A Negationless Interpretation of Intuitionistic Axiomatic Theories: Higher-order Arithmetic

Victor N. Krivtsov*

Abstract

This work is a sequel to our [14]. It is shown how Theorem 6 of [14], dealing with the translatability of **HA** (Heyting's arithmetic) into negationless arithmetic **NA**, can be extended to the case of intuitionistic arithmetic in higher types.

Introduction

The main purpose of the present paper is to extend Theorem 6 of [14], asserting the translatability of **HA** into negationless arithmetic **NA**, to the case of intuitionistic higher-order arithmetic **HAH** and a formalization (denoted by **NAH**) of negationless higher-order arithmetic to be described in Section 1.

In Section 2 we consider the connection of **NAH** with a version of intuitionistic higher-order arithmetic, the system $\overline{\mathbf{HAH}}$. An extension of the translatability result to these systems turns out easy to perform, since the only essential distinction between $\overline{\mathbf{HAH}}$ and **NAH** is that the former is based on **HA** (instead of **NA**).

In Section 3 we study the relationship between $\overline{\mathbf{HAH}}$ and **HAH** and then prove the main result on the translatability of **HAH** into **NAH** (Theorem 2). The idea underlying the proposed interpretation of **HAH** in $\overline{\mathbf{HAH}}$ is similar to that sketched by Beth in [1, page 439], see also [5, page 185].

An important point to note is that Theorem 2 can be extended, in the obvious way, to many other intuitionistic and constructive systems based

*Department of Mathematical Logic, Faculty of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia. (e-mail address: `victor@lpcs.math.msu.ru`)

on **HA** (and their negationless counterparts); for instance, this is true for intuitionistic second-order arithmetic, systems of intuitionistic analysis with species variables (see, e. g., [15]), and various extensions of **HAH**, including the formalization of Bishop's constructive mathematics [2], as given in [6].

Finally, in Addendum, we show that our formalization of negationless predicate logic, the system **NPC**, is equivalent to a modification of the $\Sigma-$free fragment of Nelson's sequent calculus $\mathbf{P_1}$ [17]. This implies that all the results of this paper, as well as those of [14], can be restated for the corresponding systems formulated as sequent calculi.

Throughout the paper $:=$ stands for the definition symbol and $\equiv$ for syntactical identity of expressions. Given a formal system $\underline{S}$, we use $Fm(\underline{S})$ to denote the set of all formulas in $\underline{S}$. For the notations not explained here and below we refer to [14].

1 Negationless higher-order arithmetic NAH

We first describe the underlying system for **NAH**, the system $\mathbf{NAH_1}$.

Types are defined inductively as follows:

(a) $\mathbf{0}$ is a type (the type of the natural numbers);

(b) if $\sigma_1, \ldots, \sigma_n$ are types, then $\sigma_1 \times \ldots \times \sigma_n$ is a type (the product type of $\sigma_1, \ldots, \sigma_n$);

(c) if σ is a type, then so is $[\sigma]$, the power type of σ.

The letters $\tau, \sigma, \sigma_1, \sigma_2, \ldots$ will be used to denote types.

The *height* of a type σ, in symbols $h(\sigma)$, is defined inductively by:

$h(\mathbf{0}) := 0$;

$h(\sigma_1 \times \ldots \times \sigma_n) := \max\,\{h(\sigma_i) : 1 \leq i \leq n\} + 1$;

$h([\sigma]) := h(\sigma) + 1$.

The language of $\mathbf{NAH_1}$ contains, in addition to the symbols of the language of negationless arithmetic **NA**:

(a) Countably many (bound) variables $x_0^\sigma, x_1^\sigma, x_2^\sigma, \ldots$ of each type $\sigma \neq \mathbf{0}$;

(b) countably many parameters $a_0^\sigma, a_1^\sigma, a_2^\sigma, \ldots$ of each type $\sigma \neq \mathbf{0}$;

(c) for each type σ, a binary predicate symbol $\in_\sigma$.

Variables [resp. parameters] for natural numbers are now treated as variables [resp. parameters] of type $\mathbf{0}$. Intuitively, variables of type $[\sigma]$ range over *inhabited* subspecies of Sp_σ, where Sp_σ is the species of objects of type σ.

We use x, y, and z with or without type-superscripts as metavariables for the variables and a, b, c, d, and e for the parameters.

