On a dichotomy related to colourings of definable graphs in generic models

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Abstract

We prove that in the Solovay model every OD graph G on reals satisfies one and only one of the following two conditions: (I) G admits an OD colouring by ordinals; (II) there exists a continuous homomorphism of $\mathcal{G}_0$ into G, where $\mathcal{G}_0$ is a certain $\mathbf{F}_\sigma$ locally countable graph which is not R-OD colourable by ordinals in the Solovay model. If the graph G is locally countable or acyclic then (II) can be strengthened by the requirement that the homomorphism is a $1-1$ map, *i. e.* an embedding.

As the second main result we prove that $\mathbf{\Sigma}^1_2$ graphs admit the dichotomy (I) *vs.* (II) in set–generic extensions of the constructible universe L (although now (I) and (II) may be in general compatible). In this case (I) can be strengthened to the existence of a $\mathbf{\Delta}^1_3$ colouring by countable ordinals provided the graph is locally countable.

The proofs are based on a topology generated by OD sets.

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Introduction

A new direction in the classical domain of graph colouring was discovered by Kechris, Solecki, and Todorcevic [7]. They found that graphs on reals drastically change their behaviour in the case when a definable colouring is requested. For instance there exists a $\mathbf{F}_\sigma$ graph $\mathcal{G}_0$ on reals (rather a family of graphs generated by a common method), acyclic (therefore colourable by only two colours) and locally countable, which does not admit even a countable *Borel* colouring.

Moreover, it is proved in [7] that $\mathcal{G}_0$ is in a certain way minimal among all analytic graphs which are not countably Borel colourable.

To formulate this result consistently, let us recall some notation. We refer to [7] for a more substantial review.

A *graph* on a set X (typically X is a set of reals) is any set $\mathsf{G} \subseteq X^2$ such that $x \not\mathsf{G}\, x$ and $x \,\mathsf{G}\, y \longleftrightarrow y \,\mathsf{G}\, x$ for all $x, y \in X$.

A graph G is *acyclic* iff there exists no chain of the form $x_0 \,\mathsf{G}\, x_1 \,\mathsf{G}\, x_2 \,\mathsf{G} \dots \mathsf{G}\, x_n$ with $x_0 = x_n$ and $n \geq 3$. A graph G is *locally countable* iff every vertex has at most countably many neighbours in G.

A *colouring* of a graph G on X is a function c defined on X so that $x \,\mathsf{G}\, y$ implies $c(x) \neq c(y)$. A colouring c is *countable* iff $c(x) \in \omega$ for all x. In this case c is a *Borel* colouring iff every pre–image $c^{-1}(n)$ $(n \in \omega)$ is a Borel set. We shall also consider *colourings by ordinals*, meaning that $c(x) \in \mathrm{Ord}$ for all x.

Let G and G' be graphs on topological spaces X and X' respectively. A *homomorphism* of G' into G is a continuous function $H : X' \longrightarrow X$ such that $x \,\mathsf{G}'\, y \longrightarrow H(x) \,\mathsf{G}\, H(y)$ for all $x, y \in X$. An *embedding* is an injective homeomorphism, that is, a continuous $1-1$ function with the same property (but $x \,\mathsf{G}'\, y \longleftarrow H(x) \,\mathsf{G}\, H(y)$ is not requested).

One writes $\mathsf{G}' \leq_{\mathrm{c}} \mathsf{G}$ if there exists a homeomorphism of G' into G. One writes $\mathsf{G}' \sqsubseteq_{\mathrm{c}} \mathsf{G}$ if there exists an embedding of G' into G. ($_{\mathrm{c}}$ stands for "continuous".) Even $\mathsf{G}' \leq_{\mathrm{c}} \mathsf{G}$ suffices for G to inherit uncolourability properties from G'.

Let us fix a sequence $\langle \sigma_n : n \in \omega \rangle$ of binary tuples $\sigma_n \in 2^n$ such that for any $s \in 2^{<\omega}$ there exists n such that $s \subseteq \sigma_n$ (the *density*). We define $\mathcal{G}_0$, a graph on the *Cantor space* $\mathcal{D} = 2^\omega$, as follows: $x \,\mathcal{G}_0\, y$ iff there exists n such that

$$x \restriction n = y \restriction n = \sigma_n\,, \quad x(n) = 1 - y(n)\,, \quad \text{and} \quad x(k) = y(k) \;\text{ for all }\; k > n\,.$$ [1]

One easily proves that $\mathcal{G}_0$ is a locally countable acyclic $\mathbf{F}_\sigma$ graph on $\mathcal{D}$ which does not admit a countable Borel colouring. (We refer to [7], see also Subsection 3.1;

[1] The essential here properties of $\mathcal{G}_0$ do not depend on the particular choice of the tuples σ_n. However it seems unknown whether different graphs $\mathcal{G}_0$ are isomorphic.

the density is essential only for the Borel uncolourability.) Therefore any graph G such that $\mathcal{G}_0 \le_{\rm c} \mathsf{G}$ does not admit a countable Borel colouring as well. The following theorem shows that this is also a necessary condition.

Theorem 1 (Theorem 6.3 in [7]) *Every analytic graph* G *on reals satisfies one and only one of the two statements,* $\langle \mathrm{I} \rangle$ *and* $\langle \mathrm{II} \rangle$:

$\langle \mathrm{I} \rangle$ G *admits a countable Borel colouring.*

$\langle \mathrm{II} \rangle$ $\mathcal{G}_0 \le_{\rm c} \mathsf{G}$. *Moreover,* $\mathcal{G}_0 \sqsubseteq_{\rm c} \mathsf{G}$ *in each of the two cases: a)* G *is locally countable; b)* G *is acyclic.*

It is quite usual in descriptive set theory that theorems proved for objects of initial projective classes (*e. g.* for Borel sets) generalize in the Solovay model [2] on all R-OD (real–ordinal–definable) sets of reals. The following theorem is such a generalization of Theorem 1.

Theorem 2 *In the Solovay model, every* R-OD *graph* G *on reals satisfies one and only one of the two statements,* (I) *and* (II) :

(I) G *admits a* R-OD *colouring by countable ordinals.*

(II) $\mathcal{G}_0 \le_{\rm c} \mathsf{G}$. *Moreover,* $\mathcal{G}_0 \sqsubseteq_{\rm c} \mathsf{G}$ *in each of the two cases: a)* G *is locally countable; b)* G *is acyclic.*

(It is pointed out in [7] with reference to Woodin that a dichotomy of this type is a consequence of some determinacy hypothesis.)

Two open problems should be mentioned.

Problem 1. Characterize the case when G is *countably* R-OD colourable.

Problem 2. Prove Theorem 2 with (II) in the form $\mathcal{G}_0 \sqsubseteq_{\rm c} \mathsf{G}$ unconditionally. (This is also an open problem for Theorem 1.)

A partial result in concern of the latter problem will be proved in Section 5. Let a *curve* be a binary relation K satisfying at least one of the two following requirements: $x \,\mathsf{K}\, y \;\&\; x \,\mathsf{K}\, y' \longrightarrow y = y'$ or $x \,\mathsf{K}\, y \;\&\; x' \,\mathsf{K}\, y \longrightarrow x = x'$. (In other words K is the graph of a partial function, either $y = f(x)$ or $x = f(y)$.) Let a *special* graph mean: a union of the form $\mathsf{G} = \bigcup_{\xi<\omega_1} \mathsf{K}_\xi$, where each K_ξ is a curve and the map $\xi \longmapsto \mathsf{K}_\xi$ is R-OD.

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[2] By *Solovay model* we mean a generic extension $\mathrm{L}[G]$ of the constructible universe L via a certain notion of forcing $\mathcal{P}^\lambda$, as defined by Solovay [9]. See Section 1 for more information.

Theorem 3 *To prove Theorem 2 for all* R-OD *graphs with* (II) *in the unconditional form* $\mathcal{G}_0 \sqsubseteq_{\mathrm{c}} \mathsf{G}$, *it suffices to obtain this result for special graphs.*

This reduction corresponds to a result for Borel graphs (namely, reduction to the case when G is a union of countably many Borel curves), absent in [7] but discussed between A. S. Kechris and the author.

It is also quite usual that theorems on Borel sets have (suitably weaker) counterparts for sets of initial projective levels, true in **ZFC** or in a quite reasonable hypothesis of broad nature, as *e. g.* sharps or determinacy hypotheses. It has been recently found that the assumption that the universe V of all sets is a set–generic extension of the constructible universe L, helps to prove theorems on analytic ($\mathbf{\Sigma}^1_1$) equivalence relations still unproved in **ZFC** alone. (We refer to a theorem on thin $\mathbf{\Sigma}^1_1$ relations of Hjorth [3] and a dichotomy theorem of Glimm – Effros type proved for $\mathbf{\Sigma}^1_1$ relations by Kanovei [5]. It is remarkable that these results have also been proved in **ZFC** plus the sharps hypothesis (Hjorth [3], Hjorth and Kechris [4]) or the existence of a weakly compact cardinal (Friedman [1]), but a proof in **ZFC** alone is so far unknown.) The following theorem contributes in this type of results.

Theorem 4 *Suppose that the universe is a set–generic extension of the constructible universe. Then every* $\mathbf{\Sigma}^1_2$ *graph* G *on reals satisfies one* [3] *of the two following statements:*

(I′) G *admits a* R-OD *colouring by ordinals. If in addition* G *is locally countable then* G *admits a* $\mathbf{\Delta}^{\mathrm{HC}}_2$ *colouring by countable ordinals.* [4]

(II′) — *exactly as* (II) *of Theorem 2.*

Since (I′) does not seem in any way adequate in the general case, it is an important problem to obtain something better than simply OD in (I′). One more problem is to obtain matching counterexamples among $\mathbf{\Pi}^1_2$ graphs. But the most interesting would be to prove the theorem in **ZFC** (not assuming that the universe is a set–generic extension of L).

The exposition starts with a review of some facts in the theory of the Solovay and more elementary collapse models in sections 1 and 2. We also introduce a

[3] In this case (I′) and (II′) are generally speaking compatible.

