INDEPENDENCE STRUCTURES IN SET THEORY[1]

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The axioms for "independent choices" presented in van Lambalgen [1992] are strengthened here, so that they can be seen as introducing a new type of indiscernibles in set theory. The resulting system allows for the construction of natural inner models. The article is organised as follows. Section 1 introduces the axioms, some preliminary lemmas are proved and the relation with the axiom of choice is investigated. Section 0 gives a philosophical motivation for the axioms; the reader who is not interested in such matters can skip this part. In section 2 we compare the structure introduced by the axioms, here called an *independence structure*, with two constructions from model theory, indiscernibles and minimal sets. Section 3 contains the construction of inner models, while section 4 presents some concluding philosophical remarks.

0. INTRODUCTION

An important foundational problem in set theory is the indeterminacy of the power set operation, as exemplified by numerous statements concerning cardinality, measurability or the Baire property, which can be shown to be independent of ZF. As is wellknown, Gödel predicted this state of affairs already in 1948, when he wrote that the root of the trouble is that ZF admits two very different kind of models, namely, the smallest model L of constructible sets, and the less well-determined model generated by as many iterations as possible of the operation "arbitrary subset of". Gödel proposed to formalize the latter notion by means of his programme of introducing larger and larger cardinals. Indeed, it can be shown that for instance the assumption of the existence of measurable cardinals implies the existence of a non-constructible (although definable) set of reals, so that the assumption of large cardinals has an effect on our possibilities for generating arbitrary subsets of the natural numbers. However, the combination of these two ideas, cumulative hierarchy and maximality, is not without its problems. On the one hand, all sets of reals, say, are supposed to be generated at level $\omega+3$; on the other hand, sets of reals can be defined using parameters which occur only at higher levels. These parameters must be such that they do not introduce *new* sets of reals; it does not seem unreasonable to view this as a *restriction* on the higher levels. More concretely, the following arguments *pro* and *con* the continuum hypothesis seem to show that sometimes one level can be maximized only at the expense of another. The argument *con* is familiar and runs as follows: we want the power set of ω to be as rich and large as possible, so why would it be equinumerous with ω_1, which is merely the simplest kind of uncountability? Now consider, however, the following argument *pro* (due to Steel): 2^ω and a well-ordering of length ω_1 appear (as sets) at stage $\omega+2$, hence a bijection between these two would appear at stage $\omega+3$, and it would restrict the power set operation at this stage if the bijection were not included. Hence it seems that stages $\omega+2$ and $\omega+3$ cannot be maximized simultaneously.

Customarily, the axiom of choice is also justified using maximality considerations: we would like to have as many as possible functions from, e.g., $\mathcal{P}(2^\omega)$ to 2^ω, hence in particular a choice function. Below, we shall prove that there is no choice function on the power set of the reals, given certain axioms on random or independent choice. This argument will be seen to exploit the

1 This article, based on an invited lecture at the Logic Colloquium '93 in Keele, is a sequel to van Lambalgen [1992]. Apart from presenting new results, it differs from its predecessor in the following respects: (i) the presentation of the axioms is simplified, following some suggestions of Wojciech Buszkowski, (ii) the axioms have been strengthened, and (iii) the philosophical discussion has (hopefully) been improved. The article has appeared in W. Hodges et al (eds.), *Logic: from Foundations to Applications (European Logic Colloquium),* Oxford University Press 1996.

tension hinted at above: maximization at the level of reals is possible only at the expense of later levels (and conversely). Tentatively, I conclude from this that additional philosophical ideas, beyond maximization and the hierarchy, are needed to settle CH and AC (where by 'settle' I mean 'derive from a more primitive principle'). In particular, apart from the problem indicated by Gödel, namely that ZF does not distinguish between definable and 'arbitrary' subsets, we might have to take a decision concerning *which* levels to maximize.

In the following, we shall make an attempt at formalizing the concept "arbitrary subset of the integers" which is not based on the idea of maximization. This approach is suggested by Bernays [1985], who wrote that one could envisage subsets of the integers being generated by "independent choices, deciding for each integer whether it is to be included or excluded; we add to this the totality of the set so chosen". Clearly, independent choices are to be contrasted with choices determined by a law, and in a sense there are more subsets generated by independent choices than by lawlike choices. However, apart from this quantitative aspect, there is also the structural aspect: we expect that sequences generated by independent choices satisfy properties different from those satisfied by lawlike sequences. It is these structural aspects that we shall emphasize. In other words, one can discern (at least) two aspects of independence: on the one hand it is a liberalization, on the other hand it introduces structural constraints not expressible before.

Bernays goes on to say that on this conception of the generation of sets, the axiom of choice is self-evident. Indeed, what could prevent us from associating an element of A to a non-empty subset A contained in 2^ω, independently of what we do with other subsets? Referring to our discussion above we can see, however, that there is one possible constraint on the independence: the choice function g so determined must not give rise to new sets of reals, not yet present in its domain. Once we see this, it is clear how to falsify AC: show by diagonalisation that any choice function on $\mathcal{P}(2^\omega)$ would generate a new set of reals. This can be done (in an extension of ZF, of course); moreover, it will be shown that all falsifications of AC can be reduced to this diagonal form.

We have found it helpful to present the formalization with the use of a new primitive, which roughly denotes an independence relation; this is because in general the independence relation will be a class. It will be shown that the new primitive is eliminable using an assumption on the existence of certain ultrafilters, in the sense that the $\in$ - consequences of the expanded theory follow from this assumption (cf. corollary 2.2.3.).

Neither the expansion of the language of set theory, nor the attempt to axiomatize the notion of "arbitrary subset' directly, is entirely novel. For the former, one can refer to the work of Boolos [1985], who attempted to justify the ZF axioms from a description of the cumulative hierarchy in terms of the primitives 'stage' and 'earlier than', or Reinhardt's [1974], where a constant V for the universe is introduced in order to derive the existence of measurable cardinals.

In fact, the first attempt to formalize some such thing as "arbitrary choice" in set theory seems to be due to Myhill in 1962 (see Kruse [1967]). Myhill conceived of this process probabilistically and hence chose the following set up:

Let 2^ω be the set of infinite binary sequences and let λ be the product measure generated by the uniform distribution on $\{0,1\}$, i.e. λ is Lebesgue measure. Introduce a new predicate R(x), x ranging over 2^ω, which is supposed to mean that x is randomly generated (say by a fair coin toss). As an axiom Myhill proposed:

$$\lambda\{x \mid \phi(x)\} = 1 \text{ if and only if } \forall x(R(x) \to \phi(x)),$$

where R is allowed to occur in ϕ and in the schemata of ZF. It readily follows that there is no definable well-ordering of the universe: if not, we could define the least random sequence, but the complement of this singleton is a set of measure one which does not contain all random reals. In particular, it follows that $V \neq L$. (The consistency of Myhill's axiom apparently is not easy to establish; one seems to need a model where the set of Solovay random reals has measure one, and where every set of reals definable without parameters is measurable. See Stern [1985] for the construction of a model with these properties.)

We thus see that there is some connection between "arbitrary choice' and the axiom of choice (in the form of the well-ordering theorem), although not quite the connection intimated by Bernays.

This impression is corroborated when we look at a different attempt to introduce a notion of "arbitrary choice' in classical mathematics, namely Joan Moschovakis' construction (in her [1987]) of a classical version of intuitionistic lawless sequences. Intuitionistically, a lawless sequence is a sequence generated by choices which are completely unrestricted; at any stage of generation, we know only a finite intial segment of the sequence. The axiom characteristic for such sequences is Open Data: if a property A holds of a lawless sequence α, then there must exist an intial segment w of α such that for all lawless β in w, A(β). (For an exposition of the intuitionistic theory of lawless sequences, see Troelstra [1977].) Classically, this notion is inconsistent, but Moschovakis succeeded in constructing a classical model for a closely related concept: a sequence is *relatively* lawless if it forever evades description by a law. To construct this model, she needed the following set theoretical assumption: any definable well-ordering of the continuum is countable. Indeed, it can be shown that Moschovakis' axioms imply that there is no definable well-ordering of the continuum (even when real parameters are allowed).[2]

Again, we see that the presence of "arbitrary choice" has as a consequence that the choices needed to get well-orderings are to some extent difficult to perform.

Below, we shall investigate this connection in a more systematic manner.

1. THE FUNDAMENTAL RELATION

To obtain a formalization of "arbitrary choice" we shall proceed in stages.

(i) It seems very hard to characterize the independence of successive choices directly, except perhaps probabilistically. The latter approach, however, is too specific for our purposes. We therefore proceed in an indirect manner and characterize independence of successive choices by means of independence between *sequences* of choices: if a sequence x is generated by independent choices, and if we split x into two disjoint subsequences y and z in any reasonable way, then we expect that y and z will be independent. This means, for instance, that, even when we are given z as an oracle, we will not be able to detect a regularity in y.

This concept of independence is also helpful for another reason: if we generate two infinite binary sequences x and y by successive independent choices, then we also expect that x and y will be independent. To forestall possible objections, let us mention here that we shall use a very much weaker axiom: $\forall y \exists x$(x is independent from y). This says only that *eventually* we would generate an x independent from y.

We now generalize this set up slightly.

(ii) Usually (e.g. in the case of linear or algebraic independence) independence is a relation between an object and a finite set of objects. Similarly, we take independence to be a relation R(x,a) between an infinite binary sequence x and a finite set a of such sequences, or, in the general case, a finite set a of sets. The interpretation remains the same: one cannot detect regularities in x, even when using a as an oracle. Note that, if a can be a finite set of arbitrary sets, the following

[2] Actually, because of the way Moschovakis' axioms are formulated, this holds only for well-orderings defined by *restricted* formulas; see [1986] for details.

expression is now welldefined: $R(x,2^\omega)$. This move can be interpreted as putting a restriction on the possible a's that may occur. For example, we may want to consider a universe in which *only* reals are generated by independent choices; all other sets are generated from these by definable operations. In fact, on the basis of the axioms given below, one can prove: if F is a definable function (the definition may involve R and ordinals), then (*) $R(x,X) \rightarrow R(x,F(X))$. In models in which every set is (ordinal) definable from a finite set X of reals, (*) can be used to extend R from reals to arbitrary sets, as will be made clear by theorems 3.2.1. and 3.2.2. below.

(iii) Ordinals play a special role in this context. We may think of ordinals as the most 'lawlike' objects one can imagine, so that their presence or absence in the right hand argument of R should not make much of a difference. Indeed, Gödel put his as follows:

> I would think that "definability in terms of ordinals" [...] is at least an adequate formulation in an absolute sense for [the] property "being formed according to a law" as opposed to "being formed by a random choice of the elements". For in the ordinals there is certainly no element of randomness, and hence neither in sets defined in terms of them. This is particularly clear if you consider von Neumann's definition of ordinals, because it is not based on any well-ordering relation of sets, which may very well involve some random element (*Remarks before the Princeton bicentennial conference* [1990], p. 152)

In our context, Gödel's remark can be rendered formally thus:

(Gödel's law) $R(x,X) \leftrightarrow R(x,\{\alpha\}\cup X)$.

This statement is actually derivable from the axioms given below.

