Finitely Generated Magari Algebras and Arithmetic

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August 29, 1994

Abstract

Some consequences are studied of Shavrukov's theorem regarding the Magari algebras (diagonalizable algebras) that are embeddable in the Magari algebra of formal arithmetical theories. Semantic characterizations of faithfully interpretable modal propositional theories in a finite number of propositional letters are given, in particular for finitely axiomatizable ones. Supported by this theory computer aided calculations on the theories of lowest complexity in one propositional letter were executed leading to a complete list of 62 formulas that axiomatize such theories under which the 8 maximal ones of particular interest.

1 Introduction

This paper discusses Magari algebras (often called diagonalizable algebras) over a finite number of generators. Magari algebras are the algebras corresponding to the provability logic L (GL in [2], PRL in [11]). According to Solovay's theorem [12] on provability interpretations the theorems of the provability logic L are precisely those modal formulas that are provable in PA under arbitrary arithmetical interpretations (interpreting $\Box$ as the formalized provability predicate in PA). Here, we are concerned with finitely generated Magari algebras that are embeddable in the Magari algebra of Peano Arithmetic. Shavrukov [10] characterized these subalgebras as having the so-called strong disjunction property.

*Research supported by the Netherlands Organization for Scientific Research (NWO)

In this paper the terminology of propositional theories (i.e. sets of propositional modal formulas closed under Modus ponens and Necessitation) is more convenient. Rephrased in that terminology, we study those propositional theories T over L in a finite number of propositional variables that are (faithfully) interpretable in PA. Theories correspond to τ-filters in the free Magari algebras and interpretability to embeddability as a subalgebra. Interpretable theories T in $p_1, \cdots, p_n$ are those propositional theories in $p_1, \cdots, p_n$ for which there is a sequence of arithmetical sentences $A_1, \cdots, A_n$ such that an L formula ψ is an L consequence of T iff ψ^* is a theorem of PA in the arithmetical interpretation $*$ in which the atomic formula p_i is interpreted as A_i (see e.g. [12],[2] or [11]). Written out: T axiomatizes an arithmetically interpreted theory:

$$\{\psi \mid T \vdash_L \psi\} = \{\psi \mid \vdash_{PA} \psi^*(A_1, \cdots, A_n)\}.$$

The faithfully interpretable propositional theories T in L^n (i.e., L restricted to the language of $p_1, \cdots, p_n$) are according to Shavrukov the consistent recursively enumerable (r.e.) theories that satisfy *the strong disjunction property*: $T \vdash_L \Box\psi \vee \Box\chi$ implies $T \vdash_L \psi$ or $T \vdash_L \chi$. (Parenthetically: interpretable theories in infinitely many propositional variables need not be r.e.) The strong disjunction property may be thought of as being composed out of the simple disjunction property: $T \vdash_L \Box\psi \vee \Box\chi$ implies $T \vdash_L \Box\psi$ or $T \vdash_L \Box\chi$, and *ω-consistency*: $T \vdash_L \Box\psi$ implies $T \vdash_L \psi$.

An older concept to which this can be related is the concept of *exact provability* introduced in [7] (see also [3]): in the terminology used here a formula can be defined to be exactly provable if it axiomatizes an interpretable theory. That means that an exactly provable (or *exact*) formula of L is a formula ϕ which axiomatizes an arithmetically interpreted propositional theory:

$$\{\psi \mid \phi \vdash_L \psi\} = \{\psi \mid \vdash_{PA} \psi^*(A_1, \cdots, A_n)\}.$$

The object of our research is to to get an overview of exact formulas of low complexity aided by computerized calculations. For that purpose different semantic characterizations of interpretable theories and exact formulas in terms of Kripke-models have been developed which are of interest in their own right. It turns out that an important role is played by *maximal exact formulas*, i.e. exact formulas that are not implied by any other exact formulas, and, more in general, by *maximal theories with the strong disjunction property*. The characterizations of these concepts discussed in this paper make heavy use of the relationship between exactly provable formulas in provability logic and sets of finite *types* of modal formulas (see [10], first introduced as *characters* in [4]). The paper is built up as follows. After a preliminary section 2, characterizations of interpretable theories and exact formulas are given in section 3. Then maximal theories with the strong disjunction property and their relationship to maximal interpretable theories, in general, are discussed in section 4, and maximal exact formulas, in particular, in section 5. In the last section 6, it is shown how the theory was applied to calculate the 62 exact formulas in one propositional variable of modal complexity 1, and the 8 maximal ones among them.

We owe considerable thanks to V. Shavrukov for advice and for heeding us from some mistakes.

2 Preliminaries

The *provability logic* L is the modal propositional logic with as its axioms the ones of classical propositional logic as well as all formulas of the forms $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ and $\Box(\Box\phi \to \phi) \to \Box\phi$, and the inference rules Modus ponens and Necessitation. As usual $\Diamond\psi$ is defined as $\neg\Box\neg\psi$, and we will use the abreviation $\boxdot\,\phi$ for the formula $\phi \wedge \Box\phi$. Note that $\phi \vdash_L \psi$ will mean that ψ is derivable from ϕ using the axioms and rules of L including Necessitation (but not substitution). This means that $\phi \vdash_L \psi$ is equivalent to $\vdash_L \boxdot\,\phi \to \psi$. We say "*$\phi$ is interderivable with ψ*" and write $\phi \equiv \psi$ for the conjunction of $\phi \vdash_L \psi$ and $\psi \vdash_L \phi$. Note that this implies that always $\phi \equiv \boxdot\,\phi$. We reserve the terminology *"ϕ is equivalent to ψ"* for $\vdash_L \phi \leftrightarrow \psi$. From this point on we will usually abbreviate $\phi \vdash_L \psi$ to $\phi \vdash \psi$.

By its completeness theorem, L is the logic of all finite, transitive and irreflexive Kripke-models (as proved in [2] and [11]). In the sequel we will frequently use facts about the semantics of modal logic that can be found, for example, in [1]. The Kripke-models are supposed to be finite, transitive and irreflexive, unless stated otherwise. We consider our Kripke-models to be triples $\langle W, <, \Vdash \rangle$ with W its set of *worlds* or *nodes*, $<$ its *accessibility relation* and $\Vdash$ its *forcing relation.* We will sometimes use the notations: $\uparrow k = \{l \mid l > k\}$ and $\underline{\uparrow} k = \{l \mid l \geq k\}$.

Definition 1 *The level of* box nesting *of an L formula is denoted by the inductively defined function $\beta(\phi)$:*

p *atom*: $\beta(p) = 0$;

$\phi = \psi \circ \chi$**:** $\beta(\phi) = max\{\beta(\psi), \beta(\chi)\}$ *if* $\circ \in \{\wedge, \vee, \rightarrow\}$*;*

$\phi = \neg\psi$**:** $\beta(\phi) = \beta(\psi)$*;*

$\phi = \Box\psi$**:** $\beta(\phi) = \beta(\psi) + 1$.

The fragment L^n_m will be the fragment with $P^n = \{p_1, \cdots, p_n\}$ as its set of atoms and the nesting of the box operator restricted by the condition $\beta(\phi) \leq m$.

Fact 2 *The Lindenbaum algebra of* L^n_m *is a finite Boolean algebra.*

In case of a Lindenbaum algebra we will call the atoms of the Boolean algebra *irreducible* elements. Note that ϕ is irreducible in L^n_m iff ϕ is not a contradiction and, for all $\psi, \chi \in L^n_m$, $\vdash \phi \rightarrow \psi \vee \chi$ implies $\vdash \phi \rightarrow \psi$ or $\vdash \phi \rightarrow \chi$. Such a Lindenbaum algebra is, of course, always a *Magari algebra* which is defined to be a Boolean algebra with an additional operator $\Box$ that satisfies the laws $\Box(\overline{\Box\alpha} \cup \alpha) = \Box\alpha$, $\Box(\alpha \cap \beta) = \Box\alpha \cap \Box\beta$ and $\Box 1 = 1$.

