Non-finitely axiomatizable, union-free reducts of algebras of relations

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Abstract

In this paper, we show that certain subreducts of the class of representable relation algebras are not finitely axiomatizable. We do the same for sequential and cylindric algebras.

1 Introduction

Monk showed that the class RRA of representable relation algebras is not finitely axiomatizable; cf. [Mon64]. Various authors have investigated whether this negative result holds for various subreducts of RRA (see the formal definition of subreduct in Definition 2.2 below).

Andréka showed that any subreduct of RRA whose operations include union, intersection and composition is not finitely axiomatizable [And91], and that the {union, composition}-subreduct is a non-finitely axiomatizable quasivariety [And88].

On the other hand, some subreducts are finitely axiomatizable. For instance, Bredikhin and Schein [BS78] showed that the {intersection, composition}-subreduct coincides with the class of semilattice-ordered semigroups. It is easy to show that the {intersection, composition, emptyset}-subreduct is a finitely axiomatizable variety as well. Another example is the generalized subreduct with the similarity type of intersection, composition and its two residuals: see [AM94].

Recently, Hirsch and Hodkinson have applied a game-theoretic approach to various problems concerning relation algebras [HH97, HH97a, HH97b, HH97c, HH97d]. In this paper, we apply a similar strategy to prove non-finite axiomatizability results for certain subreducts of RRA. In particular, we will show that any generalized subreduct of RRA in which intersection, composition and converse are term definable is not finitely axiomatizable. Another non-finitely axiomatizable version of algebras of binary relations is the class of representable sequential algebras; see, e.g., [JM97]. As a corollary, we obtain that the union-free subreduct of representable sequential algebras is not finitely axiomatizable either. Finally, we show the same for subreducts of representable cylindric algebras of finite dimension at least 3, in which intersection, cylindrifications and substitutions are term definable.

We note that the non-finite axiomatizability of the {intersection, composition, converse}-subreduct of RRA follows from [Hai91] (although this is not stated in that paper). See also [Bre93] about the axiomatizability of the equational theories of reducts of RRA.

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2 Relation algebras

First, we recall the definition of (representable) relation algebras.

Definition 2.1 *A* relation algebra, *an* RA, *is an algebra*

$$\mathcal{A} = (A, 0, 1, \cdot, +, -, ;, \breve{}, 1')$$

such that $(A, 0, 1, \cdot, +, -)$ *is a Boolean algebra, and the following equations hold, for every* $x, y, z \in A$*:*

$(R1)$ $x ; (y ; z) = (x ; y) ; z$
$(R2)$ $(x + y) ; z = (x ; z) + (y ; z)$
$(R3)$ $x ; 1' = x$
$(R4)$ $x^{\breve{}\breve{}} = x$
$(R5)$ $(x + y)^{\breve{}} = x^{\breve{}} + y^{\breve{}}$
$(R6)$ $(x ; y)^{\breve{}} = y^{\breve{}} ; x^{\breve{}}$
$(R7)$ $x^{\breve{}} ; (-(x ; y)) \leq -y$.

We denote the class of all relation algebras by RA.

By a relation set algebra, *an* Rs, *we mean an algebra* $\mathcal{A} = (A, 0, 1, \cdot, +, -, ;, \breve{}, 1')$ *such that* $A \subseteq \mathcal{P}(W)$ *(the powerset of* W*) for some set* $W = U \times U$*,* $0 = \emptyset$*,* $1 = W$*,* $\cdot$ *is intersection,* $+$ *is union,* $-$ *is complement w.r.t.* W*,* $;$ *is relation composition,* $\breve{}$ *is relation converse, and* $1'$ *is the identity relation on* W*. More formally, for all elements* $x, y \in A$*,*

$$\begin{aligned} x ; y &= \{(u, v) \in W : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w\} \\ x^{\breve{}} &= \{(u, v) \in W : (v, u) \in x\} \\ 1' &= \{(u, v) \in W : u = v\}. \end{aligned}$$

We denote the class of relation set algebras by Rs.[1] *Given an* $\mathcal{A} \in$ Rs*,* W *and* U *as above, we call* W *the* unit *of* $\mathcal{A}$ *and* U *the* base *of* $\mathcal{A}$*.*

The class RRA *of* representable relation algebras *is defined as*

$$\mathsf{RRA} = \mathbf{SP}\mathsf{Rs}$$

— i.e., we close the class Rs *under products and isomorphic copies of subalgebras.*

Definition 2.2 *Let* $\mathcal{A} = (A, o)_{o \in \tau}$ *be an algebra of the similarity type* τ*. Let* τ' *be a set of operations whose elements are term definable by* τ *(in* $\mathcal{A}$*). By the* τ'-reduct *of* $\mathcal{A}$*, we mean the algebra* $Rd_{\tau'}\mathcal{A} = (A, o)_{o \in \tau'}$*. We call* $Rd_{\tau'}\mathcal{A}$ a generalized *reduct of* $\mathcal{A}$*, since* τ' *may not be a subset of* τ*.*

If K *is a class of algebras of the same similarity type,* $\mathbf{Rd}_{\tau'}\mathsf{K}$ *denotes the class of* τ'*-reducts of elements of* K*. The* τ'-subreduct *of* K *is defined as* $\mathbf{SRd}_{\tau'}\mathsf{K}$*: i.e., we close* $\mathbf{Rd}_{\tau'}\mathsf{K}$ *under (isomorphic copies of) subalgebras. Again, we call* $\mathbf{SRd}_{\tau'}\mathsf{K}$ *a generalized subreduct of* K*.*

It is well known that RRA is a variety, and hence a quasivariety. It follows that any generalized subreduct of RRA is a quasivariety:

$$\mathbf{SRd}_{\tau'}\mathsf{RRA} = \mathbf{SRd}_{\tau'}\mathbf{PUp}\mathsf{RRA} = \mathbf{SPUpRd}_{\tau'}\mathsf{RRA}.$$

Our first main result concerns the finite axiomatizability of such quasivarieties.

[1] We will also consider set algebras of relations in smaller signatures than this, but by default the signature will be as above.