1.1 AXIOMS

In the following, it is understood that x,y range over 2^ω. We need two preliminary definitions:

DEFINITION 1.1.1 Let $x,y \in 2^\omega$. We say that x and y are equivalent , and write $x\approx y$, if x can be obtained from y by changing finitely many coordinates. The equivalence class of x mod$\approx$ is denoted [x]. A set $B \subseteq 2^\omega$ is a *tailset* if $y \in B$ and $x\approx y$ imply $x \in B$.

⊠

DEFINITION 1.1.2 We introduce a special notation for generalized pairing and projection operations on reals. Let $<x_1,...,x_n, ...>$ be a countable sequence of reals. Then we have primitive recursive set functions π, p_k with the following properties: $\pi(<x_1,...,x_n, ...>) \in 2^\omega$ and $p_k(\pi(<x_1,...,x_n, ...>)) = x_k$., Similarly we have functions ρ^n, i_k such that for a set $\{x_1,...,x_n\}$, $\rho^n(\{x_1,...,x_n\}) \in 2^\omega$ and for $k\leq n$, $i_k(\rho^n(\{x_1,...,x_n\})) = x_k$. ⊠

DEFINITION 1.1.3 ZFR is the set theory in the language $\{\in,R\}$ which consists of ZF (with R allowed in the schemata) together with the following axioms (where X ranges over arbitrary finite sets):

R1 (Existence) $\forall X\exists x R(x,X)$

R2 (Downward monotonicity) $R(x,X) \wedge Y\subseteq X \rightarrow R(x,Y)$

R3 (Irreflexivity) $\neg R(x,\{x\})$

R4 (Steinitz exchange principle) $R(x,\{y\}\cup X) \wedge R(y,X) \rightarrow R(y,\{x\}\cup X)$

R5 (Zero-one law) Let $\phi(x,a_a,..,a_n)$ (all parameters exhibited) determine a tailset in the variable x . Then

$$\exists x(R(x,\{a_1,...,a_n\}) \wedge \phi(x,a_1,..,a_n)) \rightarrow \forall x(R(x,\{a_1,...,a_n\}) \rightarrow \phi(x,a_1,..,a_n)).$$

R6. (Tailset) $\{x \mid R(x,X)\}$ is a tailset for each X.

R7. (Splitting)

For each n, $R(x,X) \rightarrow R(i_1(x),X) \wedge \forall k(1<k<n \rightarrow R(i_k(x),\{i_1(x),..,i_{k-1}(x)\}\cup X))$.

A special case of Existence is $\exists x R(x,\emptyset)$. We shall abbreviate $R(x,\emptyset)$ by $R(x)$. ⊠

DEFINITION 1.1.4 ZFR^0 is the set theory in the language $\{\in, R\}$ which consists of ZF (with R allowed in the schemata) together with the above axioms, *with the proviso that* X *ranges over finite sets of reals and ordinals*. ⊠

It will be seen below that the choice of parameters is an important matter, so for purposes of comparison we introduce a third theory as follows:
DEFINITION 1.1.5 ZFR^{-1} is the subtheory of ZFR^0 which has the additional restriction that no parameters are allowed in the zero-one law, and where Existence is replaced by $\exists x R(x)$. ⊠

We shall see below that in the context of ZFR^0 or ZFR, $\exists x R(x)$ is equivalent to $\forall X \exists x R(x,X)$, but this is no longer true if the zero-one law is weakened.

1.2 MOTIVATION

First of all it seems reasonable to allow R in the schemata if we start from the idea that we want to maximize the number of subsets of the reals. A consequence of this stipulation is a certain self-reflexivity: for instance, it follows from lemma 1.3.4. below that an x with R(x) is not ordinal definable, even if we allow R in the definition.
As regards the axioms for R, the first three properties are obvious for any independence relation, although the reason why we did not include transitivity of $\neg R$, i.e. the statement $\neg R(x, X \cup \{y\}) \wedge \neg R(y,X) \rightarrow \neg R(x,X)$, instead of the weaker 2 is perhaps worth mentioning. Suppose x is a sequence with R(x), and let y and z be obtained by selecting the odd and even coordinates of x respectively. By Splitting we obtain: R(y), R(z), R(y,z), and by (*) plus Irreflexivity $\neg R(y,x)$ and $\neg R(z,x)$. Assuming Steinitz exchange, we get $\neg R(x,y)$, because otherwise $R(x,y) \wedge R(y) \rightarrow R(y,x)$, a contradiction. Transitivity now gives: $\neg R(z,x) \wedge \neg R(x,y) \rightarrow \neg R(z,y)$, a contradiction.
It follows that these independence axioms do no define (infinite) matroids, but a slightly weaker type of structure. For overviews of the theory of matroids, see Welsh [1976], White [1986] and Oxley [1992].
On the other hand, Steinitz exchange (sometimes called Steinitz-MacLane exchange) is an axiom for matroids that we can safely adopt. Indeed, it is almost a consequence of the remaining axioms, in the following sense:

LEMMA 1.2.1. Suppose that (for n=2) Splitting is strengthened to

$$R(z,X) \rightarrow R(i_2(z), i_1(z) \cup X) \wedge R(i_1(z), i_2(z) \cup X).$$

Then Steinitz exchange becomes derivable.
PROOFSKETCH Suppose for some x,y: $R(x, \{y\} \cup X) \wedge R(y,X) \wedge \neg R(y, \{x\} \cup X)$. Lemma 1.3.4. below implies that if $x \approx x'$, $\neg R(y, \{x\} \cup X) \leftrightarrow \neg R(y, \{x'\} \cup X)$. By the zero-one law, (*) $\forall x \forall y (R(x, \{y\} \cup X) \wedge R(y,X) \rightarrow \neg R(y, \{x\} \cup X))$. By Existence, we can choose y such that R(y,X). By strengthened Splitting and Downward Monotonicity, $R(i_1(y),X) \wedge R(i_2(y), i_1(y) \cup X) \wedge R(i_1(y), i_2(y) \cup X)$, whence by (*) $\neg R(i_1(y), i_2(y) \cup X)$, a contradiction. ⊠

Steinitz exchange is satisfied in one of the most important of the intended interpretations, namely forcing.

If $\mathcal{M}$ is a model of ZFC, and if we put (for x, $y_1,...,y_n \in 2^\omega$), R(x, $\{y_1,...,y_n\}$) if x is Solovay-random over $\mathcal{M}[y_1,...,y_n]$, then R satisfies the given axioms. To prove Steinitz, one has to use absoluteness properties of Borel sets and the Fubini theorem. (One could use Cohen-generic reals as well; in that case Steinitz reduces simply to the productlemma for forcing.)
Furthermore, it will be seen below that Steinitz has important and intuitively appealing consequences, such as 'Gödel's law' $R(x,X) \leftrightarrow R(x,\{\alpha\}\cup X)$ for ordinals α.

Lastly, the justification of the zero-one law runs as follows. Intuitively, we have only two pieces of information concerning a sequence generated by independent choices: the process which generates the sequence, and a finite initial segment of the sequence. Now suppose we fix the process, and let ϕ be a formula which does not distinguish between sequences which differ only on an initial segment; then ϕ cannot make *any* distinctions between randomly generated sequences. In other words, the zero-one law is a way of saying that we shall always consider the same process. (It is necessary for consistency to take account of the parameters; otherwise we would get a statement like $\exists x(R(x) \wedge \neg(x\approx y)) \rightarrow \forall x(R(x) \rightarrow \neg(x\approx y))$, where the antecedent is true while the succedent is a contradiction.)
It is perhaps of interest to compare this axiom to intuitionistic Open Data (cf. the introduction), the justification of which starts out in the same way. The difference is, however, that Open Data uses the intuitionistic meaning of the implication essentially: if we can *prove* that $A(\alpha)$, then this can only be done on the basis of an initial segment, so we can also *prove* that $A(\beta)$ must hold for all β which share this initial segment with α. Classically, this move is not available, so we must be content with the weaker zero-one law ('weaker' because it follows from Open Data plus an assumption concerning shift invariance of lawless sequences).
It will be seen below that, technically, the zero-one law represents the homogeneity of a forcing notion.

It is possible to add further axioms to 1-7; e.g. one can add axioms connecting independence and Lebesgue measure, thus forcing a probabilistic interpretation of independence. We shall not do so here (but see van Lambalgen [1992]), because we are interested in an algebraic core concept.

1.3 Preparatory lemmas

If a result holds, *mutatis mutandis*, in both ZFR and ZFR^0, we indicate this by prefixing the statement of the result with "(ZFR$^{(0)}$)".
In this section we collect important lemmata of ZFR$^{(0)}$, and we show that these set theories have negative consequences for the axiom of choice.
Here is a simple first lemma. The reader will have wondered why we did not include the inverse of the splitting operation, which might be called *Combination*, among our axioms. Indeed, it seems reasonable that if x is independent from X, and y is independent from $X\cup\{x\}$, then the interleaving of x and y is independent from X. It turns out that this property is derivable:

LEMMA 1.3.1. (ZFR$^{(0)}$) Let $\rho^n(x^1,..,x^n)$ be the generalized pairing function which produces a real from $x^1,..,x^n$ (cf. definition 1.1.2.). Then $R(x^n,\{x^{n-1},..,x^1\}\cup X) \wedge .. \wedge R(x^1,X) \rightarrow R(\rho^n(x^1,..,x^n),X)$.
PROOF Suppose $\exists x^1..x^n(R(x^n,\{x^{n-1},..,x^1\}\cup X) \wedge .. \wedge R(x^1,X) \wedge \neg R(\rho^n(x^1,..,x^n),X))$.
By applying the zero-one law n times, we obtain $\forall x^1..x^n((R(x^n,\{x^{n-1},..,x^1\}\cup X) \wedge .. \wedge R(x^1,X)) \rightarrow \neg R(\rho^n(x^1,..,x^n),X))$. Using Splitting and Existence one can now derive a contradiction.

$\boxtimes$

The following property is slightly more abstruse; it is called 'extensionality for R' because it corresponds to extensionality for the generalized quantfier Q which can be defined from R by

$$Qx\phi(x,a_1,..,a_n) := \forall x(R(x,\{a_1,..,a_n\}) \to \phi(x,a_1,..,a_n)).$$

LEMMA 1.3.2. (Extensionality for R) (i) If $\forall x(\phi(x,a_1,..,a_n) \leftrightarrow \psi(x,b_1,..,b_m))$, then $\forall x(R(x,\{a_1,..,a_n\}) \to \phi(x,a_1,..,a_n)) \leftrightarrow \forall x(R(x,\{b_1,..,b_m\}) \to \psi(x,b_1,..,b_m))$;
(ii) $\forall x(R(x,\{a_1,..,a_n\}\cup\{b_1,..,b_m\}) \to \phi(x,a_1,..,a_n)) \to \forall x(R(x,\{a_1,..,a_n\}) \to \phi(x,a_1,..,a_n))$.
PROOF (i) Assume $\forall x(\phi(x,a_1,..,a_n) \leftrightarrow \psi(x,b_1,..,b_m))$ and $\forall x(R(x,\{a_1,..,a_n\} \to \phi(x,a_1,..,a_n))$, but not $\forall x(R(x,\{b_1,..,b_m\}) \to \psi(x,b_1,..,b_m))$.
Let $\theta(x,b_1,..,b_m)$ be the formula defined by $\exists z{\approx}x\psi(x,b_1,..,b_m)$.
If $\exists x(R(x,\{b_1,..,b_m\}) \wedge \neg\psi(x,b_1,..,b_m))$, then also $\exists x(R(x,\{b_1,..,b_m\}) \wedge \neg\theta(x,b_1,..,b_m))$. Since θ defines a tailset, we can apply the zero-one law to obtain $\forall x(R(x,\{b_1,..,b_m\}) \to \neg\theta(x,b_1,..,b_m))$.
By Downward Monotonicity, we have $\forall x(R(x,\{a_1,..,a_n;b_1,..,b_m\}) \to \phi(x,a_1,..,a_n))$ and hence $\forall x(R(x,\{a_1,..,a_n;b_1,..,b_m\}) \to \psi(x,b_1,..,b_m))$ and $\forall x(R(x,\{a_1,..,a_n;b_1,..,b_m\}) \to \theta(x,b_1,..,b_m))$. Once more applying Downward Monotonicity we obtain a contradiction.
(ii) is obtained from (i) by taking ψ to be the conjunction of ϕ and a suitable tautology.

