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Marco Swaen

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Marco Swaen
Department of Mathematics and Computer Science
University of Amsterdam

Abstract. In this paper we study the logical and the arithmetical fragments of Martin-Löf's type theories with a weak version of the Σ-elimination rule, corresponding to the $\exists$E-rule of Natural Deduction. The principal result of this paper is that the weak, intensional version of **ML** is conservative over Intuitionistic Finite Type Arithmetic $\mathbf{HA}^{\omega}$.
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Correspondence to:

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Roetersstraat 15
1018WB Amsterdam

or

Faculteit der Wijsbegeerte
(Department of Philosophy)
Grimburgwal 10
1012GA Amsterdam

1 DEFINITION OF **IQC** AND **IQC**$^{\omega}$

Let L_0 be the language of first order predicate logic with equality. For each $n \in \mathbb{N}$ L_0 has countably many n-ary predicate symbols R_1^n, R_2^n,... and countably many function symbols F_1^n, F_2^n,... with n argument places. There are countably many individual constants and variables. We will use x,y,... as metavariables for variables; c, c',c",... as metavariables for constants; t,t',t",s,... as metavariables for terms.

Let **IQC** stand for Intuitionistic Predicate Logic with equality, based on the language L_0.

The class of ω-types (notation: TYP^{ω}) is defined inductively as follows:

$U \in TYP^{\omega}$; $I \in TYP^{\omega}$;

$\sigma, \tau \in TYP^{\omega} \Rightarrow \sigma \times \tau, \sigma \to \tau, \sigma + \tau \in TYP^{\omega}$.

The type U will be used as the basic domain of individuals, the type I is to be a singleton.

The language L^{ω} of so-called ω-sorted predicate logic extends L_0 with variables in all ω-types, and various term-operators. As metavariables for variables of type σ we will use: x^{σ}, y^{σ},.... The class of ω-terms, with their types (notation: TER^{ω}) is defined inductively as follows:

		$c_i \in U$
		$\mathbf{e} \in I$
$\sigma \in TYP^{\omega}$	$\Rightarrow$	$x^{\sigma}, x^{\sigma}_1, x^{\sigma}_2, \ldots \in \sigma$
$t_i \in U$ for $i=1,\ldots,n$	$\Rightarrow$	$F^n_j(t_1,\ldots,t_n) \in U$
$s \in \sigma$, $t \in \tau$	$\Rightarrow$	$(s,t) \in \sigma \times \tau$
$t \in \sigma \times \tau$	$\Rightarrow$	$\mathbf{p}_0 t \in \sigma$, $\mathbf{p}_1 t \in \tau$
$t \in \tau$, $\sigma \in TYP^{\omega}$	$\Rightarrow$	$\lambda x^{\sigma}.t \in \sigma \to \tau$
$t \in \sigma \to \tau$, $s \in \sigma$	$\Rightarrow$	$ts \in \tau$
$t \in \sigma$, $\tau \in TYP^{\omega}$	$\Rightarrow$	$\mathbf{k}_0(t) \in \sigma + \tau$
$t \in \sigma$, $\tau \in TYP^{\omega}$	$\Rightarrow$	$\mathbf{k}_1(t) \in \tau + \sigma$
$t \in \sigma + \tau$, $s_0(x^{\sigma}) \in \varrho$, $s_1(y^{\tau}) \in \varrho$	$\Rightarrow$	$\mathbf{D}_{xy}(t, s_0, s_1) \in \varrho$

The variables x^{σ} and y^{τ} in $\mathbf{D}_{xy}(t, s_0, s_1)$ are considered bound by $\mathbf{D}_{xy}$. Expressions of the form " $t \in \sigma$ " ,with t a term of type σ, are called typing statements. A term-equation is an expression of the form "$s = t \in \sigma$", with s and t terms of type σ. The class of prime formulas $PRIM^{\omega}$ contains all equations, typing statements and expressions of the form $R^n_i(t_1,\ldots,t_n)$ with $t_1,\ldots,t_n \in U$. The class of formulas FOR^{ω} is the smallest class containing all prime formulas and closed under the logical operators $\bot$, $\to$, $\wedge$, $\vee$, $\forall$- and $\exists$-quantification over typed variables. If convenient we will use $\forall x \in \sigma(.)$ and $\exists x \in \sigma(.)$ as alternative notations for $\forall x^\sigma(.)$ and $\exists x^\sigma(.)$ respectively.

The system **IQC**$^\omega$ of so called ω-sorted Intuitionistic Predicate Logic is founded on the following axioms and rules:

$$
\begin{array}{ll}
t \in I & \Rightarrow t = \mathbf{e} \in I \\
s \in \sigma,\ t \in \tau & \Rightarrow (\lambda x^\sigma.t)(s) = t[x^\sigma/s] \in \tau \\
s_0 \in \sigma_0,\ s_1 \in \sigma_1 & \Rightarrow \mathbf{p}_i(s_0,s_1) = s_i \in \sigma_i \quad \text{for } i=0,1 \\
t \in \sigma \times \tau & \Rightarrow (\mathbf{p}_0 t, \mathbf{p}_1 t) = t \in \sigma \times \tau \\
\mathbf{k}_0 t \in \sigma + \tau,\ s_0(x^\sigma) \in \varrho,\ s_1(y^\tau) \in \varrho & \Rightarrow \mathbf{D}_{xy}(\mathbf{k}_0 t, s_0, s_1) = s_0[x^\sigma/t] \in \varrho \\
\mathbf{k}_1 t \in \sigma + \tau,\ s_0(x^\sigma) \in \varrho,\ s_1(y^\tau) \in \varrho & \Rightarrow \mathbf{D}_{xy}(\mathbf{k}_1 t, s_0, s_1) = s_1[y^\tau/t] \in \varrho
\end{array}
$$

and the rules of natural deduction (with equality) in all ω-types, plus an extra replacement rule:

$$\frac{s_0 = s_1 \in \sigma \qquad t \in \tau}{t[x^\sigma/s_0] = t[x^\sigma/s_1] \in \tau}$$

In $\forall$E and $\exists$I typing statements are added to the premisses:

$$\frac{\forall x^\sigma\, \varphi \qquad t \in \sigma}{\varphi[x^\sigma/t]} \qquad\qquad \frac{\varphi[x^\sigma/t] \qquad t \in \sigma}{\exists x^\sigma\, \varphi}$$

2 THEOREM

IQC$^\omega$ is a conservative extension of **IQC**.

<u>Proof</u>:(Scetch) Let $\mathcal{K}$ be a Kripke model with underlying partial ordering $(K,\leqslant)$. Suppose σ and τ have been interpreted in all nodes $k \in K$, then define:

$$
\begin{array}{ll}
[\![\sigma \to \tau]\!]_k & := \{ f : \bigcup_{k' \geqslant k} [\![\sigma]\!]_{k'} \to \bigcup_{k' \geqslant k} [\![\tau]\!]_{k'} \mid \forall k' \geqslant k\ \forall a,a' \in [\![\sigma]\!]_{k'} . (f(a) \in [\![\tau]\!]_{k'} \\
 & \qquad \wedge\ \forall k'' \geqslant k'\ (k'' \Vdash a = a' \in \sigma \Rightarrow k'' \Vdash f(a) = f(a') \in \tau))\} \\
[\![\sigma \times \tau]\!]_k & := [\![\sigma]\!]_k \times [\![\tau]\!]_k \\
[\![\sigma + \tau]\!]_k & := \{0\} \times [\![\sigma]\!]_k \cup \{1\} \times [\![\tau]\!]_k
\end{array}
$$

The various operators for application, abstraction, pairing and projection can be defined to match with this definition. In this way a model $\mathcal{K}^*$ for **IQC**$^\omega$ is constructed, that agrees with K on all L_0-formulas. □

3 DEFINITION (of **ML** and **ML**$^{\vee}$)

Let **ML** be Martin-Löf's Intuitionistic Type Theory without universes as presented in [Troelstra 87]. Formulating the rules we will use t, t', t", s, s_i, ... as metavariables for terms; A, A', B, B_i ,.... as metavariables for types; φ, φ' as metavariables for the succedents of judgements. The letters Γ,Γ' stand for sequences of variable declarations of the form:

$$x_1 \in A_1,\ x_2 \in A_2, \ldots, x_n \in A_n$$

Such a sequence is called a context, if for all $i \leqslant n$ type A_i does not contain any free variables except for $x_1,\ldots,x_{i-1}$, and if $i \neq j$ implies $x_i \not\equiv x_j$. One easily proves by induction that all sequences of variable declarations occurring in derivable judgements are contexts. Let T stand for a type or a term; by T[x/t] we refer to the result of substituting t for all free occurrences of x in T. If convenient the free occurrence of variable x in T is indicated as T[x], in which case T[t] stands for the result of substituting t for x in T. The rules of **ML** are the following:

