Decidable theories of ω-layered metric temporal structures

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Abstract

This paper focuses on decidability problems for metric and layered temporal logics which allow one to model time granularity in various contexts. The decidability of pure metric (non-granular) fragments and of metric temporal logics endowed with finitely many layers has been already proved by reduction to the decidability problem of the well-known theory $S1S$. In the present work, we prove the decidability of both the theory of metric temporal structures provided with an infinite number of arbitrarily coarse temporal layers and the theory of metric temporal structures provided with an infinite number of arbitrarily fine temporal layers. The proof for the first theory is obtained by reduction to the decidability problem of an extension of $S1S$ which is proved to be the logical counterpart of the class of ω-languages accepted by systolic tree automata. The proof for the second one is done through the reduction to the monadic second-order decidable theory of k successors SkS.

1 Introduction

This paper focuses on decidability problems for metric and layered temporal logics [3, 9]. The considered logics—suitable to model time granularity in various contexts—allow one to build granular temporal models by referring to the "natural scale" in any component of the model and properly constraining the interactions between differently grained components. As pointed out in [2], the ability of providing and relating temporal representations at different 'grain levels' of the same reality is widely recognized as an important research theme for temporal logic and a major requirement for many applications. With regard to the problem of specifying and proving correctness of real-time systems, there exists a wide-ranging class of real-time systems whose components have dynamic behavior regulated by very different - even by orders of magnitude - time constants (granular systems). A good specification language must allow one to specify and verify the components of a granular system and their interactions in a simple and intuitively clear way, and thus the ability of dealing with different time granularities is a desirable feature of a specification language for granular systems, e.g., [4, 5, 6].

In [3, 9], Montanari et al. proposed a metric and layered temporal logic (MLTL for short) for specifying granular real-time systems. Metric temporal logics, e.g., [7, 12, 13], extend propositional logic with a parameterized operator of relative temporal realization. MLTL can be viewed as the combination of a number of differently-grained metric temporal logics. It replaces the flat temporal domain of metric temporal logics with a temporal universe consisting of a set of differently-grained temporal domains. Such a temporal universe identifies the temporal domains relevant to a granular system specification and defines the relations between instants belonging to different domains. To qualify formulae with respect to the temporal universe, MLTL is provided with an operator of *contextualization* that identifies the domains a given formula refers to. Within each temporal domain, it is then possible to talk about truth and falsehood of formulae at different time instants by means of a *displacement* operator. Finally, a *projection* operator can be used to constrain the relationships between formulae associated with differently-grained domains. The combined use of these operators allows one to represent a granular system by properly connecting a set of differently-grained formulae. In the simplest case, such a representation consists of the logical composition of a number of formulae referring to different temporal domains. In more complex cases, the projection operator is used to deal with nested quantifications of differently-grained temporal displacements (e.g., to specify the condition: "there exist some days during which the plant remains inactive for some hours"), or to specify the composition of differently-grained temporal displacements (e.g., to specify the condition: "in twenty seconds, five minutes will have passed from the occurrence of the fault"). A full account of syntax, semantics and axiomatization of MLTL can be found in [3, 9, 12, 13].

In order to guarantee the usefulness of metric and layered temporal logics as formal tools, it is necessary to show some basic decidability properties. The decidability problem for the pure metric (non-granular) fragment has been addressed by Alur and Henzinger in [1] which was, in fact, our starting point. They showed that, under suitable assumptions about the temporal domain and the associated operations, the validity and satisfiability problems for real-time logics extending propositional temporal logics with metric features are decidable. These problems can indeed be reduced to the decidability problem for a decidable theory: the well-known theory $S1S$. A first extension of their results, aiming at dealing with time granularity, has been presented in [10, 11]. Such an extension allows one to treat situations in which a finite number of coarsenings/refinements of the temporal domain is sufficient. The key idea to deal with the resulting finitely-layered metric temporal structures is to reformulate the decidability problem into an equivalent one relative to the finest metric component (layer). We first formally defined the theory T_{ML} of finitely-layered metric temporal structures, and the associated second-order language $\mathcal{L}^2_{ML}$. Then, we defined a computable function τ which translates each sentence ϕ of the language $\mathcal{L}^2_{ML}$ for T_{ML} into a sentence $\tau(\phi)$ of the language $\mathcal{L}^2$ underlying the theory $S1S$ so that $\tau(\phi)$ is valid (satisfiable) in $S1S$ if and only if ϕ is valid (satisfiable) in T_{ML}. The translation is actually performed in two steps: we first embed finitely-layered metric temporal structures into (flat) metric temporal structures; then, we reduce metric temporal structures to *S1S* structures. Hence, in both the original work by Alur and Henzinger and the above mentioned extension to finitely layered temporal structures, the basic tool for proving decidability properties is the theory $S1S$, and the basic engine is Büchi theorem on the decidability of regular ω-languages.

In the present work we deal with the more general case in which the underlying temporal structure consists of infinitely many temporal layers (ω-layered, k-refinable, metric temporal structures). We introduce the second-order language $\mathcal{L}^2_{\omega ML^k}$ for ω-layered (k-refinable) metric temporal structures, and show how to interpret it over different classes of structures. We first consider the case of temporal structures in which there is a finest temporal domain together with a infinite number of coarser and coarser domains (*upward unbounded layered structures*). To deal with such structures we use a more expressive theory, that we called $S1S^k$, which is a proper extension of $S1S$. The decidability of $S1S^k$ is shown using a more powerful basic engine, namely the decidability of ω-languages recognized by k-ary Systolic

Tree Automata. Such class of ω-languages has been recently proved to properly extend the class of regular ω-languages and to have the same decidability properties [14, 15]. Since all the basic closure properties of regular ω-languages hold also for systolic tree ω- languages, a direct correspondence with the above mentioned second-order theory $S1S^k$ (properly extending $S1S$) can be established. From the one hand, upward unbounded layered structures provide an interesting example of application for the decidability of systolic tree ω-languages. On the other hand, we believe that the second-order theory $S1S$ is too weak to deal with infinitely coarsening domains, and that $S1S^k$ is a somehow "minimal" theory able to deal with such a case. Successively, we deal with the problem of deciding infinitely refinable structures (*downward unbounded layered structures*). We first show that the formal tool used to decide upward unbounded layered structures is not applicable. Then, we prove that the decidability of the satisfiability (resp. validity) problem for the theory of such structures can be reduced to the decidability of the satisfiability (resp. validity) problem for SkS, the well-known monadic second-order decidable theory of k successors [16].

The paper is organized as follows. In Section 2, we provide some background knowledge about systolic and Rabin tree automata. Then, in Section 3, we define the theory $S1S^k$. In Section 4, we formally define the theories of upward (resp. downward) unbounded layered structures, and prove that they are decidable. In Section 5, we show how the basic functionalities of metric and layered temporal logic can be expressed in $\mathcal{L}^2_{\omega ML^k}$. Conclusions provide an assessment of the work, and outline the main topics of ongoing research.

2 Systolic and Rabin tree automata

2.1 Systolic tree automata on ω-words

In this section, we give a brief description of *k-ary Systolic Tree Automata* (*k-STA* for short). A detailed systematic presentation can be found in [14, 15].

Throughout this paper, let Σ denote a finite alphabet and let Σ^ω denote the set of *ω-words* over Σ. The symbols $\alpha, \beta, \ldots$ are used for ω-words and $L, L', \ldots$ for sets of ω-words. For an ω-word α, $\alpha(i)$, with $i \in \mathbb{N}$, denotes the i-th element of α, and $\alpha(m,n)$, with $m,n \in \mathbb{N}$, denotes the segment $\alpha(m) \ldots \alpha(n)$ of α. The symbol $\cdot$ denotes concatenation on strings. A systolic automaton consists of an infinite number of nodes which can be interpreted as memoryless processors. Nodes are linked among them and the resulting structure is an (infinite) leafless perfectly balanced k-ary tree. In order to process a word w, whose length $|w|$ is equal to k^i, the i-th level of the tree is chosen. Now, the automaton is fed in such a way that adjacent processors at level i-th are fed with adjacent symbols of w, and that the leftmost processor is fed with the first symbol of w. All the processors at level i-th synchronously output a symbol belonging to the *state alphabet* Q, according to the *input relation*. Each processor at level $(i-1)$-th receives k states output by its k children and it synchronously (with respect to processors at the same level) outputs a symbol belonging to Q according to the *transition relation*. Therefore, information flows bottom-up, in parallel and synchronously, level by level.

Let us introduce now a notion of step-wise systolic computation on ω-words. Consider an ω-word α. At each computation step, the automaton process a segment of α whose length increases by a factor of k step by step. In particular, an ω-word on the set of states Q stores at the i-th position the state q resulting from processing the prefix $\alpha(0, k^i - 1)$ of α. The state resulting from processing the next prefix $\alpha(0, k^{i+1} - 1)$ is obtained from q and from the states

$q_2, \ldots, q_k$ output by the systolic automaton fed with $\alpha(k^i, 2k^i - 1), \ldots, \alpha((k-1)k^i, k^{i+1} - 1)$, respectively, according to the transition relation f. In Figure 1, we graphically describe the way in which a binary systolic tree automaton processes an ω-word α. The left-hand side edge of the tree structure consists of nodes associated with states obtained by processing prefixes of α whose length is a power of k. Such a sequence of states is called a *systolic run*.

Formally, a K-ary systolic tree automata is defined as follows.

Definition 2.1 *A* systolic automaton *is a tuple* $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$, *where*

- Σ *is the finite* input alphabet*;*
- Q *is the finite set of* states*;*
- $in \subseteq \Sigma \times Q$ *is the* input relation*;*
- $f \subseteq Q^{k+1}$ *is the* transition relation*;*
- $F \subseteq Q$ *is the set of* final states.

Definition 2.2 *The* finitary *computation of an automaton $\mathcal{A}$ over a (finite) word w of length k^m is a binary relation $O_{\mathcal{A}} \subseteq \Sigma^* \times Q$ recursively defined as follows:*

- *if $|w| = 1$, then $\langle w, q \rangle \in O_{\mathcal{A}}$ if and only if $\langle w, q \rangle \in in$;*
- *if $|w| = k^m$, with $m > 0$, then $\langle w, q \rangle \in O_{\mathcal{A}}$ if and only if $\langle q_1, \ldots, q_k, q \rangle \in f$, where q_i, for $1 \leq i \leq k$, is such that $\langle w_i, q_i \rangle \in O_{\mathcal{A}}$, with $|w_i| = k^{m-1}$ and $w_1 \cdot w_2 \ldots \cdot w_k = w$.*

Definition 2.3 *A* systolic run *of $\mathcal{A}$ on an ω-word $\alpha \in \Sigma^\omega$ is an ω-word $\sigma \in Q^\omega$ such that*

- $\langle \alpha(0), \sigma(0) \rangle \in in$*;*
- $\langle \sigma(i-1), q_2, \ldots, q_k, \sigma(i) \rangle \in f$, *with* $\langle \alpha((j-1)k^{i-1}, jk^{i-1} - 1), q_j \rangle \in O_{\mathcal{A}}$, *for* $2 \leq j \leq k$.

A systolic run σ is *successful* if and only if some state of F occurs infinitely often in σ. An automaton $\mathcal{A}$ *accepts* an ω-word α if and only if there exists a successful systolic run σ on α. The ω-language recognized by $\mathcal{A}$, denoted by $\mathcal{L}_\omega(\mathcal{A})$, is the set

$$\{\alpha \in \Sigma^\omega : \mathcal{A} \text{ accepts } \alpha\}.$$

Theorem 2.4 *Let $\mathcal{L}_\omega$(k-STA) be the class of ω-languages recognized by k-ary systolic tree automata. The following properties hold:*

(i) the class of regular ω-languages is strictly contained in $\mathcal{L}_\omega$(k-STA);

(ii) the class $\mathcal{L}_\omega$(k-STA) is closed under union, intersection, projection, and complementation;

(iii) the emptiness problem for $\mathcal{L}_\omega$(k-STA) is decidable.

The proof of Theorem 2.4 is given in [14, 15].

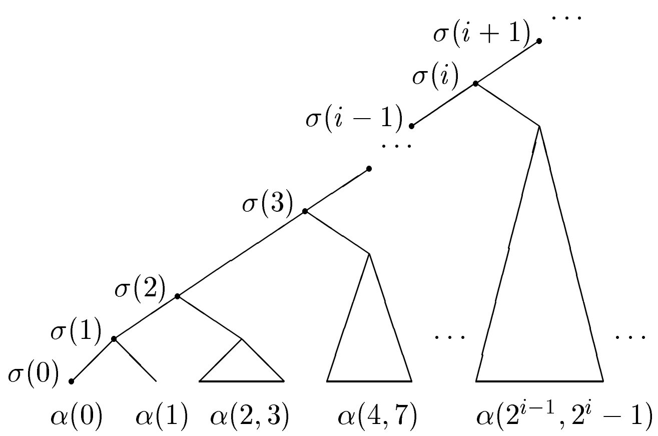

Figure 1: A systolic run σ on α.

Example 2.5 *Let us consider the* ω*-language* $L = \{a^{2^i} \cdot \{b\}^\omega : i \geq 0\}$. L *is clearly non-regular and it is recognized by the (binary) systolic tree automaton* $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ *defined as follows:*

- $\Sigma = \{a, b\}$; $Q = \{q_1, q_2, q_3\}$; $F = \{q_3\}$;
- $in = \{\langle a, q_1 \rangle, \langle b, q_2 \rangle\}$;
- $f = \{\langle q_1, q_1, q_1 \rangle, \langle q_2, q_2, q_2 \rangle, \langle q_1, q_2, q_3 \rangle, \langle q_3, q_2, q_3 \rangle\}$.

Example 2.6 *Another example of non-regular* ω*-language is* $L' = \{s_0 \cdot s_1 \cdot \ldots \cdot s_i \cdot \ldots\}$, *with* $s_i = ba^{2^i - 1}$ *for* $i \geq 0$. L' *is recognized by the (binary) systolic tree automaton* $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ *defined as follows:*

- $\Sigma = \{a, b\}$; $Q = \{q_1, q_2, q_3, q_4\}$; $F = \{q_3\}$;
- $in = \{\langle a, q_2 \rangle, \langle b, q_1 \rangle\}$;
- $f = \{\langle q_1, q_1, q_3 \rangle, \langle q_2, q_1, q_4 \rangle, \langle q_3, q_4, q_3 \rangle, \langle q_2, q_2, q_2 \rangle, \langle q_2, q_4, q_4 \rangle\}$.

In Section 3, we will define a proper extension of $S1S$ that allows us to transfer the above results from automata theory to logic.

2.2 Rabin tree automata and the theory SkS

In this section, we briefly describe Rabin tree automata, focusing on their relationships with SkS, the monadic second-order theory of k successors. An extensive presentation can be found in [16].

Given an alphabet Σ, a k-ary Σ-valued tree t is specified by its set of nodes, called the *domain* of t and denoted by $dom(t)$, and by a valuation $v_t : dom(t) \mapsto \Sigma$. The domain $dom(t)$ is a nonempty subset of $\{0, \ldots, k-1\}^*$, closed under prefixes, and such that for all $w \in \{0, \ldots, k-1\}^*$ and $i, j \in \{0, \ldots, k-1\}$, if $w \cdot j \in dom(t)$ and $i < j$, then $w \cdot i \in dom(t)$. The proper prefix relation over $\{0, \ldots, k-1\}^*$ is denoted by $<_P$. A path through t is a

maximal subset of $dom(t)$ linearly ordered by $<_P$.
For the sake of simplicity, in the sequel we will restrict our presentation to binary trees, and to the corresponding theory $S2S$ of two successors. Notation and results can be easily generalized to k-ary trees and to the corresponding theory SkS.

Let T_Σ^ω be the set of infinite Σ-valued binary trees (with domain $\{0,1\}^*$). Rabin tree automata operate over infinite tree, and are formally defined as follows:

Definition 2.7 *A* Rabin tree automaton *is a tuple* $\mathcal{A} = \langle \Sigma, Q, q_0, \Delta, \Omega \rangle$, *where*

- Σ *is the finite* input alphabet*;*
- Q *is the finite set of* states*;*
- $q_0 \in Q$ *is the* initial state*;*
- $\Delta \subseteq Q \times \Sigma \times Q \times Q$ *is the* transition relation*;*
- $\Omega = \{(L_1, U_1), \ldots, (L_n, U_n)\}$, *with* $L_i, U_i \subseteq Q$, *is a collection of* accepting pairs *of state sets.*

Unlike systolic tree automata, Rabin tree automata support a top-down computation. Given an infinite Σ-valued binary tree t, the automaton starts its computation at the root of t in the initial state q_0 and then simultaneously works down the paths of the tree level by level. The transition relation Δ determines which pairs of states (q_1, q_2) can be associated with the two children of a node, given the node's state and value. Formally, a run of a Rabin tree automaton is defined as follows.

