Free Algebras Corresponding to Multiplicative Classical Linear Logic and some Extensions.

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Abstract

In this paper, constructions of free algebras corresponding to multiplicative classical linear logic, its affine variant and their extensions with n-contraction ($n \geq 2$) are given. As an application, the cardinality problem of some one-variable linear fragments with n-contraction is solved.

1 Introduction

The topic of substructural logics, i.e. logics with restricted structural rules, already has a long tradition (witness relevance logic, BCK, Lambek calculus). However, with the birth of Girard's linear logic [3] it has regained the attention of researchers with various motivations and different traditions. An extensive survey covering the subject of substructural logics can be found in Došen, Schroeder-Heister [2].

The present paper continues our investigation of intuitionistic and classical Gentzen systems with bounded contraction. In particular, n-contraction ($n \geq 2$) is a version of the contraction rule, where $n+1$ occurrences of a formula may be contracted to n occurrences. Our motivation for exploring these systems has its roots in the observation that substituting n-contraction for full contraction in the systems considered already results in the splitting of logical operations familiar from linear logic. However, most of the systems with n-contraction do not enjoy the cut-elimination property. Thus, standard proof-theoretic techniques to investigate meta-properties of these systems are not available. In spite of that, the desire of getting better insights into the effects of bounded contraction led to a number of different papers on this topic. To start with, in [8] we showed that the linear models for n-valued Łukasiewicz logics are suitable models of n-contraction. We also presented a new complete axiomatization for this logics, essentially by means of n-contraction. Further, in [5] Hori, Ono and Schellinx extensively studied syntactic and semantical properties of extensions of the intuitionistic linear logic with knotted structural rules, resulting in cut-elimination theorems, decidability and undecidabily results as well as finite model property theorems. Briefly, knotted structural rules permit n copies of a formula to be contracted to k copies, when $n > k$, and to be weakened to k copies, when $n < k$ (with $n, k \geq 1$). Knotted structural rules can be seen as a generalization of those discussed earlier. And finally, in [7] we introduced connectification-operators for intuitionistic and classical linear algebras corresponding to linear logic and to some of its extensions with n-contraction. As a useful application of these operators we established the disjunction property for both intuitionistic and classical affine linear logics with n-contraction.

The present paper had its origin in the problem of describing the structure and determining the cardinality of one-variable fragments of some extensions of linear logic with n-contraction and weakening. First, we point out that for the intuitionistic case this task is easy. It is well-known that the one-variable fragment of intuitionistic propositional logic (in our notation $\mathbf{IPL}_1^a$) is infinite, due to the Rieger-Nishimura lattice being the Lindenbaum algebra of the one-variable fragment in question (see Troelstra, van Dalen [11]). Consider now the system, $\mathbf{IPL}_n^a$ ($n \geq 2$), of intuitionistic affine logic with n-contraction (see appendix (A.1)). We extend the Rieger-Nishimura lattice with $\star$ and $\multimap$, interpreted as meet and relative pseudocomplementation respectively. Clearly, the structure obtained is an infinite $\mathbf{IPL}_n^a$-algebra (see Prijatelj [7]). Moreover, the natural valuation of one-variable $\mathbf{IPL}_n^a$-formulas with values in this algebra is surjective. Thus, the one-variable fragment of the system $\mathbf{IPL}_n^a$ is infinite.

Far more interesting and involved is the problem of the structure and cardinality of one-variable fragments of some classical linear logic extensions with n-contraction ($n \geq 2$). The Lindenbaum algebra of the one-variable fragment of ordinary classical logic is a lattice of exactly four elements. However, in this paper we show that the Lindenbaum algebras of the one-variable fragments of purely multiplicative (i.e. tensor, par) classical linear logic extended with n-contraction and/or weakening are infinite.

Here we make a brief digression to mention some of the recent papers investigating similar questions. It is interesting to note, that each of them is supplemented by a computer program designed for a specific generation of models. In [6] de Jongh, Hendriks and Renardel de Lavalette examined the structure of finite diagrams of intuitionistic propositional logic fragments. The diagram of a fragment is nothing but the set of equivalence classes of its formulae partially ordered by the derivability relation. The latest developments of this work can be found in Hendriks [4]. And finally, in [9] Slaney showed that natural systems close to relevance logic, but weaker, have infinitely many non-equivalent Ackermann constants. The non-distributive version of relevance logic considered by Slaney, is in fact equivalent to propositional classical linear logic extended with full contraction.

In the study of the one-variable fragments of logics with n-contraction, we encounter the following obstacles. Firstly, the systems with n-contraction considered here do not enjoy cut-elimination (for a counter-example, see section 2). Thus, the usual proof-theoretic methods to examine provable equivalence of formulas in these systems are not available. The other approach to this problem, is a model theoretic one. As far as we know, there are only two classes of models suitable to mimic the effects of n-contraction in the logical systems. These are the models for n-valued Łukasiewicz logics (see appendix (A.4)) and the algebraic models given in section 2. For our purposes, Łukasiewicz models are useless, since they do not distinguish sufficiently many formulas. On the other hand, there is a difficulty with the algebraic models as well, namely, how to determine non-equivalent expressions in a partially ordered algebraic structure. In what follows, we shall give a partial answer to this, sufficient to solve the cardinality problem discussed earlier. Briefly, we construct a free algebra on one generator corresponding to the one-variable fragment of multiplicative classical affine logic with 2-contraction. Further, we elaborate some lemma's and propositions in order to show the existence of two infinite chains in the free algebra introduced. The free algebra on one generator is isomorphic to the Lindenbaum algebra of the one-variable fragment considered, yielding that this fragment is infinite. As an immediate consequence, we see that also one-variable fragments of multiplicative classical linear logic with n-contraction (for any $n \geq 2$), and its affine version are infinite.

Moreover, we wish to emphasize that our construction of the free algebra is of interest in its own right. It will be adapted to respective free algebras on one generator corresponding to multiplicative classical linear logic (a new result in this field) as well as to its affine variant, and their extensions with

n-contraction. Also, it will become evident, later on, that any of these constructions can be generalized to the corresponding free algebra on an arbitrary set of generators.

At the end of section 4, we shall indicate why our present strategy to solve the cardinality problem of one-variable fragments is not directly applicable to the systems extended with constants and/or negation.

Finally, in the last section, we consider the system obtained from propositional classical linear logic by omitting modalities and the rules for additive connectives, but adding two new axioms. We construct the free algebra on one generator corresponding to this system, which has exactly twelve elements.

We use Troelstra's notation ([10]) for the operators of linear logic.

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2 The system $\langle \star, + \rangle$-$\mathbf{CPL}_2^a$ and monoidal $\mathbf{CPL}_2^a$-algebras

A multiplicative system of affine classical linear logic with 2-contraction, $\langle \star, + \rangle$-$\mathbf{CPL}_2^a$, is given by the following axioms and rules. Throughout the sequel, Γ, Γ_1, Γ_2, Δ, Δ_1, Δ_2 denote finite multisets of formulas.

Axioms

$$A \Rightarrow A$$

Logical rules

$$L\star \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \star B \Rightarrow \Delta} \qquad\qquad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \star B, \Delta_1, \Delta_2} \quad R\star$$

$$L+ \quad \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A + B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A + B, \Delta} \quad R+$$

Structural rules

$$LW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \quad RW$$

$$LC_2 \quad \frac{\Gamma, A, A, A \Rightarrow \Delta}{\Gamma, A, A \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow A, A, A, \Delta}{\Gamma \Rightarrow A, A, \Delta} \quad RC_2$$

$$CUT \quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Note that for any formulas A, B and C, the sequent $(B+C)\star A \Rightarrow (B\star A)+C$ is derivable in $\langle \star, + \rangle$-$\mathbf{CPL}_2^a$. However, $\langle \star, + \rangle$-$\mathbf{CPL}_2^a \not\vdash (B \star A) + C \Rightarrow (B + C) \star A$. To see this, choose for A and C propositional variables p

and q respectively, and let B be arbitrary. As a counter-model of the given sequent we may take the Łukasiewicz model $\mathbf{M}_3(\llbracket\ .\ \rrbracket)$ (see appendix (A.4)), with $\llbracket p \rrbracket = 0$ and $\llbracket q \rrbracket = 1$.

Next, we shall give a counter-example for cut-elimination in the system $\langle\star,+\rangle$-$\mathbf{CPL}^a_2$.

Let p be a propositional variable in the language of $\langle\star,+\rangle$-$\mathbf{CPL}^a_2$. Then,

$$\langle\star,+\rangle\text{-}\mathbf{CPL}^a_2 \vdash p, p+p \Rightarrow (p+p)^3,$$

where $(p+p)^3$ stands for $(p+p)\star(p+p)\star(p+p)$. The derivation of the given sequent can be obtained as follows.