GENERAL RULES

ASS $$\frac{\Gamma \Rightarrow A\ \mathrm{Typ}}{\Gamma, x \in A \Rightarrow x \in A} \quad x \notin FV(\Gamma)$$

THIN $$\frac{\Gamma \Rightarrow A\ \mathrm{Typ} \qquad \Gamma, \Gamma' \Rightarrow \varphi}{\Gamma, x \in A, \Gamma' \Rightarrow \varphi} \qquad x \notin FV(\Gamma, \Gamma')$$

TYPE FORMATION RULES

NTYP $\mathbb{N}$ Typ

ΠTYP $$\frac{\Gamma \Rightarrow A\ \mathrm{Typ} \qquad \Gamma, x \in A \Rightarrow B\ \mathrm{Typ}}{\Gamma \Rightarrow \Pi x^A.B\ \mathrm{Typ}}$$

ΣTYP $$\frac{\Gamma \Rightarrow A\ \mathrm{Typ} \qquad \Gamma, x \in A \Rightarrow B\ \mathrm{Typ}}{\Gamma \Rightarrow \Sigma x^A.B\ \mathrm{Typ}}$$

+TYP $$\frac{\Gamma \Rightarrow A_0\ \mathrm{Typ} \qquad \Gamma \Rightarrow A_1\ \mathrm{Typ}}{\Gamma \Rightarrow A_0 + A_1\ \mathrm{Typ}}$$

ITYP $$\frac{\Gamma \Rightarrow t \in A \quad \Gamma \Rightarrow t' \in A}{\Gamma \Rightarrow I(A,t,t')\ \mathrm{Typ}}$$

TERM FORMATION RULES

NI $\quad \mathbf{0} \in \mathbb{N} \qquad \dfrac{\Gamma \Rightarrow t \in \mathbb{N}}{\Gamma \Rightarrow \mathbf{S}t \in \mathbb{N}}$

NE
$$\frac{\Gamma \Rightarrow t \in \mathbb{N} \quad \Gamma, x \in \mathbb{N} \Rightarrow A \text{ Typ} \quad \Gamma \Rightarrow t' \in A[x/\mathbf{0}] \quad \Gamma, x \in \mathbb{N}, y \in A \Rightarrow t'' \in A[x/\mathbf{S}x]}{\Gamma \Rightarrow \mathbf{R}_{xy}(t, t', t'') \in A[x/t]}$$

ΠI
$$\frac{\Gamma, x \in A \Rightarrow t \in B}{\Gamma \Rightarrow \lambda x^A.t \in \Pi x^A.B}$$

ΠE
$$\frac{\Gamma \Rightarrow t \in \Pi x^A.B \quad \Gamma \Rightarrow t' \in A}{\Gamma \Rightarrow tt' \in B[x/t']}$$

ΣI
$$\frac{\Gamma \Rightarrow t \in A \quad \Gamma \Rightarrow t' \in B[x/t] \quad \Gamma, x \in A \Rightarrow B \text{ Typ}}{\Gamma \Rightarrow (t, t') \in \Sigma x^A.B}$$

ΣE
$$\frac{\Gamma \Rightarrow t \in \Sigma x^A.B}{\Gamma \Rightarrow \mathbf{p}_0 t \in A} \qquad \frac{\Gamma \Rightarrow t \in \Sigma x^A.B}{\Gamma \Rightarrow \mathbf{p}_1 t \in B[x/\mathbf{p}_0 t]}$$

$+I_{1,2}$
$$\frac{\Gamma \Rightarrow t \in A_0 \quad \Gamma \Rightarrow A_1 \text{ Typ}}{\Gamma \Rightarrow \mathbf{k}_0 t \in A_0 + A_1} \qquad \frac{\Gamma \Rightarrow t \in A_1 \quad \Gamma \Rightarrow A_0 \text{ Typ}}{\Gamma \Rightarrow \mathbf{k}_1 t \in A_0 + A_1}$$

+E
$$\frac{\Gamma \Rightarrow t \in A_0 + A_1 \quad \Gamma, z \in A_0 + A_1 \Rightarrow C \text{ Typ} \quad \Gamma, x \in A_i \Rightarrow t_i \in C[z/\mathbf{k}_i t] \ \ (i=0,1)}{\Gamma \Rightarrow \mathbf{D}_{xy}(t, t_0, t_1) \in C[z/t]}$$

II
$$\frac{\Gamma \Rightarrow t = t' \in A}{\Gamma \Rightarrow \mathbf{e} \in I(A, t, t')}$$

IE
$$\frac{\Gamma \Rightarrow t'' \in I(A, t, t')}{\Gamma \Rightarrow t = t' \in A}$$

EQUALITY RULES

REFL$_{1,2,3}$

$$\frac{\Gamma \Rightarrow t \in A}{\Gamma \Rightarrow t = t \in A} \qquad \frac{\Gamma \Rightarrow t = t \in A}{\Gamma \Rightarrow t \in A} \qquad \frac{\Gamma \Rightarrow A \text{ Typ}}{\Gamma \Rightarrow A = A}$$

SYM$_{1,2}$

$$\frac{\Gamma \Rightarrow t = t' \in A}{\Gamma \Rightarrow t' = t \in A} \qquad \frac{\Gamma \Rightarrow A = A'}{\Gamma \Rightarrow A' = A}$$

TRANS$_{1,2}$

$$\frac{\Gamma \Rightarrow t = t' \in A \qquad \Gamma \Rightarrow t' = t'' \in A}{\Gamma \Rightarrow t = t'' \in A} \qquad \frac{\Gamma \Rightarrow A = A' \qquad \Gamma \Rightarrow A' = A''}{\Gamma \Rightarrow A = A''}$$

SUB

$$\frac{\Gamma, x \in A, \Gamma' \Rightarrow \varphi \qquad \Gamma \Rightarrow t \in A}{\Gamma, \Gamma'[x/t] \Rightarrow \varphi[x/t]}$$

REPL$_1$

$$\frac{\Gamma, x \in A, \Gamma' \Rightarrow B \text{ Typ} \qquad \Gamma \Rightarrow t = t' \in A}{\Gamma, \Gamma'[x/t] \Rightarrow B[x/t] = B[x/t']}$$

REPL$_2$

$$\frac{\Gamma, x \in A, \Gamma' \Rightarrow s \in B \qquad \Gamma \Rightarrow t = t' \in A}{\Gamma, \Gamma'[x/t] \Rightarrow s[x/t] = s[x/t'] \in B[x/t]}$$

REPL$_3$

$$\frac{\Gamma \Rightarrow t \in A \qquad \Gamma \Rightarrow A = A'}{\Gamma \Rightarrow t \in A'}$$

ΠCONV

$$\frac{\Gamma \Rightarrow \lambda x^A.t \in \Pi x^A.B \qquad \Gamma \Rightarrow t' \in A}{\Gamma \Rightarrow (\lambda x^A.t)t' = t[x/t'] \in B[x/t']}$$

ΣCONV$_1$

$$\frac{\Gamma \Rightarrow t \in A \qquad \Gamma \Rightarrow t' \in B[x/t] \qquad \Gamma, x \in A \Rightarrow B \text{ Typ}}{\Gamma \Rightarrow \mathbf{p}_0(t, t') = t \in A}$$

ΣCONV$_2$

$$\frac{\Gamma \Rightarrow t \in A \qquad \Gamma \Rightarrow t' \in B[x/t] \qquad \Gamma, x \in A \Rightarrow B \text{ Typ}}{\Gamma \Rightarrow \mathbf{p}_1(t, t') = t' \in B[x/t]}$$

ΣCONV$_3$

$$\frac{\Gamma \Rightarrow t \in \Sigma x^A.B}{\Gamma \Rightarrow (\mathbf{p}_0 t, \mathbf{p}_1 t) = t \in \Sigma x^A.B}$$

+CONV
$$\frac{\Gamma\Rightarrow t\in A_i \quad \Gamma,z\in A_0+A_1\Rightarrow C\ \mathrm{Typ} \quad \Gamma,y_j\in A_j\Rightarrow t_j\in C[z/\mathbf{k}_j y_j] \quad (j=0,1)}{\Gamma\Rightarrow \mathbf{D}_{y_1y_2}(\mathbf{k}_i t,t_0,t_1)=t_i[y_i/t]\in C[z/\mathbf{k}_i t]}$$