As noted in [10], the irreducible elements in L^n_{m+1} are precisely the satisfiable formulas of the form:

$$a \;=\; b \wedge \bigwedge \Diamond\gamma_i \wedge \Box \bigvee \gamma_i$$

where b is an irreducible element of the classical fragment CpL^n and the γ_i range over some set of irreducible elements of L^n_m.

Facts 3 *Writing* A^n_m *for the set of irreducible (equivalence classes of) formulas in* L^n_m *we have in every* L^n *Kripke-model* K *for every node* k*:*

1. k *forces exactly one of the elements of* A^n_m.

2. *If* $\alpha \in A^n_m$ *is forced by* k*,*
   then, for every L^n_m *formula* ϕ*:*$k \Vdash \phi \;\;\Leftrightarrow\;\; \vdash \alpha \rightarrow \phi$.

Definition 4 *The* $\alpha \in A^n_m$ *such that* $k \Vdash \alpha$ *will be denoted by* $\phi^n_m(k)$.

Fact 5 *In any* L^n *Kripke-model* K, $\phi^n_{m+1}(k)$ *depends for any* $k \in K$ *only on the set of atoms forced at* k *and the* $\phi^n_m(k')$ *of the successors* k' *of* k *in* K.

Definition 6 *Let* k *be a node in an* L^n *Kripke-model* K*. Then the* n, m-type *of* k, $t^n_m(k)$, *is defined by:*

- $t_0^n(k) = \langle \{ p \in P^n \mid k \Vdash p \}, \emptyset \rangle$;
- $t_{m+1}^n(k) = \langle \{ p \in P^n \mid k \Vdash p \}, \{ t_m^n(l) \mid l \in K \ \& \ k < l \} \rangle$.

The set of all such n, m-types is written T_m^n.

Although we prefer this semantic definition of n, m-type in the context of this paper, we agree that the distinction between the semantic n, m-types and the $\phi_m^n(k)$ (essentially what is called the *n, m-type of k* in [10]) is just a matter of point of view.

If $t = \langle Q, X \rangle$, we write $j_0(t)$ for Q and $j_1(t)$ for X. The following facts are easily verified and reflect the close relationship between the $t_m^n(k)$ and $\phi_m^n(k)$.

Facts 7 *Let K be an L^n Kripke-model and k, l be elements of K.*

1. *k is of exactly one of the types in T_m^n.*
2. *If $t \in T_m^n$ is the type of both k and l, then for every L_m^n formula ϕ: $k \Vdash \phi \quad \Leftrightarrow \quad l \Vdash \phi$.*
3. *The $n, m{+}1$-type of k depends only on the set of atoms forced at k and the n, m-types of the successors of k.*

Facts 3, 5 and 7 combine into:

Fact 8 *The set of n, m-types corresponds exactly with the set A_m^n of irreducible elements of L_m^n in the sense that:*

$$\forall l \in K(\, l \Vdash \phi_m^n(k) \quad \Leftrightarrow \quad t_m^n(l) = t_m^n(k))$$

Another useful, easily verifiable fact is the following:

Fact 9 *The $n, m{+}1$-type t of $k \in K$ uniquely determines the n, m-type of $k \in K$. We write $t {\restriction} m$ for this n, m-type.*

Lemma 8 immediately leads to:

Corollary 10 *If K is a Kripke model such that each n, m-type occurs exactly once in K, then the subsets of K correspond exactly with the equivalence classes of L_m^n. That is, the following function Ω is an isomorphism:*

$$\Omega(\phi) = \{ k \in K \mid k \Vdash \phi \}.$$

It can be proved that such an *exact model*, in which each subset corresponds to a formula and vice versa, exists for each L^n_m.

Theorem 11 *For each n and m there exists an exact Kripke-model K for L^n_m, i.e., for each $U \subseteq K$, there is a formula ϕ in L^n_m such that $\{k \in K \mid k \Vdash \phi\} = U$, and K is n,m-complete, in the sense that for all $\phi, \psi \in L^n_m$, $\{k \in K \mid k \Vdash \phi\} = \{k \in K \mid k \Vdash \psi\}$ iff $\vdash \phi \leftrightarrow \psi$.*

Proof. We apply the so-called Henkin method to the (up to equivalence) finite set of formulas in L^n_m, which is closed under taking subformulas. This gives one a Kripke-model with the maximal consistent sets as its worlds, with $\Gamma < \Delta$ defined by: for each $\Box\gamma \in \Gamma$, both $\Box\gamma$ and γ are elements of Δ, and $\Gamma \Vdash p_i$ by: $p_i \in \Gamma$. The maximal consistent sets can be replaced by their conjunctions which are exactly the irreducible elements of L^n_m. A subset of the model corresponds then to a disjunction of irreducibles, i.e. an arbitrary formula of L^n_m. Obviously, non-equivalent formulas are forced on different subsets of the model. $\dashv$

In general, unlike in the case of intuitionistic propositional logic (see [8], [6]), not all the exact models of L^n_m are isomorphic. For the infinite fragment L^n there is no such exact model in which all subsets determine a formula, but there is a canonical (infinite) model which is n-complete. Here we baptise this canonical n-complete model ExL^n. This model gives considerable insight into the free Magari algebra over n generators. Its construction (which was first given in [5] and [9]) is given presently (somewhat sloppily but, we hope, clearly) after the introduction of the notion of *depth* of a node.

Definition 12 *Let K be a Kripke-model and $k \in K$. The* depth *of k, $\delta(k)$, is defined by:*

- $\delta(k) = 0$ *if k is a terminal node;*
- $\delta(k) = max\{\delta(l) \mid k < l\} + 1$ *otherwise.*

Definition 13 *ExL^n with its $<$ and $\Vdash$ is defined as the union of inductively defined ExL^n_m for $m \in \omega$.*

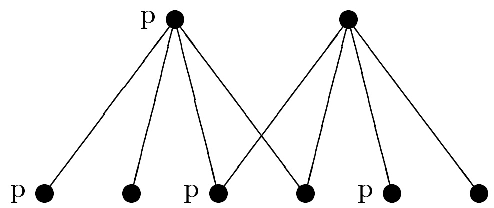
- *$ExL^n_0 = \wp(\{p_1, \cdots, p_n\})$, the elements of ExL^n_0 are all $<$-incomparable, and $Q \Vdash p \;\Leftrightarrow\; p \in Q$;*

- $ExL^n_{m+1} = \{\langle Q, X\rangle \mid Q \subseteq \{p_1, \cdots, p_n\}$,
$X \subseteq \bigcup_{i\leq m} ExL^n_i, X \cap ExL^n_m \neq \emptyset$, X closed upwards$\}$,
$\langle Q, X\rangle < Y \Leftrightarrow Y \in X$, *and* $\langle Q, X\rangle \Vdash p \Leftrightarrow p \in Q$;

- $ExL^n = \bigcup_{i\in\omega} ExL^n_i$.

The above definition is such that ExL^n_i will contain the elements of ExL^n of depth exactly i. This immediately brings along with it that ExL^n is conversely well-founded. It is obvious from the construction that each n, m-type will be realized by some $k \in ExL^n$. The latter ensures the n-completeness of ExL^n. It is convenient to us to execute most of our constructions inside this model. For many of these constructions this would not have been necessary.

Example 14 *The construction of ExL^1_0 and ExL^1_1 yields:*



Let us write $\Box^0\phi = \phi$ and $\Box^{n+1}\phi = \Box\Box^n\phi$. The following facts about the nodes of ExL^n will be useful in the sequel.

