A Negationless Interpretation of Intuitionistic Axiomatic Theories: Arithmetic and Analysis

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Abstract

The present work contains an axiomatic treatment of some parts of the version of intuitionistic mathematics advocated by G.F.C. Griss, the so-called "negationless intuitionistic mathematics" (see, e.g., [7],[12],[13, 8.2.], [14]).

Formal systems **NPC**, **NA**, and $\mathbf{FIM^N}$ for negationless predicate logic, arithmetic, and analysis are proposed. Our Theorem 6 in Section 2 asserts the translatability of Heyting's arithmetic **HA** into **NA**. This result can in fact be extended to a large class of intuitionistic theories based on **HA** and their negationless counterparts. For instance, in Section 3 this is shown for Kleene's system of intuitionistic analysis **FIM** and our $\mathbf{FIM^N}$.

Dedicated to the centenary of the birth of George Francois Cornelis Griss (1898–1953)

Introduction

The principal tenet of the restricted version of intuitionistic mathematics advocated by G.F.C. Griss, aka negationless mathematics, consists in that only *inhabited properties* (viz. those realizable on some mathematical objects) are admissible in intuitionistic mathematics. Indeed, in connection with Brouwer's definition of species[1] Griss wrote [8, page 108]:

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[1] cf. [1]: "... at every stage of the construction of mathematics, properties which can be supposed to hold for previously obtained mathematical concepts, may be introduced as new mathematical concepts called *species*."

> "Whatever are the properties that can be supposed? What other criterion could there be than 'to hold for mathematical conceivabilities already obtained'? In the definition of species the words "can be supposed" should be replaced by "are known". *One should restrict oneself in intuitionistic mathematics to mathematical conceivabilities and properties of those mathematical conceivabilities and one should not make suppositions of which one does not know whether it is possible to fulfil them.*"

This viewpoint is directly related to his criticism of hypothetical reasoning[2] and, hence, the use of negation [7, page 675]:

> "On philosophic grounds I think the use of the negation in intuitionistic mathematics has to be rejected. Proving that something is not right, i.e. proving the incorrectness of a supposition, is no intuitive method. For one cannot have a clear conception of a supposition that eventually proves to be a mistake. Only construction without the use of negation has some sense in intuitionistic mathematics."

An informal development of some parts of negationless mathematics was given by Griss in his [7] – [10].

The main difficulty in formalizing the negationless standpoint is that it requires a specific form of logic. In fact, the philosophical motivation for negationless mathematics suggests that the corresponding formal system P for predicate logic should meet the following two requirements.

First, the use of the logical constants in P must agree with the intended negationless interpretation; thus besides the rejection of negation certain restrictions on the use of disjunction and implication should be taken into account (cf. [12],[13, 8.2.],[14]).

Secondly, for any derivation Σ in P and any formula occurrence ϕ in Σ, the existential closure of ϕ, as well as that of any *well-formed part of* ϕ (see Definition 4 below), should be derivable in P (the so-called *syntactical non-nullity criterion*, cf. [24]).

Formalizations of predicate logic intended to agree with Griss' ideas were proposed by Vredenduin [30], Gilmore [6], Valpola [29], and Nelson [24], [25], see also [21] and [22].

With respect to the above-mentioned requirements, Nelson's sequent calculus $\mathbf{P_1}$ [24] appears to give the most satisfactory representation of negationless reasoning. However, the logical rules of inference in $\mathbf{P_1}$ are rather

[2] For an earlier criticism concerning the problem of interpreting constructively the implication statement $A \to B$, where A is false, see Freudenthal's paper [5] (cf. [11]).

complex, which creates serious difficulties in the study of this system, let alone axiomatic theories based on $\mathbf{P_1}$, by means of the usual proof-theoretical methods. It thus seems desirable to have a more convenient, from the viewpoint of structural proof theory, axiomatic formulation of negationless predicate logic. To this end we state, in Section 1, another formal system **NPC** , which is a system of natural deduction featuring the agreement of the notion of derivation in it with the intended interpretation of *quantified implication* [24]. Namely, a derivation Σ in **NPC** is defined as a pair $<\Sigma_1, \Sigma_2>$ of derivations (each of them understood in the usual sense), where Σ_1 proves the non-nullity of the conjunction of the open assumptions of Σ_2, see Definition 1 below.

The rest of Section 1 is devoted to proving the following basic properties of **NPC**: The syntactical non-nullity criterion (Theorem 1); conservativity with respect to Heyting's predicate logic (Theorem 2); the completeness relative to (a negationless version of) validity in all intuitionistic structures (Theorem 3).

Note that extensions of **NPC** to systems including *quantified disjunction* (see [24]) were introduced and studied at length in our [18]. For the purposes of the present paper, especially for those of Sections 2 and 3, however, it is sufficient (and even preferable!) to consider only a disjunction-free fragment of negationless predicate logic.

Besides formalizing negationless theories based on **NPC**, we make it clear in what sense can the traditional, going back to Brouwer and Heyting, intuitionistic mathematics be justified from the negationless viewpoint and how much of it can be reconstructed according to the conception of negationless intuitionistic mathematics outlined by Griss (these two basic inquiries are strongly interrelated [3]).

In Section 2 we describe a formal system **NA** for negationless arithmetic and prove the central statement on the translatability of Heyting's arithmetic **HA** into **NA** (Theorem 6). In Section 3 we extend this theorem to Kleene's system of intuitionistic analysis **FIM** [16] and the corresponding negationless theory $\mathbf{FIM^N}$ (Theorem 8).

It should be emphasized that the results of Sections 2 and 3 on translatability are true in a rather more general context than that of intuitionistic arithmetic and analysis. Using the same technique as in the proof of Theorem 6, similar results can be obtained with respect to *current* intuitionistic formal theories containing the first-order arithmetic. Namely, it can be shown that for any theory T of this kind there exists its negationless counterpart T^N such that T and T^N are syntactically translatable into each other and

[3]López-Escobar [20, page 26] states the former one as follows: "Hilbert did not want to be expelled from the paradise created by Cantor, we should at least be able to prove that there is no need to live in the purgatory created by Griss."

thus turn out to be essentially equivalent. [4]

This paper is a modified and somewhat improved version of one part of the author's doctoral thesis [18].

I wish to express my sincere thanks to Professor David Nelson for his interest and encouragement.

1 First-order predicate calculus NPC

Throughout the following all arguments on the metalevel are intended to be intuitionistically valid.

We shall presuppose some familiarity with the basic notions of natural deduction, as can be found in [26] and [27].

We first describe an auxiliary system $\mathbf{NPC_1}$.

The language of $\mathbf{NPC_1}$ is to contain:

(a) Countably many (bound) variables $x_0, x_1, x_2, \ldots$;
(b) countably many individual parameters $a_0, a_1, a_2, \ldots$;
(c) predicate symbols $P_0, P_1, P_2, \ldots$ (it is assumed that there is at least one predicate symbol P_0);
(d) function symbols $f_0, f_1, f_2, \ldots$;
(e) individual constants $c_0, c_1, c_2, \ldots$;
(f) logical constants & (and), $\forall$ (for all), $\exists$ (there exists), and quantified implication $\rightarrow_{\bar{x}}$.

Terms (denoted by $t, t_1, t_2, \ldots$) are defined in the usual way.

Notations. We use x, y, and z as metavariables for the variables, a, b, and c for the parameters, $\bar{a}$ for the finite sequences of parameters, $\bar{t}$ for the finite sequences of terms. Throughout this section $\tau(i)$ denotes the arity of P_i ($i = 0, 1, 2, \ldots$). In what follows $:=$ is used for the definition symbol and $\equiv$ for syntactical identity of expressions.

Let E be any expression, let $\bar{a} \equiv a_{n_1}, \ldots, a_{n_m}$ be a sequence of distinct parameters, and let $\bar{\sigma} \equiv \sigma_1, \ldots, \sigma_m$ be a sequence of any symbols. We denote by $E(\bar{a}/\bar{\sigma})$ the result of simultaneous substitution of $\sigma_1, \ldots, \sigma_m$ for the

[4] Further extensions of Theorem 6 to intuitionistic theories with species variables, e.g., **HAS** (intuitionistic second-order arithmetic), will be considered in a sequel to the present work.

parameters $a_{n_1}, \ldots, a_{n_m}$ in E. If no confusion can arise, we often write $E(\bar{\sigma})$ instead of $E(\bar{a}/\bar{\sigma})$.