Applying $R\star$ to two axiom instances $p+p \Rightarrow p+p$ yields $(p+p)^{(2)} \Rightarrow (p+p)^2$. Another application of $R\star$ to the latter sequent and $p+p \Rightarrow p+p$ yields $(p+p)^{(3)} \Rightarrow (p+p)^3$, and so, with LC_2, applied next, gives $(p+p)^{(2)} \Rightarrow (p+p)^3$. On the other hand, RW applied to $p \Rightarrow p$ yields $p \Rightarrow p^{(2)}$ and with $R+$ $p \Rightarrow p+p$. Cutting the latter sequent with $(p+p)^{(2)} \Rightarrow (p+p)^3$ results in $p, p+p \Rightarrow (p+p)^3$, with $A^{(k)} \equiv A, A, \ldots, A$, i.e. k copies of formula A in the derivation above.

However, it is easy to check that the sequent $p, p+p \Rightarrow (p+p)^3$ has no cut-free derivation in the system considered. Moreover, note that this is also a counter-example for cut-elimination in the one-variable fragment of $\langle\star,+\rangle$-$\mathbf{CPL}^a_2$.

We now introduce monoidal $\mathbf{CPL}^a_2$-algebras corresponding to this system.

Definition 2.1 $\mathbf{X} = \langle X, \star, +, \leq\rangle$ *is a monoidal* $\mathbf{CPL}^a_2$*-algebra, if:*

- *(I)* $\langle X, \star\rangle$ *and* $\langle X, +\rangle$ *are commutative monoids*[1]*;*
- *(II)* $\leq$ *is a partial order on* X*, satisfying the following clauses for all* $x, y, z \in X$*:*
  - *(i)* $x \star y \leq x$ *and* $x \leq x + y$*, corresponding to weakening;*
  - *(ii)* $x \star x \leq x \star x \star x$ *and* $x + x + x \leq x + x$*, corresponding to* 2*-contraction;*
  - *(iii)* $(y+z) \star x \leq (y \star x) + z$*, i.e. sub-commutativity of* $\star$ *and* $+$*;*
  - *(iv) if* $x \leq y$*, then* $x \star z \leq y \star z$ *and* $x + z \leq y + z$*, i.e. monotonicity of* $\star$ *and* $+$ *with respect to* $\leq$*.*

[1] also referred to as semigroups

This formulation of a monoidal $\mathbf{CPL}_2^a$-algebra will turn out to be particularly suitable for our purpose, that is to construct the corresponding free algebra on one generator. Note that (II)(iii) of definition 2.1 covers precisely the derivability of the sequent $(B+C)\star A \Rightarrow (B\star A)+C$ in $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. Also, it is useful to know and easy to check that in a $\mathbf{CPL}_2^a$- algebra (defined in appendix (A.3)) sub-commutativity of monoidal operations is just equivalent to adjointness (i.e. clause (5) in the definition of a $\mathbf{CPL}_2^a$-algebra).

Further, a monoidal $\mathbf{CPL}_2^a$-model, $\mathbf{M}$, is a pair $\langle \mathbf{X}, [\![\ .\ ]\!]\rangle$, where $\mathbf{X}$ is a monoidal $\mathbf{CPL}_2^a$-algebra and $[\![\ .\ ]\!]$ is a valuation defined in a usual way (see appendix (A.3)). For validity of a $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$-sequent $\Gamma \Rightarrow \Delta$ (see appendix (A.3)) in a given monoidal $\mathbf{CPL}_2^a$-model $\mathbf{M}$, we use the standard notation: $\models_{\mathbf{M}} \Gamma \Rightarrow \Delta$.

Soundness and completeness hold for the system and models under consideration:

Proposition 2.2 *Given a $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$- sequent $\Gamma \Rightarrow \Delta$,*

$$\textit{if}\ \ \langle\star,+\rangle\text{-}\mathbf{CPL}_2^a \vdash \Gamma \Rightarrow \Delta, \ \ \textit{then}\ \ \models_{\mathbf{M}} \Gamma \Rightarrow \Delta,$$

for every monoidal $\mathbf{CPL}_2^a$-model $\mathbf{M}$.

Proof: By induction on the length of a derivation of $\Gamma \Rightarrow \Delta$.

To show soundness of CUT the use of sub-commutativity of $\star$ and $+$, i.e. (II)(iii) of the definition of a monoidal $\mathbf{CPL}_2^a$-algebra, is essential. The rest of the proof is straightforward. ◇

Proposition 2.3 *There exists a monoidal $\mathbf{CPL}_2^a$-model $\mathbf{M}_L$, such that*

$$\textit{if}\ \ \models_{\mathbf{M}_L} \Gamma \Rightarrow \Delta, \ \ \textit{then}\ \ \langle\star,+\rangle\text{-}\mathbf{CPL}_2^a \vdash \Gamma \Rightarrow \Delta,$$

for any $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$-sequent $\Gamma \Rightarrow \Delta$.

Proof: Clearly, $\mathbf{M}_L = \langle L, [\![\ .\ ]\!]\rangle$, where $L = \langle F/\equiv, \star', +', \leq'\rangle$ is the Lindenbaum algebra of the system $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. The Lindenbaum algebra considered is obtained by the standard definition for linear logic systems. We emphasize, however, that the partial order $\leq'$ in $F/\equiv$ is given by:
for any formulas A and B, $[A]_\equiv \leq' [B]_\equiv$ iff $\langle\star,+\rangle$-$\mathbf{CPL}_2^a \vdash A \Rightarrow B$. Clearly, then $A \equiv B$ iff $[A]_\equiv \leq' [B]_\equiv$ and $[B]_\equiv \leq' [A]_\equiv$.
It is now easy to check that, L, indeed, is a monoidal $\mathbf{CPL}_2^a$-algebra. The rest of the proof is standard (see Troelstra [10]). ◇

3 A construction of the free monoidal $\mathbf{CPL}_2^a$-algebra on one generator

A double commutative monoid is a triple $\langle S, \star, + \rangle$, where S is a non-empty set, $\star$ and $+$ are binary associative and commutative operations in S.

We shall now give an explicit construction of the free double commutative monoid $\mathcal{P}(\gamma) = \langle P, \star, + \rangle$ generated by $\{\gamma\}$. First, we shall define for any $d = 0, 1, \ldots, \infty$ the sets $P_d^\star$, (i.e. $\star$-$+$-polynomials of depth d, with a principal operation $\star$, when $d \geq 1$), and their duals P_d^+, inductively on the depth of a $\star$-$+$-polynomial.
(1) The only polynomial of depth 0 is the generator γ, so we put
$P_0^\star = P_0^+ = \{\gamma\}$.
(2) For $d \geq 1$:
(i) Let $P_d^\star$ be the set of all formal products

$$(p_1) \star \cdots \star (p_k), \quad (k \geq 2),$$

such that

$$p_j \in P_{d_j}^+ \ \ (j = 1, \ldots, k) \quad \text{and} \quad \max_{j=1}^{k} d_j = d - 1.$$

(ii) Let P_d^+ be the set of all formal sums

$$(p_1) + \cdots + (p_k), \quad (k \geq 2),$$

such that

$$p_j \in P_{d_j}^\star \ \ (j = 1, \ldots, k) \quad \text{and} \quad \max_{j=1}^{k} d_j = d - 1.$$

The operations $\star$ and $+$ between $\star$-$+$-polynomials are defined in the obvious way.

We now define an equivalence relation, $\simeq$, in $P_d^\star$ and in P_d^+ inductively by:

(1) $\gamma \simeq \gamma$;

(2) for $d \geq 1$:
$(p_1) \star \cdots \star (p_k) \simeq (p'_1) \star \cdots \star (p'_s)$ iff $k = s$ and for some permutation θ of $\{1, \ldots, k\}$, $p_j \simeq p'_{\theta(j)}$ $(j = 1, \ldots, k)$. And analogously, $(p_1) + \cdots + (p_k) \simeq (p'_1) + \cdots + (p'_s)$ iff $k = s$ and for some permutation θ of $\{1, \ldots, k\}$, $p_j \simeq p'_{\theta(j)}$ $(j = 1, \ldots, k)$.

Moreover, since

$$p \simeq p' \quad \text{implies} \quad p \star q \simeq p' \star q \quad \text{and} \quad p + q \simeq p' + q,$$

for all $\star$-$+$-polynomials q, clearly $\simeq$ is a congruence relation on the algebra of $\star$-$+$-polynomials.

