A simple inductive argument to compute more Kleinberg sequences under the Axiom of Determinacy

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Abstract. A characteristic feature of infinitary combinatorics under the Axiom of Determinacy is the existence of sequences of partition cardinals, called *Kleinberg sequence*. It is well known that there are lots of Kleinberg sequences below $\aleph_{\varepsilon_0}$, but the exact values of their elements is still unknown. In this note, we give a simple inductive argument that allows to compute the Kleinberg sequences corresponding to the ω_1-cofinal measures on the odd projective ordinals without doing any detailed ultrapower analyses.

In this note, we shall be working in the system $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}$ where AD is the *Axiom of Determinacy* stating that all perfect information games with two players and ω many rounds are determined. We refer to the standard textbook [Ka94, §§ 27–32] for information about the basic developments of set theory in this particular axiomatic setting.

One remarkable consequence of AD is the existence of an interesting combinatorial structure on uncountable cardinals. By a 1973 result of Martin, AD implies the existence of so-called *strong partition cardinals* [Ka94, Theorem 28.12]: under AD, $\aleph_1$ is a strong partition cardinal and the club filter $\mathcal{C}$ on $\aleph_1$ is a normal measure. Moreover, Kleinberg (1977) has proved that strong partition cardinals κ together with a normal measure μ on κ generate a sequence $\langle\kappa_n^\mu\,;\,n\geq 1\rangle$ of partition cardinals (called a *Kleinberg sequences*; *cf.* Theorem 1), and has computed the sequence derived from Martin's mentioned result: $\kappa_n^{\mathcal{C}}=\aleph_n$.

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Nowadays, we know much more about infinitary combinatorics under AD than in 1977, and it was mainly the work of Steve Jackson [Ja88,Ja99b] that gave us many more strong partition cardinals and normal measures below $\aleph_{\varepsilon_0}$. By Kleinberg's result, each pair of these gives rise to a Kleinberg sequence. (*Cf.* [Lö02a, § 2] or [Lö02b, § 3.1] for brief surveys of definitions and results in this area.)

At the moment, we lack a uniform way of computing these Kleinberg sequences. Partial results, heavily resting on the Jackson-Khafizov analysis from [JaKh∞] have been obtained by the second author in [Lö02a], but since a general Jackson-Khafizov analysis for all cardinals below $\aleph_{\varepsilon_0}$ is still a *desideratum*, these results can not be generalized, and in general, computations of Kleinberg sequences need a detailed analysis of all ultrapowers involved, including ultrapowers that do not even occur in the Kleinberg sequence, but fill the gaps between the cardinals in the Kleinberg sequence.

In this note, we give a very simple inductive argument to compute the Kleinberg sequence of the ω_1-cofinal measure on the odd projective ordinals based on the *Ultrapower Shifting Lemma* of [Lö02a] (*cf.* Theorem 2; [Lö02a, Lemma 2.7]).

1 An abstract combinatorial computation

In this section, we shall give an abstract computation based on the *Ultrapower Shifting Lemma.* We shall not be using AD in this section, so all results are $\mathsf{ZF} + \mathsf{DC}$-theorems. We define the iterated successor operation on cardinals κ as follows by transfinite recursion:

- $\kappa^{(0)} = \kappa$,
- $\kappa^{(\alpha+1)} = (\kappa^{(\alpha)})^+$ for all ordinals α, and
- $\kappa^{(\lambda)} = \bigcup\{\kappa^{(\alpha)}\,;\,\alpha \in \lambda\}$ for limit ordinals λ.

A cardinal κ is called a **strong partition cardinal** (in Erdős arrow notation: $\kappa \to (\kappa)^\kappa$) if for every partition $F : [\kappa]^\kappa \to 2$ of the increasing functions from κ to κ exists a homogeneous set $H \subseteq \kappa$ of cardinality κ, *i.e.* $\mathrm{Card}(F``[H]^\kappa) = 1$. Note that the existence of a strong partition cardinal violates the Axiom of Choice (*cf.* [Ka94, Proposition 7.1]).

If μ is a measure on κ and α is an ordinal, then we write α^κ/μ for the (Mostowski-collapse of the) ultrapower of α with respect to μ. Since we assume DC, α^κ/μ is an ordinal.

Theorem 1 (Kleinberg). Let κ be a strong partition cardinal cardinal, let μ be a normal measure on κ and let $\kappa_1^\mu := \kappa$ and $\kappa_{n+1}^\mu := (\kappa_n)^\kappa/\mu$. Then

1. κ_1^μ and κ_2^μ are measurable,
2. for all $n \geq 2$, $\mathrm{cf}(\kappa_n^\mu) = \kappa_2^\mu$,
3. all κ_n^μ are Jónsson cardinals, and
4. $\sup\{\kappa_n^\mu\,;\, n \geq 1\}$ is a Rowbottom cardinal.
5. Moreover, if $\kappa^\kappa/\mu = \kappa^+$, then $\kappa_{n+1}^\mu = (\kappa_n^\mu)^+$ for all $n \in \omega$.