Definition 2.8 *A* run *of a Rabin tree automaton* $\mathcal{A}$ *on* $t \in T_\Sigma^\omega$ *is a map* $\sigma : dom(t) \mapsto Q$ *such that*

- $\sigma(\epsilon) = q_0$*;*
- *for each* $w \in dom(t)$, $(\sigma(w), v_t(w), \sigma(w \cdot 0), \sigma(w \cdot 1)) \in \Delta$.

The automaton accepts the tree if there is a run built up in this way which is successful. The notion of *successful run* of a Rabin tree automaton is formally defined as follows.

Definition 2.9 *Let* $t \in T_\Sigma^\omega$ *and* r *be a run of a* Rabin tree automaton $\mathcal{A} = \langle \Sigma, Q, q_0, \Delta, \Omega \rangle$ *on* t. *The run* r *is* successful *if for all paths* π *through* t, *there exists* $i \in \{1, \ldots, n\}$ *such that* $In(r|\pi) \cap L_i = \emptyset$ *and* $In(r|\pi) \cap U_i \neq \emptyset$, *where* $r|\pi$ *denotes the restriction of the run* r *to the path* π *and* $In(r|\pi)$ *returns the set of states that occur infinitely many times in* $r|\pi$.

A tree $t \in T_\Sigma^\omega$ is *accepted* by Rabin tree automaton $\mathcal{A}$ if there exists a successful run of $\mathcal{A}$ on it. A set $T \subseteq T_\Sigma^\omega$ is *Rabin recognizable* if it consists of the trees accepted by a Rabin tree automaton.

Theorem 2.10 *Let* $T(\mathcal{A})$ *be the set of trees accepted by a Rabin tree automaton* $\mathcal{A}$. *The following properties hold:*

(i) the emptiness problem for Rabin tree automata is decidable;

(ii) for any Rabin tree automaton $\mathcal{A}$, *there is a Rabin tree automaton* $\mathcal{A}'$ *recognizing* $T_\Sigma^\omega \setminus T(\mathcal{A})$.

Let us consider now the logical counterpart of Rabin tree automata. We first introduce a representation of trees as model-theoretic structures. Let Σ be the alphabet $\{0,1\}^n$. A tree $t \in T_\Sigma^\omega$ can be codified by a model $\underline{t}$ of the form:

$$\underline{t} = \langle \{0,1\}^*, \epsilon, succ_0, succ_1, <_P, P_1, \ldots, P_n \rangle,$$

where $succ_0$ and $succ_1$ are the two successor functions over $\{0,1\}^*$, with $succ_0(w) = w \cdot 0$ and $succ_1(w) = w \cdot 1$, $<_P$ is the proper prefix relation over $\{0,1\}^*$, and $P_1, \ldots, P_n$ are subsets of $\{0,1\}^*$, where for each $1 \leq i \leq n$, $w \in P_i$ if and only if the i-th component of $v_t(w) = 1$.

Definition 2.11 *The (monadic) second-order theory of two successors $S2S$ includes individual variables $x, y, \ldots$ and first-order variables $p_1, p_2, \ldots$, ranging over elements and subsets of $\{0,1\}^*$, respectively. Terms are built up from the individual variables and the constant ϵ by applying the successor functions $succ_0$ and $succ_1$. Atomic formulae are of the forms $t = t'$, $t <_P t'$, and $p(t)$, where t, t' are terms and p is a first-order variable. Arbitrary formulae are obtained from atomic ones by using boolean connectives and the quantifier $\exists, \forall$ over both individual and first-order variables.*

Notice that ϵ and $<_P$ can actually be defined in terms of $succ_0$ and $succ_1$, and thus removed.

Let $\phi(p_1, \ldots p_n)$ be a $S2S$-formula, where at most the n first-order variables $p_1, \ldots, p_n$ occur free, and $\underline{t}$ be a tree model. We write $\underline{t} \models \phi(p_1, \ldots p_n)$ if ϕ is satisfied in $\underline{t}$, with P_i as interpretation of p_i (for $1 \leq i \leq n$). Moreover, let $T(\phi) = \{t \in T_\Sigma^\omega : \underline{t} \models \psi(p_1, \ldots p_n)\}$. If $T = T(\phi)$ for some $S2S$-formula ϕ, T is said definable in $S2S$.

The following theorem links the notions of $S2S$-definable and Rabin recognizable.

Theorem 2.12 *A set $T \subseteq T_\Sigma^\omega$ is definable in $S2S$ if and only if T is Rabin recognizable.*

Putting together Theorem 2.10 and Theorem 2.12, it immediately follows that:

Corollary 2.13 *The theory $S2S$ is decidable.*

3 The theory $S1S^k$

In Section 2.1, we introduced systolic ω-languages from an operational point of view. In this section, we will define a suitable (decidable) extension of the second-order theory of one successor $S1S$, called $S1S^k$, which can be viewed as the logical counterpart of the operational definition of systolic ω-languages. In the next section, we will show how $S1S^k$ can be used to decide the theory of ω-layered, k-refinable, metric temporal structures consisting of an infinite number of arbitrarily coarse temporal domains.

$S1S^k$ extends the sequential calculus $S1S$ by adding a unary function symbol $\stackrel{k}{\leftarrow}$, called *power function*, and a unary predicate symbol L. For any natural number $x > 0$, the power function computes the natural number $x - x'$, where x' is the least power of k (with non-null coefficient) in the k-ary representation of x. The predicate L holds for a natural number x if and only if the least power of k (with non-null coefficient) in the k-ary representation of x has coefficient $k - 1$.

Definition 3.1 *The* power function $\stackrel{k}{\leftarrow}: \mathbb{N}^+ \rightarrow \mathbb{N}$ *is such that* $y = \stackrel{k}{\leftarrow}(x)$ *if and only if*

$$x = a_n k^n + a_{n-1} k^{n-1} + \ldots + a_m k^m,$$
$$\text{with } 0 \leq a_i \leq k-1,\ m \leq n \text{ and } a_m \neq 0, \text{ and}$$
$$y = a_n k^n + a_{n-1} k^{n-1} + \ldots + (a_m - 1) k^m.$$

The predicate L holds at x if and only if

$$x = a_n k^n + a_{n-1} k^{n-1} + \ldots + a_m k^m,$$
$$\text{with } 0 \leq a_i \leq k-1,\ m \leq n \text{ and } a_m = k-1.$$

It is worth noting that the predicate "*is a power of k*" can be easily expressed in terms of the power function $\overset{k}{\leftarrow}$ as follows: x "*is a power of k*" if and only if $0 = \overset{k}{\leftarrow}(x)$. Hence, $S1S^k$ is at least as expressive as the well-known extension of $S1S$ with the predicate "*is a power of k*".

Let $\mathcal{L}^2_{S1S^k}$ be the second-order language with (uninterpreted) predicate symbols, the unary function symbol $\overset{\mathtt{k}}{\leftarrow}$, the unary predicate symbol $\mathtt{L}$, the binary predicate symbol $\leq$, and quantification over individual variables and unary predicate symbols. We interpret $\mathcal{L}^2_{S1S^k}$ over the natural numbers $\mathbb{N}$, with $\overset{\mathtt{k}}{\leftarrow}$ being interpreted as the power function $\overset{k}{\leftarrow}$, $\mathtt{L}$ as the predicate L and $\leq$ as the usual linear ordering. $\mathcal{L}^2_{S1S^k}$ is the language underlying $S1S^k$.

In the following, we will restrict our attention to formulae devoid of free individual variables. Therefore, given a formula $\phi \in \mathcal{L}^2_{S1S^k}$, with free predicate symbols $\mathtt{p}_1, \ldots, \mathtt{p}_n$, an interpretation $\mathcal{I}$ for ϕ specifies n sets $p_1^{\mathcal{I}}, \ldots, p_n^{\mathcal{I}} \subseteq \mathbb{N}$. The interpretation $\mathcal{I}$ can be equivalently viewed as an infinite sequence of subsets $s_i \subseteq \{p_1, \ldots, p_n\}$, with $i \geq 0$, such that $p_j \in s_i$ if and only if $i \in p_j^{\mathcal{I}}$.

Let us now get a finite alphabet Σ, with $|\Sigma| = 2^n$, and define a bijection $cod : 2^{\{p_1,\ldots,p_n\}} \mapsto \Sigma$ mapping each $s \subseteq \{p_1, \ldots, p_n\}$ into a symbol $cod(s)$ of Σ. On the one hand, such a correspondence allows us to represent any interpretation $\mathcal{I}$ for ϕ by an ω-word α over Σ such that $\alpha(i) = a$, with $a \in \Sigma$, if and only if $a = cod(\{p_{i_1}, \ldots, p_{i_m}\})$ (with $m \leq n$ and $p_{i_j} \in \{p_1, \ldots, p_n\}$, for $1 \leq j \leq m$), $i \in p^{\mathcal{I}}$ for all $p \in \{p_{i_1}, \ldots, p_{i_m}\}$, and $i \notin p^{\mathcal{I}}$ for all $p \in \{p_1, \ldots, p_n\} \setminus \{p_{i_1}, \ldots, p_{i_m}\}$. On the other hand, each ω-word α naturally induces an interpretation as formally stated by the following definition.

Definition 3.2 *An ω-word $\alpha \in \Sigma^\omega$* induces *an interpretation $\underline{\alpha}$ (the* canonical interpretation under *α) having the form*

$$\underline{\alpha} = \langle \mathbb{N}, \overset{k}{\leftarrow}, L, \leq, (Q_a)_{a \in \Sigma} \rangle,$$

where $\langle \mathbb{N}, \leq \rangle$ is the structure of natural numbers with the usual linear ordering, $\overset{k}{\leftarrow}$ is the power function, $L \subseteq \mathbb{N}$ such that for all $x \in \mathbb{N}$, $x \in L$, $\mathtt{L}$ holds at x, and Q_a (for $a \in \Sigma$) is the set $\{i \in \mathbb{N} : \alpha(i) = a\}$.

For any given formula ϕ, with free predicate symbols $\mathtt{p}_1, \ldots, \mathtt{p}_n$ (or, equivalently, for any finite alphabet Σ, with $|\Sigma| = 2^n$), we can thus define the "interpreted" theory $S1S^k_\Sigma$, whose language is defined as follows.

Definition 3.3 *For any given alphabet Σ, the language for the interpreted theory $S1S^k_\Sigma$ is built up as follows:*

- terms *are freely constructed from first-order variables by application of the power function $\overset{k}{\leftarrow}$;*

- atomic formulae *are of the form* $L(t)$, $X(t)$, $Q_a(t)$ *(for* $a \in \Sigma$*), and* $t \leq t'$*, where* t *and* t' *are terms and* X *is a predicate variable;*
- $S1S^k_\Sigma$-formulae *are freely constructed from atomic formulae by using the usual boolean connectives, and the quantifiers* $\exists$ *and* $\forall$ *acting on both individual and predicate variables.*

Let ϕ be a $S1S^k_\Sigma$-formula. We write $\phi(p_1, \ldots p_m)$ to indicate that at most the m predicate variables $p_1, \ldots, p_m$ occur free in ϕ. Formulae without free variables are called *sentences*. Given $\alpha \in \Sigma^\omega$ and a sentence ϕ, we write $\underline{\alpha} \models \phi$ if ϕ is satisfied in $\underline{\alpha}$ (the notion "ϕ is satisfied in $\underline{\alpha}$" is standard and is not formally defined here). The *ω-language defined by* a $S1S^k_\Sigma$- sentence ϕ is $\mathcal{L}(\phi) = \{\alpha \in \Sigma^\omega : \underline{\alpha} \models \phi\}$.

Let us prove now that the extended calculus $S1S^k$ can be viewed as the logical counterpart of systolic ω-languages. This result is a (non-trivial) generalization of the one given in [14, 15] for $k = 2$. Some preliminary definitions are needed.

First, notice that the power function, together with the predicate L, allow us to associate a set of natural numbers, structured as a perfectly balanced k-ary tree, with each natural number. Actually, we will associate such trees only with natural numbers having 1 as the least non-null coefficient in their k-ary representation (notice that for $k = 2$, this condition is true for every natural number (greater than 0), and thus a binary tree is associated with each natural number).

Formally, given a natural number

$$y = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_m k^m, \text{ with } m > 0 \text{ and } a_m = 1$$

the j-th child x of y, with $0 \leq j \leq k-1$, is the natural number

$$x = a_n k^n + a_{n-1}k^{n-1} + \ldots + jk^m + 1k^{m-1}.$$

As an example, the trees associated with numbers 8 (for $k = 2$) and 9 (for $k = 3$) are shown in Figure 2, where we assumed that the 0-th child is the leftmost child in the representation of the tree (hereafter, we will always make this assumption). It is worth noting that the structure imposed in this way on the considered subset of natural numbers is analogous to the "structure" of a systolic computation, i.e. a systolic run (cf. Figure 1), on an ω-word.

By exploiting the power function $\overset{k}{\leftarrow}$ and the predicate L, we are able to define $k-1$ auxiliary predicates L_j, with $1 \leq j \leq k-1$, such that $L_j(x)$ holds if and only if j is the least non-null coefficient in the k-ary representation of x (notice that $L_{k-1} = L$). For $j = 1, \ldots, k-1$, $\mathtt{L_j(x)}$ can be defined as follows:

$$\mathtt{L_j(x)} \text{ iff } \exists \mathtt{x_{k-1}}, \ldots, \mathtt{x_j}(\mathtt{L(x_{k-1})} \land \mathtt{x} = \mathtt{x_j} \land \bigwedge_{\mathtt{i=j+1}}^{\mathtt{k-1}} \mathtt{x_{i-1}} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x_i})). \tag{1}$$

It is not difficult to see that for any pair of natuaral numbers x and y such that both $L_1(x)$ and $L_1(y)$ hold, x is the 0-th child of y ($y \overset{0}{\rightarrow} x$ for short) if and only if

$$x = max\{w : w < y \land \overset{k}{\leftarrow} (w) = \overset{k}{\leftarrow} (y)\}.$$

Therefore, $y \overset{0}{\rightarrow} x$ can be defined as follows:

$$\mathtt{L_1(y)} \land \mathtt{L_1(x)} \land \mathtt{x} < \mathtt{y} \land \overset{\mathtt{k}}{\leftarrow} (\mathtt{y}) = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x}) \land \tag{2}$$

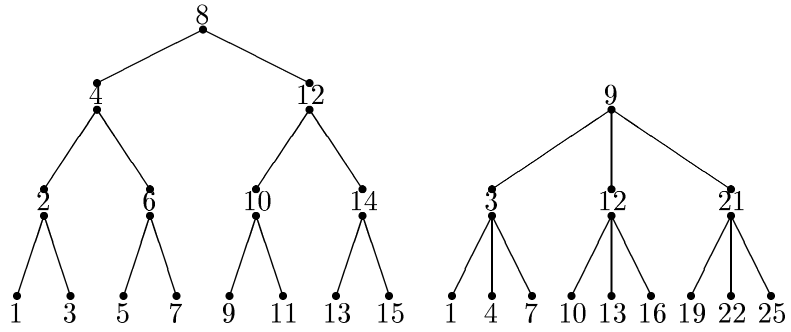

Figure 2: The binary trees associated with numbers 8 (for $k = 2$) and 9 (for $k = 3$).

$$\forall \mathtt{w}((\mathtt{L}_1(\mathtt{w}) \wedge \mathtt{w} < \mathtt{y} \wedge \overset{\mathtt{k}}{\leftarrow} (\mathtt{y}) = \overset{\mathtt{k}}{\leftarrow} (\mathtt{w})) \rightarrow \mathtt{w} \leq \mathtt{x}).$$