Put now $P_0 = P_0^\star/\simeq = P_0^+/\simeq$ and $P_d = (P_d^\star/\simeq)\bigcup(P_d^+/\simeq)$, for any $d \geq 1$, and take

$$P = \bigcup_{d=0}^{\infty} P_d.$$

Finally, we define $\mathcal{P}(\gamma) = \langle P, \star, + \rangle$, with $\star$ and $+$ being induced from $\star$ and $+$ on $\star$-$+$-polynomials.

Since $\simeq$ is a congruence relation on the algebra of $\star$-$+$-polynomials, the operations $\star$ and $+$ are, indeed, well-defined on $\simeq$-equivalence classes. Moreover, note that associativity and commutativity are built in elements of P. It is now easy to see, using standard arguments (Birkhoff [1]), that $\mathcal{P}(\gamma)$ is indeed, a free double commutative monoid on one generator γ. Next, observe the following important property of the free $\star$-$+$- algebra.

Fact 3.1 *By the construction of $\mathcal{P}(\gamma)$, it follows that every $x \in P_d$ $(d \geq 1)$, has a unique decomposition, up to a permutation of factors or summands, in the following sense:*

$$(i) \quad if \quad x \in P_d^\star/\simeq \quad (for \;\; some \;\; d = 1, \ldots, \infty), \quad then$$

$$x = x_1 \star \cdots \star x_k, \;\; (k \geq 2), \quad with \quad x_j \in \bigcup_{i=0}^{d-1} (P_i^+/\simeq) \quad (j = 1, \ldots, k);$$

$$(ii) \quad if \quad x \in P_d^+/\simeq \quad (for \;\; some \;\; d = 1, \ldots, \infty), \quad then$$

$$x = x_1 + \cdots + x_k, \;\; (k \geq 2), \quad with \quad x_j \in \bigcup_{i=0}^{d-1} P_i^\star/\simeq \quad (j = 1, \ldots, k).$$

Our intention, in what follows, is to construct the free monoidal $\mathbf{CPL}_2^a$-algebra on one generator from $\mathcal{P}(\gamma)$. The construction is carried out in four steps.

(I) A binary relation $\leq_0$ is introduced in P, by clauses (i)-(iv) being satisfied for all $x, y, z \in P$:

  (i) $x \leq_0 x$;

(ii) $x \star y \leq_0 x$ and $x \leq_0 x + y$;

(iii) $x \star x \leq_0 x \star x \star x$ and $x + x + x \leq_0 x + x$;

(iv) $(y + z) \star x \leq_0 (y \star x) + z$.

(II) We construct the monotonic closure of $\leq_0$, as follows.

A sequence of binary relations $\{\leq_n\}_{n=0}^{\infty}$ in P is defined inductively by:
for every $n \geq 1$, let $\leq_n$ be the extension of $\leq_{n-1}$, determined by:
for all $x, y, z \in P$, if $x \leq_{n-1} y$, then $x \star z \leq_n y \star z$ and $x + z \leq_n y + z$.
We define

$$\leq_\infty = \bigcup_{n=0}^{\infty} (\leq_n),$$

i.e. given $x, y \in P$, $x \leq_\infty y$ iff $x \leq_n y$, for some $n \geq 0$.

(III) Let $\leq$ be the transitive closure of $\leq_\infty$ in P, i.e. given $x, y \in P$, $x \leq y$ iff there exists a chain $x \leq_\infty x_1 \leq_\infty \cdots \leq_\infty x_k \leq_\infty y$ in P.

With the following two lemma's we shall justify monotonicity of $\star$ and $+$ with respect to $\leq$ in P and show that $\leq$ is a preorder on P.

Lemma 3.2 *The operations $\star$ and $+$ are monotone with respect to $\leq_\infty$ in P.*

Proof: Assume $x \leq_\infty y$ for some $x, y \in P$. By definition, there is $n \geq 0$, such that $x \leq_n y$. Take any $z \in P$. Then, $x \star z \leq_{n+1} y \star z$ and $x + z \leq_{n+1} y + z$. Hence, $x \star z \leq_\infty y \star z$ and $x + z \leq_\infty y + z$. $\diamond$

Lemma 3.3 *The relation $\leq$ is a preorder on P. Moreover, the operations $\star$ and $+$ are monotone with respect to $\leq$ in P.*

Proof: Indeed, $\leq$ is reflexive and transitive, due to I(i) and (III) above.
To show monotonicity, assume $x \leq y$ for some $x, y \in P$. Hence there is a chain in P, $x \leq_\infty x_1 \leq_\infty \cdots \leq_\infty x_k \leq_\infty y$. Then for any $z \in P$, $x \star z \leq_\infty x_1 \star z \leq_\infty \cdots \leq_\infty x_k \star z \leq_\infty y \star z$ and $x + z \leq_\infty x_1 + z \leq_\infty \cdots \leq_\infty x_k + z \leq_\infty y + z$ are again chains in P due to lemma 3.2. Therefore, $x \star z \leq y \star z$ and $x + z \leq y + z$. $\diamond$

In order to show the existence of two infinite chains in the algebraic structure $\langle P, \star, +, \leq \rangle$ constructed so far, and, consequently, also in the free monoidal $\mathbf{CPL}_2^a$-algebra, we continue with two key-lemma's.

Lemma 3.4 *Given $n \geq 0$ and $z \in P$, if $\gamma \leq_n z$, then either $z = \gamma$ or $z = \gamma + z'$, for some $z' \in P$.*

Proof: Assume $\gamma \leq_n z$, for some $n \geq 0$ and $z \in P$. Note, first, that $\gamma \leq_n z$ iff $\gamma \leq_0 z$. It is now easy to see that either I(i) or I(ii) of definition of $\leq_0$ may occur, yielding $z = \gamma$ or $z = \gamma + z'$ for some $z' \in P$. And we are done. $\diamond$

Lemma 3.5 *Given $n \geq 0$ and $y, z \in P$, if $\gamma + y \leq_n z$, then $z = \gamma + z'$, for some $z' \in P$, such that one of the clauses below is satisfied:*
(i) $y \leq_n z'$;
(ii) $z' = \gamma$;
(iii) $z' = \gamma + z''$, for some $z'' \in P$.

Proof: By induction on n.

(1) ($n = 0$):
Assume $\gamma + y \leq_0 z$, for some $y, z \in P$. The following three cases are to be considered.

If I(i) occurs, then $z = \gamma + y$, and hence $z' = y$, yielding $y \leq_0 z'$.

If I(ii) occurs, then $z = \gamma + y + u$, for some $u \in P$. Hence, $z' = y + u$ and thus, indeed, $y \leq_0 z'$.

If I(iii) occurs, then two subcases occur.

(a) $y = \gamma + \gamma$ and $z = \gamma + \gamma$, resulting in $z' = \gamma$.
(b) $y = \gamma + \gamma + u + u + u$ and $z = \gamma + \gamma + u + u$, for some $u \in P$. Therefore, $z' = \gamma + u + u$, and we are done.

(2) ($n \geq 1$):
Assume lemma 3.5 to hold for $k = 0, 1, \ldots, (n-1)$. Assume further that for some $y, z \in P$, $\gamma + y \leq_n z$ and $\gamma + y \not\leq_{n-1} z$. This means that there are $a, b, c \in P$, satisfying

$$(M) \quad a \leq_{n-1} b,$$

and hence, $a + c \leq_n b + c$, such that $\gamma + y = a + c$, and $z = b + c$. Now, due to fact 3.1, the following possibilities are to be distinguished.

(i) $a = \gamma$, hence $z = b + y$. Then, by (M) above, $\gamma \leq_{n-1} b$. Using lemma 3.4 we get $b = \gamma$ or $b = \gamma + b'$, for some $b' \in P$. Thus, $z = \gamma + y$ or $z = \gamma + b' + y$, yielding $z' = y$ or $z' = b' + y$. And hence, $y \leq_n z'$.

(ii) $c = \gamma$. By (M), this gives $a+\gamma \leq_n b+\gamma$, and hence, $y = a \leq_n b = z'$.

(iii) $a = \gamma + u$ (for some $u \in P$), resulting in $y = u + c$. Now by (M) $\gamma + u \leq_{n-1} b$, therefore we can use induction hypothesis and get $b = \gamma + b'$ with one of the options

(O) $u \leq_{n-1} b'$ or $b' = \gamma$ or $b' = \gamma + b''$, for some $b'' \in P$.

Since $z = b + c = \gamma + b' + c$, we get $z' = b' + c$. We finally have to consider each option in (O).

If $u \leq_{n-1} b'$, then $y = u + c \leq_n b' + c = z'$, and we are done.

If $b' = \gamma$ or $b' = \gamma + b''$, then $z' = \gamma + c$ or $z' = \gamma + b'' + c$, and we are done again.

