On a Spector ultrapower of the Solovay model*

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Abstract

We prove that a Spector–like ultrapower extension $\mathfrak{N}$ of a countable Solovay model $\mathfrak{M}$ (where all sets of reals are Lebesgue measurable) is equal to the set of all sets constructible from reals in a generic extension $\mathfrak{M}[\alpha]$ where α is a random real over $\mathfrak{M}$. The proof involves an almost everywhere uniformization theorem in the Solovay model.

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Introduction

Let $\mathfrak{U}$ be an ultrafilter in a transitive model $\mathfrak{M}$ of **ZF**. Assume that an ultrapower of $\mathfrak{M}$ via $\mathfrak{U}$ is to be defined. The first problem we meet is that $\mathfrak{U}$ may not be an ultrafilter in the universe because not all subsets of the index set belong to $\mathfrak{M}$.

We can, of course, extend $\mathfrak{U}$ to a true ultrafilter, say $\mathfrak{U}'$, but this may cause additional trouble. Indeed, if $\mathfrak{U}$ is a special ultrafilter in $\mathfrak{M}$ certain properties of which were expected to be exploit, then most probably these properties do not transfer to $\mathfrak{U}'$; assume for instance that $\mathfrak{U}$ is countably complete in $\mathfrak{M}$ and $\mathfrak{M}$ itself is countable. Therefore, it is better to keep $\mathfrak{U}$ rather than any of its extensions in the universe, as the ultrafilter.

If $\mathfrak{M}$ models **ZFC**, the problem can be solved by taking the *inner* ultrapower. In other words, we consider only those functions $f : I \longrightarrow \mathfrak{M}$ (where $I \in \mathfrak{M}$ is the carrier of $\mathfrak{U}$) which belong to $\mathfrak{M}$ rather than *all* functions $f \in \mathfrak{M}^I$, to define the ultrapower. This version, however, depends on the axiom of choice in $\mathfrak{M}$; otherwise the proofs of the basic facts about ultrapowers (e. g. Łoś' theorem) will not work.

The "choiceless" case can be handled by a sophisticated construction of SPECTOR [1991], which is based on ideas from both forcing and the ultrapower technique. As presented in KANOVEI and VAN LAMBALGEN [1994], this construction proceeds as follows. One has to add to the family of functions $\mathcal{F}_0 = \mathfrak{M}^I \cap \mathfrak{M}$ a number of new functions $f \in \mathfrak{M}^I$, $f \notin \mathfrak{M}$, which are intended to be choice functions whenever we need such in the ultrapower construction.

In this paper, we consider a very interesting choiceless case: $\mathfrak{M}$ is a Solovay model of **ZF** plus the principle of dependent choice, in which all sets of reals are Lebesque measurable, and the ultrafilter $\mathfrak{L}$ on the set I of Vitali degrees of reals in $\mathfrak{M}$, generated by sets of positive measure.

1 On a.e. uniformization in the Solovay model

In this section, we recall the uniformization properties in a Solovay model. Thus let $\mathfrak{M}$ be a countable transitive Solovay model for Dependent Choices plus "all sets are Lebesgue measurable", as it is defined in SOLOVAY [1970], – the *ground model*. The following known properties of such a model will be of particular interest below.

Property 1 [True in $\mathfrak{M}$]
$\mathbb{V} = \mathbb{L}(\text{reals})$; *in particular every set is real–ordinal–definable.* □

To state the second property, we need to introduce some notation.

Let $\mathcal{N} = \omega^\omega$ denote the Baire space, the elements of which will be referred to as *real numbers* or *reals*..

Let P be a set of pairs such that $\operatorname{dom} P \subseteq \mathcal{N}$ (for instance, $P \subseteq \mathcal{N}^2$). We say that a function f defined on $\mathcal{N}$ *uniformizes* P *a.e.* (almost everywhere) iff the set

$$\{\alpha \in \operatorname{dom} P : \langle \alpha, f(\alpha) \rangle \notin P\}$$

has null measure. For example if the projection $\operatorname{dom} P$ is a set of null measure in $\mathcal{N}$ then any f uniformizes a.e. P, but this case is not interesting. The interesting case is the case when $\operatorname{dom} P$ is a set of full measure, and then f a.e. uniformizes P iff for almost all α, $\langle \alpha, f(\alpha) \rangle \in P_\alpha$.