Analogously, for any pair of natuaral numbers x and y such that both $L_1(x)$ and $L_1(y)$ hold, x is the j-th child of y ($y \overset{j}{\rightarrow} x$ for short), with $1 \leq j \leq k-1$, if and only if

$$L_j(\overset{k}{\leftarrow} (x)) \wedge y = (\overset{k}{\leftarrow})^j(x).$$

Therefore, $y \overset{j}{\rightarrow} x$, with $0 < j \leq k-1$) can be defined as follows

$$\mathtt{L}_1(\mathtt{y}) \wedge \mathtt{L}_1(\mathtt{x}) \wedge (\exists \mathtt{x}_\mathtt{j}, \ldots, \mathtt{x}_1(\mathtt{x}_\mathtt{j} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x}) \wedge \tag{3}$$

$$\mathtt{L}_\mathtt{j}(\mathtt{x}_\mathtt{j}) \wedge \mathtt{x}_1 = \mathtt{y} \wedge \bigwedge_{\mathtt{i}=2}^{\mathtt{j}} \mathtt{x}_{\mathtt{i}-1} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{x}_\mathtt{i}))).$$

Theorem 3.4 *An ω-language is definable in $S1S^k$ if and only if it belongs to $\mathcal{L}_\omega$(k-STA).*

Proof. ($\Leftarrow$) Let $\mathcal{A} = \langle \Sigma, Q, in, f, F \rangle$ be a systolic automaton. We prove that there exists a sentence ϕ such that $\mathcal{L}(\phi) = \mathcal{L}_\omega(\mathcal{A})$. In order to simulate a systolic run, we define a partial function from $\mathbb{N}$ to Q that associates a state with each natural number x for which $L_1(x)$ holds. Formally, for any state q, let $p_q \subseteq \mathbb{N}$ be the set of natural numbers the state q is associated with. Moreover, for $q \neq q'$, $p_q \cap p_{q'} = \emptyset$ (cf. subformula (4) of the definition of ϕ). Let us consider a systolic run on an ω-word α. First, the leaves of the hierarchical structure on $\mathbb{N}$ imposed by the set of relations $\overset{j}{\rightarrow}$, with $0 \leq j \leq k-1$, act as input nodes. More precisely, a natural number x is an input node if and only if $\mathtt{Input(x)}$ holds, where $\mathtt{Input(x)}$ stands for:

$$\mathtt{L}_1(\mathtt{x}) \wedge \neg \exists \mathtt{y}(\mathtt{x} = \overset{\mathtt{k}}{\leftarrow} (\mathtt{y})).$$

A state q is associated with an input node x if and only if

$$\langle \alpha(x-1), q_1 \rangle, \langle \alpha(x), q_2 \rangle, \ldots, \langle \alpha(x+k-2), q_k \rangle \in in \text{ and } \langle q_1, \ldots, q_k, q \rangle \in f.$$

(cf. subformula (5) of the definition of ϕ). Notice that in such a way we associate with each leaf the state resulting from a computation step over the k states output by the systolic automaton fed with a substring of lenght k, and not the state that the input relation in associates with each input symbol. Next, if the natural numbers $x_1, \ldots, x_k$ are the children of z and $q_1, \ldots, q_k$ are the states associated with them, then q can be associated with z if $\langle q_1, \ldots, q_k, q\rangle \in f$ (cf. subformula (6) of the definition of ϕ). Finally, if the node x is a power of k, then the state associated with x is a state which results from processing the prefix $\alpha(0, kx-1)$ of α. So, the acceptance condition can be expressed by requiring that an infinite number of powers of k is related with a final state (cf. subformula (7) of the definition of ϕ). A sentence ϕ such that $\mathcal{L}(\phi) = \mathcal{L}_\omega(\mathcal{A})$ is the existential closure, with respect to first-order variables of the form p_q (with $q \in Q$), of the following formula:

$$\bigwedge_{\mathtt{q} \neq \mathtt{q'}} \neg \exists \mathtt{y}\, (\mathtt{p_q(y)} \wedge \mathtt{p_{q'}(y)}) \wedge \tag{4}$$

$$\forall \mathtt{x} \left(\mathtt{Input(x)} \rightarrow \left(\bigwedge_{\mathtt{a_1,\ldots,a_k} \in \Sigma} \left(\bigwedge_{\mathtt{i=1}}^{\mathtt{k}} \mathtt{Q_{a_i}(x+i-2)} \right) \rightarrow \bigvee_{\{\mathtt{q}:\langle \mathtt{q_1,\ldots,q_k,q}\rangle \in \mathtt{f}, \langle \mathtt{a_j,q_j}\rangle \in \mathtt{in}, 1 \leq \mathtt{j} \leq \mathtt{k}\}} \mathtt{p_q(x)} \right) \right) \wedge \tag{5}$$

$$\forall \mathtt{z} \left(\bigwedge_{\mathtt{q_1,\ldots,q_k} \in \mathtt{Q}} \left(\bigwedge_{\mathtt{i=0}}^{\mathtt{k-1}} (\mathtt{z} \xrightarrow{\mathtt{i}} \mathtt{q_i} \wedge \mathtt{p_{q_i}(x_i)}) \rightarrow \bigvee_{\{\mathtt{q}:\langle \mathtt{q_1,\ldots,q_k,q}\rangle \in \mathtt{f}\}} \mathtt{p_q(z)} \right) \right) \wedge \tag{6}$$

$$\bigvee_{\mathtt{q} \in \mathtt{F}} \forall \mathtt{x} \exists \mathtt{y}\, (\mathtt{x} < \mathtt{y} \wedge \mathtt{0} \overset{\mathtt{k}}{=\leftarrow} (\mathtt{y}) \wedge \mathtt{p_q(y)}). \tag{7}$$

($\Rightarrow$) The opposite implication is proved by exploiting the same technique used in [16]. For technical convenience, predicate symbols Q_a, for $a \in \Sigma$, are replaced by free set variables. So, formulae $\phi(p_1, \ldots, p_n)$ are considered, where no symbol Q_a occurs, that are interpreted over ω-words over the special alphabet $\{0,1\}^n$. If $\alpha \in (\{0,1\}^n)^\omega$, then $\underline{\alpha} \models p_k(x)$ if and only if the x-th symbol of α has 1 in its k-th component. For a suitable n, symbols in Σ can be binary encoded, and any atomic formula $Q_a(x)$ can be replaced by a finite conjunction including either $p_k(x)$ and $\neg p_k(x)$, for $1 \leq k \leq n$. Hence, given a sentence ϕ, the thesis can be proved for the corresponding formula $\phi(p_1, \ldots, p_n)$ interpreted over ω-words over $\{0,1\}^n$. We preliminary reduce each formula $\phi(p_1, \ldots, p_n)$ of $S1S^k$ to an equivalent formula of a formalism simpler than $S1S^k$, denoted by $S1S^k_0$, whose terms are the first-order variables of the original formalism and whose atomic formulae take one of the following forms:

1. $p_i \subseteq p_j$ ("p_i is a subset of p_j");
2. $Succ(p_i, p_j)$ ("p_i, p_j are the singletons $\{x\}$, $\{y\}$, resp., with $x+1 = y$");
3. $Power(p_i, p_j)$ ("p_i, p_j are the singletons $\{y\}$, $\{x\}$, resp., with $y \overset{2}{=\leftarrow} (x)$");
4. $L(p)$ ("p is a singleton $\{x\}$ and $L(x)$").

The reduction from $S1S^k$ to $S1S^k_0$ is a trivial extension of the reduction from $S1S$ to $S1S_0$ (we refer to [16] for the details). We show now, by induction on the structure of the $S1S^k_0$-formula $\phi(p_1, \ldots, p_n)$, that there exists a k-STA $\mathcal{A}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}(\phi(p_1, \ldots, p_n))$.
(Base case). Let us consider the atomic formula $\phi(p) = L(p)$. An automaton $\mathcal{A} = \langle \Sigma, Q, in, f, F\rangle$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}(\phi)$ can be defined as follows:

- $\Sigma = \{0, 1\}$; $Q = \{0, 1, 2\}$ and $F = \{2\}$;
- in is the identity function;
- $f = \{\langle q_1, \ldots, q_k, 0\rangle : q_i = 0 \text{ for } 1 \leq i \leq k\} \cup$
  $\{\langle q_1, \ldots, q_{k-1}, 1, 2\rangle : q_i = 0 \text{ for } 1 \leq i \leq k-1\} \cup$
  $\{\langle 1, q_2, \ldots, q_k, 1\rangle : q_i = 0 \text{ for } 2 \leq i \leq k\} \cup$
  $\{\langle q_1, \ldots, q_k, 2\rangle : q_i = 2 \text{ for some } 1 \leq i \leq k \text{ and } q_j = 0 \text{ for all } j \neq i\}$.

It is possible to check that $\alpha \in \mathcal{L}_\omega(\mathcal{A})$ if and only if there exists one (and only one) i such that $L(i)$ holds and $\alpha(i) = 1$, and $\alpha(j) = 0$, for all $j \neq i$.
Let us consider the atomic formula $\phi(p_1, p_2) = Power(p_1, p_2)$. An automaton $\mathcal{A} = \langle \Sigma, Q, in, f, F\rangle$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}(\phi)$ can be defined as follows:

- $\Sigma = \{0, 1\}^2$; $Q = \{0, 1\}^2 \cup \{1\}$ and $F = \{1\}$;
- in is the identity function;
- $f = \{\langle q_1, \ldots, q_k, \langle 0, 0\rangle\rangle : q_i = \langle 0, 0\rangle \text{ for } 1 \leq i \leq k\} \cup$
  $\{\langle\langle 1, 0\rangle, q_2, \ldots, q_k, \langle 1, 0\rangle\rangle : q_i = \langle 0, 0\rangle \text{ for } 2 \leq i \leq k\} \cup$
  $\{\langle\langle 0, 1\rangle, q_2, \ldots, q_k, \langle 0, 1\rangle\rangle : q_i = \langle 0, 0\rangle \text{ for } 2 \leq i \leq k\} \cup$
  $\{\langle q_1, \ldots, q_k, 1\rangle : q_i = \langle 1, 0\rangle, q_{i+1} = \langle 0, 1\rangle, \text{ for some } 1 \leq i \leq k-1 \text{ and } q_j = \langle 0, 0\rangle$
  $\text{for all } j \neq i, j \neq i+1\} \cup$
  $\{\langle q_1, \ldots, q_k, 1\rangle : q_i = 1 \text{ for some } 1 \leq i \leq k \text{ and } q_j = \langle 0, 0\rangle, , \text{ for all } j \neq i\}$.

It is easy to check that $\alpha \in \mathcal{L}_\omega(\mathcal{A})$ if and only if $\alpha(i) = \langle 1, 0\rangle$ and $\alpha(j) = \langle 0, 1\rangle$, for i and j such that $i = \overset{k}{\leftarrow} (j)$, and $\alpha(k) = \langle 0, 0\rangle$, for all k such that $k \neq i$ and $k \neq j$.
Any other kind of atomic $S1S_0^k$-formula ϕ is a $S1S_0$-formula. Therefore, by Büchi Theorem, $\mathcal{L}(\phi)$ is a regular ω-language and, by Theorem 2.4, we have that $\mathcal{L}(\phi) \in \mathcal{L}_\omega(k\text{-}STA)$.
(Inductive step). For the inductive step, it suffices to consider the connectives $\neg$ and $\vee$, and the existential quantifier $\exists$. The thesis for these three cases is apparent by closure of $\mathcal{L}_\omega(k\text{-}STA)$ under complementation, union, and projection, respectively (cf. Theorem 2.4). $\dashv$

Corollary 3.5 *The theory $S1S^k$ is decidable.*

Proof. It follows immediately from the decidability of the emptiness problem for $\mathcal{L}_\omega(k\text{-}STA)$ (cf. Theorem 2.4) and Theorem 3.4. $\dashv$

4 Decidable theories of ω-layered temporal structures

In this section, we discuss two theories of ω-layered metric temporal structures, namely, the theory of ω-layered structures consisting of an infinite number of *arbitrarily coarse* infinite temporal domains (called upward unbounded layered structures), and the theory of ω-layered structures consisting of an infinite number of *arbitrarily fine* infinite temporal domains (called downward unbounded layered structures). We first introduce the second-order language $\mathcal{L}^2_{\omega LM^k}$ for ω-layered (k-refinable) metric temporal structures, and show how it can be interpreted over the class of upward unbounded layered structures as well as over the class of downward unbounded ones. Then, we prove that the theory of upward unbounded (k-refinable) layered structures is decidable, through its reduction to the theory $S1S^k$. Successively, we first show that $S1S^k$ cannot be exploited to prove the decidability of the theory of downward unbounded (k-refinable) layered structures, and then we prove that such a theory is decidable, through its reduction to SkS.

4.1 The basic language for ω-layered metric temporal structures

Let $\mathcal{L}^2_{\omega LM^k}$ be the second-order language for ω-layered (k-refinable) metric temporal structures, including individual variables, the binary function symbol $\downarrow$, (uninterpreted) unary predicate symbols, the binary relational symbol $\leq$, and quantification of individual variables and (uninterpreted) unary predicate symbols. We restrict ourselves to formulae that contain no free individual variables. The language $\mathcal{L}^2_{\omega LM^k}$ is formally defined as follows.

Definition 4.1 (Basic language) *Let V and $\mathcal{V}$ be sets of individual and first-order variable symbols, respectively. Terms and formulae of $\mathcal{L}^2_{\omega LM^k}$ are built up as follows:*

- (Terms) *First-order variables $\mathtt{x} \in \mathtt{V}$ are terms. If $\mathtt{t}$ is a term, then $\downarrow(\mathtt{j},\mathtt{t})$, with $0 \leq j \leq k-1$, is a term.*
- (Atomic formulae) *If $\mathtt{t}$ and $\mathtt{t}'$ are terms and $\mathtt{p} \in \mathcal{V}$, then $\mathtt{p}(\mathtt{t})$ and $\mathtt{t} \leq \mathtt{t}'$ are atomic formulae.*
- (Formulae) *Atomic formulae are formulae. If ϕ and ψ are formulae, $\mathtt{x} \in \mathtt{V}$ and $\mathtt{p} \in \mathcal{V}$, then $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$, $\exists\mathtt{x}\phi$, $\exists\mathtt{p}\phi$, $\forall\mathtt{x}\phi$ and $\forall\mathtt{p}\phi$ are formulae.*

In Sections 4.2 and 4.3, we will show how such a language can be interpreted over upward and downward unbounded layered structures, respectively.

4.2 Decidability of upward unbounded layered structures

4.2.1 The theory of upward unbounded layered structures.

In the following, we first formally define upward unbounded layered structures, and then show how to interpret $\mathcal{L}^2_{\omega LM^k}$ over them.

Definition 4.2 (Upward unbounded layered structure) *An upward unbounded ω-layered k-refinable metric temporal structure is a triplet $\langle \bigcup_{i\geq 0} T^i, \downarrow, \leq \rangle$, where*

- *$\{T^i\}_{i\geq 0}$ are pairwise disjoint enumerable sets;*
- *$\downarrow: \{0,\dots k-1\} \times \bigcup_{i\geq 1} T^i \mapsto \bigcup_{i\geq 0} T^i$ is a diadic function such that*
  - *$\downarrow|_{T^i}: \{0,\dots k-1\} \times T^i \mapsto T^{i-1}$, for $i>0$;*
  - *for each $t \in T^{i-1}$ (with $i>0$), there exist $t' \in T^i$ and $0 \leq j \leq k-1$ such that $t = \downarrow(j,t')$;*
- *$\leq$ is a total ordering of $\bigcup_{i\geq 0} T^i$ such that*
  - *$\langle T^i, \leq|_{T^i \times T^i} \rangle$ is isomorphic to $\langle \mathbb{N}, \leq \rangle$, for each $i \geq 0$;*
  - *$\downarrow(0,t) < t$ and $t < \downarrow(j,t)$, for $1 \leq j \leq k-1$;*
  - *$\downarrow(j,t) < \downarrow(j+1,t)$, for $0 \leq j \leq k-2$;*

- $\downarrow(k-1,t) <\downarrow(0,+_i 1(t))$, *for all* $i \geq 1$ *and* $t \in T^i$,

where $\downarrow|_{T^i}$ *and* $\leq|_{T^i \times T^i}$ *denote the restrictions of* $\downarrow$ *to* T^i *and of* $\leq$ *to* $T^i \times T^i$, *respectively,* $t < t'$ *is a shorthand for* $t \leq t'$ *and* $t' \not\leq t$, *and for all* $i \geq 0$ *and* $t \in T^i$, $+_i 1(t)$ *denotes the element* $t' \in T^i$ *such that* $t < t'$ *and* $\neg\exists t''(t'' \in T^i \wedge t < t'' < t')$.