We call the sequence $\langle \kappa_n^\mu\,;\, n \geq 1\rangle$ the **Kleinberg sequence derived from** μ.

Proof. Cf. [Kl77]. q.e.d.

Theorem 2 (Ultrapower Shifting Lemma). Let β and γ be ordinals and let μ be a κ-complete ultrafilter on κ with $\kappa^\kappa/\mu = \kappa^{(\gamma)}$. If for all cardinals $\kappa < \nu \leq \kappa^{(\beta)}$

- either ν is a successor and $\mathrm{cf}(\nu) > \kappa$,
- or ν is a limit and $\mathrm{cf}(\nu) < \kappa$,

then $(\kappa^{(\beta)})^\kappa/\mu \leq \kappa^{(\gamma+\beta)}$.

Proof. Cf. [Lö02a, Lemma 2.7]. q.e.d.

Lemma 3. Let $\kappa < \lambda$ be cardinals, μ a measure on κ and $\mathrm{cf}(\lambda) > \kappa$. Then $\mathrm{cf}(\lambda^\kappa/\mu) = \mathrm{cf}(\lambda)$.

Proof. "$\leq$": For $\alpha < \lambda$ let $c_\alpha : \kappa \to \lambda$ be the constant function $c_\alpha(\xi) = \alpha$. We shall show that $\{[c_\alpha]_\mu\,;\, \alpha \in \lambda\}$ is cofinal in λ^κ/μ:

Let $f \in \lambda^\kappa$ be arbitrary. Since $\mathrm{cf}(\lambda) > \kappa$, the range of the function f is bounded in λ, *i.e.*, there is an $\alpha^* \in \lambda$ such that $\{f(\xi)\,;\, \xi \in \kappa\} \subseteq \alpha^*$. Then $[f]_\mu < [c_{\alpha^*}]_\mu$.

"$\geq$": Now let $X \subseteq \lambda^\kappa/\mu$ be a cofinal subset. If $\xi \in X$, there is some $\alpha \in \lambda$ such that $\xi \leq [c_\alpha]_\mu$ by the above argument. Let α_ξ be the least such ordinal. We claim that $A := \{\alpha_\xi\,;\, \xi \in X\}$ is a cofinal subset of λ: Let $\gamma \in \lambda$ be arbitrary. Since X was cofinal, pick some $\xi_\gamma \in X$ such that $\xi_\gamma > [c_\gamma]_\mu$. But then, $\alpha_{\xi_\gamma} \in A$ with $\alpha_{\xi_\gamma} > \gamma$. So, A is cofinal in λ. But $\mathrm{Card}(A) \leq \mathrm{Card}(X)$, so $\mathrm{cf}(\lambda) \leq \mathrm{cf}(\lambda^\kappa/\mu)$. q.e.d.

Theorem 4. Let κ be a strong partition cardinal and μ_0 and μ_1 be normal ultrafilters on κ with $\kappa^\kappa/\mu_0 = \kappa^+$ and $\kappa^\kappa/\mu_1 = \kappa^{(\omega+1)}$. For all $\beta < \omega^2$, assume that $(\kappa^{(\beta)})^\kappa/\mu_1$ is a cardinal.

Then for all $\xi < \omega^2$, the following equalities hold:

- $(\kappa^{(\xi)})^\kappa/\mu_1 = \kappa^{(\omega+1+\xi)}$, and
- $\mathrm{cf}(\kappa^{(\xi+1)}) = \begin{cases} \kappa^+ & \text{if } \xi \text{ is a successor or zero, or} \\ \kappa^{(\omega+1)} & \text{if } \xi > 0 \text{ is a limit.} \end{cases}$

Proof. By Kleinberg's Theorem 1 (1.), (2.) and (5.), we have

$$\mathrm{cf}(\kappa^{(n+1)}) = \kappa^+$$

for $n \in \omega$. Also, for all limit ordinals $\lambda < \omega^2$, the cofinality of $\kappa^{(\lambda)}$ is ω. We denote these facts by (IH$_*$).

We proceed by induction on ξ, using the following induction hypothesis:[1]

$$(\mathrm{IH}_\xi)\quad \left[\begin{array}{l} \text{For all } \alpha \leq \xi, \text{ the following two conditions hold:} \\ 1.\ (\kappa^{(\alpha)})^\kappa/\mu_1 = \kappa^{(\omega+1+\alpha)}, \\ 2.\ \mathrm{cf}(\kappa^{(\omega+1+\alpha)}) := \begin{cases} \omega & \text{if } \alpha > 0 \text{ is a limit,} \\ \kappa^+ & \text{if } \alpha \text{ is 1 or a double successor, or} \\ \kappa^{(\omega+1)} & \text{if } \alpha \neq 1 \text{ is zero or a single successor.} \end{cases} \end{array}\right.$$

Obviously, if (IH$_*$) and all (IH$_\xi$) (for $\xi < \omega^2$) hold, the theorem is proved.

