Institute for Language, Logic and Information

PROVABILITY LOGICS FOR RELATIVE INTERPRETABILITY

Dick de Jongh
Frank Veltman

ITLI Prepublication Series
for Mathematical Logic and Foundations ML-88-03

University of Amsterdam

Institute for Language, Logic and Information
Instituut voor Taal, Logica en Informatie

PROVABILITY LOGICS FOR RELATIVE INTERPRETABILITY

Dick de Jongh
Department of Mathematics and Computer Science
University of Amsterdam
Frank Veltman
Department of Philosophy
University of Amsterdam

MSC-1980 classification 03B45/03F30
Key Words and phrases: Provability logic, interpretability, completeness proofs, fixed points

Received July 1988

To be published in the Proceedings of the Heyting Conference
Varna, September 1988

Correspondence to:

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Roetersstraat 15
1018WB Amsterdam

or

Faculteit der Wijsbegeerte
(Department of Philosophy)
Grimburgwal 10
1012GA Amsterdam

0. Introduction.

In this paper the system **IL** of modal logic for relative interpretability described in Visser (1988) is studied. In **IL** formulae $A \rhd B$ (read: A *interprets* B) are added to the provability logic **L** with as their intended interpretation in (arithmetical) theories T: T + B is relatively interpretable in T + A. The system has been shown to be sound with respect to such arithmetical interpretations(Švejdar 1983, Montagna 1984, Visser 1986, 1988P).

With respect to priority of parentheses we treat $\rhd$ as $\rightarrow$. As axioms for **IL** we take the usual axioms $\Box A \rightarrow \Box\Box A$ and $\Box(\Box A \rightarrow A) \rightarrow \Box A$ (*Löb's Axiom*) for the provability logic **L** and its rules, modus ponens and necessitation, plus the axioms:

(1) $\Box(A \rightarrow B) \rightarrow (A \rhd B)$

(2) $(A \rhd B) \wedge (B \rhd C) \rightarrow (A \rhd C)$

(3) $(A \rhd C) \wedge (B \rhd C) \rightarrow (A \vee B \rhd C)$

(4) $(A \rhd B) \rightarrow (\Diamond A \rightarrow \Diamond B)$

(5) $\Diamond A \rhd A$

Furthermore, we will consider the following extensions of **IL**:

ILM = **IL** + M, where M is the following axiom

$$(A \rhd B) \rightarrow (A \wedge \Box C \rhd B \wedge \Box C)$$

ILP = **IL** + **P**, where **P** is the following axiom

$$(A \rhd B) \rightarrow \Box(A \rhd B)$$

We will write $\vdash_{IL}$ for derivability in **IL**, similarly for the other systems, but sometimes we may leave the subscript off.

The object of the whole study, undertaken together with Smoryński and Visser is to obtain for the standard formal systems an analogon of Solovay's theorem: which are the interpretability logics corresponding to **PA**, **GB** etc? The provability logics of all these systems are the same as Solovay's Theorem shows, but the interpretability logics are not. The logic **ILP** has been proved by Smoryński and Visser to be complete in this sense with respect to **GB** and other finitely axiomatizable systems. Conjectures are that **ILM** is the logic of **PA** and other essentially reflexive systems. A third system

ILW = **IL** + **W**, where **W** is the following axiom

$$(A \rhd B) \to (A \rhd B \wedge \Box\neg A)$$

is weaker than both other logics, and is conjectured to embody the principles common to all "reasonable" arithmetics. For more details one should consult Visser's paper in this volume.

In this paper we restrict ourselves to purely modal properties of the systems in question. In section 1 the semantics for the different logics is described. In section 2 the fixed point theorem of **L** is extended to **IL** (this result was reached in cooperation with Visser). In the remaining sections modal completeness theorems are proved for the systems **IL**, **ILP** and **ILM**. The logics also turn out to have the finite model property, so decidability is a consequence. We are still working on a completeness proof for **ILW**.

1. Semantics[1].

It is a well-known fact that the modal logic **L** is complete with respect to the **L**-*frames* $\langle W,R\rangle$, which consist of a set of worlds W together with a transitive conversely well-founded relation R.

1.1 Definition. If $\langle W,R\rangle$ is a partially ordered set and $w\in W$, then $W[w] = \{w'\in W \mid wRw'\}$.

1.2 Definition. An ***IL-frame*** is a **L**-frame $\langle W,R\rangle$ with an additional relation S_w, for each $w\in W$, which has the following properties:

(i) S_w is a relation on $W[w]$,

(ii) S_w is reflexive and transitive,

(iii) if $w', w''\in W[w]$ and $w'Rw''$, then $w'S_w w''$,

We will often write S for $\{S_w \mid w\in W\}$

---

[1] The original question to provide the system **ILM** with a semantics was due to Albert Visser. Also afterwards he was a continuous source of inspiration and he heeded us from several mistakes.

1.3 Definition. An ***IL-model*** is given by a IL-frame $\langle W, R, S \rangle$ combined with a forcing relation with the clauses:

$u \Vdash \Box A \Leftrightarrow \forall v(uRv \Rightarrow v \Vdash A)$

$u \Vdash A \rhd B \Leftrightarrow \forall v(uRv \text{ and } v \Vdash A \Rightarrow \exists w(vS_u w \text{ and } w \Vdash B))$.

1.4 Definition.

(a) We write $F \vDash A$ iff $F = \langle W, R, S \rangle$, and for every $\Vdash$ on F, $w \Vdash A$ for each $w \in W$;

(b) If $\mathcal{K}$ is a class of frames, we write $\mathcal{K} \vDash A$ iff $F \vDash A$ for each $F \in \mathcal{K}$.

(c) $\mathcal{K}_W$ is the class of frames satisfying
(iv) for any w, the converse of $R \circ S_w$ is wellfounded

(d) $\mathcal{K}_M$ is the class of frames satisfying
(iv') if $uS_w vRz$, then uRz

(e) $\mathcal{K}_P$ is the class of frames satisfying
(iv") if $uS_w v$, then $uS_{w'} v$ for any w' such that wRw', $w'Ru$.

1.5 Lemma (*Soundness*).

(a) For each A, if $\vdash_{IL} A$, then $F \vDash A$.