The *formulas* in $\mathbf{NPC_1}$ are defined inductively as follows:

(i) If P_i is a predicate symbol and $t_1, \ldots, t_{\tau(i)}$ are terms, then $P^i(t_1, \ldots, t_{\tau(i)})$ is an atomic formula;
(ii) every atomic formula is a formula;
(iii) if A and B are formulas, then $A \,\&\, B$ is a formula;
(iv) if A is a formula, then so are $\forall x A(a/x)$ and $\exists x A(a/x)$, provided x does not occur in A ;
(v) if A and B are formulas, $\bar{a}$ a non-empty sequence of distinct parameters, $\bar{x}$ a sequence [of the same length as $\bar{a}$] of variables which do not occur in A or B, then $A(\bar{a}/\bar{x}) \rightarrow_{\bar{x}} B(\bar{a}/\bar{x})$ is a formula.

Henceforth, the letters A, B, and C (with or without subscripts) will be metavariables for formulas. For any formula A, $\exists A$ will denote its existential closure.

We use the standard abbreviations for iterated quantifiers:
If $\bar{x} \equiv x_{n_1}, \ldots, x_{n_m}$, then $\forall \bar{x} := \forall x_{n_1} \ldots \forall x_{n_m}$ and $\exists \bar{x} := \exists x_{n_1} \ldots \exists x_{n_m}$.

The *axioms* of $\mathbf{NPC_1}$ are the existential closures of $P_i(\bar{a})$, where $\bar{a}$ is a sequence of $\tau(i)$ distinct parameters ($i = 0, 1, 2, \ldots$).

The logical rules of inference:

Introduction rules (I-rules)

&I) $$\frac{A \quad B}{A \,\&\, B}$$

$\rightarrow$I) $$\frac{\exists \bar{x} A(\bar{x}) \qquad \begin{matrix}[A(\bar{a})] \\ B(\bar{a})\end{matrix}}{A(\bar{x}) \rightarrow_{\bar{x}} B(\bar{x})}$$

$\forall$I) $$\frac{A(a)}{\forall x A(x)}$$

$\exists$I) $$\frac{A(t)}{\exists x A(x)}$$

Elimination rules (E-rules)

&E) $$\frac{A \,\&\, B}{A} \qquad \frac{A \,\&\, B}{B}$$

$\rightarrow$E$_1$) $$\frac{A(\bar{t}) \quad A(\bar{x}) \rightarrow_{\bar{x}} B(\bar{x})}{B(\bar{t})}$$

$\rightarrow$E$_2$) $$\frac{A(\bar{x}) \rightarrow_{\bar{x}} B(\bar{x})}{\exists \bar{x} A(\bar{x})}$$

$\forall$E) $$\frac{\forall x A(x)}{A(t)}$$

$\exists$E) $$\frac{\exists x A(x) \qquad \begin{matrix}[A(a)] \\ C\end{matrix}}{C}$$

It is understood that the standard conventions about the proper parameter a in $\forall$I and $\exists$E are observed, see [27, II.1.2.1., II.1.2.4.]. In $\rightarrow$I, it is assumed that the parameters in $\bar{a}$ must not occur in the assumptions $B(\bar{a})$ depends on except the assumptions of the form $A(\bar{a})$, and that the parameters in $\bar{a}$ may occur only in and above the right premiss.

The notions of *derivation* and *derivability* for $\mathbf{NPC_1}$ are defined as usual. A part of a derivation Σ in $\mathbf{NPC_1}$ which itself is a derivation is called a *subderivation of* Σ.

Some notational conventions. For technical reasons we use $\mathcal{T}$ as a denotation for the existential closure of $P_0(\bar{a})$, where $\bar{a}$ is a sequence of $\tau(0)$ distinct parameters. If $\Gamma = \{A_1, \ldots, A_n\}$ is a non-empty finite set of formulas, then $\&\Gamma$ denotes the formula $A_1 \& \ldots \& A_n$; if $\Gamma = \emptyset$, then we put $\&\Gamma := \mathcal{T}$.

Definition 1. Let S be an arbitrary system of natural deduction (in the sense of [26, page 23]) based on predicate logic $\mathbf{NPC_1}$. We define a system $[S]$ which differs from S only in the notion of derivation. Namely, a *derivation in* $[S]$ *of a formula* A *from a set of formulas* Γ is a pair of derivations $<\Sigma_1, \Sigma_2>$ in S such that Σ_2 is a derivation of A from Γ and Σ_1 a derivation of $\exists(\&\Gamma_{\Sigma_2})$, where Γ_{Σ_2} denotes the set of open assumptions of Σ_2. We say that A *is derivable in* $[S]$ *from* Γ if and only if there exists a derivation in $[S]$ of A from Γ ; in case $\Gamma = \emptyset$, we will say simply that A *is derivable in* $[S]$ (in symbols, $[S] \vdash A$).

In what follows S is referred to as the *underlying system for* $[\,S]$.

Observe that A is derivable in $[S]$ if and only if A is derivable in S.

Definition 2. If A, B are formulas in S (S as in the above definition), then A and B are S*-equivalent* just in case $\{B\} \vdash_S A$ and $\{A\} \vdash_S B$.

Remark 1. It is easy to verify that the logical constants $\forall$ and $\&$ are definable in $\mathbf{NPC_1}$, in the sense that:

(a) $\forall x A(x)$ and $\mathcal{T} \rightarrow_x A(x)$ are $\mathbf{NPC_1}$-equivalent;
(b) $A \& B$ and $A(a/x) \rightarrow_x B(a/x)$ are $\mathbf{NPC_1}$-equivalent, provided a does not occur in A or B ;
(c) the rules for $\forall$ and $\&$ are derivable for these defined logical constants.

Definition 3. $\mathbf{NPC} := [\mathbf{NPC_1}]$.

Definition 4. Given any formula A in $\mathbf{NPC_1}$, we define the *well-formed parts* (w.f.p.) of A inductively as follows:

(i) A is a w.f.p. of A ;
(ii) if $B \,\&\, C$ is a w.f.p. of A, then so are B and C ;
(iii) if $\mathrm{Q}xB(x)$ is a w.f.p. of A, then so is $B(a)$, where Q is either $\forall$ or $\exists$, a the first parameter not occurring in $\mathrm{Q}xB(x)$;
(iv) for any sequence $\bar{x}$ of n variables, if $B(\bar{x}) \rightarrow_{\bar{x}} C(\bar{x})$ is a w.f.p. of A, then so are $B(\bar{a})$ and $C(\bar{a})$, where $\bar{a}$ is the sequence of the first n parameters which do not occur in $B(\bar{x}) \rightarrow_{\bar{x}} C(\bar{x})$.

Lemma 1. *For any formula A in* $\mathbf{NPC_1}$ *and any B which is a well-formed part of A: If $\exists A$ is derivable in* $\mathbf{NPC_1}$ *, then so is $\exists B$.*

Proof. By induction on A, using Definition 4.
■

Lemma 2. *Let Σ be a derivation in* $\mathbf{NPC_1}$ *with the set of open assumptions Γ_Σ , and suppose that $\exists(\&\Gamma_\Sigma)$ is derivable in* $\mathbf{NPC_1}$*. Then, for any subderivation Π of Σ with the set of open assumptions Γ_Π , the formula $\exists(\&\Gamma_\Pi)$ is derivable in* $\mathbf{NPC_1}$.

Proof. This is easily verified by induction on the number of those formula occurrences in Σ which do not occur in Π ; the induction steps are treated according to the inferences in Σ with the end-formula of Π as a premiss.
■

Remark 2. Actually, in view of the above-mentioned conventions about proper parameters, the conclusion of Lemma 2 can be strengthened to: $\mathbf{NPC_1} \vdash \exists(\&\Gamma)$, where Γ is the set of *all* formula occurrences in Σ.

The following statement is an immediate consequence of Lemmas 1, 2.

Theorem 1 (*syntactical non-nullity of* **NPC**). *Let $<\Sigma_1, \Sigma_2>$ be a derivation in* **NPC** *, let A be any formula occurrence in either Σ_1 or Σ_2 , and let C be any well-formed part of A. Then* $\mathbf{NPC} \vdash \exists C$.

Notations. We use **IQC** [resp. $\mathbf{IQC}^-$] for the natural-deduction version of Heyting's predicate calculus as presented in [27] [resp. the negation-free fragment of **IQC**], and $\mathbf{IQC_1}$ [resp. $\mathbf{IQC_1^-}$] for the system obtained from **IQC** [resp. $\mathbf{IQC}^-$] by adding all the axioms of **NPC**. (It is understood that the atomic formulas in **IQC** are those in **NPC**.)