Terms (of given types) are defined inductively as follows:

(a) Every term of **HA** is a term of type $\mathbf{0}$;

(b) every parameter a^σ $(\sigma \neq \mathbf{0})$ is a term of type σ;

(c) if $t_1^{\sigma_1}, \ldots, t_n^{\sigma_n}$ are terms, then $(t_1, \ldots, t_n)$ is a term of type $\sigma_1 \times \ldots \times \sigma_n$;

(d) if t is a term of type $\sigma_1 \times \ldots \times \sigma_n$, then $p_i^n t$ is a term of type σ_i, for $1 \leq i \leq n$.

Henceforth $r, t, t_1, t_2, \ldots$ (with or without type-superscripts) will be metavariables for terms.

Formulas are constructed from atomic formulas of the form $t = r$, $t \# r$ (t and r are of type $\mathbf{0}$), and $t^\sigma \in_\sigma r^{[\sigma]}$ by means of $\&, \forall, \exists$, and $\rightarrow_{\bar{x}}$ (the quantified implication, see [14, Section 1]).

We use the letters A, B, C, etc. to denote formulas. $A(\bar{x}) \leftrightarrow_{\bar{x}} B(\bar{x})$ is treated as an abbreviation for $(A(\bar{x}) \rightarrow_{\bar{x}} B(\bar{x})) \,\&\, (B(\bar{x}) \rightarrow_{\bar{x}} A(\bar{x}))$.

Later on we will write $\in$ instead of $\in_\sigma$, as the subscript is always clear from the context.

Equality for power types and product types is defined by:

$$t_1^{[\sigma]} = t_2^{[\sigma]} := (x \in t_1) \leftrightarrow_x (x \in t_2),$$

$$t_1^{\sigma_1 \times \ldots \times \sigma_n} = t_2^{\sigma_1 \times \ldots \times \sigma_n} := \&\{p_i^n t_1 = p_i^n t_2 : 1 \leq i \leq n\}.$$

Definition 1. The system $\mathbf{NAH_1}$ is based on many-sorted version of $\mathbf{NPC_1}$, see [14], together with the following axioms and inference rules:

(i) The (non-logical) axioms and rules of $\mathbf{NA_1}$ with the rule of induction Ind for all formulas in $\mathbf{NAH_1}$;

(ii) $p_i^n(t_0, \ldots, t_{n-1}) = t_i$, for $i < n$;

(iii) extensionality rules

$$\text{EXT}_\sigma \quad \frac{t_1^\sigma = t_2^\sigma \quad t_1^\sigma \in r^{[\sigma]}}{t_2^\sigma \in r^{[\sigma]}} \;;$$

(iv) comprehension rules

$$\text{CR}_\sigma^{\text{N}} \quad \frac{\exists y^\sigma A(y)}{\exists x^{[\sigma]}(y \in x \leftrightarrow_y A(y))} \; (x \text{ does not occur in } A(y)\,);$$

(v) non-nullity axioms

$$\text{G}_\sigma \quad \forall x^{[\sigma]} \exists y^\sigma (y \in x).$$

This completes the description of $\mathbf{NAH_1}$.

Observe that the rule

$$\frac{t_1 = t_2 \quad A(t_1)}{A(t_2)}$$

is derivable in $\mathbf{NAH_1}$; this is easily proved by induction on A.

Remark 1. Recall that the basic tenet of Griss' point of view consists in that only (hereditary) inhabited species are admissible in intuitionistic mathematics [11]. This requirement is reflected in the comprehension rules $\text{CR}_\sigma^{\text{N}}$ and the axioms G_σ.

Definition 2. $\mathbf{NAH} := [\mathbf{NAH_1}]$.

Theorem 1 (*syntactical non-nullity of* $\mathbf{NAH}$). *Let* $<\Sigma_1, \Sigma_2>$ *be a derivation in* $\mathbf{NAH}$*, let* A *be any formula occurrence in either* Σ_1 *or* Σ_2*, and let* C *be any well-formed part of* A*. Then* $\mathbf{NAH} \vdash \exists C$.

Proof. Quite similar to that of Theorem 1 in [14].
■

In what follows we use **HAH** to denote a natural-deduction version of intuitionistic higher-order arithmetic with product types, power types, and a type of natural numbers satisfying Heyting' arithmetic **HA** with full induction, see [20, pages 168, 169]. The extensionality rules of **HAH** are EXT_σ (in the language of **HAH**); the comprehensions axioms CA_σ are of the form:

$$\exists x^{[\sigma]} \forall y^\sigma (y \in x \leftrightarrow A(y))$$

(x does not occur in $A(y)$).