Facts 15

1. $\delta(k) = m \quad\Leftrightarrow\quad k \Vdash \neg\Box^m\bot \wedge \Box^{m+1}\bot$;

2. *If* $\delta(k), \delta(l) \leq m$, $m > 0$ *and* $t^n_m(k) = t^n_m(l)$, *then* $k = l$;

3. *If* $\delta(k) = m$ *and* $l \Vdash \phi^n_m(k) \wedge \neg\Box^m\bot \wedge \Box^{m+1}\bot$, *then* $k = l$.

These facts suggest a kind of normal form for the irreducible formulas corresponding to the elements of ExL^n.

Definition 16 *Let* $k \in ExL^n$ *and assume* $\delta(k) = m$.
Then $\phi^n(k) = \phi^n_m(k) \wedge \neg\Box^m\bot \wedge \Box^{m+1}\bot$.

From the n-completeness of ExL^n we conclude that for $k \in ExL^n$ the $\phi^n(k)$ are the irreducible elements in L^n. However, from the counter-examples in [4] we know there is no nice bound for the m one has to use when one wants to write a formula ϕ as a disjunction of irreducible formulas of L^n.

3 Interpretable theories and exact formulas in L^n

As stated in the introduction, Shavrukov's theorem in [10] gives a characterization of the exactly provable formulas in L (or *exact formulas* for short).

Fact 17 *A formula $\phi \in L$ is exact iff ϕ is not a contradiction and has the* strong disjunction property (*is* s.d.):

$$\forall \psi, \chi \in L\,(\phi \vdash \Box\psi \vee \Box\chi \quad \Rightarrow \quad \phi \vdash \psi \quad or \quad \phi \vdash \chi)$$

The property in this fact is called *steady* by Shavrukov [10]. Whether a formula $\phi \in L^n_m$ is steady or not, Shavrukov [10] also proved, depends only on its behavior with regard to other formulas in L^n_m:

Fact 18 *A formula $\phi \in L^n_m$ is exact iff ϕ is not a contradiction and is s.d. for formulas in L^n_m:*

$$\forall \psi, \chi \in L^n_m(\phi \vdash \Box\psi \vee \Box\chi \quad \Rightarrow \quad \phi \vdash \psi \quad or \quad \phi \vdash \chi)$$

For a simple proof of this last fact see [15]. We will transform this characterization of exactness into a semantic one. That will turn out not to work for interpretable theories in general. The characterization will apply however to the maximal interpretable theories of the next section.

Definition 19 $\omega^n(\phi) = \{k \in ExL^n \mid k \Vdash \boxdot\, \phi\}$.
If T is a propositional theory in L^n, then $\omega^n(T) = \{k \in ExL^n \mid k \Vdash T\}$.

Obviously $\omega^n(\phi)$ and $\omega^n(T)$ will always be *closed upwards* in the sense that, if e.g. $k \in \omega^n(\phi)$ and $k < l$, then $l \in \omega^n(\phi)$.

Theorem 20 *A formula $\phi \in L^n$ is exact iff $\omega^n(\phi)$ is non-empty and* extendible downwards, *i.e.* $\forall k, l \in \omega^n(\phi)\, \exists h \in \omega^n(\phi)\, (\, h < k \,\&\, h < l\,)$.

Proof. $\Rightarrow$: Let ϕ be an exact formula in L^n. As ϕ unequals the contradiction by definition, we have $\omega^n(\phi) \neq \emptyset$ by the completeness of ExL^n. To prove the second condition, let $k, l \in \omega^n(\phi)$, and $\phi^n(k), \phi^n(l)$ be (representatives of) the irreducible classes in L^n corresponding to k and l. Assume that, if $h \in \omega^n(\phi)$ and $h < k$, then $h \not< l$. Then, again by the completeness of ExL^n, we would have $\phi \vdash \Diamond\phi^n(k) \to \Box\neg\phi^n(l)$, or equivalently $\phi \vdash \Box\neg\phi^n(k) \vee \Box\neg\phi^n(l)$. As ϕ is supposed to be exact, ϕ would either prove $\neg\phi^n(k)$ or $\neg\phi^n(l)$, in contradiction with the assumption that $k, l \in \omega^n(\phi)$. Hence, there should be an $h \in \omega^n(\phi)$ such that $h < k$ and $h < l$.
$\Leftarrow$: Let $\psi, \chi \in L^n$ and $\phi \vdash \Box\psi \vee \Box\chi$, and assume there are $k, l \in \omega^n(\phi)$ such that $k \not\Vdash \psi$ and $l \not\Vdash \chi$. By the last condition of the theorem, there is an $h \in \omega^n(\phi)$ such that $h < k$ and $h < l$. As we would then have $h \not\Vdash \Box\phi \vee \Box\psi$, we obtain a contradiction. Hence, we proved that $\phi \vdash \psi$ or $\phi \vdash \chi$. $\dashv$

By the completeness of L non-interderivable ϕ and ψ give rise to distinct $\omega^n(\phi)$ and $\omega^n(\psi)$. This is in general not so for theories. An example is the theory axiomatized by p on the one hand, and the theory T_1 axiomatized by $\Box^m\bot \to p$ for each m, on the other. The sets $\omega^1(p)$ and $\omega^1(T_1)$ are the same, consisting of all nodes that together with all their successors force p, but clearly the theories are not: p is not a consequence of T_1. Similarly, the theory $T_2 = \Box^m\bot \to \Box p \vee \Box\neg p$ for each m can be shown to have the strong disjunction property. But not all pairs of nodes in $\omega^1(T_2)$ have a common predecessor in ExL^1, because $\omega^1(T_2)$ consists of those nodes that together with all their successors force p and those nodes that together with all their successors don't force p. This shows that the semantic characterization of exactness does not generalize to interpretability of non-finitely axiomatizable theories, at least if one doesn't freely use infinite models. There is a restricted class of theories that does respect the characterization.

Definition 21 *T has the* finite model property (f.m.p.) *iff $T \nvdash \phi$ implies that there is a finite Kripke-model for T on which ϕ is falsified.*

Fact 22 *T has the* finite model property (f.m.p.) *iff*
$\{\phi \mid \textit{for each } k \in \omega^n(T),\ k \Vdash \phi\} = T.$

If T has the f.m.p., then T is determined uniquely by $\omega^n(T)$. Also the semantic characterization of exactness immediately generalizes by the same proof.

Theorem 23 *A theory T in L^n with the f.m.p. is interpretable iff $\omega^n(T)$ is non-empty and extendible downwards.*

Later we will encounter the finite model property in a syntactic form.

Lemma 24 *An L theory T has the f.m.p. iff, for each ϕ,*
$T \vdash \phi \quad \Leftrightarrow \quad$ *for each* $n \in \omega,\ T \vdash \Box^n \bot \to \phi$.

Proof. $\Rightarrow$: Assume T has the f.m.p. First let $T \nvdash \phi$. By its f.m.p. T has a finite model with a node k falsifying ϕ. Supposing k has depth m ensures that $T \nvdash \Box^{m+1}\bot \to \phi$. The other direction is trivial.
$\Leftarrow$: If $T \nvdash \phi$, then the right hand side implies that $T \nvdash \Box^n \bot \to \phi$ for some n, i.e., there is a model for T of depth $n-1$ falsifying ϕ, a finite model. $\dashv$

One sees from the proof that actually the negative formulation of the right hand side of the above lemma is the natural one.

Note that the $\omega^n(\phi)$ of an exact $\phi \in L^n$ is infinite by the conditions of the characterization. On the other hand there is a simple correspondence between such an infinite set and a finite set of n,m-types in ExL^n:

Definition 25 *Let ϕ be an L^n formula. Then* $T^n_m(\phi) = \{t^n_m(k) \mid k \Vdash \boxdot\, \phi\}$.

