerged briefly in section 3, during the discussion of generalized quantifiers and query languages in computer science. It will also occupy most of section 7 on 'dynamic semantics'. At least in the more linguistic tradition, however, the first computational analysis of this kind had to do with so-called 'procedural semantics', thinking of expressions as coming with certain algorithms for their successful evaluation. E.g., van Benthem 1986 introduced 'semantic automata' for generalized quantifiers. In a more mathematical setting, Moschovakis 1991 even proposes to equate evaluation algo-

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[13]Note that the 'second-order version' of an ordinary quantifier Q is sensitive to the choice of defining formula – equivalent formulas may yield versions with different properties. When are they the same?

[14]A similar proposal was made in Colban 1991 using a different notion of measure, which however (as Colban noted) reduced to a universal property of Q and hence was axiomatizable by the result of Doets mentioned earlier.

rithms with Fregean 'senses', as opposed to the earlier-mentioned static 'reference' of quantifier expressions. This general perspective turns out to be firmly related to a long technical tradition.

Quantifiers on finite structures can be coded as sets of words. This is particularly simple in the monadic case. A binary word $w_1 \ldots w_n$ corresponds to an n element structure with one unary predicate: a 1 is in the predicate, a 0 is not. So a type <1> quantifier Q corresponds to a set (language) W_Q of such words. Similarly, an arbitrary monadic quantifier Q corresponds to a set W_Q of k-letter words for a suitable k. Note that since we assume ISOM here these languages will be *permutation-closed*: the order inside a string does not matter.

We can try to classify familiar classes of quantifiers in terms of the computational complexity of the corresponding languages. Or, in the other direction, we can try to find logical characterizations in terms of quantifiers of familiar complexity classes. Here are the known results in the monadic case:

(a) (van Benthem 1987) Let Q be a type <1> quantifier, or a type <1,1> quantifier satisfying conservativity and extension.[15] Then, (i) W_Q is recognized by an acyclic finite automaton iff Q is first-order definable; (ii) W_Q is recognized by a push-down automaton iff Q is definable (as a binary relation between natural numbers) in additive arithmetic.

(b) (M. Mostowski 1993a) Let Q be a monadic quantifier, or a resumption of one. Then W_Q is recognized by a finite automaton iff Q is definable in $L_{\omega\omega}(\mathbf{D})$, where **D** is the class of divisibility quantifiers D_n, $n = 2,3,\ldots$ ($D_nA \Leftrightarrow |A|$ is divisible by n).

For non-monadic quantifiers one can use another representation, with ternary words. For example, a type <2> quantifier corresponds to a set of words

$$w_{11} \ldots w_{1n}\#w_{21} \ldots w_{2n}\# \ \ldots \ \#w_{n1} \ldots w_{nn}$$

(w_{ij} is 0 or 1, # is a separator). This word encodes a binary structure with universe $\{1,\ldots,n\}$ and with the predicate that holds of (i,j) iff $w_{ij} = 1$. For this representation order does seem to matter, and in fact the order in which the elements of the universe are presented to the recognizing device must be a part of the structures in the quantifiers for some of the characterization results in the following list (those marked with *):

quantifier Q	*language* W_Q	*by*
*first-order definable	computable with concurrent parallel random access machine in constant time	Immermann 1989
*definable by transitive closure quantifiers	Turing computable in non-deterministic LOGSPACE	Immermann 1989
*definable in fixed point logic FP	Turing computable in PTIME	Immermann 1986 and Vardi 1982

[15]Equivalently, a relativization of a type <1> quantifier.

definable in first-order logic with extra predicates (Σ^1_1 definable)	Turing computable in non-deterministic PTIME	Fagin 1974
definable in first-order logic with extra predicates and sorts	partially Turing computable (recursively enumerable)	(Trakhtenbrot 1950)
definable and co-definable in first-order logic with extra predicates and sorts	Turing computable (recursive)	(Trakhtenbrot 1950)[16]

We noted in section 3 that the use of ordered structures is not really motivated from a logical point of view, and that it is an interesting research problem to try to find corresponding characterizations without the order. (Nevertheless, from a linguistical or psychological point of view, the idea of some arbitrary but inevitable semantic 'surveying trajectory' is not without its attractions.) Going in the other direction, there are lots of classes of quantifiers that one might want to characterize computationally, for example,

(i) classes of type <1> quantifiers determined by some set of natural numbers, such as the class of P_n, $n = 2,3,\ldots$, where $P_nA \iff |A|$ is a power of n.