Theorem 2.3 *Let* K *be a generalized subreduct of* RRA *such that intersection, relation composition, and converse are term definable in* K*. Then*

1. K *is not axiomatizable by any finite set of first-order sentences and*
2. *the equational theory of* K *is not finitely based.*

Proof of Theorem 2.3: In [HH97b], a sequence of finite, integral relation algebras is defined such that none of the algebras is representable (i.e., it is not isomorphic to an RRA) while any non-trivial ultraproduct of the algebras is representable. (This is implicit in [HH97b], but analogous results in connection with Maddux's notion of n-dimensional relational basis are explicit.) We will use a similar construction here, to show that the $\{\cdot, ;, \breve{}\}$-reducts are not representable either (Lemma 3.1), while the ultraproduct is representable (Lemma 5.1). By Łoś' theorem [Hodg93, Theorem 9.5.1], this is enough to show that K is not finitely axiomatizable in first-order logic. Further, we will show that there are valid equations that fail in the above non-representable algebras (Lemma 3.1), establishing that the equational theory is not finitely axiomatizable either. ∎

The next three sections are devoted to make the above proof complete by proving Lemmas 3.1 and 5.1.

3 The rainbow construction

In this section we define relation algebras $\mathcal{A}_n$ $(n \in \omega)$, and show that their $\{\cdot, ;, \breve{}\}$-reduct is not representable.

Let n be any natural number. We define $\mathcal{A}_n$ as the full relation algebra with the following atoms:

- identity: $1'$,
- greens: $\mathsf{g}_i \quad (0 \leq i \leq 2^n)$,
- whites: $\mathsf{w}, \mathsf{w}_{ij} \quad (0 \leq i \leq j \leq 2^n)$,
- yellow: y,
- black: b,
- reds: $\mathsf{r}_i \quad (0 < i < 2^n)$.

All the atoms are self-converse. A triple (x, y, z) of atoms is said to be an *inconsistent triangle* if $x \cdot (y ; z) = y \cdot (z ; x) = z \cdot (x ; y) = 0$. Then, using additivity, composition is determined by the following set of inconsistent triangles:

(green,green,green)
(yellow,yellow,yellow)
(green,green,white)
(yellow,yellow,black)
$(\mathsf{r}_i,\mathsf{r}_j,\mathsf{r}_k)$ unless $i + j = k$ or $i + k = j$ or $j + k = i$
$(\mathsf{g}_i,\mathsf{g}_{i+1},\mathsf{r}_j)$ unless $j = 1$
$(\mathsf{g}_i,\mathsf{y},\mathsf{w}_{jk})$ unless $i \in \{j, k\}$,

where, e.g., (green,green,white) stands for: $g\,;g'\cdot w = g\,;w\cdot g' = w\,;g\cdot g' = 0$ for all green atoms g, g' and white atoms w. We also require that $(x, y, 1')$ is inconsistent for all distinct atoms x, y.

It is not difficult to check that $\mathcal{A}_n$ is a relation algebra. In fact, all the axioms but $(R1)$ are straightforward to check. An easy way to prove that $(R1)$ is satisfied as well is to show that the existential player can survive two rounds in the game played using atomic networks on $\mathcal{A}_n$ (see Definition 4.1), and Claim 5.2 below shows that she can do this.

Let $\mathcal{B}_n$ denote the $\{\cdot, ;, ^\smile\}$-reduct of $\mathcal{A}_n$.

Lemma 3.1 *For any $n \in \omega$, $\mathcal{A}_n$ is not in* RRA. *In fact, the $\{\cdot, ;, ^\smile\}$-reduct $\mathcal{B}_n$ is not representable either. Moreover, for every $n \in \omega$, there is an equation valid in set algebras that fails in $\mathcal{B}_n$.*

Proof: Towards a contradiction, let us assume that there is an isomorphism h from $\mathcal{B}_n$ to a set algebra of relations of similarity type $\{\cdot, ;, ^\smile\}$.

Since $\mathsf{w} \not\leq 0$, there is $(u, v) \in h(\mathsf{w})$ such that $(u, v) \notin h(0)$. Because $\mathsf{w} \leq \mathsf{g}_i\,;\mathsf{y}$, we see that for every $0 \leq i \leq 2^n$ there exists u_i such that $(u, u_i) \in h(\mathsf{g}_i)$ and $(u_i, v) \in h(\mathsf{y})$. Since $\mathsf{g}_i{}^\smile = \mathsf{g}_i$ in $\mathcal{B}_n$, $(u_i, u) \in h(\mathsf{g}_i)$; and similarly, $(v, u_i) \in h(\mathsf{y})$.

Now $(u_i, u_{i+1}) \in h(\mathsf{g}_i\,;\mathsf{g}_{i+1}\cdot\mathsf{y}\,;\mathsf{y}) = h(\mathsf{r}_1)$ for every $0 \leq i < 2^n$. By $\mathsf{g}_i\,;\mathsf{g}_{i+2}\cdot\mathsf{y}\,;\mathsf{y}\cdot\mathsf{r}_1\,;\mathsf{r}_1 = \mathsf{r}_2$, for every $i < 2^n - 1$, $(u_i, u_{i+2}) \in h(\mathsf{r}_2)$. In particular, $(u_0, u_2) \in h(\mathsf{r}_2)$ and $(u_{2^n-2}, u_{2^n}) \in h(\mathsf{r}_2)$. By induction, we get that $(u_0, u_{2^{n-1}}) \in h(\mathsf{r}_{2^{n-1}})$ and $(u_{2^{n-1}}, u_{2^n}) \in h(\mathsf{r}_{2^{n-1}})$. Then we have $(u_0, u_{2^n}) \in h(\mathsf{g}_0\,;\mathsf{g}_{2^n}\cdot\mathsf{y}\,;\mathsf{y}\cdot\mathsf{r}_{2^{n-1}}\,;\mathsf{r}_{2^{n-1}}) = h(0)$. Since $(u, u_0) \in h(\mathsf{g}_0)$ and $(u_{2^n}, v) \in h(\mathsf{y})$, we get that $(u, v) \in h(\mathsf{g}_0\,;0\,;\mathsf{y}) = h(0)$. But we assumed that $(u, v) \notin h(0)$. We have our contradiction.