$\boxtimes$

As an example of the use of extensionality for R, we show that the existence axiom can be considerably weakened:

LEMMA 1.3.3. Extensionality for R implies $\exists xR(x) \to \forall X\exists xR(x,X)$.
PROOF Choose X. We have $\forall x(x{\neq}x \leftrightarrow x{\neq}x \vee X{\neq}X)$, hence by extensionality for R $\forall x(R(x) \to x{\neq}x) \leftrightarrow \forall x(R(x,X) \to x{\neq}x \vee X{\neq}X)$. $\exists xR(x)$ implies $\exists x(R(x) \wedge x{=}x)$, hence also $\exists x(R(x,X) \wedge x{=}x \wedge X{=}X)$, that is, $\exists xR(x,X)$. $\boxtimes$

So we see that the presence of parameters in the zero-one law has non-trivial consequences. Observe, though, that even in the weak theory ZFR^{-1} there will be countably many mutually independent sequences (by Splitting).

THEOREM 1.3.1. ($ZFR^{(0)}$) Let $\phi(x,\alpha,X)$ be a formula in $\{\in,R\}$, where all the parameters are exhibited: x ranges over 2^ω, $\alpha \in$ ORD, X either a finite set of reals and ordinals (ZFR^0), or a finite set of arbitrary sets (ZFR). Then $\forall\kappa[\forall\alpha{<}\kappa\forall x(R(x,X\cup\{\alpha\}) \to \phi(x,\alpha,X)) \to \forall x(R(x,X) \to \forall\alpha{<}\kappa\phi(x,\alpha,X))]$. In particular we have Gödel's law: $\forall x(R(x,X) \to R(x,X\cup\{\alpha\}))$.
PROOFSKETCH In the proof of theorem 2.9 of van Lambalgen [1992] it is shown that, using R, one can define a generalized quantifier Q as follows: $Qx\psi(x,y_1,..,y_n) := \forall x(R(x,\{y_1,..,y_n\}) \to \psi(x,y_1,..,y_n))$ (all parameters exhibited). Q thus defined satisfies the Friedman axioms; cf. also theorem 2.2.2. below. From theorem 2.3 of van Lambalgen [1992] it follows that for a generalized quantifier Q satisfying the Friedman axioms, $\forall\kappa[\forall\alpha{<}\kappa Qx\phi(x,\alpha,X) \to Qx\forall\alpha{<}\kappa\phi(x,\alpha,X))]$. Spelling this out using the definition of Q yields the theorem.

$\boxtimes$

Domenico Zambella has observed that if we include Gödel's law among the axioms, the zero-one law can be taken to be an axiom, instead of a schema. We first replace each formula by a Δ_0

formula, possibly with additional parameters V_α; clearly we may replace the V_α by α itself. By Gödel's law, we need not include α among the parameters in R. Now use a truth definition for Δ_0 formulas; the zero-one law applied to this instance yields all instances. Hence ZFR is finitely axiomatizable over ZF.

The following property seems to be a desirable feature of any independence relation:

LEMMA 1.3.4. ($ZFR^{(0)}$) Let F be a function which is ordinal definable in $\{\in, R\}$. Then $R(x,X) \rightarrow R(x,F(X))$.

PROOF Suppose first that F is definable in $\{\in,R\}$. If $\exists x(R(x,X) \wedge \neg R(x,F(X)))$, then by the zero-one law, $\forall x(R(x,X) \rightarrow \neg R(x,F(X)))$. By R2 also $\forall x(R(x,XF(X)) \rightarrow \neg R(x,F(X)))$; however, this conflicts with Downward Monotonicity. The general case (F is definable in $\{\in,R\}$ from ordinals $\alpha_1,...,\alpha_n$) can be reduced to this case using Gödel's law.

REMARK Almost the same argument works if we use extensionality for R instead of the zero-one law. ⊠

It will be seen that this unprepossessing lemma is an important tool. In fact it is so important that it deserves a name of its own; we call it the 'function lemma". We shall first use it to relate the above definition of the independence relation to the definition given in van Lambalgen [1992]. There, R was assumed to be a relation of indefinite arity, i.e. a union of relations of ordered n+1-tuples (where the first argument is a real), whereas in the new set-up R is a relation between a real and an arbitrary finite set. Given an independence relation R in the former sense, we can define a relation R' as a relation between reals and ordered n-tuples by:

$$\langle x,a_1...a_n\rangle \in R \text{ if and only if } \langle x,\langle a_1...a_n\rangle\rangle \in R'$$

and conversely, so it suffices to consider relations between reals and ordered n-tuples. For these we can prove:

LEMMA 1.3.5. (ZFR) $R(x,\langle a_1,...,a_n\rangle) \leftrightarrow R(x,\{a_1,...,a_n\})$.

PROOF At first sight it would seem that the right-to-left direction needs a definable choice function to create an ordered n-tuple out of a set of n elements; somewhat surprisingly, this is not so. The implication from left to right follows because the function which maps $\langle a_1,...,a_n\rangle$ to $\{a_1,...,a_n\}$ is definable, so we can use the function lemma. For the converse direction, let $F(\{a_1,...,a_n\})$ be the definable function which gives the (finite) set of all ordered n-tuples on $\{a_1,...,a_n\}$. By the function lemma, $R(x,F(\{a_1,...,a_n\}))$. Since $\{\langle a_1,...,a_n\rangle\} \subseteq F(\{a_1,...,a_n\})$, we can invoke Downward Monotonicity to obtain $R(x,\{\langle a_1,...,a_n\rangle\})$. Now apply a definable 'non-choice' function to $\{\langle a_1,...,a_n\rangle\}$; the function lemma then gives $R(x,\langle a_1,...,a_n\rangle)$. ⊠

There is a corresponding result for ZFR^0, but here we have to be a bit careful, since strictly speaking $R(x,\langle a_1,...,a_n\rangle)$ is not an admissible expression ($\langle a_1,...,a_n\rangle$ is not a set of reals). We therefore use a definable generalized pairing function π, which constructs a single real out of reals $a_1,...,a_n$, together with definable inverses. We then have

LEMMA 1.3.6. (ZFR^0) $R(x,\{a_1,...,a_n\}) \leftrightarrow R(x,\pi\{a_1,...,a_n\})$.

PROOF First observe that $\pi\{a_1,...,a_n\}$ is well-defined since we have a definable linear order on $\{a_1,...,a_n\}$. The direction from left to right is immediate from the function lemma. Conversely, using the definable inverses, we can construct $\{a_1,...,a_n\}$ from $\pi\{a_1,...,a_n\}$, so we may again invoke the function lemma. ⊠

Hence the approaches chosen by van Lambalgen [1992] and the one adopted here are equivalent. Backed by these lemmas, we shall henceforth be rather careless about the distinction between finite sets and tuples.
The final lemma of this subsection shows that R is in a sense large:

LEMMA 1.3.7. ($ZFR^{(0)}$) For any X, $\{x \mid R(x,X)\}$ is uncountable.
PROOF Suppose $\{x \mid R(x,X)\}$ is countable. Code $\{x \mid R(x,X)\}$ into a single real y. By Existence, there is z such that $R(z,X\cup\{y\})$. By the function lemma and Irreflexivity, $z \notin \{x \mid R(x,X)\}$, a contradiction. ⊠

1.4 CONSISTENCY

The following results were proved in van Lambalgen [1992] with the help of simple forcing techniques. For a sketch of the proofs, see footnote 2.

THEOREM 1.4.1 Suppose ZFC is consistent. (i) ZFR^0 + AC is consistent (hence also ZFR^{-1} + AC) (ii) ZFR + DC is consistent. ⊠

We shall see in section 3 that the models used to prove consistency are essentially recaptured as inner models of the respective theories.

1.5 $ZFR^{(0)}$ AND AC

We now come to the connection between ZFR and the axiom of choice. Using the function lemma and Gödel's law, it is easy to derive the non-existence of a well-ordering on 2^ω. For suppose $F: \alpha \to 2^\omega$ is a bijection from an ordinal onto the reals; by R1, there exists x with R(x,F); by Gödel's law, for all $\beta<\alpha$, $R(x,\beta F)$, hence using the function lemma and the application function, $R(x,F(\beta))$. By irreflexivity, it follows that $x\neq F(\beta)$ for all $\beta<\alpha$, a contradiction. However, there is also a simple proof using only Existence, Irreflexivity and the function lemma, which highlights the conceptual issues involved.

THEOREM 1.5.1. (ZFR) There is no choice function on the power set of the reals.
PROOF Let g be any choice function on the power set of the reals. Then $\{x \mid R(x,g)\}$ is a set by the separation axiom; moreover, it is nonempty in virtue of the existence axiom. Hence we can apply g to this set, that is we have $R(g\{x \mid R(x,g)\},g)$. Define a function F by: for any choice function g on $\mathcal{P}(2^\omega)$, $F(g) = g\{x \mid R(x,g)\}$. F is $\{\in,R\}$ - definable, so, by the function lemma, R(x,g) implies R(x,F(g)), which in turn implies $x\neq F(g)$ by Irreflexivity; however, we have just seen that R(F(g),g), a contradiction. ⊠

The root of the trouble is that, using g, we can define a set of reals (non-empty in virtue of Existence),*which has to belong to the domain of* g. This constrains the allegedly independent determination of the values of g so that g cannot be a choice function on the whole of its domain. One can also view this proof as a counterpart to the conceptual indeterminateness of CH, sketched in the introduction: choice functions on $\mathcal{P}(2^\omega)$ are formed at level $\omega+3$, the (relevant part of the) relation R at level $\omega+4$; we now have to decide which level to maximize. One could argue that, naturally, the lowest levels have priority here (although the temporal metaphor is dangerous here, and could easily lead to an abandonment of classical logic); this would be an argument for the existence of a choice function on $\mathcal{P}(2^\omega)$, and against the continuum hypothesis. However, in view of the following result, it would also be a argument against a definable well-ordering of 2^ω (since R restricted to reals can be formed at level $\omega+2$).