By the results in the previous section this is a sound definition and, in consequence, in particular every exact L^n_m formula corresponds uniquely to a finite set of n,m-types. Grantedly, this definition is somewhat unnatural by the insertion of the $\boxdot$. The reason for this is, of course, that we are interested in the types of the nodes in $\omega^n(\phi) = \{k \in ExL^n \mid k \Vdash \boxdot\ \phi\}$.

Lemma 26 *Let ϕ and ψ be L^n_m formulas.*
Then $T^n_m(\phi) = T^n_m(\psi)$ implies $\phi \equiv \psi$.

Proof. Let $T^n_m(\phi) = T^n_m(\psi)$, and assume $k \Vdash \boxdot\ \phi$. Then $t^n_m(k) \in T^n_m(\phi) = T^n_m(\psi)$. Hence $t^n_m(k) = t^n_m(k')$ for some k' that forces $\boxdot\,\psi$. So, $k' \Vdash \psi$ and, since $\psi \in L^n_m$, $k \Vdash \psi$. The other direction is the same. $\dashv$

Lemma 27 *Let ϕ be an L^n_m formula.*
Then $T^n_m(\phi) = \{t{\restriction}m \mid t \in T^n_{m+1}(\phi)\}$.

Proof. Obvious, considering fact 9. $\dashv$

Lemma 28 *For each L^n formula ϕ and each m there is a finite upwardly closed subset K of $\omega^n(\phi)$ such that the elements of K exactly realize $T^n_m(\phi)$.*

Proof. Just take any finite subset of $\omega^n(\phi)$ such that its elements exactly realize $T^n_m(\phi)$. The upward closure of this set will do, because its elements also force $\boxdot\,\phi$. $\dashv$

To find the sets of n,m-types suitable for exact formulas ϕ, we have to translate the conditions on the $\omega^n(\phi)$ of exact ϕ into conditions on the underlying set of n,m-types. For example, for a finite $T^n_m(\phi)$ to correspond to an infinite $\omega^n(\phi)$, it is necessary that some type in $T^n_m(\phi)$ can have itself as a successor. To describe this kind of reflexivity we introduce the notion of a *reflexive type.*

Definition 29 *A term $t \in T^n_m$ is called* reflexive *if $t{\restriction}m \in j_1(t)$.*

The following theorem is related to lemma 5.13 of [10].

Theorem 30 *A formula $\phi \in L^n_m$ with $m > 0$ is exact iff there is a type $t \in T^n_m(\phi)$ such that $j_1(t) = T^n_{m-1}(\phi)$, which, of course, makes t a reflexive type.*

Proof. $\Rightarrow$: Let $\phi \in L^n_m$ be an exact formula. Note that $T^n_{m-1}(\phi)$ is a finite set of types. Let $K \subset \omega^n(\phi)$ be finite and closed upwards such that $\{t^n_m(k) \mid k \in K\} = T^n_m(\phi)$, as guaranteed to exist by lemma 28. According to theorem 20 we can find an $h \in \omega^n(\phi)$ below all of the elements of K. By lemma 27 this h must have a type as required.
$\Leftarrow$: Assume ϕ and t to fulfill the conditions given. As $T^n_m(\phi) \neq \emptyset$, also $\omega^n(\phi) \neq \emptyset$. Suppose $k, l \in \omega^n(\phi)$. Let K be a finite upwardly closed subset of $\omega^n(\phi)$ such that $k, l \in K$ and $\{t^n_{m-1}(k') \mid k' \in K\} = T^n_{m-1}(\phi)$ (compare lemma 28). Consider a world h just below this K such that $\{p \in P^n \mid h \Vdash p\} = j_0(t)$. It will be clear that $t^n_m(h) = t$ and (since ϕ is assumed to be an L^n_m formula) this proves $h \in \omega^n(\phi)$. Of course, $h < k$ and $h < l$, so the conditions of theorem 20 apply to $\omega^n(\phi)$. $\dashv$

The theory developed in this article has enabled us to calculate the exact formulas in L^1_1. This will be explained in more detail in the last section. It will be shown that already in this very first small fragment there are 62 non-interderivable members. It turned out that it was worthwile to single out the 8 maximal elements of these 62. More general than maximal exactness the concept of maximal s.d. theory turns out to be in itself an interesting one. The next section will be devoted to this last concept and the one after it to maximal exact formulas.

4 Maximal theories with the strong disjunction property and maximal interpretable theories

In this section it turns out to be fruitful to not restrict one's attention to r.e. s.d. theories, but also include non-r.e. ones, i.e. non-interpretable theories.

Definition 31

- *A theory T is* maximal s.d. *if T is consistent and s.d. and, for no consistent U with $T \subset U$, U is s.d.;*
- *An L^n formula ϕ is* maximal exact *if ϕ is exact and, for each exact L^n formula ψ such that $\psi \vdash \phi$, also $\phi \vdash \psi$;*
- *A theory T is* ω-consistent *if $T \vdash \Box\phi$ implies $T \vdash \phi$;*
  A theory T is maximal ω-consistent *if T is ω-consistent and, for no U with $T \subset U$, U is ω-consistent;*
- *A theory T is* of infinite height (i.h.) *if, for no $n \in \omega$, $T \vdash \Box^n \bot$;*
  A theory T is maximal i.h. *if T is i.h. and, for no U with $T \subset U$, U is i.h.*
- *A theory T is* maximal interpretable *if T is interpretable and, for no U with $T \subset U$, U is interpretable.*

Note that the concept of a maximal interpretable theory is a very natural one. Such a theory, and in particular also a maximal exact formula, describes a maximal condition that arithmetical sentences can be required to satisfy.

Lemma 32 *Each i.h. theory is contained in a maximal i.h. theory.*

Proof. By Zorn's lemma. ⊣

Lemma 33 *T is maximal i.h. iff, for each ϕ, either $T \vdash \phi$, or, for some $n \in \omega$, $T \cup \{\phi\} \vdash \Box^n \bot$.*

Proof. $\Rightarrow$: Assume ϕ is such that not $T \vdash \phi$, and, for no $n \in \omega$, $T \cup \{\phi\} \vdash \Box^n \bot$. Then T can be properly extended to an i.h. theory containing ϕ which means that T is not maximal i.h.
$\Leftarrow$: Assume T is not maximal i.h., i.e., for some i.h. U, $T \subset U$. Let ϕ be an element of U, but not of T. Then not $T \vdash \phi$, but not $T \cup \{\phi\} \vdash \Box^n \bot$ for any n either, because for no n, $U \vdash \Box^n \bot$. ⊣

Lemma 34 *If T is s.d., then T is ω-consistent. If T is ω-consistent, then T is i.h.*

Proof. Trivial. $\dashv$

Lemma 35 *If T is a maximal i.h. theory, then T is s.d. and hence ω-consistent.*

Proof. Let T be maximal i.h. To show that T is s.d., assume $T \vdash \Box\phi \vee \Box\psi$. Now apply lemma 33 to both ϕ and ψ to obtain that, either $T \vdash \phi$ or, for some $n \in \omega$, $T \cup \{\phi\} \vdash \Box^n\bot$, and the same for ψ. It is sufficient to exclude the possibility that $T \cup \{\phi\} \vdash \Box^n\bot$ for some n as well as $T \cup \{\phi\} \vdash \Box^m\bot$ for some m. In that case it would follow that $T \vdash \Box\phi \rightarrow \Box^{n+1}\bot$ and $T \vdash \Box\psi \rightarrow \Box^{m+1}\bot$ which implies $T \vdash \Box^{max(n+1,m+1)}\bot$, contradicting the infinite height of T. $\dashv$

Lemma 36 *T is maximal i.h. iff T is maximal ω-consistent iff T is maximal s.d.*

Proof. From the previous lemmas. $\dashv$

Corollary 37

1. *Each s.d. theory in L^n is contained in a maximal s.d. theory.*
2. *If T is r.e. and maximal s.d., then T is recursive.*
3. *Not each interpretable theory in L^n is contained in a maximal interpretable theory.*

Proof. Part 1 is trivial. Part 2 follows from lemmas 36 and 33. Part 3 follows from part 2 by considering two recursively inseperable r.e. sets of natural numbers A and B and then constructing the theory

$$U = \{\Box^{n+1}\bot \wedge \neg\Box^n\bot \rightarrow p \mid n \in A\} \cup \{\Box^{n+1}\bot \wedge \neg\Box^n\bot \rightarrow \neg p \mid n \in B\}.$$

The theory U is s.d., since any two models can be joined by a new root taking care only that the new root may need a specific valuation for p, and r.e., and hence interpretable. A maximal recursive extension of U would seperate A and B and, therefore, cannot exist. $\dashv$

One would expect it to be easy to prove that each maximal interpretable theory has to be maximal s.d., but we have not been able to show this, so we leave it as an open question.