Property 2 [True in $\mathfrak{M}$]
Any set $P \in \mathfrak{M}$, $P \subseteq \mathcal{N}^2$, *can be uniformized a.e. by a Borel function.* (This implies the Lebesgue measurability of all sets of reals, which is known to be true in $\mathfrak{M}$ independently.) □

This property can be expanded (with the loss of the condition that f is Borel) on the sets P which do not necessarily satisfy $\operatorname{dom} P \subseteq \mathcal{N}$.

Theorem 3 *In* $\mathfrak{M}$, *any set* P *with* $\operatorname{dom} P \subseteq \mathfrak{M}$ *admits an a.e. uniformisation.*

Proof Let P be an arbitrary set of pairs such that $\operatorname{dom} P \subseteq \mathcal{N}$ in $\mathfrak{M}$. Property 1 implies the existence of a function $D : (\mathrm{Ord} \cap \mathfrak{M}) \times (\mathcal{N} \cap \mathfrak{M})$ onto $\mathfrak{M}$ which is $\in$-definable in $\mathfrak{M}$.

We argue in $\mathfrak{M}$. Let, for $\alpha \in \mathcal{N}$, $\xi(\alpha)$ denote the least ordinal ξ such that

$$\exists \gamma \in \mathcal{N} \, [\langle \alpha, D(\xi, \gamma) \rangle \in P].$$

(It follows from the choice of D that $\xi(\alpha)$ is well defined for all $\alpha \in \mathcal{N}$.) It remains to apply Property 2 to the set $P' = \{\langle \alpha, \gamma \rangle \in \mathcal{N}^2 : \langle \alpha, D(\xi(\alpha), \gamma) \rangle \in P\}$. □

2 The functions to get the Spector ultrapower

We use a certain ultrafilter over the set of Vitali degrees of reals in $\mathfrak{M}$, the initial Solovay model, to define the ultrapower.

Let, for $\alpha, \alpha' \in \mathcal{N}$, $\alpha \mathtt{\ vit\ } \alpha'$ if and only if $\exists\, m \,\forall\, k \ge m\ (\alpha(k) = \alpha'(k))$, (the *Vitali equivalence*).

- For $\alpha \in \mathcal{N}$, we set $\underline{\alpha} = \{\alpha' : \alpha' \mathtt{\ vit\ } \alpha\}$, the *Vitali degree* of α.
- $\underline{\mathcal{N}} = \{\underline{\alpha} : \alpha \in \mathcal{N}\}$; i, j denote elements of $\underline{\mathcal{N}}$.

As a rule, we shall use *underlined* characters $\underline{f}$, $\underline{F}$, ... to denote functions defined on $\underline{\mathcal{N}}$, while functions defined on $\mathcal{N}$ itself will be denoted in the usual manner.

Define, in $\mathfrak{M}$, an ultrafilter $\mathcal{L}$ over $\underline{\mathcal{N}}$ by: $\underline{X} \subseteq \underline{\mathcal{N}}$ belongs to $\mathcal{L}$ iff the set $X = \{\alpha \in \mathcal{N} : \underline{\alpha} \in \underline{X}\}$ has full Lebesgue measure. It is known (see e.g. van Lambalgen [1992], Theorem 2.3) that the measurability hypothesis implies that $\mathcal{L}$ is κ-complete in $\mathfrak{M}$ for all cardinals κ in $\mathfrak{M}$.

One cannot hope to define a good $\mathcal{L}$-ultrapower of $\mathfrak{M}$ using only functions from $\mathcal{F}_0 = \{\underline{f} \in \mathfrak{M} : \operatorname{dom}\underline{f} = \underline{\mathcal{N}}\}$ as the base for the ultrapower. Indeed consider the identity function $\mathfrak{i} \in \mathfrak{M}$ defined by $\mathfrak{i}(i) = i$ for all $i \in \underline{\mathcal{N}}$. Then $\mathfrak{i}(i)$ is nonempty for all $i \in \underline{\mathcal{N}}$ in $\mathfrak{M}$, therefore to keep the usual properties of ultrapowers we need a function $\underline{f} \in \mathcal{F}_0$ such that $\underline{f}(i) \in i$ for almost all $i \in \underline{\mathcal{N}}$, but Vitali showed that such a choice function yields a nonmeasurable set.