The non-representability of $\mathcal{B}_n$ is witnessed by the following equation. Let $\rho_{i,j}$ stand for $\mathsf{g}_i\,;\mathsf{g}_j\cdot\mathsf{y}\,;\mathsf{y}$. We define $\rho(k, k+2^l)$ by induction:

$$\rho(k, k+1) = \rho_{k,k+1} \text{ and } \rho(k, k+2^{l+1}) = \rho(k, k+2^l)\,;\rho(k+2^l, k+2^{l+1})\cdot\rho_{k,k+2^{l+1}}.$$

Let σ_n be $\mathsf{w}\cdot\prod\{(\mathsf{g}_i\cdot\mathsf{g}_i{}^\smile)\,;(\mathsf{y}\cdot\mathsf{y}^\smile) : 0 \leq i \leq 2^n\}$ and τ_n equal $\mathsf{w}\cdot\mathsf{g}_0\,;\rho(0, 2^n)\,;\mathsf{y}$. The equation e_n is defined as $\sigma_n = \sigma_n\cdot\tau_n$. It is easy to check that e_n is valid in set algebras. On the other hand, the argument we used above to prove that $\mathcal{B}_n$ is not representable shows that e_n fails in $\mathcal{B}_n$. ∎

It remains to show that any non-trivial ultraproduct of the $\mathcal{A}_n$ $(n \in \omega)$ is representable.

4 The game

We recall from [HH97] the definition of a game connected to representability.

Definition 4.1 *Let $\mathcal{A}$ be a relation-type algebra.*

1. *A* pre-network *is a complete directed finite graph with edges labeled by elements of $\mathcal{A}$: i.e., $N = (E_N, \ell_N)$, where $E_N = U_N \times U_N$ for some finite non-empty set U_N, the base of N, and $\ell_N : E_N \longrightarrow A$ is a map assigning an element of $\mathcal{A}$ to each edge.*

   A pre-network is a network *if it also satisfies, for every $x, y, z \in U_N$,*

   (a) $\ell_N(x, y) \leq 1'$ *iff* $x = y$,

   (b) $\ell_N(x, y)\,;\ell_N(y, z)\cdot\ell_N(x, z) \neq 0$.

A pre-network is called atomic *if all the edges are labeled by atoms of $\mathcal{A}$. If no confusion is likely, we will omit the subscript N.*

Given two pre-networks N, N', we write $N \subseteq N'$ if every edge of N is an edge of N' and, for every edge (x,y) of N, $\ell_{N'}(x,y) \leq \ell_N(x,y)$.

2. *Let $n \in \omega$. We define a game $G_n(\mathcal{A})$ between two players, $\forall$ (male), and $\exists$ (female). They build a finite chain $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_n$ of pre-networks in the following way. N_0 is any consistent triangle, i.e., a network such that $|U_{N_0}| \leq 3$. We regard N_0 as being chosen by $\forall$ before the game starts. In each round i $(0 \leq i < n)$,*

   - *$\forall$ chooses an edge (x,y) from N_i and elements $r, s \in A$,*
   - *$\exists$ responds with a pre-network $N_{i+1} \supseteq N_i$ such that one of the following holds:*
     – $\exists$ rejects: N_{i+1} is the same as N_i except that $\ell_{N_{i+1}}(x,y) = \ell_{N_i}(x,y) \cdot -(r\,;s)$,
     – $\exists$ accepts: the nodes of N_{i+1} are those of N_i, plus a possibly new one, z, and the labels on edges of N_{i+1} satisfy the following:
       - *$\ell_{N_{i+1}}(x,z) = r$,*
       - *$\ell_{N_{i+1}}(z,y) = s$,*
       - *$\ell_{N_{i+1}}(x,y) = \ell_{N_i}(x,y) \cdot r\,;s$.*

   $\exists$ wins a match of the game $G_n(\mathcal{A})$ if every N_i $(0 \leq i \leq n)$ is a network. We say that $\exists$ has a winning strategy *if she can win all matches.*

   The atomic *game $G_n^a(\mathcal{A})$ is defined by requiring that all the elements r, s chosen by $\forall$ are atoms, and that each N_i is an atomic pre-network.*

The following proposition from [HH97d] provides us with a sufficient condition for representability of atomic relation algebras.

Proposition 4.2 *Let $\mathcal{A}$ be an atomic relation algebra. Then $\exists$ has a winning strategy in $G_n^a(\mathcal{A})$ for all $n \in \omega$ iff $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra.*[2] *Hence, because* RRA *is elementary, if $\exists$ has a winning strategy in $G_n^a(\mathcal{A})$ for all $n \in \omega$, then $\mathcal{A}$ is representable.*

5 The ultraproduct

Note that $\mathcal{A}_n$ is an atomic RA for every $n \in \omega$, whence any ultraproduct $\mathcal{A}$ of them is an atomic RA as well.

Lemma 5.1 *Any non-trivial ultraproduct $\mathcal{A}$ of the $\mathcal{A}_n$'s (over ω) is in* RRA.

[2] A complete representation of a relation algebra $\mathcal{B}$ is an isomorphism from $\mathcal{B}$ to a representable relation algebra that preserves arbitrary meets and joins whenever they exist in $\mathcal{B}$. However, we will not need this notion in this paper.

Proof: First we show that $\exists$ can survive arbitrarily long games on a "large" set (occurring in the ultrafilter) of algebras. The "ultraproduct" of these strategies will enable her to win arbitrarily long (in fact, ω-long) games on the ultraproduct. Thus, by Proposition 4.2, the ultraproduct will be representable.

Claim 5.2 *Let $l \in \omega$. $\exists$ has a winning strategy for $G^a_l(\mathcal{A}_n)$ for cofinitely many algebras $\mathcal{A}_n$ $(n \in \omega)$.*

Proof: Let n be large enough — say, $n \geq l$.

Let us assume that we are in the pth $(0 \leq p < l)$ round and that an atomic network N_p is already constructed.

We define the important notion of a *red block.* Suppose that u, v are distinct nodes of N_p and that $\ell_{N_p}(u,v) \neq \mathsf{w}_{ij}$ for any i, j. Let

$$W = \{w : w \text{ a node of } N_p,\ \ell_{N_p}(u,w) \text{ is green, and } \ell_{N_p}(v,w) = \mathsf{y}\}.$$

Assume that $|W| \geq 2$. Also assume that W can be linearly ordered $(w_1 < \ldots < w_q)$ in the following way: the map f from $\{1, \ldots, q\}$ into the set $2^n + 1$ of indices of green atoms given by $\ell_{N_p}(u, w_i) = \mathsf{g}_{f(i)}$ for every $1 \leq i \leq q$, satisfies $f(i) < f(j)$ for every $1 \leq i < j \leq q$. Note that the color of every (w_i, w_j) must be red.

In such a situation, we will call the subnetwork N' of N_p with base $\{u, v\} \cup W$ a *red block* with *center* (u, v). We will say that w_i and w_{i+1} are *neighbors,* and that the *distance* of w_i from w_j is $|f(j) - f(i)|$.

We now state the following induction hypothesis.