Open Question 38 *Is each maximal interpretable theory maximal s.d.?*

Lemma 39 *If T is maximal s.d., then T has the f.m.p.*

Proof. Assume $T \nvdash \phi$ for a maximal s.d. theory T. By lemma 33, for some $n \in \omega$, $T \vdash \phi \rightarrow \Box^n \bot$. Löb's theorem implies then that $T \nvdash \Box^{n+1}\bot \rightarrow \phi$. Now apply lemma 24. $\dashv$

The next theorem uses the above to provide a semantic characterization of maximal s.d. theories.

Theorem 40 *T in L^n is maximal s.d. iff T has the f.m.p. and, for each $k \in \omega^n(T)$, there is an $m \in \omega$ such that for all $l \in \omega^n(T)$ of depth $\geq m$, $l < k$.*

Proof. $\Rightarrow$: Let T in L^n be maximal s.d. That T has the f.m.p. is lemma 39, so it suffices to check the second point. For that purpose, consider $k \in \omega^n(T)$. Obviously $\neg\phi^n(k)$ is not a theorem of T. So, for some $m \in \omega$, $T \vdash \boxdot\neg\phi^n(k) \rightarrow \Box^m\bot$. Equivalently, $T \vdash \neg\Box^m\bot \rightarrow \phi^n(k) \vee \Diamond\phi^n(k)$. This implies that, for all $l \in \omega^n(T)$ of depth $\geq m{+}1$, $l < k$, since $\phi^n(k)$ uniquely describes k.
$\Leftarrow$: It is sufficient to check the condition of lemma 33 under the assumption that the right hand side of the present theorem is satisfied by T. So, assume $T \nvdash \phi$. As T has the f.m.p., there is a finite model for T that falsifies ϕ, given as $\underline{\uparrow} k$ for some $k \in ExL^n$. Since, for some $m \in \omega$, all nodes with depth $\geq m$ in $\omega^n(T)$ are below k, $T \vdash \neg\Box^m\bot \rightarrow \Diamond\neg\phi$. By contraposition, $T \vdash \boxdot\phi \rightarrow \Box^m\bot$ which was to be proved. $\dashv$

A natural question concerning L^1 is whether there are any maximal interpretable theories that are symmetric with respect to p and $\neg p$. Rosser-sentences are interpretations of maximal exact formulas (see the last section), but they are not examples of such symmetry, since $p \rightarrow \Box p$ is in the theory of the Σ form of the Rosser sentence, and $\neg p \rightarrow \Box\neg p$ in the dual theory. It is easy to see that no s.d. theory can contain both $p \rightarrow \Box p$ and

$\neg p \to \Box\neg p$ without containing p or $\neg p$ itself (use the excluded middle). In the next section it will become clear that an exact formula in p cannot be symmetric in p and $\neg p$, and maximally exact. So, an example will necessarily have an infinite axiomatization.

Example 41 *A symmetric maximal interpretable theory in p.*
Consider the subset K of ExL^1 defined as follows:
$K_0 = \{\{p\}, \emptyset\}$, $K_{m+1} = \{\langle\{p\}, \bigcup_{i\leq m} K_i\rangle, \langle\emptyset, \bigcup_{i\leq m} K_i\rangle\}$, $K = \bigcup_{m\in\omega} K_m$
The theory $T = \{\phi \mid k \Vdash \phi \ \mathit{for\ all}\ k \in K\}$ is maximal s.d. (it is easily checked that it satisfies the conditions of theorem 40) and symmetric. An axiomatization of T is given by:

$$\Diamond p \leftrightarrow \Diamond\neg p$$
$$\Diamond(p \wedge \Diamond p) \leftrightarrow \Diamond(\neg p \wedge \Diamond p)$$
$$\Diamond(p \wedge \Diamond(p \wedge \Diamond p)) \leftrightarrow \Diamond(\neg p \wedge \Diamond(p \wedge \Diamond p))$$
$$\cdots$$

This axiomatization is somewhat more perspicuous than the equivalent one formulated with the $\Box$: $\Box p \leftrightarrow \Box\neg p$, $\cdots$. A sentence interpreting such a theory in PA is a kind of symmetric Rosser-sentence which can, of course, neither be Σ^0_1, nor Π^0_1. Note, in fact, that $p \to \Box p$ as well as $\neg p \to \Box\neg p$ are strongly non-provable for such a sentence, in the sense that e.g., $\Box(p \to \Box p) \to \Box\Box\bot$ is provable for it. (Note also that $\Box(p \to \Box p) \to \Box\bot$ couldn't be provable for an arithmetic sentence.)

Proof that the axiomatization is complete for T. Assume $T \nvdash \phi$. There exists a (possibly infinite) Kripke-model $\bar{K}$ satisfying T on which ϕ is falsied. It suffices to show that each $1,m$-type realized in $\bar{K}$ can be realized in $K'_m = \bigcup_{i\leq m} K_i$. For $1,0$-types this is obvious. Assume some $1,m{+}1$-type t is realized in $\bar{K}$. All types in $j_1(t)$ are, by induction hypothesis, realized in K'_m. Assume that of the elements in K'_m realizing types in $j_1(t)$, k is in the K_i with highest index i. It will be obvious that the axioms of T imply that there has to be a type present in $j_1(t)$ that is realized by the (only) other element of K_i. This implies that t is realized by an element of K_{i+1}, which will do. $\dashv$

5 Maximal exact formulas

This section will be devoted to maximal exact formulas. First we will have to sharpen our semantic characterization of exact formulas. Let us exploit the relationship between irreducible classes and semantic types to write $\phi^n_m(t)$ for the $\phi^n_m(k)$ with $t^n_m(k) = t$.

Definition 42 *Let C be a set of n,m-types. Then $\phi^n_m(C) = \bigvee\{\phi^n_m(t) \mid t \in C\}$.*

Recall that $T^n_m(\phi) = \{t^n_m(k) \mid k \Vdash \boxdot\, \phi\}$.