(b) For $S = W, M, P$, respectively,
$F \vDash \mathbf{ILS} \Leftrightarrow F \in \mathcal{K}_S$ (ILS *characterizes* $\mathcal{K}_S$).

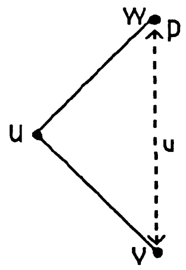
(c) For $S = W, M, P$, respectively, if $\vdash_{ILS} A$, then $\mathcal{K}_S \vDash A$.

Proof. Straightforward. ⊠

In Sections 3 and following completeness will be proved for the three systems **IL**, **ILP** and **ILM**. Actually, **ILP** will be proved complete with respect to the more restricted class of frames frames in which S_w and $S_{w'}$ are identical on the intersection of their domains. We will keep writing **ILS** if we want leave open which system we are aiming at.

1.6 Example. For each of the systems above, $\nvdash \neg((p \rhd \neg p) \wedge (\neg p \rhd p))$.

Proof. The following is a countermodel:



In the above picture only the "extra" arrows for S_u are indicated. Note that in an arithmetical interpretation such a formula would be what is called an *Orey-sentence* (see e.g. Visser 1986). Note also that one could make this model into one in which S_u is antisymmetric; however, the procedure would make the model infinite. ⊠

In the case of provability logic validity on trees is equivalent to validity on L-frames. In the case of interpretability logic this is not generally the case.

1.7 Proposition. The formula $\Box(p \to \neg q \wedge \Box\neg q) \wedge (p \rhd q) \to (p \rhd q \wedge \Box\bot)$ is valid on all **ILM**-models on trees, but $\mathcal{K}_{\mathbf{ILM}} \nvDash \Box(p \to \neg q \wedge \Box\neg q) \wedge (p \rhd q) \to (p \rhd q \wedge \Box\bot)$ and hence $\nvdash_{\mathbf{ILM}} \Box(p \to \neg q \wedge \Box\neg q) \wedge (p \rhd q) \to (p \rhd q \wedge \Box\neg\bot)$.

Proof. Left to the reader. ⊠

Of course the usual procedure for "stretching out" a partially ordered model into a tree works in this case. The point is that property (iv') will get lost: it will no longer generally hold that, if $w'S_w w''Ru$, then $w'Ru$; the only thing one can say of u then is that it will have a forcing relation identical to that of some successor of w', and hence the resulting model will no longer be an **ILM**-model in our sense. For **IL**, **ILW** and **ILP**, on the other hand, one can restrict oneself to tree models.

2. Fixed points.

From the fact that **IL** is an extension of **L** it is obvious that to prove the existence of explicit fixed points in **IL** it is actually sufficient to find a fixed point for $A(p)\rhd B(p)$, i.e. to find a formula C such that $\vdash_{IL} C\leftrightarrow A(C)\rhd B(C)$. For, after that we can proceed as in the standard proof for L (see Smoryński 1985). One might conjecture that $C=A(\top)\rhd B(\top)$ would do the trick, and in fact that formula does work for **ILM** (as the reader may check). However, for **IL** a more complicated formula is necessary: $C = A(\top)\rhd B(\Box\neg A(\top))$. (The even more complicated, but more symmetric formula $A(\Box\neg A(\top))\rhd B(\Box\neg A(\top))$ is equivalent to C and therefore works too.) We will give a semantic proof (see Visser 1988 for a syntactic proof[2]). Of course, the present proof does need the completeness of IL proved in section 3. To establish the fixed point property we have to show:

2.1 Lemma.

$w\Vdash A(\top)\rhd B(\Box\neg A(\top)) \Leftrightarrow$

$w\Vdash A(A(\top)\rhd B(\Box\neg A(\top)))\rhd B(A(\top)\rhd B(\Box\neg A(\top)))$.

Proof. We first establish some simple general facts, for arbitrary w. If we give them without comment their proof is trivial. We write $u\Vdash_{max}A$ iff $u\Vdash A$ and $\forall v(uRv\Rightarrow v\nVdash A)$, and we write $w\underline{R}u$ for wRu or $w=u$.

(1) $w\Vdash D\rhd E \Leftrightarrow \forall u(wRu\wedge u\Vdash_{max}D\Rightarrow\exists v(uS_w v\wedge v\Vdash E))$;

(2) if $w\Vdash\Box D$ and wRu, then $u\Vdash\Box D$;

(3) if $w\Vdash_{max}D$, then $w\Vdash\Box\neg D$;

(4) if $w\Vdash\Box\neg D$, then, if $w\underline{R}u$, then $u\Vdash D\rhd E$;

(5) if $w\Vdash_{max}D$, then, if $w\underline{R}u$, then $u\Vdash D\rhd E$;

(6) if $w\Vdash_{max}D$, then $w\Vdash_{max}A(\top) \Leftrightarrow w\Vdash_{max}A(D\rhd E)$; by (5), as $w\Vdash$ can only be depend on $u\Vdash$ for u with $w\underline{R}u$, since e.g. $wRvRv'S_vu$ implies wRu by Def.1.2.(i);

(7) if $w\Vdash_{max}A(\top)$, then $w\Vdash_{max}A(A(\top)\rhd E)$, by (6);

[2] The fixed point theorem was established first semantically for **ILM** by the authors. Then Visser obtained the theorem syntactically, also for the more complicated case of **IL**.

(8) if $w \Vdash_{max} A(A(T) \rhd E)$, then $w \Vdash A(T)$;
for assume $w \Vdash_{max} A(A(T) \rhd E)$, then, by (7), $wRu \Rightarrow u \Vdash \neg A(T)$, hence, for all u with $w\underline{R}u$, $u \Vdash A(T) \rhd E$. As in (6), $w \Vdash A(T)$ follows from $w \Vdash A(A(T) \rhd E)$;

(9) if $w \Vdash D \rhd E$, then, for all u with $w\underline{R}u$, if $u \Vdash \Box\neg E$, then $u \Vdash \Box\neg D$;

(10) if $w \Vdash_{max} E$, then, for all u with $w\underline{R}u$, $u \Vdash \Box\neg D \Leftrightarrow u \Vdash D \rhd E$;

(11) if $w \Vdash_{max} E$, then, for all u with $w\underline{R}u$,
$u \Vdash B(\Box\neg D) \Leftrightarrow u \Vdash B(D \rhd E)$;