ICONV
$$\frac{t''\in I(A,t,t')}{t''=\mathbf{e}\in I(A,t,t')}$$

NCONV_1
$$\frac{\Gamma,x\in\mathbb{N}\Rightarrow A\ \mathrm{Typ} \quad \Gamma\Rightarrow t'\in A[x/\mathbf{0}] \quad \Gamma,x\in\mathbb{N},y\in A\Rightarrow t''\in A[x/\mathbf{S}x]}{\Gamma\Rightarrow \mathbf{R}_{xy}(\mathbf{0},t',t'')=t'\in A[x/\mathbf{0}]}$$

NCONV_2
$$\frac{\Gamma\Rightarrow t\in\mathbb{N} \quad \Gamma,x\in\mathbb{N}\Rightarrow A\ \mathrm{Typ} \quad \Gamma\Rightarrow t'\in A[x/\mathbf{0}] \quad \Gamma,x\in\mathbb{N},y\in A\Rightarrow t''\in A[x/\mathbf{S}x]}{\Gamma\Rightarrow \mathbf{R}_{xy}(\mathbf{S}t,t',t'')=t''[x,y/t,\mathbf{R}_{xy}(t,t',t'')]\in A[x/\mathbf{S}t]}$$

In the sequel we will use $\lambda x\in A.t$, $\Pi x\in A.B$ and $\Sigma x\in A.B$ as alternative notations for $\lambda x^A.t$, $\Pi x^A.B$ and $\Sigma x^A.B$ respectively.
The rules of ΣE and ΣCONV can equivalently be formulated as follows:

$$\frac{\Gamma\Rightarrow t\in\Sigma x^A.B \qquad \Gamma,x\in A,y\in B\Rightarrow s\in C[z/(x,y)] \qquad \Gamma,z\in\Sigma x^A.B\Rightarrow C\ \mathrm{Typ}}{\Gamma\Rightarrow s[x,y/\mathbf{p}_0 t,\mathbf{p}_1 t]\in C[z/t]}$$

$$\frac{\Gamma\Rightarrow (t_0,t_1)\in\Sigma x^A.B \quad \Gamma,x\in A,y\in B\Rightarrow s\in C[z/(x,y)] \qquad \Gamma,z\in\Sigma x^A.B\Rightarrow C\ \mathrm{Typ}}{\Gamma\Rightarrow s[x,y/\mathbf{p}_0(t_0,t_1),\mathbf{p}_1(t_0,t_1)]=s[x,y/t_0,t_1]\in C[z/t]}$$

In this form ΣE resembles the ∃E-rule of Natural Deduction. In the so-called weak version of ΣE variable z is not allowed to occur in C, in close correspondence to the variable condition in the ∃E-rule of Natural Deduction. In **ML** this condition can be implemented by changing the third premiss into:

$\Gamma\Rightarrow C\ \mathrm{Typ}$,

since in general if $\Gamma\Rightarrow\varphi$ is derivable in **ML**, all free variables of φ are declared in Γ, so $x,y\in FV(C)$ would imply that x and y are declared in Γ, but then $\Gamma,x\in A,y\in B$ in the second premis cannot be a context. The weak versions of ΣE and ΣCONV are denoted by $\Sigma E^\vee$ and $\Sigma CONV^\vee$ respectively.
In a similar way the weak versions of +E and +CONV are defined as

$+E^\vee$
$$\frac{\Gamma\Rightarrow t\in A+B \qquad \Gamma,x\in A\Rightarrow s_0\in C \qquad \Gamma,y\in B\Rightarrow s_1\in C \qquad \Gamma\Rightarrow C\ \mathrm{Typ}}{\Gamma\Rightarrow \mathbf{D}_{xy}(t,s_0,s_1)\in C}$$

$$+\mathrm{CONV}^{\vee}\quad \frac{\Gamma\Rightarrow \mathbf{k}_i t\in A_0+A_1 \qquad \Gamma, z_0\in A_0\Rightarrow s_0\in C \qquad \Gamma, z_1\in A_1\Rightarrow s_1\in C \qquad \Gamma\Rightarrow C\ \mathrm{Typ}}{\Gamma\Rightarrow \mathbf{D}_{xy}(\mathbf{k}_i t, s_0, s_1)=s_i[z_i/t]\in C}$$

By **ML**$^{\vee}$ we denote the result of weakening the Σ- and +-rules in **ML** in the indicated sense.

4 DEFINITION (of **MLP** and **MLP**$^{\vee}$)
We define a predicate logic version of Martin-Löf's type theory, called **MLP**. This system is to comprise all the rules for Σ-, Π-, +- and I- types, the general rules for equality and conversion plus the following extra rules:

I_0TYP $\qquad I_0$ Typ

$$I_0E \qquad \frac{\Gamma\Rightarrow t\in I_0 \qquad \Gamma\Rightarrow A\ \mathrm{Typ}}{\Gamma\Rightarrow \mathbf{f}_A(t)\in A}$$

UTYP $\qquad$ U Typ

$$\mathrm{UI} \qquad c_i\in U \qquad \frac{\Gamma\Rightarrow t_1\in U \quad \dots \quad \Gamma\Rightarrow t_n\in U}{\Gamma\Rightarrow F_i^n(t_1,\dots,t_n)\in U}$$

$$R_i^n\mathrm{TYP} \qquad \frac{\Gamma\Rightarrow t_1\in U \quad \dots \quad \Gamma\Rightarrow t_n\in U}{\Gamma\Rightarrow R_i^n(t_1,\dots,t_n)\ \mathrm{Typ}}$$

$$R_i^n\mathrm{CONV} \qquad \frac{\Gamma\Rightarrow t\in R_i^n(t_1,\dots,t_n)}{\Gamma\Rightarrow t=\mathbf{e}\in R_i^n(t_1,\dots,t_n)}$$

Let **MLP**$^{\vee}$ denote the result of weakening the Σ and + rules as indicated above for the case of **ML**.

5 DEFINITION (of the embedding $^+$)
Via the principle of "Formulas-as-Types" (see e.g. [Howard 80]), **IQC** in a canonical way is embedded in **MLP**$^{\vee}$. L_0-terms are all available in **MLP**$^{\vee}$ in the sense that for any term t with free variables $x_1,\dots,x_n$, we have

$$\mathbf{MLP}^{\vee}\vdash x_1\in U,\dots,x_n\in U \Rightarrow t\in U,$$

whereas any L_0-formula φ is represented by a type φ^+ in **MLP**$^\vee$ such that if A contains only $x_1,\ldots,x_n$ free then

$$\mathbf{MLP}^\vee \vdash x_1 \in U,\ldots,x_n \in U \Rightarrow \varphi^+ \text{ Typ}.$$

The type φ^+ is defined inductively as follows:

$(R_i^n(t_1,\ldots,t_n))^+$	$:= R_i^n(t_1,\ldots,t_n)$	
$(t=t')^+$	$:= I(U,t,t')$	
$(\bot)^+$	$:= I_0$	
$(\varphi\wedge\psi)^+$	$:= \Sigma x^{\varphi+}.\psi^+$	with $x\notin FV(\varphi)\cup FV(\psi)$
$(\varphi\rightarrow\psi)^+$	$:= \Pi x^{\varphi+}.\psi^+$	with $x\notin FV(\varphi)\cup FV(\psi)$
$(\varphi\vee\psi)^+$	$:= \varphi^+ + \psi^+$	
$(\exists x\varphi)^+$	$:= \Sigma x^U.\varphi^+$	
$(\forall x\varphi)^+$	$:= \Pi x^U.\varphi^+$	

The embedding $^+$ is easily extended to **IQC**$^\omega$

for ω-types:

$(\sigma\times\tau)^+$	$:= \Sigma x^{\sigma+}.\tau^+$
$(\sigma\rightarrow\tau)^+$	$:= \Pi x^{\sigma+}.\tau^+$
$(\sigma+\tau)^+$	$:= \sigma^+ + \tau^+$
$(I)^+$	$:= I(U,c_0,c_0)$

for ω-terms:

t^+ is the result of replacing all variables x^σ in t by $x^{\sigma+}$

and for L^ω-formulas: as for L_0-formulas plus

$(R_i^n(t_1,\ldots,t_n))^+$	$:= R_i^n(t_1^+,\ldots,t_n^+)$
$(\exists x^\sigma\varphi)^+$	$:= \Sigma x^{\sigma+}.\varphi^+$
$(\forall x^\sigma\varphi)^+$	$:= \Pi x^{\sigma+}.\varphi^+$

That $^+$ indeed yields an embedding of **IQC**$^\omega$ is expressed in the following theorem.