Lemma 43 *If $\phi \in L^n_m$, then $\phi \equiv \phi^n_m(T^n_m(\phi))$.*

Proof. Immediate from lemma 26. $\dashv$

Lemma 44 *If $C \subseteq T^n_m$ $(m > 0)$, then $C = T^n_m(\phi)$ for an exact formula $\phi \in L^n_m$ iff*

1. *there is a finite upwardly closed $K \subseteq ExL^n$ such that $C = \{t^n_m(k) \mid k \in K\}$ (we will call C* upwardly closed realizable*);*
2. *there is $t \in C$ such that $\forall t' \in C\, (t'{\restriction}(m-1) \in j_1(t))$.*

Moreover, in that case $\phi \equiv \phi^n_m(C)$. Such a type t will be called a enveloping type for *C.*

Proof. $\Rightarrow$: If $\phi \in L^n_m$ is exact, then $T^n_m(\phi)$ will have the required property 1 by the definition of $T^n_m(\phi)$, property 2 by theorem 30, and satisfies the final requirement by lemma 43.
$\Leftarrow$: We prove that $\phi^n_m(C)$ is an exact formula. To apply theorem 30 to $\phi^n_m(C)$, we have to find an appropriate reflexive n,m-type. By the assumption on C, there is an n,m-type $t \in C$ such that $\forall t' \in C\, (t'{\restriction}(m-1) \in j_1(t))$. Let K be the upwardly closed realization of the types in C as assumed in the first conditon of this lemma. Note that K realizes precisely the $n,m-1$-types in $\{t'{\restriction}(m-1) \mid t' \in C\}$ (compare lemma 27). Let k be a (new) root immediately below K such that k forces exactly the elements of $j_0(t)$. Then $t^n_m(k) = t$. So, $k \Vdash \boxdot\, \phi^n_m(C)$ and, hence, t is a member of $T^n_m(\phi^n_m(C)) = C$ and a type appropriate for the application of theorem 30. $\dashv$

We will prove that the maximal exact classes in L^n correspond to what we will call *tail models* in ExL^n. Clearly this is a result that to a large extent is bound to the particular model ExL^n.

Definition 45 *$K \subset ExL^n$ is called a tail model iff:*

1. *K is closed upwards;*
2. *there is an $m \in \omega$ such that $\{k \in K \mid \delta(k) \geq m\}$ is linearly ordered by $<$ and all nodes of this set force the same atoms.*

Our definition of tail model slightly differs from the one in [14], in that Visser's tail models are equipped with a minimal (infinite-depth) element.

Definition 46 *If $k \in ExL^n$, then $\uparrow k \downarrow$ is the tail model consisting of $\uparrow k$ equipped with a tail descending from k with forcing of the atoms defined as on k.*

Lemma 47 *If $\phi \in L^n_m$, $k \in \omega^n(\phi)$ and k has a reflexive n, m-type, then $\uparrow k \downarrow \subseteq \omega^n(\phi)$.*

Proof. First note that all elements of the tail have the same n, m-type as k. Hence, all these nodes force ϕ, and consequently $\boxdot \phi$. $\dashv$

Lemma 48 *If $K \subset ExL^n$ is a tail model, then $K = \omega^n(\phi)$ for some ϕ in L^n.*

Proof. Let K be $\uparrow k \downarrow$, k having depth m. Let θ be the conjunction of the atoms and negated atoms as they are forced on k. Then $K = \omega^n(\phi)$ for ϕ defined as the conjunction of $\Box^{m+1}\bot \to \bigvee\{\phi^n_m(k') \mid k \leq k'\}$ and $\neg\Box^{m+1}\bot \to \theta \wedge \Diamond\phi^n(k)$. $\dashv$

Theorem 49 *If $\phi \in L^n$, then ϕ is maximal exact in L^n iff $\omega^n(\phi)$ is a tail model in ExL^n.*

proof. $\Leftarrow$: Assume $\omega^n(\phi)$ is a tail model and $\phi \in L^n_m$. Since $\omega^n(\phi)$ is infinite and $T^n_{m-1}(\phi)$ finite it is obvious that the tail has to contain elements appropriate for an application of theorem 30. This shows that ϕ has to be exact. Assume ψ to be an exact formula such that $\psi \vdash \phi$, i.e., such that $\omega^n(\psi) \subseteq \omega^n(\phi)$. Then, because $\omega^n(\psi)$ is non-empty and extendible downwards it has to contain the tail elements from a certain node onwards, and, because it it is closed upwards it has to contain all other elements of $\omega^n(\phi)$, which means that ϕ and ψ are interderivable. Hence, ϕ is maximal exact.

$\Rightarrow$: Let $\phi \in L^n_m$ be maximal exact. Assume $t \in T^n_m(\phi)$ is a reflexive type with the properties guaranteed to exist by theorem 30. Now, take as in the proof of lemma 44($\Leftarrow$) $k \in \omega^n(\phi)$ with n, m-type t such that $t^n_m(\upharpoonleft k) = T^n_m(\phi)$. By lemma 47, $\upharpoonleft k \downarrow \subseteq \omega^n(\phi)$. By lemma 48, there exists a ψ with $\omega^n(\psi) = \upharpoonleft k \downarrow \subseteq \omega^n(\phi)$. Since ϕ was asssumed to be maximal exact, this means $\phi \equiv \psi$ and the tail model constructed is $\omega^n(\phi)$. $\dashv$

From this theorem the remark we made that maximal exact formulas in p cannot be symmetric with regard to p and $\neg p$ becomes immediately obvious. The tail is always asymmetric. We follow with some more properties and problems concerning maximal exact formulas.

Theorem 50 *If a formula $\phi \in L^n_m$ is maximal exact, then there is precisely one reflexive type t in $T^n_m(\phi)$. Moreover, $T^n_{m-1}(\phi) = j_1(t)$.*

Proof. The last part follows immediately from theorem 30. Assume $\phi \in L^n_m$ with $m > 0$ is maximal exact. Assume s and s' to be two distinct n, m-types in $T^n_m(\phi)$. If k and k' in $\omega^n(\phi)$ realize s and s' respectively, then, by lemma 47, $\upharpoonleft k \downarrow$ and $\upharpoonleft k' \downarrow$ are two distinct tail models within $\omega^n(\phi)$. This contradicts the fact that $\omega^n(\phi)$ is a tail model. $\dashv$

Examples of non-maximal exact L^1_1 formulas with exactly one reflexive $1, 1$-type will be given in the table in the last section. It is unclear to us whether exact L^n_m formulas that are maximal with respect to L^n_m formulas are maximal overall.

Definition 51 *An exact L^n_m formula ϕ is called m-maximal exact iff, for all exact $\psi \in L^n_m$ such that $\psi \vdash \phi$, $\psi \equiv \phi$.*

It will turn out in the last section that the 1-maximal exact formulas in L^1 are maximal exact. The fact that, by lemma 26, L^n_m formulas are determined by their n, m-types leads to the following insight.

Fact 52 *The m-maximal exact L^n_m-formulas are the ones with a set of types C that contains exactly one reflexive n, m-type t and for which C is minimal upwardly closed realizable, in the sense that, C is upwardly closed realizable, but this is not the case for any proper subset of C containing t.*

A conjecture is that the set of n, m-types of an arbitrary exact L^n_m formula is the union of the sets of types of the m-maximal exact L^n_m formulas from which ϕ is derivable. That such a union always is the set of types of an exact formula if at least an enveloping type is present follows immediately from the next lemma.

Lemma 53 *If C is the union of sets $C_1, \cdots, C_k$ of n, m types corresponding to exact L^n_m formulas $\phi_1, \cdots, \phi_k$ with an enveloping type t common to all $C_1, \cdots, C_k$, then there exists a $\phi \in L^n_m$ such that $T^n_m(\phi) = C$.*

Proof. It suffices to note that, if $K_1, \cdots, K_k$ are upwardly closed realizations of $C_1, \cdots, C_k$, then $K_1 \cup \cdots \cup K_k$ is an upwardly closed realization of C, and then to apply lemma 44. ⊣