Notice that, for each $t \in T^i$, with $i \geq 1$, and $0 \leq j \leq k-1$, $\downarrow(\mathtt{j},\mathtt{t})$ associates with t the $(j+1)$-th element of its k-decomposition with respect to T^{i-1}.

Let $\mathcal{I}$ be an interpretation of the language $\mathcal{L}^2_{\omega LM^k}$ over upward unbounded layered structures. As usual, let us denote by $c^{\mathcal{I}}$ the element of the domain $\bigcup_{i \geq 0} T^i$ associated with the constant symbol c by $\mathcal{I}$. This notation is extended in a natural way to ground terms and atoms.

We restrict ourselves to interpretations $\mathcal{I}$ that satisfy the following conditions:

$$(\downarrow(\mathtt{j},\mathtt{t}))^{\mathcal{I}} = \begin{cases} \downarrow(j, t^{\mathcal{I}}) & \text{if } t^{\mathcal{I}} \in \bigcup_{i>0} T^i \\ \perp & \text{otherwise;} \end{cases}$$

$$(\leq)^{\mathcal{I}} = \leq,$$

where $\perp$ stands for *undefined*.

We now define the notion of satisfaction of a formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ by an interpretation $\mathcal{I}$. As usual, the definition is recursive on the structure of the formula.

Definition 4.3 (Satisfiability relation) *Let* $\mathcal{I}$ *be an interpretation for the language* $\mathcal{L}^2_{\omega LM^k}$, *and let* $\mu : V \rightarrow \bigcup_{i \geq 0} T^i$ *and* $\nu : \mathcal{V} \rightarrow 2^{\bigcup_{i \geq 0} T^i}$ *be the valuations of individual and first-order variables, respectively. We write* $(\phi)^{\mathcal{I},\mu,\nu} = true$ *to indicate that* $\mathcal{I}$ *satisfies* ϕ *under* μ *and* ν. *The satisfiability relation is formally defined as follows:*

- $(\mathtt{p}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = true \Leftrightarrow t^{\mathcal{I},\mu} \in \nu(p)$;
- $(\mathtt{t_1} \leq \mathtt{t_2})^{\mathcal{I},\mu,\nu} = true \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \leq^{\mathcal{I}}$;
- *boolean connectives and quantifiers are dealt with in the usual way.*

Notice that, from the definition of satisfiability relation, it follows that (atomic) formulae evaluate to false whenever at least one of their arguments evaluates to $\perp$.

Since we are interested in formulae ϕ that contain no individual variables, two distinct interpretations may differ from each other only in the values they assign to free first-order variables. Formally, let ϕ be a formula of $\mathcal{L}^2_{\omega LM^k}$, with free predicate symbols $\mathtt{p_1}, \ldots, \mathtt{p_m}$, that contains no free individual variables. An interpretation $\mathcal{I}$ for ϕ, under valuations μ and ν for individual and first-order variables, is given by m sets $p_1^{\mathcal{I}}, \ldots, p_m^{\mathcal{I}} \subseteq \bigcup_{i \geq 0} T^i$, where, for all $t \in \bigcup_{i \geq 0} T^i$ and $1 \leq j \leq m$, $t \in p_j^{\mathcal{I}}$ if and only if $\mathtt{p_j}$ holds at t.

4.2.2 Mapping $\mathcal{L}^2_{\omega LM^k}$ into $\mathbf{S1S^k}$.

In this section, we prove that the theory of upward unbounded layered structures is decidable, by defining a translation function τ that maps $\mathcal{L}^2_{\omega LM^k}$-formulae into equisatisfiable $S1S^k$-formulae.

First of all, observe that upward unbounded layered structures may differ from each other only in the names of their elements, that is, they are isomorphic. Formally, given a pair of upward unbounded layered structures $\mathcal{T}' = \langle \bigcup_{i\geq 0} T'^i, \downarrow', \leq' \rangle$ and $\mathcal{T}'' = \langle \bigcup_{i\geq 0} T''^i, \downarrow'', \leq'' \rangle$, the mapping $f : \bigcup_{i\geq 0} T'^i \to \bigcup_{i\geq 0} T''^i$ that associates the j-th element of T'^i with the j-th element of T''^i is a bijection which preserves projection and ordering. Therefore, it follows that a formula ϕ is satisfiable under an interpretation $\mathtt{p}_1^{\mathcal{I}}, \ldots \mathtt{p}_\mathtt{m}^{\mathcal{I}} \subseteq \bigcup_{\mathtt{i}\geq 0} \mathtt{T'^i}$ (where $\mathtt{p}_1, \ldots, \mathtt{p}_\mathtt{m}$ are the free predicates symbols occurring in ϕ) if and only if ϕ is satisfiable under the interpretation $\mathtt{f}(\mathtt{p}_1^{\mathcal{I}}), \ldots, \mathtt{f}(\mathtt{p}_\mathtt{m}^{\mathcal{I}}) \subseteq \bigcup_{\mathtt{i}\geq 0} \mathtt{T''^i}$. This property allows us to replace the class of upward unbounded layered structures by a suitable single *concrete* structure which can be easily encoded into $S1S^k$.

Definition 4.4 *The* concrete *upward unbounded layered structure is the triplet:*

$$\mathcal{C} = \langle \bigcup_{i\geq 0} T^{i\mathcal{C}}, \downarrow^{\mathcal{C}}, \leq^{\mathcal{C}} \rangle,$$

where

- $T^{i\mathcal{C}} = \{k^i + nk^{i+1} : n \geq 0\}$;
- $\downarrow^{\mathcal{C}}: \{0, \ldots, k-1\} \times \bigcup_{i\geq 1} T^{i\mathcal{C}} \to \bigcup_{i\geq 0} T^{i\mathcal{C}}$ *maps the pair* $(j, k^i + nk^{i+1})$ *into* $k^{i-1} + (j + nk)k^i$*, for* $0 \leq j \leq k-1$ *and* $i \geq 0$*;*
- $\leq^{\mathcal{C}}$ *is the restriction to* $\bigcup_{i\geq 0} T^{i\mathcal{C}}$ *of the usual ordering on natural numbers.*

In particular, from the above definition, it follows that the least element of the i-th layer $0_i^{\mathcal{C}}$ is equal to k^i and that the successor function $+_i^{\mathcal{C}}1$ maps $k^i + nk^{i+1}$ into $k^i + (n+1)k^{i+1}$, for each $i \geq 0$.

The following proposition proves that the *concrete* upward unbounded layered structure is well-defined.

Proposition 4.5 *$\mathcal{C}$ is an upward unbounded layered structure.*

Proof. For $i, j \geq 0$, with $i \neq j$, $T^{i\mathcal{C}}$ and $T^{j\mathcal{C}}$ are disjoint. In fact, if $t \in T^{i\mathcal{C}}$ and the k-ary representation of t is $a_n k^n + \ldots + a_0 k^0$, then $a_i = 1$ and $a_l = 0$ for $0 \leq l < i$. For $t = k^i + nk^{i+1}$, we have that:

- $\downarrow^{\mathcal{C}} (0, t) = k^{i-1} + nk^{i+1} < k^i + nk^{i+1} = t$;
- $t = k^i + nk^{i+1} < k^{i-1} + (j + nk)k^i = \downarrow^{\mathcal{C}} (j, t)$, for $j > 0$;
- $\downarrow^{\mathcal{C}} (j, t) = k^{i-1} + (j + nk)k^i < k^{i-1} + (j + 1 + nk)k^i = \downarrow^{\mathcal{C}} (j+1, t)$;
- $\downarrow^{\mathcal{C}} (k-1, t) = k^{i-1} + (k - 1 + nk)k^i < k^{i-1} + (n+1)k^{i+1} = \downarrow^{\mathcal{C}} (0, +_i^{\mathcal{C}}(t))$.

$\dashv$

It is worth noting that, for $k = 2$, there exists a unique concrete structure, while alternative definitions are possible for $k > 2$.

The next step consists in showing that the function $\downarrow^{\mathcal{C}}$ can be expressed in $S1S^k$ by defining a suitable mapping $\tau : \mathcal{L}^2_{\omega LM^k} \to S1S^k$. Such a mapping is inductively defined as follows:

- if ϕ is an atomic formula devoid of any occurrence of terms of the form $\downarrow(\mathtt{j},\mathtt{t})$, then
  $\tau(\phi) = \phi$;
- if ϕ is an atomic formula and $\downarrow(\mathtt{j_1},\mathtt{x_1}),\ldots,\downarrow(\mathtt{j_n},\mathtt{x_n})$ are the n innermost occurrences of $\downarrow$ in ϕ, with $n \leq 2$, then
  $\tau(\phi) = \tau(\phi[\mathtt{z_1}\backslash\downarrow(\mathtt{j_1},\mathtt{x_1}),\ldots,\mathtt{z_n}\backslash\downarrow(\mathtt{j_n},\mathtt{x_n})]) \wedge \bigwedge_{\mathtt{1\leq i\leq n}} \mathtt{x_i} \overset{\mathtt{j_i}}{\rightarrow} \mathtt{z_i}$,
  where $\mathtt{z_1},\ldots,\mathtt{z_n}$ are fresh variables and $\mathtt{x} \overset{\mathtt{j}}{\rightarrow} \mathtt{z}$ is a shorthand for the formula of Equation 2 (if $j = 0$) or of Equation 3 (if $j > 0$);
- if $\phi = \neg\psi$, then
  $\tau(\phi) = \neg\tau(\psi)$;
- if $\phi = \psi \wedge \theta$ (resp. $\phi = \psi \vee \theta$), then
  $\tau(\phi) = \tau(\psi) \wedge \tau(\theta)$ (resp. $\tau(\phi) = \tau(\psi) \vee \tau(\theta)$);
- if $\phi = \exists\mathtt{x}\psi$ (resp. $\forall\mathtt{x}\psi$), then
  $\tau(\phi) = \exists\mathtt{x}(\mathtt{L_1}(\mathtt{x}) \wedge \tau(\psi))$ (resp. $\tau(\phi) = \forall\mathtt{x}(\mathtt{L_1}(\mathtt{x}) \rightarrow \tau(\psi))$;
- if $\phi = \exists\mathtt{p}\psi$ (resp. $\forall\mathtt{p}\psi$), then
  $\tau(\phi) = \exists\mathtt{p}(\forall\mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{L_1}(\mathtt{y})) \wedge \tau(\psi))$
  (resp. $\tau(\phi) = \forall\mathtt{p}(\forall\mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{L_1}(\mathtt{y})) \rightarrow \tau(\psi))$),
  where $\mathtt{L_1}$ is a shorthand for the formula of Equation 1.

Lemma 4.6 *For any $\phi \in \mathcal{L}^2_{\omega LM^k}$, ϕ is satisfiable with respect to the concrete structure $\mathcal{C}$ if and only if $\tau(\phi)$ is satisfiable with respect to $\langle \mathbb{N}, \overset{k}{\leftarrow}, L, \leq\rangle$.*

Proof. Hereinafter, let $\mathcal{I}$ and $\mathcal{J}$ denote interpretations for $\mathcal{L}^2_{\omega LM^k}$ and $S1S^k$, respectively. For the sake of simplicity, we assume that the sets of individual and first-order variables of $\mathcal{L}^2_{\omega LM^k}$ and $S1S^k$ coincide.

($\Rightarrow$) Let ϕ be a formula, $\mathcal{I}$ be an intepretation for $\mathcal{L}^2_{\omega LM^k}$, μ be the valuation of individual variables, and ν be the valuation for first-order variables, and suppose that $(\phi)^{\mathcal{I},\mu,\nu} = true$. We prove that there exist an interpretation $\mathcal{J}$ for $S1S^k$ and a valuation μ' such that $(\tau(\phi))^{\mathcal{J},\mu',\nu} = true$, where μ' differs from μ at most in the valuation of the fresh variables introduced by τ. The proof is by induction on the structure of ϕ.

(Base case). Let ϕ be an atomic formula. The proof is by induction on the maximum number n of nested occurrences of a term having the form $\downarrow(j,t)$ in ϕ (*nesting degree*).

Let $n = 0$. By the definition of τ, $\tau(\phi) = \phi$. Since $\bigcup_{i\geq 0} T^{i\mathcal{C}} \subseteq \mathbb{N}$ and $\tau(\phi)$ $(= \phi)$ does not include any occurrence of the function $\downarrow$, it immediately follows that $(\tau(\phi))^{\mathcal{J},\mu,\nu} = true$, for any interpretation $\mathcal{J}$.

Let $n > 0$. Let $\downarrow(j_1,x_1),\ldots,\downarrow(j_m,x_m)$ be the m innermost occurrences of $\downarrow$ in ϕ and let $z_1,\ldots,z_m$ be the corresponding m fresh variables. Let μ' be the map such that $\mu'(z_i) = (\downarrow(j_i,x_i))^{\mathcal{I},\mu}$, for $1 \leq i \leq m$, and $\mu' = \mu$ elsewhere. If $(\phi)^{\mathcal{I},\mu,\nu} = true$, then the formula $\phi' = \phi[z_1\backslash\downarrow(j_1,x_1),\ldots,z_m\backslash\downarrow(j_m,x_m)]$ is an atomic formula with nesting degree $n-1$ such that $(\phi')^{\mathcal{I},\mu',\nu} = true$. By the induction hypothesis, there exist $\mathcal{J}$ and μ'' such that $\tau(\phi')^{\mathcal{J},\mu'',\nu} = true$, where μ'' coincides with μ' at least on all individual variables occurring in ϕ'. Now, it is easy to check that $(\bigwedge_{1\leq i\leq m} x_i \overset{j_i}{\rightarrow} z_i)^{\mathcal{I},\mu'',\nu} = true$, and hence $(\tau(\phi))^{\mathcal{I},\mu'',\nu} = true$.

(Inductive step). The proofs for the boolean connectives $\phi = \neg\psi$, $\phi = \psi\wedge\theta$ and $\phi = \psi\vee\theta$ are straight forward, and thus omitted.

As for the quantifiers, let us consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists x\psi$. The other quantifications are dealt with in a similar way.

Suppose that $(\phi)^{\mathcal{I},\mu,\nu} = true$. Since $\phi = \exists x\psi$, this means that there exists $\overline{\mu}$ that differs from μ at most in the value it assigns to x such that $(\psi)^{\mathcal{I},\overline{\mu},\nu} = true$. By the induction hypothesis, there exist $\mathcal{J}$, μ', and ν' such that $(\tau(\psi))^{\mathcal{J},\mu',\nu'} = true$ and $\mu'(x) = \overline{\mu}(x)$. From $\mu'(x) = \overline{\mu}(x)$, it follows that $(L_1(x))^{\mathcal{J},\mu',\nu'} = true$, and therefore $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.

($\Leftarrow$) Let ϕ be a formula, $\mathcal{J}$ be an intepretation for $S1S^k$, μ be the valuation of individual variables, and ν be the valuation for first-order variables, and suppose that $\tau(\phi)^{\mathcal{J},\mu,\nu} = true$. We prove that there exist an interpretation $\mathcal{I}$ for $\mathcal{L}^2_{\omega LM^k}$, and two valuations μ' and ν' such that $(\phi)^{\mathcal{I},\mu',\nu'} = true$, where μ' and ν' are defined as follows:

$$\mu'(x) = max\{v : v \in \bigcup_{i \geq 0} T^{i\mathcal{C}}, v \leq \mu(x)\} \tag{8}$$

$$\nu'(p) = \{v : \overline{v} \in \nu(p)\} \wedge v = max\{\overline{\overline{v}} : \overline{\overline{v}} \leq \overline{v}\}\}. \tag{9}$$

As for the opposite implication, the proof is by induction on the structure of ϕ.

(Base case). Let ϕ be an atomic formula. The proof is by induction on the nesting degree of ϕ.

If $n = 0$, then ϕ has either the form $p(x)$ or the form $x \leq x'$. From the definitions of μ' and ν', it immediately follows that $(\phi)^{\mathcal{I},\mu',\nu'} = true$, for any interpretation $\mathcal{I}$.