Thus at least we have to add to $\mathcal{F}_0$ a new function $\underline{f}$, not an element of $\mathfrak{M}$, which satisfies $\underline{f}(i) \in i$ for almost all $i \in \underline{\mathcal{N}}$. Actually it seems likely that we have to add a lot of new functions, to handle similar situations, including those functions the existence of which is somehow implied by the already added functions. A general way how to do this, extracted from the exposition in Spector [1991], was presented in Kanovei and van Lambalgen [1994]. However in the case of the Solovay model the a.e. uniformization theorem (Theorem 3) allows to add essentially a single new function, corresponding to the $\mathfrak{i}$-case considered above.

The generic choice function for the identity

Here we introduce a function $\mathfrak{r}$ defined on $\underline{\mathcal{N}} \cap \mathfrak{M}$ and satisfying $\mathfrak{r}(i) \in i$ for all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$. $\mathfrak{r}$ will be generic over $\mathfrak{M}$ for a suitable notion of forcing.

The notion of forcing is introduced as follows. In $\mathfrak{M}$, let $\mathbb{P}$ be the set of all functions p defined on $\underline{\mathcal{N}}$ and satisfying $p(i) \subseteq i$ and $p(i) \ne \emptyset$ for all i.[1] (For example $\mathfrak{i} \in \mathbb{P}$.) We order $\mathbb{P}$ so that p is stronger than q iff $p(i) \subseteq q(i)$ for all i. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, G defines a function $\mathfrak{r}$ by

$$\mathfrak{r}(i) = \text{the single element of } \textstyle\bigcap_{p \in G} p(i)$$

[1] Or, equivalently, the collection of all sets $X \subseteq \mathcal{N}$ which have a nonempty intersection with every Vitali degree. Perhaps this forcing is of separate interest.

for all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$. Functions $\mathfrak{r}$ defined this way will be called $\mathbb{P}$-*generic over* $\mathfrak{M}$. Let us fix such a function $\mathfrak{r}$ for the remainder of this paper.

The set of functions used to define the ultrapower

We let $\mathfrak{F}$ be the set of all superpositions $f \circ \mathfrak{r}$ where[2] $\mathfrak{r}$ is the generic function fixed above while $f \in \mathfrak{M}$ is an arbitrary function defined on $\mathcal{N} \cap \mathfrak{M}$. Notice that in particular any function $\underline{f} \in \mathfrak{M}$ defined on $\underline{\mathcal{N}} \cap \mathfrak{M}$ is in $\mathfrak{F}$: take $f(\alpha) = \underline{f}(\underline{\alpha})$.

To see that $\mathfrak{F}$ can be used successfully as the base of an ultrapower of $\mathfrak{M}$, we have to check three fundamental conditions formulated in Kanovei and van Lambalgen [1994].

Proposition 4 [Measurability] *Assume that* $E \in \mathfrak{M}$ *and* $f_1, ..., f_n \in \mathfrak{F}$. *Then the set* $\{i \in \underline{\mathcal{N}} \cap \mathfrak{M} : E(f_1(i), ..., f_n(i))\}$ *belongs to* $\mathfrak{M}$.

Proof By the definition of $\mathfrak{F}$, it suffices to prove that $\{i : \mathfrak{r}(i) \in E\} \in \mathfrak{M}$ for any set $E \in \mathfrak{M}$, $E \subseteq \mathcal{N}$. By the genericity of $\mathfrak{r}$, it remains then to prove the following in $\mathfrak{M}$: for any $p \in \mathbb{P}$ and any set $E \subseteq \mathcal{N}$, there exists a stronger condition q such that, for any i, either $q(i) \subseteq E$ or $q(i) \cap E = \emptyset$. But this is obvious. $\square$

Corollary 5 *Assume that* $V \in \mathfrak{M}$, $V \subseteq \mathcal{N}$ *is a set of null measure in* $\mathfrak{M}$. *Then, for* $\mathfrak{L}$*-almost all* i, *we have* $\mathfrak{r}(i) \notin V$.