Definition 5. We define a translation * which associates with every formula A in **NPC** a formula A^* in **IQC** as follows:

(a) For A atomic, $A^* \equiv A$;
(b) $*$ commutes with $\&, \forall$, and $\exists$;
(c) $(A(\bar{a}/\bar{x}) \rightarrow_{\bar{x}} B(\bar{a}/\bar{x}))^* \equiv \exists \bar{x} A^*(\bar{a}/\bar{x}) \,\&\, \forall \bar{x}(A^* \rightarrow B^*)(\bar{a}/\bar{x})$;
(d) if Γ is a set of formulas in **NPC**, then $\Gamma^* := \{A^* : A \in \Gamma\}$.

Let Δ stand for $\{A^* : A$ is a formula in **NPC**$\}$, and let, for any formula A in **IQC**, $\mathrm{Sf}(A)$ denote the set of subformulas of A. Then we obtain, as an easy corollary to Definition 5:

Lemma 3.

(a) *If* $A \,\&\, B \in \Delta$, *then* $A \in \Delta$;
(b) *if* $A \,\&\, B \in \mathrm{Sf}(C)$ *and* $C \in \Delta$, *then* $A \,\&\, B \in \Delta$;
(c) *if* $\exists x A(x) \in \Delta$, *then, for any term* t, $A(t) \in \Delta$;
(d) *if* $\exists x A(x) \in \mathrm{Sf}(C)$ *and* $C \in \Delta$, *then* $\exists x A(x) \in \Delta$;
(e) *if* $\forall x A(x) \in \Delta$, *then, for any term* t, $A(t) \in \Delta$;
(f) *if* $A(t) \in \Delta$, *then* $\forall x A(x) \in \Delta$;
(g) *if* $C \in \Delta$, *then* C *contains neither* $\perp$ (*absurdity*) *nor* $\vee$;
(h) $A \rightarrow B \notin \Delta$;
(i) *if* $A \rightarrow B \in \mathrm{Sf}(C)$ *and* $C \in \Delta$, *then* $A \in \Delta$ *and* $B \in \Delta$.

The following lemma shows that $\mathbf{IQC_1}$ is a conservative extension of $\mathbf{NPC_1}$, via the translation $*$.

Lemma 4. *For any formula* A *and any set of formulas* Γ *in* $\mathbf{NPC_1}$: A *is derivable in* $\mathbf{NPC_1}$ *from* Γ *if and only if* A^* *is derivable in* $\mathbf{IQC_1}$ *from* Γ^*.

Proof. From left to right: This follows easily by induction on the length of derivations in $\mathbf{NPC_1}$. From right to left: Let A be a formula and Γ a set of formulas in $\mathbf{NPC_1}$. In view of the *normal form theorem* for **IQC** (see [27, 3.5.1.]), it suffices to show that any *expanded normal derivation* ([27, 3.3.3.]) Π^* in $\mathbf{IQC_1}$ of A^* from Γ^* can be transformed into a derivation Π of A from Γ in $\mathbf{NPC_1}$. The proof goes by induction on the number of formula occurrences in Π^*. We look at the two less obvious cases where Π^* ends with an application of $\rightarrow$E and &I.

Case (1): The last inference in Π^* is an application of $\rightarrow$E. Then Π^* has the form

$$\dfrac{\begin{matrix}\Sigma^* \\ A_1\end{matrix} \qquad \begin{matrix}\Sigma_1 \\ A_1 \rightarrow A^*\end{matrix}}{A^*}$$

By the *subformula principle* (see [27, 3.5.1.]), there exists $C \in \Delta$ such that $A_1 \to A^* \in \mathrm{Sf}(C)$. Thus, by (i) of Lemma 3, we have $A_1 \in \Delta$. Again by the subformula principle, it follows that there are only two possible cases to consider where the last inference in Σ_1 is: (1a) an application of $\exists$E ; (1b) an application of $\forall$E.

Subcase (1a): Π^* has the form

$$\dfrac{\begin{array}{c}\Sigma^*\\ A_1\end{array}\quad \dfrac{\begin{array}{c}\Sigma_2^*\\ \exists x A_2(x)\end{array}\quad \begin{array}{c}[A_2(a)]\\ \vdots\\ A_1 \to A^*\end{array}}{A_1 \to A^*}}{A^*}$$

Consider the following reduction of Π^* (which replaces the expanded normal derivation in question by an expanded normal one):

$$\left.\dfrac{\begin{array}{c}\Sigma_2^*\\ \exists x A_2(x)\end{array}\quad \dfrac{\begin{array}{c}\Sigma^*\\ A_1\end{array}\quad \begin{array}{c}[A_2(a)]\\ \vdots\\ A_1 \to A^*\end{array}}{A^*}}{A^*}\quad\right\} \equiv \Sigma_3^*$$

Then, by the subformula principle and Lemma 3(d), $\exists x A_2(x) \in \Delta$; hence, by Lemma 3(c), we get $A_2(a) \in \Delta$. Apply the induction hypothesis to Σ_2^* and Σ_3^*.

Subcase (1b): Π^* has the form

$$\dfrac{\begin{array}{c}\Sigma^*\\ A_1(\bar{t})\end{array}\quad \dfrac{\dfrac{\begin{array}{c}\Sigma_2\\ \forall \bar{x}(A_1 \to A^*)(\bar{x})\end{array}}{\text{several applications of }\forall\mathrm{E}}}{(A_1 \to A^*)(\bar{t})}}{A^*(\bar{t})}$$

where it is assumed that the last inference in Σ_2 is not an application of $\forall$E.

By the subformula principle, there are only two possible cases to consider where the last inference in Σ_2 is: (1ba) an application of $\exists$E ; (1bb) an application of &E.

Subcase (1ba): This is proved in the same way as (1a) above.

Subcase (1bb): By the subformula principle, Σ_2 has the form

$$\dfrac{\begin{array}{c}\Sigma_3^*\\ \exists\bar{x}A_1(\bar{x})\,\&\,\forall\bar{x}(A_1 \to A^*)(\bar{x})\end{array}}{\forall\bar{x}(A_1 \to A^*)(\bar{x})}$$

Apply the induction hypothesis to Σ^* and Σ_3^*.

Case (2): The last inference in Π^* is an application of &I. Then Π^* has the form

$$\dfrac{\begin{array}{cc}\Sigma_1^* & \Sigma_2\\ A_1 & A_2\end{array}}{A_1\,\&\,A_2}$$

Observe that by Lemma 3(a), $A_1 \in \Delta$. As the case where $A_2 \in \Delta$ is proved trivially from the induction hypothesis, assume that $A_2 \notin \Delta$. Then there exist formulas B_1, B_2 in $\mathbf{NPC_1}$ such that $A_1 \equiv \exists\bar{x}B_1^*(\bar{x})$ and $A_2 \equiv \forall\bar{x}(B_1^* \to B_2^*)(\bar{x})$. Moreover, since Π^* is in expanded normal form, Σ_2 must be of the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[B_1^*(\bar{b})]\\ \Sigma_3^*\\ B_2^*(\bar{b})\end{array}}{(B_1^* \to B_2^*)(\bar{b})}}{\text{several applications of }\forall\text{I}}}{\forall\bar{x}(B_1^* \to B_2^*)(\bar{x})}$$

Apply the induction hypothesis to Σ_1^* and Σ_3^*.
■

Note that the proof of Lemma 4 is constructive, i.e., a derivation Π of A from Γ in $\mathbf{NPC_1}$ is effectively constructed from the given expanded normal derivation Π^* in $\mathbf{IQC_1}$ of A^* from Γ^*.

As an immediate consequence of Lemma 4 we have:

Theorem 2. *For any formula A and any set of formulas Γ in* **NPC**: *A is derivable in* **NPC** *from Γ if and only if there exists a finite, possibly empty $\Gamma_1 \subseteq \Gamma$ such that $\exists(\&\Gamma_1)^*$ is derivable in* $\mathbf{IQC_1}$ *and A^* is derivable in* $\mathbf{IQC_1}$ *from Γ_1^*.*

Remark 3. (i) It is easy to see that the above proof of Lemma 4 implies (a version of) the normal form theorem for $\mathbf{NPC_1}$, and hence also for **NPC**.