Induction hypothesis: For every red block $N' \subseteq N_p$ with center (u, v) and base $\{u, v, w_1, \ldots, w_q\}$ in the above notation, and for every $1 \leq i < j \leq k \leq q$,

1. $\ell_{N_p}(w_i, w_j) = \mathsf{r}_{f(j)-f(i)}$ if $f(j) - f(i) \leq 2^{l-p}$,
2. $\ell_{N_p}(w_i, w_j) = \mathsf{r}_t$ for some $t \leq 2^{l-1} + \ldots + 2^{l-(j-i)}$,
3. $\ell_{N_p}(w_i, w_j) = \mathsf{r}_t$ and $\ell_{N_p}(w_j, w_k) = \mathsf{r}_s$ imply $\ell_{N_p}(w_i, w_k) = \mathsf{r}_{t+s}$.

Note that $q \leq p + 1$, since in each round at most one new point is created, and that the largest index on a red atom (to label (w_1, w_q)) is at most $2^{l-1} + \ldots + 2^{l-p}$.

Let us assume that in the pth round player $\forall$ plays (u, v, y, z) for some edge (u, v) of N_p and atoms y, z of $\mathcal{A}_n$ and that $\ell_{N_p}(u, v) = x$.

If $x \cdot y\,; z = 0$, then $\exists$ rejects $\forall$'s proposal — i.e., she defines $N_{p+1} = N_p$. If $x \leq y\,; z$ and there is a point w in N_p such that $\ell_{N_p}(u, w) = y$ and $\ell_{N_p}(w, v) = z$, then again $\exists$ lets $N_{p+1} = N_p$. Note that this covers the case when either y or z is the identity $1'$.

Otherwise $\exists$ extends N_p by a new point w and lets $\ell_{N_{p+1}}(u, w) = \ell_{N_{p+1}}(w, u) = y$, $\ell_{N_{p+1}}(w, v) = \ell_{N_{p+1}}(v, w) = z$ and $\ell_{N_{p+1}}(w, w) = 1'$ — note that this is well defined, since $u = v$ implies that $x = 1'$, whence $y = z$. She defines the labels for the remaining edges (w, w') as follows (she will label an edge with the same atom as the atom labelling its converse edge; we will not bother to mention this from now on).

$\exists$'s strategy is to choose a white w_{ij} whenever it is possible: i.e., if labeling (w, w') by w_{ij} ensures that the triangles (w, w', u) and (w, w', v) are consistent. If this fails, she tries to use black, b. If this is impossible, too, then she uses a red color r_i. She chooses the index i carefully so that $\forall$ will not be able to create a contradiction during the l rounds of the game. We have the following cases.

Case 1: $\{y, z\} \neq \{\mathsf{g}_i, \mathsf{y}\}$ for any i. An easy argument using case distinction shows that, for any w', $\exists$ can choose $\ell_{N_{p+1}}(w', w)$ to be either w_{jj}; or, if this creates an inconsistent triangle among (w, w', u) and (w, w', v), w_{jk} for some distinct j, k; or if this creates an inconsistent triangle too, b. It is easy to check that this yields an atomic network, since no triangle that involves either two white edges, or two black edges, or a white and a black edge, can be inconsistent. Further, N_{p+1} satisfies the induction hypothesis, since no new red block has been created.

Case 2: $y = \mathsf{g}_i$ and $z = \mathsf{y}$. If (u, w') is not green, she can let $\ell_{N_{p+1}}(w', w) = \mathsf{w}_{ii}$ if (w', v) is not green, or $\ell_{N_{p+1}}(w', w) = \mathsf{w}_{ij}$ if (w', v) has color g_j. Otherwise, if (w', v) is not yellow, she plays $\ell_{N_{p+1}}(w', w) = \mathsf{b}$.

The hard case is when $\ell_{N_p}(u, w') = \mathsf{g}_j$ and $\ell_{N_p}(w', v) = \mathsf{y}$. We can assume that $\mathsf{g}_i \neq \mathsf{g}_j$, otherwise $\exists$ did not extend N_p. First let us assume that w' is the only point besides w such that (u, w') is green and (w', v) is yellow. Then $\exists$ lets $\ell_{N_{p+1}}(w', w) = \mathsf{r}_k$, where

$$k = \begin{cases} |i - j| & \text{if } |i - j| \leq 2^{l-p-1} \\ 2^{l-p-1} & \text{if } |i - j| > 2^{l-p-1}. \end{cases}$$

That is, $k = \min\{|i - j|, 2^{l-p-1}\}$. Since $i \neq j$, such a red color r_k clearly exists.

Now let

$$W = \{w' : w' \text{ a node of } N_p,\ (u, w') \text{ is green, and } (w', v) \text{ is yellow}\}.$$

We claim that if $|W| > 1$ then $(u, v, w_1, \ldots, w_q)$ forms a red block with center (u, v).

To prove the claim, assume that $|W| > 1$. Let W be enumerated in the order $w_1 < w_2 < \ldots < w_q$ so that there is f such that $\ell_{N_p}(u, w_j) = \mathsf{g}_{f(j)}$ and $f(j) \leq f(k)$ whenever $j < k$ (cf. the definition of red block). Note that $q \leq p$, and that (w_j, w_k) must be red for every distinct $j, k \leq q$.

Let w_r be that element of W that was added to N_p before any other element of W was created. First, we show that any other w_j was added to the network after u and v, and that the red color for each (w_j, w_k) was chosen by $\exists$.

To see this, consider the rectangle (u, w_j, w_k, v), where $1 \leq j < k \leq q$ and $j, k \neq r$, and assume to the contrary that, say, u was added after w_j, w_k, and v. (The case where v was added after w_j, w_k, u is similar.) Since $\exists$ never plays green, $\ell_{N_p}(u, w_j)$ and $\ell_{N_p}(u, w_k)$ were chosen by $\forall$. Thus, he played $(w_j, w_k, \mathsf{g}_{f(j)}, \mathsf{g}_{f(k)})$ (or possibly its mirror image $(w_k, w_j, \mathsf{g}_{f(k)}, \mathsf{g}_{f(j)})$) in the round when u was created. Since $\ell_{N_p}(v, w_j)$ and $\ell_{N_p}(v, w_k)$ are yellow, $\mathsf{w}_{f(j)f(k)}$ was a possible choice for $\exists$ as a color for (u, v). Then, according to her strategy, she chose $\mathsf{w}_{f(j)f(k)}$. We assumed that the green g_i played by $\forall$ in the pth round is distinct from the other greens $\mathsf{g}_{f(j)}$ on (u, w_j) (otherwise $\exists$ did not have to extend N_p). But $(u, v, \mathsf{g}_i, \mathsf{y})$ would have been rejected by $\exists$ ($(\mathsf{g}_i, \mathsf{y}, \mathsf{w}_{f(j)f(k)})$ is inconsistent), which is a contradiction.