If $n > 0$, then from $\tau(\phi)^{\mathcal{J},\mu,\nu} = true$ it follows that both

$$(\tau(\phi[z_1 \backslash \downarrow (j_1, x_1), \ldots, z_m \backslash \downarrow (j_m, x_m)]))^{\mathcal{J},\mu,\nu} = true$$

and

$$(\bigwedge_{1 \leq i \leq m} x_i \xrightarrow{j_i} z_i)^{\mathcal{J},\mu,\nu} = true.$$

By the induction hypothesis, it follows that there exists $\mathcal{I}$ such that $(\phi[z_1 \backslash \downarrow (j_1, x_1), \ldots, z_m \backslash \downarrow (j_m, x_m)])^{\mathcal{I},\mu',\nu'} = true$. Since $(x_i \xrightarrow{j_i} z_i)^{\mathcal{J},\mu,\nu} = true$, for $1 \leq i \leq m$, it follows that $\mu(x_i), \mu(z_i) \in \bigcup_{i \geq 0} T^{i\mathcal{C}}$ and thus $\mu'(x_i) = \mu(x_i)$, $\mu'(z_i) = \mu(z_i)$ and $\mu'(z_i) = (\downarrow (j_i, x_i))^{\mathcal{I},\mu'}$. This allows us to conclude that $(\phi)^{\mathcal{I},\mu',\nu'} = true$.

(Inductive step). The proofs for the boolean connectives are straight forward, and thus omitted. As for the quantifiers, we only consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists x\psi$. Let $(\tau(\phi))^{\mathcal{J},\mu,\nu} = true$. By the induction hypothesis, it follows that there exists $\mathcal{I}$ such that $(\psi)^{\mathcal{I},\mu',\nu'} = true$, with μ' and ν' defined as in Equation 8 and Equation 9, respectively. Since $(L_1(x))^{\mathcal{J},\mu,\nu} = true$, by the definition of μ' we have that $\mu'(x) \in \bigcup_{i \geq 0} T^{i\mathcal{C}}$ and thus $(\exists x\psi)^{\mathcal{I},\mu',\nu'} = true$. $\dashv$

Theorem 4.7 *The theory of upward unbounded layered structures is decidable.*

Proof. It follows immediately from Lemma 4.6 and the decidability of the theory $S1S^k$ (cf. Theorem 3.5). $\dashv$

4.3 Decidability of downward unbounded layered structures

In the following sections, we define the theory of downward unbounded layered structures, and prove its decidability by reduction to the decidable theory SkS.

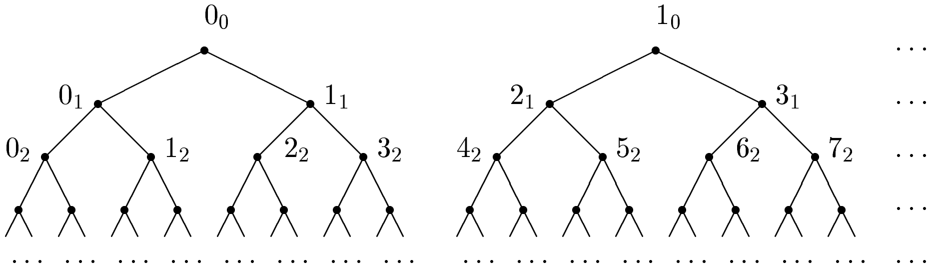

Figure 3: A downward unbounded (2-refinable) structure (i_j stands for $(+_j1)^i(0_j)$).

4.3.1 The theory of downward unbounded layered structures.

As for the upward case, we first formally define downward unbounded layered structures, and then show how to interpret $\mathcal{L}^2_{\omega LM^k}$ over them.

Definition 4.8 (Downward unbounded layered structure) *A downward unbounded ω-layered k-refinable metric temporal structure is a triplet $\langle \bigcup_{i\geq 0} T^i, \downarrow, \leq \rangle$, where*

- $\{T^i\}_{i\geq 0}$ *are pairwise disjoint enumerable sets;*
- $\downarrow$: $\{0, \ldots k-1\} \times \bigcup_{i\geq 0} T^i \mapsto \bigcup_{i\geq 0} T^i$ *is a diadic function such that*
  - $\downarrow\!|_{T^i}$: $\{0, \ldots k-1\} \times T^i \mapsto T^{i+1}$, *for* $i \geq 0$;
  - *for each* $t \in T^i$ *(with* $i \geq 1$*), there exists* $t' \in T^{i-1}$ *such that* $t = \downarrow(j, t')$;
- $\leq$ *is a total ordering of* $\bigcup_{i\geq 0} T^i$ *such that*
  - $\langle T^i, \leq\!|_{T^i \times T^i} \rangle$ *is isomorphic to* $\langle \mathbb{N}, \leq \rangle$, *for each* $i \geq 0$;
  - $t < \downarrow(j, t)$, *for* $0 \leq j \leq k-1$;
  - $\downarrow(j, t) < \downarrow(j+1, t)$, *for* $0 \leq j \leq k-2$;
  - *if* $t < t'$, *then* $\downarrow(j, t) < t'$, *for* $0 \leq j \leq k-1$,

  where $\downarrow\!|_{T^i}$, $\leq\!|_{T^i \times T^i}$, *and* $+_i1$ *are defined as in Definition 4.2.*

An example of downward unbounded layered structure is shown in Figure 3.

From Definition 4.8, we can easily derive the truth of the following proposition:

Proposition 4.9 *For any* $t \in T^i$,

- $\downarrow(j+1, t) = +_{i+1}1(\downarrow(j, t))$
- $\downarrow(0, +_i1(t)) = +_{i+1}1(\downarrow(k-1, t))$.

Interpretations $\mathcal{I}$ of the language $\mathcal{L}^2_{\omega LM^k}$ over downward unbounded layered structures differ from the interpretations over upward unbounded layered structures only in the definition of the semantic clause for $\downarrow$:

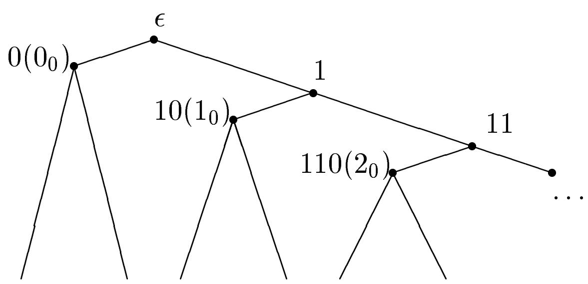

Figure 4: The encoding of a downward unbounded (2-refinable) structure into $\{0,1\}^\star$.

$(\downarrow(\mathtt{j},\mathtt{t}))^{\mathcal{I}} = \downarrow(\mathtt{j},\mathtt{t}^{\mathcal{I}})$.

Unlike the case of upward unbounded layered structures, the interpretation of $(\downarrow(\mathtt{j},\mathtt{t}))$ over downward unbounded layered structures is indeed always defined.

The satisfability relation is defined exactly as in the case of upward unbounded layered structures.

4.3.2 Mapping $\mathcal{L}^2_{\omega LM^k}$ into SkS.

In this section, we prove that the theory of downward unbounded layered structures is decidable, by showing that any formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ can be transformed into an equisatisfiable formula of SkS. In particular, for each $0 \leq j \leq k-1$, the function $\downarrow(j,.)$ acts as $succ_j$ (i.e. the j-th successor of SkS), and the ordering $\leq$ of $\mathcal{L}^2_{\omega LM^k}$ can be expressed by using the prefix order $\leq_P$ of SkS. In Figure 4, we show how a downward unbounded layered structure can be encoded into the domain of interpretation of SkS (i.e., $\{0,\dots,k-1\}^\star$).

More formally, we consider a downward unbounded layered structure

$$\mathcal{R} = \langle \bigcup_{i\geq 0} T^{i\mathcal{R}}, \downarrow^{\mathcal{R}}, \leq^{\mathcal{R}} \rangle,$$

where

- $T^{i\mathcal{R}} = \{v \cdot 0 \cdot w : v \in \{k-1\}^\star, w \in \{0,\dots,k-1\}^\star, |w| = i\}$;
- $\downarrow^{\mathcal{R}}(j,v) = v \cdot j (= succ_j(v))$, for all $v \in \bigcup_{i\geq 0} T^{i\mathcal{R}}$;
- $\leq^{\mathcal{R}}$ is the restriction to $\bigcup_{i\geq 0} T^{i\mathcal{R}}$ of the lexicographic ordering over $\{0,\dots,k-1\}^\star$.

It is easy to verify that $\mathcal{R}$ is a downward unbounded layered structure.

Proposition 4.10 *Let $\mathcal{S} = \langle \bigcup_{i\geq 0} T^i, \downarrow, \leq \rangle$ be a downward unbounded layered structure. For each $i \geq 0$, $t \in T^i$, there exists $j \in \mathbb{N}$ such that $t = (+_i 1)^j(0_i)$. The map $cod : \bigcup_{i\geq 0} T^i \mapsto \bigcup_{i\geq 0} T^{i\mathcal{R}}$ such that*

$$cod(t) = \begin{cases} (k-1)^j \cdot 0 & \text{if } i = 0; \\ (k-1)^q \cdot 0 \cdot r & \text{otherwise,} \end{cases}$$

where $j = k^i q + r$, with $0 \leq r < k$, is an isomorphism of downward unbounded layered structures.

The proof is straight forward, and thus omitted.

In order to map a formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ into an equisatisfiable formula of SkS, we define two auxiliary predicates $\mathtt{R}$ and $\mathtt{Dom}$ of SkS. These predicates are defined in such a way that $\mathtt{R(x)}$ holds if and only if $x^{\mathcal{I}} \in \{k-1\}^\star$, and $\mathtt{Dom(x)}$ holds if and only if $x^{\mathcal{I}} \in \bigcup_{i \geq 0} T^{i\mathcal{R}}$ (which is a proper subset of $\{0, \ldots, k-1\}^\star \setminus \{k-1\}^\star$).
Formally, $\mathtt{R(x)}$ is a shorthand for the formula:

$$\begin{aligned}\exists p(p(\epsilon) \wedge p(x) \wedge \forall y, z(\quad & p(y) \rightarrow p(succ_{k-1}(y)) \wedge \\ & (p(y) \wedge \textstyle\bigvee_{i=0}^{k-1} succ_i(z) = y) \rightarrow p(z) \wedge \\ & (p(z) \wedge p(y)) \rightarrow \neg \textstyle\bigvee_{i=0}^{k-2} succ_i(z) = y)),\end{aligned}$$

while $\mathtt{Dom(x)}$ is a shorthand for the formula:

$$\begin{aligned}\exists p(p(x) \wedge \forall y, z(\quad & R(y) \rightarrow (\neg p(y) \wedge p(succ_0(y)) \wedge \textstyle\bigwedge_{j=1}^{k-1} \neg p(succ_j(y))) \wedge \\ & p(y) \rightarrow \textstyle\bigwedge_{j=0}^{k-1} p(succ_j(y)) \wedge \\ & (p(y) \wedge \neg R(z) \wedge \textstyle\bigvee_{j=1}^{k-1} succ_j(z) = y) \rightarrow p(z)))\end{aligned}$$

The transformation of a formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ into a formula of SkS is defined as follows:

if $\phi = \mathtt{t} \leq \mathtt{t}'$, then
  $\tau(\phi) = \mathtt{Dom(t_1)} \wedge \mathtt{Dom(t_2)} \wedge (\mathtt{t_1} \leq_{\mathtt{P}} \mathtt{t_2} \vee$
    $\exists x(\bigvee_{l=0}^{k-1}(succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2))),$
  where $\mathtt{t_1}$ and $\mathtt{t_2}$ are the terms obtained from $\mathtt{t}$ and $\mathtt{t}'$, respectively, by replacing any occurrence of a term of the form $\downarrow(j, t'')$ by $succ_j(t'')$;
if $\phi = \mathtt{p(t)}$, then
  $\tau(\phi) = \mathtt{Dom(t')} \wedge \mathtt{p(t')}$,
  where $\mathtt{t}'$ is the term obtained from $\mathtt{t}$ by replacing any occurrence of a term $\downarrow(j, t'')$ by $succ_j(t'')$;
if $\phi = \neg\psi$, then
  $\tau(\phi) = \neg\tau(\psi)$;
if $\phi = \psi \wedge \theta$ (resp. $\phi = \psi \vee \theta$), then
  $\tau(\phi) = \tau(\psi) \wedge \tau(\theta)$ (resp. $\tau(\phi) = \tau(\psi) \vee \tau(\theta)$);
if $\phi = \exists \mathtt{x}\psi$ (resp. $\forall \mathtt{x}\psi$), then
  $\tau(\phi) = \exists \mathtt{x}(\mathtt{Dom(x)} \wedge \tau(\psi))$ (resp. $\tau(\phi) = \forall \mathtt{x}(\mathtt{Dom(x)} \rightarrow \tau(\psi))$;
if $\phi = \exists \mathtt{p}\psi$ (resp. $\forall \mathtt{p}\psi$), then
  $\tau(\phi) = \exists \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{Dom(y)}) \wedge \tau(\psi))$
  (resp. $\tau(\phi) = \forall \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{Dom(y)}) \rightarrow \tau(\psi)))$.

The relationship between the original formula $\phi \in \mathcal{L}^2_{\omega LM^k}$ and the resulting formula $\phi' \in SkS$ is formally stated by the following lemma.

Lemma 4.11 *For any $\phi \in \mathcal{L}^2_{\omega LM^k}$, ϕ is satisfiable with respect to the class of downward unbounded layered structures if and only if $\tau(\phi)$ is satisfiable with respect to $\langle\{0, \ldots, k-1\}^*, \epsilon, succ_0, \ldots, succ_{k-1}, <_P\rangle$.*

Proof. The proof is by induction on the structure of ϕ. Hereinafter, let $\mathcal{I}$ and $\mathcal{J}$ denote interpretations for the languages $\mathcal{L}^2_{\omega LM^k}$ and SkS, respectively. For the sake of simplicity, we assume that the sets of individual and first-order variables of $\mathcal{L}^2_{\omega LM^k}$ and SkS coincide.

($\Rightarrow$) Let $\mathcal{I}$ be an intepretation for $\mathcal{L}^2_{\omega LM^k}$, μ be a valuation of individual variables, and ν be a valuation of first-order variables, and suppose that $(\phi)^{\mathcal{I},\mu,\nu} = true$. We prove that there exists an interpretation $\mathcal{J}$ for SkS such that if we take the valuations μ' and ν', with $\mu'(x) = cod(\mu(x))$, for any individual variable x, and $\nu'(p) = \{cod(t) : t \in \nu(p)\}$, for any first-order variable p, then $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.
(Base case). The only non-trivial case is $\phi = t \leq t'$. If $(t)^{\mathcal{I},\mu,\nu} \leq (t')^{\mathcal{I},\mu,\nu}$, then there are three possible cases:

(1) $(t)^{\mathcal{I},\mu,\nu}$ is an ancestor of $(t')^{\mathcal{I},\mu,\nu}$. This implies that there are $v_1, \ldots, v_n \in \bigcup_{i \geq 0} T^i$ such that $v_1 = (t)^{\mathcal{I},\mu,\nu}$, $v_n = (t')^{\mathcal{I},\mu,\nu}$, and $v_{i+1} =\downarrow (j_i, v_i)$, for each $1 \leq i \leq n-1$. Let t_1 and t_2 be the terms obtained from t and t' by replacing any occurrence of the operator $\downarrow (j,.)$ by the j-th successor of SkS. Since cod is an isomorphism, we have that $(t_1)^{\mathcal{J},\mu',\nu'} = cod((t)^{\mathcal{I},\mu,\nu})$ and $(t_2)^{\mathcal{J},\mu',\nu'} = cod((t)^{\mathcal{I},\mu,\nu}) \cdot j_1 \cdot \ldots \cdot j_{n-1}$. From the definitions of cod and Dom, it immediately follows that both $(Dom(t_1))^{\mathcal{J},\mu',\nu'}$ and $(Dom(t_2))^{\mathcal{J},\mu',\nu'}$ hold. Moreover, from the definition of $\leq_P$, it is easy to see that also $(t_1)^{\mathcal{J},\mu',\nu'} \leq_P (t_2)^{\mathcal{J},\mu',\nu'}$ holds, and thus $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.

(2) There exists $v \in \bigcup_{i \geq 0} T^i$ such that v is an ancestor of both $(t)^{\mathcal{I},\mu,\nu}$ and $(t')^{\mathcal{I},\mu,\nu}$. If we take the valuation μ'', which differs from μ' at most in the value $cod(v)$ it assigns to x, we have that $(\bigvee_{l=0}^{k-1} succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2)^{\mathcal{J},\mu'',\nu'} = true$, where t_1 and t_2 are defined as in (1). It follows that $(\exists x(\bigvee_{l=0}^{k-1} succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2))^{\mathcal{J},\mu',\nu'} = true$, and thus the thesis.