Proof By the proposition, the set $I = \{i : \mathfrak{r}(i) \in V\}$ belongs to $\mathfrak{M}$. Suppose that, on the contrary, $I \in \mathfrak{L}$. Then $A = \{\alpha : \underline{\alpha} \in I\}$ is a set of full measure. On the other hand, since $\mathfrak{r}(i) \in i$, we have $A \subseteq \bigcup_{\beta \in V} \underline{\beta}$, where the right–hand side is a set of null measure because V is such a set, contradiction. $\square$

Proposition 6 [Choice] *Let* $f_1, ..., f_n \in \mathfrak{F}$ *and* $W \in \mathfrak{M}$. *There exists a function* $\underline{f} \in \mathfrak{F}$ *such that, for* $\mathfrak{L}$*-almost all* $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$, *it is true in* $\mathfrak{M}$ *that*

$$\exists\, x\, W(f_1(i), ..., f_n(i), x) \quad \longrightarrow \quad W(f_1(i), ..., f_n(i), \underline{f}(i)) \,.$$

Proof This can be reduced to the following: given $W \in \mathfrak{M}$, there exists a function $\underline{f} \in \mathfrak{F}$ such that, for $\mathfrak{L}$-almost all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$,

$$\exists\, x\, W(\mathfrak{r}(i), x) \quad \longrightarrow \quad W(\mathfrak{r}(i), \underline{f}(i)) \tag{$*$}$$

in $\mathfrak{M}$.

[2] To make things clear, $f \circ \mathfrak{r}(i) = f(\mathfrak{r}(i))$ for all i .

We argue in $\mathfrak{M}$. Choose $p \in \mathbb{P}$. and let $p'(i) = \{\beta \in p(i) : \exists\, x\, W(\beta, x)\}$, and $X = \{i : p'(i) \neq \emptyset\}$. If $X \notin \mathcal{L}$ then an arbitrary $\underline{f}$ defined on $\underline{\mathcal{N}}$ will satisfy $(*)$, therefore it is assumed that $X \in \mathcal{L}$. Let

$$q(i) = \begin{cases} p'(i) & \text{iff} \quad i \in X \\ p(i) & \quad\text{otherwise} \end{cases}$$

for all $i \in \underline{\mathcal{N}}$; then $q \in \mathbb{P}$ is stronger than p. Therefore, since $\mathfrak{r}$ is generic, one may assume that $\mathfrak{r}(i) \in q(i)$ for all i.

Furthermore, DC in the Solovay model $\mathfrak{M}$ implies that for every $i \in X$ the following is true: there exists a function ϕ defined on $q(i)$ and such that $W(\beta, \phi(\beta))$ for every $\beta \in q(i)$. Theorem 3 provides a function Φ such that for almost all α the following is true: the value $\Phi(\alpha, \beta)$ is defined and satisfies $W(\beta, \Phi(\alpha, \beta))$ for all $\beta \in q(\underline{\alpha})$. Then, by Corollary 5, we have

$$\text{for all } \beta \in q(\underline{\mathfrak{r}(i)}), \quad W(\beta,\, \Phi(\mathfrak{r}(i), \beta)\,)$$

for almost all i. However, $\underline{\mathfrak{r}(i)} = i$ for all i. Applying the assumption that $\mathfrak{r}(i) \in q(i)$ for all i, we obtain $W(\underline{\mathfrak{r}(i)},\, \Phi(\mathfrak{r}(i), \mathfrak{r}(i))\,)$ for almost all i. Finally the function $\underline{f}(i) = \Phi(\mathfrak{r}(i), \mathfrak{r}(i))$ is in $\mathcal{F}$ by definition. $\square$

Proposition 7 [Regularity] *For any* $\underline{f} \in \mathcal{F}$ *there exists an ordinal* $\xi \in \mathfrak{M}$ *such that for* $\mathcal{L}$*-almost all* i, *if* $\underline{f}(i)$ *is an ordinal then* $\underline{f}(i) = \xi$.

Proof To prove this statement, assume that $\underline{f} = f \circ \mathfrak{r}$ where $f \in \mathfrak{M}$ is a function defined on $\mathcal{N}$ in $\mathfrak{M}$.