(12) if $w \Vdash_{max} E$, then, for all u with $w\underline{R}u$,
$u \Vdash_{max} B(\Box\neg D) \Leftrightarrow u \Vdash_{max} B(D \rhd E)$;

(13) if $w \Vdash_{max} B(\Box\neg D)$, then $w \Vdash_{max} B(D \rhd B(\Box\neg D))$;

(14) if $w \Vdash_{max} B(D \rhd B(\Box\neg D))$, then $w \Vdash_{max} B(\Box\neg D)$;
for assume $w \Vdash_{max} B(D \rhd B(\Box\neg D))$, then, by (13), $w \Vdash \Box\neg B(\Box\neg D)$. So, by (4), for all u with $w\underline{R}u$, $u \Vdash D \rhd B(\Box\neg D) \Leftrightarrow u \Vdash \Box\neg D$; so $w \Vdash_{max} B(\Box\neg D)$.

Now we establish the main claim:
$\Rightarrow$: Let $w \Vdash A(T) \rhd B(\Box\neg A(T))$. Assume wRu and $u \Vdash_{max} A(A(T) \rhd B(\Box\neg A(T)))$. By (8) $u \Vdash A(T)$. So, for some v with $uS_w v$, $v \Vdash B(\Box\neg A(T))$. We may just as well assume $v \Vdash_{max} B(\Box\neg A(T))$, as $uS_w vRv'$ implies $uS_w v'$ by def. 1.2(iii). By (13) this implies $v \Vdash B(A(T) \rhd B(\Box\neg A(T)))$.
$\Leftarrow$: Let $w \Vdash A(A(T) \rhd B(\Box\neg A(T))) \rhd B(A(T) \rhd B(\Box\neg A(T)))$. Assume wRu, $u \Vdash_{max} A(T)$. By (7) $u \Vdash A(A(T) \rhd B(\Box\neg A(T)))$. So, for some v with $uS_w v$, $v \Vdash B(A(T) \rhd B(\Box\neg A(T)))$. Again we may assume that $v \Vdash_{max} B(A(T) \rhd B(\Box\neg A(T)))$, and (14) gives us $v \Vdash B(\Box\neg A(T))$. ⊠

For completeness sake we formulate the explicit fixed point theorem.

2.2 Theorem. For each IL-formula $A(p, q_1, \ldots, q_n)$ in which p occurs only modalized (i.e. all occurrences of p are under some $\Box$ or $\rhd$) there is a provably unique IL-formula $B(q_1, \ldots, q_n)$ such that $\vdash_{IL} A(B(q_1, \ldots, q_n), q_1, \ldots, q_n) \leftrightarrow B(q_1, \ldots, q_n)$.

3. Modal completeness: preliminaries.

The usual method in modal logic for obtaining completeness proofs is to construct directly or indirectly the necessary countermodels by taking maximal consistent sets of the logic under consideration as the worlds of the model (without necessarily one consistent set standing for only one world) and providing this set of worlds with an appropriate relation R. This method cannot be applied here, since the logic is not compact: one does not generally get conversely well-founded models on an infinite set of worlds. The solution is to restrict the maximal consistent sets to subsets of some finite set of formulae. Such a so-called adequate set has to be rich enough to handle the truth definition, and hence has to be closed under the forming of subformulae and single negations. Furthermore, for each particular logic, additional requirements on the adequate set will be needed to be able to apply the axioms.

3.1 Definition. An *adequate* set of formulae is a set Φ which fulfills the following conditions:

(i) Φ is closed under the taking of subformulae,

(ii) if $B\in\Phi$, and B is no negation, then $\neg B\in\Phi$

(iii) $\bot\rhd\bot\in\Phi$.

(iv) if $B\rhd C\in\Phi$, then also $\Diamond B, \Diamond C\in\Phi$

(iv) if B as well as C is an antecedent or a consequent of some $\rhd$-formula in Φ, then $B\rhd C\in\Phi$.

Obviously, each finite set Γ of formulae is contained in a finite adequate set Φ.

3.2 Definition. Let Γ and Δ be two maximal **ILS**-consistent subsets of some finite adequate Φ. Then

$\Gamma\prec\Delta \iff$ for each $\Box A\in\Gamma$, $\Box A, A\in\Delta$, and for some $\Box A\notin\Gamma$, $\Box A\in\Delta$

Whenever $\Gamma\prec\Delta$, we say that Δ is a *successor* of Γ.

3.3 Lemma. Let Γ_0 be a maximal **ILS**-consistent subsets of some finite adequate Φ. Let W_{Γ_0} be the smallest set such that

(i) $\Gamma_0\in W$;

(ii) if $\Delta \in W$ and Δ' is a maximal **ILS**-consistent subset of Φ such that $\Delta \prec \Delta'$, then $\Delta' \in W$.

Then

(i) $\prec$ is transitive and irreflexive on W_{Γ_0}.

(ii) For each $\Gamma \in W_{\Gamma_0}$, $\Box A \in \Gamma \iff A \in \Delta$ for every Δ such that $\Gamma \prec \Delta$.

Proof. As in the case of L. (i) is trivial, and so is $\Rightarrow$ of (ii). For $\Leftarrow$ of (ii) one needs Löb's axiom. ⊠

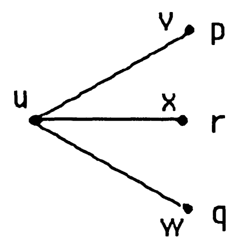
One might think that this means that, in essence, the completeness problem for **ILS** reduces to defining relations $\preccurlyeq_\Delta$ on W_{Γ_0} such that

(i) $\preccurlyeq$ has all the properties of the relation S in $\mathcal{K}_S$

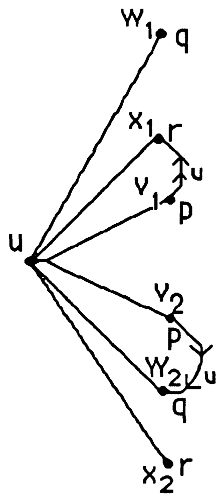
(ii) For each Γ in W_{Γ_0}, $B \rhd C \in \Gamma$ iff for every Δ such that $\Gamma \prec \Delta$ and $B \in \Delta$ there is some Δ' with $\Delta \preccurlyeq_\Gamma \Delta'$ and $C \in \Delta'$.