(3) There exist $v, v' \in \bigcup_{i \geq 0} T^i$ such that v is an ancestor $(t)^{\mathcal{I},\mu,\nu}$ and v' is an ancestor of $(t')^{\mathcal{I},\mu,\nu}$, with $v \leq v'$. If we take the valuation μ'', which differs from μ' at most in the value $(k-1)^m$ it assigns to x, with $cod(v) = (k-1)^m \cdot 0$, then we have that $(\bigvee_{l=0}^{k-1} succ_l(x) \leq_P t_1 \wedge \bigvee_{s=l+1}^{k-1} succ_s(x) \leq_P t_2)^{\mathcal{J},\mu'',\nu'} = true$, where t_1 and t_2 are defined as in (1), and thus the thesis.

(Inductive step). The proofs for the boolean connectives $\phi = \neg\psi$, $\phi = \psi \wedge \theta$ and $\phi = \psi \vee \theta$ are straight forward, and thus omitted.
As for the quantifiers, let us consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists x \psi$. The other quantifications are dealt with in a similar way. Assume that $(\phi)^{\mathcal{I},\mu,\nu} = true$. This means that there exists $\overline{\mu}$ that differs from μ at most in the value it assigns to x such that $(\psi)^{\mathcal{I},\overline{\mu},\nu} = true$. By the induction hypothesis, there exists $\mathcal{J}$ such that $(\tau(\psi))^{\mathcal{J},\overline{\mu}',\nu'} = true$. Now, by construction, $\overline{\mu}'(x) = cod(\overline{\mu}(x))$, and thus $(Dom(x))^{\mathcal{J},\overline{\mu}',\nu'} = true$. This allows us to conclude that $(\tau(\phi))^{\mathcal{J},\mu',\nu'} = true$.

($\Leftarrow$) The proof of the opposite implication is only sketched. Assume that $(\tau(\phi))^{\mathcal{J},\mu,\nu} = true$, for given $\mathcal{J}, \mu$, and ν. It can be easily proved, by induction on the structure of ϕ, that there exist two valuations $\overline{\mu}$ and $\overline{\nu}$ such that, for each individual variable x and each first-order variable p, $\overline{\mu}(x) \in cod(\bigcup_{i \geq 0} T^i)$, $\overline{\nu}(p) \subseteq cod(\bigcup_{i \geq 0} T^i)$, and $(\tau(\phi))^{\mathcal{J},\overline{\mu},\overline{\nu}} = true$. Let us take two valuations μ' and ν' such that, for each individual variable x and each first-order variable p, $\mu'(x) = cod^{-1}(\overline{\mu}(x))$ and $\nu'(p) = \{cod^{-1}(w) : w \in \overline{\nu}(p)\}$. It is not difficult to show, by induction on the structure of ϕ, that there exists $\mathcal{I}$ such that $(\phi)^{\mathcal{I},\mu',\nu'} = true$. $\dashv$

Theorem 4.12 *The theory of downward unbounded layered structures is decidable.*

Proof. It follows immediately from Lemma 4.11 and the decidability of the theory SkS (cf. [16]). $\dashv$

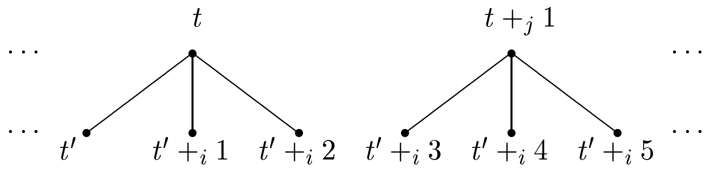

Figure 5: A 3-refinable temporal structure.

4.4 On the ordering relations

In this section, we briefly discuss the basic features of the total ordering relation $\leq$ defined over the domain $\bigcup_{i\geq 0} T^i$ of downward (resp. upward) unbounded layered structures.

In both the downward and the upward case, the relation $\leq$ is defined in such a way that the pair $\langle \bigcup_{i\geq 0} T^i, \leq \rangle$ as well as each pair $\langle T^i, \leq|_{T^i \times T^i} \rangle$, with $i \geq 0$, are isomorphic to the pair $\langle \mathbb{N}, \leq \rangle$. Moreover, in both cases the zero element for $\langle \bigcup_{i\geq 0} T^i, \leq \rangle$ coincides with the zero element for $\langle T^0, \leq|_{T^0 \times T^0} \rangle$.

It is worth noting that we cannot use the same definition of $\leq$ for both classes of structures: ordering upward (resp. downward) unbounded layered structures as downward (resp. upward) ones, we indeed would have no zero element for $\langle \bigcup_{i\geq 0} T^i, \leq \rangle$. This is due to the fact that the topologies of upward and downward unbounded layered structures are intrinsically different.

A number of natural orderings over the domain of both downward and upward unbounded layered structures can be easily defined on the basis of the given one.

As an example, consider a possible interpretation of differently-grained temporal domains as different ways of partitioning the real (rational) time axis. Accordingly, we can interpret each time instant of a given domain as a suitable time interval over such an axis. More precisely, let each time instant be associated with a time interval open on the left and closed on the right. We can easily define an ordering relation $\leq_e$ over time instants such that for all $t, t' \in \bigcup_{i\geq 0} T^i$, $t \leq_s t'$ if and only if the ending point of the interval corresponding to t precedes the ending point of the interval for t' with respect to the standard ordering relation over real (rational) numbers. For k-refinable structures, this implies that any time instant t coincides with its rightmost child $\downarrow (k-1, t)$ and strictly precedes all the others, namely, $\downarrow (0, t)$, $\ldots$, $\downarrow (k-2, t)$. In Figure 5, we depict (a little part of) a 3-refinable structure, where the time instant t (resp. $+_j 1(t)$), belonging to the domain T^j, is decomposed into the three time instants t', $+_i 1(t')$, and $(+_i 1)^2(t')$ (resp. $(+_i 1)^3(t')$, $(+_i 1)^4(t')$, and $(+_i 1)^5(t')$), belonging to the finer domain T^i. For the sake of readability, for each $k \geq 0$, $n > 1$, and x, we denote $(+_k 1)^n(x)$ by $x +_k n$.

Given the temporal structure of Figure 5, the interpretation that maps each time instant into a time interval open on the left and closed on the right is represented in Figure 6.

Similarly, we can associate each time instant with a time interval closed on the left and open on the right, and define an ordering relation $\leq_s$ over time instants such that for all $t, t' \in \bigcup_{i\geq 0} T^i$, $t \leq_s t'$ if and only if the starting point of the interval corresponding to t precedes the starting point of the interval for t'. Accordingly, any time instant t coincides

$$\ldots \; (t, \qquad\qquad\qquad\qquad t +_j 1\,]\,(t +_j 1, \qquad\qquad \ldots$$

$$\ldots \; (t', \quad t' +_i 1]\,(t' +_i 1, t' +_i 2\,]\,(t' +_i 2, t' +_i 3\,]\,(t' +_i 3, t' +_i 4\,] \quad \ldots$$

Figure 6: A possible interval interpretation - case 1.

$$\ldots \; [\,t, \qquad\qquad\qquad\qquad t +_j 1)\,[\,t +_j 1, \qquad\qquad \ldots$$

$$\ldots \; [t', \quad t' +_i 1)\,[t' +_i 1, t' +_i 2\,)\,[t' +_i 2, t' +_i 3\,)\,[t' +_i 3, t' +_i 4\,) \quad \ldots$$

Figure 7: A possible interval interpretation - case 2.

with its leftmost child $\downarrow (0,t)$ and strictly precedes all the others, namely, $\downarrow (1,t)$, $\ldots$, $\downarrow (k-1,t)$. Such an interpretation for the temporal structure given in Figure 7 is graphically depicted in Figure 7.

Both downward and upward unbounded layered structures allow one to define the ordering relations $\leq_e$ and $\leq_s$ in terms of the basic ordering relation $\leq$.

Let us consider first downward unbounded layered structures. The relation $\leq_e$ can be defined as follows:

$$\mathtt{t} \leq_{\mathtt{e}} \mathtt{t}'$$

$$\text{iff}$$

$$\mathtt{ancestor}(\mathtt{t}',\mathtt{t}) \vee (\neg\mathtt{ancestor}(\mathtt{t}',\mathtt{t}) \wedge \neg\mathtt{ancestor}(\mathtt{t},\mathtt{t}') \wedge \mathtt{t} \leq \mathtt{t}') \vee \mathtt{ancestor}_{\mathtt{k-1}}(\mathtt{t},\mathtt{t}'),$$

where $\mathtt{ancestor}(\mathtt{t}',\mathtt{t})$ stands for the fact that t belongs to the tree rooted at t'. In formulae:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}') \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{w}(\mathtt{p}(\mathtt{w}) \rightarrow \bigwedge_{\mathtt{i=0}}^{\mathtt{k-1}} \mathtt{p}(\downarrow (\mathtt{i},\mathtt{w}))) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{z} \neq \mathtt{t}' \wedge$$

$$\wedge \mathtt{p}(\mathtt{z}) \wedge \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow (\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{w}(\bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow (\mathtt{i},\mathtt{w}) = \mathtt{t}' \rightarrow \neg\mathtt{p}(\mathtt{w})) \wedge$$

$$\wedge \forall \mathtt{z}(\neg\exists \mathtt{w} \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow (\mathtt{i},\mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}' \rightarrow \neg\mathtt{p}(\mathtt{z})),$$

and $\mathtt{ancestor}_{\mathtt{k-1}}(\mathtt{t},\mathtt{t}')$ stands for the fact that t can be reached from t' following its rightmost branch ($(k-1)$-th children). In formulae:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}') \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{w}(\mathtt{p}(\mathtt{w}) \rightarrow \mathtt{p}(\downarrow (\mathtt{k}-1,\mathtt{w}))) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{p}(\mathtt{z}) \wedge \mathtt{p}(\mathtt{w}) \wedge \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow (\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow$$

$$\rightarrow \downarrow (\mathtt{k}-1,\mathtt{w}) = \mathtt{z}) \wedge \forall \mathtt{w},\mathtt{z}(\mathtt{z} \neq \mathtt{t}' \wedge \mathtt{p}(\mathtt{z}) \wedge \bigvee_{\mathtt{i=0}}^{\mathtt{k-1}} \downarrow (\mathtt{i},\mathtt{w}) = \mathtt{z} \rightarrow \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{w}(\downarrow (\mathtt{k}-1,\mathtt{w}) = \mathtt{t}' \rightarrow$$

$$\to \neg \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow (\mathtt{i}, \mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}_1 \to \neg \mathtt{p}(\mathtt{z}))).$$

while the relation $\leq_s$ is simply defined as:

$$\mathtt{t} \leq_{\mathtt{s}} \mathtt{t}' \ \text{ iff } \ \mathtt{t} \leq \mathtt{t}' \vee \mathtt{ancestor}_0(\mathtt{t}', \mathtt{t}),$$

where $\mathtt{ancestor}_0(\mathtt{t}', \mathtt{t})$ stands for the fact that t can be reached from t' following its leftmost branch (0-th children), and it can be obtained from $\mathtt{ancestor}_{\mathtt{k}-1}(\mathtt{t}, \mathtt{t}')$ by substituting $\mathtt{0}$ for $\mathtt{k} - \mathtt{1}$ everywhere. In formulae:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}') \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{w}(\mathtt{p}(\mathtt{w}) \to \mathtt{p}(\downarrow (\mathtt{0}, \mathtt{w}))) \wedge \forall \mathtt{w}, \mathtt{z}(\mathtt{p}(\mathtt{z}) \wedge \mathtt{p}(\mathtt{w}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow (\mathtt{i}, \mathtt{w}) = \mathtt{z} \to$$

$$\to \downarrow (\mathtt{0}, \mathtt{w}) = \mathtt{z}) \wedge \forall \mathtt{w}, \mathtt{z}(\mathtt{z} \neq \mathtt{t}' \wedge \mathtt{p}(\mathtt{z}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow (\mathtt{i}, \mathtt{w}) = \mathtt{z} \to \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{w}(\downarrow (\mathtt{0}, \mathtt{w}) = \mathtt{t}' \to$$

$$\to \neg \mathtt{p}(\mathtt{w})) \wedge \forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow (\mathtt{i}, \mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}_1 \to \neg \mathtt{p}(\mathtt{z}))).$$

The definitions of $\leq_e$ and $\leq_s$ for upward unbounded layered structures are exactly the same, except for the fact that the definitions of $\mathtt{ancestor}$, $\mathtt{ancestor}_{\mathtt{k}-1}$, and $\mathtt{ancestor}_0$ do not include the last conjunct:

$$\forall \mathtt{z}(\neg \exists \mathtt{w} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow (\mathtt{i}, \mathtt{w}) = \mathtt{z} \wedge \mathtt{z} \neq \mathtt{t}' \to \neg \mathtt{p}(\mathtt{z})).$$

Many other natural orderings over the domain of layered structures can be defined on the basis of the basic ordering relation $\leq$. A further example will be given in the next section.

5 High-level languages for ω-layered temporal structures

In this section, we demonstrate that the key features of metric and layered temporal logic can actually be expressed in $\mathcal{L}^2_{\omega LM^k}$. To this end, we first define a high-level second-order language for the theory of upward (resp. downward) unbounded layered structures, including suitable primitives for modeling contextualization, projection, and displacement, and then show that the resulting language is in fact as expressive as the basic one.

5.1 The language for upward unbounded layered structures

Let $\mathcal{UL}^2_{\omega LM^k}$ be the high-level second-order language for the theory of upward unbounded layered structures, including individual variables, the constant symbols $\mathtt{0_i}$ (local zero elements), the unary function symbols $+_{\mathtt{i}}\mathtt{1}$ (local successors), the binary function symbol $\downarrow$ (projection), (uninterpreted) unary predicate symbols, the unary predicate symbols $\mathtt{T^i}$ (vertical contextualizations or y-contextualizations) and $\Delta_{\mathtt{i}}$ (horizontal contextualizations or x-contextualizations), the binary relational symbols $\leq$ (ordering), $\preceq$ (approximate ordering) and $\equiv_{\mathtt{i},\mathtt{j}}$ (congruences), and quantification of individual variables and (uninterpreted) unary predicate symbols (with $i \geq 0$). We restrict ourselves to formulae that contain no free individual variables. $\mathcal{UL}^2_{\omega LM^k}$ is formally defined as follows.

Definition 5.1 (Language) *Let V and $\mathcal{V}$ be sets of individual and first-order variable symbols, respectively. Terms and formulae of $\mathcal{UL}^2_{\omega LM^k}$ are built up as follows:*

- (Terms) *First-order variables $\mathtt{x} \in \mathtt{V}$ and constants $\mathtt{0_i}$ are terms. If $\mathtt{t}$ is a term, then $+_\mathtt{i}\mathtt{1(t)}$ and $\downarrow(\mathtt{j},\mathtt{t})$, with $0 \leq j \leq k-1$, are terms.*
- (Atomic formulae) *If $\mathtt{t}$ and $\mathtt{t}'$ are terms and $\mathtt{p} \in \mathcal{V}$, then $\mathtt{T^i(t)}$, $\mathtt{\Delta_i(t)}$, $\mathtt{p(t)}$, $\mathtt{t} \leq \mathtt{t}'$, $\mathtt{t} \preceq \mathtt{t}'$ and $\mathtt{t} \equiv_{\mathtt{i,j}} \mathtt{t}'$ are atomic formulae.*
- (Formulae) *Atomic formulae are formulae. If ϕ and ψ are formulae, $\mathtt{x} \in \mathtt{V}$ and $\mathtt{p} \in \mathcal{V}$, then $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$, $\exists \mathtt{x}\phi$, $\exists \mathtt{p}\phi$, $\forall \mathtt{x}\phi$ and $\forall \mathtt{p}\phi$ are formulae.*

An intepretation $\mathcal{I}$ for the language $\mathcal{L}^2_{\omega LM^k}$ over upward unbounded layered temporal structures is also an interpretation for the high-level language $\mathcal{UL}^2_{\omega LM^k}$ if and only if it satisfies the following further conditions:

$\mathtt{0}_\mathtt{i}^\mathcal{I} = t$, where t is the (unique) element of T^i such that $\forall t'(t' \in T^i \rightarrow t \leq t')$;

$$(+_\mathtt{i}\mathtt{1(t)})^\mathcal{I} = \begin{cases} +_i1(t^\mathcal{I}) & \text{if } t^\mathcal{I} \in T^i \\ \perp & \text{otherwise,} \end{cases}$$

$(\mathtt{T^i})^\mathcal{I} = T^i$;

$(\mathtt{\Delta_i})^\mathcal{I} = \{(+_j1)^i(0_j^\mathcal{I}) : j \geq 0\}$;

$(\preceq)^\mathcal{I} = \{(t,t') : \mathit{ancestor}(t,t') \vee \mathit{ancestor}(t',t) \vee t \leq t'\}$,
where, for all $t, t' \in \bigcup_{i \geq 0} T^i$, $\mathit{ancestor}(t,t')$ if and only if $\exists j(t = \downarrow(j,t') \vee \exists j(t'' = \downarrow(j,t') \wedge \mathit{ancestor}(t,t'')))$;

$(\equiv_{\mathtt{i,j}})^\mathcal{I} = \{(t,t') : t \in T^i \wedge t' \in T^i \wedge t = (+_i1)^m(0_i^\mathcal{I}) \wedge t' = (+_i1)^n(0_i^\mathcal{I}) \wedge m \equiv_j n\}$.