(ii) Using a classical reasoning, it is easy to give another, model-theoretic proof of Lemma 4 taking into account that the restriction of a Kripke model for $\mathbf{IQC_1}$ to the sentences in Δ provides, via the translation *, a Kripke style model for $\mathbf{NPC_1}$, and that the (strong) completeness of $\mathbf{NPC_1}$ for the modelling thus obtained is provable along the same lines as the (strong) completeness theorem for **IQC** relative to Kripke models.

We now indicate another consequence of Lemma 4, a variant of completeness theorem for **NPC** with respect to validity in all intuitionistic structures. For the sake of simplicity, we restrict attention to the case of *pure* first-order predicate logic, without function symbols and without individual constants.

Let A be a formula in **IQC**, all of whose predicate letters occur in the list $P_{n_1}, \ldots, P_{n_p}$. Let D range over (inhabited) species of individuals, R_{n_i} over subspecies of $D^{\tau(n_i)}$. Then A is said to be *valid in all intuitionistic structures*, written $Val(A)$, iff $\forall D \forall R_{n_1} \ldots \forall R_{n_p} A^D(R_{n_1}, \ldots, R_{n_p})$ holds intuitionistically, where $A^D(R_{n_1}, \ldots, R_{n_p})$ is obtained from A by relativizing all quantifiers to D and substituting R_{n_i} for P_{n_i}, respectively (cf. [17]).

Let $\underline{Val}$ be defined as Val but with the range of R_{n_i} restricted to inhabited species, and let, for any formula A in $\mathbf{NPC_1}$, $\underline{Val}^*(A)$ stand for $\underline{Val}(A^*)$.

Then the following statement holds intuitionistically.

Theorem 3 (*completeness and soundness*). *For any formula A and any set of formulas Γ in* **NPC**: *A is derivable in* **NPC** *from Γ if and only if there exists a finite, possibly empty $\Gamma_1 \subseteq \Gamma$ such that $\underline{Val}^*(\&\Gamma_1(\bar{a}/\bar{x}) \to_{\bar{x}} A(\bar{a}/\bar{x}))$, provided $\bar{a}$ contains all the parameters occurring in A or the formulas in Γ_1.*

Proof. From left to right: By induction on derivations in $\mathbf{NPC_1}$. From right to left: Assume $\underline{Val}^*(\&\Gamma_1(\bar{a}/\bar{x}) \to_{\bar{x}} A(\bar{a}/\bar{x}))$. Since $\mathbf{IQC}^-$ is complete relative to Val, see e.g. [4, page 288], $\mathbf{IQC_1^-}$ is complete relative to $\underline{Val}$. Hence $(\&\Gamma_1(\bar{a}/\bar{x}) \to_{\bar{x}} A(\bar{a}/\bar{x}))^*$ is derivable in $\mathbf{IQC_1^-}$, and thus, by Lemma 4, $\mathbf{NPC_1} \vdash \&\Gamma_1(\bar{a}/\bar{x}) \to_{\bar{x}} A(\bar{a}/\bar{x})$. Therefore, by $\to \mathrm{E}_1$ and $\to \mathrm{E}_2$, A is derivable in **NPC** from Γ.

■

Corollary 1. *For any formula A in* **NPC**: *A is derivable in* **NPC** *if and only if $\underline{Val}^*(A)$.*

Remark 4. It can be shown (this will be done in the sequel to the present work) that **NPC** is a subsystem of a sequent calculus $\overline{\mathbf{P_1}}$ which results from Nelson's system $\mathbf{P_1}$ by adding all the axioms of **NPC** as initial sequents (viz. sequents of the form $\rightarrow A$, where A is an axiom of **NPC**) and the inference rule

$$\frac{\Gamma, \exists x A(x) \rightarrow \mathcal{T}}{\Gamma, A(a) \rightarrow \mathcal{T}}$$

where it is assumed that a does not occur in the upper sequent. Namely, if a formula A is derivable in **NPC** from a finite (possibly empty) set of formulas $\{A_1, \ldots, A_n\}$, than the the sequent $A_1, \ldots, A_n \rightarrow A$ is derivable in $\overline{\mathbf{P_1}}$. Therefore the soundness of **NPC** holds both relative to the realizability definition in [24] and to the interpretation in terms of constructions as presented in [19].

Theorem 4 (*semantical non-nullity*). *Let* $<\Sigma_1, \Sigma_2>$ *be a derivation in* **NPC** *, let* A *be any formula occurrence in either* Σ_1 *or* Σ_2 *, and let* C *be any well-formed part of* A*. Then* $\underline{Val}^*(\exists C)$.

Proof. Immediate, combining Theorem 1 with the 'soundness part' of Theorem 3.

■

Remark 5. All the results of this section are extended to many-sorted first-order predicate logic in the obvious way.

2 Negationless arithmetic NA

We first describe the underlying system $\mathbf{NA_1}$.

The language of $\mathbf{NA_1}$ is to contain: Variables $x_0, x_1, x_2, \ldots$ and parameters $a_0, a_1, a_2, \ldots$ for natural numbers; two binary predicate symbols P_0 and P_1 for equality and apartness [late on we write $a = b$, resp. $a \# b$ instead of $P_0(a, b)$, resp. $P_1(a, b)$] ; a single individual constant 0 (zero); a unary function symbol S (successor); function symbols for all primitive recursive functions [the function symbols for the primitive recursive functions used in the text are standard].

Terms (denoted by r, s, and t) are defined as usual.

Formulas are constructed from atomic formulas by means of &, $\forall$, $\exists$, and $\rightarrow_{\bar{x}}$.

The *axioms* and *rules* of $\mathbf{NA_1}$ consist of:

(i) The axioms and rules of $\mathbf{NPC_1}$ (in the language of $\mathbf{NA_1}$);

(ii) defining equations for all primitive recursive functions, see [15, §44];

(iii) the following axioms and rules for equality, successor, and apartness:

(a) $\forall x(x = x)$,

(b) $\forall x(Sx \# 0)$,

(c) $\dfrac{t \# r}{Sr \# St}$,

(d) $\dfrac{St \# Sr}{t \# r}$,

(e) $\dfrac{St = Sr}{t = r}$,

(f) $\dfrac{t = s \quad A(t)}{A(s)}$,

(R1) $\dfrac{x \# 0 \to_x (t(x) \# 0) \quad t(s) = 0}{s = 0}$,

(R2) $\dfrac{t(x) = 0 \to_x (x = r) \quad s \# r}{t(s) \# 0}$,

(R3) $\dfrac{t(s) = 0 \quad t(r) \# 0}{s \# r}$,

(R4) $\dfrac{t \# 0}{\exists x(t = Sx)}$;

(iv) the rule of induction

$$\text{Ind} \quad \dfrac{A(0) \quad \begin{array}{c}[A(a)] \\ A(Sa)\end{array}}{A(t)}$$

with the standard restrictions concerning the proper parameter a.

Definition 6. $\mathbf{NA} := [\mathbf{NA_1}]$.

Theorem 5 (*syntactical non-nullity of* **NA**). *Let* $<\Sigma_1, \Sigma_2>$ *be a derivation in* **NA** , *let* A *be any formula occurrence in either* Σ_1 *or* Σ_2 , *and let* C *be any well-formed part of* A. *Then* $\mathbf{NA} \vdash \exists C$.

Proof. Similar to that of Theorem 1.
■

We use **HA** for a natural-deduction version of Heyting's arithmetic. **HA** is supposed to be based on **IQC** with the predicate symbol $=$, the individual constant 0, the function symbol S, and the function symbols for all primitive recursive functions. The non-logical axioms are: $\forall x(x = x)$, $\forall x \neg(Sx = 0)$, and the defining equations for the primitive recursive functions, supposed to be the same as in **NA**; the non-logical rules consist of (e), (f), and Ind of **NA** (the last two w.r.t. the language of **HA**).

For convenience, we assume that the formulas in **HA** are constructed from atomic formulas (i.e., equations between terms) by means of &, $\rightarrow$, $\forall$, and $\exists$. Thus $\bot$, $\neg$, $\leftrightarrow$, and $\vee$ are treated as abbreviations defined by $\bot := S0 = 0$, $\neg A := A \rightarrow \bot$, $A \leftrightarrow B := (A \rightarrow B) \,\&\, (B \rightarrow A)$, and $A \vee B := \exists x[(x = 0 \rightarrow A) \,\&\, (\neg(x = 0) \rightarrow B)]$.

Both in $\mathbf{NA_1}$ and **HA**:

$(= t, r) := \mathrm{sg}(|t - r|)$;

$(\# t, r) := \overline{\mathrm{sg}}(|t - r|)$.