The situation is not as simple as that. Before we continue with the the completeness proofs, we will give an example to make this clear.

3.4 Example. It will be obvious that $\nvdash_{ILS} (p \rhd q \vee r) \rightarrow (p \rhd q) \vee (p \rhd r)$. Now, take Γ_0 to be a maximal **ILS**-consistent set in Φ that contains $p \rhd q \vee r$, $\neg(p \rhd q)$, and $\neg(p \rhd r)$, as well as the formulae $\Box\Box\bot$, $\Box(p \vee q \vee r)$, $\Box\neg(p \wedge q)$, $\Box\neg(q \wedge r)$, and $\Box\neg(p \wedge r)$. It is then clear that the resulting W_{Γ_0} will look as follows:



It will also be clear that no S_u can be defined on this model in such a way that $u \Vdash p \rhd q \vee r, \neg(p \rhd q), \neg(p \rhd r)$. By doubling W[u] however an appropriate model can be obtained (the arrows give the additional S_u-relations not given by R):



To overcome this problem, our strategy in the next section will be to multiply the maximal **ILS**-consistent sets by indexing them with finite sequences of formulae. We write $\tau \subseteq \tau'$ iff the finite sequence τ is a (not necessarily proper) initial segment of the finite sequence τ'; we write $*$ for concatenation, and, if $w = \langle \Gamma, \tau \rangle$, we write $(w)_0$ for Γ and $(w)_1$ for τ.

Using these pairs we set aside, for each world w and each appropriate formula C, a specific set of the successors of w indexed by C (the so-called critical C-successors of w) to provide the counterexamples to the formulae $B \rhd C$ that must be falsified in w. We will restrict the relation S_w so that it does not "leave" this set of C-critical successors. Speaking intuitively, the C-critical successors of w will be the ones that contain no formula B that "asks for" C (where B is an antecedent and C the consequent of a $\rhd$-formula in w). The next two lemmas show that this whole idea is feasible. The first one says that indeed a counterexample can be found, when needed: for each $\neg(B \rhd C)$ in w a C-critical successor with B in it can be found. The second one says that we will be allowed to restrict the S_w to the C-critical successors: if $A \rhd D$ is a member of w, and A is a member of a C-critical successor of w, then another C-critical successor of w with D in it can be found.

3.5 Definition. Let Γ and Δ be maximal **ILS**-consistent subsets of some given adequate Φ. Then Δ is a C-*critical* successor of Γ iff

(i) $\Gamma \prec \Delta$;

(ii) $\neg A, \Box\neg A \in \Delta$ for each A such that $A \rhd C \in \Gamma$.

Note that successors of C-critical successors of Γ are C-critical successors of Γ.

3.6 Lemma. Suppose Γ is maximal **ILS**-consistent in Φ and $\neg(B \rhd C) \in \Gamma$; then there exists a C-critical successor Δ of Γ, maximal **ILS**-consistent in Φ, such that $B \in \Delta$.

Proof. Take Δ to be a maximal **ILS**-consistent extension of

$\{D, \Box D \mid \Box D \in \Gamma\} \cup \{\neg A, \Box\neg A \mid A \rhd C \in \Gamma\} \cup \{B, \Box\neg B\}$

Note first that the adequacy of Φ insures that all the formulae of Δ are indeed available. Secondly, note that if such a Δ exists, it is indeed a C-critical successor of Γ: the fact that

$$\{D, \Box D \mid \Box D \in \Gamma\} \cup \{\Box\neg B\} \subseteq \Delta$$

makes it a successor of Γ, and the fact that

$$\{\neg A, \Box\neg A \mid A \rhd C \in \Gamma\} \subseteq \Delta$$

makes it C-critical.

Now, if no such Δ exists, then there are $A_1, \dots, A_m$ and $D_1, \dots, D_k$ with $A_1 \rhd C, \dots, A_m \rhd C \in \Gamma$, $\Box D_1, \dots, \Box D_k \in \Gamma$ such that

$$D_1, \dots, D_k, \Box D_1, \dots, \Box D_k, \neg A_1, \dots, \neg A_m, \Box\neg A_1, \dots, \Box\neg A_m, B, \Box\neg B \vdash \bot.$$

Or, equivalently:

$$D_1, \dots, D_k, \Box D_1, \dots, \Box D_k, \neg(A_1 \vee \dots \vee A_m), \Box\neg(A_1 \vee \dots \vee A_m), B, \Box\neg B \vdash \bot$$

This would mean that:

$$D_1, \dots, D_k, \Box D_1, \dots, \Box D_k, B, \Box\neg B \vdash A_1 \vee \dots \vee A_m \vee \Diamond(A_1 \vee \dots \vee A_m).$$

In other words:

$$D_1, \dots, D_k, \Box D_1, \dots, \Box D_k \vdash B \wedge \Box\neg B \rightarrow A_1 \vee \dots \vee A_m \vee \Diamond(A_1 \vee \dots \vee A_m).$$

Since **IL** contains **L**:

$$\Box D_1, \dots, \Box D_k \vdash \Box(B \wedge \Box\neg B \rightarrow A_1 \vee \dots \vee A_m \vee \Diamond(A_1 \vee \dots \vee A_m))$$

By axiom (1):

$$\Box D_1, \dots, \Box D_k \vdash B \wedge \Box\neg B \rhd A_1 \vee \dots \vee A_m \vee \Diamond(A_1 \vee \dots \vee A_m)$$

In view of particularly the axioms (5) and (3) we have that

$$\vdash A_1 \vee \dots \vee A_m \vee \Diamond(A_1 \vee \dots \vee A_m) \rhd A_1 \vee \dots \vee A_m.$$

So, by axiom (2):

$$\Box D_1, \dots, \Box D_k \vdash B \wedge \Box\neg B \rhd A_1 \vee \dots \vee A_m$$

Given that $A_1 \rhd C, \dots, A_m \rhd C \in \Gamma$, we also have $\Gamma \vdash A_1 \vee \dots \vee A_m \rhd C$ (apply axiom (3)), and so by axiom (2):

$$\Gamma \vdash B \wedge \Box\neg B \rhd C$$

Now, it is not difficult to see that

$\vdash B \rhd B \wedge \Box \neg B$

(To that purpose, note first that

$\vdash (B \wedge \Box \neg B) \vee \Diamond (B \wedge \Box \neg B) \rhd B \wedge \Box \neg B$. Secondly, since **ILS** contains **L**, $\vdash \Box (B \rightarrow (B \wedge \Box \neg B) \vee \Diamond (B \wedge \Box \neg B))$. So by axiom (1), $\vdash B \rhd (B \wedge \Box \neg B) \vee \Diamond (B \wedge \Box \neg B)$. Combining these two facts we find $\vdash B \rhd B \wedge \Box \neg B$.)