Definition 3. We denote by $\overline{\mathbf{HAH}}$ the system obtained from **HAH** by adding the axioms G_σ and replacing CA_σ by the following comprehension rules CR_σ :

$$\frac{\exists y^\sigma A(y)}{\exists x^{[\sigma]}(y \in x \leftrightarrow A(y))} \ (x \text{ does not occur in } A(y)\,).$$

For convenience, we assume that $\perp$, $\neg A$, $A \leftrightarrow B$, and $A \vee B$ are abbreviations, both in **HAH** and $\overline{\mathbf{HAH}}$, for $S0 = 0$, $A \to \perp$, $(A \to B)\,\&\,B \to A)$, and $\exists x[(x = 0 \to A)\&(\neg(x = 0) \to B)]$, respectively.

2 The relationship between NAH and $\overline{\mathbf{HAH}}$

In this section we will prove an extension of Theorem 6 of [14] to the case of the systems **NAH** and $\overline{\mathbf{HAH}}$.

We first define a translation * (cf. [14, Definition 7]) which associates with every formula A in **NAH** a formula A^* in $\overline{\mathbf{HAH}}$.

Definition 4.

(a) $(t = r)^* \equiv t = r$, for any terms t and r of type $\mathbf{0}$;

(b) $(t \,\#\, r)^* \equiv (\,\#\, t, r) = 0$, for any terms t and r of type $\mathbf{0}$;

(c) $(t \in r)^* \equiv t \in r$;

(d) * commutes with $\&, \forall$, and $\exists$;

(e) $(A(\bar{a}/\bar{x}) \to_{\bar{x}} B(\bar{a}/\bar{x}))^* \equiv \exists \bar{x} A^*(\bar{a}/\bar{x})\,\&\,\forall \bar{x}(A^* \to B^*)(\bar{a}/\bar{x})$;

(f) if Γ is a set of formulas in **NAH**, then $\Gamma^* := \{A^* : A \in \Gamma\}$.

For any formula A in either $\overline{\mathbf{HAH}}$ or **NAH** we let:
$v_1(A) :=$ max $\{h(\sigma) : \sigma$ is a type of a semi-term occurring in $A\}$;
$v_2(A) :=$ the number of the logical constants in A.
[A *semi-term* is an expression like a term, except that it may contain occurrences of (bound) variables, see, e.g., [18].]

Definition 5 (cf. [14, Definition 8]). For any formula A in $\overline{\mathbf{HAH}}$ [resp. **NAH**] and any parameter a of type $\mathbf{0}$ not occurring in A, we define a formula A_a in $\overline{\mathbf{HAH}}$ [resp. **NAH**] by induction on lexicographical ordering of the pairs of the form $< v_1(B),\, v_2(B) >$, where B is a formula in $\overline{\mathbf{HAH}}$ [resp. **NAH**], as follows:

(a) $(t = r)_a \equiv (= t, r) \cdot \mathrm{sg}(a) = 0$, for any terms t and r of type $\mathbf{0}$;

(b) $(t \,\#\, r)_a \equiv (\,\# \,t, r) \cdot \mathrm{sg}(a) = 0$ (in the case of $\mathbf{NAH}$), for any terms t and r of type $\mathbf{0}$;

(c) $(t^\sigma \in r^{[\sigma]})_a \equiv \exists x((b = t)_a \,\&\, b \in r)(b/x)$,

where $(b = t) \in Fm(\overline{\mathbf{HAH}})$ [resp. $Fm(\mathbf{NAH})$], b does not occur in $(t \in r)_a$, and x is the first variable of type σ not occurring in $(b = t)_a$ or A;

(d) the mapping associating A_a with A commutes with each logical constant.