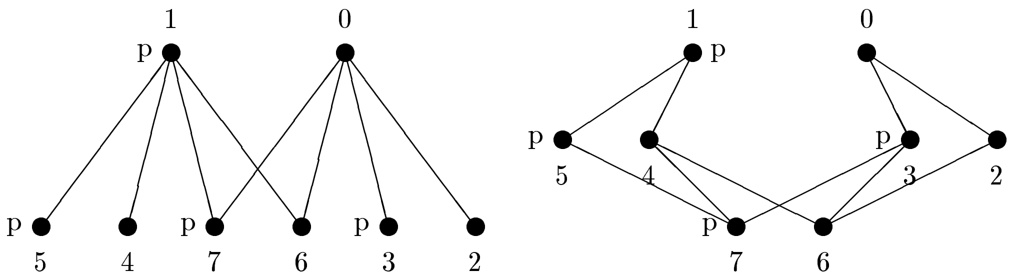
It is certainly not true that any union of types of m-maximal exact formulas is the set of n, m-types of some exact L^n_m formula. A counter-example is provided by the sets of types belonging to p and to $\neg p$, both 1-maximal exact formulas, which cannot be combined to an exact formula, even for $m = 1$. A common enveloping type is needed, and is obviously not available for p and $\neg p$ (see section 6). Another, related fact worth mentioning is that any exact formula ϕ in L^n is determined by the maximal exact formulas from which it is derivable. This can be derived from the observation that each downward chain in $\omega^n(\phi)$ will ultimately only contain nodes k with reflexive types, and that $\uparrow k \downarrow = \omega^n(\psi)$ for some maximal exact formula from which ϕ is derivable, combined with the observation that, if for L^n_m formulas ϕ and ψ, $\omega^n(\phi)$ and $\omega^n(\psi)$ are distinct, then there is a node with a reflexive n, m-type in the one, but not in the other.

Certainly this does not generalize to interpretable theories. The s.d. theory T_1 axiomatized by $\Box^n \bot \to p$ for each n that was introduced after theorem 20 provides a counter-example. Its only maximal s.d. extension is the one axiomatized by p.

6 Calculating exact formulas

In this section the calculation of the exact formulas in L^1_1 will be discussed. It will be shown that already in this very first small fragment there are 62 non-interderivable members with 8 maximal elements. Of the next fragment L^1_2 even the cardinality of the set of maximal exact elements has eluded us so far. The fragment L^1_3 is already definitely too large to attack in this manner.

To calculate the exact formulas in L_1^1 we worked with the set of $1,1$-types. To start with, there are eight $1,1$-types enumerated in T_1^1 that may be ordered in such a fashion that the result is an exact model (this notion was introduced after corollary 10). There are more possibilities, but here are two exact models of L_1^1, the left one coinciding with the first two levels of ExL^1.



In the chosen exact model (in our case the left one) the logical connectives that are used to form formulas correspond with set-theoretic operations which are given as input to the algorithm. Systematically then all L_1^1 formulas are generated with their truth sets, throwing out any formulas the truth set of which has occurred before. Also the non-reflexive types $0, 1, 3$ and 4 are distinguished from the reflexive $2, 5, 6$ and 7. This enables the algorithm to check whether the upward closure of the truth set of the formula has an enveloping type (see lemma 43). If such is the case, then the formula and the upward closure of its truth set is noted down in the list of exact formulas, if at least the upward closure of the truth set has not occurred before. The list of exact L_1^1 formulas that came out will be given at the end of this section. We first give the 1-maximal exact formulas among them (the ones with minimal sets of types) as they were generated by the program, preceded by their sets of types:

1. $\{1,5\}$	3. $\{0,1,7\}$	5. $\{1,4,7\}$	7. $\{0,3,7\}$
2. $\{0,2\}$	4. $\{0,1,6\}$	6. $\{1,4,6\}$	8. $\{0,3,6\}$

1. p	5. $(p \vee \Box p) \wedge (p \wedge \Box\bot \vee \neg(\Box\neg p \vee p \wedge \Box p))$
2. $\neg p \wedge \Box\neg p$	6. $p \wedge \Box\bot \vee \neg(p \vee \Box\neg p)$
3. $\Box\bot \vee p \wedge \neg(\Box p \vee \Box\neg p)$	7. $(p \vee \Box\bot) \wedge \neg(p \wedge \Box p)$
4. $\Box\bot \vee \neg(\Box\neg p \vee (p \vee \Box p))$	8. $(p \to \Box\neg p \wedge \neg\Box\bot) \wedge ((\Box p \vee \Box\neg p) \to (p \vee \Box\bot))$

For the bracketing $\wedge$ has a higher priority than $\vee$. Of course, which formula from an interderivability class is given by the program depends on the settings of the program like the priority rule for the connectives to be used and a selection on the shortest formula made. We will give the formulas a more informative form.

1. p
2. $\neg p$
3. $(\Box p \to \Box\bot) \wedge (\Box\neg p \to \Box\bot) \wedge (\neg p \to \Box\neg p)$
4. $(\Box p \to \Box\bot) \wedge (\Box\neg p \to \Box\bot) \wedge (p \to \Box p)$
5. $p \leftrightarrow \Diamond\neg p \vee \Box\bot$
6. $p \leftrightarrow \Box\neg p$
7. $p \leftrightarrow \neg\Box p$
8. $p \leftrightarrow \Box\neg p \wedge \neg\Box\bot$

Formulas 1 and 2 correspond to *provable* and *refutable* sentences in PA. Formulas 6 and 7 can be (faithfully) interpreted by *Gödel-sentences* and their duals in PA. Similarly, formulas 3 and 4 correspond to *Rosser-sentences* and their duals in PA. The only small surprise is formed by formula 8 and its dual 5. It is easy to see that 8 is interderivable with $p \leftrightarrow \Box\Box\bot \wedge \neg\Box\bot$ and then, of course, 5 with $p \leftrightarrow (\Box\Box\bot \to \Box\bot)$. These two formulas are not L^1_1, but can apparently interderivably be given as such. Note also that, by the fixed point theorem of L (see e.g., [11], [2]), there is no surprise in the fact that in the equivalences 5 to 8 the p in the right hand side can be eliminated in favor of the $\bot$, but only in the fact that using p one can push down the complexity.

The reader will enjoy providing the tail models that show that these 1-maximal exact formulas are actually the maximal exact L^1_1 formulas. As mentioned above we do not know whether such a state of affairs holds for $m > 1$.

The 62 exact L^1_1 formulas that the program churned out will be given below without comment and with only very slight editing. They will be given with their sets of types that show that each such set is a union of sets of types of 1-maximal formulas. Again, as mentioned above, we do not know whether this is generally the state of affairs for $m > 1$.

1: $p \to p$	{ 0 1 2 3 4 5 6 7 }
2: $p \to \Box\bot$	{ 0 1 2 4 6 }
3: $p \vee \Box\bot$	{ 0 1 3 5 7 }
4: $p \to \Box p$	{ 0 1 2 4 5 6 }