[4] HC is the set of all hereditarily countable sets. The "boldface upright" $\mathbf{\Sigma}^{\mathrm{HC}}_n$ means Σ_n-definability in HC with reals being admitted as parameters. The "lightface slanted" $\mathit{\Sigma}^{\mathrm{HC}}_n$ will mean parameter–free Σ_n-definability in HC. Π-notions are introduced in the same way, while Δ in any form means simultaneously Σ and Π .

topology generated by OD sets (a counterpart of the Gandy – Harrington topology) and study an associated forcing.

Section 3 reveals the nature of the dichotomy. Concentrating on OD graphs, we consider the case when the given OD graph G satisfies the property that every real belongs to an OD G-discrete set X, and prove that this leads to (I) in the Solovay model. (A set X is G-*discrete* iff G does not intersect $X \times X$.)

Sections 4 and 5 handle the case when <u>not</u> every real x belongs to an OD G-discrete set. We prove that then $\mathcal{G}_0 \le_{\rm c} \mathsf{G}$, and moreover $\mathcal{G}_0 \sqsubseteq_{\rm c} \mathsf{G}$ provided G is locally countable or acyclic. This part of the proof of Theorem 2 utilizes an important splitting construction of Kechris *et al.* [7], but technically realized so that the Choquet games (as in [7]) are not involved.

To explain the ideas behind the proof of Theorem 4, let κ be the L-cardinal of the notion of forcing $\mathsf{P} \in \mathrm{L}$ such that the universe V is a P-generic extension of L. Let G be in fact a $\varSigma^1_2$ graph. Assume, for the sake of simplicity, that V still contains an inaccessible cardinal $\Omega > \kappa$.

Then V can be embedded into a vider universe, V^*, which is a Solovay model over L, so that $\Omega = {\omega_1}^{\mathrm{V}^*}$. (In fact a suitable collapse generic extension will be taken as V^*.) The dichotomy (I) *vs.* (II) holds in V^* by Theorem 2.

If (II) holds in V^* then we obtain (II′) in V by an absoluteness argument based on the Shoenfield theorem. If (I) holds in V^* then we get (I′) in V using the fact that (I) appears in V^* in the form of a statement somehow relativized to L. This reasoning is presented in Section 6.

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Notation

We shall use the *Cantor set* $\mathcal{D} = 2^\omega$ rather than the Baire space $\mathcal{N} = \omega^\omega$ as the principal space in this paper. Elements of $\mathcal{D}$ will be called *reals* below.

In the rest we shall follow the ordinary notation. Sometimes the F-image $\{F(x) : x \in X\}$ of a set X will be denoted by $F\,"X$.

V will denote the universe of all sets, L the constructible universe.

We shall use sans serif characters like G to denote graphs and other binary relations. On the other hand, the math italic G will denote generic sets.

By *dense* we shall always mean: *open dense*.

1 Solovay and collapse models

This section reviews notation related to and some properties of the Solovay model and collapse generic models. (The latter type will be used in the proof of Theorem 4.)

For an ordinal α, $\mathcal{P}_\alpha = \alpha^{<\omega} = \bigcup_{n\in\omega}\alpha^n$ denotes the forcing to collapse α down to ω. It is ordered as follows: $u \le v$ (u is *stronger* than v) iff $v \subseteq u$.

For $\lambda \in \mathrm{Ord}$, we let $\mathcal{P}_{<\lambda}$ ($\mathcal{P}^\lambda$ in Solovay [9]) be the "finite support" product of all sets $\mathcal{P}_\alpha$, $\alpha < \lambda$. In other words, $\mathcal{P}_{<\lambda}$ is the set of all functions p such that $\mathrm{dom}\, p$ is a finite subset of λ and $p(\alpha) \in \mathcal{P}_\alpha$ for each $\alpha < \lambda$, $\alpha \in \mathrm{dom}\, p$, with the order: $p \le q$ iff $\mathrm{dom}\, q \subseteq \mathrm{dom}\, p$ and $p(\alpha) \le q(\alpha)$ for all $\alpha \in \mathrm{dom}\, q$.

1.1 Solovay's "Σ-lemma"

It is known that the Solovay model is seen from any subclass of the form $\mathrm{L}[x]$, where x is a real or a set definable from a real, in one and the same regular way. This remarkable uniformity is partially inherited by collapse extensions. The following lemma stands behind this crucial property.

Lemma 5 *Suppose that $\mathfrak{M}$ is a transitive model of* **ZFC**, *$P \in \mathfrak{M}$ is a p. o. set, and $G \subseteq P$ is P-generic over $\mathfrak{M}$. Let $S \in \mathfrak{M}[G]$, $S \subseteq \mathrm{Ord}$. There exists a set $\Sigma \subseteq P$, $\Sigma \in \mathfrak{M}[S]$ such that $G \subseteq \Sigma$ and G is Σ-generic over $\mathfrak{M}[S]$.*

Thus $\mathfrak{M}[G]$ is a generic (not necessarily P-generic) extension of any submodel $\mathfrak{M}[S]$, $S \subseteq \mathrm{Ord}$.)

Proof (extracted from the proof of Lemma 4.4 in [9]).

Let $\mathbf{S}$ be a name for S in the language of the P-forcing over $\mathfrak{M}$.

Define a sequence of sets $A_\alpha \subseteq P$ ($\alpha \in \mathrm{Ord}$) by induction on α in $\mathfrak{M}[S]$.

(A1) $p \in A_0$ iff, for some $\sigma \in \mathrm{Ord}$, either $\sigma \in S$ but p P-forces $\sigma \notin \mathbf{S}$ over $\mathfrak{M}$, or $\sigma \notin S$ but p P-forces $\sigma \in \mathbf{S}$ over $\mathfrak{M}$.

(A2) $p \in A_{\alpha+1}$ iff there exists a dense in P below p set $D \in \mathfrak{M}$, $D \subseteq A_\alpha$.

(A3) If α is a limit ordinal then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

The following properties of these sets are easily verifiable: first, if $p \in A_\alpha$ and $q \le p$ in P then $q \in A_\alpha$; second, if $\beta < \alpha$ then $A_\beta \subseteq A_\alpha$.

Since each A_α is a subset of P, it follows that $A_\delta = A_{\delta+1}$ for an ordinal $\delta \in \mathfrak{M}$. We put $\Sigma = P \setminus A_\delta$. Thus Σ can be seen as the set of all conditions $p \in P$

which do not force something about $\mathbf{S}$ which contradicts a factual information about S. We prove that Σ is as required. This involves two auxiliary facts.

(Σ1) $G \subseteq \Sigma$.

(Σ2) *If* $D \in \mathfrak{M}$ *is a dense subset of* P *then* $D \cap \Sigma$ *is dense in* Σ.

To prove (Σ1) assume on the contrary that $G \cap A_\gamma \neq \emptyset$ for some γ. Let γ be the least such an ordinal. Clearly γ is not limit and $\gamma \neq 0$; let $\gamma = \alpha + 1$. Let $p \in A_\gamma \cap G$. Since G is generic, item (A2) implies $G \cap A_\alpha \neq \emptyset$, contradiction. ($\Sigma$2) is easy: if $p \in \Sigma$ then $p \notin A_{\delta+1}$; therefore by (A2) $D \cap A_\delta$ is *not* dense in P below p, so there exists $q \le p, q \in D \setminus A_\delta$ – then $q \in \Sigma$.

We prove that G is Σ-generic over $\mathfrak{M}[S]$. Suppose, towards contradiction, that $D \cap G = \emptyset$ for a dense in Σ set $D \in \mathfrak{M}[S]$, $D \subseteq \Sigma$.

Since $D \in \mathfrak{M}[S]$, there exists an $\in$-formula $\Phi(x, y)$ containing only sets in $\mathfrak{M}$ as parameters and such that $\Phi(S, y)$ holds in $\mathfrak{M}[S]$ iff $y = D$.

Let $\Psi(G')$ be the conjunction of the following formulas:

(1) $S' = \mathbf{S}[G']$ (the interpretation of the "term" $\mathbf{S}$ via G') is a set of ordinals, and there exists unique $D' \in \mathfrak{M}[S']$ [5] such that $\Phi(S', D')$ holds in $\mathfrak{M}[S']$;

(2) this set D' is a dense subset of Σ' where $\Sigma' \subseteq P$ is the set obtained by applying our definition of Σ for $S = S'$ within $\mathfrak{M}[S']$;

(3) $D' \cap G' = \emptyset$.

Then $\Psi(G)$ is true in $\mathfrak{M}[G]$ by our assumptions. Let $p \in G$ P-force $\Psi(G)$ over $\mathfrak{M}$. Then $p \in \Sigma$ by (Σ1). By the density there exists a condition $q \in D$, $q \le p$. Les us consider a Σ-generic over $\mathfrak{M}[S]$ set $G' \subseteq \Sigma$ containing q. Then G' is also P-generic over $\mathfrak{M}$ by (Σ2). We observe that $\mathbf{S}[G'] = S$ because $G' \subseteq \Sigma$. It follows that D' and Σ' (as is the description of Ψ) coinside with resp. D and Σ. In particular $q \in D' \cap G'$, a contradiction because p forces (3). $\square$

Corollary 6 *If* $P = \mathcal{P}_\lambda$ *in the lemma for an ordinal* $\lambda \in \mathfrak{M}$ *then* $\mathfrak{M}[G]$ *is* $\mathcal{P}_\alpha$*-generic extension of* $\mathfrak{M}[S]$ *for an ordinal* $\alpha \le \lambda$.

It is allowed that $\alpha = 1$, to include in particular the case $\mathfrak{M}[S] = \mathfrak{M}[G]$. Note that α, generally speaking, depends on G: different parts of Σ may have different branching poperties.

[5] Let us assume, to avoid irrelevant complications, that $\mathfrak{M}$ satisfies the following axiom: $\exists X\,(\mathrm{V} = \mathrm{L}[X])$; this makes $D' \in \mathfrak{M}[S']$ a legitimate formula. The lemma will be applied below only in this particular case, but the proof could be modified to fix the general case.

Proof We observe that Σ is, in $\mathfrak{M}[S]$, a subtree of the full λ-branching tree $\mathcal{P}_\lambda = \lambda^{<\omega}$. This allows to derive the result by ordinary reasoning. $\square$

1.2 "Weak" sets in the models

As usual, L denotes the constructible universe. By Ω-*Solovay model axiom* and Ω-*collapse model axiom* we shall understand the hypotheses:

Ω-`SM` : Ω is inaccessible in L and the universe is a $\mathcal{P}_{<\Omega}$-generic extension of L.