(iv) $c = \gamma+u$ (for some $u \in P$), yielding $y = a+u$. Hence, $z = \gamma+u+b$ and so $z' = u + b$. Due to (M), we get $y = a + u \leq_n b + u = z'$, and we are done.

This completes the proof. $\diamond$

Proposition 3.6 *Given $y, z \in P$, the following holds:*
(1) if $\gamma + y \leq z$, then $z = \gamma + z'$ for some $z' \in P$, such that one of the clauses below is satisfied:

(i) $y \leq z'$;

(ii) $z' = \gamma$;

(iii) $z' = \gamma + z''$, for some $z'' \in P$.

(2) if $\gamma \leq z$, then either $z = \gamma$ or $z = \gamma + z'$, for some $z' \in P$.

Proof: Given $x, y \in P$ and $k \in \mathbf{N}$, we introduce the following notation: $x \leq^{(k)} y$ if there is a chain $x \leq_\infty a_1 \leq_\infty \ldots \leq_\infty a_k \leq_\infty y$ in P. Clearly, to prove the present proposition it is enough to prove its (k)-version, for every $k \in \mathbf{N}$, where every occurrence of $\leq$ in the statement above is replaced by $\leq^{(k)}$. We are now going to prove both statements of the (k)-version, i.e. $(1^{(k)})$ and $(2^{(k)})$, simultaneously by induction on k.

(i) The case $k = 0$ is covered by lemma 3.5 and by lemma 3.4 respectively.

(ii) For $k \geq 1$, assume $\gamma + y \leq^{(k)} z$. Hence, there is $n \geq 0$ and a chain

$$\gamma + y \leq_n a_1 \leq_n \cdots \leq_n a_k \leq_n z.$$

Using lemma 3.5, we get $a_1 = \gamma + a'_1$, for some $a'_1 \in P$, with the options

(O) $y \leq_n a'_1$ or $a'_1 = \gamma$ or $a'_1 = \gamma + a''_1$, for some $a''_1 \in P$.

Since $\gamma + a'_1 \leq^{(k-1)} z$, we can use induction hypothesis $(1^{(k-1)})$ yielding $z = \gamma + z'$ (for some $z' \in P$), with either $a'_1 \leq^{(k-1)} z'$ or $z' = \gamma$ or $z' = \gamma + z''$ (for some $z'' \in P$). If the last two possibilities occur, we are done.

So, assume (A): $a'_1 \leq^{(k-1)} z'$. Now by (O), there are three cases to be considered.

If $y \leq_n a'_1$, then $y \leq^{(k)} z'$, due to (A), and we are done.

If $a'_1 = \gamma$ or $a'_1 = \gamma + a''_1$, then (A) gives $\gamma \leq^{(k-1)} z'$ or $\gamma + a''_1 \leq^{(k-1)} z'$. Now, we can use induction hypotheses $(2^{(k-1)})$ and $(1^{(k-1)})$ and get $z' = \gamma$ or $z' = \gamma + z''$, for some $z'' \in P$, and we are done.

Part $(2^{(k)})$ of the proposition under consideration can be proved similarly.

This completes the proof.

◇

By duality the following proposition can be established.

Proposition 3.7 *Given $y, z \in P$, the following holds true:*
(1) if $z \leq \gamma \star y$, then $z = \gamma \star z'$, for some $z' \in P$, such that one of the clauses below is satisfied:

(i) $z' \leq y$;

(ii) $z' = \gamma$;

(iii) $z' = \gamma \star z''$, *for some* $z'' \in P$.

(2) if $z \leq \gamma$, then either $z = \gamma$ or $z = \gamma \star z'$, for some $z' \in P$.

Consider now the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ of elements of P, given inductively by:

$$x_0 = \gamma + \gamma \ , \ \ y_0 = \gamma \star (\gamma + \gamma) \ , \ \ x_{n+1} = y_n + \gamma \ , \ \ y_{n+1} = x_{n+1} \star \gamma.$$

Due to I(ii) of the definition of $\leq_0$ and monotonicity of $\star$ and $+$ with respect to $\leq$ (see lemma 3.3), it is easy to see that:

$$x_{n+1} \leq x_n \quad and \quad y_{n+1} \leq y_n \quad for \;\; all \;\; n \in \mathbf{N}.$$

In order to show, later on, that none of these two chains collapses in the free monoidal $\mathbf{CPL}_2^a$-algebra, we need one more lemma.

Lemma 3.8 *Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be as above. Then, $x_n \not\leq x_{n+1}$ and $y_n \not\leq y_{n+1}$, for any $n \in \mathbf{N}$.*

Proof: By induction on n.

(1) ($n = 0$:) we want to prove
(i) $\gamma + \gamma \not\leq (\gamma \star (\gamma + \gamma)) + \gamma$
(ii) $\gamma \star (\gamma + \gamma) \not\leq ((\gamma \star (\gamma + \gamma)) + \gamma) \star \gamma$.
Assume, first, $\gamma + \gamma \leq (\gamma \star (\gamma + \gamma)) + \gamma$. By proposition 3.6(1), we get $\gamma \leq \gamma \star (\gamma + \gamma)$. This contradicts proposition 3.7(1), and we are done.
Assume, next, $\gamma \star (\gamma + \gamma) \leq ((\gamma \star (\gamma + \gamma)) + \gamma) \star \gamma$. By proposition 3.7(1), we get $\gamma + \gamma \leq (\gamma \star (\gamma + \gamma)) + \gamma$, contradicting (i) above.

(2) ($n \geq 1$:)
(i) Assume $x_n \leq x_{n+1}$, i.e. $y_{n-1} + \gamma \leq y_n + \gamma$. By proposition 3.6(1), we get $y_{n-1} \leq y_n$, contradicting the induction hypothesis.
(ii) Finally, assume $y_n \leq y_{n+1}$, i.e. $x_n \star \gamma \leq x_{n+1} \star \gamma$. Using proposition 3.7(1), we get $x_n \leq x_{n+1}$, which is in contradiction with the first part of the lemma, already proved by 2(i).

$\diamond$

(IV) Next, we define a congruence relation $\equiv$ on $\langle P, \star, +, \leq \rangle$ by:

$$\text{for any} \quad x, y \in P, \; x \equiv y \quad \text{iff} \quad x \leq y \quad \text{and} \quad y \leq x.$$

Clearly, due to lemma 3.3, $\equiv$ is, indeed, a congruence relation on the structure considered. We now define

$$\mathcal{F}(\gamma) = \langle P/\equiv, \star', +', \leq' \rangle,$$

with $\star'$,$+'$, and $\leq'$ being induced from P.

Since $\equiv$ is a congruence relation, $\star'$ and $+'$ are well-defined in $\mathcal{F}(\gamma)$. Moreover, $\leq'$ is a partial order on $\mathcal{F}_(\gamma)$. Taking into account also the given constructions of P and $\leq$ on P, it is now easy to see that $\mathcal{F}(\gamma)$ is a monoidal $\mathbf{CPL}_2^a$-algebra. Moreover, $\mathcal{F}(\gamma)$ is the free $\mathbf{CPL}_2^a$-algebra on one generator γ (see Birkhof [1]). To verify the latter, we have to show that for every monoidal $\mathbf{CPL}_2^a$-algebra $\mathbf{X}$ and for every $x \in X$, there exists a unique morphism $f : \mathcal{F}(\gamma) \to \mathbf{X}$, such that $f([\gamma]) = x$. Define, for any $q \in P$, $f([q]) = q(x)$, where $q(x)$ is the evaluation of the polynomial q at x in X. To see, first, that f is well-defined (i.e. independent of representatives) we have to show that for any $q', q \in P$, if $q' \equiv q$, then $q'(x) = q(x)$ in X. In fact, it is enough to prove that for all $n \in \mathbf{N}$, if $q' \leq_n q$, then $q'(x) \leq q(x)$ in X. A dilligent reader is now invited to work out the proof following the definition of $\leq_n$ by induction on n. From the above, it follows that f preserves the partial order. Hence, indeed, f is a morphism between the two algebras considered, since it also satisfies the following: for any $q, q' \in P$, $f([q \circ q']) = (q \circ q')(x) = q(x) \circ q'(x) = f([q]) \circ f([q'])$, with $\circ \in \{\star, +\}$. Also, $f([\gamma]) = x$, by definition. And finally, the uniqueness of f follows directly from f being a morphism and $f([\gamma]) = x$. Thus, $\mathcal{F}(\gamma)$ is the free monoidal $\mathbf{CPL}_2^a$-algebra on one generator γ, and isomorphic to the Lindenbaum algebra of the one-variable fragment of $\langle \star, + \rangle$-$\mathbf{CPL}_2^a$.