6 THEOREM (Soundness)

(i) Let $t\in\sigma$ in **IQC**$^\omega$ with $FV(t)=\{x_1\in\sigma_1,\ldots,x_n\in\sigma_n\}$, then
MLP$^\vee$ $\vdash x_1\in\sigma_1^+,\ldots, x_n\in\sigma_n^+ \Rightarrow t^+\in\sigma^+$.

(ii) Let $\varphi, \varphi_1,\ldots\varphi_m$ be L^ω-formulas with free variables $x_1\in\sigma_1,\ldots,x_n\in\sigma_n$, such that in **IQC** $\{\varphi_1,\ldots\varphi_m\}\vdash\varphi$
then there is a term t such that
MLP$^\vee$ $\vdash x_1\in\sigma_1^+,\ldots, x_n\in\sigma_n^+, y_1\in\varphi_1^+,\ldots, y_m\in\varphi_m^+ \Rightarrow t\in\varphi^+$.

Proof:(i) First we note that for all $\sigma\in TYP^\omega$:

$$\mathbf{MLP}^\vee \vdash \sigma \text{ Typ}.$$

Now we prove (i) by induction on the formation of t in $\mathbf{IQC}^{\omega}$. For instance suppose $t \equiv \mathbf{p}_1 t' \in \tau$ where $t' \in \sigma \times \tau$, then by induction hypothesis we have

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+ \Rightarrow t'^+ \in \sigma \times \tau$$

Furthermore:

$$\mathbf{MLP}^{\vee} \vdash \Rightarrow \tau \text{ Typ}$$

so by THIN:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+ \Rightarrow \tau \text{ Typ}$$

and by ASS:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, x \in \sigma, y \in \tau \Rightarrow y \in \tau$$

Now apply ΣE:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+ \Rightarrow y[x,y/\mathbf{p}_0 t', \mathbf{p}_1 t'] \in \tau$$

Since $y[x,y/\mathbf{p}_0 t', \mathbf{p}_1 t'] \equiv \mathbf{p}_1 t' \equiv t^+ \in \tau$:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+ \Rightarrow t^+ \in \tau$$

(ii) By induction on the length of the derivation in $\mathbf{IQC}^{\omega}$. For instance the case of $(\rightarrow I)$:

Suppose $\psi \rightarrow \varphi$ is derived from $\varphi_1, \ldots, \varphi_m$ by an application of $(\rightarrow I)$, then by induction assumption we have a term t such that:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+, z \in \psi^+ \Rightarrow t \in \varphi^+$$

Now apply (ΠI) yielding:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+ \Rightarrow \lambda z \in \psi^+ . t \in \Pi z \in \psi^+ . \varphi^+$$

By definition $(\psi \rightarrow \varphi)^+ \equiv \Pi z \in \psi^+ . \varphi^+$ so we have

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+ \Rightarrow \lambda z \in \psi^+ . t \in (\psi \rightarrow \varphi)^+$$

The case of(∀E):

Suppose $\varphi[s]$ derived from $\varphi_1, \ldots \varphi_m$ by an application of (∀E), then by induction hypothesis there is a term t such that

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+ \Rightarrow t \in \Pi x^{\sigma+} . \varphi^+,$$

and if $x_1 \in \sigma_1, \ldots, x_i \in \sigma_i, z_1 \in \tau_1, \ldots, z_k \in \tau_k$ are the free variables of s then

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_i \in \sigma_i^+, z_1 \in \tau_1^+, \ldots, z_k \in \tau_k^+ \Rightarrow s^+ \in \sigma^+$$

now by a series of THIN applications the contexts can be made compatible:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, z_1 \in \tau_1, \ldots, z_k \in \tau_k, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+ \Rightarrow t \in \Pi x^{\sigma+} . \varphi^+,$$

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, z_1 \in \tau_1, \ldots, z_k \in \tau_k, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+ \Rightarrow s^+ \in \sigma^+,$$

then ΠE yields:

$$\mathbf{MLP}^{\vee} \vdash x_1 \in \sigma_1^+, \ldots, x_n \in \sigma_n^+, z_1 \in \tau_1, \ldots, z_k \in \tau_k, y_1 \in \varphi_1^+, \ldots, y_m \in \varphi_m^+ \Rightarrow t s^+ \in \varphi[x/s]^+ . \quad \square$$

Conversely **MLP**$^{\smile}$ is conservative over **IQC**$^{\omega}$ (and over **IQC**, by Theorem 2), as we will demonstrate by a translation of **MLP**-judgements into L^{ω}-formulas, based on a distinction between two aspects of **MLP**-judgements. Let $\Gamma \Rightarrow t \in A$ be a judgement; from one viewpoint the judgement tells us that term t has a certain form specified by its type, for instance that t is a pair if A is a Σ-type. This aspect will be expressed by a collapse $|.|$ of **MLP**-types and -terms into ω-types and ω-terms (definition 8). On the other hand the judgement $\Gamma \Rightarrow t \in A$ declares A inhabited, i.e. a provable proposition; this propositional aspect of **MLP**-types is expressed by a translation * into L^{ω}-formulas (definition 11).

7 LEMMA

In **IQC**$^{\omega}$ all types are inhabited.

Proof: Define:

$$\begin{aligned}
\mathbf{t}_U &:= c_0\\
\mathbf{t}_I &:= \mathbf{e}\\
\mathbf{t}_{\sigma\times\tau} &:= (\mathbf{t}_\sigma, \mathbf{t}_\tau)\\
\mathbf{t}_{\sigma\to\tau} &:= \lambda x^\sigma.\mathbf{t}_\tau\\
\mathbf{t}_{\sigma+\tau} &:= k_0\mathbf{t}_\sigma
\end{aligned}$$

Then by induction one easily proves that $\mathbf{IQC}^{\omega} \vdash t_\sigma \in \sigma$. □

8 DEFINITION (of the collapse $|.|$)

The translation of **MLP**$^{\smile}$-types and -terms into ω-types and -terms, regarding the form of the term in the judgement is defined inductively. Strictly spoken types and terms are translated together with the context they have been formed on: in order to translate the free variables we need to know of what type they have been declared.

$$\begin{aligned}
|U| &:= U\\
|R^n_i(t_1,\dots,t_n)| &:= I\\
|I(A,s,t)| &:= I\\
|I_0| &:= I\\
|\Sigma x^A.B| &:= |A| \times |B|\\
|\Pi x^A.B| &:= |A| \to |B|\\
|A+B| &:= |A| + |B|
\end{aligned}$$

Note that $|A|$ does not depend on subterms occurring in A.

$\|c_i\|$	$:= c_i$
$\|F^n_i(t_1,\ldots,t_n)\|$	$:= F^n_i(\|t_1\|,\ldots,\|t_n\|)$
$\|\mathbf{e}\|$	$:= \mathbf{e}$
$\|(t_0,t_1)\|$	$:= (\|t_0\|, \|t_1\|)$
$\|\mathbf{p}_i t\|$	$:= \mathbf{p}_i \|t\|$ for $i=0,1$
$\|\lambda x^A.t\|$	$:= \lambda x^{\|A\|}.\|t\|$
$\|ts\|$	$:= \|t\|\|s\|$
$\|\mathbf{k}_i(t)\|$	$:= \mathbf{k}_i \|t\|$
$\|\mathbf{D}_{xy}(t,s_0,s_1)\|$	$:= \mathbf{D}_{xy}(\|t\|,\|s_0\|,\|s_1\|)$
$\|\mathbf{f}_A(t)\|$	$:= \mathbf{t}_{\|A\|}$
$\|x\|$	$:= x \in \|A\|$ if $x \in A$ in the context

9 LEMMA

(i) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow A$ Typ
then $|A| \in TYP^{\omega}$

(ii) if $\mathbf{MLP}^{\vee} \vdash \Gamma, x \in A, \Gamma' \Rightarrow B$ Typ and $\mathbf{MLP} \vdash \Gamma \Rightarrow s \in A$
then $|B[x/s]| \equiv |B| [x^{|A|}/|s|]$

(iii) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow A=B$
then $|A| \equiv |B|$

Proof: (i) by induction on the length of the derivation. (ii): Trivial since subterms do not affect the translation of the type they occur in.
(iii) Only two $\mathbf{MLP}^{\vee}$-rules can serve to introduce a type equation, namely $REFL_3$ and $REPL_1$ (definition 3). The type equation, once it has been introduced, can only pass SUB, SYM_2, $TRANS_2$ or THIN. Suppose $\mathfrak{D}$ is a derivation tree of $\Gamma \Rightarrow A=B$, and let subtree $\mathfrak{D}^{o}$ consist of all type equations directly connected to the final conclusion of $\mathfrak{D}$, then $\mathfrak{D}^{o}$ starts with applications of either $REFL_3$ or $REPL_1$ succeeded by applications of SUB, SYM_2, $TRANS_2$ and THIN. Now (iii) is easily proved by induction on the maximal length of the branches in $\mathfrak{D}^{o}$. □