Finally, by applying axiom (2) once more, it follows from $\Gamma \vdash B \wedge \Box \neg B \rhd C$ and $\vdash B \rhd B \wedge \Box \neg B$ that

$\Gamma \vdash B \rhd C$

This contradicts the consistency of Γ. ⊠

3.7 Lemma. Suppose $B \rhd C \in \Gamma$ and let Δ be an E-critical successor of Γ with $B \in \Delta$. Then there is an E-critical successor Δ' of Γ with $C \in \Delta'$.

Proof. Suppose there is not such a Δ'. Then there would be $\Box D_1, \ldots, \Box D_n \in \Delta$, and $F_1 \rhd E, \ldots, F_k \rhd E \in \Delta$ such that

$$D_1, \ldots, D_n, \Box D_1, \ldots, \Box D_n, \neg F_1, \ldots, \neg F_k, \Box \neg F_1, \ldots, \Box \neg F_k, C \vdash \bot$$

and, therefore,

$$D_1, \ldots, D_n, \Box D_1, \ldots, \Box D_n \vdash C \rightarrow F_1 \vee \ldots \vee F_k \vee \Diamond (F_1 \vee \ldots \vee F_k)$$

which as before implies:

$$\Box D_1, \ldots, \Box D_n \vdash C \rhd F_1 \vee \ldots \vee F_k.$$

By axiom (2), $B \rhd C \in \Gamma$ implies that $\Gamma \vdash B \rhd F_1 \vee \ldots \vee F_k$ and, by axiom (3), $\Gamma \vdash B \rhd E$. Given the adequacy conditions, this can be strengthened to $B \rhd E \in \Gamma$. Since Δ is an E-critical successor of Γ, this implies $\neg B \in \Delta$, and we have arrived at a contradiction, since it is assumed that $B \in \Delta$. ⊠

4.The Modal completeness of IL.

In this section we just have to carefully adjoin sequences to the maximal IL-consistent sets and see that the intuitive ideas of the previous section can be set to work properly.

4.1 Theorem (*Completeness and decidability of* IL) If $\nvdash_{IL} A$, then there is a finite IL-model K such that $K \nvDash A$.

Proof. Take some finite adequate set Φ containing $\neg A$. Let Γ be a maximally consistent subset of Φ containing $\neg A$.

Now, set W_Γ to be the smallest set of pairs $\langle \Delta, \tau \rangle$, where τ is a finite sequence of formulae from Φ, that fulfills the following requirements:

(i) $\langle \Gamma, \langle\rangle \rangle \in W_\Gamma$

(ii) If $\langle \Delta, \tau \rangle \in W_\Gamma$, then $\langle \Delta', \tau \rangle \in W_\Gamma$ for every successor Δ' of Δ;

(iii) If $\langle \Delta, \tau \rangle \in W_\Gamma$, then $\langle \Delta', \tau * \langle C \rangle \rangle \in W_\Gamma$ for every C-critical successor Δ' of Δ.

W_Γ is finite. (For every Δ, the number of successors of Δ is finite. Moreover, if $\Delta \prec \Delta'$, the number of successors of Δ' is smaller than the number of successors of Δ.)

Observation: If $\langle \Delta, \tau \rangle \in W_\Gamma$ and E occurs in τ, then $\neg E \in \Delta$.

Proof: Show with induction on the construction of W_Γ that if $\langle \Delta, \tau \rangle \in W_\Gamma$ and E occurs in τ then $\neg E, \Box\neg E \in \Delta$.

Define R on W_Γ as follows:

wRw' iff $(w)_0 \prec (w')_0$ and $(w)_1 \subseteq (w')_1$.

It is easy to check that R has all the properties required.

Finally, let uS_wv apply if (I) and (II) hold:

(I) $u, v \in W_\Gamma[w]$;

(II) $(u)_1 = (v)_1 = (w)_1$, or $(u)_1 = (w)_1 * \langle C \rangle * \tau$ and $(v)_1 = (w)_1 * \langle C \rangle * \sigma$ for some C, σ and τ.

We leave it to the reader to check that under this definition S_w will have the required properties:

We are now ready to define

$w \Vdash p$ iff $p \in (w)_0$,

and prove that

for each $A \in \Phi$, $w \Vdash A$ iff $A \in (w)_0$.

The only interesting case to look at in the inductive proof is the one that A is $B \rhd C$, i.e. we have to show that

$$B \rhd C \in (w)_0 \iff \forall u (wRu \wedge B \in (u)_0 \Rightarrow \exists v (uS_wv \wedge C \in (v)_0)):$$

$\Leftarrow$: Suppose $B \rhd C \notin (w)_0$. Then $\neg(B \rhd C) \in (w)_0$. We must show that $\exists u (wRu \wedge B \in (u)_0 \wedge \forall v (uS_wv \rightarrow \neg C \in (v)_0))$. Let Δ be as in lemma 3.6 with $(w)_0$ as Γ, and take u to be $\langle \Delta, (w)_1 * \langle C \rangle \rangle$. Consider any v such that uS_wv. Then C occurs in $(v)_1$. By the observation above, $\neg C \in (v)_0$.

$\Rightarrow$: Suppose $B \rhd C \in (w)_0$. Consider any u such that wRu and $B \in (u)_0$. Let us first assume that $(u)_1 = (w)_1 * \langle E \rangle * \tau$. In that case we can apply lemma 3.7 for $\Gamma = (w)_0$ and $\Delta = (u)_0$ to obtain an E-critical successor Δ' of Γ with $C \in \Delta'$. It suffices now to take $v = \langle \Delta', (w)_1 * \langle E \rangle \rangle$. It is clear that v fulfills all requirements to make $uS_w v$.