COROLLARY 1.5.1. (ZFR^0) There is no choice function on $\mathcal{P}(2^\omega)$ which is definable with real and ordinal parameters.
PROOF Copy the proof of the previous theorem, with $R(x,\{s_1, ...,s_n\})$ instead of $R(x,g)$, where the s_i are the real parameters used in defining the choice function (the ordinals can be disregarded by Gödel's law). ⊠

Observe that the requirements on g can be weakened to being a choice function on the uncountable subsets of $\mathcal{P}(2^\omega)$; in view of lemma 1.3.7., $\{x \mid R(x,g)\}$ is uncountable. So we have to ask ourselves: is it plausible that for this g there exists x such that $R(x,g)$? The next result shows that this question is *essentially* connected to the validity of the axiom of choice:

THEOREM 1.5.2. Let T be the theory consisting of ZF + the axiom of choice for well-ordered sets + "there is no well-ordering of the reals". In T one can interpret an extension of ZF using the relation R, where R is allowed in the schemata and satisfies R1,2,3,4, extensionality for R (cf. lemma 1.3.2.) and Gödel's law.
For a proof, see van Lambalgen [1992], corollary 2.15. ⊠

In other words, if there is no well-ordering of the reals, then one can always prove this by means of the argument given in theorem 1.5.1. (modulo the axiom of choice for well-ordered sets). By the same token, if one doesn't believe in the existence of an independence relation satisfying the axioms given above, then one is bound to believe in the existence of a well-ordering of the reals.

The last result shows that even in the weak theory ZFR^{-1} one can derive the non-existence of a definable well-ordering of the reals:

THEOREM 1.5.3. (ZFR^{-1} + DC) There is no definable well-ordering of the reals.
PROOF If x is a real, let [x] denote the equivalence class of x mod≈. The definable well-ordering can be lifted to a definable well-ordering on the set of equivalence classes. Let this well-ordering be <. Define a predicate S by

$$S(x) \text{ if } R(x) \wedge \forall y(R(y) \rightarrow [y] \geq [x]).$$

We must have $\exists x S(x)$, otherwise there would exist an infinite descending chain. By definition, also $\exists x(R(x) \wedge S(x))$.
But S defines a tailset, so $\forall x(R(x) \rightarrow S(x))$.
By the weak existence axiom, $\exists x R(x)$. By Splitting, we can find x^0, x^1 such that $R(x^0)$ and $R(x^1,x^0)$, whence also $R(x^1)$. It follows that we must have

$$R(x^0) \wedge R(x^1) \rightarrow [x^1] \geq [x^0],$$

and

$$R(x^1) \wedge R(x^0) \rightarrow [x^0] \geq [x^1],$$

so that $x^0 \approx x^1$. However, in that case by the tailset property and the fact that $R(x^1,x^0)$, also $R(x^0,x^0)$, which contradicts Irreflexivity. ⊠

COROLLARY 1.5.3. (ZFR^{-1} + DC) The axiom of constructibility is false. ⊠

2. MODEL-THEORETIC ANALOGUES

In the present section we show that the independence relation bears a strong resemblance to two constructions in model theory: indiscernibles and minimal sets (see Hodges [1993], chapters 11 and 4, respectively). More precisely, in the context of ZFR, R plays the role of a set of (total) indiscernibles (or rather a family of such sets), whereas in the context of ZFR^0, R can be seen rather as giving a kind of minimal set. For the duration of this section, we shall disregard axiom 7, the splitting property. The reason is that for example indiscernibles do not have inner structure, so for purposes of comparison we shall disregard the inner structure given by Splitting as well. For ease of exposition we begin with ZFR.

2.1 INDEPENDENCE STRUCTURES AND INDISCERNIBLES

Under the assumption of a measurable cardinal, the uncountable cardinals form a set of order indiscernibles over *L*. We now show that ZFR can be viewed as a theory introducing (equivalence classes of) reals as indiscernibles, with R playing the role of the linear order.
Observe that R can unambiguously be defined for equivalence classes of reals by $R([x],a) := R(x,a)$; the tailset property ensures that this definition is independent of the choice of a representative. For equivalence classes occurring in the right hand argument of R we have the following result:

LEMMA 2.1.1. $R(x,[y]) \leftrightarrow R(x,y)$.
PROOF The direction from right to left follows from the function lemma. For the converse direction, observe that $R(x,y) \leftrightarrow \forall z{\approx}y R(x,z)$ (by the function lemma) and hence $R(x,y) \leftrightarrow \forall z{\in}[y]R(x,z)$. Applying extensionality for R (lemma 1.3.2.) we obtain $\forall x(R(x,y) \to R(x,y)) \leftrightarrow \forall x(R(x,[y]) \to \forall z{\in}[y]R(x,z))$, whence $\forall x(R(x,[y]) \to R(x,y))$.

⊠

As a consequence, we can show that Steinitz can be lifted to equivalence classes:

LEMMA 2.1.2. $R([x],\{[y]\}\cup X) \wedge R([y],X) \to R([y],\{[x]\}\cup X)$.
PROOF Suppose $R([x],\{[y]\}\cup X) \wedge R([y],X)$, then by the previous lemma $R(x,\{y\}\cup X) \wedge R(y,X)$. By Steinitz $R(y,\{x\}\cup X)$, whence $R([y],\{[x]\}\cup X)$ by the tailset property and lemma 2.1.1. ⊠

Consider a formula $\phi(z_1, ...,z_n,a)$, where the variables z_i range over equivalence classes of reals. Let $[x_1]$, ..., $[x_n]$ and $[y_1]$, ...,$[y_n]$ be sequences of equivalence classes satisfying $R([x_1],a)$, ..., $R([x_n],[x_1]...[x_{n-1}]a)$ and $R([y_1],a)$, ..., $R([y_n],[x_1]...[y_{n-1}]a)$ respectively. We claim that in this case, $\phi([x_1], ..., [x_n],a) \leftrightarrow \phi([y_1], ..., [y_n],a)$.
For assume $\phi([x_1], ..., [x_n],a)$, then $\exists x_1...\exists x_n(R(x_1,a) \wedge..\wedge R(x_n,x_1...x_{n-1}) \wedge \phi([x_1], ..., [x_n],a))$.
Since for fixed $x_1, ...,x_{n-1}$, $R(x_1,a) \wedge..\wedge R(x_n,x_1...x_{n-1}) \wedge \phi([x_1], ..., [x_n],a)$ defines a tailset in x_n, we can apply the zero-one law to obtain $\forall x_n(R(x_1,a) \wedge..\wedge R(x_n,x_1...x_{n-1}) \to \phi([x_1], ..., [x_n],a))$.
Continuing in this way, we get $\forall x_1...\forall x_n(R(x_1,a) \wedge..\wedge R(x_n,x_1...x_{n-1}) \to \phi([x_1], ..., [x_n],a))$, hence also $\forall x_1...\forall x_n(R([x_1],a) \wedge..\wedge R([x_n],[x_1]...[x_{n-1}]) \to \phi([x_1], ..., [x_n],a))$. By the hypothesis on $[y_1]$, ...,$[y_n]$, it follows that we must have $\phi([y_1], ..., [y_n],a)$.

This definition works in a set theory based on axioms 1-3,5 and 6. If in addition Steinitz exchange holds, we obtain a structure which resembles total indiscernibles.

DEFINITION 2.1.1 (i) Let X be a finite set. A set theory admits an *independence structure for* X if there exists a (class) relation R satisfying axioms 1-3, such that for all formulas $\phi(z_1, ..., z_n, X)$:

$$R(x_1,X) \wedge .. \wedge R(x_n,x_1..x_{n-1}X) \wedge R(y_1,X) \wedge .. \wedge R(y_n,y_1..y_{n-1}X) \rightarrow$$
$$(\phi(x_1, ..., x_n,X) \leftrightarrow \phi(y_1, ..., y_n,X)).$$

(ii) A set theory admits a weak *independence structure* if it admits an independence structure for X equal to the empty set.

(iii) A set theory admits an *independence structure* if for every finite set X, it admits an independence structure for X.

(iv) A (weak) independence structure is *total* if in addition R satisfies the Steinitz principle.

⊠

Hence ZFR admits a total independence structure, while ZFR^0 admits (at least) a weak total independence structure. If we assume the axiom of choice for well-ordered sets, the non-existence of a well-ordering of the reals leads to an independence structure.
For a simple application, consider

THEOREM 2.1.1. (ZFR) There exists an uncountable set of pairs without choice function.
PROOF Let $A = \{\{[x],[y]\} \mid x$ not equivalent to y mod$\approx\}$. Suppose g is a choice function for A; choose [x],[y] with R([x],g) and R([y],[x]g). By R3, $[x] \neq [y]$, hence $\{[x],[y]\} \in A$. Suppose that $g\{[x],[y]\} = [x]$. By Steinitz and the zero-one law , it then follows that also $g\{[x],[y]\} = [y]$, a contradiction. ⊠

It follows that $2^\omega/\approx$ cannot be linearly ordered, hence the usual concept of order indiscernibles would not be applicable here. It therefore seems reasonable to consider independence structures as a generalization of indiscernibles to cases where AC fails, and where there is no natural linear order.

2.2 EXISTENCE OF INDEPENDENCE STRUCTURES

We reformulate some results of van Lambalgen [1992], to connect the existence of (total) independence structures with the existence of certain ultrafilters.

DEFINITION 2.2.1. An ultrafilter U on $2^\omega/\approx$ is a *Fubini* ultrafilter if it satisfies for A contained in $2^\omega/\approx \times 2^\omega/\approx$: $\{x \mid A_x \in U\} \in U$ if and only if $\{y \mid A^y \in U\} \in U$, where A_x, A^y are, respectively, the vertical and horizontal sections of A.

If all sets of reals are measurable, there is a canonical way to obtain Fubini ultrafilters, as follows. Let F be the filter of full subsets of 2^ω. By the Kolmogorov zero-one law, for every tailset A 2^ω, either $A \in F$ or $A^c \in F$. By Fubini's theorem, for every A contained in $2^\omega \times 2^\omega$, $\{x \mid A_x \in F\} \in F$ if and only if $\{y \mid A^y \in F\} \in F$. Define a function f on 2^ω by $f(x) = [x]$, then U defined by $A \in U$ if and only if $f^{-1}(A) \in F$ is a Fubini ultrafilter on $2^\omega/\approx$.
The following is in the folklore; for a proof see, e.g., van Lambalgen [1992], theorem 2.3.

THEOREM 2.2.1. A Fubini ultrafilter is κ - complete, for every cardinal κ. ⊠

The importance of the Fubini property for filters was stressed by Harvey Friedman (cf Steinhorn [1985a,b)], who used it to give a complete axiomatisation of the notion of 'almost all' in Borel structures. In van Lambalgen [1992] it was shown that filters satisfying the Fubini property correspond in a natural sense to independence relations; more particularly, we have the following theorem:

THEOREM 2.2.2.(i) In ZFR (with R5 replaced by extensionality for R), a filter on 2^ω satisfying the Fubini property can be defined; (ii) In ZF+"there exists a filter on 2^ω satisfying the Fubini property", one can define an independence relation satisfying axioms 1-4 and extensionality for R.
SKETCH OF PROOF OF (i) Define F by $C \in F$ iff $\forall x(R(x,\{C\}) \rightarrow x \in C)$. Now we can verify that, e.g., for all y, $\{z \mid y \neq z\} \in F$: $R(x,\{z \mid y \neq z\})$ implies $R(x,\{y\})$ by the function lemma , which implies $x \neq y$ by irreflexivity. The full proof can be found in van Lambalgen [1992].