So far, we have seen that all but at most one w_j were created after u and v, and that the color of (u, v) cannot be any indexed white. Since $\exists$ never plays green or yellow, the reason for adding w_j to the network was that $\forall$ played $(u, v, \mathsf{g}_{f(j)}, \mathsf{y})$. Then it was $\exists$'s right (and duty) to

choose the color for any other edge involving w_j. In particular, she chose the red for (w_j, w_k) (if w_k was already constructed). The above argument also shows that we can assume that the green colors on (u, w_j) $(0 \leq j \leq q)$ are all different — otherwise, $\exists$ did not create w_j when $\forall$ played $(u, v, \mathsf{g}_{f(j)}, \mathsf{y})$. Thus, $(u, v, w_1, \ldots, w_q)$ is indeed a red block, as claimed.

So the network $\{u, v, w_1, \ldots, w_q\}$ must satisfy the induction hypothesis. We claim next that $\exists$ can find appropriate red colors for each (w, w_j) $(1 \leq j \leq q)$ such that conditions 1–3 of the induction hypothesis hold (when we replace p by $p+1$).

Indeed, let $w_j \in W$ be such that $|i - f(j)|$ is minimal. If $|i - f(j)| \leq 2^{l-p-1}$, then she lets $\ell_{N_{p+1}}(w_j, w) = \mathsf{r}_{|i-f(j)|}$. If $|i - f(j)| > 2^{l-p-1}$, then she lets $\ell_{N_{p+1}}(w_j, w) = \mathsf{r}_{2^{l-p-1}}$. $\exists$ labels the other edges (w_k, w) by using a red atom indexed by the sum (if $w_k < w_j$ and $f(j) < i$, or $w_j < w_k$ and $i < f(j)$) or the difference (if $w_j < w_k$ and $f(j) < i$, or $w_k < w_j$ and $i < f(j)$) of the indices of the reds on (w_k, w_j) and (w_j, w). It can be checked that these red colors exist, and conditions 1–3 above hold for the red block $(u, v, w_1, \ldots, w, \ldots, w_q)$. This ends our proof of the claim.

It remains to show that the induction hypothesis holds for *any* red block N' of N_{p+1}. First, note that if a red block satisfied the induction hypothesis for p (in the previous round) then it satisfies the induction hypothesis for $p+1$ as well. We make the following observation about "new" red blocks that are not red blocks of N_p (cf. above): $\exists$ plays a red color on an edge (w, w') only if there is an edge (u, v) such that (u, w') is green, (w', v) is yellow, and $\forall$ plays $(u, v, \mathsf{g}_i, \mathsf{y})$ so that $\exists$ is forced to extend the network with w and label (u, w) with g_i and (w, v) with y. This implies that if we have a new red block, then its center must be the edge (u, v) played by $\forall$. Thus, the only possible new red block has one of the following forms:

- $(u, v, w_1, \ldots, w, \ldots, w_q)$, if $(u, v, w_1, \ldots, w_q)$ was a red block in N_p, and if $\forall$ played $(u, v, \mathsf{g}_i, \mathsf{y})$. By the previous paragraph, this red block satisfies the induction hypothesis.
- (u, v, w, w') or (u, v, w', w), if $\ell_{N_p}(u, v) \neq \mathsf{w}_{ij}$ for any i, j, if $\{x \in N_p : \ell_{N_p}(u, v)$ is green and $\ell_{N_p}(u, v) = \mathsf{y}\} = \{w'\}$, and if $\forall$ played $(u, v, \mathsf{g}_i, \mathsf{y})$. By the argument at the start of case 2, red blocks of this form also satisfy the induction hypothesis.

Case 3: $y = \mathsf{y}$ and $z = \mathsf{g}_i$. This case is completely analogous to case 2, and we omit the details.

The largest index on red colors used by $\exists$ so far is at most $2^{l-1} + 2^{l-2} + \ldots + 2^{l-p-1} < 2^l$, since, in the kth round, she labeled an edge (w, w') of neighboring points w, w' with r_j such that $j \leq 2^{l-k-1}$. Thus, in the remaining rounds of the game, $\forall$ cannot force her to use a non-existing red r_i $(i \geq 2^n)$. In any red block, if the distance $|f(j) - f(k)|$ between two points w_j and w_k is "small", i.e., smaller than 2^{l-p-1}, then she used $\mathsf{r}_{|f(j)-f(k)|}$ to label (w_j, w_k). Thus, in the remaining rounds, she has enough indices between 1 and $|f(j) - f(k)|$ to label any edge (w_j, w) and (w, w_k) "inserted" into (w_j, w_k). This shows that she can survive l rounds without arriving at the impossible task to use a non-existing red color. Claim 5.2 is proved. ∎

We now finish the proof of the lemma. Since $\exists$ can survive arbitrarily long games on a large set (i.e., included in the non-principal ultrafilter) of algebras, she can achieve this in the ultraproduct as well. Indeed, the winning strategies in $G_l^a(\mathcal{A}_n)$ provide her with a winning strategy in $G_l^a(\mathcal{A})$, as follows.

Assume that a finite network N is already defined and $\forall$ plays an edge (x, y) with color a (for some atom $a \in A$) and atoms b, c of $\mathcal{A}$. Note that every atom d of the ultraproduct $\mathcal{A}$ is an

equivalence class of an ω-sequence $(d_i : i \in \omega)$, with each d_i an atom of $\mathcal{A}_i$. For every $i \in \omega$, one can define a pre-network N_i in the following way. The base of N_i is that of N, and the color of every edge of N_i is an atom d_i of $\mathcal{A}_i$ such that the color d of this edge in N is the equivalence class of $(d_i : i \in \omega)$. It is easy to check that

$$\{i \in \omega : N_i \text{ is a network}\}$$

is contained in the ultrafilter.