We emphasize, however that our construction of $\mathcal{F}(\gamma)$ has a much wider scope, establishing also the free algebras (on one generator) corresponding to the multiplicative fragment of classical linear logic (i.e. tensor, par fragment without constants and modalities), in our notation $\langle \star, + \rangle$-$\mathbf{CPL}$, its affine version, $\langle \star, + \rangle$-$\mathbf{CPL}^a$, and their extensions with n-contraction ($n \geq 2$): $\langle \star, + \rangle$-$\mathbf{CPL}_n$ and $\langle \star, + \rangle$-$\mathbf{CPL}_n^a$ respectively. To be specific, we state the following fact.

Fact 3.9 *A construction of the free algebra, on one generator γ, corresponding to $\langle \star, + \rangle$- $\mathbf{CPL}$, $\langle \star, + \rangle$-$\mathbf{CPL}^a$, $\langle \star, + \rangle$-$\mathbf{CPL}_n$ and $\langle \star, + \rangle$-$\mathbf{CPL}_n^a$, ($n \geq 2$), can be obtained from the given construction of $\mathcal{F}(\gamma)$ respectively by:*

(a) omitting I(ii),(iii);

(b) omiting I(iii);

(c) omitting I(ii) and replacing I(iii) by I(iii)':

$$x^n \leq_0 x^{n+1} \quad \textit{and} \quad (n+1)x \leq_0 nx,$$

where x^n and nx denote n copies of x in the product and in the sum respectively;

(d) replacing I(iii) with I(iii)', as above.

Besides, note that the construction of $\mathcal{F}(\gamma)$ and its variants, in the corollary above, can easily be generalized to the corresponding free algebras generated by an arbitrary set of generators.

4 One-variable fragments of multiplicative classical linear logic with n-contraction are infinite

We are now ready to prove the existence of the two infinite chains in $\mathcal{F}(\gamma)$, with its application to the one-variable fragment of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$.

Theorem 4.1 *(i) $\mathcal{F}(\gamma)$ is infinite.*

(ii) There are infinitely many provably non-equivalent formulas of one-variable in the system $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$.

Proof:

(i) Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be the sequences of elements of P, as given above. Then, clearly,

$$[x_{n+1}]_{\equiv} \leq' [x_n]_{\equiv} \quad and \quad [y_{n+1}]_{\equiv} \leq' [y_n]_{\equiv}, \quad for \ \ all \ \ n \in \mathbf{N}.$$

Now by lemma 3.8, we may conclude that $x_{n+1} \not\equiv x_n$ and $y_{n+1} \not\equiv y_n$, thus proving the existence of two strictly decreasing infinite chains in $\mathcal{F}(\gamma)$.

(ii) Let p be a single propositional variable in the language of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. Take now the monoidal $\mathbf{CPL}_2^a$-model $\mathbf{M} = \langle\mathcal{F}(\gamma), [\![\ .\ ]\!]\rangle$ and put $[\![p]\!] = \gamma$. Since $\mathbf{M}$ is the Lindenbaum model of the fragment considered, we are done by (i) of the present theorem.

$\diamond$

Figure 1, illustrates a part of $\mathcal{F}(\gamma)$ with the first two elements of each of the two strictly decreasing infinite chains: $x_0 = 2\gamma$, $x_1 = \gamma + (\gamma \star 2\gamma)$, $y_0 = \gamma \star 2\gamma$ and $y_1 = \gamma \star (\gamma + (\gamma \star 2\gamma))$. Moreover, for any $\star$-$+$-polynomial p of γ, $2p$ and p^2 stand for $p+p$ and $p \star p$ respectively. From now on, we will use these abbreviations whenever convenient.

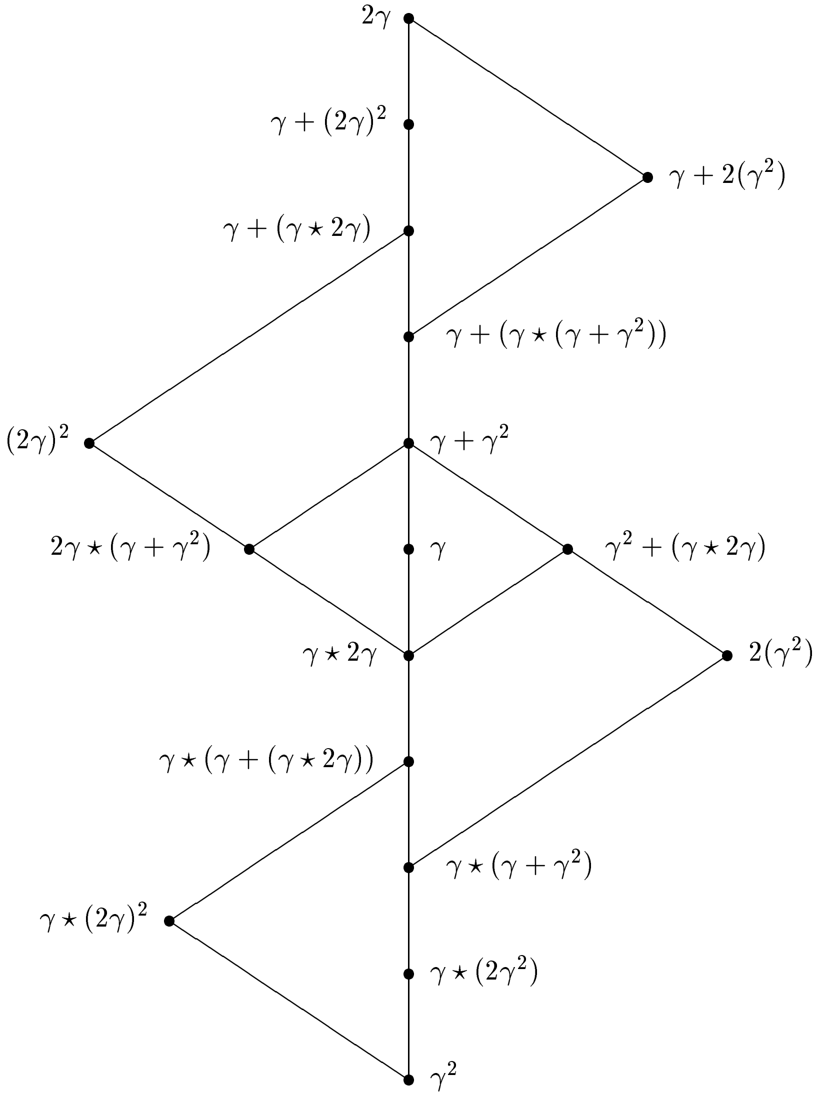

Figure 1: A part of $\mathcal{F}(\gamma)$.

Explicit constructions of the two infinite sequences $\{F_n\}_{n=0}^{\infty}$ and $\{G_n\}_{n=0}^{\infty}$ of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$ provably non-equivalent formulas of one-variable can be obtained by analogy with the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ given above. To be specific:

$$F_0 = p + p \ , \ G_0 = p \star (p+p) \ , \ F_{n+1} = G_n + p, \ , \ G_{n+1} = F_{n+1} \star p,$$

where p is the single propositional variable in the language of $\langle\star,+\rangle$-$\mathbf{CPL}_2^a$. Finally, observe that the respective sets of derivable sequents (i.e. theorems) in $\mathbf{CPL}_2^a$, $\mathbf{CPL}_n^a$, $\mathbf{CPL}_n$, $(n \geq 2)$, $\mathbf{CPL}^a$, and $\mathbf{CPL}$ are in the following relation:

$$Th(\mathbf{CPL}_2^a) \supseteq Th(\mathbf{CPL}_n^a) \supseteq Th(\mathbf{CPL}_n) \supseteq Th(\mathbf{CPL}),$$

$$\text{and also,} \quad Th(\mathbf{CPL}_2^a) \supseteq Th(\mathbf{CPL}^a).$$

This fact, clearly, remains valid for purely multiplicative fragments of the systems considered. Thus, the two infinite sequences of formulas $\{F_n\}_{n=0}^{\infty}$ and $\{G_n\}_{n=0}^{\infty}$ are also provably non-equivalent in the multiplicative fragment of any of the systems $\mathbf{CPL}_n^a$, $\mathbf{CPL}_n$, $(n \geq 2)$, $\mathbf{CPL}^a$ and $\mathbf{CPL}$.
We can now sum up the results obtained in this section in the following

Corollary 4.2 *There are infinitely many provably non-equivalent formulas of one-variable in any of the systems:* $\langle\star,+\rangle$- $\mathbf{CPL}_n^a$, $\langle\star,+\rangle$-$\mathbf{CPL}_n$ $(n \geq 2)$, $\langle\star,+\rangle$-$\mathbf{CPL}^a$ *and* $\langle\star,+\rangle$- $\mathbf{CPL}$.