Ω-`Clps` : Ω is a limit cardinal in L and the universe is a $\mathcal{P}_\Omega$-generic extension of L.

A set x will be called Ω-*"weak" over* $\mathfrak{M}$ iff x belongs to a $\mathcal{P}_\alpha$-generic extension of $\mathfrak{M}$ for some $\alpha < \Omega$. We define

$$\mathcal{W}[S] = \{x \in \mathcal{D} : x \text{ is } \Omega\text{-``weak'' over } \mathrm{L}[S]\},$$

the set of all reals "weak" over a class $\mathrm{L}[S]$. As usual $\mathcal{W} = \mathcal{W}[\emptyset]$.

Lemma 7 *Assume* Ω-`SM`. *Then all reals are* Ω-*"weak" over* L. $\square$

Lemma is not true in the assumption Ω-`Clps`, of course. This observation outlines the difference in the development of the Solovay and collapse models: usually properties of *all* reals in the former remain true for "weak" reals in the latter.

Proposition 8 *Assume* Ω-`SM` (*resp. assume* Ω-`Clps`). *Then* $\omega_1 = \Omega$ (*resp.* $\omega_1 = \Omega^+$, *the next cardinal in* L). *Furthermore, suppose that a set* $S \subseteq \mathrm{Ord}$ *is* Ω-*"weak" over* L. *Then*

1. Ω *is inaccessible in* $\mathrm{L}[S]$ *and the universe* V *is a* $\mathcal{P}_{<\Omega}$-*generic extension of* $\mathrm{L}[S]$. (*Resp.* Ω *is a limit cardinal in* $\mathrm{L}[S]$ *and* V *is a* $\mathcal{P}_\Omega$-*generic extension of* $\mathrm{L}[S]$.)
2. *If* Φ *is a sentence containing only sets in* $\mathrm{L}[S]$ *as parameters then* Λ (*the empty function*) *decides* Φ *in the sense of* $\mathcal{P}_{<\Omega}$ (*resp. in the sense of* $\mathcal{P}_\Omega$) *as a forcing notion over* $\mathrm{L}[S]$.
3. *If a set* $X \subseteq \mathrm{L}[S]$ *is* $\mathrm{OD}[S]$ *then* $X \in \mathrm{L}[S]$.

($\mathrm{OD}[S]$ = S–*ordinal definable*, that is, definable by an $\in$-formula containing S and ordinals as parameters.)

Proof *Item 1.* By definition, S belongs to a $\mathcal{P}_\alpha$-generic extension of L for some $\alpha < \Omega$. In the Ω-`Clps` case, the universe V is a $\mathcal{P}_\gamma$-generic extension of $\mathrm{L}[S]$ for some $\gamma \le \Omega$, by Corollary 6. But $\gamma < \Omega$ is impossible: indeed otherwise Ω simply remains uncountable in V, contradiction with Ω-`Clps`.

Let us consider the Ω-`SM` case. We have $\mathrm{V} = \mathrm{L}[G]$; G being a $\mathcal{P}_{<\Omega}$-generic set over L. Obviously $S \in \mathrm{L}[x]$ for a real x. It follows (Corollary 3.4.1 in [9]) that there exists an ordinal $\lambda < \Omega$ such that S belongs to $\mathrm{L}[G_{\le\lambda}]$ for some $\lambda < \Omega$, where $G_{\le\lambda} = G \cap \mathcal{P}_{\le\lambda}$ is $\mathcal{P}_{\le\lambda}$-generic over L. Therefore by Lemma 4.3 in [9], $\mathrm{L}[G_{\le\lambda}]$ also is a $\mathcal{P}_\lambda$-generic extension of L. Finally $\mathrm{L}[G_{\le\lambda}]$ is a $\mathcal{P}_\gamma$-generic extension of $\mathrm{L}[S]$ for some $\gamma \le \lambda$ by Corollary 6.

On the other hand, $\mathrm{V} = \mathrm{L}[G]$ is a $\mathcal{P}_{\ge\lambda+1}$-generic extension of $\mathrm{L}[G_{\le\lambda}]$, so that totally V is a $(\mathcal{P}_\gamma \times \mathcal{P}_{\ge\lambda+1})$-generic extension of $\mathrm{L}[S]$. But the product can be easily proved to include a dense subset order isomorphic to $\mathcal{P}_{<\Omega}$. This ends the proof of item 1 of the proposition.

Items 2 and 3. Easily follow from item 1. □

2 Reals in Solovay and collapse models

In this section, a useful coding of reals and sets of reals in generic models of the type we consider, is introduced. After this we study a topology on reals in the models, close to the Gandy – Harrington topology.

2.1 The coding

If a set $G \subseteq \mathcal{P}_\alpha$ is $\mathcal{P}_\alpha$-generic over a transitive model $\mathfrak{M}$ ($\mathfrak{M}$ is a set or a class) then $f = \bigcup G$ maps ω onto α, so that α is countable in $\mathfrak{M}[G] = \mathfrak{M}[f]$. Functions $f : \omega \longrightarrow \alpha$ obtained this way will be called *$\mathcal{P}_\alpha$-generic over $\mathfrak{M}$.*

We let $\mathbb{F}_\alpha(\mathfrak{M})$ be the set of all $\mathcal{P}_\alpha$-generic over $\mathfrak{M}$ functions $f \in \alpha^\omega$. We further define $\mathbb{F}_\alpha[S] = \mathbb{F}_\alpha(\mathrm{L}[S])$ and $\mathbb{F}_\alpha = \mathbb{F}_\alpha(\mathrm{L}) = \mathbb{F}_\alpha[\emptyset]$.

Recall that *reals* are elements of $\mathcal{D} = 2^\omega$ in this research.

Let $\alpha \in \mathrm{Ord}$. $\mathbb{T}_\alpha$ will denote the set of all "terms" — indexed sets of the form $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle$ such that $t_n \subseteq \mathcal{P}_\alpha$ for each n. "Terms" $t \in \mathbb{T}_\alpha$ can be used to code functions $C : \alpha^\omega \longrightarrow \mathcal{D} = 2^\omega$; namely, for every $f \in \alpha^\omega$ we define $x = \mathrm{C}_t(f) \in \mathcal{D}$ by: $x(n) = 1$ iff $f \restriction m \in t_n$ for some m.

We put $\mathbb{T} = \bigcup_{\alpha<\Omega} \mathbb{T}_\alpha$. (This definition makes sense only when Ω is fixed by the context, *e.g.* provided we suppose Ω-`SM` or Ω-`Clps`. Recall that $\Omega = \omega_1$ assuming Ω-`SM` but $\Omega < \omega_1$ assuming Ω-`Clps`.)

Suppose that $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle \in \mathbb{T}_\alpha$, $u \in \alpha^{<\omega} = \mathcal{P}_\alpha$, $\mathfrak{M}$ arbitrary. We define $\mathsf{X}_{tu}(\mathfrak{M}) = \{\mathsf{C}_t(f) : u \subset f \in \mathbb{F}_\alpha(\mathfrak{M})\}$ and $\mathsf{X}_t(\mathfrak{M}) = \mathsf{X}_{t\Lambda}(\mathfrak{M}) = \mathsf{C}_t\text{"}\mathbb{F}_\alpha(\mathfrak{M})$. In particular, let $\mathsf{X}_t[S] = \mathsf{X}_t(\mathrm{L}[S])$ and $\mathsf{X}_t = \mathsf{X}_t[\emptyset] = \mathsf{X}_t(\mathrm{L})$; the same for X_{tu}.

We recall that $\mathcal{W}[S]$ is the set of all reals Ω-"weak" over $\mathrm{L}[S]$.

Proposition 9 *Assume that either* Ω-`SM` *or* Ω-`Clps` *holds in the universe. Let* $S \subseteq \mathrm{Ord}$ *be* Ω-*"weak" over* L. *Then*

1. *If* $\alpha < \Omega$, $F \subseteq \mathbb{F}_\alpha[S]$ *is* $\mathrm{OD}[S]$, *and* $f \in F$, *then there exists* $m \in \omega$ *such that each* $f' \in \mathbb{F}_\alpha[S]$ *satisfying* $f' \restriction m = f \restriction m$ *belongs to* F.
2. *For each* $x \in \mathcal{W}[S]$ *there exist* $\alpha < \Omega$, $f \in \mathbb{F}_\alpha[S]$, *and* $t \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$ *such that* $x = \mathsf{C}_t(f)$.
3. *Every* $\mathrm{OD}[S]$ *set* $X \subseteq \mathcal{W}[S]$ *is a union of sets of the form* $\mathsf{X}_t[S]$, *where* $t \in \mathbb{T} \cap \mathrm{L}[S]$. *Conversely* $\mathsf{X}_t[S] \subseteq \mathcal{W}[S]$ *whenever* $t \in \mathbb{T} \cap \mathrm{L}[S]$.
4. *Assume that* $\alpha < \Omega$, $u \in \alpha^{<\omega}$, *and* $t \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$. *Then each* $\mathrm{OD}[S]$ *set* $X \subseteq \mathsf{X}_{tu}[S]$ *is a union of sets of the form* $\mathsf{X}_{tv}[S]$, *where* $u \subseteq v \in \alpha^{<\omega}$.

(Take notice that $\mathcal{W}[S] = \mathcal{D}$, all reals, provided Ω-`SM`, but $\mathcal{W}[S]$ is a proper subset of $\mathcal{D}$ in the assumption of Ω-`Clps`.)

Proof *Item 1.* We observe that $F = \{f' \in \alpha^\omega : \Phi(S, f')\}$ for an $\in$-formula Φ. Let $\Psi(S, f')$ denote the formula: "Λ P-forces $\Phi(S, f')$ over the universe", where P is the forcing $\mathcal{P}_{<\Omega}$ in the case Ω-`SM`, and $\mathcal{P}_\Omega$ otherwise, so

$$F = \{f' \in \alpha^\omega : \Psi(S, f') \text{ is true in } \mathrm{L}[S, f']\}$$

by Proposition 8 (items 1, 2). Thus, since $f \in F \subseteq \mathbb{F}_\alpha[S]$, there exists $m \in \omega$ such that the restriction $u = f \restriction m$ (then $u \in \alpha^{<\omega} = \mathcal{P}_\alpha$) $\mathcal{P}_\alpha$-forces $\Psi(S, \mathbf{f})$ over $\mathrm{L}[S]$, where $\mathbf{f}$ is the name of the α-collapsing function. The m is as required.