Now $\exists$ considers those particular matches in the games $G_l^a(\mathcal{A}_i)$ $(i \in \omega)$ where $\forall$ plays $(x, y) \in N_i$ and $b_i, c_i \in A_i$ such that b, c are the equivalence classes of $(b_i : i \in \omega), (c_i : i \in \omega)$. If the set

$$S_0 = \{i \in \omega : \exists \text{ rejects } \forall\text{'s proposal}\}$$

is in the ultrafilter, then she rejects in the game $G_l^a(\mathcal{A})$ as well. If the complement $\omega \setminus S_0$ of this set is in the ultrafilter, then she considers two of its subsets: S_1 is the set of those indices where she is not forced to extend the network, and S_2 is the set of those indices where she is forced to extend the network. If S_1 is in the ultrafilter, then she does not have to extend the network N, as in N, there are (x, z) and (z, y) such that $b = \ell_N(x, z)$ and $c = \ell_N(z, y)$. If S_2 is contained in the ultrafilter, then she can extend the network by using the atoms of $\mathcal{A}$ determined by the equivalence classes of the elements she uses in the games $G_l^a(\mathcal{A}_n)$. This completes her move in response to $\forall$ in this round. Her move in the next round (and in subsequent rounds) is decided in much the same way, but note that she will be *continuing* with her winning strategy already in progress in the games $G_l^a(\mathcal{A}_n)$ for a large set (in the ultrafilter) of indices n: either S_0, S_1, or S_2. The algebras with indices not in this set can be discarded. See [HH97d, Lemma 16] for more details. ∎

6 Sequential algebras

In this section we show a corollary of the main theorem for subreducts of representable sequential algebras.

Definition 6.1 *An algebra* $\mathcal{A} = (A, 0, 1, \cdot, +, -, ;, \lhd, \rhd, 1')$ *is* a representable sequential algebra, *if*

- $A \subseteq \mathcal{P}(W)$ *for some transitive and reflexive relation* W,
- $(A, 0, 1, \cdot, +, -)$ *is a Boolean set algebra,*
- $;$ *is relation composition,*
- $1'$ *is the identity relation on* W,
- *and for all* x, y,

$$\begin{aligned} x \rhd y &= \{(u, v) \in W : (w, u) \in x, (w, v) \in y \text{ for some } w\} \\ x \lhd y &= \{(u, v) \in W : (v, w) \in y, (u, w) \in x \text{ for some } w\}. \end{aligned}$$

The class of representable sequential algebras is a variety, [JM97], but it is not finitely axiomatizable (a result of Andréka and van Karger, [vK94]). We show that non-finite axiomatizability holds already for a fragment of the language.

Corollary 6.2 *The* $\{\cdot, ;, 1', \rhd\}$*-subreduct of the class of representable sequential algebras is not finitely axiomatizable.*

Proof: We show that the $\{\cdot, ;, 1', \rhd\}$-reducts of the relation algebras $\mathcal{A}_n$ $(n \in \omega)$ are not representable. Indeed, define $x \rhd y$ as $x^\smile ; y$ and assume that there is an isomorphism from $(A_n, \cdot, ;, \rhd, 1')$ onto a set algebra of binary relations. Then $(A_n, \cdot, ;, ^\smile)$ must be representable as well, since we can define $x^\smile$ as $x \rhd 1'$ — a contradiction.

On the other hand, the sequential-reduct of the ultraproduct is representable, since $x \rhd y$ and $x \lhd y$ are definable as $x^\smile ; y$ and $x ; y^\smile$, respectively. ∎

7 Further plans

A challenging problem is whether identity is enough instead of converse to obtain non-finite axiomatizability in Theorem 2.3.

8 Appendix: cylindric algebras

In this section we will apply a similar idea to prove non-finite axiomatizability results for algebras of relations of higher arity.

Now we recall the definition of (representable) cylindric algebras.

Definition 8.1 *Let α be an ordinal. A* cylindric algebra of dimension α, a CA_α, *is an algebra*

$$\mathcal{A} = (A, 0, 1, \cdot, +, -, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha}$$

such that $(A, 0, 1, \cdot, +, -)$ is a Boolean algebra, and the following equations hold, for every $x, y \in A$ and $i, j, k < \alpha$:

$(C1)$ $\mathsf{c}_i(x + y) = \mathsf{c}_i x + \mathsf{c}_i y$
$(C2)$ $x \leq \mathsf{c}_i x$
$(C3)$ $\mathsf{c}_i - \mathsf{c}_i x = -\mathsf{c}_i x$
$(C4)$ $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$
$(C5)$ $\mathsf{d}_{ii} = 1$ *and* $\mathsf{d}_{ij} = \mathsf{d}_{ji}$
$(C6)$ $\mathsf{d}_{ik} = \mathsf{c}_j(\mathsf{d}_{ij} \cdot \mathsf{d}_{jk})$ *if* $j \notin \{i, k\}$
$(C7)$ $\mathsf{d}_{ij} \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \leq x$.

We denote the class of all cylindric algebras of dimension α by CA_α.

By a cylindric set algebra of dimension α, a Cs_α, *we mean an algebra $\mathcal{A} = (A, 0, 1, \cdot, +, -, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha}$ such that $A \subseteq \mathcal{P}({}^\alpha U)$ for some set U, $0 = \emptyset$, $1 = {}^\alpha U$, $\cdot$ is intersection, $+$ is union, $-$ is complement w.r.t. ${}^\alpha U$, c_i is the ith cylindrification, and d_{ij} is the diagonal element identifying the ith and jth coordinates. That is, the unit ${}^\alpha U$ of a Cs_α is the set of α-long sequences of elements of U, and the extra-Boolean operations have the following interpretations. Let $s \equiv_i t$ iff $(\forall j \neq i) s(j) = t(j)$. Then for each element $x \in A$ and $i, j < \alpha$,*

$$\begin{aligned} \mathsf{c}_i x &= \{s \in {}^\alpha U : s \equiv_i t \text{ for some } t \in x\} \\ \mathsf{d}_{ij} &= \{s \in {}^\alpha U : s(i) = s(j)\}. \end{aligned}$$

The class RCA_α *of* representable cylindric algebras of dimension α *is defined as*

$$\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha,$$

i.e., we close the class Cs_α under products and isomorphic copies of subalgebras.