(ii) Henkin quantifiers

(iii) more generally, for some class of upward monotone quantifiers, its closure under branching.[17]

The connection between definability of a quantifier and the computational complexity of the corresponding language thus seems to deserve more systematic investigation. For instance, the analysis can be extended to countably infinite models. A quantifier consisting of countable models corresponds to a 'language' which is a set of infinite binary words, i.e., of reals. The known results include

quantifier Q	*'language' W_Q*	*by*
definable in the countable admissible fragment L_A	A-recursive	Barwise 1969
definable in $L_{\omega_1\omega}$	hyperarithmetic in a real	Barwise 1972b

This can even be continued to uncountable models by considering set theoretic criteria on the 'language'.

7. Quantifiers in dynamic settings

Computational aspects of natural language have inspired a more general move towards what is currently called 'dynamic semantics', where one emphasizes, instead of traditonal

[16]The idea for this and the previous result can be found in Trakhtenbrot 1950, though the specific conclusion about definability with extra predicates and sorts is due to Väänänen and was communicated to us by him.

[17]Note that characterizing a class **Q** of quantifiers is not trivially the same as characterizing the quantifiers definable in, say, $L_{\omega\omega}(\mathbf{Q})$.

'truth conditions', the *changes in* human *information states* brought about by processing/understanding various linguistic expressions. Dynamic features have been investigated most extensively in connection with anaphora (cf. Kamp 1984, Heim 1983, Groenendijk and Stokhof 1991), defaults (Veltman 1991) and presuppositions (Beaver 1994): cf. van Benthem, Muskens and Visser 1994 for an extensive survey of systems, results and issues. This line is actually one instance of a more general move in cognitive science: cf. the theory of informational updates and belief revision in Gärdenfors 1988. In this light, it seems natural to ask which dynamic aspects (if any) naturally occur with generalized quantifiers. There are at least three levels where one can look for these.

First and most traditionally, the dynamics of anaphora naturally involve quantified NPs, and hence, a marriage between dynamic semantics for anaphora and generalized quantifier logics seems an obvious project. There have been several proposals for achieving this, starting with Chierchia 1991, and continuing through van den Berg 1991, van Eyck and de Vries 1992 to Fernando 1993, Kanazawa 1993a,b. What these papers achieve is dynamic semantics for unary generalized quantifiers in terms of testing some fixed assignment (in most cases) and changing the current assignment of evaluation (in some). A natural question in this context is what becomes of the earlier denotational constraints in the static case, such as Conservativity or various forms of Monotonicity. Prima facie, these can now fail (this was even the original point of the famous 'donkey sentences' from Geach 1968, which have inspired so much research in this area). Kanazawa 1993a shows how to turn notions and results from the original static quantifier theory into constraints on reasonable dynamic extensions. E.g., dynamic monotonicity serves to prune the possible range of dynamic quantifiers corresponding to a given static one. Moreover, he finds that Conservativity can actually occur in two plausible dynamic versions, which still collapse in the static case. These observations are instances of a general 'transfer theory' that one would like to see. In the same realm, Kanazawa 1993b characterizes, amongst others, those dynamic quantifiers for which the so-called 'proportion problem' cannot arise.

But quantifiers can also interact with other dynamic phenomena in compositional interpretation. One example are the polyadic constructions of section 4. Intuitively, there are various processing strategies for quantified components in a sentence, some more 'sequential' (leading to standard Fregean iterations) and others more 'parallel', leading to branching or other genuinely polyadic constructs. So far, there has been no systematic dynamic analysis of polyadics with any explanatory value, which might provide a deeper underpinning for the mathematical possibilities that we found in the above. Another form of dynamics in interpretation is the changing of contexts or 'local domains of quantification' during the evaluation of sentences. What a noun phrase like "all girls" refers to may be subject to continuous contextual modification. This phenomenon was signalled in the logical tradition in Westerståhl 1984, and some proposals for a corresponding dynamic logic of 'domain change' may be found in van Benthem and Cepparello 1994. (Note, for instance, how the usual Conservativity for determiners may be understood dynamically as an instruction constraining general domain change.) Similarly, recent work on anaphora and quantification aiming to formalize the notion of an E-type anaphor (Francez and Lappin 1993, Jackson 1994, van der Does 1993, 1994) also engages in 'domain dynamics', whereby each occurrence of a quantifier is restricted by an appropriate context set. This can be modelled using dependent objects, e.g., variables labelled by formulas.