If $(u)_1 = (w)_1$, then all we know is that $(w)_0 \prec (u)_0$. Note, however, that every successor of Γ is a $\bot$-critical successor of Γ. (By axiom (4), $\vdash F \rhd \bot \rightarrow \neg F$; hence if $F \rhd \bot \in \Gamma$, then $\neg F \in \Gamma$, and therefore $\neg F, \Box \neg F \in \Delta$ for every Δ such that $\Gamma \prec \Delta$. So we can apply lemma 3.7 for $\Gamma = (w)_0$, $\Delta = (u)_0$, and $E = \bot$, in order to obtain a ($\bot$-critical) successor Δ' of Γ with $C \in \Delta'$. Take $v = \langle \Delta', (w)_1 \rangle$. ⊠

5. The modal completeness of ILP.

5.1 Definition. A set Φ of formulae is ILP-*adequate* iff

(i) Φ is adequate in the sense of definition 3.1

(ii) if $B \rhd C \in \Phi$, then also $\Box(B \rhd C) \in \Phi$.

Obviously, each finite set Γ of formulae is contained in a finite ILP-adequate set Φ.

5.2 Theorem (*Completeness and decidability of* ILP). If $\nvdash_{ILP} A$, then there is a finite ILP-model K such that $K \nVdash A$.

Proof. Take some finite adequate set Φ containing $\neg A$. Let Γ be a maximally consistent subset of Φ containing $\neg A$.

In constructing the model, we multiply the maximal ILP-consistent sets similarly as with IL while at the same time transforming the model into a tree in the standard manner. The purpose of making the model into a tree is insuring that a unique immediate predecessor exists for each world. We can then, in determining S_w "from" a successor u of w use the immediate predecessor w' of u instead of w itself. The latter is reasonable, since formulas $B \rhd C$ are persistent along $\prec$-chains in this logic. A world in the model will be a sequence of pairs

$$\langle\langle \Gamma_0, \tau_0 \rangle, \ldots, \langle \Gamma_{n-1}, \tau_{n-1} \rangle, \langle \Gamma_n, \tau_n \rangle\rangle$$

More precisely, W_Γ is built up according to the following clauses:

(i) $\langle\langle\Gamma,\langle\rangle\rangle\rangle\in W_\Gamma$

(ii) If $\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle\rangle\in W_\Gamma$, and Δ is a successor of Γ then also $\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle,\langle\Delta,\tau_n\rangle\rangle\in W_\Gamma$;

(iii) If $\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle\rangle\in W_\Gamma$ and Δ is a C-critical successor of Γ, then also $\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle,\langle\Delta,\tau_n*\langle C\rangle\rangle\rangle\in W_\Gamma$

If $w=\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle\rangle\in W_\Gamma$, we write $\Delta_w=\Gamma_n$ and $\tau_w=\tau_n$.

We next define R on W_Γ as follows: wRw' iff w is a proper initial segment of w'. Thus, R is transitive and irreflexive. More importantly, every world different from $\langle\langle\Gamma,\langle\rangle\rangle\rangle$ has precisely one immediate R-predecessor.

Note that that the model will treat $\Box$ properly.

We are now ready to define uS_wv as applying if (I) and (II) hold:

(I) wRu, and for every w', if w'Ru then w'Rv

(II) $\tau_u\subseteq\tau_v$

It is easy to check that under this definition S_w will have the required properties.

Next we define

$w\Vdash p$ iff $p\in\Delta_w$,

and prove that

for each $A\in\Phi$, $w\Vdash A$ iff $A\in\Delta_w$.

Again, the only interesting case to look at in the inductive proof is the one that A is $B\rhd C$, i.e. we have to show that

$B\rhd C\in\Delta_w \iff \forall u(wRu\wedge B\in\Delta_u \Rightarrow \exists v(uS_wv\wedge C\in\Delta_v))$:

$\Leftarrow$: Suppose $B\rhd C\notin\Delta_w$. Then $\neg(B\rhd C)\in\Delta_w$. We must show that $\exists u(wRu\wedge B\in\Delta_u\wedge\forall v(uS_wv\rightarrow\neg C\in\Delta_v))$.

Assume $w=\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle\rangle$. Let Δ be as in lemma 3.6 with Γ_n as Γ. Take u to be $\langle\langle\Gamma_0,\tau_0\rangle,\ldots,\langle\Gamma_n,\tau_n\rangle,\langle\Delta',\tau_n*\langle C\rangle\rangle\rangle$ with the Δ' given by that lemma.

Consider any v such that uS_wv. Then C occurs in τ_v. As in the previous case, it is easy to see that this means that $\neg C\in\Delta_v$.

$\Rightarrow$: Suppose $B\rhd C\in\Delta_w$ and wRu with $B\in\Delta_u$. Let w' the(!) immediate predecessor of u. Note that axiom P and the ILP-adequacy of Φ insure that $B\rhd C\in\Delta_{w'}$.

Let us first assume that $\tau_u=\tau_{w'}*\langle E\rangle$. In that case we can apply lemma 3.7 with $\Gamma=\Delta_{w'}$ and $\Delta=\Delta_u$ to obtain an E- critical successor Δ' of Γ with $C\in\Delta'$. It suffices now to take $v=w'*\langle\Delta',\tau_u\rangle$. It is clear that v fulfills all requirements to make uS_wv.

If, on the other hand, $\tau_u = \tau_{w'}$, then all we know is that $\Delta_{w'} \prec \Delta_u$. Recall however that every successor is a $\bot$-critical successor. So, here too, we can apply lemma 3.7 for $\Gamma = \Delta_{w'}$, $\Delta = \Delta_u$, and $E = \bot$, in order to obtain a ($\bot$-critical) successor Δ' of Γ with $C \in \Delta''$. Take $v = w' * \langle \Delta'', \tau_u \rangle$. ⊠

5.3 Corollary (to the proof of theorem 5.2). **ILP** is complete with respect to the frames in which, if wRw', then $S_{w'} = S_w \upharpoonright W[w']$.
Proof. It is clear from the proof that, in the model constructed $u S_w v$ iff $u S_{w'} v$ for the immediate predecessor w' of w. ⊠

The corollary means that we can take the S-relation in **ILP** to be a rigid relation, essentially independent of w.

