The first order formulas preserved under ultrafilter extensions are not recursively enumerable

Balder ten Cate

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In this short note, we prove that the first-order formulas preserved under ultrafilter extensions are not recursively enumerable. All relevant definitions and notations can be found in [1].

Let ϕ_{inf} be any satisfiable FO sentence that is preserved under ultrafilter extensions and that only has infinite models.[1] Let R be a binary relation symbol not occuring in ϕ_{inf}, let P be a unary predicate not occuring in ϕ_{inf} and let

$$\theta \;=\; \phi_{inf}^{\neg P} \wedge \forall xy.(x \neq y \rightarrow Rxy) \wedge (\exists x.(Px \wedge Rxx) \leftrightarrow \exists x.(\neg Px \wedge Rxx))$$

Note that we use the familiar notation for relativisation of a FO formula by a (negated) unary predicate.

Lemma 1 *If $\mathfrak{M} \models \theta$ and P denotes an infinite set in $\mathfrak{M}$, then $\mathfrak{ue}\mathfrak{M} \models \theta$.*

Proof: By assumption, ϕ_{inf} is preserved under ultrafilter extensions. The second conjunct of θ, i.e., $\forall xy.(x \neq y \rightarrow Rxy)$ is also preserved under ultrafilter extension, since it is modally definable using global modality. Finally, consider the third conjunct of θ. From the fact that $\mathfrak{M} \models \forall xy.(x \neq y \rightarrow Rxy)$, we can derive that the denotation of R in $\mathfrak{ue}\mathfrak{M}$ includes all pairs of ultrafilters (u, v) such that v is a non-principal ultrafilter. In particular, each non-principal ultrafilter in $\mathfrak{ue}\mathfrak{M}$ is R-connected to itself. Both the denotation of P and its complement are infinite in $\mathfrak{M}$, hence admit non-principal ultrafilters. It follows that $\mathfrak{ue}\mathfrak{M} \models \exists x.(Px \wedge Rxx) \wedge \exists x.(\neg Px \wedge Rxx)$. □

Lemma 2 *If $\mathfrak{M}$ is infinite and P denotes a finite set in $\mathfrak{M}$, then there is an $\{R\} \cup REL(\phi_{inf})$-variant $\mathfrak{M}'$ of $\mathfrak{M}$ such that $\mathfrak{M}' \models \theta$ and $\mathfrak{ue}\mathfrak{M}' \not\models \theta$.*

Proof: By the Löwenheim-Skolem theorem, ϕ_{inf} has a model $\mathfrak{N}$ that has the same cardinality as $\mathfrak{M}$. Let $\mathfrak{M}'$ be obtained from $\mathfrak{M}$ by letting R denote the inequality relation on the domain, and by copying from $\mathfrak{N}$ the interpretation of the relation symbols occuring in ϕ_{inf}. Clearly, $\mathfrak{M}' \models \theta$. Moreover, as we will now show, $\mathfrak{ue}\mathfrak{M} \not\models \theta$.

Since P denotes a finite set, the submodel of $\mathfrak{ue}\mathfrak{M}'$ consisting of the ultrafilters satisfying P constitutes an isomorphic copy of the submodel of $\mathfrak{M}'$ consisting of the points satisfying P. Since $\mathfrak{M}' \models \neg\exists x.(Px \wedge Rxx)$, it follows that $\mathfrak{ue}\mathfrak{M}' \models \neg\exists x.(Px \wedge Rxx)$. On the other hand, by similar arguments as in the proof of Lemma 1, one can show that $\mathfrak{ue}\mathfrak{M}' \models \exists x.(\neg Px \wedge Rxx)$, using the fact that there are infinitely many points in $\mathfrak{M}'$ satisfying $\neg P$. It follows that $\mathfrak{ue}\mathfrak{M}' \not\models \theta$. □

Theorem 1 *Let ϕ be any FO formula preserved under ultrafilter extensions, not containing any of the relation symbols $\{R\} \cup REL(\phi_{inf})$. Let Q be a fresh unary predicate. Then the following are equivalent:*

1. *ϕ has a model in which P denotes a finite set.*
2. *$\theta \wedge \phi^Q$ is not preserved under ultrafilter extensions.*

[1] Such formulas exist, since there are elementary modal logics without the finite model property. An example is $\forall xy.(R_{\rightarrow}xy \leftrightarrow R_{\leftarrow}yx) \wedge \forall x.\exists^{=1}y.(xR_{\rightarrow}y) \wedge \forall x.\exists^{\leq 1}y.(xR_{\leftarrow}y) \wedge \forall x.\exists y.(Sxy \wedge \neg\exists z.(R_{\leftarrow}yz))$, which expresses a modally definable frame property and has only infinite models.