For each $i \geq 0$, $\mathtt{0_i}$ denotes the origin of the domain T^i and $+_\mathtt{i}\mathtt{1}$ is interpreted as the successor function over T^i. y-contextualizations $\mathtt{T^i(t)}$ and x-contextualizations $\mathtt{\Delta_i(t)}$ respectively restrict the range of possible values of the term t, constraining it to be interpreted over the domain T^i and over the set of elements at distance i from the origin of the domain they belong to. It is worth noting that, for each $i \geq 0$, the set of elements satisfying $\mathtt{\Delta_i}$, together with (the obvious restriction of) $\leq$, is isomorphic to the structure $\langle \mathbb{N}, \leq \rangle$. In this respect, x-contextualization and y-contextualization act in a perfectly symmetric way. For each $t, t' \in \bigcup_{i \geq 0} T^i$, $\preceq$ relates t to t' whenever $\mathtt{t} \leq \mathtt{t}'$ holds or either t' belongs to the projection of t over the domain t' belongs to or t belongs to the projection of t' over the domain t belongs to. Finally, for all $i \geq 0$ and $j \geq 2$, $\equiv_{\mathtt{i,j}}$ denotes the congruence modulo-j over T^i.

Example 5.2 We describe a property whose specification exploits the expressive power of the operators of x-contextualization and projection of upward unbounded layered structures, and cannot be expressed in the theory of finitely-layered ones. For the sake of simplicity, we take $k = 2$. Consider a process of decay, where the elapsing time between two successive occurrences of a given phenomenon exponentially increases. It can be modeled in $\mathcal{UL}^2_{\omega LM^k}$ as follows. Let $\mathtt{p}$ and $\mathtt{q}$ be two predicates that respectively detect occurrences and not occurrences of the considered phenomenon. With respect to each temporal domain, we assume that the phenomenon occurs at the first two time instants (that is, with delay 0); the phenomenon is not detected at the next time instant; it occurs again at the fourth instant (that is, with delay 2^0); then it is not detected for 2 time instants, but it surely happens at least at the end of an interval of length 2 (that is, with delay greater than or equal to 2^1 and less than 2^2); then it is not detected for 4 time instants, but it surely happens at least at the end of an interval of length 4 (that is, with delay greater than or equal to 2^2 and less than 2^3); and so on. In such a way, if an occurrence of the phenomenon is detected at a time instant $i \in T^j$, then it occurs at least in its rightmost child, while if it is not detected at $i \in T^j$, then it is not detected at each of its children.

Formally, the process is specified by the following $\mathcal{UL}^2_{\omega LM^k}$-formula:

$$\forall \mathtt{x}((\Delta_0(\mathtt{x}) \vee \Delta_1(\mathtt{x})) \rightarrow \mathtt{p}(\mathtt{x}) \wedge \Delta_2(\mathtt{x}) \rightarrow \mathtt{q}(\mathtt{x}) \wedge \mathtt{p}(\mathtt{x}) \leftrightarrow \neg \mathtt{q}(\mathtt{x}) \wedge$$

$$\wedge \mathtt{q}(\mathtt{x}) \rightarrow (\mathtt{q}(\downarrow(\mathtt{0},\mathtt{x})) \wedge \mathtt{q}(\downarrow(\mathtt{1},\mathtt{x}))) \wedge \mathtt{p}(\mathtt{x}) \rightarrow \mathtt{p}(\downarrow(\mathtt{1},\mathtt{x}))).$$

It is easy to verify that each model of the above formula associates with each layer T^j an ω-sequence of the form $s_0 \cdot s_1 \cdot \ldots \cdot s_i \cdot \ldots$, where $s_0 = \{p\}$ and

$$s_i = \{p\} \cdot \{q\}^{2^{i-1}} \cdot w_i, \text{ with } w_i \in \{p,q\}^*, |w_i| = 2^{i-1} - 1.$$

It is possible to prove that the set of ω-sequences fulfilling the above condition is a non-regular ω-language over the alphabet $\Sigma = \{p,q\}$ belonging to the class $\mathcal{L}_\omega(k\text{-}STA)$.

The notion of satisfaction of a formula $\phi \in \mathcal{UL}^2_{\omega LM^k}$ by an interpretation $\mathcal{I}$ is a straightforward generalization of the corresponding notionf for $\mathcal{UL}^2_{\omega LM^k}$.

Definition 5.3 (Satisfiability relation) *Let $\mathcal{I}$ be an interpretation for the language $\mathcal{UL}^2_{\omega LM^k}$, and let μ and ν be the valuations of individual and first-order variables, respectively. The satisfiability relation is formally defined as follows:*

- $(\mathtt{p}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = \mathit{true} \Leftrightarrow t^{\mathcal{I},\mu} \in \nu(p)$;
- $(\mathtt{T}^\mathtt{i}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = \mathit{true} \Leftrightarrow t^{\mathcal{I},\mu} \in T^{i^{\mathcal{I}}}$;
- $(\Delta_\mathtt{i}(\mathtt{t}))^{\mathcal{I},\mu,\nu} = \mathit{true} \Leftrightarrow t^{\mathcal{I},\mu} \in \Delta_i^{\mathcal{I}}$;
- $(\mathtt{t}_1 \leq \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = \mathit{true} \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \leq^{\mathcal{I}}$;
- $(\mathtt{t}_1 \preceq \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = \mathit{true} \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \preceq^{\mathcal{I}}$;
- $(\mathtt{t}_1 \equiv_{\mathtt{i},\mathtt{j}} \mathtt{t}_2)^{\mathcal{I},\mu,\nu} = \mathit{true} \Leftrightarrow (t_1^{\mathcal{I},\mu}, t_2^{\mathcal{I},\mu}) \in \equiv_{i,j}^{\mathcal{I}}$;
- *boolean connectives and quantifiers are dealt with in the usual way.*

We conclude this section by proving that $\mathcal{UL}^2_{\omega LM^k}$ is as expressive as $\mathcal{L}^2_{\omega LM^k}$. More precisely, we show that for each $i \geq 0$, the constant $\mathtt{0}_\mathtt{i}$, the successor function $+_\mathtt{i}\mathtt{1}$, the vertical and horizontal contextualizations $\mathtt{T}^\mathtt{i}$ and $\Delta_\mathtt{i}$, the congruences $\equiv_{\mathtt{i},\mathtt{j}}$, with $j \geq 2$, and the approximate ordering $\preceq$ can be expressed in terms of $\downarrow$ and $\leq$.

Let $\phi \in \mathcal{UL}^2_{\omega LM^k}$. The following steps transforms ϕ into an equisatisfiable formula $\phi' \in \mathcal{L}^2_{\omega LM^k}$.

Step 1: removal of constants $\mathtt{0}_\mathtt{i}$.

For each $i \geq 0$, we replace all the occurrences of $\mathtt{0_i}$ (if any) by a fresh variable $\mathtt{x_i}$ and add to the resulting formula the conjunct:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_{i-1}} (\forall \mathtt{y} \ \ (\mathtt{x_0} \leq \mathtt{y}) \wedge \bigwedge_{\mathtt{j=1}}^{\mathtt{i}} \downarrow (\mathtt{0}, \mathtt{x_j}) = \mathtt{x_{j-1}}).$$

Step 2: removal of successor functions $+_\mathtt{i}\mathtt{1}$.

Since terms may contain nested occurrences of successor functions, we cannot remove all their occurrences in a single step. At each single step, we can only remove the innermost occurrences of successor functions, that is, each occurrence of $+_\mathtt{i}\mathtt{1}$, with $i \geq 0$, applied to a term $\mathtt{t}$ built up using first-order variables and the binary function $\downarrow$ only.

Let $+_\mathtt{i}\mathtt{1}(\mathtt{t})$ be a term of the above specified form. We replace it by a fresh variable $\mathtt{x_i}$ and add to the resulting formula the conjunct:

$$\mathtt{T^i}(\mathtt{x_i}) \wedge \mathtt{T^i}(\mathtt{t}) \wedge \mathtt{t} < \mathtt{x_i} \wedge \forall \mathtt{y} (\mathtt{T^i}(\mathtt{y}) \wedge \mathtt{t} < \mathtt{y} \rightarrow \mathtt{x_i} \leq \mathtt{y}).$$

We iterate such a replacement until the resulting formula is devoid of occurrences of successor functions.

Step 3: removal of y-contextualizations $\mathtt{T^i}$.

We replace each occurrence of atomic formulae of the form $\mathtt{T^i}(\mathtt{t})$, with $i \geq 0$, by the formula:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_i} (\mathtt{t} = \mathtt{x_i} \wedge \forall \mathtt{y} \ \ \neg \downarrow (\mathtt{0}, \mathtt{x_0}) = \mathtt{y} \wedge \bigwedge_{\mathtt{j=1}}^{\mathtt{i}} (\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}} \downarrow (\mathtt{l}, \mathtt{x_j}) = \mathtt{x_{j-1}})).$$

For $i = 0$, the above formula reduces to:

$$\exists \mathtt{x_0} (\mathtt{t} = \mathtt{x_0} \wedge \forall \mathtt{y} \ \ \neg \downarrow (\mathtt{0}, \mathtt{x_0}) = \mathtt{y}).$$

Step 4: removal of x-contextualizations $\mathtt{\Delta_i}$.

The removal of x-contextualizations $\mathtt{\Delta_i}$ is performed in two steps. First, for each $i > 0$, we replace each occurrence of atomic formulae of the form $\mathtt{\Delta_i}(\mathtt{t})$ by a formula which contains an atomic formula of the form $\mathtt{\Delta_0}(\mathtt{x})$; then, we remove from the resulting formula each occurrence of atomic formulae of the form $\mathtt{\Delta_0}(\mathtt{t})$ (possibly added at the previous step).

For each $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. We replace each occurrence of atomic formulae of the form $\mathtt{\Delta_i}(\mathtt{t})$ by the formula:

$$\exists \mathtt{x_{n+1}}, \ldots, \mathtt{x_0} (\mathtt{\Delta_0}(\mathtt{x_{n+1}}) \wedge \mathtt{x_0} = \mathtt{t} \wedge \bigwedge_{\mathtt{j=0}}^{\mathtt{n}} \downarrow (\mathtt{a_j}, \mathtt{x_{j+1}}) = \mathtt{x_j}).$$

Then, we replace each occurrence of atomic formulae of the form $\mathtt{\Delta_0}(\mathtt{t})$ by the formula:

$$\exists \mathtt{p} (\mathtt{p}(\mathtt{0_0}) \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{y}, \mathtt{z} (\mathtt{p}(\mathtt{y}) \wedge \downarrow (\mathtt{0}, \mathtt{z}) = \mathtt{y} \rightarrow \mathtt{p}(\mathtt{z})) \wedge$$

$$\forall \mathtt{y}, \mathtt{z} (\mathtt{p}(\mathtt{y}) \wedge \mathtt{p}(\mathtt{z}) \rightarrow \bigwedge_{\mathtt{i=1}}^{\mathtt{k-1}} \neg \downarrow (\mathtt{i}, \mathtt{z}) = \mathtt{y})),$$

and then we remove the constant 0_0 by executing the operations specified in step 1.

Step 5: removal of congruences $\equiv_{\mathtt{i,j}}$.

We replace each occurrence of atomic formulae of the form $\mathtt{t_1} \equiv_{\mathtt{i,j}} \mathtt{t_2}$, with $i \geq 0$ and $j \geq 2$, by the formula:

$$\bigvee_{\mathtt{l=0}}^{\mathtt{j-1}} (\mathtt{Cong_i^{l,j}}(\mathtt{t_1}) \wedge \mathtt{Cong_i^{l,j}}(\mathtt{t_2}))$$

where $\mathtt{Cong_i^{m,n}}(\mathtt{t})$, with $m < n$, stands for:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{t}) \wedge \mathtt{p}((+_\mathtt{i}1)^\mathtt{m}(0_\mathtt{i})) \wedge \bigwedge_{\mathtt{l=0,l\neq m}}^{\mathtt{n-1}} \neg \mathtt{p}((+_\mathtt{i}1)^\mathtt{l}(0_\mathtt{i})) \wedge$$

$$\wedge \forall \mathtt{y}(\mathtt{T^i}(\mathtt{y}) \wedge (+_\mathtt{i}1)^\mathtt{n}(0_\mathtt{i}) \leq \mathtt{y} \rightarrow (\mathtt{p}(\mathtt{y}) \leftrightarrow \exists \mathtt{z}(\mathtt{p}(\mathtt{z}) \wedge (+_\mathtt{i}1)^\mathtt{n}(\mathtt{z}) = \mathtt{y})))),$$

and then we remove the occurrences of 0_i, $+_i 1$ and T^i by executing the operations specified in steps 1, 2 and 3, respectively.

Step 6: removal of the approximate ordering $\mathtt{t_1} \preceq \mathtt{t_2}$.

We replace each occurrence of atomic formulae of the form $\mathtt{t_1} \preceq \mathtt{t_2}$ by the formula:

$$\neg \mathtt{ancestor}(\mathtt{t_1}, \mathtt{t_2}) \wedge \neg \mathtt{ancestor}(\mathtt{t_2}, \mathtt{t_1}) \rightarrow \mathtt{t_1} \leq \mathtt{t_2},$$

where $\mathtt{ancestor}(\mathtt{t_1}, \mathtt{t_2})$ is defined as in Section 4.4.

Step 7: universal closure.

The formula ϕ' is the universal closure of the formula obtained by the sequential execution of steps 1-6.

The relationship between the original formula $\phi \in \mathcal{UL}^2_{\omega LM^k}$ and the resulting formula $\phi' \in \mathcal{SUL}^2_{\omega LM^k}$ is formally stated by the following lemma.

Lemma 5.4 *Let ϕ be an $\mathcal{UL}^2_{\omega LM^k}$-formula and ϕ' be the $\mathcal{SUL}^2_{\omega LM^k}$-formula obtained from ϕ after the execution of the transformation steps 1-7. ϕ is satisfiable if and only if ϕ' is satisfiable.*

The proof of the Lemma 5.4 is straight forward, and thus omitted.

5.2 The language for downward unbounded layered structures

The formulae of the high-level second-order language $\mathcal{DL}^2_{\omega LM^k}$ for the theory of downward unbounded layered structures differ from the ones of $\mathcal{UL}^2_{\omega LM^k}$ only for the fact that atomic formulae having the form $\mathtt{\Delta_i}(\mathtt{t})$ are replaced by atomic formulae having the form $\mathtt{\Delta_{i,j}}(\mathtt{t})$, for $i, j \geq 0$.

Interpretations $\mathcal{I}$ of $\mathcal{DL}^2_{\omega LM^k}$ differ from $\mathcal{UL}^2_{\omega LM^k}$-interpretations only in the definition of the semantic clause for $\downarrow$, and in the replacement of the semantic clause for Δ_i by the following one:

$(\mathtt{\Delta_{i,j}})^{\mathcal{I}} = \{(+_k 1)^i((\downarrow)^k(0, (+_0 1)^j(0_0^{\mathcal{I}}))) : k \geq 0\}$.

This semantic clause states that for any $k \geq 0$ and any $t \in T^k$, $\Delta_{i,j}$ holds at t if and only if t is at distance i from the (unique) element $t' \in T^k$ belonging to the leftmost branch of the tree rooted at the (unique) element $t'' \in T^0$ which is at distance j from 0_0. It is easy to see that the predicate Δ_i of $\mathcal{UL}^2_{\omega LM^k}$ is equivalent to $\Delta_{i,0}$.