We emphasize, however, that the corollary above presents a non-trivial result only with respect to the systems with n-contraction. The systems $\mathbf{CPL}^a$ and $\mathbf{CPL}$, as well as their respective multiplicative fragments, all enjoy cut-elimination (as opposed to the systems with n-contraction). Therefore, for these systems the statement of corollary 4.2 can be obtained very easily by purely syntactic reasoning.

A natural question, arising at this point, is whether the results, obtained so far, can be extended to the systems which include also constants, negation and additive connectives. We shall now briefly comment on this problem.

Consider first the system $\mathbf{CPL}_2^a$ (see appendix (A.2)) in the absence of additive connectives. We claim that a monoidal $\mathbf{CPL}_2^a$-algebra, extended with 0, 1 and $\sim$ satisfying, for any $x \in X$, the clauses:

(I)(i) $x \star 1 = x$ and $x + 0 = x$;

(I)(ii) $\sim\sim x = x$;

(I)(iii) $x \star \sim x = 0$ and $x + \sim x = 1$.

is a partially ordered structure satisfying all clauses of the definition of a $\mathbf{CPL}_2^a$-algebra with exception of (2) (see appendix (A.3)). We refer to this algebraic structure, $\mathbf{X} = \langle X, \sim, \star, +, \leq, 0, 1\rangle$, as $\langle \sim, \star, +\rangle$-$\mathbf{CPL}_2^a$-algebra. To verify our claim, it only remains to be shown that clause (5) of the definition of a $\mathbf{CPL}_2^a$-algebra (i.e. adjointness) is derivable in a $\langle \sim, \star, +\rangle$-$\mathbf{CPL}_2^a$-algebra. We proceed as follows.

(a) Assume $x \star y \leq z$, for some $x, y, z \in X$. Then, $x = x \star 1 = x \star (y + \sim y) \leq (x \star y) + \sim y \leq z + \sim y = \sim y + z$, due to (I)(i), (I)(iii), commutativity of $\star$, (II)(iii), assumption (a), (II)(iv), commutativity of $+$ and transitivity of $\leq$ respectively.

(b) Assume $x \leq \sim y + z$, for some $x, y, z \in X$. Then, $x \star y \leq (\sim y + z) \star y \leq (\sim y \star y) + z = 0 + z = z$, due to assumption (b), (II)(iv), (II)(iii), commutativity of $\star$, (I)(iii), (I)(i) and transitivity of $\leq$ respectively.

We shall prove below that adjointness is, in fact, equivalent to sub-commutativity of $\star$ and $+$ in the algebra considered. Given $x, y \in X$, $y \star x \leq y \star x$ by reflexivity of $\leq$. Due to adjointness, we get $y \leq \sim x + (y \star x)$. Thus, for any $z \in X$, $y + z \leq \sim x + ((y \star x) + z)$, due to (II)(iv) and associativity of $+$. Using adjointness, once again, gives $(y + z) \star x \leq (y \star x) + z$, and we are done.

Due to the presence of involution in $\langle \sim, \star, +\rangle$- $\mathbf{CPL}_2^a$-algebras a form of de Morgan duality can be expressed in them. Since this fact will be used in the course of the next section we prove it below.

Lemma 4.3 *In any $\langle \sim, \star, +\rangle$-$\mathbf{CPL}_2^a$-algebra $\mathbf{X} = \langle X, \sim, \star, +, \leq, 0, 1\rangle$ the following pairs of operators are dual to each other: $(\sim, \sim)$, $(\star, +)$ and $(0, 1)$.*

Proof: Clearly $\sim$ is dual to itself by (I)(ii).
Next, we want to show, using anti-symmetry of $\leq$, that

$$\sim (x \star y) = (\sim x) + (\sim y).$$

(1) $\sim (x \star y) \leq (\sim x) + (\sim y)$ iff $1 \leq \sim\sim (x \star y) + (\sim x) + (\sim y) = ((x \star y) + (\sim x)) + \sim y$, due to commutativity of $\star$, (I)(i), adjointness, (I)(ii) and associativity of $+$. Indeed, $((x \star y) + \sim x) + \sim y \geq ((x + \sim x) \star y) + \sim y = (1 \star y) + \sim y = y + \sim y = 1$, by (II)(iii), (I)(iii), commutativity of $\star$, (I)(i)

and (I)(iii).

(2) $(\sim x) + (\sim y) \leq\sim (x \star y)$ can be proved analogously, and so we are done. The duality of 0 and 1 is now a trivial matter. $\diamond$

One can extend the given construction of the free monoidal algebra $\mathcal{F}(\gamma)$ with the respective clauses for 0, 1 and involution. However, in this case fact 3.1 fails, due to each of the additional clauses I(i)-I(iii) in the algebra considered. Consequently, all lemma's and propositions, established for proving the existence of two infinite chains in $\mathcal{F}(\gamma)$, fail too. This means that our present strategy, to prove the existence of the two infinite chains in the free algebra considered, is not directly applicable to any of the extended systems discussed above. Retaining positive attitude, in spite of that, one should search for a suitable conservativity theorem. However, due to non-eliminability of cut in the underlying syntactic systems, we are again missing the proof-theoretic techniques to achieve that. Thus, it seems more promising to find a faithful embedding of the free monoidal $\mathbf{CPL}_2^a$-algebra in the free algebra corresponding to the system $\mathbf{CPL}_2^a$. However, this still remains to be achieved.

We conclude this section with the conjecture that the one-variable fragment of the system $\mathbf{CPL}_2^a$ (and consequently, of any weaker system in the sense of corollary 4.2) is infinite.

5 A computation of the free algebra on one generator corresponding to the one-variable fragment of $\langle\sim,\star,+\rangle$-$\mathbf{CPL}_2^{a\,>}$

In this section, we give an example of a computation of the free algebra on one generator corresponding to the one-variable fragment of the system $\langle\sim,\star,+\rangle$-$\mathbf{CPL}_2^{a\,>}$ discussed below. We show, in what follows that the free algebra mentioned has exactly twelve elements.

Firstly, consider the system obtained from $\mathbf{CPL}_2^a$ (see appendix (A.2)) by omitting rules for additive connective but adding the following two axiom-schemes:

$$A \Rightarrow A \star (A + A) \;\text{ and }\; A + A \Rightarrow (A + A) \star (A + A).$$

We shall refer to this system as $\langle\sim,\star,+\rangle$-$\mathbf{CPL}_2^{a\,>}$.

Later on, it will become clear that essentially the two additional axioms above force the Lindenbaum algebra of the one-variable fragment considered to be finite, more precisely, to have exactly twelve elements.

By duality the following sequents are derivable in the system just introduced:

$$\vdash A + (A \star A) \Rightarrow A$$

and

$$\vdash (A \star A) + (A \star A) \Rightarrow A \star A.$$

Due to the obvious derivability of the converse of each of these sequents the following formulas are provably equivalent in $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^{a\,>}$:
$A \Longleftrightarrow A \star (A + A)$, $A + A \Longleftrightarrow (A + A) \star (A + A)$, $A + (A \star A) \Longleftrightarrow A$ and $(A \star A) + (A \star A) \Longleftrightarrow A \star A$, for any formula A.

Next, we introduce $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^{a\,>}$-algebras, $\mathbf{X} = \langle X, \sim, \star, +, \leq, 0, 1\rangle$, corresponding to the system under consideration. Clearly, this is just a $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^a$-algebra, introduced at the end of the previous section, extended by an additional clause (corresponding to the two new axioms):

(0) $x \leq x \star (x + x)$ and $x + x \leq (x + x) \star (x + x)$, for any $x \in X$.

The soundness and the completeness theorem for the system and models considered here can be established by a Lindenbaum construction.

The next lemma is crucial for our purpose, i.e. to compute the free algebra on one generator.

Lemma 5.1 *In any $\langle\sim, \star, +\rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra $\mathbf{X} = \langle X, \sim, \star, +, \leq, 0, 1\rangle$ the following clauses are satisfied for any $x, y \in X$:*

(ne0) $(x + y) \star \sim x \leq y$ *and* $y \leq (x \star y) + \sim x$;

(ne1) $0 \leq x \leq 1$;

(e2) $x \star 0 = 0$ *and* $x + 1 = 1$;

(e3) $x \star (x + x) = x$ *and* $x + (x \star x) = x$;

(e4) $(x + x)^2 = x + x$ *and* $x^2 + x^2 = x^2$;

(e5) $x \star x \star x = x \star x$ *and* $x + x + x = x + x$;

(ne6) *if* $x \star x \leq y$, *then* $x \star x \star y = x \star x$
and
if $y \leq x + x$, *then* $x + x + y = x + x$;

(e7) $((\sim x)+(\sim x))\star x=(x+x)\star\sim x$
and
$(x\star x)+\sim x=((\sim x)\star(\sim x))+x$;

(e8) $(x^2+(\sim x)^2)^2=x^2+(\sim x)^2$
and
$2((2x)\star 2(\sim x))=(2x)\star 2(\sim x)$.