By analogy with Definition 5, we define a translation * which associates with every formula A in **NA** a formula A^* in **HA**.

Definition 7.

(a) $(t = r)^* \equiv (t = r)$;
(b) $(t \# r)^* \equiv (\# t, r) = 0$;
(c) * commutes with &, $\forall$, and $\exists$;
(d) $(A(\bar{a}/\bar{x}) \rightarrow_{\bar{x}} B(\bar{a}/\bar{x}))^* \equiv \exists \bar{x} A^*(\bar{a}/\bar{x}) \,\&\, \forall \bar{x}(A^* \rightarrow B^*)(\bar{a}/\bar{x})$;
(e) if Γ is a set of formulas in **NA**, then $\Gamma^* := \{A^* : A \in \Gamma\}$.

Definition 8. For any formula A in **HA** (in **NA**) and any parameter a not occurring in A, we define a formula A_a in **HA** (in **NA**) by letting:

(a) $(t = r)_a \equiv (= t, r) \cdot \mathrm{sg}(a) = 0$;
(b) $(t \# r)_a \equiv (\# t, r) \cdot \mathrm{sg}(a) = 0$ (in the case of **NA**);
(c) the mapping associating A_a with A commutes with each logical constant.

Definition 9. Let A be a formula in **HA**, and let a be a parameter not occurring in A. We associate with A a formula A_a^+ in **NA** which is defined inductively by:

(a) $(t = r)_a^+ \equiv (t = r)_a$;
(b) ${}_a^+$ commutes with &, $\forall$, and $\exists$;
(c) $(B \to C)_a^+ \equiv (B_a^+)_b(b/x) \to_x (C_a^+)_b(b/x)$,

where it is supposed that b does not occur in B_a^+ or C_a^+, and x is the first variable not occurring in either B_a^+, C_a^+, or A.

If Γ is a set of formulas in **HA**, a a parameter not occurring in the formulas in Γ, then $\Gamma_a^+ := \{A_a^+ : A \in \Gamma\}$.

Definition 10. For any formula A in **HA** and any parameter a not occurring in A, we put $A^+ := A_a^+(a/S0)$.

The rest of this section is devoted to proving the following

Theorem 6 (*the relationship between* **HA** *and* **NA**). (I). *For any formula A in* **HA**:

(1) **HA** $\vdash A \leftrightarrow (A^+)^*$;
(2) **HA** $\vdash A$ *if and only if* **NA** $\vdash A^+$.

(II).*For any formula A in* **NA**:

(3) *A and $(A^+)^*$ are* $\mathbf{NA_1}$*-equivalent*;
(4) **NA** $\vdash A$ *if and only if* **HA** $\vdash A^*$.

In order to prove the theorem we first derive some lemmas.

Lemma 5. *The following formulas are derivable in* **NA**.

(i) $\forall x(0 \dot{-} x = 0)$;
(ii) $\forall x(x \dot{-} x = 0)$;
(iii) $\forall x \forall y(\mathrm{pd}(x \dot{-} y) = \mathrm{pd}(x) \dot{-} y)$;
(iv) $\forall x \forall y(x \cdot y = y \cdot x)$.

Proof. By direct verification.
■

Lemma 6. *The following rules are derivable in* $\mathbf{NA_1}$.

(1) $\dfrac{\mathrm{sg(t)} = 0}{t = 0}$; (4) $\dfrac{\overline{\mathrm{sg}}(|t - r|) = 0}{t \# r}$;

(2) $\dfrac{t + r = 0}{t = 0 \,\&\, r = 0}$; (5) $\dfrac{t \# r}{\overline{\mathrm{sg}}(|t - r|) = 0}$;

(3) $\dfrac{|t - r| = 0}{t = r}$.

Proof. (1): By (R4) and axioms (b), (f), we get $x \# 0 \to_x (\mathrm{sg}(x) \# 0)$. Thus $\mathrm{sg}(t) = 0$ implies $t = 0$, by (R1).

(2): First observe that from (R4), (b), and (f) it follows $x \# 0 \to_x (x = S\mathrm{pd}(x))$. Suppose now $t + r = 0$ and $b \# 0$ with b not occurring in t. Then $b = S\mathrm{pd}(b)$, hence $t + b = t + S\mathrm{pd}(b) = S(t + \mathrm{pd}(b))$, and so $t + b \# 0$. Therefore, $x \# 0 \to_x (t + x \# 0)$. By (R1), this implies $r = 0$, hence also $t = 0$.

(3): It is clearly sufficient to show that the formula $A(a) :=$

$$|x - a| = 0 \to_x (x = a)$$

is derivable in $\mathbf{NA_1}$. We argue by applying the rule of induction Ind w.r.t. a. Basis: $\mathbf{NA_1} \vdash A(0)$, by (i) of Lemma 5. Induction step: Suppose $|c - Sb| = 0$. Then, in view of (2), $c \dot{-} Sb = 0$ and $Sb \dot{-} c = 0$. Hence also $\mathrm{pd}(c \dot{-} b) = 0$, and thus, by (iii) of Lemma 5, $\mathrm{pd}(c) \dot{-} b = 0$. Further, by (R3) applied to $Sb \dot{-} c = 0$ and $(Sb \dot{-} 0) \# 0$, we get $c \# 0$. Thus, as in the proof of (2) above, $c = S\mathrm{pd}(c)$; therefore, $Sb \dot{-} S\mathrm{pd}(c) = 0$, hence $\mathrm{pd}(Sb \dot{-} \mathrm{pd}(c)) = 0$, and with (iii) of Lemma 5, $b \dot{-} \mathrm{pd}(c) = 0$. Thus [since $\mathrm{pd}(c) \dot{-} b = 0$] we have $|\mathrm{pd}(c) - b| = 0$, and therefore (induction hypothesis) $\mathrm{pd}(c) = b$. Hence, $c = Sb$, because $c = S\mathrm{pd}(c)$. We thus have shown that $|c - Sb| = 0$ implies $c = Sb$. Therefore $A(Sb)$ holds, by (ii) of Lemma 5.

(4): By (R3), using (ii) of Lemma 5.

(5): By (R2) applied to $|x - r| = 0 \to_x (x = r)$ and $t \# r$, we get $|t - r| \# 0$. Therefore, $\overline{\mathrm{sg}}(|t - r|) = 0$, by (R4).

■

Lemma 7. *Let A be a formula in* $\mathbf{NA}$, *and let b be a parameter not occurring in A. Then*

(i) *A and $A_b(b/S0)$ are* $\mathbf{NA_1}$ *-equivalent;*

(ii) *$A_b(b/S0)$ and $A_b(b/Sa)$ are* $\mathbf{NA_1}$ *-equivalent, for any parameter a not occurring in A;*

(iii) $\mathbf{NA} \vdash A_b(b/0)$;
(iv) A *and* $\forall x A_b(b/x)$ *are* $\mathbf{NA_1}$ *-equivalent*;
(v) $\mathbf{HA} \vdash (A^*)_b \leftrightarrow (A_b)^*$.

Proof. (i): By induction on A, using Lemma 6.
(ii), (iii), and (v): By a straightforward induction on A.
(iv): By (i)–(iii).
■

Lemma 8. *Let A be a formula in* $\mathbf{HA}$*, and let b be a parameter not occurring in A. Then*

(i) $\mathbf{HA} \vdash A \leftrightarrow A_b(b/S0)$;
(ii) $\mathbf{NA} \vdash A_b^+(b/0)$.

Proof. (i): Straightforward, by induction on A.
(ii): By induction on A, using (iii) and (iv) of Lemma 7.
■

Lemma 9. *For any formula A and any set of formulas Γ in* $\mathbf{NA_1}$: *If A is derivable in* $\mathbf{NA_1}$ *from Γ, then A^* is derivable in* $\mathbf{HA}$ *from Γ^*.*

Proof. By a straightforward induction on derivations in $\mathbf{NA}_1$.
■

Lemma 10. *For any formula A in* $\mathbf{HA}$, $\mathbf{HA} \vdash A \leftrightarrow (A^+)^*$.