Item 2. By definition x belongs to an $\mathcal{P}_\alpha$-generic extension of $\mathrm{L}[S]$, for some ordinal $\alpha < \Omega$. Thus $x \in \mathrm{L}[S, f]$ where $f \in \mathbb{F}_\alpha[S]$. Let $\mathbf{x}$ be a name for x. It remains to define $t_n = \{u \in \mathcal{P}_\alpha : u \ \mathcal{P}_\alpha\text{-forces } \mathbf{x}(n) = 1\}$ in $\mathrm{L}[S]$.

Item 3. Let $x \in X$. We use item 2 to get an ordinal $\alpha < \Omega$, a generic function $f \in \mathbb{F}_\alpha[S]$, and a "term" $t \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$ such that $x = \mathsf{C}_t(f)$. Applying item 1 to the set $F = \{f' \in \mathbb{F}_\alpha[S] : \mathsf{C}_t(f') \in X\}$ and the given function f, we obtain a forcing condition $u = f \restriction m \in \mathcal{P}_\alpha$ $(m \in \omega)$ such that $x \in \mathsf{X}_{tu}[S] \subseteq X$. Finally the set $\mathsf{X}_{tu}[S]$ is equal to $\mathsf{X}_{t'}[S]$ for some other $t' \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$.

Item 4. Similar to the previous item. □

2.2 The ordinal definable topology

It occurs that the topology generated by OD sets in the Solovay and collapse models has a strong semblance of the Gandy – Harrington topology (in a simplified form because the specific Σ^1_1 technique becomes obsolete). In particular, the topology is strongly Choquet. However we shall not utilize this property. The treatment will be organized in a rather forcing–like way, in the manner of Miller [8].

A set X will be called OD-*1st-countable* if the OD power set $\mathcal{P}^{\rm OD}(X) = \mathcal{P}(X) \cap \mathrm{OD}$ is at most countable. In this case, $\mathcal{P}^{\rm OD}(X)$ has only countably many different OD subsets because it is a general property of the Solovay model that $\mathcal{P}^{\rm OD}(\mathcal{X})$ is countable for any countable OD set $\mathcal{X} \subseteq \mathrm{OD}$.

Lemma 10 *Assume that either* Ω-`SM` *or* Ω-`Clps` *holds in the universe. Let* $\alpha < \Omega$ *and* $t \in \mathbb{T}_\alpha \cap \mathrm{L}$. *Then* $X = \mathsf{X}_t$ *is* OD-*1st-countable.*

Proof By Proposition 9 every OD subset of X is uniquely determined by an OD subset of $\mathcal{P}_\alpha = \alpha^{<\omega}$. Since each OD set $S \subseteq \mathcal{P}_\alpha$ is constructible (Proposition 8), we obtain an OD map $h : \alpha^+$ onto $\mathcal{P}^{\rm OD}(X)$, where α^+ is the least L-cardinal bigger than α. Therefore $\mathcal{P}^{\rm OD}(X)$ has at most α^{++}-many OD subsets. It remains to notice that $\alpha^{++} < \Omega$ follows from either of Ω-`SM` or Ω-`Clps`. $\square$

Let $\mathbb{X} = \{X \subseteq \mathcal{W} : X$ is OD and nonempty $\}$, where $\mathcal{W} = \mathcal{W}[\emptyset]$.

(We recall that $\mathcal{W} = \mathcal{D} =$ all reals in the assumption Ω-`SM` but $\mathcal{W} \subsetneqq \mathcal{D}$ in the assumption Ω-`Clps`, consisting of all reals Ω-"weak" over L.)

Let us consider $\mathbb{X}$ as a forcing notion (smaller sets are stronger conditions). We say that a set $G \subseteq \mathbb{X}$ is OD-*generic in* $\mathbb{X}$ iff it nonempty intersects every dense OD subset of $\mathbb{X}$. [6]

Lemma 11 *Assume that* Ω-`SM` *or* Ω-`Clps` *holds. If a set* $G \subseteq \mathbb{X}$ *is* OD-*generic then the intersection* $\bigcap G$ *is a singleton* $\{x\} = \{x_G\}$ *and* $x_G \in \mathcal{W}$.

Proof It follows from the density and Proposition 9 that G contains a set X of the form $X = \mathsf{X}_t$ for some $t \in \mathbb{T}_\alpha$, $\alpha < \Omega$. Note that by Proposition 9 (item 4) for any dense in $\mathcal{P}_\alpha = \alpha^{<\omega}$ OD set $D \subseteq \mathcal{P}_\alpha$ and any $u \in \alpha^{<\omega}$ the set

$$\mathcal{X} = \{Y \in \mathbb{X} : X \cap Y = \emptyset\} \cup \{\mathsf{X}_{tv} : v \in D \ \&\ u \subseteq v\}$$

is OD and dense in $\mathbb{X}$. This leads to a function $f \in \mathbb{F}_\alpha$ such that $\mathsf{X}_{t\, f\restriction m} \in G$ for all $m \in \omega$, since the family of all OD subsets of $\mathcal{P}_\alpha$ is countable (see above).

[6] This can be transformed to an ordinary forcing over L in the assumption Ω-`SM`. Of course formally $\mathbb{X} \notin \mathrm{L}$, but $\mathbb{X}$ is OD order isomorphic to a partially ordered set in L.

Let us show that $x = \mathsf{C}_t(f) \in \bigcap G$. Let $Y \in G$. The set D of all $u \in \mathcal{P}_\alpha$ such that either $\mathsf{X}_{tu} \subseteq Y$ or $\mathsf{X}_{tu} \cap Y = \emptyset$ is dense in $\mathcal{P}_\alpha$ by Proposition 9 (item 4), and obviously OD. Therefore there exists a number m such that $u = f \restriction m \in D$. Then $\mathsf{X}_{tu} \in G$ by the choice of f, so that $\mathsf{X}_{tu} \cap Y = \emptyset$ is impossible. Thus $\mathsf{X}_{tu} \subseteq Y$ and $x = \mathsf{C}_t(f) \in Y$, as required. □

3 The dichotomy

Let us start the proof of the dichotomy of Theorem 2 in the assumption Ω-SM. In parallel, consideration of the collapse (*i. e.* $\mathcal{P}_\Omega$-generic) models will be carried out, in order to prepare the proof of the dichotomy of Theorem 4 afterwards.

This line of arguments takes this section (mainly devoted to the case when a colouring exists) and two next sections (where we define a homeomorphism or embedding of $\mathcal{G}_0$ into the gived graph). During the reasoning we shall assume that the following hypotheses is satisfied:

(‡) either Ω-SM or Ω-Clps holds, and G is an OD graph on reals satisfying $\mathsf{G} \subseteq \mathcal{W}_{(2)}$, where $\mathcal{W}_{(2)}$ is the two–dimentional copy of $\mathcal{W}$,

$$\mathcal{W}_{(2)} = \{\langle x, y\rangle \in \mathcal{D}^2 : \langle x, y\rangle \text{ is } \Omega\text{-“weak” over } \mathrm{L}\}.$$

The Ω-SM part of (‡) will work towards the proof of Theorem 2; note that $\mathcal{W}_{(2)} = \mathcal{D} \times \mathcal{D}$ in the assumption Ω-SM (all reals and pairs of reals are “weak”).

The Ω-Clps part will be involved in the proof of Theorem 4. Notice that $\mathcal{W}_{(2)} \subseteq \mathcal{W} \times \mathcal{W}$ but it is not clear whether $\mathcal{W}_{(2)} \subsetneqq \mathcal{W} \times \mathcal{W}$ in the assumption Ω-Clps. (We do not know how to find a non–“weak” pair of “weak” reals.)

3.1 Incompatibility

We prove first of all that the requirements (I) and (II) of Theorem 2 are incompatible in the assumption of Ω-SM. Suppose on the contrary that G satisfies both (I) and (II). Then $\mathcal{G}_0$ itself admits a R-OD colouring $c : \mathcal{D} \longrightarrow \omega_1$.

We get the contradiction following an argument for Borel colourings in [7]. Each set $X_\alpha = c^{-1}(\alpha)$ ($\alpha < \omega_1$) is an OD set of reals, and $\mathcal{D} = \bigcup_{\alpha<\omega_1} X_\alpha$. By the known properties of the Solovay model, some X_α is not meager. Then X_α is co-meager on a set of the form $\mathcal{D}_w = \{x \in \mathcal{D} : w \subseteq x\}$, where $w \in 2^{<\omega}$. By the choice of the sequence of tuples $\sigma_n \in 2^{<\omega}$ (see Introduction), we have $w \subseteq \sigma_n$ for some n. Then X_α is co-meager on $\mathcal{D}_{\sigma_n}$. Let H be the automorphism of $\mathcal{D}$ defined by: $H(x) = y$, where $y(k) = x(k)$ for all $k \neq n$, but $y(n) = 1 - x(n)$. Then $Y = H^{-1}(X_\alpha \cap \mathcal{D}_{\sigma_n})$ is co-meager on $\mathcal{D}_{\sigma_n}$. Let $x \in X_\alpha \cap \mathcal{D}_{\sigma_n} \cap Y$. Then $y = H(x) \in X_\alpha$, but $x \,\mathcal{G}_0\, y$, contradiction.

3.2 The dichotomy

We recall that a set X is G-*discrete* iff $G \cap (X \times X) = \emptyset$. The following theorem will imply the dichotomy results of theorems 2, 3, and 4.

Theorem 12 *Assume* $(\ddagger)$. *Then*

(i) *If* $\underline{\textit{every}}$ *real* $x \in \mathcal{W}$ *belongs to an* OD G-*discrete set then* G *admits an* OD *colouring by countable ordinals.*

(ii) *If* $\underline{\textit{not every}}$ *real* $x \in \mathcal{W}$ *belongs to an* OD G-*discrete set then* $\mathcal{G}_0 \le_{\mathrm{c}} \mathsf{G}$. *Moreover, we have* $\mathcal{G}_0 \sqsubseteq_{\mathrm{c}} \mathsf{G}$ *in each of the three cases*:

  (a) G *is locally countable*;

  (b) G *is acyclic*;

  (c) $x \,\mathsf{G}\, y$ *implies* $x \notin \mathrm{L}[y]$ & $y \notin \mathrm{L}[x]$.