⊠

Parallel to the construction of a Fubini ultrafilter outlined above, we can strengthen this result to:

THEOREM 2.2.3. (i) In ZFR, a Fubini ultrafilter on $2^\omega/\approx$ can be defined; (ii) In ZF+"there exists a Fubini ultrafilter on $2^\omega/\approx$", one can define an independence relation satisfying the full independence axioms. ⊠

COROLLARY 2.2.3. The $\in$ - consequences of ZFR are axiomatized by ZF + "there exists a Fubini ultrafilter on $2^\omega/\approx$". ⊠

Hence the new primitive R is not really essential, although, at least to the author, the intuition behind the independence relation is clearer than that behind the ultrafilter.
From the theorem above it follows that a set theory admits a total independence structure if one of the following assumptions is satisfied: all sets of reals are measurable, all sets of reals satisfy the Baire property (use Kuratowski - Ulam instead of Fubini; cf. Oxtoby [1980], chapter 15), or, of course, the axiom of determinacy. However, weaker (but less elegant) assumptions suffice, e.g. the existence of a translation invariant extension of Lebesgue measure to all sets of reals, which is κ-additive for every κ. In fact, this is how the consistency of ZFR + DC is proved in van Lambalgen [1992][2].
To obtain an independence structure (not necessarily total) it suffices to postulate the existence of an ultrafilter which is κ-complete for every κ; for an interesting example of this, the reader may consult Henle [1984].
A very interesting aspect of the equivalence of independence structures and the existence of suitable ultrafilters, is the possibility to construct ultrapowers of the universe by means of Spector's technique of extended ultrapowers (cf. [1988]. [1991]). Los' theorem is almost equivalent to the

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3 Let us briefly sketch a consistency proof here. First one adds ω_1 Solovay random reals to a model $\mathcal{M}$of V=L; let the extension be $\mathcal{M}[G]$. On $\mathcal{M}[G]$, R can be defined explicitly using Borel codes. From R, define a generalized quantifier as follows: $Qx\phi(x,\alpha_1...\alpha_k,y_1...y_n)$ iff $\forall x(R(x,y_1...y_n) \rightarrow \phi(x,\alpha_1...\alpha_k,y_1...y_n))$ .Since this quantifier is extensional, it determines a filter; in other words, whether a set which is ordinal definable from real parameters is in the quantifier, is independent of the definition of this set (this is where the crucial 'extensionality for R' comes in; it is a consequence of the homogeneity of the forcing notion). Moreover, Q itself is hereditarily ordinal definable. We may now consider the submodel $HOD(2^\omega)$ of $\mathcal{M}[G]$; on this model, we can define R explicitly using Q.
As the construction of the inner model for ZFR will make clear, it is possible to use a simpler construction: add ω generic reals to a model of V=L and take the $HOD(2^\omega)$ part. In this case, however, the generic extension itself is not a model of ZFR^0 +AC.

axiom of choice (cf. Spector [1988], theorem 1), but Spector managed to overcome this obstacle to the construction of an elementary embedding by means of a forcing argument. One may now ask how the Fubini property reflects on the ultrapower and the elementary embedding; this question will be treated in a sequel to this paper.

2.3 MATROIDS?

As was observed in section 1, total independence structures will in general not be matroids, because the transitivity axiom $\neg R(x, X\cup\{y\}) \wedge \neg R(y,X) \rightarrow \neg R(x,X)$ fails in the presence of axioms concerning splittings of random sequences such as R7. In fact, it will be shown below that in $ZFR^{(0)}$, transitivity is already inconsistent with R1-6. It was proved in van Lambalgen [1992] that it is consistent (modulo the consistency of an intuitionistic set theory) to add transitivity of R to ZFR if one weakens the zero-one law to extensionality for R. Hence it would seem that one can apply the theory of matroids to set theory in this way. However, we have seen that the axiom of choice fails in this context, and as far as the author knows there does not exist an interesting theory of infinite matroids in the absence of AC (cf. Oxley [1992]). Intuitively, transitivity of R excludes closure of R under non-trivial operations, so it is no surprise that transitivity is inconsistent with axioms 1 to 6 (since 6 is precisely a closure property); however, we were unable to find an easy proof of the inconsistency which exploits this intuition directly. First we need a

LEMMA 2.3.1. ($ZFR^{(0)}$) If transitivity of R is added to 1-6, then the ultrafilter defined in the previous theorem will satisfy: if $B^c \in U$ and g: $2^\omega/\approx \rightarrow P(2^\omega/\approx)$ defines an indexed family of sets such that for $z \in B$, $g(z)^c \in U$, then $\bigcap\{g(z)^c \mid z \in B\} \in U$.[3]

PROOF If we consider ZFR, it suffices to show the corresponding property for the filter F on 2^ω defined by

$$C \in F \text{ iff } \forall x(R(x,\{C\}) \rightarrow x \in C);$$

i.e. we have to show if $B^c \in F$ and g: $2^\omega \rightarrow P(2^\omega)$ defines an indexed family of sets such that for $z \in B$, $g(z)^c \in F$, then $\bigcap\{g(z)^c \mid z \in B\} \in F$.

In the case of $ZFR^{(0)}$, we must replace {C} in R by the real parameters defining C.

(Observe that the property holds for the filter of co-countable sets; obviously there is no *ultra*filter of co-countable sets, and we can similarly derive a contradiction here.)

Assume $R(x, \{\bigcap\{g(z)^c \mid z \in B\}\})$; we have to show that $x \in \bigcap\{g(z)^c \mid z \in B\}$. If for all $z \in B$, $R(x, \{g,z\})$, then also for all $z \in B$, $R(x,\{g(z)\})$, hence $x \in g(z)$ and we are done. Now suppose that for some $z \in B$, we would have $\neg R(x, \{g,z\})$. If $z \in B$, then $z \notin B^c$, so $\neg R(z, B^c)$, hence by the function lemma $\neg R(z, B)$.

By transitivity, $\neg R(x, \{g,B\})$, hence again by the function lemma , $\neg R(x, \{\bigcap\{g(z)^c \mid z \in B\}\})$, a contradiction. ⊠

THEOREM 2.3.1. ($ZFR^{(0)}$) Transitivity for R fails.

Proof Argue in $ZFR^{(0)}$ plus transitivity. The proof is somewhat roundabout because we cannot assume AC. Let $LLN(p) := \{x \in 2^\omega \mid$ the limiting relative frequency of 1 in x equals p$\}$. We have either $LLN(^1/_2)/\approx \in U$ or $(LLN(^1/_2)/\approx)^c \in U$. Suppose first that $LLN(^1/_2)/\approx \in U$. Let h be an injection from $LLN(^1/_2)$ to LLN(0) obtained by interposing long blocks of 0's; then h and h^{-1} will preserve $\approx$. It follows that h is welldefined on $LLN(^1/_2)/\approx$; similarly, h^{-1} is welldefined on

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[4] Behind this lemma and theorem 2.2.2. there is a general correspondence between properties of quantifiers and (in)dependence relations. In fact, generalized quantifiers stand to (in)dependence relations as modal operators stand to orderings. See for example van Lambalgen [1991] and Alechina and van Lambalgen [1994].

h"LLN($^1/2$)/≈ ⊆ LLN(0)/≈ (here, h" denotes the forward image of h). Since (LLN($^1/2$)/≈)^c ∉ U, also h"LLN($^1/2$)/≈∉ U. Clearly, for all x ∈ h"LLN($^1/2$)/≈, {h^{-1}(x)}^{c} ∈ U, but (LLN($^1/2$)/≈)^c = ⋂ {{h^{-1}(x)}^{c} | x ∈ h"LLN($^1/2$)/≈} ∉ U, in contradiction with lemma 2.3.1.
If, on the other hand, (LLN($^1/2$)/≈)^c ∈ U, we use a fixed sequence z ∈ LLN($^1/2$) to construct an injection from LLN($^1/2$)^c into LLN($^1/2$), and then proceed as before. ⊠

2.4. INDEPENDENCE STRUCTURES AND MINIMAL SETS

There is a slightly different way to view the preceding material, in terms of an analogy to minimal sets and structures.

DEFINITION 2.4.1. Let $\mathcal{A}$ be a structure for a language $\mathcal{L}$, X a subset of the domain of $\mathcal{A}$. We say that an element a is in the algebraic closure of X, and write 'a ∈ acl(X)' if there is a formula ϕ such that the following two conditions hold:
(i) $\mathcal{A} \models \phi(a,X)$
(ii) $\{ c \in \mathcal{A} \mid \mathcal{A} \models \phi(c,X)\}$ is finite.
Strictly speaking, of course, acl should be indexed by $\mathcal{A}$. ⊠
It is easy to see that acl satisfies the following properties: X ⊆ acl(X), X ⊆ acl(Y) implies acl(X) ⊆ acl(Y), and a ∈ acl(X) implies a ∈ acl(Z) for a finite Z ⊆ X. Algebraic closure is especially well-behaved on *minimal sets*:

DEFINITION 2.4.2. Let $\mathcal{A}$ be a structure for a language $\mathcal{L}$. An infinite definable set X of elements of $\mathcal{A}$ is called *minimal* if for every formula ϕ of $\mathcal{L}$, possibly with parameters from $\mathcal{A}$, either $X \cap \{x \mid \mathcal{A} \models \phi(x)\}$ or $X \cap \{x \mid \mathcal{A} \models \neg\phi(x)\}$ is finite. $\mathcal{A}$ itself is a *minimal structure* if the domain of $\mathcal{A}$ is a minimal set. ⊠

LEMMA 2.4.1. Let $\mathcal{A}$ be a structure for a language $\mathcal{L}$, and let X be a minimal subset of $\mathcal{A}$ definable from U. On X, acl satisfies the Steinitz exchange property in the following sense: for any W ⊇ U and a,b ∈ X,

$$a \in acl(W \cup \{b\}) \text{ and } a \notin acl(W) \text{ implies } b \in acl(W \cup \{a\}).$$

In particular, if $\mathcal{A}$ is a minimal structure, we have that for any finite W in the domain of $\mathcal{A}$, a ∈ acl(W∪{b}) and a ∉ acl(W) implies b ∈ acl(W∪{a}). ⊠

Let $\mathcal{A}$ be a minimal structure. On $\mathcal{A}$, define a generalized quantifier as follows: for a finite set X contained in the domain of $\mathcal{A}$,

$$Qx\phi(x,X) \text{ if and only if } \{ x \mid \mathcal{A} \models \phi(x,X)\} \text{ is cofinite.}$$

Then we have Qxϕ(x,X) if and only if $\forall x(x \notin acl(X) \rightarrow \neg\phi(x,X))$. It follows from lemma 2.4.1. and the remark preceding definition 2.4.2. that Q then satisfies the following properties:

LEMMA 2.4.2. All formulas may contain parameters from $\mathcal{A}$. Assume $\mathcal{A}$ is infinite.
(0) $\neg Qx(x \neq x)$
(1) $Qx\phi(x) \vee Qx\neg\phi(x)$
(2) $Qx(x \neq y)$
(3) $Qx\phi \wedge \forall x(\phi \rightarrow \psi) \rightarrow Qx\psi$
(4) $Qx\phi \wedge Qx\psi \rightarrow Qx(\phi \wedge \psi)$
(5) $QxQy\phi \rightarrow QyQx\phi$
(6) $\forall yQx\phi \wedge Qy\forall x\phi \rightarrow Qx\forall y\phi$

PROOF (5) corresponds to Steinitz exchange, (6) corresponds to the property $X \subseteq acl(Y)$ implies $acl(X) \subseteq acl(Y)$ (i.e. transitivity of acl); in the quantifier setting, it expresses that a finite union of finite sets is finite. ⊠