10 LEMMA

(i) if $\mathbf{MLP}^{\vee} \vdash \Gamma, x \in A, \Gamma' \Rightarrow t \in B$
and $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow s \in A$
then $|t[x/s]| \equiv |t| [x^{|A|}/|s|]$

(ii) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow t \in B$
then $\mathbf{IQC}^{\omega} \vdash \forall x_1^{|A_1|} \ldots \forall x_n^{|A_n|} (|t| \in |B|)$

Proof:(i) We define a class ET of terms that certainly comprises all **MLP**-terms, and for which the demonstrandum can easily be established

$$c_i,\ \mathbf{e},\ x \in ET$$

$$t_0,\dots t_n \in ET \;\Rightarrow\; F_i^n(t_1,\dots t_n) \in ET$$

$$t,t',t'' \in ET \;\Rightarrow\; \lambda x^A.t,\ tt',\ (t,t'), \mathbf{p}_0 t,\ \mathbf{p}_1 t,\ \mathbf{k}_0(t),\ \mathbf{k}_1(t),\ \mathbf{D}_{xy}(t,t',t''), \mathbf{f}_A t \in ET$$

for any A **MLP**-type

(ii) By induction on the length of the derivation of $\Gamma \Rightarrow t \in B$; in case of a term equation $\Gamma \Rightarrow s = t \in A$ we demand $\Gamma \Rightarrow s \in A$ and $\Gamma \Rightarrow t \in A$ to satisfy the induction hypothesis. As an example we will treat the case of SUB with $\varphi \equiv s \in B$ (Definition 3).

Let Γ stand for $x_1 \in A_1,\dots,x_n \in A_n$ and Γ' for $y_1 \in B_1,\dots,y_m \in B_m$.

By induction hypothesis:

(1) $\forall x_1^{|A_1|} \dots \forall x_n^{|A_n|}\ \forall x^{|A|}\ \forall y_1^{|B_1|} \dots \forall y_m^{|B_m|}(\ |s| \in |B|\ )$ and

(2) $\forall x_1^{|A_1|} \dots \forall x_n^{|A_n|}(\ |t| \in |A|\ )$

Now suppose $x_1 \in |A_1|, \dots, x_n \in |A_n|$, then $\forall E$ applied on (2) yields $|t| \in |A|$ and on (1)

$$(\forall y_1 \in |B_1| \dots \forall y_m \in |B_m|(\ |s| \in |B|\ ))\ [x^{|A|}/|t|],$$

now apply Lemma 9 (ii)

$$\forall y_1 \in |B_1[x/t]| \dots \forall y_m \in |B_m[x/t]|(\ |s|\ [x^{|A|}/|t|] \in |B|\ [x^{|A|}/|t|]\ )$$

and by Lemma 10 (i)

$$\forall y_1 \in |B_1[x/t]| \dots \forall y_m \in |B_m[x/t]|(\ |s[x/t]| \in |B[x^{|A|}/t]|).$$

Now quantify over $x_1 \in |A_1|, \dots, x_n \in |A_n|$ so:

$$\forall x_1 \in |A_1| \dots \forall x_n \in |A_n|\ \forall y_1 \in |B_1[x/t]| \dots \forall y_m \in |B_m[x/t]|(\ |s[x/t]| \in |B[x/t]|)$$

The case of $+E^{\vee}$:

By induction hypothesis we have:

(3) $\forall x_1 \in |A_1| \dots \forall x_n \in |A_n|(|t| \in |B_0 + B_1|);$

(4) $\forall x_1 \in |A_1| \dots \forall x_n \in |A_n|\ \forall y_i \in |B_i|(|t_i| \in |C|)$ for $i=0,1$

So if $x_1 \in |A_1|,\dots,x_n \in |A_n|$ then $|t| \in |B_0| + |B_1|$, $|t_i|[y_i^{|B_i|}] \in |C|$ for $i=0,1$

therefore $\mathbf{IQC}^{\omega} \vdash |\mathbf{D}_{y_0 y_1}(t,t_0,t_1)| \equiv \mathbf{D}_{y_0 y_1}(|t|, |t_0|, |t_1|) \in |C|$,

then by the $\forall I$-rule:

$$\mathbf{IQC}^{\omega} \vdash \forall x_1^{|A_1|} \dots \forall x_n^{|A_n|}(|\ \mathbf{D}_{y_0 y_1}(t,t_0,t_1)| \in |C|). \qquad \square$$

11 DEFINITION (of the translation *)

The translation of $\mathbf{MLP}^{\vee}$-types into L^{ω}-formulas, concerning the fact that the type involved is inhabited, is defined inductively as follows

$$U^* := c_0 = c_0 \in U$$

$$R^n_i(t_1,\dots,t_n)^* := R^n_i(|t_1|,\dots,|t_n|)$$

$I(A,s,t)^*$:= $A^* \wedge |s|=|t| \in |A|$
I_0^* := $\bot$
$(A+B)^*$:= $A^* \vee B^*$
$(\Sigma x^A.B(x))^*$:= $\exists x^{|A|}(A^* \wedge B(x)^*)$
$(\Pi x^A.B(x))^*$:= $\forall x^{|A|}(A^* \rightarrow B(x)^*)$

Note that in the last two cases x cannot occur free in A^*

12 LEMMA

(i) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow A$ Typ,
then A^* is an L^{ω}-formula and $FV(A^*) \subseteq FV(A)$;

(ii) if $\mathbf{MLP}^{\vee} \vdash \Gamma, x \in A, \Gamma' \Rightarrow B$ Typ and $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow s \in A$,
then $B[x/s]^* \equiv B^*[x^{|A|}/|s|]$.

Proof: Similar to the proof of the previous lemmas. □

13 PROPOSITION

(i) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow t \in A$
then $\mathbf{IQC}^{\omega} \vdash \forall x_1^{|A_1|}(A_1^* \rightarrow \ldots \forall x_n^{|A_n|}(A_n^* \rightarrow |t| \in |A| \wedge A^*)\ldots)$

(ii) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow s=t \in A$
then $\mathbf{IQC}^{\omega} \vdash \forall x_1^{|A_1|}(A_1^* \rightarrow \ldots \forall x_n^{|A_n|}(A_n^* \rightarrow (|s|=|t| \in |A|) \wedge A^*)\ldots)$

(iii) if $\mathbf{MLP}^{\vee} \vdash \Gamma \Rightarrow A=B$
then $\mathbf{IQC}^{\omega} \vdash \forall x_1^{|A_1|}(A_1^* \rightarrow \ldots \forall x_n^{|A_n|}(A_n^* \rightarrow (A^* \leftrightarrow B^*))\ldots)$

Proof: (i), (ii) and (iii) are proved simultaneously by induction on the length of the derivation in $\mathbf{MLP}^{\vee}$. We will only spell out the more interesting cases.

(I_0E) By induction hypothesis:

$$\forall x_1^{|A_1|}(A_1^* \rightarrow \ldots \forall x_n^{|A_n|}(A_n^* \rightarrow |t| \in I \wedge \bot).$$

Assume $x_1 \in |A_1|$, $A_1^*, \ldots, x_n \in |A_n|$ and A_n^*, then after a series of $\forall$E and $\rightarrow$E

$$|t| \in I \wedge \bot,$$

so by $\bot$E of $\mathbf{IQC}^{\omega}$ we can conclude:

$$A^*.$$

By Lemma 7: $|\mathbf{f}(t)| := \mathbf{t}_{|A|} \in |A|$, so after a series of $\forall$I and $\rightarrow$I:

$$\forall x_1^{|A_1|}(A_1^* \rightarrow \ldots \forall x_n^{|A_n|}(A_n^* \rightarrow |\mathbf{f}_A(t)| \in |A| \wedge A^*)$$

The part of the context in the application of a rule that remains unaltered, can be treated as has been done just now, therefore in the sequel we will omit Γ.

(ΠI) Translation of antecedent and succedent are almost the same except for the appearance of the statement $|\lambda x^A.t| \in |\Pi x^A.B|$, the derivability of which has been established in Lemma 10 (ii).

(ΠE) By induction hypothesis: $|t| \in |\Pi x^A.B| \wedge (\Pi x^A.B)^*$ i.e.