Proof: First, suppose $\mathfrak{M} \models \phi$ for some model $\mathfrak{M}$ in which P denotes a finite set. Let $\mathfrak{N}$ be any infinite model extending $\mathfrak{M}$, in which all relation symbols still have the original denotation, and in which Q denotes the domain of $\mathfrak{M}$. Then $\mathfrak{N} \models \phi^Q$ and P still denotes a finite set in $\mathfrak{N}$. Moreover, since $\{R\} \cup REL(\phi_{inf})$ do not occur in ϕ, by Lemma 2 we may assume that $\mathfrak{N} \models \theta$ and $\mathfrak{ue}\mathfrak{N} \not\models \theta$. It follows that $\mathfrak{N} \models \theta \wedge \phi^Q$ and $\mathfrak{ue}\mathfrak{N} \not\models \theta \wedge \phi^Q$. In other words, $\theta \wedge \phi^Q$ is not preserved under ultrafilter extensions.

Next, suppose ϕ does *not* have a model in which P denotes a finite set. In other words, ϕ^Q implies that P denotes an infinite set. It follows by Lemma 1 that θ is preserved under ultrafilter extensions of models satisfying ϕ^Q. Since ϕ^Q itself is also preserved under ultrafilter extensions, we conclude that $\theta \wedge \phi^Q$ is preserved under ultrafilter extensions. □

Corollary 1 *Let ϕ be any FO formula preserved under ultrafilter extensions, not containing any of the relation symbols $\{R\} \cup REL(\phi_{inf})$. Then the following are equivalent:*

1. *ϕ has a finite model*
2. *$\theta \wedge \neg\phi^P \wedge \exists x.Px$ is not preserved under ultrafilter extensions*

This would immediately prove that preservation under ultrafilter extensions is a non-recursively enumerable problem, if it weren't for the requirement that ϕ is preserved under ultrafilter extensions. We will have to prove it by hand, using a reduction from a tiling problem. More precisely, we use the *periodic* tiling problem, i.e., given a finite set of tiles $T_1, \ldots, T_n$ with corresponding matching conditions, is there a tiling of $\mathbb{Z} \times \mathbb{Z}$ that periodically repeats itself in both directions? In other words, is it possible to tile a finite $k \times m$ rectangle such that the top and bottom side match, as well as the left and right side? This problem is recursively enumerable but not co-recursively enumerable.

Given tiles $T_1, \ldots, T_n$ with corresponding matching conditions, let ϕ_{grid} be the conjunction of the following formulas (where we use $T_1, \ldots, T_n$ as unary relation symbols, and R_h, R_v as binary relation symbols).

$$\forall x.\big(\exists^{=1} y.R_h(x,y) \wedge \exists^{=1} y.R_v(x,y)\big) \tag{1}$$

$$\forall xyz.(R_h(x,y) \wedge R_v(x,z) \rightarrow \exists u.(R_v(y,u) \wedge R_h(z,u)) \tag{2}$$

$$\forall x.\Big(\bigvee_{1\leq k\leq n} T_k(x) \quad \wedge \bigwedge_{1\leq k<l\leq n} \neg\big(T_k(x) \wedge T_l(x)\big)\Big) \tag{3}$$

$$\forall xy.(R_h(x,y) \rightarrow \text{HMATCH}(x,y)) \tag{4}$$

$$\forall xy.(R_v(x,y) \rightarrow \text{VMATCH}(x,y)) \tag{5}$$

Here, HMATCH and VMATCH stand for quantifier free formulas that express the horizontal resp. vertical matching conditions. Each of these formulas is preserved under ultrafilter extensions, since they all express modally definable elementary frame properties.

Proposition 1 *$T_1, \ldots, T_n$ periodically tile $\mathbb{Z} \times \mathbb{Z}$ iff $GRID$ is satisfiable in a finite model.*

It follows by Corollary 1 that $T_1, \ldots, T_n$ periodically tiles $\mathbb{Z} \times \mathbb{Z}$ iff $\theta \wedge GRID^P \wedge \exists x.Px$ is *not* preserved under ultrafilter extensions. It follows that the problem of deciding whether a FO formula is preserved under ultrafilter extensions is not recursively enumerable.

References

[1] P. Blackburn, M. de Rijke and Y. Venema. *Modal Logic.* Cambridge University Press, Cambridge, UK, 2001.