We argue in $\mathfrak{M}$. Consider an arbitrary $p \in \mathbb{P}$. We define a stronger condition p' as follows. Let $i \in \underline{\mathcal{N}}$. If there does not exist $\beta \in p(i)$ such that $f(\beta)$ is an ordinal, we put $p'(i) = p(i)$ and $\xi(i) = 0$. Otherwise, let $\xi(i) = \xi$ be the least ordinal ξ such that $f(\beta) = \xi$ for some $\beta \in p(i)$. We set $p'(i) = \{\beta \in p(i) : f(\beta) = \xi(i)\}$.

Notice that $\xi(i)$ is an ordinal for all $i \in \underline{\mathcal{N}}$. Therefore, since the ultrafilter $\mathcal{L}$ is κ-complete in $\mathfrak{M}$ for all κ, there exists a single ordinal $\xi \in \mathfrak{M}$ such that $\xi(i) = \xi$ for almost all i .

By genericity, we may assume that actually $\mathfrak{r}(i) \in p'(i)$ for all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$. Then ξ is as required. $\square$

The ultrapower

Let $\mathfrak{N} = \mathrm{Ult}_{\mathcal{L}}\, \mathcal{F}$ be the ultrapower. Thus we define:

- $f \approx g$ iff $\{i : f(i) = g(i)\} \in \mathcal{L}$ for $f,\, g \in \mathcal{F}$;
- $[f] = \{g : g \approx f\}$ (the $\mathcal{L}$-*degree* of f);

- $[f] \in^* [g]$ iff $\{i : f(i) \in g(i)\} \in \mathcal{L}$;
- $\mathfrak{N} = \{[f] : f \in \mathcal{F}\}$, equipped with the above defined membership $\in^*$.

Theorem 8 *$\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$ via the embedding which associates $x^* = [\underline{\mathcal{N}} \times \{x\}]$ with any $x \in \mathfrak{M}$. Moreover $\mathfrak{N}$ is wellfounded and the ordinals in $\mathfrak{M}$ are isomorphic to the $\mathfrak{M}$-ordinals via the mentioned embedding.*

Proof See KANOVEI and VAN LAMBALGEN [1994]. □

Comment. Propositions 4 and 6 are used to prove the Łoś theorem and the property of elementary embedding. Proposition 7 is used to prove the wellfoundedness part of the theorem.

3 The nature of the ultrapower

Theorem 8 allows to collapse $\mathfrak{N}$ down to a transitive model $\widehat{\mathfrak{N}}$; actually $\widehat{\mathfrak{N}} = \{\widehat{X} : X \in \mathfrak{N}\}$ where

$$\widehat{X} = \{\widehat{Y} : Y \in \mathfrak{N} \text{ and } Y \in^* X\}.$$

The content of this section will be to investigate the relations between $\mathfrak{M}$, the initial model, and $\widehat{\mathfrak{N}}$, the (transitive form of its) Spector ultrapower. In particular it is interesting how the superposition of the "asterisk" and "hat" transforms embeds $\mathfrak{M}$ into $\widehat{\mathfrak{N}}$.

Lemma 9 *$x \longmapsto \widehat{x^*}$ is an elementary embedding $\mathfrak{M}$ into $\widehat{\mathfrak{N}}$, equal to identity on ordinals and sets of ordinals (in particular on reals).*

Proof Follows from what is said above. □

Thus $\widehat{\mathfrak{N}}$ contains all reals in $\mathfrak{M}$. We now show that $\widehat{\mathfrak{N}}$ also contains some new reals. We recall that $\mathfrak{r} \in \mathcal{F}$ is a function satisfying $\mathfrak{r}(i) \in i$ for all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$.

Let $\mathbf{a} = \widehat{[\mathfrak{r}]}$. Notice that by Łoś $[\mathfrak{r}]$ is a real in $\mathfrak{N}$, therefore $\mathbf{a}$ is a real in $\widehat{\mathfrak{N}}$.