(1) $|t| \in |A| \to |B| \;\wedge\; \forall x^{|A|}(A^* \to B^*)$

and (2) $|s| \in |A| \wedge A^*$

Thus:

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{|t|\in|A|\to|B|} \qquad \overset{(2)}{|s|\in|A|}}{|t|(|s|)\in|B|}}{|t(s)|\in|B[x/s]|} \qquad \dfrac{\dfrac{\dfrac{\overset{(1)}{\forall x^{|A|}(A^*\to B^*)} \qquad \overset{(2)}{|s|\in|A|}}{A^*\to B^*[x/|s|]} \qquad \overset{(2)}{A^*}}{B^*[x/|s|]}}{B[x/s]^*}}{|t(s)|\in|B[x/s]| \wedge B[x/s]^*}$$

where the broken underlinings in the second but last line indicate a reformulation according to the fact that $|B|$ does not depend on the subterms of B and to lemma 9 (ii).

($\Sigma E^{\vee}$) By induction hypothesis:

(1) $|t|\in|A|\times|B| \wedge \exists x^{|A|}(A^*\wedge B^*)$;

(2) $\forall x^{|A|}(A^*\to\forall y^{|B|}(B^*\to|s|\in|C|\wedge C^*))$;

(3) $x,y \notin FV(C)$.

We have to show:

(4): $|s[x,y/\mathbf{p}_0 t,\mathbf{p}_1 t]|\in|C| \wedge C^*$.

It suffices to derive C^* from (1) and (2) in $\mathbf{IQC}^\omega$ since $|s[x,y/\mathbf{p}_0 t,\mathbf{p}_1 t]| \in |C|$ already has been proved in Lemma 10 (ii).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(i)}{A^*\wedge B^*}}{B^*} \qquad \dfrac{\dfrac{\overset{\text{lemma 7}}{\mathbf{t}_{|B|}\in|B|} \qquad \dfrac{\dfrac{\overset{(i)}{A^*\wedge B^*}}{A^*} \qquad \dfrac{\overset{(i)}{x\in|A|} \qquad \overset{(2)}{\forall x^{|A|}(A^*\to\forall y^{|B|}(B^*\to|s|\in|C|\wedge C^*))}}{A^*\to\forall y^{|B|}(B^*\to|s|\in|C|\wedge C^*)}\,\forall E}{\forall y^{|B|}(B^*\to|s|\in|C|\wedge C^*)}\to E}{(B^*\to|s|\in|C|\wedge C^*)[y/\mathbf{t}_{|B|}]}\,\forall E}{B^*\to(|s|\in|C|)[y/\mathbf{t}_{|B|}]\wedge C^*}}{(|s|\in|C|)[y/\mathbf{t}_{|B|}]\wedge C^*}\to E}{C^*} \qquad \overset{(1)}{\exists x^{|A|}(A^*\wedge B^*)}}{C^*}\,(i)\,\exists E$$

The hypotheses marked with (i) are discharged in the last $\exists$E-application.

Note that in the last rule application (3) is essentially needed. □

14 LEMMA

Let φ be an L^ω-formula then $\mathbf{IQC}^\omega \vdash \varphi^{+*} \leftrightarrow \varphi$.

Proof: One easily checks the following:

(1) $t \equiv |t^+|$ for $t \in TER^\omega$;

(2) $|\sigma^+| \equiv \sigma$ for $\sigma \in TYP^\omega$;

(3) $\mathbf{IQC}^\omega \vdash \sigma^{+*}$ for $\sigma \in TYP^\omega$.

Therefore we have:

$$(s=t\in\sigma)^{+*} \equiv I(\sigma^+, s^+, t^+)^* \equiv |s^+|=|t^+|\in|\sigma^+| \wedge \sigma^{+*} \leftrightarrow s=t\in\sigma;$$

so:

$$\mathbf{IQC}^\omega \vdash (s=t\in\sigma)^{+*} \leftrightarrow s=t\in\sigma;$$

and:

$$R^n_i(t_1,\ldots,t_n)^{+*} \equiv R^n_i(t_1^+,\ldots,t_n^+)^* \equiv R^n_i(|t_1^+|,\ldots,|t_n^+|) \equiv R^n_i(t_1,\ldots,t_n).$$

Now the lemma is proved by induction on the complexity of φ. As an example we will treat the case of $\varphi\to\psi$.

By definition:

$$(\varphi\to\psi)^{+*} \equiv (\Pi x\in\varphi^+.\psi^+)^* \equiv \forall x\in|\varphi^+|(\varphi^{+*}\to\psi^{+*}).$$

Variable x does not occur in φ^{+*} nor in ψ^{+*}; according to Lemma 7 $\mathbf{t}_{|\varphi^+|}\in|\varphi^+|$, so by $\forall$E

$$\mathbf{IQC}^\omega \vdash (\varphi\to\psi)^{+*} \to (\varphi^{+*}\to\psi^{+*}),$$

and conversely by $\forall$I

$$\mathbf{IQC}^\omega \vdash (\varphi\to\psi)^{+*} \leftarrow (\varphi^{+*}\to\psi^{+*}).$$ □

15 THEOREM

Let $\varphi_1,\ldots,\varphi_m$ and φ be L^ω-formulas with free variables $x_1^{\sigma_1},\ldots,x_n^{\sigma_n}$, then

(1) $\{\varphi_1,\ldots,\varphi_m\}\vdash\varphi$ in $\mathbf{IQC}^\omega$

iff

(2) for some t: $\mathbf{MLP}^{\vee}\vdash x_1\in\sigma_1^+,\ldots,x_n\in\sigma_n^+, y_1\in\varphi_1^+,\ldots,y_m\in\varphi_m^+ \Rightarrow t\in\varphi^+$.

Proof: "$\Rightarrow$" Theorem 6.

"$\Leftarrow$" Suppose (2), and assume $\varphi_1,\ldots,\varphi_m$, which, according to Lemma 14 is quite the same as assuming $\varphi^{+*},\ldots\varphi^{+*}$, then by Proposition 13 from:

$\forall x_1^{\sigma_1}(\sigma_1^{+*}\to\ldots\to\forall y_1\in|\varphi_1^+|(\varphi_1^{+*}\to\ldots\forall y_m\in|\varphi_m^+|(\varphi_m^{+*}\to|t|\in|\psi^+|\wedge\psi^{+*})\ldots)$

Since $\mathbf{IQC}^\omega\vdash\sigma_i^{+*}$ this implies

$\forall x_1^{\sigma_1}\ldots\forall x_n^{\sigma_n}(\forall y_1\in|\varphi_1^+|(\varphi_1^{+*}\to\ldots\forall y_m\in|\varphi_m^+|(\varphi_m^{+*}\to|t|\in|\psi^+|\wedge\psi^{+*})\ldots);$

assuming $x_1 \in \sigma_1, \ldots, x_n \in \sigma_n$

$(\forall y_1 \in |\varphi_1^+|(\varphi_1^{+*} \to \ldots \forall y_m \in |\varphi_m^+|\,(\varphi_m^{+*} \to |t| \in |\psi^+| \wedge \psi^{+*}) \ldots)$.

Assume $\varphi_1^{+*}, \ldots \varphi_m^{+*}$ and apply $\forall$E on $\mathbf{t}_{|\varphi^+|} \in |\varphi_i^+|$ (by Lemma 7) alternating with $\to$E on φ_i^{+*}

$|t| \in |\psi^+| \wedge \psi^{+*}$

Thus ψ can be derived from $\varphi_1, \ldots, \varphi_m$ in $\mathbf{IQC}^\omega$. □

16 COROLLARY

$\mathbf{MLP}^\vee$ is conservative over **IQC**.

Proof: Combining Theorem 15 with Theorem 2. □

17 REMARK

The translation used in our proof reflects a model theoretical construction. Let $\mathcal{K}$ be an L^ω-Kripke model; let $[\![\sigma]\!]'_k$ and $[\![t[\vec{a}]]\!]'_k$ stand for $\mathcal{K}$'s interpretation of type σ and term $t \in TER^\omega$ under assignment $\vec{a}$ in node k respectively, then for **MLP**-type A and **MLP**-term t on context $x_1 \in A_1, \ldots, x_n \in A_n$ define:

$[\![A[\vec{a}]]\!]_k \quad := \{ x \in [\![|A|]\!]'_k : k \Vdash A^*[\vec{a}] \}$;

$[\![t[\vec{a}]]\!]_k \quad := [\![\, |t|\, [\vec{a}]\,]\!]'_k$.

where $\vec{a} = (a_1, \ldots, a_n)$ with $a_i \in [\![A_i[a_1, \ldots, a_{i-1}]]\!]_k$ for $i \leqslant n$. In this way a type A is interpreted very crudely; $[\![A]\!]$ is either $|A|$ or empty depending on whether A^* is a true or not.