We see from the material in sections 2.2. and 2.3. that R defines a structure which satisfies slightly weaker properties:

DEFINITION 2.4.3. A structure $\mathcal{A}$ is *a.e.-minimal* if there exists a generalized quantifier Q on $\mathcal{A}$ which satisfies:
(0) $\neg Qx(x \neq x)$
(1) $Qx\phi(x) \vee Qx\neg\phi(x)$
(2) $Qx(x \neq y)$
(3) $Qx\phi \wedge \forall x(\phi \rightarrow \psi) \rightarrow Qx\psi$
(4) $Qx\phi \wedge Qx\psi \rightarrow Qx(\phi \wedge \psi)$
(5) $QxQy\phi \rightarrow QyQx\phi$

LEMMA 2.4.3. In ZFR$^{(0)}$, $2^{\omega}/\approx$ is a.e.-minimal, but it cannot satisfy (6).
SKETCH OF PROOF For ZFR, define the quantifier as in section 2.2. For ZFR^0, define Q as

$$Qx\phi(x,[y_1]...[y_n]) \text{ if } \forall x(R(x,y_1...y_n) \rightarrow \phi(x,[y_1]...[y_n]));$$

the function lemma and Gödel's law ensure that this definition is independent of the choice of representatives. The arguments in section 2.2. show that we obtain an a.e.-minimal structure in this way.
To see that $2^{\omega}/\approx$ cannot satisfy (6), i.e. that *no* Q can be defined on $2^{\omega}/\approx$ which satisfies (0) - (6), argue as follows in ZFR$^{(0)}$: suppose there were such a Q, then by theorems 2.9(ii) and 2.11(ii) of van Lambalgen [1992] (for ZFR and ZFR^0 respectively), there is also an R' satisfying axioms 1 to 6 plus transitivity for $\neg R$ (cf. footnote 3). Theorem 2.3.1. now yields a contradiction.
⊠

Of course, it still remains to be seen whether a.e.-minimal structures have an independent interest comparable to that of minimal structures.

3. INNER MODELS

The consistency of the various theories involving the independence relation is proved using well-known forcing constructions yielding an explicit definition of R. The purpose of this section is to prove a kind of converse: as inner models of ZFR$^{(0)}$ we obtain structures which strongly resemble the forcing constructions used to prove consistency.

In the following we shall often use expressions of the form "R(x,g)" where g is a countable sequence of reals. By this we shall mean $R(x,\pi(g))$ where π is a definable coding; the function lemma shows that this convention does not depend upon the particular coding chosen. All axioms except R2 (downward monotonicity) hold for the extended notion; R2 holds if we restrict ourselves to ordinal definable subsets of g, and this is all we need below.

3.1. AN INNER MODEL FOR ZFR^0 + AC.

DEFINITION 3.1.1. An *independent sequence* is a function $f:\alpha \rightarrow 2^{\omega}$, where α is an ordinal, such that for all $\beta < \alpha$, $R(f(\beta), f|\beta)$. The *length* of an independent sequence f is the domain of f.[4]

⊠

We use the following notion of relative constructibility:

DEFINITION 3.1.2. Let A be a set, then $L(A)$ is defined inductively as follows:

$L_0(A) = \{A\} \cup trcl(A)$

$L_{\alpha+1}(A) = Def(L_\alpha(A))$

$L_\alpha(A) = \bigcup\{L_\gamma(A) \mid \gamma < \alpha\}$ for Lim α;

here, Def(X) denotes the collection of sets definable (with parameters) in the structure $<X,\in>$. Alternatively, $Def(L_\alpha(A)) = cl(L_\alpha(A) \cup \{L_\alpha(A)\})$, where cl denotes closure under the Gödel operations. ⊠

THEOREM 3.1.1. $L(A)$ satisfies ZF; if trcl(A) has a well-ordering in $L(A)$, $L(A)$ also satisfies AC.

⊠

As a consequence, if f is an independent sequence of length ω_1, $L(f) \models$ ZFC. The main result of this section is

THEOREM 3.1.2. Let f be an independent sequence of length ω_1. There exists an interpretation of R in $L(f)$ such that $L(f) \models ZFR^0 + AC$.

This result would not hold were we to replace f by an independent sequence of length ω. The proof proceeds in two steps. First we show how to interpret ZFR^0 minus Splitting in $L(f)$, and then we modify the interpretation of R to incorporate Splitting as well. The first step is based upon the following fundamental fact about relative constructibility:

LEMMA 3.1.1. Let y be a real in $L(f)$. Then there exist ordinals $\alpha,\beta < \omega_1$ such that $y \in L_\alpha(<f(\gamma) : \gamma < \beta>)$.
The proof is by standard Löwenheim-Skolem techniques. ⊠

Now choose X; let $\alpha,\beta < \omega_1$ be such that $X \in L_\alpha(<f(\gamma) : \gamma < \beta>)$. It follows that $X \in L(<f(\gamma) : \gamma < \beta>)$; but this universe has a well-ordering definable from $<f(\gamma) : \gamma < \beta>$, hence there is a constructible function F and an ordinal δ such that $X = F(\delta,<f(\gamma) : \gamma < \beta>)$ on $L(f)$. Henceforth we shall write $f|\beta$ for $<f(\gamma) : \gamma < \beta>$.
Motivated by this lemma, the function lemma and Gödel's law, we choose the following interpretation S for R in $L(f)$:

DEFINITION 3.1.3. S(x,X) if for some countable g contained in f and a Δ_0 operation F $\exists y(x \approx y \wedge y \in range(f)\backslash range(g) \wedge X = F(g))$. ⊠

Clearly, S is consistent with R in the following sense: for $x,X \in L(f)$, $L(f) \models S(x,X)$ implies R(x,X) (this follows from the function lemma). If we also knew the converse, the validity of ZFR^0 in $L(f)$ would be fairly immediate, but this is in general false (e.g. if V is not equal to $L(f)$).

THEOREM 3.1.3. $L(f) \models ZFR^0$, when R is interpreted as S.

---

5 This concept is also borrowed from model theory; see Hodges [1993], p. 168.

The key point of the proof is checking the validity of the zero-one law; the rest of the axioms will then be seen to follow easily. In the following, we shall usually disregard ordinals in R by tacit appeal to Gödel's law. The proof of the 0- law consists of a series of reflection arguments. This is necessary because the zero-one law in V applies to formulas with real and ordinal parameters only, hence f itself is not an admissible parameter. Choose a formula ϕ. First, we can find countable ordinals β,δ such that $L_\beta(f|\delta)$ satisfies 'enough of ZFC', $f|\delta$ is uncountable in $L_\beta(f|\delta)$ and $L(f)$ satisfies the zero-one law for ϕ (for all parameters) if and only if $L_\beta(f|\delta)$ satisfies the zero-one law for ϕ (for all parameters). *From now on, until the end of the proof of lemma 3.1.3, we work in the universe* $L_\beta(f|\delta)$*, which we shall continue to call* $L(f)$ *in order not to proliferate notation.*

LEMMA 3.1.2. Suppose $L(f) \models \exists x(S(x,X) \wedge \phi(x,X,\tau))$, where $\phi(x,X,\tau)$ defines a tailset in the variable x, X is a finite set of reals and τ a finite sequence of ordinals (countable ordinals when viewed from outside). Then in $L(f)$, $\{x \mid x \in \text{range}(f) \wedge \phi(x,X,\tau)\}$ is uncountable.

PROOF Suppose not; let $\delta < \omega_1^{L(f)}$ be such that

$$L(f) \models \text{"}\{x \mid x \in \text{range}(f) \wedge \phi(x,X,\tau)\} \subseteq \text{range}(f|\delta)\text{"}.$$

Let $X = F(h)$, where F is constructible and $h \subseteq f$ is countable such that for some x with $\exists y(x \approx y \wedge y \in \text{range}(f)\backslash\text{range}(h))$,

$$L(f) \models \phi(x,X,\tau).$$

Let α,γ (where $\delta < \gamma < \omega_1^{L(f)}$; and $\alpha<\omega_1$ by our convention concerning $L(f)$) be such that $X,h,x,\tau \in L_\alpha(f|\gamma)$ and for all $z \in 2^\omega \cap L_\alpha(f|\gamma)$,

$$L_\alpha(f|\gamma) \models \phi(z,X,\tau) \text{ iff } L(f) \models \phi(z,X,\tau).$$

(This can be achieved by a combination of reflection and Löwenheim-Skolem arguments.)

We may suppose that $x = f(\beta)$ for some $\beta < \gamma$. Choose $\beta' > \delta$ such that $f(\beta')$ is outside range(h) (there will be such a β' if we choose γ large enough). Then we may write

$$L_\alpha(\langle\beta,f(\beta)\rangle \cup \langle\beta',f(\beta')\rangle \cup \text{rem}(f|\gamma)) \models \phi(f(\beta),X,\tau),$$

where $\text{rem}(f|\gamma)$ denotes the remainder of $f|\gamma$, i.e. $f|\gamma$ minus $\{\langle\beta,f(\beta)\rangle \cup \langle\beta',f(\beta')\rangle\}$.

Hence from the point of view of V we have

$$R(f(\beta),\text{rem}(f|\gamma)) \wedge R(f(\beta'),f(\beta)\text{rem}(f|\gamma)) \wedge$$
$$L_\alpha(\langle\beta,f(\beta)\rangle\cup\langle\beta',f(\beta')\rangle\cup\text{rem}(f|\gamma))\models \phi(f(\beta),X,\tau),$$

whence by Steinitz and the zero-one law in V,

$$L_\alpha(\langle\beta,f(\beta)\rangle \cup \langle\beta',f(\beta')\rangle \cup \text{rem}(f|\gamma)) \models \phi(f(\beta'),X,\tau).$$

It follows that $L(f) \models \phi(f(\beta'),X,\tau)$, a contradiction. ⊠

LEMMA 3.1.3. The zero-one law holds in $L(f)$.

PROOF We use the same conventions as in the previous lemma. Suppose $L(f) \models \exists x(S(x,X) \wedge \phi(x,X,\tau))$. It suffices to show for all $h \in L(f)$, where $h \subseteq f$ countable such that for some constructible F, $X = F(h)$:

$$L(f) \models \forall x(\exists y(x \approx y \wedge y \in \text{range}(f)\backslash\text{range}(h)) \rightarrow \phi(x,X,\tau)).$$

Fix h and choose α,γ (where $\gamma < \omega_1^{L(f)}$) such that

(i) $h,X,\tau \in L_\alpha(f|\gamma)$, h is countable in $L_\alpha(f|\gamma)$

(ii) $L_\alpha(f|\gamma) \models$ "$\{x \mid \phi(x,X,\tau)\}$ is uncountable" and

(iii) $L(f) \models \forall x(\exists y(x \approx y \wedge y \in \text{range}(f)\backslash\text{range}(h)) \rightarrow \phi(x,X,\tau))$ if and only if $L_\alpha(f|\gamma) \models \forall x(\exists y(x \approx y \wedge y \in \text{range}(f|\gamma)\backslash\text{range}(h)) \rightarrow \phi(x,X,\tau))$.