The problem is to account for the dependency among domains and the interaction of this dependency with the binding properties of quantifiers. For example, should free variables in the labels always be brought under the scope of the relevant quantifiers, or can dependencies be resolved in a different way?

Further, quantified expressions may also triggers actual changes in the construction of semantic models (the preceding form of dynamics merely concerned 'zooming in' on certain parts of a fixed domain). For instance, an existental quantifier *some A* may be an instruction to *add* a new object to the domain satisfying this or that property, much as in the construction of a Beth-style semantic tableau. Again, no systematic theory of the latter form of quantifier dynamics exists so far. What would probably be needed is a more principled account of various functions served by linguistic utterances: model checking, model construction, querying, etcetera.

Finally, here is a more technical question that is high on the agenda right now. Can dynamic generalized quantifiers be obtained from the standard theory via systematic 'dynamic lifts', say in the style of Dekker 1993? Also, there are many more standard technical questions of axiomatization and expressive power to be asked in connection with a mathematical marriage between generalized quantifiers and dynamic logic, yielding systems of the form 'PDL(Q)' (cf. Harel 1984).

8. Quantifiers and modal logic

There are many analogies between quantifiers and modal operators, as has often been pointed out. Notably, modal diamonds are like existential quantifiers and modal boxes like universal ones. This fact underlies the usual translations from modal logic to standard logical formalisms (cf. van Benthem 1984). But there is more to this analogy. For instance, in its present-day manifestation, Modal Logic is a very general theory of (restricted) quantifier patterns with their model theory and proof theory (cf. Venema 1991, de Rijke 1993). And hence, many more analogies have emerged. For instance, van der Hoek and de Rijke 1993 use modal techniques to find direct axiomatizations and definability results for numerical quantifiers *at least n*, as well as *most*, within a syllogistic context. The latter paper also suggests applications of 'small' generalized quantifier formalisms to so-called 'terminological languages' in the field of knowledge representation.

Likewise, Ben Shalom 1994 shows how the central modal semantic invariance of 'bisimulation' may be discerned underneath notions and results in the standard theory of unary quantifiers. For this purpose, she analyzes the following general definition schema:

$$\mathbf{M}, s \models o_Q\phi \qquad \text{iff} \quad Q_W \{s' \mid sRs'\}\ \{s' \mid \mathbf{M}, s' \models \phi\}\ .$$

Various equivalences then turn out to hold between standard modal results and quantifier properties. For instance, the modality o_Q is invariant under bisimulation if and only if the generalized quantifier Q is a Boolean combination of the first-order quantifiers $\exists$ and $\forall$. Evidently, this is just the start for a more extensive elaboration of analogies between generalized quantifier theory and modal logic. For instance, what would be a preservation theorem w.r.t. bisimulation for the whole language EL(Q)?

Perhaps a deeper contact between the two perspectives has arisen in recent work on generalized semantics for first-order quantifiers, inspired by the earlier tradition of cylindric algebra (cf. the cylindric modal algebra of Venema 1991 and the 'modal state semantics' of van Benthem 1994). Here, the general pattern of interpretation for quantified expressions becomes as follows:

$\mathbf{M}, a \models Qx\phi$ iff there exists an assignment b such that (1) $R_x\, ab$ and (2) $\mathbf{M}, b \models \phi$

More generally, assignments may become abstract states here, while the restricting relations R_x can vary too. This semantics produces a decidable minimal base logic, on top of which different kinds of quantifiers may be defined (which would all be collapsed to $\exists$ in the standard semantics). For instance, combinations $QxQy\,\phi$ will not be equivalent to $QyQx\phi$, and neither of them is equivalent to the binary tuple quantifier $Qxy\phi$. (All these equivalences would express existential conditions on the behaviour of the restricting relations, and the availability of states in the universe). Thus, one can model many classes of generalized quantifiers by varying the behaviour of these restriction models, where exact correspondences may be obtained using techniques from Modal Logic. For instance, simple principles suppressing 'vacuous quantification' like $QxQx\phi \leftrightarrow Qx\phi$ or $Qx\neg Qx\phi \leftrightarrow \neg Qx\phi$ will now become modal S5-axioms stating that R_x is a Euclidean relation.