By assumption, $(\kappa^{(0)})^\kappa/\mu_1 = \kappa^\kappa/\mu_1 = \kappa^{(\omega+1)}$ and from Theorem 1 (1.), we know that this is a regular cardinal, so (IH$_0$) holds.

For the successor step $\xi \mapsto \xi + 1$ assume that (IH$_\xi$) holds. Let us look at the Ultrapower Shifting Lemma 2 with $\gamma = \omega + 1$ and

[1] An ordinal γ is a double successor is there is some δ such that $\gamma = \delta + 2$. An ordinal is a single successor if it's a successor but not a double successor; equivalently, if it is the successor of a limit ordinal.

$\beta = \xi + 1$. Since $\xi < \omega^2$, we have $\xi + 1 < \omega + 1 + \xi$, so (IH$_\xi$) and (IH$_*$) allows us to apply Lemma 2 and get:

$$\begin{array}{rcll} \kappa^{(\omega+1+(\xi+1))} & \geq & (\kappa^{(\xi+1)})^\kappa/\mu_1 & \text{(Lemma 2)} \\ & > & (\kappa^{(\xi)})^\kappa/\mu_1 & \\ & = & \kappa^{(\omega+1+\xi)}. & \text{(IH}_\xi\text{)} \end{array}$$

Since $(\kappa^{(\xi+1)})^\kappa/\mu_1$ is a cardinal (by assumption) lying in the interval between $\kappa^{(\omega+1+\xi)}$ and its successor, we get

$$(\kappa^{(\xi+1)})^\kappa/\mu_1 = \kappa^{(\omega+1+(\xi+1))}.$$

We shall now compute the cofinality of $\kappa^{(\omega+1+(\xi+1))}$ in order to check that (IH$_{\xi+1}$) holds:

Case 1: $\xi < \omega$. In this case, $\mathrm{cf}(\kappa^{(\xi+1)}) = \kappa^+ > \kappa$ by (IH$_*$). So, we can apply Lemma 3 to $\lambda := \kappa^{(\xi+1)}$. Thus

$$\begin{array}{rcll} \mathrm{cf}(\kappa^{(\omega+1+(\xi+1))}) & = & \mathrm{cf}((\kappa^{(\xi+1)})^\kappa/\mu_1) & \\ & = & \mathrm{cf}(\kappa^{(\xi+1)}) & \text{(Lemma 3)} \\ & = & \kappa^+. & \text{(IH}_*\text{)} \end{array}$$

Case 2: $\omega \leq \xi < \omega^2$. In this case, there is an ordinal $\alpha < \xi$ such that $\xi + 1 = \omega + 1 + \alpha$, and the following equivalences hold:

$$(*) \left[\begin{array}{rcl} \alpha \text{ is 1 or a double successor} & \iff & \xi \text{ is a successor,} \\ \alpha \neq 1 \text{ is zero or a single successor} & \iff & \xi \text{ is a limit.} \end{array} \right.$$

Now, by (IH$_\xi$), we get that $\mathrm{cf}(\kappa^{(\xi+1)}) = \mathrm{cf}(\kappa^{(\omega+1+\alpha)}) > \kappa$. So, again applying Lemma 3 to $\lambda := \kappa^{(\xi+1)}$, we get

$$\begin{aligned} \mathrm{cf}(\kappa^{(\omega+1+(\xi+1))}) &= \mathrm{cf}((\kappa^{(\xi+1)})^\kappa/\mu_1) \\ &= \mathrm{cf}(\kappa^{(\xi+1)}) && \text{(by Lemma 3)} \\ &= \mathrm{cf}(\kappa^{(\omega+1+\alpha)}), \end{aligned}$$

thus by $(*)$

$$\mathrm{cf}(\kappa^{(\omega+1+(\xi+1))}) = \begin{cases} \kappa^+ & \text{if } \xi \text{ is a successor, and} \\ \kappa^{(\omega+1)} & \text{if } \xi \text{ is a limit.} \end{cases}$$