We shall demonstrate, after quite a short proof of statement (i), how this theorem implies theorems 2 and 3. Statement (ii) of the theorem will need much more efforts; the proof takes the next two sections.

Proof of item (i) of Theorem 12. In principle the assumption immedialely leads to an OD colouring of G by ordinals. Indeed there exists an $\in$-definable map $\phi : \mathrm{Ord}$ onto OD. For each real x, we let $c(x)$ be the least ordinal γ such that $x \in \phi(\gamma)$ and $\phi(\gamma)$ is G-discrete. Then c is an OD colouring of G by ordinals.

A more difficult problem is to get a colouring by *countable* ordinals.

The set $\mathbb{T} \cap \mathrm{L}$ is obviously a constructible set of cardinality Ω in L. (Notice that $\Omega = \omega_1$ in the case Ω-`SM` and $\Omega < \omega_1$ in the case Ω-`Clps`.) Let us fix an $\in$-definable enumeration $\mathbb{T} \cap \mathrm{L} = \{t_\alpha : \alpha < \Omega\}$. It follows from Proposition 9 that every OD set of reals is a union of sets of the form $Z_\alpha = \mathsf{X}_{t_\alpha}$. Now one defines an OD colouring c of G by countable ordinals, setting $c(x)$ to be the least $\alpha < \omega_1$ such that $x \in Z_\alpha$ and Z_α is a G-discrete set. $\dashv$

Remark. Let G be a Σ^1_n graph $(n \ge 2)$ in Theorem 12, item (i), in the Ω-`SM` version. Then G admits a Δ^{HC}_n colouring. Indeed one easily sees that the relation "X_t is G-discrete" is Π^1_n, hence Π^{HC}_{n-1}. The enumeration of terms can be chosen to be Δ^{HC}_1. An elementary computation shows that then c is Δ^{HC}_n.

3.3 Applications

Proof of Theorem 2. Consider a R-OD graph G on reals in the assumption Ω-`SM`. We shall assume that G is in fact an OD graphs. (The "boldface" case, when G is $\mathrm{OD}[p]$ for a real p, can be handled similarly: p uniformly enters the reasoning as a parameter. In particular L changes to $\mathrm{L}[p]$ in the definition of $\mathcal{W}$, similarly OD changes to $\mathrm{OD}[p]$ in the definition of the topology and the set $\mathbb{X}$ in Subsection 2.2, *etc.*)

Since clearly $\mathcal{W} = \mathcal{D}$, all reals, in the assumption Ω-$\mathtt{SM}$, we have $(\ddagger)$. This allows to apply Theorem 12, immediately getting the dichotomy of Theorem 2 for the given graph G. □

Proof of Theorem 3. Let us consider, in the assumption Ω-$\mathtt{SM}$, a R-OD graph G. As above, we shall suppose that G is actually an OD graph. Let us divide G into the two parts: $\mathsf{G}' = \{\langle x,y\rangle \in \mathsf{G} : x \in \mathrm{L}[y] \text{ or } y \in \mathrm{L}[x]\}$, and $\mathsf{G}'' = \mathsf{G} \setminus \mathsf{G}'$.

It is asserted that G' is a special OD graph.

Indeed we have to present G' as the union of an OD ω_1-sequence of curves. Let, for $\alpha < \omega_1$, $\mathsf{K}_{\alpha 0}$ be the set of all pairs $\langle x,y\rangle \in \mathsf{G}$ such that x is the α-th real in the sense of the y-constructible wellordering of $\mathrm{L}[y]$. Let $\mathsf{K}_{\alpha 1}$ be the set of all pairs $\langle x,y\rangle \in \mathsf{G}$ such that y is the α-th real in the sense of the x-constructible wellordering of $\mathrm{L}[x]$. Then by definition G' is the union of all these sets, all of them are curves, and they can be ordered in an OD ω_1-sequence.

Thus by the assumption of Theorem 3 the special subgraph G' satisfies the dichotomy (I) *vs.* $\mathcal{G}_0 \sqsubseteq_{\mathrm{c}} \mathsf{G}$ of Theorem 2.

On the other hand G'', the other subgraph, satisfies (ii)c of Theorem 12 by definition. It follows that G'' satisfies the same dichotomy, as required. □

4 The homomorphism

In this section, we present the proof of item (ii) of Theorem 12 in the version $\mathcal{G}_0 \le_{\mathrm{c}} \mathsf{G}$. Thus it is assumed that either Ω-$\mathtt{SM}$ or Ω-$\mathtt{Clps}$ holds. We consider an OD graph G on reals, and suppose that the set

$$A_0 = \{x \in \mathcal{W} : x \text{ does not belong to an OD } \mathsf{G}\text{-discrete set}\}$$

is nonempty. A continuous homomorphism H of $\mathcal{G}_0$ into G will be defined.

Sinse A_0 is obviously an OD set, there exists, by Proposition 9, a set $A \subseteq A_0$ of the form $A = \mathsf{X}_t$, where $t \in \mathbb{T}_\alpha \cap \mathrm{L}$ for some $\alpha < \Omega$.

It follows from Lemma 10 that the set $\mathbb{X}_A = \{X \in \mathbb{X} : X \subseteq A\}$ contains only countably many OD subsets in the universe. Let $\{\mathcal{X}_n : n \in \omega\}$ be a (not necessarily OD) enumeration of all dense subsets of $\mathbb{X}_A$.

In general for any OD-1st-countable set Q let us fix once and for all an enumeration $\{\mathcal{X}_n(Q) : n \in \omega\}$ of all dense OD subsets in $\mathcal{P}^{\mathrm{OD}}(Q) \setminus \{\emptyset\}$, where, we recall, $\mathcal{P}^{\text{OD}}(Q) = \mathcal{P}(Q) \cap \text{OD}$ is the OD part of the power set of Q. No uniformity on Q is assumed in the choice of the enumerations.

We recall that a sequence of binary tuples $\sigma_n \in 2^n$ is fixed by the definition of $\mathcal{G}_0$, see Introduction. Let $m \in \omega$. By a *crucial pair* in 2^m we shall understand any pair $\langle u, v\rangle$ of tuples $u,\, v \in 2^m$ such that $u = \sigma_k{}^\wedge 0^\wedge w$ and $v = \sigma_k{}^\wedge 1^\wedge w$, for some $k < m$ and $w \in 2^{m-k-1}$.

We observe that the (directed) graph of crucial pairs in 2^m is a *tree*: each pair of $u,\, v \in 2^m$ is connected in 2^m by a unique (non–self–intersecting) chain of crucial pairs $u = u_0 \frown u_1 \frown u_2 \frown \dots \frown u_{n-1} \frown u_n = v$, where $u' \frown v'$ means that either $\langle u', v'\rangle$ or $\langle v', u'\rangle$ is a crucial pair.

We shall define a set $X_u \subseteq A$ for every $u \in 2^{<\omega}$, and a binary relation Q_{uv} for each crucial pair $\langle u, v\rangle$ satisfying the following requirements:

C-1: each X_u $(u \in 2^m)$ is an OD nonempty subset of A and $X_u \in \mathcal{X}_m$;

C-2: $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0,\, 1$.

C-3: if $\langle u, v\rangle$ is a crucial pair then $\mathsf{Q}_{uv} \subseteq \mathsf{G}$ and Q_{uv} is a nonempty OD set ;

C-4: if $\langle u, v\rangle$ is a crucial pair then $\mathsf{Q}_{u^\wedge i,\, v^\wedge i} \subseteq \mathsf{Q}_{uv}$ for any $i = 0,\, 1$;

C-5: if $\langle u, v\rangle$ is a crucial pair then $X_u\, \mathsf{Q}_{uv}\, X_v$, that is,
$\forall\, x \in X_u\, \exists\, y \in X_v\, (x\, \mathsf{Q}_{uv}\, y)$ and $\forall\, y \in X_v\, \exists\, x \in X_u\, (x\, \mathsf{Q}_{uv}\, y)$;

C-6: if $m \geq 1$ and $u = \sigma_{m-1}{}^\wedge 0$, $v = \sigma_{m-1}{}^\wedge 1$ (a special type of crucial pairs) then Q_{uv} is an OD-1st-countable set ;

C-7: for any crucial pair of $u = \sigma_n{}^\wedge 0^\wedge w$ and $v = \sigma_n{}^\wedge 1^\wedge w$ in 2^m $(n < m \in \omega)$, $\mathsf{Q}_{uv} \in \mathcal{X}_m(\mathsf{Q}_{\sigma_n{}^\wedge 0,\, \sigma_n{}^\wedge 1})$.

We can assume without any loss of generality that $\mathsf{Q}_{uv} \subseteq X_u \times X_v$ for all crucial pairs (for if not replace the relations by the intersections with $X_u \times X_v$).

Suppose that such a system of sets X_u and relations Q_{uv} has been defined. It follows from C-1 and C-2 that every sequence of the form $\langle X_{a \restriction m} : m \in \omega\rangle$ (where $a \in \mathcal{D} = 2^\omega)$) is OD-generic in $\mathbb{X}$, therefore the intersection $\bigcap_m X_{a \restriction m}$ is a singleton, say $H(a)$, by Lemma 11, and H continuously maps $\mathcal{D}$ into $\mathcal{D}$.

Let us prove that H is a homomorphism $\mathcal{G}_0$ into G, that is, $a\, \mathcal{G}_0\, b$ implies $H(a)\, \mathsf{G}\, H(b)$ for all $a,\, b \in \mathcal{D}$. By definition $a\, \mathcal{G}_0\, b$ means that, for some k, we have for instance $a = \sigma_k{}^\wedge 0^\wedge z$ and $b = \sigma_k{}^\wedge 1^\wedge z$ where $z \in 2^\omega$. It follows from C-6, C-7, C-4 that the sequence of sets $Q_m = \mathsf{Q}_{\sigma_k{}^\wedge 0^\wedge (z \restriction m),\, \sigma_k{}^\wedge 1^\wedge (z \restriction m)}$ $(m \in \omega)$ is OD-generic in the two–dimentional "copy" of $\mathbb{X}$, so that the intersection $\bigcap_m Q_m$ is a singleton. It follows from C-5 that the singleton is equal to $\langle H(a), H(b)\rangle$. We conclude that $\langle H(a), H(b)\rangle \in \mathsf{G}$ because $Q_0 = \mathsf{Q}_{\sigma_k{}^\wedge 0,\, \sigma_k{}^\wedge 1} \subseteq \mathsf{G}$ by C-3. Thus $H(a) \mathrel{\mathsf{G}} H(b)$ as required.