We define the operation *substitution* s^i_j as follows: for every distinct $i, j < \alpha$, $\mathsf{s}^i_j x = \mathsf{c}_i(x \cdot \mathsf{d}_{ij})$, while $\mathsf{s}^i_i x = x$. Note that in a cylindric set algebra with base U,

$$\mathsf{s}^i_j x = \{s \in {}^\alpha U : s \equiv_i t \text{ for some } t \in x \text{ such that } t(i) = t(j)\}$$

for distinct i, j. The operation of composition ; is defined as

$$x ; y = \mathsf{c}_2(\mathsf{s}^1_2 \mathsf{c}_2 x \cdot \mathsf{s}^0_2 \mathsf{c}_2 y)$$

and, using one extra coordinate, we can define permutation p_{ij} as well: for distinct i, j,

$$\mathsf{p}_{ij} x = \mathsf{s}^k_j \mathsf{s}^j_i \mathsf{s}^i_k \mathsf{c}_k x$$

where $k \notin \{i, j\}$ and

$$\mathsf{p}_{ii} x = x.$$

Theorem 8.2 *Let $2 < \alpha < \omega$ and K be a generalized subreduct of RCA_α such that intersection, cylindrifications and substitutions are term definable in K. Then K is not finitely axiomatizable by first-order sentences.*

We will use the same strategy and a similar construction as in the proof of Theorem 2.3.

Proof of Theorem 8.2: Again we will use an idea of [HH97d].

First we define a class of colored graphs (closely related to the atom structures of the relation algebras of Section 3). Using these graphs we will define finite cylindric algebras: roughly speaking, an atom will be a surjective map from α to a graph. We will show that the {intersection, cylindrifications, substitutions}-reducts of these algebras are not representable. On the other hand, using games will show that any non-trivial ultraproduct of the algebras is a representable cylindric algebra.

First, for every natural number n, we define a class of colored graphs. The colors will have a similar role as in the case of relation algebras. White had two roles, and this is reflected here by introducing a new shade of white: ivory.[3]

Let n be a fixed natural number. An *n-colored graph* is an undirected graph Γ such that every irreflexive edge ($(u, v) \in \Gamma$ such that $u \neq v$) of Γ is colored by a unique edge color and some $(\alpha - 1)$-tuples have a unique color, too. The edge colors are:

- greens: $\mathsf{g}_i \quad (0 \leq i \leq 2^n)$,
- yellows: $\mathsf{y}_i \quad (1 \leq i \leq \alpha - 2)$,
- blacks: $\mathsf{b}_i \quad (1 \leq i \leq \alpha - 2)$,
- reds: $\mathsf{r}_i \quad (1 \leq i < 2^n)$,
- ivory: i.

The colors for $(\alpha - 1)$-tuples are:

[3]The proof of the theorem for dimension 3 is simpler: we need only the colors from the relation algebra case and those n-colored graphs (cf. below) that do not contain any inconsistent triangle.

- whites: $\mathsf{w}_S \quad (S \subseteq 2^n + 1)$.

We will write $\Gamma(x, y)$ and $\Gamma(a_1, \ldots, a_{\alpha-1})$ for the colors of the edge (x, y) and of the $(\alpha-1)$-tuple $(a_1, \ldots, a_{\alpha-1})$, respectively.

We define *colored graph embedding* in the obvious way: an injective map that preserves all edges and colors, where defined, in both directions.

Let $0 \leq i \leq 2^n$ and let Γ be an n-colored graph consisting of α nodes, $x_0, x_1, \ldots, x_{\alpha-2}, y$. We call Γ an *i-cone* if $\Gamma(x_0, y) = \mathsf{g}_i$, and for every $1 \leq j \leq \alpha - 2$, $\Gamma(x_j, y) = \mathsf{y}_j$, and no other edge of Γ is colored green or yellow. The *apex* of the cone is y, its *base* is the ordered $(\alpha-1)$-tuple $(x_0, \ldots, x_{\alpha-2})$. The *tint* of the cone is i.

We will consider special n-colored graphs.

Definition 8.3 *The class $\mathcal{G}_n$ consists of all n-colored graphs Γ with the following properties.*

1. *Γ is a complete graph.*
2. *Γ contains no triangles of the following types:*
   - *(green,green,green)*
   - *(yellow,yellow,yellow)*
   - *(green,green,ivory)*
   - $(\mathsf{y}_i, \mathsf{y}_i, \mathsf{b}_i) \quad (1 \leq i \leq \alpha - 2)$
   - $(\mathsf{r}_i, \mathsf{r}_j, \mathsf{r}_k) \quad$ *unless $i + j = k$ or $i + k = j$ or $j + k = i$*
   - $(\mathsf{g}_i, \mathsf{g}_{i+1}, \mathsf{r}_j) \quad$ *unless $j = 1$.*
3. *If $a_0, \ldots, a_{\alpha-2} \in \Gamma$ are distinct and no edge (a_i, a_j) $(0 \leq i < j \leq \alpha - 2)$ is colored green or yellow, then the sequence $(a_0, \ldots, a_{\alpha-2})$ is colored a unique shade w_S of white. No other $(\alpha-2)$-tuples are colored white.*
4. *For every i-cone $(0 \leq i \leq 2^n)$ in $\mathcal{G}_n$ with base $(x_0, \ldots, x_{\alpha-2})$, $(x_0, \ldots, x_{\alpha-2})$ is colored by a unique shade of white, w_S, such that $i \in S$.*

Clearly, $\mathcal{G}_n$ is closed under isomorphism and under induced subgraphs.

We are ready to define the cylindric algebras $\mathcal{C}_n$ for every $n \in \omega$. Let $n \in \omega$ be fixed. Consider the class of surjective maps

$$a : \alpha \longrightarrow \Gamma_a$$

for every $\Gamma_a \in \mathcal{G}_n$. We define an equivalence relation $\sim$ on these surjective maps:

$$\begin{aligned} a \sim b \iff\ & a(i) = a(j) \text{ iff } b(i) = b(j), \\ & \text{and } \Gamma_a(a(i), a(j)) = \Gamma_b(b(i), b(j)), \text{ if defined,} \\ & \text{and } \Gamma_a(a(k_0), \ldots, a(k_{\alpha-2})) = \Gamma_b(b(k_0), \ldots, b(k_{\alpha-2})), \text{ if defined,} \end{aligned}$$

for all $i, j \in \alpha$ and $(\alpha-1)$-tuples k of elements of α. Write $[a]$ for the $\sim$-equivalence class of a.