Current questions in this area concern the choice of natural sets of defining conditions for quantifiers, and obtaining logics with desirable combinations of meta-properties (including decidability). Moreover, on the modal analogy, one may define quantifiers of higher arities, such as variants of temporal *since* and *until*, and study their properties in a similar vein.

Finally, a somewhat similar relational semantics with a modal-style axiomatization has been proposed for first-order quantifiers in Alechina and van Benthem 1993, but this time, with a dependence relation directly on objects, rather than between assignments. Here, one sets

$\mathbf{M}, a_1,\ldots,a_k \models Qx\phi(x, y_1,\ldots,y_k)$ iff there exists some object a such that $Ra,a_1,\ldots,a_k$ with $\mathbf{M}, a,a_1, \ldots,a_k \models \phi(x, y_1,\ldots,y_k)$

The precise connection with the preceding view is not yet clear, but will probably involve invariance principles destroying the individual identity of variables. Alechina 1993 extends this analysis to binary generalized quantifiers, enlarging the modal analogy to one with Conditional Logic, and providing applications to default reasoning. Precise reductions and questions of axiomatization remain to be explored.

9. Proof theory of generalized quantifiers

Generalized quantifiers are usually thought of as a typically semantics-generated notion, but historically, their first use was in systems of inference, namely, in the traditional Syllogistic. And also more generally, reasoning with quantifiers has served as a paradigm for what may be called *natural logic* in human languages (cf. Sanchez Valencia 1991). Thus, the technical issues of axiomatization for quantifier logics raised in previous sections are very much to the point. Nevertheless, there are further aspects here. For instance, natural inferential divisions in human languages need not coincide with those found in the usual generalized quantifier formalisms (which take all of first-order logic for granted at the outset), and a more careful hierarchy of 'sub-mechanisms' may be found in van Benthem 1993, asking for separate descriptions of, e.g., pure syllogistic, Boolean monotonicity reasoning, and higher forms of quantified inference.

Another relevant topic from the earlier theory of unary generalized quantification retains its relevance too (van Benthem 1984). When doing so-called *inverse logic*, one starts from some inferential patterns (potentially) occurring in natural language, and then asks whether any generalized quantifiers exist exemplifying these, or just these. Questions of inverse logic have been studied extensively for unary quantifiers (cf. Westerståhl 1989), but they remain largely unexplored for polyadics or collectives.

But one can also strike out from inside the core of mathematical logic. Generalized quantifiers have traditionally posed a challenge to the famous Brouwer-Heyting-Kolmogorov view of logical Proof Theory. Unlike the usual (constructive) connectives and first-order quantifiers, quantifiers like *most* or even *exactly one* do not seem to admit of perspicuous adequate introduction- and elimination-rules. (For some authors, this has even been a strong argument against their logical worthiness, witness Sommers 1982.) Indeed, there exist characterization results by Zucker, Prawitz and Schroeder-Heister which seem to imply that only the first-order standard quantifiers are amenable to this style of analysis, which would make generalized quantifiers proof-theoretic oddities. This challenge has been met in several ways. One is that of Sundholm 1991, where the usual set-theoretic truth conditions for generalized quantifiers are transcribed into a Martin-Löf-style type theory, whose rules for the standard quantifiers $\exists, \forall$ employed in those definitions will then provide an indirect proof-theoretic treatment after all. (Ranta 1991 applies a more elaborate proof-theoretic program in this vein to natural language.) A more radical innovation in the proof-theoretic treatment of generalized quantifiers may be found in recent work by van Lambalgen 1991.

In its current manifestation, the latter approach may be characterized by the slogan "generalized quantifiers from substructural logics". The usual sequent calculi for predicate logic have a hidden structural rule (usually inside the introduction rule for the universal quantifier), to the effect that variables do not have an identity of their own, but serve to mark positions only. One can make this structural rule explicit, and then consider proof systems where it is absent, allowing variables to have a separate identity. Of the many ways in which this can happen, one of particular importance to generalized quantifiers is the case where variables depend on other variables. Such systems yield complete Gentzen axiomatizations for the quantifiers *for many*, *for uncountably many* and *for almost all* (in Friedman's sense). Moreover, van Lambalgen has shown that only certain of these logics lead to systems that allow of Cut Elimination, thereby providing a new angle upon what might be considered 'natural' generalized quantifiers. The resulting range of questions will be clear. Basically, all of classical Proof Theory may be rethought in the presence of generalized quantifiers in a rule format with appropriate variable restrictions. It seems plausible that this viewpoint is closely related to the semantic use of 'assignment restrictions' mentioned in the modal perspective of section 8. (For instance, by leaving out certain assignments in a model, one can force variables to 'work together' and hence become 'dependent'.) It might also be usefully applied to the dependent objects needed for the dynamic formalizations of E-type anaphora mentioned in section 7. Likewise, there seem to be interesting connections with systems of cylindric algebra, but the precise situation is not yet clear.