Construction of the sets and relations

To begin with, we note that, since A (see above) is a nonempty OD set, it contains a subset $X \subseteq A$, $X \in \mathfrak{X}_0$. We take such a set X as X_Λ.

Suppose that $m \ge 1$, and the construction of sets X_s and relations Q_{st} (where $\langle s, t\rangle$ is a crucial pair) has been completed for all $s,\, t \in 2^{<m}$. Let us define the sets X_u and relations Q_{uv} for $u,\, v \in 2^m$, the next level.

At the beginning, we set $X_{s^\wedge i} = X_s$ for all $s \in 2^{m-1}$ and $i = 0, 1$, and $\mathsf{Q}_{s^\wedge i\,,\,t^\wedge i} = \mathsf{Q}_{st}$ for all $i = 0, 1$, and any crucial pair $\langle s, t\rangle$ in 2^{m-1}. This definition gives the initial versions of sets X_u $(u \in 2^m)$ and relations Q_{uv} ($\langle u, v\rangle$ being a crucial pair in 2^m, still with the exception of the crucial pair of $u_0 = \sigma_{m-1}{}^\wedge 0$ and $u_1 = \sigma_{m-1}{}^\wedge 1$.) The sets X_u and relations Q_{uv} will be reduced, in several steps, to meet requirements C-1 through C-7.

Step 1. This step is devoted to the "new" crucial pair of $u_0 = \sigma_{m-1}{}^\wedge 0$ and $u_1 = \sigma_{m-1}{}^\wedge 1$ in 2^m. At the moment $\mathsf{Q}_{u_0 u_1}$ has not yet been defined.

The set $X = X_{u_0} = X_{u_1} = X_{\sigma_{m-1}}$ is obviously a nonempty OD subset of A.

Notice that the OD set $\mathsf{C} = \mathsf{G} \cap (X \times X)$ is nonempty because X is not G-discrete (being a subset of A_0). Using Proposition 9 (item 3) and Lemma 10 (in the two–dimentional version), we get a nonempty OD-1st-countable OD set $\mathsf{Q} \subseteq \mathsf{C}$. We finally put $\mathsf{Q}_{u_0 u_1} = \mathsf{Q}$ and $B_0 = \operatorname{dom} \mathsf{Q}$, $B_1 = \operatorname{ran} \mathsf{Q}$, so that B_0 and B_1 are OD subsets of $X_{\sigma_{m-1}}$, $\mathsf{Q}_{u_0 u_1}$ is an OD relation and an OD-1st-countable set (so C-6 is satisfied), and $B_0 \mathrel{\mathsf{Q}_{u_0 u_1}} B_1$.

Step 2. We reduce the current system of sets X_u to the system of still OD and nonempty sets $B_u \subseteq X_u$ such that $B_{u_0} = B_0$, $B_{u_1} = B_1$, and C-5 is satisfied. Let $u \in 2^m$ be other than u_0 or u_1. Then u can be connected in 2^m with some u_i ($i = 0, 1$) by a unique non-self-intersecting chain of crucial pairs $u_i = v_0 \frown v_1 \frown v_2 \frown \dots \frown v_n = u$ so that u_{1-i} does not appear in the chain. We define $B^0 = B_i$ and then, by induction, $B^{l+1} = \{y \in X_{v_{l+1}} : \exists\, x \in B^l\ (x \mathrel{\mathsf{Q}_l} y)\}$ for all $l < n$, where $\mathsf{Q}_l = \mathsf{Q}_{v_l v_{l+1}}$ or resp. $\mathsf{Q}_l = (\mathsf{Q}_{v_{l+1} v_l})^{-1}$ provided $\langle v_l, v_{l+1}\rangle$ or resp. $\langle v_{l+1}, v_l\rangle$ is a crucial pair. The final set B^n can be taken as B_u.

Step 3. We have a system of OD nonempty sets B_u $(u \in 2^m)$ and OD relations Q_{uv} ($\langle u, v\rangle$ being a crucial pair in 2^m) which satisfies all of C-1 through C-6, still with the exception of C-7 and the requirement $X_u \in \mathfrak{X}_m$ in C-1.

To satisfy the latter let us fix $u \in 2^m$. By the density of $\mathfrak{X}_m$ there exists a nonempty OD set $B \subseteq B_u$ such that $B \in \mathfrak{X}_m$. Using the construction of Step

2, we obtain a system of OD nonempty sets $B'_u \subseteq B_u$ keeping the mentioned properties of the sets B_u and in addition satisfying $B'_u = B$.

We iterate this reduction consecutively 2^m times (the number of elements in 2^m), getting a system of nonempty OD sets $C_u \subseteq B_u$ which satisfies all of C-1 through C-6, still with the exception of C-7. Let us take intersections of the form $\mathsf{Q}_{uv} \cap (X_u \times X_v)$ as the "new" Q_{uv}.

Step 4. At this step, further reductions are executed, to fulfill C-7. We consider a crucial pair of $u' = \sigma_n{}^\wedge 0^\wedge w$ and $v' = \sigma_n{}^\wedge 1^\wedge w$ $(n < m)$ in 2^m. Let the "new" $\mathsf{Q}_{u'v'}$ be an OD subset of the "old" $\mathsf{Q}_{u'v'}$ which belongs to $\mathcal{X}_m(\mathsf{Q}_{\sigma_n{}^\wedge 0,\, \sigma_n{}^\wedge 1})$. The "new" sets $C'_{u'}$ and $C'_{v'}$ are defined by

$$C'_{u'} = \{x \in C_{u'} : \exists\, y \in C_{v'}\, (x \,\mathsf{Q}_{u'v'}\, y)\}\,, \quad C'_{v'} = \{y \in C_{v'} : \exists\, x \in C_{u'}\, (x \,\mathsf{Q}_{u'v'}\, y)\}\,,$$

so that still $C'_{u'} \,\mathsf{Q}_{u'v'}\, C'_{v'}$. Then we restrict the sets C_u $(u \in 2^m)$ using the method described at Step 2. Running this construction consecutively for all crucial pairs $\langle u', v' \rangle$ in 2^m, we obtain finally a system of sets X_u and relations Q_{uv} which satisfies all of C-1 through C-7.

This ends the inductive construction of sets X_u and relations Q_{uv}, and completes the proof of statement (ii) of Theorem 12 in the version $\mathcal{G}_0 \le_{\mathrm{c}} \mathsf{G}$. $\dashv$

5 The embedding

This section proves the "moreover" part of item (ii) of Theorem 12 where $\mathcal{G}_0 \sqsubseteq_{\mathrm{c}} \mathsf{G}$ is requested. We shall also prove Theorem 3. We still suppose $(\ddagger)$, consider an OD graph G, and continue to assume that the set A_0 (see the beginning of Section 4) is nonempty.

The common denominator is to modify the splitting construction described in Section 4 so that the resulting function H is a $1-1$ map. The following additional requirement is clearly sufficient.

C-8: if $s \in 2^{m-1}$ $(m \ge 1)$ then $X_{s^\wedge 0} \cap X_{s^\wedge 1} = \emptyset$.

Let us demonstrate how the construction of Section 4 can be improved to meet this additional requirement, in the subcases (ii)a, (ii)b, (ii)c of Theorem 12. The result for (ii)c will be used to accomplish the proof of Theorem 3.

5.1 Acyclic case

Let us assume that G is an acyclic graph. The construction of Section 4 expands by one more step.

Step 5. At the moment we have a system of sets X_u and relations Q_{uv} which satisfies all of C-1 through C-7. Let us re–denote the sets, putting $D_u = X_u$. The aim is to reduce sets D_u $(u \in 2^m)$ once again, to satisfy also C-8.

Suppose that $s \in 2^{m-1}$ and $u_0 = s^\wedge 0$, $u_1 = s^\wedge 1$. Then u_0 can be connected with u_1 in 2^m by a unique sequence of crucial pairs $u_0 = v_0 \frown v_1 \frown \dots \frown v_n = u_1$ where, by the choice of u_0 and u_1, n is necessarily an odd number. It follows from C-5 that there exists a system of points $x_k \in D_{v_k}$ such that $x_k \, \mathsf{Q}_k \, x_{k+1}$ for all $k < n$, where $\mathsf{Q}_k = \mathsf{Q}_{v_k v_{k+1}}$ or resp. $\mathsf{Q}_k = (\mathsf{Q}_{v_{k+1} v_k})^{-1}$ provided $\langle v_k, v_{k+1} \rangle$ or resp. $\langle v_{k+1}, v_k \rangle$ is a crucial pair.

Take notice that $x_0 \neq x_n$ because G is acyclic and n is odd, in particular $n \neq 2$. Therefore there exist disjoint OD sets $D'_i \subseteq D_{u_i}$ $(i = 0, 1)$ such that $x_0 \in D'_0$ and $x_n \in D'_1$. Now the construction of Step 2 in Section 4 allows to obtain a system of nonempty OD sets $X_u \subseteq D_u$ $(u \in 2^m)$ which keeps C-1 through C-7 and satisfies $X_{u_0} \subseteq D'_0$ and $X_{u_1} \subseteq D'_1$, so $X_{u_0} \cap X_{u_1} = \emptyset$.

Iterating this procedure for all relevant pairs in 2^m, we get finally a system at level m satisfying all of C-1 through C-7, and in addition C-8, as required.

5.2 Locally countable case

The case when G is a locally countable graph needs another approach. We let a *1–1 curve* be a binary relation K which is formally (the graph of) a $1-1$ map.