A corresponding result for **ML** with regard to $\mathbf{HA}^\omega$ can be proved along the same lines. In Definition 5 add

$0^+ \quad := \mathbf{0}$

$(st)^+ \quad := \mathbf{S}(t^+)$

$\mathbf{r}_{xy}(t, t_0, t_1)^+ \quad := \mathbf{R}_{xy}(t^+, t_0^+, t_1^+)$

For |.| and * add:

$|\mathbb{N}| \quad := \mathbb{N}$

$|\mathbf{0}| \quad := 0$

$|\mathbf{S}t| \quad := \mathbf{S}|t|$

$|\mathbf{R}_{xy}(t, t_0, t_1)| \quad := \mathbf{r}_{xy}(|t|, |t_0|, |t_1|)$

$\mathbb{N}^* \quad := 0 = 0 \in \mathbb{N}$

18 THEOREM
ML$^{\vee}$ is conservative over **HA**$^{\omega}$. □

In his 1980 publication Diller gives a version of Intuitionistic Type Theory (in the sequel referred to as **DL**) that in many respects has nicer properties than **ML**. Diller also defines a restricted system (which we here will call **DL**^{R}), that he states, "is essentially a natural deduction version of N-**HA**$^{\omega}$ "(N-**HA**$^{\omega}$ is equivalent to our **HA**$^{\omega}$). We will use our method to prove:

19 THEOREM
Let $\varphi_1,\ldots,\varphi_m$ and φ be **HA**$^{\omega}$-formulas with free variables $x_1^{\sigma_1},\ldots,x_n^{\sigma_n}$, then the following are equivalent:
(i) $\{\varphi_1,\ldots,\varphi_m\}\vdash\varphi$ in **HA**$^{\omega}$
(ii) for some t with $FV(t)=\{x_1\epsilon\sigma_1^{+},\ldots,x_n\epsilon\sigma_n^{+},y_1\epsilon\varphi_1^{+},\ldots,y_m\epsilon\varphi_m^{+}\}$:
$$\mathbf{DL}^{R}\vdash t\epsilon\varphi^{+}.$$

In Diller's original version **DL** the so called condition on variables replaces the use of contexts. It is not clear however that the effect of the condition on variables completely covers the use of contexts in **ML**. If contexts are used in the formulation of Diller's rules, giving rise to a version say $\mathbf{DL}_C$, the restricted system $\mathbf{DL}_C^{R}$ easily is embedded in **ML**$^{\vee}$, so that Theorem 18 yields Theorem 19 for $\mathbf{DL}_C^{R}$. If the condition on variables is used in stead of contexts, we have to do a little more work in adapting the proof of Theorem 18 to the peculiarities of **DL**^{R}.

20 DEFINITION (of $\mathbf{DL}_C$ and $\mathbf{DL}_C^{R}$)
In $\mathbf{DL}_C$ there are no rules for +-types, the formation rules for the other types are identical to those in **ML**; the term-formation for I-types however is essentially different:

$$\text{II}\quad \frac{\Gamma\Rightarrow t\epsilon A}{\Gamma\Rightarrow \mathbf{r}t\epsilon I(A,t,t)}$$

$$\text{IE}\quad \frac{\Gamma\Rightarrow t\epsilon B[x/s]\qquad \Gamma\Rightarrow t'\epsilon I(A,s,s')\qquad \Gamma,x\epsilon A\Rightarrow B\ Typ}{\Gamma\Rightarrow \mathbf{i}(t,t',s,s')\epsilon B[x/s']}$$

In $\mathbf{DL}_C$ a distinction is made between conversion (denoted as $\Gamma\Rightarrow t\geqslant t'\epsilon A$

for terms, and as $\Gamma \Rightarrow A \geqslant A'$ for types) and equality (i.e. $\Gamma \Rightarrow s \in I(A,t,t')$). In the rules for I-types we see that equality does not automatically lead to convertibility as it does in the IE rule of **ML**. Conversion in $\mathbf{DL_C}$ is generated by the same reduction rules as in **ML**, except for the IRED-rule and is reflexive (REFL), transitive (TRANS) and obeys replacement, i.e:

$$\text{IRED} \quad \frac{\Gamma \Rightarrow t \in B[x/s] \qquad \Gamma \Rightarrow s = s \in A \qquad \Gamma, x \in A \Rightarrow B \text{ Typ}}{\Gamma \Rightarrow \mathbf{i}(t, \mathbf{r}s, s, s) \geqslant t \in B[x/s]}$$

$$\text{REPL}_{1,2} \quad \frac{\Gamma \Rightarrow t \geqslant t' \in A \qquad \Gamma, x \in A \Rightarrow s \in B}{\Gamma \Rightarrow s[x/t] \geqslant s[x/t'] \in B[x/t]} \qquad \frac{\Gamma \Rightarrow t \geqslant t' \in A \qquad \Gamma, x \in A \Rightarrow B \text{ Typ}}{\Gamma \Rightarrow B[x/t] \geqslant B[x/t']}$$

Term-equations (i.e. statements of the form $\Gamma \Rightarrow t \geqslant t' \in A$) only appear as premiss in TRANS or REPL, in the latter case resulting in another term-equation or a type-equation. In their turn type-equations can only occur as premiss in TRANS and in the *Ersetzungsregeln*:

$$\text{ERS}_{1,2} \quad \frac{\Gamma \Rightarrow t \in A \qquad \Gamma \Rightarrow A \geqslant B}{\Gamma \Rightarrow t \in B} \qquad \frac{\Gamma \Rightarrow t \in A \qquad \Gamma \Rightarrow A \leqslant B}{\Gamma \Rightarrow t \in B}$$

Note that in $\mathbf{DL_C}$ convertibility $t \geqslant t' \in A$ implies equality i.e. $s \in I(A,t,t')$ for some term s: if $t \geqslant t' \in A$ then $\mathbf{r}t \in I(A,t,t')$ via ERS and REPL, whereas there is no way to derive $t \geqslant t' \in A$ from $s \in I(A,t,t')$, since term equations can only be introduced by application of reduction rules.

Define FIN the class of Finite Types as the smallest class such that :

$$N \in FIN;$$

$$\sigma, \tau \in FIN \Rightarrow \sigma \times \tau, \sigma \to \tau \in FIN,$$

and let PRT stand for the collection of $\mathbf{DL_C}$-terms all of whose subterms are of finite type.

In the restricted system $\mathbf{DL_C}^R$ extra conditions prohibit the formation and use of types that are not interpretable as $\mathbf{HA}^\omega$-formulas; in ΣTYP and ΠTYP the variable to be bound should either be of finite type or absent, resulting in either a genuine ∃/∀-quantification or a ∧/→-formation. In case of ΠE and ΣI for genuine quantification only PRT-terms are permitted as argument and as witness respectively. For the same reason the term $t \in N$ in NE should be PRT:

$$\text{NE}\quad \frac{\Gamma\Rightarrow t\in N \quad \Gamma\Rightarrow t_0\in A[\mathbf{0}] \qquad \Gamma,x\in\mathbb{N},y\in A[x]\Rightarrow t_1\in A[x/\mathbf{S}x]}{\Gamma\Rightarrow R_{xy}(t,t_0,t_1)\in A[x/t]}\ t\in PRT$$

The rule of ΣE is weakened just as in **ML**$^{\vee}$. Finally IE is restricted to equations:

$$\text{IE}'\quad \frac{\Gamma\Rightarrow t\in I(B,t_0,t_1)[x/s] \qquad \Gamma\Rightarrow t'\in I(A,s,s') \qquad \Gamma,x\in A\Rightarrow I(B,t_0,t_1)\ Typ}{\Gamma\Rightarrow \mathbf{i}(t,t',s,s')\in I(B,t_0,t_1)\ [x/s']}$$

The latter restriction however is inessential, as Diller states, since the general IE rule can easily be derived by induction from the restricted version. As in the case of **ML**$^{\vee}$ we have Soundness:

21 THEOREM (Soundness)
Let $\varphi, \varphi_1,\ldots,\varphi_m$ be $\mathbf{HA}^{\omega}$-formulas with free variables $x_1\in\sigma_1,\ldots,x_n\in\sigma_n$ such that in $\mathbf{HA}^{\omega}$: $\{\varphi_1,\ldots,\varphi_m\}\vdash\varphi$, then
for some t:

$$\mathbf{DL}_C{}^R\vdash x_1\in\sigma_1{}^+,\ldots,x_n\in\sigma_n{}^+,y_1\in\varphi_1{}^+,\ldots,y_m\in\varphi_m{}^+\Rightarrow t\in\varphi^+.$$