Proof: Note, first, that clauses (ne0), (ne1) and (ne6) of the lemma are not in equational form. In fact, these clauses are just needed, together with the clauses of definition of a $\langle\sim,\star,+\rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra, to verify some of the equational clauses above, and in the sequel. Due to lemma 4.3, it is enough to show just one of the dual statements in each of the clauses above.

(ne0) follows from (II)(iii), (I)(iii), commutativity of $+$, (I)(i) and transitivity of $\leq$;

(ne1) is due to (II)(i), commutativity of $+$ (commutativity of $\star$) and (I)(i);

(e2) is due to commutativity of $\star$, (II)(i), (ne1) above and anti-symmetry of $\leq$;

(e3) follows from (0), (II)(i) and anti-symmetry of $\leq$;

(e4) can be obtained by (0), (II)(i) and anti-symmetry of $\leq$;

(e5) is due to associativity of $\star$, (II)(i), (II)(ii) and anti-symmetry of $\leq$;

(ne6) To verify this clause, assume $x\star x\leq y$ for some $x,y\in X$. Then we get $x\star x=x\star x\star x\star x\leq x\star x\star y\leq x\star x$, due to associativity of $\star$, (e5) above (used twice), assumption, (II)(iv), (II)(i) and anti-symmetry of $\leq$;

(e7) is obtained as follows:
$((\sim x)+(\sim x))\star x=((\sim x)+(\sim x))\star x\star(x+x)\leq(\sim x)\star(x+x)=(x+x)\star\sim x$, due to (e3), (ne0) above, and commutativity of $\star$. Analogously, one can establish the inequality in the other direction, and hence, by anti-symmetry of $\leq$, we are done.

(e8) can be obtained as follows:
first, due to (e4) above we have $x^2+x^2=x^2$ and $(\sim x)^2+(\sim x)^2=(\sim x)^2$. Therefore, $(x^2+(\sim x)^2)^2=((x^2+(\sim x)^2)+(x^2+(\sim x)^2))^2=$

$= (x^2 + (\sim x)^2) + (x^2 + (\sim x)^2) = x^2 + (\sim x)^2$, due to associativity and commutativity of $+$ and (e4) above.
This completes the proof.

$\diamond$

We are now ready to compute the free $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra on one generator corresponding to the one-variable fragment of $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$. For this purpose we proceed as follows.

First, we take an arbitrary object γ. Then, we generate $\sim$-$\star$- $+$-polynomials of γ inductively on the number of occurrences of operations in them. By using clauses of the definition of a $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra and of lemma 5.1, it is easy to check (as in the proof above) that each of $\sim$-$\star$-$+$-polynomials of γ is equal (in any $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra) to one of the twelve representatives, listed below:

$$\gamma,\ \ \gamma^2,\ \ 2\gamma,\ \ \sim \gamma,\ \ 0(= \gamma \star \sim \gamma)\ \ 1(= \gamma + \sim \gamma)\ \ (\sim \gamma)^2,\ \ 2(\sim \gamma),$$

$$2(\gamma) \star \sim \gamma,\ \ (\sim \gamma)^2 + \gamma,\ \ 2(\gamma) \star 2(\sim \gamma),\ \ \gamma^2 + (\sim \gamma)^2.$$

As a by-product we get that these $\sim$-$\star$-$+$-polynomials form the partially ordered structure, illustrated in figure 2.

Let now P be the set of all $\sim$-$\star$-$+$-polynomials of γ, and hence, $P/=$ the set of the corresponding twelve equivalence classes. Then it is straightforward to check (consult figure 2) that $\mathcal{F}_{12}(\gamma) = \langle P/=, \sim', \star', +' \leq' \rangle$, indeed, is a $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra, establishing the following

Theorem 5.2 *$\mathcal{F}_{12}(\gamma)$ is the free $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra on one generator γ.*

Note that each of the two infinite chains, constructed in section 3, in this case, collapses into:

$$x_n = \gamma + \gamma \quad \text{and} \quad y_n = \gamma, \quad \text{for all} \quad n \in \mathbf{N}.$$

Thus, the free algebra on γ corresponding to the positive one-variable fragment of $\langle \sim, \star, + \rangle$-$\mathbf{CPL}_2^{a\,>}$ is just the 3-element chain: $\gamma^2 \leq \gamma \leq 2\gamma$. Actually, this is due to the two additional axioms in the system and in the algebra considered.

Going back to the topic of this section, we shall state an immediate consequence of theorem 5.2.

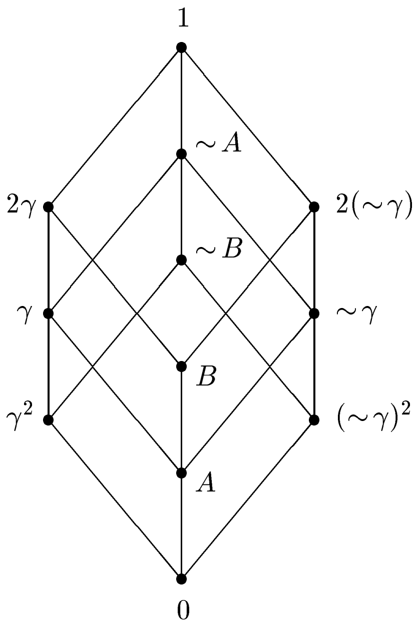

Figure 2: $\mathcal{F}_{12}(\gamma)$, with A, $\sim A$, B and $\sim B$ denoting $2(\gamma)\star \sim \gamma$, $(\sim \gamma)^2 + \gamma$, $2(\gamma) \star 2(\sim \gamma)$, and $\gamma^2 + (\sim \gamma)^2$ respectively.

Corollary 5.3 *The Lindenbaum algebra of the one-variable fragment of* $\langle\sim,\star,+\rangle$-$\mathbf{CPL}_2^{a\,>}$ *has exactly twelve elements.*

Further, note that the free $\langle\sim,\star,+\rangle$-$\mathbf{CPL}_2^{a\,>}$-algebra, $\mathcal{F}_{12}(\gamma)$, already has a built in lattice structure. Moreover, one can read off the given structure (see figure 2) that the following dual equalities are imposed on meets ($\sqcap$) and joins ($\sqcup$) of any elements $x, y \in P/=$:

$$x \sqcap y = (x + \sim y) \star y \quad \text{and,} \quad x \sqcup y = (x \star \sim y) + y.$$

Again, a straightforward checking shows that the lattice structure considered is compatible with the given monoidal structure. That is to say, distributivity of $\star$ and $+$ over $\sqcup$ and $\sqcap$ respectively, indeed, is satisfied in the free algebra on γ extended with a lattice structure. In what follows we shall refer to it as $\mathcal{F}_{12}^{\diamond}(\gamma) = \langle P^{\diamond}/=, \sim', \star', +', \sqcap', \sqcup', 0, 1\rangle$. Moreover, it is easy to see that $\mathcal{F}_{12}^{\diamond}(\gamma)$ corresponds to the one-variable fragment of the full system $\mathbf{CPL}_2^a$ (see appendix (A.2)) extended with, the two axioms specified at the very beginning of this section and, the following two in addition:

$$A \sqcap B \Longleftrightarrow (A + \sim B) \star B \quad \text{and} \quad A \sqcup B \Longleftrightarrow (A \star \sim B) + B.$$

We shall call this system $\mathbf{CPL}_2^{a\,\diamond}$.

The story above leads to one more

Corollary 5.4 *The Lindenbaum algebra of the one-variable fragment of* $\mathbf{CPL}_2^{a\,\diamond}$ *has exactly twelve elements.*

To conclude this paper we give one more remark. Clearly, explicit constructions of the twelve elements of the Lindenbaum algebra of the one-variable fragment of $\mathbf{CPL}_2^{a\,\diamond}$ can be obtained by analogy with the elements of $P^{\diamond}/=$. It is interesting to note that precisely these elements (i.e. classes of one-variable formulas) can be distinguished by the 3-valued Łukasiewics models (see appendix (A.4)).