Proof. This follows easily by induction on A, using Lemmas 7–9.
■

Lemma 11. *For any formula A in* $\mathbf{NA}$*, A and $(A^*)^+$ are* $\mathbf{NA_1}$ *-equivalent.*

Proof. By induction on A. The case A is atomic follows from Lemma 6. Since both * and $^+$ commute with $\&, \forall, \exists$, the induction steps corresponding to the cases where $A \equiv B \,\&\, C, \forall x A(x), \exists x A(x)$ are trivial. Let $A \equiv B(\bar{a}/\bar{x}) \rightarrow_{\bar{x}} C(\bar{a}/\bar{x})$. Then $(A^*)^+$ has the form

$$\exists \bar{x}(B^*)^+(\bar{a}/\bar{x}) \,\&\, \forall \bar{x}[((B^*)^+(\bar{a}))_y \rightarrow_y ((C^*)^+(\bar{a}))_y](\bar{a}/\bar{x}) \qquad (*)$$

It is easily seen that A is derivable in $\mathbf{NA_1}$ from $(A^*)^+$, by induction hypothesis and Lemma 7(i). Conversely, suppose A. Then, by $\rightarrow \mathrm{E}_2$ and the induction hypothesis, $\exists \bar{x}(B^*)^+(\bar{a}/\bar{x})$. Hence by the hypotheses,

$$(B^*)^+(\bar{a}/\bar{x}) \rightarrow_{\bar{x}} (C^*)^+(\bar{a}/\bar{x});$$

thus with (iv) of Lemma 7,

$$\forall y[((B^*)^+(\bar{a}/\bar{x}))_y \to_{\bar{x}} ((C^*)^+(\bar{a}/\bar{x}))_y].$$

This implies (*), by (iii) of Lemma 7.
■

Lemma 12. *Let A be a formula in* **HA***, let Γ be a finite, possibly empty set of formulas in* **HA***, and let a be a parameter not occurring in A or any formula in Γ. If A is derivable in* **HA** *from Γ, then A^+_a is derivable in* $\mathbf{NA_1}$ *from Γ^+_a.*

Proof. By induction on derivations in **HA**. (Throughout the proof "$\vdash$" refers to the derivability relation in $\mathbf{NA_1}$.) We consider only the case of axiom $\forall x \neg(Sx = 0)$ and the induction steps corresponding to applications of $\to$ I and $\perp_{\mathrm{I}}$ (the intuitionistic absurdity rule); the other cases are easy to verify.

Suppose that Σ is a derivation in **HA** of A from Γ.

Case 1: Σ ends with an application of $\to$ I. Then Σ has the form

$$\dfrac{\begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{B \to C}\ .$$

We have to show that

$$\Gamma^+_a \vdash (B^+_a)_b(b/x) \to_x (C^+_a)_b(b/x). \qquad (**)$$

By the induction hypothesis we get $\Gamma^+_a, B^+_a \vdash C^+_a$. Thus by (ii) of Lemma 8 $\vdash F$, where $F := (\&\Gamma^+_a \,\&\, B^+_a)(a/x) \to_x C^+_a(a/x)$. Further, by (iv) of Lemma 7,

$$\Gamma^+_a, (B^+_a)_b \vdash (\&\Gamma^+_a \,\&\, B^+_a)_b$$

and $F \vdash F_b$. Thus also $\vdash F_b$. This implies, by $\to \mathrm{E}_1$ applied to $(\&\Gamma^+_a \,\&\, B^+_a)_b$ and F_b,

$$\Gamma^+_a, (B^+_a)_b \vdash (C^+_a)_b$$

and, hence, (**), by (iii) of Lemma 7.

Case 2: Σ ends with an application of $\perp_{\mathrm{I}}$. We have to show that $\Gamma^+_a \vdash A^+_a$, so suppose Γ^+_a. Then, by the induction hypothesis, $\perp^+_a$, i.e., $(= S0, 0)\cdot \mathrm{sg}(a) = 0$.

Since $(= S0, 0) = S0$ holds, also $a = 0$ (by (iv) of Lemma 5 and (1) of Lemma 6). Therefore, by (ii) of Lemma 8, A^+_a.

Axiom $\forall x \neg(Sx = 0)$. We have to show that $\vdash (\forall x \neg(Sx = 0))^+_a$, i.e., $\vdash B$ where B has the form

$$\forall x[((Sx = 0)^+_a)_b(b/y) \rightarrow_y ((S0 = 0)^+_a)_b(b/y)].$$

Observe that, for any parameter c, $((Sc = 0)^+_a)_b$ and $\mathrm{sg}(a) \cdot \mathrm{sg}(b) = 0$ are $\mathbf{NA_1}$ -equivalent (by (i), (iv) of Lemma 5). This implies that

$$((Sc = 0)^+_a)_b \vdash ((S0 = 0)^+_a)_b.$$

Hence, by (iii) of Lemma 7, $\vdash B$.
■

Proof of Theorem 6.

(1). Lemma 10.
(3). Lemma 11.
(4). From left to right: Lemma 9. From right to left: By Lemma 12 and (3).
(2). From left to right: By Lemma 12. From right to left: By (4) and (1).

■

Corollary 2. *Let A be a formula in* **HA** *not containing $\rightarrow$. Then* $\mathbf{HA} \vdash A$ *if and only if* $\mathbf{NA} \vdash A$.

Corollary 3 (*explicit definability*). *For any sentences, i.e., formulas with no parameters, $\exists x A(x)$ and $A(\bar{x}) \rightarrow_{\bar{x}} B(\bar{x})$ in* **NA**:

(i) *If* $\mathbf{NA} \vdash \exists x A(x)$, *then, for some numeral $\bar{n}$,* $\mathbf{NA} \vdash A(\bar{n})$;
(ii) *if* $\mathbf{NA} \vdash A(\bar{x}) \rightarrow_{\bar{x}} B(\bar{x})$ *and $\bar{x} \equiv x_{p_1}, \ldots, x_{p_m}$, then, for some numerals $\bar{n}_1, \ldots, \bar{n}_m$,* $\mathbf{NA} \vdash A(\bar{n}_1, \ldots, \bar{n}_m)$.

Corollary 4. *Let $<\Sigma_1, \Sigma_2>$ be a derivation in* **NA**, *let A be any formula occurrence in either Σ_1 or Σ_2, and let $C(a_{p_1}, \ldots, a_{p_m})$ be any well-formed part of A, with all its parameters among $a_{p_1}, \ldots, a_{p_m}$. Then, for some numerals $\bar{n}_1, \ldots, \bar{n}_m$,* $\mathbf{NA} \vdash C(\bar{n}_1, \ldots, \bar{n}_m)$.

Proof. Immediate, combining (i) of Corollary 3 with Theorem 6.
■

3 Negationless analysis

In this section we will describe briefly an extension of Theorem 6 to the case of Kleene's system of intuitionistic analysis **FIM** [16] and the corresponding system of negationless analysis $\mathbf{FIM^N}$. It should be noted that the same techniques is also applicable to many other axiomatizations of intuitionistic analysis, such as **CS** of Kreisel and Troelstra, or Kreisel's theory of lawless sequences **LS** (for a description of these systems see, e.g., [28]), or their variants containing no species variables.

We begin with a formulation of negationless primitive recursive analysis $\mathbf{PrAn^N}$ with the underlying system $\mathbf{PrAn_1^N}$. For notations not explained below we refer to [28, 1.9.].

The language of $\mathbf{PrAn_1^N}$ contains, in addition to the symbols of $\mathbf{NA_1}$, unary function variables (denoted by τ, χ, ψ, ...), unary function parameters α, β, γ, ... , the application operator Ap, the recursor R, and the abstraction operator λ.

(Numerical) terms t, r, ... and functors (= function terms) are defined inductively by:

(a) Every term in $\mathbf{NA_1}$ is a term;
(b) function parameters and unary function symbols are functors;
(c) if ϕ is a functor, t a term, then $\mathrm{Ap}(\phi,\, t)$ [abbreviated as $\phi(t)$, or even ϕt] is a term;
(d) if $t(a)$ is a term, then $\lambda x.\, t(a/x)$ is a functor;
(e) if t, r are terms, ϕ a functor, then $\mathrm{R}(t, \phi, r)$ is a term.

$\mathbf{PrAn_1^N}$ is based on $\mathbf{NA_1}$ with the two sorts of variables (and parameters), the rule of λ -conversion

$$\text{CON} \qquad \lambda x.\, r(a/x)(t) = r(a/t),$$

and the defining axioms for the recursor

$$\text{REC} \quad \begin{cases} \mathrm{R}(t, \phi, 0) = t, \\ \mathrm{R}(t, \phi, St) = \phi j(\mathrm{R}(t, \phi, r), r) \end{cases}$$

(j a standard primitive recursive pairing function).

Definition 11. $\mathbf{PrAn^N} := [\mathbf{PrAn_1^N}]$.