An atom of $\mathcal{C}_n$ is any $[a]$ for some map $a : \alpha \longrightarrow \Gamma_a$, where $\Gamma_a \in \mathcal{G}_n$. We define, for every $i, j \in \alpha$,

$$\mathsf{d}_{ij} = \{[a] : a(i) = a(j)\}$$

and

$$\mathsf{c}_i[a] = \{[b] : \text{for some } c \in [a], b(j) = c(j) \text{ for all } j \neq i\}.$$

We let $\mathcal{C}_n$ be the α-dimensional cylindric-type algebra whose universe is the powerset of the set of atoms, each $[a]$ being identified with the singleton $\{[a]\}$, the diagonals d_{ij} being defined as above, and the cylindrifications on non-atomic elements being defined by additivity. It is not hard to check that $\mathcal{C}_n$ is in fact a cylindric algebra. Indeed, the structure

$$(At(\mathcal{C}_n), E_{ij}, T_i)_{ij,\in\alpha}$$

defined by

$$E_{ij} = \{[a] : a(i) = a(j)\}$$

and

$$[a]T_i[b] \Longleftrightarrow \text{for some } c \in [a], b(j) = c(j) \text{ for all } j \neq i$$

turns out to be a cylindric atom structure, cf. [HMT] 2.7.38, 2.7.40.

Lemma 8.4 *For any $n \in \omega$, $\mathcal{C}_n$ is not in RCA_α. Further, its $\{\cdot, \mathsf{c}_i, \mathsf{s}^i_j\}_{i,j<\alpha}$-reduct $\mathcal{B}_n$ is not representable either.*

Proof: We will use the same idea as in the relation algebra case. The reason for the non-representability of the $\mathcal{C}_n$ is that its relation-algebraic reduct contains the non-representable relation algebra $\mathcal{A}_n$. Further, this holds for the given reducts of $\mathcal{A}_n$ and $\mathcal{C}_n$ as well.

We can "copy" the non-representability proof from the relation algebra section as follows. Assume that there is an isomorphism h from $\mathcal{B}_n$ onto an algebra of α-ary relations. Consider the following element of $\mathcal{B}_n$: $[a_0]$ is (the equivalence class of) the map from α onto the n-colored graph $\Gamma = (v_0, u_0, u_1, \ldots, u_{\alpha-2})$ where $\Gamma(v_0, u_0) = \mathsf{g}_0$, for every $1 \leq i \leq \alpha - 2$, $\Gamma(v_0, u_i) = \mathsf{y}_i$ and $\Gamma(u_0, \ldots, u_{\alpha-2}) = \mathsf{w}_{2^n+1}$. Let $[a_1]$ be (the equivalence class of) the map from α onto the n-colored graph $\Gamma = (v_1, u_0, u_1, \ldots, u_{\alpha-2})$ where $\Gamma(v_1, u_0) = \mathsf{g}_1$, for every $1 \leq i \leq \alpha - 2$, $\Gamma(v_1, u_i) = \mathsf{y}_i$ and $\Gamma(u_0, \ldots, u_{\alpha-2}) = \mathsf{w}_{2^n+1}$. Clearly, $[a_0]T_0[a_1]$, whence once we have a sequence in $h[a_0]$ — for the sake of simplicity let us assume that this is $(v_0, u_0, \ldots, u_{\alpha-2})$ — then we must have $(v_1, u_0, \ldots, u_{\alpha-2}) \in h[a_1]$. Now let us consider the sequence $(v_0, v_1, u_1, \ldots, u_{\alpha-2})$. Since h preserves $\cdot$, c_i and s^i_j $(i, j \in \alpha)$, we have that

$$(v_0, v_1, u_1, \ldots, u_{\alpha-2}) \in h([a_0]\,;\mathsf{p}_{01}[a_1] \cdot \prod\{\mathsf{p}_{1i}[a_0]\,;\mathsf{s}^i_0\mathsf{p}_{01}[a_1] : 2 \leq i \leq \alpha - 2\}.$$

Hence the edge (v_0, v_1) must have color r_1. Continuing this way we can arrive at the same contradiction as in the relation algebra case. ∎

It remains to show that any non-trivial ultraproduct of the $\mathcal{C}_n$'s is representable.

In [HH97d], two kinds of games are defined. The first is the modification of the game on relation algebras for cylindric algebras, and it is stated that an atomic cylindric algebra has a complete representation iff the existential player has a winning strategy in the ω-long game. The second type of game is formulated using colored graphs. The two games are equivalent in the sense that $\exists$ has a winning strategy in the cylindric game played on $\mathcal{C}_n$ iff she has a winning strategy in the n-colored graph game.

We recall the definition of the n-colored graph game from [HH97d].

Definition 8.5 *Let $\mathcal{G}_n$ be the subclass of n-colored graphs defined above.*

The game G_l^n $(l \leq \omega)$ is defined as follows. The two players, $\forall$ and $\exists$, build a chain of elements of $\mathcal{G}_n$: $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_l$ if l is finite, or $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ if $l = \omega$.

$\Gamma_0 \in \mathcal{G}_n$ is arbitrary with $|\Gamma_0| = \alpha$. In each subsequent round i $(0 \leq i < l)$,

- *$\forall$ chooses a graph Φ from $\mathcal{G}_n$ with $|\Phi| = \alpha$, a single node $\beta \in \Phi$ and a colored graph embedding $\lambda : \Phi \setminus \{\beta\} \longrightarrow \Gamma_i$.*
- *$\exists$ responds, if she can, with a finite colored graph $\Gamma_{i+1} \in \mathcal{G}_n$ and embeddings $\mu : \Gamma_i \longrightarrow \Gamma_{i+1}$ and $\nu : \Phi \longrightarrow \Gamma_{i+1}$ such that $\mu \circ \lambda$ and ν agree on $\Phi \setminus \{\beta\}$.*

$\exists$ wins a match of the game G_l if she survives each round. We say that $\exists$ has a winning strategy *if she can win all matches.*

Lemma 8.6 *Any non-trivial ultraproduct $\mathcal{C}$ of the $\mathcal{C}_n$'s is in RCA_α.*

Proof: Again we prove that $\exists$ can survive arbitrarily long matches in cofinitely many n-colored graph games. By the equivalence of n-colored graph games and cylindric games, she can achieve this in cylindric games as well.

Claim 8.7 *Let l be an arbitrary fixed element of ω. $\exists$ has a winning strategy for cofinitely many n's in the games G_l^n.*

Proof: The combination of the proof of Proposition 23 in [HH97d] and the proof of the same claim for relation algebras works. ∎

Again, the above winning strategies provide her with a winning strategy in the game played on the ultraproduct. Then, in a similar fashion as in the case of relation algebras, it follows that the ultraproduct $\mathcal{C}$ of $\mathcal{C}_n$ $(n \in \omega)$ is representable. ∎

This finishes the proof of the theorem. ∎

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