A Appendices

A.1 Systems of affine intuitionistic linear logic with n-contraction

For any $n \geq 2$, an intuitionistic system of affine propositional linear logic with n-contraction, $\mathbf{IPL}_n^a$, is given by the following axioms and rules. Throughout the below, Λ denotes the empty multiset, Φ denotes either an occurrence of an $\mathbf{IPL}_n^a$-formula or the empty multiset, and $\Gamma, \Gamma_1, \Gamma_2$ stand for finite multisets of $\mathbf{IPL}_n^a$-formulas.

Axioms

$$A \Rightarrow A \qquad 0 \Rightarrow \Lambda$$

Logical rules

$$L\star \ \frac{\Gamma, A, B \Rightarrow \Phi}{\Gamma, A \star B \Rightarrow \Phi} \qquad\qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \star B} \ R\star$$

$$L \multimap \ \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2, B \Rightarrow \Phi}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Phi} \qquad\qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \ R \multimap$$

$$L\sqcap \ \frac{\Gamma, A_i \Rightarrow \Phi}{\Gamma, A_1 \sqcap A_2 \Rightarrow \Phi} \ (i = 1, 2) \qquad\qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B} \ R\sqcap$$

$$L\sqcup \ \frac{\Gamma, A \Rightarrow \Phi \qquad \Gamma, B \Rightarrow \Phi}{\Gamma, A \sqcup B \Rightarrow \Phi} \qquad\qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \sqcup A_2} \ (i = 1, 2) \ R\sqcup$$

Structural rules

$$LW \ \frac{\Gamma \Rightarrow \Phi}{\Gamma, A \Rightarrow \Phi} \qquad\qquad \frac{\Gamma \Rightarrow \Lambda}{\Gamma \Rightarrow A} \ RW$$

$$LC_n \ \frac{\Gamma, A^{(n+1)} \Rightarrow \Phi}{\Gamma, A^{(n)} \Rightarrow \Phi}$$

where $A^{(k)} \equiv A, A, \ldots, A$, i.e. k copies of formula A.

$$CUT \quad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2, A \Rightarrow \Phi}{\Gamma_1, \Gamma_2 \Rightarrow \Phi}$$

Remark: A non-involutive negation can be defined by $\sim A = A \multimap 0$.

A.2 Systems of affine classical linear logic with n-contraction

For any $n \geq 2$, a classical system of affine propositional linear logic with n-contraction, $\mathbf{CPL}_n^a$, is given by the following axioms and rules. Throughout the below, Λ denotes the empty multiset and Γ, Γ_1, Γ_2, Δ, Δ_1, Δ_2 stand for finite multisets of $\mathbf{CPL}_n^a$-formulas.

Axioms

$$A \Rightarrow A \qquad 0 \Rightarrow \Lambda \qquad \Lambda \Rightarrow 1$$

Logical rules

$$L\sim \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \sim A \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \; \sim A, \Delta} \quad R\sim$$

$$L\star \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \star B \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \star B, \Delta_1, \Delta_2} \quad R\star$$

$$L+ \quad \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A + B \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A + B, \Delta} \quad R+$$

$$L\sqcap \quad \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_1 \sqcap A_2 \Rightarrow \Delta} \quad (i = 1, 2) \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \sqcap B, \Delta} \quad R\sqcap$$

$$L\sqcup \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \sqcup B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_1 \sqcup A_2, \Delta} \quad (i = 1, 2) \quad R\sqcup$$

Structural rules

$$LW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \quad RW$$

$$LC_n \quad \frac{\Gamma, A^{(n+1)} \Rightarrow \Delta}{\Gamma, A^{(n)} \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow A^{(n+1)}, \Delta}{\Gamma \Rightarrow A^{(n)}, \Delta} \quad RC_n$$

where $A^{(k)} \equiv A, A, \ldots, A$, i.e. k copies of formula A.

$$CUT \quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Remark: A linear implication can be defined by $A \multimap B = \sim A + B$.

A.3 Algebraic models for $\mathbf{CPL}_n^a$

Definition A.3.1 $\mathbf{X} = \langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1\rangle$ *is a* $\mathbf{CPL}_n^a$*-algebra, if:*

(1) $\langle X, \star, 1\rangle$ *and* $\langle X, +, 0\rangle$ *are commutative monoids with units* 1 *and* 0 *respectively;*

(2) $\langle X, \sqcap, \sqcup, 0, 1\rangle$ *is a lattice with bottom* 0 *and top* 1*;*

(3) $\sim$ *is involution, i.e.* $\sim\sim x = x$ *for all* $x \in X$*;*

(4) $\star$ *and* $+$ *are monotone with respect to the lattice order* $\leq$*, i.e. for all* $x, y, z \in X$*, if* $x \leq y$*, then* $x \star z \leq y \star z$ *and* $x + z \leq y + z$*;*

(5) for all $x, y, z \in X$*,* $x \star y \leq z$ *if and only if* $x \leq \sim y + z$*;*

(6) for all $x \in X$*,* $x^n \leq x^{n+1}$ *and* $(n+1)x \leq nx$*, where* $x^k = x \star \cdots \star x$ *and* $kx = x + \cdots + x$ *with* k *copies of* x *respectively.*

Remark: Note that, a $\mathbf{CPL}_n^a$-algebra is just a classical linear algebra (provided$\multimap$ is taken as primitive while $\top$, $\sim$ and $+$ are defined in a usual way, see Troelstra [10]), satisfying in addition:

- $\bot = 0$ and $\top = 1$ (corresponding to weakening);
- clause (6) (corresponding to n-contraction).

Definition A.3.2 $\mathbf{M} = \langle \mathbf{X}, [\![\ .\ ]\!] \rangle$ *is a* $\mathbf{CPL}_n^a$*-model, if:*

(1) $\mathbf{X}$ *is a* $\mathbf{CPL}_n^a$*-algebra;*

(2) $[\![\ .\ ]\!]$ *is a valuation satisfying:*

(i) $[\![P]\!] \in X$*, for every propositional variable* P*;*

(ii) $[\![0]\!] = 0$*,* $[\![1]\!] = 1$*;*

$[\![\ .\ ]\!]$ *is extended to* $\mathbf{CPL}_n^a$*-formulas inductively by:*
$[\![\sim A]\!] = \sim [\![A]\!]$ *and* $[\![A \bullet B]\!] = [\![A]\!] \bullet [\![B]\!]$*, with* $\bullet \in \{\star, +, \sqcap, \sqcup\}$*;*
Moreover, a $\mathbf{CPL}_n^a$*-sequent* $A_1, \ldots, A_k \Rightarrow B_1, \ldots, B_m$ *is valid in* $\mathbf{M}$ *if and only if* $[\![A_1]\!] \star \cdots \star [\![A_k]\!] \leq [\![B_1]\!] + \cdots + [\![B_m]\!]$.

A.4 Models for n-valued Łukasiewicz logics

Definition A.4.1 $\mathbf{M}_n([\![\ .\ ]\!])$*, a model for* n*-valued Łukasiewicz logic, consists of:*

(1) a valuation $[\![\ .\ ]\!]$ *assigns to each propositional variable* p *an element of the set* $S_n = \{\frac{k}{n-1} \mid k = 0, 1, \ldots, n-1\}$*;*

(2) $[\![\ .\ ]\!]$ *is extended to arbitrary formulas (in the language* $\{\sim, \sqcap, \sqcup, \star, +, \multimap\}$*) inductively by:*

(i) $[\![\sim A]\!] = 1 - [\![A]\!]$*;*

(ii) $[\![A \sqcap B]\!] = \min\{[\![A]\!], [\![B]\!]\}$*;*

(iii) $[\![A \sqcup B]\!] = \max\{[\![A]\!], [\![B]\!]\}$*;*

(iv) $[\![A \star B]\!] = \max\{[\![A]\!] + [\![B]\!] - 1, 0\}$*;*

(v) $[\![A + B]\!] = \min\{[\![A]\!] + [\![B]\!], 1\}$*;*

(vi) $[\![A \multimap B]\!] = \min\{1 - [\![A]\!] + [\![B]\!], 1\}$.

(3) $[\![\ .\ ]\!]$ *is extended to arbitrary sequent* $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_j$ *by:*

$$[\![A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_j]\!] = [\![\sim A_1 + \cdots + \sim A_m + B_1 + \cdots + B_j]\!].$$

A given sequent $\Gamma \Rightarrow \Delta$ *is valid in* $\mathbf{M}_3([\![\ .\ ]\!])$ *if and only if* $[\![\Gamma \Rightarrow \Delta]\!] = 1$*. Moreover, a sequent* $\Gamma \Rightarrow \Delta$ *is* 3*-valid if and only if* $[\![\Gamma \Rightarrow \Delta]\!] = 1$ *for every valuation* $[\![\ .\ ]\!]$.

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