Hence it suffices to show that $L_\alpha(f|\gamma) \models \forall y \in \text{range}(f|\gamma)\backslash\text{range}(h)\phi(y,X,\tau))$ (here we use the fact that ϕ defines a tailset). Argue in $L_\alpha(f|\gamma)$. Observe that, since $\{x \mid \phi(x,X,\tau)\}$ is uncountable, there must be at least one $y \in \text{range}(f|\gamma)\backslash\text{range}(h)$ such that $\phi(y,X,\tau)$ (by (i) and (ii)), hence applying the same permutation argument as in the previous lemma we see that $\phi(y,X,\tau)$ for *all* $y \in$

range($f|\gamma$)\range(h).
Applying (iii) we may conclude that $L(f) \vDash \forall x(\exists y(x \approx y \wedge y \in \text{range}(f)\backslash\text{range}(h)) \rightarrow \phi(x,X,\tau))$.

⊠

We now return to the original model L(f)

LEMMA 3.1.4. L(f) satisfies the Existence axiom.

PROOF Each X is constructible from a countable independent sequence, whereas f is uncountable.

⊠

LEMMA 3.1.5. L(f) satisfies Downward Monotonicity.

PROOF Suppose $L(f) \vDash S(x,X) \wedge Y \subseteq X$. Let g,F be such that X = F(g) and $x \notin$ range(g). Clearly Y is constructible from g and an ordinal, hence S(x,Y). ⊠

LEMMA 3.1.6. L(f) satisfies the Steinitz exchange axiom.

PROOF Suppose $S(x,X\cup\{y\}) \wedge S(y,X)$, but not $S(y,X\cup\{x\})$. By the zero-one law in L(f), we must have $L(f) \vDash \forall x\forall y(S(x,X\cup\{y\}) \wedge S(y,X) \rightarrow \neg S(y,X\cup\{x\}))$. Let X = F(g) and let H be constructible such that $H(g\cup\{y\}) = X\cup\{y\}$ uniformly for $y \notin$ range(g). Choose x,y $\in$ range(f) such that $x \notin \text{range}(g)\cup\{y\}$ and $y \notin$ range(g). Then $S(x,X\cup\{y\}) \wedge S(y,X)$, whence $\neg S(y,X\cup\{x\})$. However, since $H(g\cup\{x\}) = X\cup\{x\}$, $\neg S(y,X\cup\{x\})$ implies in particular $y \in \text{range}(g)\cup\{x\}$, a contradiction. ⊠

The tailset property of S is satisfied by definition. We now show how to extend the construction so as to include Splitting.

Let i_k be the projection function of definition 1.1.1. Observe that $x \approx i_k(y)$ implies that $\exists y'\approx y(x = i_k(y'))$ hence it suffices to consider the closure of S under the splitting operation.

LEMMA 3.1.7. Let f be an independent sequence of length ω_1, and let f_k be obtained from f by the following operation: each x in range(f) is replaced by the k-tuple $\langle i_1(x),..,i_k(x)\rangle$. Then f_k is again an independent sequence and $L(f) = L(f_k)$.

PROOF The first statement follows from Splitting in V. The second statement holds because the operations of splitting and combining are constructible. ⊠

The final interpretation of R is now defined as follows:

DEFINITION 3.1.4. $S_{0,k}(x,X) := S(x,X)$, where S is the relation given in definition 3.1.3.; $S_{n+1,k}(x,X) := \exists y(S_{n,k}(y,X) \wedge (y = x \vee \exists l\leq k\ (i_l(y) = x)))$, $S_k(x,X) := \exists n S_{n,k}(x,X)$, $R(x,X) := \exists k S_k(x,X)$. ⊠

Observe that each $S_{1,k}$ can be viewed as being the S of definition 3.1.3. relative to the independent sequence f_k instead of f, so that the $S_{n,k}$ correspond to gradual modifications of the original f. By lemma 3.1.7. it follows that the results proved above for S hold as well for the $S_{n,k}$.

LEMMA 3.1.8. R in L(f) satisfies Existence, Downward Monotonicity, Irreflexivity, Splitting and Tailset.

PROOF Trivial. ⊠

LEMMA 3.1.9. The zero-one law holds in L(f).

PROOF Suppose for some x $L(f) \vDash R(x,X) \wedge \phi(x,X,\tau)$, where ϕ defines a tailset. Choose y with

$R(y,X)$ and let n,k be such that $S_{n,k}(y,X)$ and $S_{n,k}(x,X)$. We then have $L(f) \models S_{n,k}(x,X) \wedge \phi(x,X,\tau)$, whence by lemma 3.1.7. and lemma 3.1.3., $L(f) \models \forall x(S_{n,k}(x,X) \rightarrow \phi(x,X,\tau))$ and in particular $L(f) \models \phi(y,X,\tau)$. ($\phi$ may contain R, but lemma 3.1.3. is applicable because R is defined from S.) It follows that $L(f) \models \forall x(R(y,X) \rightarrow \phi(y,X,\tau))$.

⊠

LEMMA 3.1.10. Steinitz holds in $L(f)$.

PROOF Suppose for some x,y, $L(f) \models R(x,X\cup\{y\}) \wedge R(y,X) \wedge \neg R(y,X\cup\{x\})$. Choose n,k large enough so that $S_{n,k}(x,X\cup\{y\}) \wedge S_{n,k}(y,X)$, then by the zero-one law for $S_{n,k}$, $L(f) \models \forall x\forall y(S_{n,k}(x,X\cup\{y\}) \wedge S_{n,k}(y,X) \rightarrow \neg R(y,X\cup\{x\}))$.

However, $\neg R(y,X\cup\{x\})$ implies $\forall n,k\neg S_{n,k}(y,X\cup\{x\})$, so we obtain a contradiction with Existence and Steinitz for $S_{n,k}$. ⊠

This completes the proof of theorem 3.1.2. ⊠

$L(f)$ resembles a generic extension of L with ω_1 Cohen or Solovay reals. When we add axioms to ZFR^0 connecting R and Lebesgue measure, as is done in van Lambalgen [1992], $L(f)$ *is* this Solovay extension.

THEOREM 3.1.3. Let f,g be independent sequences of length ω_1, then $L(f)$ is elementarily equivalent to $L(g)$.

PROOF

3.2 INNER MODELS FOR ZFR.

In this section we present two models for ZFR, constructed inside ZFR+DC. Unfortunately we were not able to construct models where DC holds as well, i.e. true inner models; this problem seems to be rather difficult (cf. the analogous problem for ZF+DC+'all sets of reals are Lebesgue measurable'). We shall not give the proof that Splitting holds in the models, since these are in both cases analogous to lemmas 3.1.8-10.

The first model will use sets hereditarily ordinal definable from range(f) in the following sense:

DEFINITION 3.2.1. For a set A, HOD(A) is the class of sets ordinal definable from A and a finite number of elements of A.⊠

Let f be a countable independent sequence (cf. definition 3.1.1.); let [f] denote the sequence of equivalence classes $<[f(n)] \mid n \in \omega>$. Let $\mathcal{M}$ denote the class HOD(range(f)).

THEOREM 3.2.1. (ZFR+DC) $\mathcal{M}$ satisfies ZFR.

PROOF The proof is somewhat roundabout; R will be constructed from a suitable ultrafilter as indicated in section 2.

We first define a generalized quantifier Q as follows:

> let $\phi(z,\text{range}(f),y_1,..y_n,\tau)$ be a formula where z runs over equivalence classes of reals, the y_i are reals, and τ is a finite sequence of ordinals. Put
>
> $Qz\phi(z,\text{range}(f),y_1,..y_n,\tau)$ if
>
> $$\text{range}([f]) - \{[y_i] \mid i\leq n\} \subseteq \{z \mid \phi(z,\text{range}(f),y_1,..y_n,\tau)\}.$$

Q is ordinal definable from range(f). We show in ZFR+DC that Q satisfies the axioms given in definition 2.4.3. The validity of $\neg Qz(z\neq z)$, $Qz(z\neq[y])$ and $Qz\phi \wedge Qz\psi \rightarrow Qz(\phi \wedge \psi)$ is trivial.
Suppose $\neg Qz\phi(z,\text{range}(f),y_1,..y_n,\tau)$, then there exists $z \in \text{range}([f])$, $z \neq [y_i]$ $(i\leq n)$, such that $\phi(z,\text{range}(f),y_1,..y_n,\tau)$. Choose $z' \in \text{range}([f])$, $z' \neq [y_i]$ $(i\leq n)$. A by now familiar permutation argument in ZFR+DC shows that also $\phi(z',\text{range}(f),y_1,..y_n,\tau)$ (here it is essential that f is countable). Hence $Qz\phi(z,\text{range}(f),y_1,..y_n,\tau)$. It follows that Q is monotone and in particular extensional.
To verify the Fubini property $QzQz'\phi \rightarrow Qz'Qz\phi$, observe that

$$QzQz'\phi(z,z',\text{range}(f),y_1,..y_n,\tau) \Rightarrow$$
$$\Rightarrow \text{range}([f]) - \{[y_i] \mid i\leq n\} \subseteq \{z \mid Qz'\phi(z,z',\text{range}(f),y_1,..y_n,\tau)\} \Rightarrow$$
$$\Rightarrow [\text{range}([f]) - \{[y_i] \mid i\leq n\} \subseteq$$
$$\subseteq \Big\{z \mid (\text{range}([f]) - \{[y_i] \mid i\leq n\} \cup \{z\} \subseteq \{z' \mid \phi(z,z',\text{range}(f),y_1,..y_n,\tau)\})\Big\}].$$

In other words, if $z,z' \in \text{range}([f]) - \{[y_i] \mid i\leq n\}$ and $z \neq z'$, then $\phi(z,z',\text{range}(f),y_1,..y_n,\tau)$. Since $\neq$ is symmetric, also $Qz'Qz\phi(z,z',\text{range}(f),y_1,..y_n,\tau)$.

It now follows that Q induces a Fubini ultrafilter U in $\mathcal{M}$: we may define $A \in U$ if for some $y_1,..,y_n$ and τ, $A = \{z \mid \phi(z,\text{range}(f),y_1,..y_n,\tau)\} \wedge Qz\phi(z,\text{range}(f),y_1,..,y_n,\tau)$. *It follows from the extensionality of* Q *that* U *is well-defined.*
In $\mathcal{M}$, let $\theta(z,a_1,..,a_k,\alpha)$ be a formula which enumerates all sets of equivalence classes ordinal definable from parameters $a_1,..,a_k \in \mathcal{M}$. Define $R(x,a_1,..,a_k)$ by

$$R(x,a_1,..,a_k) \text{ if } \bigwedge\nolimits_\alpha \{z \mid \theta(z,a_1,..,a_k,\alpha)\} \in U \rightarrow \theta([x],a_1,..,a_k,\alpha).$$

The proof of theorem 2.2.3. (based on van Lambalgen [1992], theorem 2.9.) then shows that R satisfies the independence axioms.⊠

The model constructed in the next theorem is intuitively more appealing, since R is built up from below, starting from an independent sequence. It shows roughly that in this situation it suffices to require that R satisfies the function lemma and Gödel's axiom, since the validity of the ZFR axioms is then more or less automatic. For sets A, *L*(A) was defined in definition 3.1.2.

THEOREM 3.2.2. (ZFR+DC) Let f be a countable independent sequence, then $L(\text{range}(f)) \models$ ZFR.