For the limit step, let $0 < \lambda < \omega^2$ be a limit ordinal. Note that this implies that for some $\alpha < \lambda$, we have that $\omega + \alpha = \lambda$. We now assume (IH_η) for $\eta < \lambda$, and write $(\mathrm{IH}_{<\lambda})$ for this assumption. In particular (since $\alpha < \lambda$), we know the cofinalities of all cardinals between κ and $\kappa^{(\omega+1+\alpha)} \geq \kappa^{(\omega+\alpha)} = \kappa^{(\lambda)}$. This allows us to apply the Ultrapower Shifting Lemma 2 for $\gamma = \omega + 1$ and $\beta = \lambda$:

$$\begin{array}{rcll} \sup\{\kappa^{(\omega+1+\eta)}\,;\,\eta<\lambda\} & = & \sup\{(\kappa^{(\eta)})^\kappa/\mu_1\,;\,\eta<\lambda\} & (\mathrm{IH}_{<\lambda}) \\ & \leq & (\kappa^{(\lambda)})^\kappa/\mu_1 & \\ & \leq & \kappa^{(\omega+1+\lambda)} & (\text{Lemma 2}) \\ & = & \sup\{\kappa^{(\omega+1+\eta)}\,;\,\eta<\lambda\}. & \end{array}$$

This establishes $(\kappa^{(\lambda)})^\kappa/\mu_1 = \kappa^{(\omega+1+\lambda)}$. The claim about the cofinality of $\kappa^{(\omega+1+\lambda)}$ is trivial for a limit ordinal $\lambda < \omega^2$. q.e.d.

2 Applications to infinitary combinatorics under the Axiom of Determinacy

We now move to the applications of the abstract Theorem 4 under AD. If $\lambda < \kappa$ are regular cardinals, the **λ-cofinal measure on κ** is defined to be the filter generated by sets of the type

$$\{\alpha \in \kappa\,;\, \alpha \in C \;\&\; \mathrm{cf}(\alpha) = \lambda)\}$$

for some closed unbounded subset C of κ. We write $\mathcal{C}^\lambda_\kappa$ for this filter. The projective ordinals are defined as follows:[2]

$$\boldsymbol{\delta}^1_n := \sup\{\alpha\,;\,\text{there is a } \boldsymbol{\Delta}^1_n \text{ prewellordering of } \mathbb{R} \text{ of length } \alpha\}.$$

The following theorem is a summary of work due to Kleinberg, Kunen, Martin and Jackson:

Theorem 5. Assume ZF+DC+AD. Let $\mathbf{e}_0 := 0$ and $\mathbf{e}_{n+1} := \omega^{(\omega^{\mathbf{e}_n})}$.

- If $\lambda < \boldsymbol{\delta}^1_{2n+1}$ is regular, then $\mathcal{C}^\lambda_{\boldsymbol{\delta}^1_{2n+1}}$ is a normal measure on $\boldsymbol{\delta}^1_{2n+1}$,
- for all n, the ordinal ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}} = \boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$,
- $\boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathbf{e}_n+1}$, and

[2] For more details, *cf.* [Ka94, § 30].

– $\boldsymbol{\delta}^1_{2n+1}$ is a strong partition cardinal.

Proof. For Jackson's work, we refer the reader to [Ja88,Ja99b]. More detailed references for the older results underlying this summarizing theorem can be found in [Lö02a, Fact 2.5]; *cf.* also [Ke78]. q.e.d.

Theorem 6. Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. Assume furthermore that

– ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}} = \aleph_{\mathbf{e}_n+\omega+1}$, and that
– for all $\xi < \omega^2$, the ordinal ${\aleph_{\mathbf{e}_n+\xi}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$ is a cardinal.

Then for each $m \in \omega$, the cardinal $\aleph_{\mathbf{e}_n+\omega\cdot m+1}$ is Jónsson, and $\aleph_{\mathbf{e}_n+\omega^2}$ is Rowbottom.

Proof. Let $\mu_0 := \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$ and $\mu_1 := \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$ and let $\kappa_m := \kappa^{\mu_1}_m$ be the elements of the Kleinberg sequence derived from μ_1, *i.e.*, $\kappa_{m+1} = (\kappa_m)^\kappa/\mu_1$.

By Theorem 5 and the assumptions, all requirements of Theorem 4 are met, and so we can inductively read off the values of

$$\begin{aligned} \kappa_1 &= \boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathbf{e}_n+1}, \\ \kappa_{m+1} &= (\kappa_m)^{(\omega+1)} \text{ (for } m \geq 1), \end{aligned}$$

and so

$$\kappa_{m+1} = \aleph_{\mathbf{e}_n+\omega\cdot m+1}.$$

Now the theorem follows directly from Kleinberg's Theorem 1.
q.e.d.

Note that as a direct corollary of Theorem 6, we get the Jónsson cardinals $\aleph_{\omega\cdot m+1}$ that were computed in [Lö02a, Corollary 3.3], however, –as opposed to the proof in [Lö02a]– the proof given here does not refer to the full Jackson-Khafizov analysis of ultrapowers.

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