10. Further topics

The preceding sections have identified the main loci of current research. Nevertheless, there are still various other linguistic issues that may drive logical research in the area. We conclude by mentioning a few.

(1) In natural language, there are still other forms of quantification besides those over individuals and collectives. In particular, there is a pervasive duality between so-called 'count nouns' calling for the above counting quantifiers, and so-called 'mass terms' calling for measuring quantifiers (cf. Lønning 1994). No systematic theory of the latter form of quantification has been developed so far.

(2) Moreover, it is well-known that the standard determiner pattern of quantification is not the only quantificational tool of natural languages. Even in English, quantification may also be expressed by adverbial constructions, such as "the boys all played the game" or "the boys mostly played the game". In many languages of the world, the latter type of construction is indeed the dominant one. No mathematical analysis of these alternative possibilities has taken place yet.

(3) Many types of lexical expression exhibit behaviour which is closely related to quantifiers, such as conditionals, temporal adverbials, or even dynamic connectives (cf. van Benthem 1986, Löbner 1987, ter Meulen 1994, Lapierre 1991, van Benthem and Cepparello 1994). The resulting model-theoretic and proof-theoretic analogies have hardly been touched systematically.

(4) Linguistic quantifiers do not operate in isolation. One can study their contacts with various other linguistic mechanisms. A case in point is the pervasive semantic phenomenon of partiality. There is a relatively mechanical generalization from the standard theory to the case of partial models, but also some more intriguing questions. (Cf. van Benthem 1988, van Eyck 1991 for some first explorations.)

(5) More generally, the contribution of generalized quantifiers to sentence meanings arises in interaction with various compositional mechanisms. Examples of these are anaphora (cf. section 7 on dynamic semantics), various operators switching between collective and distributive predicates (cf. van der Does 1992a), or monotonicity and general Boolean inference (Sanchez Valencia 1991, van Benthem 1991). In each case, one wants a combined account telling us how the two systems cooperate in order to produce the correct sentence meanings. One example of this is the 'monotonicity calculus' of van Benthem 1986, which predicts polarity of predicate occurrences in complex sentences, at least, assuming standard Fregean iteration. No extensions have taken place yet to the various forms of polyadic composition mentioned in section 4. Moreover, no simple calculi have been found yet for other semantic properties of interest, such as Conservativity.

(6) The preceding perspective also leads to clear-cut mathematical questions. One may study generalized quantifiers in richer logical environments than the usual first-order bases, notably, that of a Boolean typed lambda calculus (cf. van Benthem 1991). As it turns out, many open technical questions emerge then. For instance, in order to prove not just 'soundness' but also 'completeness' for a natural monotonicity calculus with arbitrary generalized quantifiers, one would need a Lyndon Theorem for positive occurrences with respect to upward monotonicity in this setting. So far, only partial results in this direction have been found (van Benthem to appear, Spaan 1993). More generally, the model theory of generalized quantifiers in Boolean typed lambda calculus remains to be developed, as a more realistic reflection of how natural languages function.

(7) Our presentation has largely presupposed standard semantics for generalized quantifiers. But of course, there are other styles of modelling, at least in principle, which might carry their own particular insights. Two such examples are algebraic semantics (cf. Németi 1991) and game-theoretical semantics (cf. Hintikka and Kulas 1984).

(8) Although generalized quantifiers belong to the latest technical tool kit of modern logic, they also reflect one of the most traditional subjects in the field, being syllogistic subject-predicate structure, dating back long before the Boolean and Fregean Revolution. There are a few historical studies in the field (cf. Sanchez Valencia 1991, and recently Hodges 1993 on monotonicity reasoning), but much more remains to be explored.

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