In the following proof we shall consider the X in R(x,X) to be a finite sequence; lemma 1.3.5. shows that this assumption is harmless. We construct R inductively parallel to the inductive definition of *L*(range(f)):

<u>$\gamma = 0$</u>: for some $x'\approx x$, $x' \in \text{range}(f)$, $y_1,...,y_n \in L_0(\text{range}(f)) = \text{range}(f)\cup\{\text{range}(f)\}$: suppose $y_n = \text{range}(f)$ and τ is a finite sequence of ordinals, then
$R_0(x,y_1...y_n\tau)$ if $x' \neq y_i$ for $i<n$;

<u>Succ(γ)</u>: if $\gamma = \alpha+1$, $x,y_1,...,y_n \in L_{\alpha+1}(\text{range}(f)) = \text{Def}(L_\alpha(\text{range}(f))) = \text{cl}(L_\alpha(\text{range}(f))\cup\{L_\alpha(\text{range}(f))\})$: choose $z_1,...,z_n$ such that for some Δ_0 operations $F_1,...,F_n$: if $y_i = F_i(z_i)$, then
$R_{\alpha+1}(x,y_1...y_n\tau)$ if for some $x'\approx x$, $R_\alpha(x',z_1...z_n\tau)$;

<u>Lim(γ)</u>: $R_\alpha(x,y_1...y_n\tau) = \exists\beta<\gamma R_\beta(x,y_1...y_n\tau)$;

Lastly, $R(x,y_1...y_n\tau)$ if $\exists\gamma R_\gamma(x,y_1...y_n\tau)$.

Observe that $L_\alpha(\text{range}(f))$ is a possible argument, since it is obtained by applying a Δ_0 operation to $<\alpha,\text{range}(f)>$. The definition also shows that the order in which the parameters appear is immaterial (use the appropriate Gödel operations).

LEMMA 3.2.1. $\{<x,y_1,...y_n,\tau> \mid L(\text{range}(f)) \models R_\alpha(x,y_1...y_n\tau) \wedge x \in \text{range}(f) \wedge \bigwedge (y_i \in Y_i) \wedge \tau \in T\}$ is a set in $L(\text{range}(f))$ for $Y_i,T \in L(\text{range}(f))$.

LEMMA 3.2.2. Existence is valid in $L(\text{range}(f))$.
PROOF By induction on the parameters occurring in the right argument of R.

LEMMA 3.2.3. Irreflexivity and Tailset hold in $L(\text{range}(f))$.
PROOF Trivial.

LEMMA 3.2.4. Downward Monotonicity holds in $L(\text{range}(f))$.
PROOF Let X,Y be sequences, Y a subsequence of X, and suppose $R(x,X)$. We can obtain Y from X and a finite set of finite ordinals, hence $R(x,Y)$.

LEMMA 3.2.5. The zero-one law is satisfied in $L(\text{range}(f))$.
PROOF By induction on the (non-ordinal) parameters occurring in the right argument of R.
Suppose $L(\text{range}(f)) \models \exists x(R(x,y\{\text{range}(f)\}\tau) \wedge \phi(x,y,\{\text{range}(f)\},\tau))$, where ϕ determines a tailset, and $y \in \text{range}(f)$.
Say $L(\text{range}(f)) \models \phi(f(2),f(1),\{\text{range}(f)\},\tau))$; a permutation argument using the zero-one law in ν then shows that for any n not equal to 1, $L(\text{range}(f)) \models \phi(f(n),f(1),\{\text{range}(f)\},\tau))$, hence also $L(\text{range}(f)) \models \forall x(R(x,y\{\text{range}(f)\}\tau) \rightarrow \phi(x,y,\{\text{range}(f)\},\tau))$.
Suppose $L(\text{range}(f)) \models \exists x(R(x,y\{\text{range}(f)\}\tau) \wedge \phi(x,y,\{\text{range}(f)\},\tau))$ for $y \in L_{\alpha+1}(\text{range}(f))$.
Choose x such that $L(\text{range}(f)) \models R(x,y\{\text{range}(f)\}\tau) \wedge \phi(x,y,\{\text{range}(f)\},\tau)$ and $z \in L_\alpha(\text{range}(f))$, F an absolute function such that $y = F(z)$ and $R(x,z\{\text{range}(f)\}\tau)$.
By the induction hypothesis, $L(\text{range}(f)) \models \forall x(R(x,z\{\text{range}(f)\}\tau) \rightarrow \phi(x,F(z),\{\text{range}(f)\},\tau))$.
Now choose *any* x with $R(x,y\{\text{range}(f)\}\tau)$, then there are $v \in L_\alpha(\text{range}(f))$ and an absolute function G such that $y = G(v)$ and $R(x,v\{\text{range}(f)\}\tau)$.
By lemma 3.2.4., $L(\text{range}(f)) \models \forall x'(R(x',zv\{\text{range}(f)\}\tau) \rightarrow \phi(x',F(z),\{\text{range}(f)\},\tau))$, hence also $L(\text{range}(f)) \models \forall x'(R(x',zv\{\text{range}(f)\}\tau) \rightarrow \phi(x',G(v),\{\text{range}(f)\},\tau))$. The induction hypothesis says that the zero-one law holds for parameters of level α, so we may use lemma 1.3.2.(ii), which is a consequence of the zero-one law.
Hence $L(\text{range}(f)) \models \forall x'(R(x',v\{\text{range}(f)\}\tau) \rightarrow \phi(x',G(v),\{\text{range}(f)\},\tau))$ and by our choice of v,G:

$$L(\text{range}(f)) \models \phi(x,y,\{\text{range}(f)\},\tau).$$

LEMMA 3.2.6. Steinitz exchange is satisfies in $L(\text{range}(f))$.
PROOF Suppose $L(\text{range}(f)) \models \exists x\exists y(R(x,X\cup\{y\}) \wedge R(y,X) \wedge \neg R(y,X\cup\{x\}))$. The zero-one law implies that $L(\text{range}(f)) \models \forall x\forall y(R(x,X\cup\{y\}) \wedge R(y,X) \rightarrow \neg R(y,X\cup\{x\}))$. An easy induction on X shows that this is impossible.

4. CONCLUDING REMARKS

We have seen that the independence axioms lead to inner models which very much resemble well-known forcing models: an extension of L with ω_1 generic reals in the case of ZFR^0, and a symmetric submodel of a generic extension in the case of ZFR.

Technically, this can be viewed as abstracting a certain algebraic structure, here baptized an *independence structure*, from forcing extensions. This algebraic structure has features in common both with indiscernibles and with minimal sets. It is interesting that the move from a generic extension to a symmetric submodel now corresponds to allowing a more inclusive class of parameters. Of course, all kinds of intermediate cases are possible: say, allowing sequences of reals of length κ (this would correspond to generically adding κ^+ reals), or allowing all well-ordered sequences of reals, ...

We now have to face the question whether these axioms are plausible. To put this question in other words: do these axioms further explicate our intuitive concept of set?

Gödel would answer decidedly negative. After the failure of his 'square' axioms for the continuum he wrote to Tarski:

> My confidence that $2^{\aleph_0} = \aleph_2$ has of course somewhat been shaken. But it still seems plausible to me. One reason is that I don't believe in any kind of irrationality such as, e.g. random sequences in any absolute sense. (Gödel [1990], 175)

Now what was attempted above is precisely a theory of 'random sequences in an absolute sense'. The situation is similar to the theory of large cardinals: Cantor's Absolute cannot be comprehended mathematically, but it can be approximated by introducing larger and larger cardinals. Analogously, it seems hard to make mathematical sense of absolutely random sequences, but we may view them as determining an ideal which can be approximated by introducing sequences with stronger independence properties. In this spirit, three theories were introduced above, which, by enlarging the class of parameters allowed in the independence relation, describe sequences which are better approximations to the unattainable ideal. Actually, we believe that behind all these extensions there is an informal reflection principle (just as in the case of large cardinals, cf. Reinhardt [1974]), but so far we have been unable to find a satisfactory formulation. (But there is at least a formal parallel with the theory of large cardinals, since independence structures also give rise interesting ultrafilters.)

A parenthetical remark: the quotation from Gödel poses the intriguing question, how to conclude $2^{\aleph_0} = \aleph_2$ from the non-existence of absolutely random sequences. Perhaps theorem 1.5.2. provides a partial solution to this question, since it shows that from the non-existence of random sequences satisfying ZFR one can prove that the cardinal of the reals is an aleph. One might speculate that the non-existence of a stronger notion of random sequence will give more detailed information about the size of the continuum.

What do these results presented in the previous sections tell us about the axiom of choice? The *communis opinio* concerning AC seems to be that it is unproblematic once one does not demand that the choice function whose existence is postulated, is definable. In a way this is somewhat strange; after all, one proves the consistency of AC precisely by considering the definable version. By contrast, allowing a modicum of arbitrary choice seems to make the kind of choices demanded by the axiom of choice more difficult: in ZFR^0 there cannot be choice functions definable from reals and ordinals, in ZFR there can be no choice functions at all. It may be objected that the latter argument against AC is not very convincing because it really shows that AC is false in a suitable model where all sets are hereditarily ordinal definable from random reals. But suppose we continue this line of thought and argue that AC may very well be true if we also allow arbitrary choice at higher levels, not just for the sole purpose of choosing subsets of ω. Let g be a choice function on uncountable sets of reals and suppose that R has an extension such that R(g,x) for all reals x. (A single real can contain only negligible information about a function on uncountable sets of reals.) Now take a real such that R(x); if Steinitz holds for R, we have R(x,g) and $\{y \mid R(y,g)\}$ is uncountable, which is all that is needed to make the argument against AC work.

So we see that in many ways the relation between different levels in the cumulative hierarchy is of importance if we want to decide statements like AC or CH. We already met two instances of this: one cannot in general simultaneously maximize two levels of the hierarchy, and the separation axiom can also be read as a restriction upon the parameters that may occur in the separating formula. Now we encounter a third instance: what can we say about the relation of informational (in)dependence between levels? If we allow that a choice function g on uncountable sets of reals is independent of a real x (either R(x,g) or R(g,x)), we are lost: AC fails.

One might retort that, since x is in the transitive closure of g, how could x be independent of g? Doesn't this go counter to the idea that sets are completely determined by their elements?

However, R was introduced as a relation of *informational* (as opposed to ontological) independence: R(x,g) means that we cannot extract a significant piece of information concerning x from g, using only lawlike objects. Obviously, if all sets are lawlike, there will not be an x such that R(x,g); but if there also are 'arbitrary' sets, this possibility seems to be open. Note that, if one accepts the presence of parameters in the zero-one law, the question of the existence of a real x such that R(x,g) boils down to deciding the truth of $\exists x R(x)$. Even if we disallow parameters (i.e. work in the theory ZFR^{-1}), there is no definable choice function on the reals, so one needs only very weak assumptions to falsify the axiom of constructibility.

We leave the matter here, and only draw two tentative conclusions: first, if the analysis of the concept of independence given in this article is accepted, then it seems that the axiom of choice is concerned rather with intricate patterns of *dependence*, and, second, deciding statements which are at present undecidable apparently requires axioms which determine the relationship between consecutive levels in the hierarchy.

We close with some remarks on possible extensions. First, this analysis of sets generated by arbitrary choices does not seem to be restricted to subsets of the integers. To formulate the analogue of the zero-one law for sets of higher types, one would have to find a suitable group of transformations to replace the group of rational translations on the reals used in axioms 5 and 6. Second, one could perhaps allow classes (or classes of classes, ...) to occur as arguments in R, so that expressions R(x,ORD) or R(x,*V*) would be well-formed.

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