6. The modal completeness of ILM.

The completeness proof for **ILM** is rather more complicated then the ones for the completeness of **IL** and **ILP**. The first problem arises from the fact that to be able to apply the characteristic axiom $(A \rhd B) \rightarrow (A \wedge \Box C \rhd B \wedge \Box C)$ we are forced to add the consequent of this formula to the adequate set, whenever we have the antecedent.

6.1 Definition. An **ILM**-*adequate* set of formulae is a set Φ which fulfills the conditions:
(i) Φ is closed under the taking of subformulae,
(ii) if B and $C \in \Phi$, then for each Boolean combination D of B and C there is a formula **ILM**-equivalent to D in Φ,
(iii) $\bot \rhd \bot \in \Phi$,
(iv) if $B \rhd C \in \Phi$, then also formulae **ILM**-equivalent to $\Diamond B, \Diamond C \in \Phi$,
(v) if both B and C are antecedent or consequent of some $\rhd$-formula in Φ, then $B \rhd C \in \Phi$,
(vi) if $B \rhd C, D \in \Phi$, then there is a formula **ILM**-equivalent to $B \wedge \Box D \rhd C \wedge \Box D$ in Φ.

With this definition it is, of course, not at all obvious that each finite set is contained in a finite adequate one. The problem in keeping things finite is that with $B\wedge\Box D \rhd C\wedge\Box D$ also $\Diamond(B\wedge\Box D)$ and $\Diamond(C\wedge\Box D)$ will have to be an element of Φ and will do its work via clause (vi) generating new formulae in the adequate set, e.g. $B\wedge\Box D\wedge\Box\neg(B\wedge\Box D) \rhd C\wedge\Box D\wedge\Box\neg(B\wedge\Box D)$. We have to show that this does not lead to an infinite regress: after a while the process starts delivering formulae equivalent to ones which have occurred previously. A little thought will convince the reader that the next lemma shows just that.

6.2 Lemma. Starting with a finite set of formulae $\Diamond B_1,\ldots,\Diamond B_n$, and and closing off under the operation of taking $\Diamond(B_i\wedge\Box\neg B_j)$ (adding each new $\Diamond$-formula to the stock) leads to a finite set of L-equivalence classes of formulae.

Proof. By induction on n. In the case that there is only one formula $\Diamond B$ the process stops immediately, because $\Diamond(B\wedge\Box\neg B)$ is L-equivalent to $\Diamond B$.

Assume the validity of the lemma for n starting formulae and apply the closing off procedure to $\Diamond B_1,\ldots,\Diamond B_{n+1}$. The formulae obtained will be of the forms $\Diamond(B_i\wedge\Box\neg D_1\wedge\ldots\wedge\Box\neg D_k)$ $(1\leqslant i\leqslant n+1)$. For each of these classes we have to show that they contain only a finite number of equivalence classes. We restrict ourselves to the case that $i=1$.

By the induction hypothesis there can be only finitely many formulae $\Diamond(B_1\wedge\Box\neg D_1\wedge\ldots\wedge\Box\neg D_k)$ in which the formula B_1 has not been used in the construction of $D_1,\ldots,D_k$. Now consider a formula $\Diamond(B_1\wedge\Box\neg D_1\wedge\ldots\wedge\Box\neg D_k)$ in which B_1 has been used. This formula is L-equivalent to $\Diamond(B_1\wedge\Box\neg B_1\wedge\Box\neg D_1\wedge\ldots\wedge\Box\neg D_k)$. We now use the fact that

$$\vdash_L \Box\neg B_1\rightarrow\Box(B_1\leftrightarrow\bot) \text{ and } \vdash_L \Box\neg B_1\rightarrow\Box\ldots\Box(B_1\leftrightarrow\bot)$$

From this it easily follows from the presence of $\Box\neg B_1$ in $\Diamond(B_1\wedge\Box\neg B_1\wedge\Box\neg D_1\wedge\ldots\wedge\Box\neg D_k)$, that in each of the $D_1,\ldots,D_k$ occurring in that formula B_1 can L-equivalently be replaced by $\bot$. Now, each of the D_i is built up in such a manner that B_1 occurs only in the context $\Box\neg(B_1\wedge\ldots)$. This means that after replacing B_1 by $\bot$ we get a tautology, which can be left out altogether. We end up with a formula $\Diamond(B_1\wedge\Box\neg E_1\wedge\ldots\wedge\Box\neg E_m)$ in which each of the E_i has been constructed according to procedure from $B_2, \ldots, B_{n+1}$. We already concluded that there can be only finitely many of such formulae. ⊠

6.3 Theorem (*Completeness and decidability of* **ILM**) If $\nvdash_{\mathbf{ILM}} A$, then there is a finite **ILM**-model K such that $K \nvDash A$.

Proof. Take some finite **ILM**-adequate set Φ containing $\neg A$. Let Γ be a maximal **ILM**-consistent subset of Φ containing $\neg A$. Unfortunately, we need more worlds than present in the W_Γ used in the proofs for **IL** and **ILP**.

This time we set W_Γ to be the collection of all pairs $\langle \Delta, \tau \rangle$, with

(i) $\Gamma \prec \Delta$ or $\Gamma = \Delta$

(ii) τ is a finite sequence of formulae from Φ, the length of which does not exceed the the depth[3] of Γ minus the depth of Δ. (So, Γ is only paired off with the empty sequence).

Clearly, W_Γ is finite. Note that the sequence τ in a pair $\langle \Delta, \tau \rangle$ provides no longer sufficient information on the "C-critical" status of Δ.

Define R on W_Γ as follows:

$$wRw' \text{ iff } (w)_0 \prec (w')_0 \text{ and } (w)_1 \subseteq (w')_1 .$$

It is easy to check that R has all the properties required.

We say that u is a C-critical R-successor of w if $(u)_0$ is a C-critical successor of $(w)_0$ and $(u)_1 = (w)_1 * \langle C \rangle * \tau$.

Let $uS_w v$ apply if (I)–(IV) hold:

(I) $u, v \in W_\Gamma[w]$;

(II) $(u)_1 \subseteq (v)_1$

(III) for each A such that $\Box A \in (u)_0$ also $\Box A \in (v)_0$.

(IV) if u is a C-critical R-successor of w, then v is a C-critical R-successor of w.