Lemma 13 *Assume* Ω-`SM` *or* Ω-`Clps`. *Let* G *be a locally countable* OD *graph. Then* G *is a union of* $\aleph_1$*-many* OD *1–1 curves.*

Proof For any real x, the set $G(x) = \{y : x \, \mathsf{G} \, y\}$ of all G-neighbours of x is a countable $\mathrm{OD}[x]$ set of reals. It is known from the theory of the Solovay and collapse models that then $G(x) \subseteq \mathrm{L}[x]$. So $x \, \mathsf{G} \, y$ implies $x \in \mathrm{L}[y]$ and $y \in \mathrm{L}[x]$. Thus $\mathsf{G} = \bigcup_{\alpha, \beta < \omega_1} \mathsf{K}_{\alpha\beta}$ where $\mathsf{K}_{\alpha\beta}$ is the set of all pairs $\langle x, y \rangle \in \mathsf{G}$ such that x is the α-th real in the sense of the x-constructible wellordering of $\mathrm{L}[y]$ and y is the β-th real in the sense of the y-constructible wellordering of $\mathrm{L}[x]$. □

The lemma allows to carry out the construction in Section 4 so that, in addition to requirements C-1 through C-7 the following is satisfied:

C-9: if $u = \sigma_{m-1}{}^\wedge 0$, $v = \sigma_{m-1}{}^\wedge 1$, then $X_u \cap X_v = \emptyset$ and Q_{uv} is a $1-1$ curve.

Let us prove that this implies C-8. We have to check that $X_{s^\wedge 0} \cap X_{s^\wedge 1} = \emptyset$ for all $s \in 2^{m-1}$. We recall that $\sigma_{m-1} \in 2^{m-1}$ too, therefore as above there exists a unique chain of crucial pairs $\sigma_{m-1} = s_1 \frown s_2 \frown \dots \frown s_n = s$ in 2^{m-1}.

We prove by induction on k that $X_{s_k{}^\wedge 0} \cap X_{s_k{}^\wedge 1} = \emptyset$.

The result for $k = 1$ immediately follows from C-9.

Let us suppose that $X_{s_k{}^\wedge 0} \cap X_{s_k{}^\wedge 1} = \emptyset$ and prove $X_{s_{k+1}{}^\wedge 0} \cap X_{s_{k+1}{}^\wedge 1} = \emptyset$.

One of the pairs $\langle s_k, s_{k+1} \rangle$ and $\langle s_{k+1}, s_k \rangle$ is crucial; let us assume that this is the first one. Then $\mathsf{K} = \mathsf{Q}_{s_k, s_{k+1}}$ is a $1-1$ map of X_{s_k} onto $X_{s_{k+1}}$ by C-9. It follows that the pairs $\langle s_k{}^\wedge i, s_{k+1}{}^\wedge i \rangle$ $(i = 0, 1)$ are crucial and the relations $\mathsf{K}_i = \mathsf{Q}_{s_k{}^\wedge i, s_{k+1}{}^\wedge i}$ are restrictions of K. Thus the sets $X_{s_{k+1}{}^\wedge 0}$ and $X_{s_{k+1}{}^\wedge 1}$ are K-images of resp. $X_{s_k{}^\wedge 0}$ and $X_{s_k{}^\wedge 1}$. We now conclude that $X_{s_{k+1}{}^\wedge 0} \cap X_{s_{k+1}{}^\wedge 1} = \emptyset$ because $X_{s_k{}^\wedge 0} \cap X_{s_k{}^\wedge 1} = \emptyset$ and K is $1-1$.

5.3 Reduction to unions of curves

Let us consider subcase (ii)c of Theorem 12. Thus G is supposed not to contain pairs $\langle x, y \rangle$ such that $x \in \mathrm{L}[y]$ or $y \in \mathrm{L}[x]$. Having this in mind, let us show how the construction of Section 4 can be modified to provide C-8 in addition to requirements C-1 through C-7.

Taking the reasoning in the "acyclic" case (Subsection 5.1) as the pattern, we see that the following is sufficient. Suppose that $s \in 2^{m-1}$, $u_0 = s^\wedge 0$, $u_1 = s^\wedge 1$. Let $u_0 = v_0{}^\frown v_1{}^\frown \dots{}^\frown v_n = u_1$ be the unique sequence of crucial pairs which connects u_0 with u_1 in 2^m. We have to prove the existence of a system of points $x_k \in D_{v_k}$ satisfying $x_0 \neq x_n$ and $x_k \, \mathsf{Q}_k \, x_{k+1}$ for all $k < n$, where $\mathsf{Q}_k = \mathsf{Q}_{v_k v_{k+1}}$ or resp. $\mathsf{Q}_k = (\mathsf{Q}_{v_{k+1} v_k})^{-1}$ provided $\langle v_k, v_{k+1} \rangle$ or resp. $\langle v_{k+1}, v_k \rangle$ is a crucial pair.

To prove this assertion, it suffices to check the following more elementary fact: if $\langle u, v \rangle$ is a crucial pair and $x \in X_u$ (resp. $y \in X_v$) then there exist at least two points $y \in X_v$ (resp. $x \in X_u$) such that $x \, \mathsf{Q}_{uv} \, y$. To prove the fact note that at least one such a point y exists by C-5. Hence if it were unique it would be $\mathrm{OD}[x]$, therefore $y \in \mathrm{L}[x]$, contradiction because $\mathsf{Q}_{uv} \subseteq \mathsf{G}$.

□ (Of Theorem 12)

6 $\mathbf{\Sigma}^1_2$ graphs in generic model

This section presents the proof of Theorem 4. The first subsection shows how to convert (II) to an absolute statement.

6.1 Narrow homomorphisms and embeddings

Dealing with $\mathbf{\Sigma}^1_2$ graphs, we would like to have the absoluteness of the statements: $\mathcal{G}_0 \le_{\mathrm{c}} \mathsf{G}$ and $\mathcal{G}_0 \sqsubseteq_{\mathrm{c}} \mathsf{G}$. This does lot look easily available because the direct computation shows that the statements are described by Σ^1_4 formulas, far from the Shonfield level. However, two levels can be reduced, at the cost of some specification of the nature of homomorphisms and embeddings involved.

Let us fix a reasonable coding system of countable ordinals by reals. Let $|z|$ denote the ordinal coded by z, and $\mathcal{O} = \{z : z \text{ codes an ordinal}\}$; $\mathcal{O}$ is a Π^1_1 set of reals. Let us once and for all associate with every Σ^1_2 graph G the representation in the form $\mathsf{G} = \bigcup_{\alpha<\omega_1} \mathsf{G}_\alpha$, where the relation $x \in \mathsf{G}_{|z|}$ is Π^1_1 uniformly on $z \in \mathcal{O}$ (then the components G_α are $\mathbf{\Pi}^1_1$ of course). In other words there exists a Π^1_1 relation π_G such that

$$\mathsf{G}_\alpha(x,y) \longleftrightarrow \pi_\mathsf{G}(x,y,z) \quad \text{whenever} \quad z \in \mathcal{O} \text{ and } |z| = \alpha < \omega_1 .$$

(Roughly $\pi_\mathsf{G}(x,y,z)$ says that $z \in \mathcal{O}$ and we have $\psi(x,y,a)$ where a is the $|z|$-th real in the sense of the Gödel wellordering of $\mathrm{L}[x,y]$ and $\psi(x,y,a)$ is a Π^1_1 formula such that $\exists\, a\, \psi(x,y,a)$ defines G. The Shoenfield absoluteness is used to ensure that the whole G is captured by the decomposition.)

Let G be a Σ^1_2 graph. A *narrow homomorphism* of $\mathcal{G}_0$ into G will be any homomorphism (see Introduction) H satisfying the property that, for some ordinal $\alpha_0 < \omega_1$, we have, for all $a,\, b \in 2^\omega$, $a\, \mathcal{G}_0\, b \longrightarrow \langle H(a), H(b)\rangle \in \bigcup_{\alpha<\alpha_0} \mathsf{G}_\alpha$. The notion of a *narrow embedding* has the same meaning.

We shall write $\mathcal{G}_0 \le_{\rm cn} G$ (or resp. $\mathcal{G}_0 \sqsubseteq_{\rm cn} G$) iff there exists a continuous narrow homomorphism (resp. embedding) of $\mathcal{G}_0$ into G.

Lemma 14 *Let G be a Σ^1_2 graph on reals. Then $\mathcal{G}_0 \le_{\rm cn} \mathsf{G}$ and $\mathcal{G}_0 \sqsubseteq_{\rm cn} \mathsf{G}$ are Σ^1_2 relations.*

Proof Direct compuration. $\square$

The lemma explains the advantage of the "narrow" concepts. However we have to demonstrate that they are not too narrow to violate Theorem 12.

Proposition 15 *Assume that G is a subgraph of a Σ^1_2 graph G^* in Theorem 12. Then the relations $\mathcal{G}_0 \le_{\rm c} \mathsf{G}$ and $\mathcal{G}_0 \sqsubseteq_{\rm c} \mathsf{G}$ in* (ii) *of Theorem 12 can be changed to resp. $\mathcal{G}_0 \le_{\rm cn} \mathsf{G}^*$ and $\mathcal{G}_0 \sqsubseteq_{\rm cn} \mathsf{G}^*$.*

Proof Let us come back to Step 1 of the construction described in Section 4. We defined there, for the crucial pair of $u_0 = \sigma_{m-1}{}^\wedge 0$ and $u_1 = \sigma_{m-1}{}^\wedge 1$ in 2^m, a certain nonempty OD set $\mathsf{Q} \subseteq \mathsf{G}$. Since $\mathsf{G} \subseteq \mathsf{G}^*$, there exists an ordinal $\gamma_m < \omega_1$ and a nonempty OD set $\mathsf{Q}' \subseteq \mathsf{Q}$ such that $\mathsf{Q}' \subseteq \bigcup_{\alpha<\gamma_m} {\mathsf{G}^*}_\alpha$. We take now this relation Q' as the actual Q.

After the whole construction is finished in this modernized way, the ordinal $\alpha_0 = \sup_{m\in\omega} \gamma_m$ will witness that the obtained homomorphism (or embedding, in the modifications used in Section 5) is narrow. $\square$

6.2 Extending the universe

The proof of Theorem 4 includes the operation of embedding of the universe V for which the theorem is being proved in a larger universe, say V^*, "good" in the sense that it satisfies the dichotomy of Theorem 2. To facilitate this argument, let us pretend that V is is fact a countable model in a vider **ZFC** universe. But this is not a serious restrictive assumption — all that we use it for is the existence of certain generic extensions of L, the constructible subuniverse of V. Alternatively, this can be carried out using Boolean valued extensions, which does not need any extra hypothesis but would reduce the quality of presentation.