5: $p \wedge \Box p$	$\{ 1\ 5 \}$
6: $p \vee \Box p$	$\{ 0\ 1\ 3\ 4\ 5\ 7 \}$
7: $p \rightarrow \Box\neg p$	$\{ 0\ 1\ 2\ 3\ 4\ 6 \}$
8: $p \vee \Box\neg p$	$\{ 0\ 1\ 2\ 3\ 5\ 7 \}$
9: $\neg p \wedge \Box\neg p$	$\{ 0\ 2 \}$
10: $\Box\bot \vee \neg\Box p$	$\{ 0\ 1\ 2\ 3\ 6\ 7 \}$
11: $\Box\neg p \rightarrow \Box p$	$\{ 0\ 1\ 4\ 5\ 6\ 7 \}$
12: $p \wedge \Box p \rightarrow \Box\bot$	$\{ 0\ 1\ 2\ 3\ 4\ 6\ 7 \}$
13: $p \wedge \Box\neg p \rightarrow \Box\bot$	$\{ 0\ 1\ 2\ 4\ 5\ 6\ 7 \}$
14: $\Box\neg p \rightarrow p \wedge \Box\bot$	$\{ 1\ 4\ 5\ 6\ 7 \}$
15: $\Box p \rightarrow (p \vee \Box\bot)$	$\{ 0\ 1\ 2\ 3\ 5\ 6\ 7 \}$
16: $\Box\neg p \rightarrow (p \vee \Box\bot)$	$\{ 0\ 1\ 3\ 4\ 5\ 6\ 7 \}$
17: $\Box p \rightarrow \neg p \wedge \Box\bot$	$\{ 0\ 2\ 3\ 6\ 7 \}$
18: $\Box\neg p \vee (p \rightarrow \Box p)$	$\{ 0\ 1\ 2\ 3\ 4\ 5\ 6 \}$
19: $\Box\neg p \vee p \wedge \neg\Box p$	$\{ 0\ 1\ 2\ 3\ 7 \}$
20: $\Box\neg p \vee (p \vee \Box p)$	$\{ 0\ 1\ 2\ 3\ 4\ 5\ 7 \}$
21: $(p \vee \Box p) \rightarrow \Box\neg p$	$\{ 0\ 1\ 2\ 3\ 6 \}$
22: $(p \vee \Box\bot) \wedge \neg(p \wedge \Box p)$	$\{ 0\ 3\ 7 \}$
23: $\Box p \vee p \wedge \neg\Box\neg p$	$\{ 0\ 1\ 4\ 5\ 7 \}$
24: $(p \vee \Box\neg p) \rightarrow \Box p$	$\{ 0\ 1\ 4\ 5\ 6 \}$
25: $\neg(p \wedge \Box p) \wedge (p \vee \Box\neg p)$	$\{ 0\ 2\ 3\ 7 \}$
26: $p \wedge \Box\bot \vee \neg(p \vee \Box\neg p)$	$\{ 1\ 4\ 6 \}$
27: $p \wedge \Box p \vee \neg(p \vee \Box\neg p)$	$\{ 1\ 4\ 5\ 6 \}$
28: $\Box\bot \vee \neg(p \vee \Box p)$	$\{ 0\ 1\ 2\ 6 \}$
29: $\Box\bot \vee p \wedge \neg\Box p$	$\{ 0\ 1\ 3\ 7 \}$
30: $p \wedge (\Box p \vee \Box\neg p) \rightarrow \Box\bot$	$\{ 0\ 1\ 2\ 4\ 6\ 7 \}$
31: $(\Box p \vee \Box\neg p) \rightarrow (p \vee \Box\bot)$	$\{ 0\ 1\ 3\ 5\ 6\ 7 \}$
32: $\Box\bot \vee \neg(p \vee \Box\neg p)$	$\{ 0\ 1\ 4\ 6 \}$
33: $\Box\bot \vee p \wedge \neg\Box\neg p$	$\{ 0\ 1\ 5\ 7 \}$
34: $\Box\bot \vee \neg(\Box p \vee \Box\neg p)$	$\{ 0\ 1\ 6\ 7 \}$
35: $(p \vee \Box p) \wedge (p \wedge \Box p \rightarrow \Box\bot)$	$\{ 0\ 1\ 3\ 4\ 7 \}$
36: $\Box\bot \vee \neg(\Box\neg p \vee p \wedge \Box p)$	$\{ 0\ 1\ 4\ 6\ 7 \}$
37: $(p \vee \Box\neg p) \wedge (p \wedge \Box\neg p \rightarrow \Box\bot)$	$\{ 0\ 1\ 2\ 5\ 7 \}$
38: $\Box\bot \vee \neg(\Box p \vee p \wedge \Box\neg p)$	$\{ 0\ 1\ 2\ 6\ 7 \}$
39: $(p \vee \Box p) \wedge (\Box\neg p \rightarrow p \wedge \Box\bot)$	$\{ 1\ 4\ 5\ 7 \}$
40: $p \wedge \Box\bot \vee \neg(\Box\neg p \vee p \wedge \Box p)$	$\{ 1\ 4\ 6\ 7 \}$
41: $(p \vee \Box p) \rightarrow \Box p \wedge (p \vee \Box\bot)$	$\{ 0\ 1\ 2\ 5\ 6 \}$
42: $(\Box p \vee \Box\neg p) \rightarrow \Box p \wedge (p \vee \Box\bot)$	$\{ 0\ 1\ 5\ 6\ 7 \}$
43: $\Box\bot \vee (\Box p \rightarrow p) \wedge \neg(p \wedge \Box\neg p)$	$\{ 0\ 1\ 2\ 5\ 6\ 7 \}$
44: $(p \vee \Box\neg p) \rightarrow \Box\neg p \wedge (p \vee \Box\bot)$	$\{ 0\ 1\ 3\ 4\ 6 \}$
45: $\Box\bot \vee \neg\Box p \wedge (\Box\neg p \rightarrow p)$	$\{ 0\ 1\ 3\ 6\ 7 \}$
46: $\Box\bot \vee \neg(p \wedge \Box p) \wedge (\Box\neg p \rightarrow p)$	$\{ 0\ 1\ 3\ 4\ 6\ 7 \}$
47: $(p \vee \Box p) \rightarrow \Box\neg p \wedge \neg(p \wedge \Box\bot)$	$\{ 0\ 2\ 3\ 6 \}$
48: $(\Box p \vee \Box\neg p) \rightarrow (p \vee \Box\bot) \wedge \neg(p \wedge \Box p)$	$\{ 0\ 3\ 6\ 7 \}$
49: $(p \vee \Box p) \rightarrow (\Box\neg p \vee p \wedge \Box p)$	$\{ 0\ 1\ 2\ 3\ 5\ 6 \}$
50: $(p \vee \Box\neg p) \rightarrow (\Box p \vee p \wedge \Box\neg p)$	$\{ 0\ 1\ 3\ 4\ 5\ 6 \}$
51: $\Box\neg p \vee (p \vee \Box p) \wedge \neg(p \wedge \Box p)$	$\{ 0\ 1\ 2\ 3\ 4\ 7 \}$
52: $\Box\bot \vee p \wedge \neg(\Box p \vee \Box\neg p)$	$\{ 0\ 1\ 7 \}$
53: $(p \vee \Box\neg p) \wedge (p \wedge (\Box p \vee \Box\neg p) \rightarrow \Box\bot)$	$\{ 0\ 1\ 2\ 7 \}$
54: $\Box p \vee (p \vee \Box\neg p) \wedge \neg(p \wedge \Box\neg p)$	$\{ 0\ 1\ 2\ 4\ 5\ 7 \}$
55: $\Box\bot \vee \neg(\Box\neg p \vee (p \vee \Box p))$	$\{ 0\ 1\ 6 \}$
56: $\Box\neg p \wedge (p \vee \Box\bot) \vee \neg(\Box\neg p \vee (p \vee \Box p))$	$\{ 0\ 1\ 3\ 6 \}$
57: $(p \vee \Box p) \wedge (p \wedge (\Box p \vee \Box\neg p) \rightarrow \Box\bot)$	$\{ 0\ 1\ 4\ 7 \}$
58: $\Box p \wedge (p \vee \Box\bot) \vee \neg(\Box\neg p \vee (p \vee \Box p))$	$\{ 0\ 1\ 5\ 6 \}$
59: $(\Box\neg p \vee (p \vee \Box p)) \wedge (p \wedge (\Box p \vee \Box\neg p) \rightarrow \Box\bot)$	$\{ 0\ 1\ 2\ 4\ 7 \}$

60: $(p \vee \Box p) \wedge (p \wedge \Box\bot \vee \neg(\Box\neg p \vee p \wedge \Box p))$ $\{\ 1\ 4\ 7\ \}$
61: $(\Box\neg p \vee (p \vee \Box p)) \rightarrow (p \vee \Box\bot) \wedge (\Box p \vee \Box\neg p)$ $\{\ 0\ 1\ 3\ 5\ 6\ \}$
62: $(p \rightarrow \Box\neg p \wedge \neg\Box\bot) \wedge ((\Box p \vee \Box\neg p) \rightarrow (p \vee \Box\bot))$ $\{\ 0\ 3\ 6\ \}$

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