We denote by **PrAn** the corresponding system of intuitionistic primitive recursive analysis [the description of **PrAn** is quite similar to that of $\mathbf{PrAn_1^N}$; $\mathbf{NA_1}$ is to be replaced by **HA**].

We shall use some standard abbreviations, both in $\mathbf{PrAn}_1^{\mathbf{N}}$ and $\mathbf{PrAn}$:

$t > r := \overline{\mathrm{sg}}(t \dot{-} r) = 0;$
$(\alpha)_a := \lambda y.\ \alpha j(a, y);$
$\gamma = (\alpha \mid \beta) := \forall x \exists z[S(\gamma(x)) = \alpha(<x> \ast \overline{\beta} z)].$

Below **FIM** is supposed to be axiomatized by adding to **PrAn** the following rules:

$$\mathrm{AC}_{01} \quad \frac{\forall x \exists \chi A(x, \chi)}{\exists \chi \forall x A(x, (\chi)_x)} \ ;$$

$$\mathrm{BI}_{\mathrm{D}} \quad \frac{\mathcal{B}_1 \quad \mathcal{B}_2 \quad \mathcal{B}_3}{\mathcal{B}} \ ,$$

where

$\mathcal{B}_1 \equiv \forall \chi \exists ! x A(\overline{\chi} x),$
$\mathcal{B}_2 \equiv \forall x (A(x) \to B(x)),$
$\mathcal{B}_3 \equiv \forall x (\forall y B(x \ast <y>) \to B(x)),$
$\mathcal{B} \ \equiv A(<>)$

(BI_{D} , decidable bar induction, corresponds to $^{\mathrm{x}}$26.3c of [16]);

$$\mathrm{AC}_{11} \quad \frac{\forall \chi \exists \tau A(\chi, \tau)}{\exists \tau [\tau \in \mathrm{K}_0 \,\&\, \forall \chi \exists \psi (\psi = (\tau \mid \psi) \& A(\chi, \psi))]} \ ,$$

where $\tau \in \mathrm{K}_0$ abbreviates

$\forall y \forall x [\tau(x) > 0 \to \tau(x \ast y) = \tau(x)] \,\&\, \forall \chi \exists x (\tau(\overline{\chi} x) > 0).$

Thus $\mathbf{FIM} := \mathbf{PrAn} + \mathrm{AC}_{01} + \mathrm{BI}_{\mathrm{D}} + \mathrm{AC}_{11}$, which is essentially a natural-deduction version of the system of intuitionistic analysis described in [16].

Corresponding to BI_{D} and AC_{11} we consider the following two rules w.r.t. the language of $\mathbf{PrAn}^{\mathbf{N}}$:

$$\mathrm{BI}_{\mathrm{D}}^{\mathrm{N}} \quad \frac{\mathcal{F}_1 \quad \mathcal{F}_2 \quad \mathcal{F}_3}{\mathcal{F}} \ ,$$

where

$\mathcal{F}_1 \equiv \forall \chi \exists ! x A(\overline{\chi} x),$
$\mathcal{F}_2 \equiv A(x) \to_x B(x),$

$\mathcal{F}_3 \equiv \forall y B(x * <y>) \rightarrow_x B(x)$,
$\mathcal{F} \ \equiv A(<>)$,

and where $\exists! x A(\overline{\chi} x) \equiv \exists x[A(\overline{\chi} y) \rightarrow_y (y = x)]$;

$$\mathrm{AC}^{\mathrm{N}}_{11} \quad \frac{\forall \chi \exists \tau A(\chi, \tau)}{\exists \tau[\tau \in \mathrm{K}^{\mathrm{N}}_0 \& \forall \chi \exists \psi(\psi = (\tau \mid \chi) \& A(\chi, \psi))]} ,$$

where $\tau \in \mathrm{K}^{\mathrm{N}}_0$ abbreviates

$\forall y[\tau(x) > 0 \rightarrow_x (\tau(x * y) = \tau(x))] \& \forall \chi \exists x(\tau(\overline{\chi} x) > 0)$.

Now we define $\mathbf{FIM}^{\mathbf{N}}_{\mathbf{1}}$ as $\mathbf{PrAn}^{\mathbf{N}}_{\mathbf{1}} + \mathrm{AC}_{01} + \mathrm{BI}^{\mathrm{N}}_{\mathrm{D}} + \mathrm{AC}^{\mathrm{N}}_{11}$ (AC_{01} w.r.t. the language of $\mathbf{PrAn}^{\mathbf{N}}$).

Definition 12. $\mathbf{FIM}^{\mathbf{N}} := [\mathbf{FIM}^{\mathbf{N}}_{\mathbf{1}}]$.

Similarly to Theorem 5 one can prove the following.

Theorem 7 (*syntactical non-nullity of* $\mathbf{PrAn}^{\mathbf{N}}$ *and* $\mathbf{FIM}^{\mathbf{N}}$). *Let* $\mathbf{H}$ *be either* $\mathbf{PrAn}^{\mathbf{N}}$ *or* $\mathbf{FIM}^{\mathbf{N}}$, *let* $<\Sigma_1, \Sigma_2>$ *be a derivation in* $\mathbf{H}$, *let* A *be any formula occurrence in* Σ_1 *or* Σ_2, *and let* C *be any well-formed part of* A. *Then* $\mathbf{H} \vdash \exists C$.

We transfer literally Definitions 7–10 to the formulas in $\mathbf{PrAn}$ and $\mathbf{PrAn}^{\mathbf{N}}$, except that now [in clause (d) of Definition 7] $\bar{a}$ is treated as a sequence of arbitrary parameters and $\bar{x}$ as a sequence of variables of appropriate sorts.

Theorem 8 (*the relationship between* $\mathbf{FIM}$ *and* $\mathbf{FIM}^{\mathbf{N}}$). (I). *For any formula* A *in* $\mathbf{FIM}$:

(1) $\mathbf{FIM} \vdash A \leftrightarrow (A^+)^*$;
(2) $\mathbf{FIM} \vdash A$ *if and only if* $\mathbf{FIM}^{\mathbf{N}} \vdash A^+$.

(II). *For any formula* A *in* $\mathbf{FIM}^{\mathbf{N}}$:

(3) A *and* $(A^*)^+$ *are* $\mathbf{FIM}^{\mathbf{N}}_{\mathbf{1}}$ *-equivalent*;
(4) $\mathbf{FIM}^{\mathbf{N}} \vdash A$ *if and only if* $\mathbf{FIM} \vdash A^*$.

Proof. First note that BI_{D} is equivalent (relative to $\mathbf{PrAn}$) to the schema

$$\mathrm{BI}^{\dagger}_{\mathrm{D}} \quad \frac{\mathcal{B}_1 \quad \mathcal{B}^{\dagger}_2 \quad \mathcal{B}^{\dagger}_3}{\mathcal{B}}$$

with $\mathcal{B}_1$ and $\mathcal{B}$ as in $\mathrm{BI_D}$,

$$\mathcal{B}_2^\dagger \equiv \exists x A(x) \,\&\, \forall x(A(x) \to B(x)),$$

$$\mathcal{B}_3^\dagger \equiv \exists x \forall y B(x * <y>) \,\&\, \forall x(\forall y B(x * <y>) \to B(x)).$$

Thus it suffices to show that the theorem holds with **PrAn** + $\mathrm{AC_{01}}$ + $\mathrm{BI_D^\dagger}$ + $\mathrm{AC_{11}}$ taking place of **FIM**. This is easily verified by a straightforward generalization of the proof of Theorem 6.

■

We conclude with two applications of Theorem 8, both rather of historical interest.