We recall that V, the universe for which Theorem 4 is being proved, is assumed to be a P-generic extension of L, for a forcing notion $\mathsf{P} \in \mathrm{L}$. Let κ be the cardinal of P in L. Then $\kappa = \aleph_\xi^{\mathrm{L}}$ for an ordinal ξ.

We let $\Omega = \aleph_{\xi+\omega}^{\mathrm{L}}$ and $\Xi = \aleph_{\xi+\omega+1}^{\mathrm{L}}$. Obviously Ω and Ξ are cardinals in V.

Lemma 16 *There is a $\mathcal{P}_\Omega$-generic extension*[7] V^* *of* L *such that* $\mathrm{V} \subseteq \mathrm{V}^*$.

Proof Assume, towards contradiction, that such an extension does not exist. Then a condition $p \in \mathsf{P}$ forces over L the following: $(*)$ *there does not exist a $\mathcal{P}_\Omega$-generic extension of the universe which is also a $\mathcal{P}_\Omega$-generic extension of* L.

Let us now consider an arbitrary $\mathcal{P}_\Omega$-generic extension V^* of L. Since $\kappa = (\mathrm{card}\,\mathsf{P})^{\mathrm{L}} < \Omega$, there exists a P-generic over L set $G \in \mathrm{V}^*$ containing p. We observe that V^* is also an $\mathcal{P}_\Omega$-generic extension of $\mathrm{L}[G]$ by Proposition 8, a contradiction with the assumption that p forces $(*)$. $\square$

6.3 Proof of Theorem 4

Let us fix a Σ^1_2 graph G, defined by a parameter–free Σ^1_2 formula $\varphi(x, y)$ in V. (As usual, we concentrate on the "lightface" case.) Avoiding unnecessary complications, we let $\Phi(x, y)$ be the Σ^1_2 formula $x \neq y \;\&\; [\varphi(x, y) \vee \varphi(y, x)]$, which defines G as well, and explicitly contains the requirements of being a graph. G_Φ will denote the graph $\{\langle x, y\rangle : \Phi(x, y)\}$ in the universe considered at the moment; thus for instance G is G_Φ in V.

Let V^* be the extension given by Lemma 16; in particular V^* satisfies Ω-`Clps` and even (‡) of Section 3, together with the graph $\mathsf{G}^* = (\mathsf{G}_\Phi)^{\mathrm{V}^*}$, and includes V. All reals $x \in \mathrm{V}$ are Ω-"weak" in V^* by definition.

We now define a subgraph $\mathsf{G}' \subseteq \mathsf{G}^*$ to which Theorem 12 will be applied.

If G *is not locally countable* in V then we simply put $\mathsf{G}' = \mathsf{G}^* \cap \mathcal{W}_{(2)}$ in V^*.

[7] See definitions in Section 1.

If G *is locally countable* in V then we first put $\mathsf{G}'' = \mathsf{G}^* \cap \mathcal{W}_{(2)}$, as above, and then define G' in V^* to be the restriction of G'' on the set C of all reals $x \in \mathcal{W}$ such that the G^*-component $[x]_{\mathsf{G}^*} = \{y : x \, \mathsf{G}^* \, y\}$ is countable in V^*.

Lemma 17 *$\mathsf{G}' \subseteq \mathcal{W}_{(2)} \cap \mathsf{G}^*$ is an* OD *graph in* V^*. $\mathsf{G} = \mathsf{G}' \cap \mathrm{V}$. *If G is acyclic or locally countable in* V *then G' is acyclic or resp. locally countable in* V^*.

Proof Let us consider the equality $\mathsf{G} = \mathsf{G}' \cap \mathrm{V}$. In the first version of the definition of G' this is a direct consequence of the Shoenfield absoluteness theorem. In the locally countable version, notice that by the local countability a real $x \in \mathrm{V}$ cannot gain new G_Φ-neighbours in any extension of V (here we use the fact that Φ is Σ^1_2), therefore all reals in V remain in the domain of G'.

The property of being acyclic for a $\varSigma^1_2$ graph can be expressed by a Π^1_2 formula, therefore even G^* inherits this property from G. (It is not clear whether the local countability can be handled in this manner. This is why we changed the definition of G' in this case so that G' acquires this property directly.) □

We have two cases.

Case 2: in V^*, <u>not all</u> reals in $\mathcal{W}$ belong to OD G'-discrete sets. Then G' satisfies (II) of Theorem 2 in V^* by Theorem 12. In fact we obtain (II) in the enhanced form given by Proposition 15, that is, $\mathcal{G}_0 \le_{\mathrm{cn}} \mathsf{G}^*$ in the main part and and $\mathcal{G}_0 \sqsubseteq_{\mathrm{cn}} \mathsf{G}^*$ in the "moreover" part. Now Lemma 14 expands this property down to the graph G in V, as required.

Case 1: in V^*, <u>all</u> reals in $\mathcal{W}$ belong to OD G'-discrete sets. Then G' admits an OD colouring by countable ordinals in V^* by Theorem 12. Let $\iota(x, \alpha)$ be an $\in$-formula which defines such a colouring in V^*. It follows from Proposition 8 that, since V^* is a $\mathcal{P}_\Omega$-generic extension of V, the restriction $c = c' \restriction \mathrm{V}$ is OD in V. (Indeed each formula with sets in V as parameters true in V^* is forced by every forcing condition.) Finally c is an OD colouring of G by ordinals in V.

We have also to prove that G admits a $\varDelta^{\mathrm{HC}}_2$ colouring in V in the case when G is locally countable in V. This is the matter of the next subsection.

6.4 The locally countable case

We keep the notation and setup of Subsection 6.3 including the assumption of Case 1. We shall also suppose that G is locally countable in V.

An elementary reasoning shows that if a locally countable OD graph G admits an OD colouring by ordinals then it admits an OD colouring by *countable* ordinals. (Indeed, each G-component $[x]_{\mathsf{G}}$ involves only countably many ordinals in the colouring; one simply "dumps" all of the ordinals involved into ω_1 in the order preserving way. Notice that the $\varSigma^1_2$ graph $x \mathsf{G} y$ iff $\mathrm{L}[x]=\mathrm{L}[y]$ but $x\ne y$ shows that an OD colouring by natural numbers may not be available.)

But this argument does not seem to give a colouring definable in HC.

A more elaborate reasoning, based on the next lemma, will be used.

Lemma 18 *The following is true in* V. *For any real x there exist: an ordinal $\alpha'<\omega_1$ and a "term" $t'\in\mathbb{T}_{\alpha'}\cap\mathrm{L}_{\alpha'}$ such that the set $\mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ contains x but does not contain any G-neighbour of x.*

Using the least, in the sense of a reasonable wellordering, pair $\langle\alpha',t'\rangle$ satisfying the conclusion of the lemma as the colour for x, we get this way a $\varDelta^{\mathrm{HC}}_2$ colouring of G (in V) by countable ordinals, as required.

Proof of the lemma. Let us fix a real $x\in\mathrm{V}$. In V^*, there exist (see the proof of Theorem 12, item (i), in Section 3): an ordinal $\gamma<\Omega$ and a "term" $t\in\mathbb{T}_\gamma\cap\mathrm{L}$ such that the set $\mathsf{X}_t=\mathsf{X}_t(\mathrm{L})$ (see Subsection 2.1) contains x and is G'-discrete, so does not contain any of G'-neighbours of x. Then, since by the local countability all G^*-neighbours of x in V^* belong to V and are G-neighbours of x, we conclude that X_t does not contain any G^*-neighbour of x in V^*.

By the choice of Ω there exists an ordinal $\alpha<\Omega$, $\alpha>\gamma$ such that $t\in\mathrm{L}_\alpha$ and $\mathsf{X}_t=\mathsf{X}_t(\mathrm{L}_\alpha)$. Since V^* is a $\mathcal{P}_\Omega$-generic extension of $\mathrm{L}[x]$ by Proposition 8, the statement that $\mathsf{X}_t(\mathrm{L}_\alpha)$ contains x but does not contain any G^*-neighbour of x is $\mathcal{P}_\Omega$-forced in $\mathrm{L}[x]$.

By the local countability again, each G-neighbour of x in V belongs to $\mathrm{L}[x]$. Therefore by the Lövenheim – Scolem theorem in $\mathrm{L}[x]$ there exist ordinals $\alpha'<\Omega'<\xi<\omega_1$ (perhaps $\not<{\omega_1}^{\mathrm{L}[x]}$) and a "term" $t'\in\mathbb{T}_{\alpha'}\cap\mathrm{L}_{\alpha'}$ such that $\mathrm{L}_\xi[x]$ contains all G-neighbours of x in V, models a big enough part of **ZFC**, and is absolute for Φ (the Σ^1_2 formula which defines G in V), and it is $\mathcal{P}_{\Omega'}$-forced in $\mathrm{L}_\xi[x]$ that the set $\mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ contains x but does not contain G_Φ-neighbours of x.

Let us show that α' and t' satisfy the requirements of the lemma. Let y be an arbitrary G-neighbour of x in V; we have to prove that $y\notin\mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ in V.

Since ξ is countable in V, there exists a $\mathcal{P}_{\Omega'}$-generic extension $\mathfrak{N}$ of $\mathrm{L}_\xi[x]$. Then it is true in $\mathfrak{N}$ that the set $\mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ contains x but does not contain G_Φ-neighbours of x.

Notice that $y\in\mathfrak{N}$ because $\mathrm{L}_\xi[x]$ already contains all G-neighbours of x in V. Furthermore it is true in $\mathfrak{N}$ that y is a G_Φ-neighbour of x by the Φ-absoluteness requirement for $\mathrm{L}_\xi[x]$. Thus $y\notin\mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ in $\mathfrak{N}$. But, since α' is countable in $\mathfrak{N}$, the statement $y\in\mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ is actually a Σ^1_1 formula with some reals in $\mathfrak{N}$ as parameters. We conclude that $y \notin \mathsf{X}_{t'}(\mathrm{L}_{\alpha'})$ is true in V as well by the Mostowski absoluteness theorem, as required. □

□ (of Theorem 4)

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