<u>Proof</u>: By induction on the derivation in $\mathbf{HA}^{\omega}$. □

22 DEFINITION (of the embedding °)
The only operators of $\mathbf{DL}_C{}^R$ not available in **ML**$^{\vee}$ are the quaternary **i** operator introduced in IE and the **r** operator introduced in II. Let T be a term or a type in $\mathbf{DL}_C{}^R$, we define T^o as the result of systematically replacing all occurrences of the form $\mathbf{i}(t,t',s,s')$ in T by t^o, and of **r**t in T by **e**. Let $\Gamma\equiv x_1\in A_1,\ldots,x_n\in A_n$ be a $\mathbf{DL}_C{}^R$ context, then by Γ^o we will refer to the sequence: $x_1\in A_1{}^o,\ldots,x_n\in A_n{}^o$. Let φ be a judgement, we define:

if $\varphi\equiv A$ Typ	then $\varphi^o\equiv A^o$ Typ
if $\varphi\equiv t\in A$	then $\varphi^o\equiv t^o\in A^o$
if $\varphi\equiv t\geqslant t'\in A$	then $\varphi^o\equiv t^o=t'^o\in A^o$
if $\varphi\equiv A\geqslant A'$	then $\varphi^o\equiv A^o=A'^o$

23 PROPOSITION
If $\mathbf{DL}_C{}^R\vdash\Gamma\Rightarrow\varphi$ then $\mathbf{ML}^{\vee}\vdash\Gamma^o\Rightarrow\varphi^o$.

<u>Proof</u>: By induction on the length of the derivation in $\mathbf{DL}_C{}^R$, for instance the

case of IE:

By induction hypothesis we have in $\mathbf{ML}^{\vee}$:

(1) $\Gamma^o \Rightarrow t^o \in B[x/s]^o$

(2) $\Gamma^o \Rightarrow t'^o \in I(A,s,s')^o$

(3) $\Gamma^o, x\in A^o \Rightarrow B^o$ Typ

Then the demonstrandum is derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(2)}{\Gamma^o \Rightarrow t'^o \in I(A,s,s')^o}}{\Gamma^o \Rightarrow t'^o \in I(A^o,s^o,s'^o)}}{\Gamma^o \Rightarrow s^o = s'^o \in A^o}\,\text{IE} \qquad \overset{(3)}{\Gamma^o, x\in A^o \Rightarrow B^o \text{ Typ}}}{\Gamma^o \Rightarrow B^o[x/s^o] = B^o[x/s'^o]}\,\text{REPL}_1 \qquad \dfrac{\overset{(1)}{\Gamma^o \Rightarrow t^o \in B[x/s]^o}}{\Gamma^o \Rightarrow t^o \in B^o[x/s^o]}}{\dfrac{\Gamma^o \Rightarrow t^o \in B^o[x/s'^o]}{\Gamma^o \Rightarrow \mathbf{e}(t,t',s,s')^o \in B[x/s']^o}}\,\text{REPL}_3$$

Most other cases are completely trivial. □

24 COROLLARY

$\mathbf{DL}_C^R$ is a conservative extension of $\mathbf{HA}^\omega$.

<u>Proof</u>: One easily checks:

$\varphi^{+o} \equiv \varphi^+$ for all φ $\mathbf{HA}^\omega$-formulas

$\sigma^{+o} \equiv \sigma^+$ for all $\sigma \in FIN$

Now let $\varphi, \varphi_1, \ldots, \varphi_n$ be $\mathbf{HA}^\omega$-formulas with free variables $x_1\in\sigma_1, \ldots, x_m\in\sigma_m$ and suppose $\mathbf{DL}_C^R \vdash x_1\in\sigma_1, \ldots, x_m\in\sigma_m, y_1\in\varphi_1^+, \ldots, y_n\in\varphi_n^+ \Rightarrow t\in\varphi^+$ for some t, then by Proposition 23:

$$\mathbf{ML}^{\vee} \vdash x_1\in\sigma_1, \ldots, x_m\in\sigma_m, y_1\in\varphi_1^+, \ldots, y_n\in\varphi_n^+ \Rightarrow t^o\in\varphi^+$$

and by Theorem 18:

$$\mathbf{HA}^\omega: \{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$$

□

25 DEFINITION (of **DL** and $\mathbf{DL}^R$)

The only difference between **DL** and $\mathbf{DL}_C$ and between $\mathbf{DL}^R$ and $\mathbf{DL}_C^R$ is the use of contexts. In **DL** no contexts are used, instead all variables bear their type as a superscript. Variables that occur in the type of another variable should not be bound, this is expressed in the so-called condition on variables:

$x_1^{A_1},\ldots,x_n^{A_n}$ satisfy the variable condition with regard to t iff x_i does not occur in the type of any free variable of t other than $x_{i+1},\ldots,x_n$. (notation: $cov(x_1^{A_1},\ldots,x_n^{A_n};t)$)

The condition on variables occurs in those rules where **ML** demands certain variables to stand at the end of the context, as is the case in ΠTYP and ΣTYP with $cov(x^A;B)$, ΠI with $cov(x^A;t)$ and $cov(x^A;B)$, ΣI with $cov(x^A;B)$, $\Sigma E^{\vee}$ with $cov(x^A,y^B;t)$ and NE with $cov(x^N,y^A;t'')$.

Both |.| and * are defined as in the case of $\mathbf{HA}^\omega$ (Theorem 18). The collapse |.| is defined as before, and along the same lines one proves:

26 LEMMA

If $\mathbf{DL}^R \vdash t \in A$

then $\mathbf{HA}^\omega \vdash |t| \in |A|$. □

27 THEOREM (Soundness)

Let $\varphi, \varphi_1,\ldots,\varphi_m$ be $\mathbf{HA}^\omega$-formulas with free variables $x_1\in\sigma_1,\ldots,x_n\in\sigma_n$ such that in $\mathbf{HA}^\omega : \{\varphi_1,\ldots,\varphi_m\}\vdash\varphi$, then

for some t with $FV(t)\subseteq\{x_1\in\sigma_1^+,\ldots,x_n\in\sigma_n^+,y_1\in\varphi_1^+,\ldots,y_m\in\varphi_m^+\}$:

$\mathbf{DL}^R \vdash t\in\varphi^+$.

PROOF OF THEOREM 19

It suffices to show that:

If $\mathbf{DL}^R \vdash t \in A$ and $FV(t)\subseteq\{x_1^{A_1}, \ldots, x_n^{A_n}\}$

then in $\mathbf{HA}^\omega$: $\{A_1^*,\ldots,A_n^*\}\vdash A^*$

which is established by induction on the length of the derivation in $\mathbf{DL}^R$, under the following hypotheses:

if $\mathbf{DL}^R \vdash t\in A$ and $FV(t)\subseteq\{x_1^{A_1}, \ldots, x_n^{A_n}\}$

then $\mathbf{HA}^\omega \vdash A_1^* \wedge \ldots \wedge A_n^* \rightarrow A^*$

if $\mathbf{DL}^R \vdash t \geqslant t' \in A$ then $\mathbf{HA}^\omega \vdash |t| = |t'| \in |A|$

if $\mathbf{DL}^R \vdash A \geqslant B$ then $\mathbf{HA}^\omega \vdash A^* \leftrightarrow B^*$

For instance the case of $\Sigma E^{\vee}$:

$$\Sigma E^{\vee} \quad \frac{t \in \Sigma x^A.B \qquad t' \in C}{t'[x,y/\mathbf{p}_0 t, \mathbf{p}_1 t] \in C} \qquad cov(x^A,y^B,t'),\ x^A,y^B \notin FV(C)$$

The only interesting situation arises when $x^A, y^B \in FV(t')$ and $x^A, y^B \notin FV(t)$, in which case $FV(t'[x^A,y^B/\mathbf{p}_0 t, \mathbf{p}_1 t]) = FV(t') \setminus \{x^A,y^B\} \cup FV(t)$, say $FV(t'[x^A,y^B/\mathbf{p}_0 t, \mathbf{p}_1 t]) = \{x_1^{A_1}, \ldots, x_n^{A_n}\}$, then by induction hypothesis:

(1) $\{A_1^*, \ldots A_n^*\} \vdash \exists x^{|A|}(A^* \wedge B(x)^*)$,

(2) $\{A_1^*, \ldots A_n^*, A^* \wedge B(x)^*\} \vdash C^*$;

because of $cov(x^A, y^B, t')$ the variable $x^{|A|}$ does not occur free in $A_1^*, \ldots A_n^*$, nor does $x^{|A|}$ occur free in C^*, so by $\exists E$ from (1) and (2):

$$\{A_1^*, \ldots A_n^*\} \vdash C^*. \qquad \square$$

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