The following notations will be used (cf. [16, Ch. III]):

$t < r := r > t;$

$t \geq r := St > r;$

$\alpha \in \mathrm{R} := \forall x \exists y \forall z (2^x \cdot \mid 2^z \cdot \alpha(y) - \alpha(y+z) \mid < 2^{y+z})$
(α is a *Cauchy-sequence*);

$\alpha \,\#\, \beta := \exists x \exists y \forall z (2^x \cdot \mid \alpha(y+z) - \beta(y+z) \mid \geq 2^{y+z})$
(α *lies apart from* β);

$\alpha \overset{\circ}{=} \beta := \forall x \exists y \forall z (2^x \cdot \mid \alpha(y+z) - \beta(y+z) \mid < 2^{y+z})$
(α and β *coincide*);

$\alpha \not\equiv \beta := \exists x (\alpha(x) \,\#\, \beta(x)).$

A *virtual order relation* $<$ is defined by

$\alpha < \beta := \neg(\alpha \gtrdot \beta) \,\&\, \neg(\alpha \overset{\circ}{=} \beta)$, where

$\alpha \gtrdot \beta := \exists x \exists y \forall z (2^x \cdot (\alpha(y+z) \dot{-} \beta(y+z)) \geq 2^{y+z}).$

Application I: *Griss' theorem* (see [13, page 125]). This can be formulated as follows:

$$[\chi \in \mathrm{R} \,\&\, \psi \in \mathrm{R} \,\&\, (\tau \,\#\, \chi \to_\tau (\tau \,\#\, \psi))] \to_{\chi,\psi} (\chi \overset{\circ}{=} \psi).$$

The original proof of the theorem, as given by Griss in his [9], used an application of the fan theorem. Wiedmer [31] found another negationless proof, not dependent on the fan theorem, for a somewhat stronger version:

$$[\chi \in \mathrm{R} \,\&\, \psi \in \mathrm{R} \,\&\, (\tau \,\#\, \chi \to_\tau (\tau \not\equiv \psi))] \to_{\chi,\psi} (\chi \overset{\circ}{=} \psi). \qquad (***)$$

Now Theorem 8 also holds with **PrAn** [resp. $\mathbf{PrAn^N}$] taking place of **FIM** [resp. $\mathbf{FIM^N}$]. In view of this fact, one can give a quite simple metamathematical argument showing that the above version of the Griss theorem is derivable in $\mathbf{PrAn^N}$. For let A denote the formula (***); since

$$\mathbf{PrAn} \vdash \alpha \in \mathrm{R} \,\&\, \beta \in \mathrm{R} \,\&\, \neg(\alpha \# \beta) \to \alpha \mathrel{\overset{\circ}{=}} \beta$$

[this is exactly *R2.7. of [16, §14]], we have also $\mathbf{PrAn} \vdash A^*$; therefore, $\mathbf{PrAn^N} \vdash A$.

Application II: *Brouwer's examples of "essentially negative predicates"* (see [13, 8.1.]). Brouwer, in his paper [2], gave 'weak counterexamples' both to $\neg(\alpha \mathrel{\overset{\circ}{=}} \beta) \to \alpha \# \beta$ and $\alpha > \beta \to \alpha \gtrdot \beta$, "in order to estimate the significance of affirmative or negationless mathematics, the development of which is sometimes advocated".

Some comments on this should be made. [5] First, it is quite clear that neither of these counterexamples, as such, argues against negationless mathematics. Secondly, they cannot be interpreted as an argument showing that there exist "essentially negative predicates", that is, predicates not expressible negationlessly in principle. For such an interpretation involves a tacit assumption that $\alpha \# \beta$ [resp. $\alpha \gtrdot \beta$] is essentially the *only* tenable negationless analogue of $\neg(\alpha \mathrel{\overset{\circ}{=}} \beta)$ [resp. $\alpha > \beta$], while Theorem 8 suggests other negationless versions of $\neg(\alpha \mathrel{\overset{\circ}{=}} \beta)$ and $\alpha > \beta$, namely, $(\neg(\alpha \mathrel{\overset{\circ}{=}} \beta))^+$ and $(\alpha > \beta)^+$.

[5] These comments pertain equally to van Dantzig's "affirmative mathematics"[3].

References

1. L.E.J. Brouwer, *Richtlijnen der intuitionistische wiskunde*, **Indagationes Mathematicae 9** (1947), p. 197. English translation in: A. Heyting (ed.) L.E.J. Brouwer, *Collected Works, vol.1: Philosophy and Foundations of Mathematics*, North-Holland, Amsterdam, 1975, p. 477.

2. L.E.J. Brouwer, *Essentieel negatieve eigenschappen*, **Indagationes Mathematicae 10** (1948), pp. 322-323. English translation in: A. Heyting (ed.) L.E.J. Brouwer, *Collected Works, vol.1: Philosophy and Foundations of Mathematics*, North-Holland, Amsterdam, 1975, pp. 478-479.

3. D. van Dantzig, *On the principles of intuitionistic and affirmative mathematics*, **Indagationes Mathematicae 9** (1947), pp. 429-440, 506-517.

4. M.A.E. Dammett, *Elements of Intuitionism*, Clarendon Press, Oxford, 1977.

5. H. Freudenthal, *Zur intuitionistischen Deutung logischer Formeln*, **Compositio Mathematica 4** (1937), pp. 112-116.

6. P.C.G. Gilmore, *The effect of Griss' criticism of the intuitionistic logic on deductive theories formalized within the intuitionistic logic*, **Indagationes Mathematicae 15** (1953), pp. 162-174, 175-186.

7. G.F.C. Griss, *Negationless intuitionistic mathematics*, **Indagationes Mathematicae 8** (1946), pp. 675-681.

8. G.F.C. Griss, *Negationless intuitionistic mathematics II*, **Indagationes Mathematicae 12** (1950), pp. 108-115.

9. G.F.C. Griss, *Negationless intuitionistic mathematics III*, **Indagationes Mathematicae 13** (1951), pp. 193-199.

10. G.F.C. Griss, *Negationless intuitionistic mathematics IVa, IVb*, **Indagationes Mathematicae 13** (1951), pp. 452-462, 463-471.

11. A. Heyting, *Bemerkunden zu dem Aufsatz von Herrn Freudenthal "Zur intuitionistischen Deutung logischer Formeln"*, **Compositio Mathematica 4** (1937), pp. 117-118.

12. A. Heyting, *G.F.C. Griss and his negationless intuitionistic mathematics*, **Synthese 9** (1950-1951), pp. 91-96.

13. A. Heyting, *Intuitionism, an Introduction*, 3rd rev. ed., North-Holland, Amsterdam, 1971.

14. J.J. de Jongh, *Restricted forms of intuitionistic mathematics.* In: *Proceedings of the Xth International Congress of Philosophy*, August 11-18, Amsterdam, 1948, pp. 744-748, North-Holland, Amsterdam, 1949.

15. S.C. Kleene, *Introduction to Metamathematics*, North-Holland, Amsterdam, 1952.

16. S.C. Kleene and R.E. Vesley, *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions*, North-Holland, Amsterdam, 1965.

17. G. Kreisel, *Elementary completeness properties of intuitionistic logic with a note on negations of prenex formulae*, **The Journal of Symbolic Logic 23** (1958), pp. 317-330.

18. V.N. Krivtsov, *Axiomatic systems of negationless mathematics* (Russian), Ph.D. thesis, Moscow State University, Moscow, 1984.

19. E.J.K. López-Escobar, *Constructions and negationless logic*, **Studia Logica 30** (1972), pp. 7-19.

20. E.J.K. López-Escobar, *Elementary interpretations of negationless arithmetic*, **Fundamenta Mathematica 82** (1974), pp. 25-38.

21. J.K. Minichiello, *Negationless intuitionistic mathematics*, Dissertation, The George Washington University, Washington, D.C., 1968.

22. J.K. Minichiello, *An extension of negationless logic*, **Notre Dame Journal of Formal Logic 10** (1969), pp. 298-302.

23. D. Nelson, *Reviews of* [6] *and* [30], **The Journal of Symbolic Logic 21** (1956), pp. 91-93.

24. D. Nelson, *Non-null implication*, **The Journal of Symbolic Logic 31**, (1966), pp. 562-572.

25. D. Nelson, *A complete negationless system*, **Studia Logica 32** (1973), pp. 41-49.

26. D. Prawitz, *Natural Deduction. A Proof-theoretical Study*, Almquist & Wiksell, Stockholm Göteborg Uppsala, 1965.

27. D. Prawitz, *Ideas and results in proof theory.* In: J.K. Fenstad (ed.), *Proceedings of the Second Scandinavian Logic Symposium*, North-Holland, Amsterdam, 1971, pp. 237-309.

28. A.S. Troelstra, *Choice Sequences, a Chapter of Intuitionistic Mathematics*, Oxford, Clarendon Press, 1977.

29. V. Valpola, *Ein System der negationslosen Logik mit ausschliesslich realisierbaren Prädicaten*, **Acta Philosophica Fennica 9** (1955), pp. 1-247.

30. P.G.J. Vredenduin, *The logic of negationless mathematics*, **Compositio Mathematica 11** (1953), pp. 204-277.

31. E. Wiedmer, *Ein neur negationsloser Beweis eines Satzes von G.F.C. Griss*, **Indagationes Mathematicae 36** (1974), pp. 89-93.