The satisfability relation is defined exactly as in the case of $\mathcal{UL}^2_{\omega LM^k}$, except for the obvious replacement of Δ_i by $\Delta_{i,j}$.

Example 5.5 Let $\mathtt{p}$ and $\mathtt{q}$ be two predicates (denoting a pair of events or of states). We show how the properties of downward unbounded layered structures can be exploited to constrain $\mathtt{p}$ and $\mathtt{q}$ to be locally indistinguishable (resp. distinguishable). We say that two predicates $\mathtt{p}$ and $\mathtt{q}$ are *locally indistinguishable* with respect to a time instant t if both $\mathtt{p}$ and $\mathtt{q}$ hold at t, and there exists a child t' of t such that $\mathtt{p}$ and $\mathtt{q}$ are locally indistinguishable with respect to t'. Two predicates are *locally distinguishable* with respect to a time instant t if they are not locally indistinguishable with respect to it.

Formally, the condition that $\mathtt{p}$ and $\mathtt{q}$ are locally indistinguishable with respect to (a time instant t denoted by) x can be expressed as follows:

$$\exists \mathtt{r}(\mathtt{path}(\mathtt{x},\mathtt{r}) \wedge \forall \mathtt{y}(\mathtt{r}(\mathtt{y}) \rightarrow \mathtt{p}(\mathtt{y}) \wedge \mathtt{q}(\mathtt{y}))),$$

where $path(x,r)$ stands for:

$$\mathtt{r}(\mathtt{x}) \wedge \forall \mathtt{y}((\mathtt{r}(\mathtt{y}) \rightarrow \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \mathtt{r}(\downarrow(\mathtt{i},\mathtt{y}))) \wedge \forall \mathtt{i},\mathtt{j}(\mathtt{r}(\downarrow(\mathtt{i},\mathtt{y})) \wedge \mathtt{r}(\downarrow(\mathtt{j},\mathtt{y})) \rightarrow \mathtt{i}=\mathtt{j}) \wedge$$

$$\wedge \forall \mathtt{z}(\mathtt{y} \neq \mathtt{x} \wedge \mathtt{r}(\mathtt{y}) \wedge \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{z}) = \mathtt{y} \rightarrow \mathtt{r}(\mathtt{z})) \wedge (\bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{y}) = \mathtt{x} \rightarrow \neg \mathtt{r}(\mathtt{y})) \wedge$$

$$\wedge(\neg \exists \mathtt{z} \bigvee_{\mathtt{i}=0}^{\mathtt{k}-1} \downarrow(\mathtt{i},\mathtt{z}) = \mathtt{y} \wedge \mathtt{y} \neq \mathtt{x} \rightarrow \neg \mathtt{r}(\mathtt{y}))),$$

where $\forall \mathtt{i},\mathtt{j}(\mathtt{r}(\downarrow(\mathtt{i},\mathtt{y})) \wedge \mathtt{r}(\downarrow(\mathtt{j},\mathtt{y})) \rightarrow \mathtt{i}=\mathtt{j})$ is the obvious shorthand.

The condition that $\mathtt{p}$ and $\mathtt{q}$ are locally distinguishable with respect to (a time instant t denoted by) x can be expressed as follows:

$$\forall \mathtt{r}(\mathtt{path}(\mathtt{x},\mathtt{r}) \rightarrow \exists \mathtt{y}(\mathtt{r}(\mathtt{y}) \wedge ((\mathtt{p}(\mathtt{y}) \wedge \neg \mathtt{q}(\mathtt{y})) \vee (\neg \mathtt{p}(\mathtt{y}) \wedge \mathtt{q}(\mathtt{y})) \vee (\neg \mathtt{p}(\mathtt{y}) \wedge \neg \mathtt{q}(\mathtt{y}))))).$$

As in the upward unbounded case, it is possible to show that $\mathcal{DL}^2_{\omega LM^k}$ is as expressive as $\mathcal{L}^2_{\omega LM^k}$. Let $\phi \in \mathcal{DL}^2_{\omega LM^k}$. The following steps transforms ϕ into an equisatisfiable formula $\phi' \in \mathcal{SDL}^2_{\omega LM^k}$.

Step 1: removal of constants $\mathtt{0_i}$.

For each $i \geq 0$, we replace all the occurrences of $\mathtt{0_i}$ (if any) by a fresh variable $\mathtt{x_i}$ and add to the resulting formula the conjunct:

$$\exists \mathtt{x_0},\ldots,\mathtt{x_{i-1}}(\forall \mathtt{y}\ \ (\mathtt{x_0} \leq \mathtt{y}) \wedge \bigwedge_{\mathtt{j}=0}^{\mathtt{i}-1} \downarrow(\mathtt{0},\mathtt{x_j}) = \mathtt{x_{j+1}}).$$

Step 2: removal of successor functions $+_{\mathtt{i}}\mathtt{1}$.

It is exactly as in the case of upward unbounded layered structures.

Step 3: removal of y-contextualizations $\mathtt{T^i}$.

We replace each occurrence of atomic formulae of the form $\mathtt{T^i}(\mathtt{t})$, with $i \geq 0$, by the formula:

$$\exists \mathtt{x_0},\ldots,\mathtt{x_i}(\mathtt{t} = \mathtt{x_i} \wedge \forall \mathtt{y}\ \ \neg(\bigvee_{\mathtt{j}=0}^{\mathtt{k}-1} \downarrow(\mathtt{j},\mathtt{y}) = \mathtt{x_0}) \wedge \bigwedge_{\mathtt{j}=0}^{\mathtt{i}-1}(\bigvee_{\mathtt{l}=0}^{\mathtt{k}-1} \downarrow(\mathtt{l},\mathtt{x_j}) = \mathtt{x_{j+1}})).$$

Step 4: removal of x-contextualizations $\Delta_{\mathtt{i},\mathtt{j}}$.

The removal of x-contextualizations $\Delta_\mathtt{i}$ is performed recursively.
We replace each occurrence of atomic formulae of the form $\Delta_{\mathtt{0,j}}(\mathtt{t})$ by the formula:

$$\exists \mathtt{p}(\mathtt{p}((+_0\mathtt{1})^\mathtt{j}(\mathtt{0}_0)) \wedge \mathtt{p}(\mathtt{t}) \wedge \forall \mathtt{y},\mathtt{z}(\mathtt{p}(\mathtt{y}) \wedge \downarrow(\mathtt{0},\mathtt{y}) = \mathtt{z} \rightarrow \mathtt{p}(\mathtt{z})) \wedge$$
$$\forall \mathtt{y},\mathtt{z}(\mathtt{p}(\mathtt{y}) \wedge \mathtt{p}(\mathtt{z}) \rightarrow \bigwedge_{\mathtt{i}=1}^{\mathtt{k}-1} \neg \downarrow(\mathtt{i},\mathtt{z}) = \mathtt{y}))$$

where $(+_0 1)^j(0_0)$ is replaced as shown at steps 1 and 2.
For each $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. We replace each occurrence of atomic formulae of the form $\Delta_{\mathtt{i,j}}(\mathtt{t})$ by the formula:

$$\exists \mathtt{x}_{\mathtt{n+1}},\ldots,\mathtt{x}_0(\Delta_{\mathtt{0,j}}(\mathtt{x}_{\mathtt{n+1}}) \wedge \mathtt{x}_0 = \mathtt{t} \wedge (\bigwedge_{\mathtt{j}=0}^{\mathtt{n}} \downarrow(\mathtt{a}_\mathtt{j},\mathtt{x}_{\mathtt{j+1}}) = \mathtt{x}_\mathtt{j})) \vee$$
$$\bigvee_{\mathtt{l}=0}^{\lfloor \log_\mathtt{k}(\mathtt{i}) \rfloor} (\mathtt{T}_\mathtt{l}(\mathtt{t}) \wedge \Delta_{\mathtt{r}_\mathtt{l},\mathtt{j}+\mathtt{q}_\mathtt{l}}(\mathtt{t})), \ \ \mathtt{with}\ \mathtt{i} = \mathtt{q}_\mathtt{l}\mathtt{k}^\mathtt{l} + \mathtt{r}_\mathtt{l}$$

where $T_l(t)$ is replaced as shown at step 4.

Step 5: removal of congruences $\equiv_{\mathtt{i,j}}$.

It is exactly as in the case of upward unbounded layered structures.

Step 6: removal of the approximate ordering $\mathtt{t}_1 \preceq \mathtt{t}_2$.

We replace each occurrence of atomic formulae of the form $\mathtt{t}_1 \preceq \mathtt{t}_2$ by the formula:

$$\neg\mathtt{ancestor}(\mathtt{t}_1,\mathtt{t}_2) \wedge \neg\mathtt{ancestor}(\mathtt{t}_2,\mathtt{t}_1) \rightarrow \mathtt{t}_1 \leq \mathtt{t}_2,$$

where $\mathtt{ancestor}(\mathtt{t}_1,\mathtt{t}_2)$ is defined as in Section 4.4.

Step 7: universal closure.

The formula ϕ' is the universal closure of the formula obtained by the sequential execution of steps 1-6. The relationship between the original formula $\phi \in \mathcal{DL}^2_{\omega LM^k}$ and the resulting formula $\phi' \in \mathcal{SDL}^2_{\omega LM^k}$ is formally stated by the following lemma.

Lemma 5.6 *Let ϕ be an $\mathcal{DL}^2_{\omega LM^k}$-formula and ϕ' be the $\mathcal{L}^2_{\omega LM^k}$-formula obtained from ϕ after the execution of the transformation steps* 1-7. *ϕ is satisfiable if and only if ϕ' is satisfiable.*

As for Lemma 5.4, the proof is straight forward, and thus omitted.

Concluding remark. In Sections 5.1 and 5.2, we have shown how to express the basic functionalities of metric and layered temporal logic in $\mathcal{L}^2_{\omega LM^k}$. We want to point out that, however, there exist significant properties of ω-layered structures that cannot be expressed in $\mathcal{L}^2_{\omega LM^k}$. As an example, it is not possible to define a binary predicate *same_layer* such that, for all $t, t' \in \bigcup_{i \geq 0} T^i$,

$$\mathit{same\_layer}(t,t') \quad \text{iff} \quad \exists i(t \in T^i \wedge t' \in T^i).$$

The quantificational prefix $\exists i$ stands for an infinite disjunctions that cannot be expressed in $\mathcal{L}^2_{\omega LM^k}$. In fact, extending $\mathcal{L}^2_{\omega LM^k}$ with the predicate *same_layer* would make the theory of downward unbounded layered structures undecidable, a result which follows from the undecidability of the extension of $S2S$ with the *equal_level* predicate E given by $E(u,v)$ if and only if $|u| = |v|$, with $u, v \in \{0,1\}^*$ (cf. [8]). We conjecture that a similar undecidability result holds for the theory of upward unbounded ones.

Conclusions

In this chapter, we have first proved the decidability of the theory of finitely-layered metric temporal structures through its reduction to the decidable theory *S1S*. Then, we considered the case of structures provided with an infinite number of either arbitrarily coarse or arbitrarily fine temporal layers. We first showed that the satisfiability and validity problems for the theory of upward unbounded layered structures are decidable by reducing them, through coding into the theory $S1S^k$, to the decidable problem of determining whether or not the ω-language recognized by a given systolic tree automaton is empty. Then, we obtained the same result for the theory of downward unbounded layered structures through coding into the theory SkS. We are currently exploring the natural generalization to layered structures which are both upward and downward unbounded.

It is worth noting that the questions whether or not we can decide upward unbounded layered structures using Büchi automata and whether or not we can decide downward unbounded layered structures using k-ary systolic tree automata are still open questions. Our conjecture is that the proposed reductions are actually the minimal ones. More generally, we are currently investigating the relations between $S1S^i$ and SjS, with $i, j > 1$, as well as the relations between $S1S^i$ and $S1S^j$, with $i \neq j$, in order to formally characterize the relationships between their temporal logic counterparts.

Finally, notice that the above results directly hold for metric and layered temporal logics non-axiomatically defined. Indeed, we identified relevant classes of temporal structures, we defined the corresponding theories, and we showed that such theories can be reduced to decidable ones. On the other hand, axiomatic completeness follows as a by-product of decidability, even though the axioms are not produced explicitly. An explicit temporal logic axiomatic counterpart of these theories can be obtained extending a simplified variant of $TPTL$ (real-time propositional temporal logic), where state variables are replaced by time variables and $\bigcirc$ is interpreted as the successor over time, with contextual and projection operators of $MLTL$. Moreover, since the validity problem is non-elementary already for the classical first-order theory of natural numbers with linear order and monadic predicates, it is obviously non-elementary also for the considered theories. Nevertheless, we expect that temporal logic counterparts of the proposed theories corresponding to elementary, yet expressively complete, fragments of $\mathcal{L}^2_{nLM}$ and $\mathcal{L}^2_{\omega LM^k}$ can be identified.

References

[1] R. Alur and T.A. Henzinger. Real-time logics: complexity and expressiveness. *Information and Computation*, 104:35–77, 1993.

[2] J. van Benthem. Temporal Logic. In: *Handbook of Logic in Artificial Intelligence and Logic Programming, Vol. 4*, D. Gabbay, C. Hogger, and J. Robinson (eds.), Oxford University Press, 241-350, 1995.

[3] E. Ciapessoni, E. Corsetti, A. Montanari, and P. San Pietro. Embedding Time Granularity in a Logical Specification Language for Synchronous Real-Time Systems. *Science of Computer Programming*, 20:141–171, 1993.

[4] E. Corsetti, E. Crivelli, D. Mandrioli, A. Montanari, A. Morzenti, P. San Pietro, and E. Ratto. Dealing with Different Time Scales in Formal Specifications. Proc. of the 6th

International Workshop on Software Specification and Design, Como, Italy, October 1991.

[5] E. Corsetti, A. Montanari, and E. Ratto. Dealing with Different Time Granularities in Formal Specifications of Real-Time Systems. *The Journal of Real-Time Systems*, 3 (2): 191-215, June 1991.

[6] J. Fiadeiro, and T. Maibaum. Sometimes "Tomorrow" is "Sometimes" - Action Refinement in a Temporal Logic of Objects. In: Proc. ICTL'94, LNAI 827, Springer, Berlin, 48–66, 1992.

[7] R. Koymans. *Specifying Message Passing and Time-Critical Systems with Temporal Logic*. LNCS 651, Springer, Berlin, 1992. The relevant sections appeared in *Journal of Real-Time Systems*, 2:255–299, 1990.

[8] H. Läuchli and C. Savoiz. Monadic second-order definable relations on the binary tree. *Journal of Symbolic Logic*, 52:219–226, 1987.

[9] A. Montanari. A Metric and Layered Temporal Logic for Time Granularity, Synchrony and Asynchrony. In: Proc. ICTL'94 Workshop, MPI-I-94-230, Max-Planck-Institut für Informatik, Ohlbach, H.J. (ed.), 49–58, 1994.

[10] A. Montanari and A. Policriti. A decidable theory of finitely-layered metric temporal structures. Research Report in the ILLC-series, ML-96-06, University of Amsterdam, July 1996.

[11] A. Montanari and A. Policriti, Decidability results for metric and layered temporal logics. *Notre Dame Journal of Formal Logic*, 37:260–282, 1996.

[12] A. Montanari and M. de Rijke. Completeness results for two-sorted metric temporal logics. In: V.S. Alagar and M. Nivat (eds.) Proc. of the *4th International Conference on Algebraic Methodology and Software Technology*, LNCS 936, Springer, Berlin, 385–399, 1995.

[13] A. Montanari, and M. de Rijke. Two-Sorted Metric Temporal Logic. Research Report CS-R9577, CWI, Amsterdam, December 1995. Submitted for publication.

[14] A. Monti and A. Peron, Systolic tree ω-languages; Proceedings of STACS-95, LNCS 900, pp. 131–142, 1995.

[15] A. Monti and A. Peron, Systolic tree ω-languages: the operational and the logical view; Technical Report SI-95/11, Università di Roma "La Sapienza", May 1995. Submitted for publication.

[16] W. Thomas. Automata on Infinite Objects. In: *Handbook of Theoretical Computer Science*, van Leeuwen, J. (ed.), Elsevier Science Publishers B.V., 133-191, 1990.