Let us check right away that under this definition S_w will have the required properties:

(i) that $u, v \in W[w]$ if $uS_w v$, is instantaneous;

(ii) reflexivity and transitivity of S_w are easy to check;

(iii) if $u, v \in W[w]$ and uRv, then (I),(II) and (III) are immediate. As for (IV) it suffices to recall that successors of C-critical successors are C-critical.

---

[3] Γ has *depth* n if if the maximal length of a complete chain $\Gamma = \Delta_0 \prec \ldots \prec \Delta_m$ is n+1.

(iv) Suppose $w'S_w w''Ru$. We must show that $w'Ru$. That $(w')_1 \subseteq (u)_1$ is immediate. That $(w')_0 \prec (u)_0$ follows from $(w'')_0 \prec (u)_0$ combined with (III) for w', w''.

We are now ready to define $w \Vdash p$ iff $p \in (w)_0$ and prove that in that case $w \Vdash A$ iff $A \in (w)_0$, holds for each $A \in \Phi$. Again, we restrict ourselves to the case that A is $B \rhd C$, i.e. we have to show that

$$B \rhd C \in (w)_0 \iff \forall u(wRu \wedge B \in (u)_0 \Rightarrow \exists v(uS_w v \wedge C \in (v)_0)):$$

$\Leftarrow$: Suppose $B \rhd C \notin (w)_0$. Then $\neg(B \rhd C) \in (w)_0$. We must show that

$$\exists u(wRu \wedge B \in (u)_0 \wedge \forall v(uS_w v \to \neg C \in (v)_0)).$$

Let Δ be as in lemma 3.5 with $(w)_0$ as Γ, and take u to be $\langle \Delta, (w)_1 * \langle C \rangle \rangle$. Consider any v such that $uS_w v$. Since u is a C-critical R-successor of w, v will be so, too. Therefore, $\neg C \in (v)_0$.

$\Rightarrow$: Suppose $B \rhd C \in (w)_0$ and let u be such that wRu and $B \in (u)_0$. Let $\{\Box D_1, \ldots, \Box D_n\} = \{\Box D \mid \Box D \in (u)_0\}$. Note that axiom M and the adequacy of Φ insure that $(w)_0$ contains a formula equivalent to

$$B \wedge \Box D_1 \wedge \ldots \wedge \Box D_n \rhd C \wedge \Box D_1 \wedge \ldots \wedge \Box D_n.$$

Let us first assume that $(u)_1 = (w)_1 * \langle E \rangle * \tau$. In that case we can apply lemma 3.7 with $\Gamma = (w)_0$, $\Delta = (u)_0$ taking a formula equivalent to $B \wedge \Box D_1 \wedge \ldots \wedge \Box D_n \rhd C \wedge \Box D_1 \wedge \ldots \wedge \Box D_n$, rather than $B \rhd C$ itself as input. In so doing, we obtain an E-critical successor Δ' of Γ with (i) $C \in \Delta'$ and (ii) $\Box D \in \Delta'$ for each D such that $\Box D \in \Delta'$. It suffices now to take $v = \langle \Delta', (u)_1 \rangle$. It is clear that v fulfills all requirements to make $uS_w v$.

If $(u)_1 = (w)_1$, then all we know is that $(w)_0 \prec (u)_0$. Recall, however, that every successor of Δ is an $\bot$-critical successor of Δ. So we can apply lemma 3.7 for $\Gamma = (w)_0$, $\Delta = (u)_0$, $E = \bot$, and the formula $B \wedge \Box D_1 \wedge \ldots \wedge \Box D_n \rhd C \wedge \Box D_1 \wedge \ldots \wedge \Box D_n$, in order to obtain a ($\bot$-critical) successor Δ' of Γ with $C \in \Delta'$ and $\Box D \in \Delta'$ for each D such that $\Box D \in \Delta$. Take $v = \langle \Delta'', (w)_1 \rangle$. ⊠

Bibliography.

F. Montagna, 1984, Provability in finite subtheories of PA and Relative Interpretability: a Modal Investigation, *Rapporto Matematico 118,* Dipartimento di Matematica, Universita di Siena.

V. Švejdar, 1983, Modal Analysis of generalized Rosser sentences, *Journal of Symbolic Logic 48,* p. 986-999.

C. Smoryński, 1985, *Modal Logic and Self-reference,* Springer-Verlag, New York

A. Visser, 1986, Peano's Smart Children, a provability-logical study of systems with built-in consistency, *Logic Group Preprint Series* No. 14, Department of Philosophy, University of Utrecht, to be published in *The Notre Dame Journal of Formal Logic.*

A. Visser, 1988P, Preliminary Notes on Interpretability Logic, *Logic Group Preprint Series No. 29,* Department of Philosophy, University of Utrecht.

A. Visser, 1988, Notes on Interpretability Logic, This Volume.

The ITLI Prepublication Series

1986

86-01	The Institute of Language , Logic and Information
86-02 Peter van Emde Boas	A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem	Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens	A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh	Some Complete Logics for Branched Time, Part I
86-06 Johan van Benthem	Logical Syntax

1987

87-01 Jeroen Groenendijk, Martin Stokhof	Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch	Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer	Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem	Polyadic quantifiers
87-05 Víctor Sánchez Valencia	Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen	Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem	Categorial Grammar and Type Theory
87-08 Renate Bartsch	The Construction of Properties under Perspectives
87-09 Herman Hendriks	Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:

LP-88-01 Michiel van Lambalgen	Algorithmic Information Theory
LP-88-02 Yde Venema	Expressiveness and Completeness of an Interval Tense Logic
LP-88-03	Year Report 1987
LP-88-04 Reinhard Muskens	Going partial in Montague Grammar
LP-88-05 Johan van Benthem	Logical Constants across Varying Types
LP-88-06 Johan van Benthem	Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch	Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof	Context and Information in Dynamic Semantics

Mathematical Logic and Foundations:

ML-88-01 Jaap van Oosten	Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen	The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman	Provability Logics for Relative Interpretability

Computation and Complexity Theory:

CT-88-01 Ming Li, Paul M.B.Vitanyi	Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid	General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars Leen Torenvliet, Peter van Emde Boas	Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks Gerard R. Renardel de Lavalette	Computations in Fragments of Intuitionistic Propositional Logic