Lemma 10 *$\mathbf{a}$ is random over $\mathfrak{M}$.*

Proof Let $B \subseteq \mathcal{N}$ be a Borel set of null measure coded in $\mathfrak{M}$; we prove that $\mathbf{a} \notin B$. Being of measure 0 is an absolute notion for Borel sets, therefore $B \cap \mathfrak{M}$ is a null set in $\mathfrak{M}$ as well. Corollary 5 implies that for $\mathcal{L}$-almost all i, we have $\mathfrak{r}(i) \notin B$. By Łoś, $\neg\ ([\mathfrak{r}] \in^* B^*)$ in $\mathfrak{N}$. Then $\mathbf{a} \notin \widehat{B^*}$ in $\widehat{\mathfrak{N}}$. However, by the absoluteness of the Borel coding, $\widehat{B^*} = B \cap \widehat{\mathfrak{N}}$, as required. □

Thus $\widehat{\mathfrak{N}}$ contains a new real number $\mathbf{a}$. It so happens that this $\mathbf{a}$ generates all reals in $\widehat{\mathfrak{N}}$.

Lemma 11 *The reals of* $\widehat{\mathfrak{N}}$ *are exactly the reals of* $\mathfrak{M}[\mathbf{a}]$.

Proof It follows from the known properties of random extensions that every real in $\mathfrak{M}[\mathbf{a}]$ can be obtained as $F(\mathbf{a})$ where F is a Borel function coded in $\mathfrak{M}$. Since $\mathbf{a}$ and all reals in $\mathfrak{M}$ belong to $\widehat{\mathfrak{N}}$, we have the inclusion $\supseteq$ in the lemma.

To prove the opposite inclusion let $\beta \in \widehat{\mathfrak{N}} \cap \mathcal{N}$. Then by definition $\beta = \widehat{[F]}$, where $F \in \mathcal{F}$. In turn $F = f \circ \mathfrak{r}$, where $f \in \mathfrak{M}$ is a function defined on $\mathcal{N} \cap \mathfrak{M}$. We may assume that in $\mathfrak{M}$ f maps reals into reals. Then, first, by Property 2, f is a.e. equal in $\mathfrak{M}$ to a Borel function $g = B_\gamma$ where $\gamma \in \mathcal{N} \cap \mathfrak{M}$ and B_γ denotes, in the usual manner, the Borel subset (of $\mathcal{N}^2$ in this case) coded by γ. Corollary 5 shows that we have $F(i) = B_\gamma(\mathfrak{r}(i))$ for $\mathcal{L}$-almost all i. In other words, $F(i) = B_{\gamma^*(i)}(\mathfrak{r}(i))$ for $\mathcal{L}$-almost all i. By Łoś, this implies $[F] = B_{[\gamma^*]}([\mathfrak{r}])$ in $\mathfrak{N}$, therefore $\beta = B_\gamma(\mathbf{a})$ in $\widehat{\mathfrak{N}}$. By the absoluteness of Borel coding, we have $\beta \in \mathbb{L}[\gamma, \mathbf{a}]$, therefore $\beta \in \mathfrak{M}[\mathbf{a}]$.□

We finally can state and prove the principal result.

Theorem 12 $\widehat{\mathfrak{N}} \subseteq \mathfrak{M}[\mathbf{a}]$ *and* $\widehat{\mathfrak{N}}$ *coincides with* $\mathbb{L}^{\mathfrak{M}[\mathbf{a}]}(\text{reals})$, *the smallest subclass of* $\mathfrak{M}[\mathbf{a}]$ *containing all ordinals and all reals of* $\mathfrak{M}[\mathbf{a}]$ *and satisfying all the axioms of* **ZF** .

Proof Very elementary. Since $\mathbb{V} = \mathbb{L}(\text{reals})$ is true in $\mathfrak{M}$, the initial Solovay model, this must be true in $\widehat{\mathfrak{N}}$ as well. The previous lemma completes the proof. □

Corollary 13 *The set* $\mathcal{N} \cap \mathfrak{M}$ *of all "old" reals does not belong to* $\widehat{\mathfrak{N}}$.

Proof The set in question is known to be non–measurable in the random extension $\mathfrak{M}[\mathbf{a}]$; thus it would be non–measurable in $\widehat{\mathfrak{N}}$ as well. However $\widehat{\mathfrak{N}}$ is an elementary extension of $\mathfrak{M}$, hence it is true in $\widehat{\mathfrak{N}}$ that all sets are measurable. □

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