Definition 6 (cf. [14, Definition 9]). Let A be a formula in $\overline{\mathbf{HAH}}$, and let a be any parameter of type $\mathbf{0}$ which does not occur in A. We associate with A a formula A_a^+ in $\mathbf{NAH}$ demanding the following:

(a) $(t = r)_a^+ \equiv (t = r)_a$, for any terms t and r of type $\mathbf{0}$;

(b) $(t^\sigma \in r^{[\sigma]})_a^+ \equiv \exists x((b = t)_a \,\&\, b \in r)(b/x)$,

where $(b = t) \in Fm(\mathbf{NAH})$, b does not occur in $(t \in r)_a$, and x is the first variable of type σ not occurring in either $(b = t)_a$ or A;

(c) the mapping ${}_a^+$ commutes with $\&, \forall$, and $\exists$;

(d) $(B \to C)_a^+ \equiv (B_a^+)_b(b/x) \to_x (C_a^+)_b(b/x)$,

where b does not occur in B_a^+ or C_a^+, and x is the first variable of type $\mathbf{0}$ not occurring in either B_a^+, C_a^+, or A.

If Γ is a set of formulas in $\overline{\mathbf{HAH}}$, and a is a parameter of type $\mathbf{0}$ not occurring in the formulas in Γ, then $\Gamma_a^+ := \{A_a^+ : A \in \Gamma\}$.

Definition 7 (cf. [14, Definition 10]). For any formula A in $\overline{\mathbf{HAH}}$ and any parameter a of type $\mathbf{0}$ not occurring in A, we put $A^+ := A_a^+(a/S0)$.

Lemma 1 (cf. [14, Lemma 7]). *Let A be a formula in $\mathbf{NAH}$ and let b be a parameter of type $\mathbf{0}$ not occurring in A. Then*

(i) *A and $A_b(b/S0)$ are $\mathbf{NAH_1}$ -equivalent;*
(ii) *$A_b(b/S0)$ and $A_b(b/Sa)$ are $\mathbf{NAH_1}$ -equivalent, for any parameter a of type $\mathbf{0}$ not occurring in A;*
(iii) $\mathbf{NAH} \vdash A_b(b/0)$;
(iv) *A and $\forall x A_b(b/x)$ are $\mathbf{NAH_1}$ -equivalent;*

(v) $\overline{\mathbf{HAH}} \vdash (A^*)_b \leftrightarrow (A_b)^*$.

Proof. (i) – (iii), (v): By induction on lexicographical ordering of the pairs $< v_1(B), v_2(B) >$, where B is a formula in $\mathbf{NAH}$, using Lemma 7 of [14].

(iv): By (i)–(iii).

■

Lemma 2 (cf. [14, Lemma 8]). *Let A be a formula in $\overline{\mathbf{HAH}}$, and let b be a parameter of type $\mathbf{0}$ not occurring in A. Then*

(i) $\overline{\mathbf{HAH}} \vdash A \leftrightarrow A_b(b/S0)$;
(ii) $\mathbf{NAH} \vdash A_b^+(b/0)$.

Proof . (i): By induction on lexicographical ordering of the pairs $< v_1(B), v_2(B) >$, where B is a formula in $\overline{\mathbf{HAH}}$.

(ii): By induction on A, using (iii) and (iv) of Lemma 1.

■

Lemma 3 (cf. [14, Lemma 9]). *For any formula A and any set of formulas Γ in $\mathbf{NAH_1}$: If A is derivable in $\mathbf{NAH_1}$ from Γ, then A^* is derivable in $\overline{\mathbf{HAH}}$ from Γ^*.*

Proof. Straightforwardly by induction on derivations in $\mathbf{NAH_1}$.

■

Lemma 4 (cf. [14, Lemma 10]). *For any formula A in $\overline{\mathbf{HAH}}$, $\overline{\mathbf{HAH}} \vdash A \leftrightarrow (A^+)^*$.*

Proof. By induction on A, using Lemmas 1–3.

■

Lemma 5 (cf. [14, Lemma 11]). *For any formula A in $\mathbf{NAH}$, A and $(A^*)^+$ are $\mathbf{NAH_1}$ -equivalent.*

Proof. Quite similar to that of Lemma 11 in [14], using (i) of Lemma 1.

■

Lemma 6.

(i) *For any terms t^σ and $r^{[\sigma]}$ and any parameter a of type $\mathbf{0}$ occurring in neither t nor r:*

$$(t \in r)_b^+(b/Sa) \text{ and } t \in r \text{ are } \mathbf{NAH_1} \text{ -equivalent.}$$

(ii) *For any terms* t^σ *and* r^σ *and any parameter* a *of type* $\mathbf{0}$ *occurring in neither* t *nor* r:

$$(t = r)^+_b(b/Sa) \textit{ and } t = r \textit{ are } \mathbf{NAH_1} \textit{-equivalent.}$$

Proof. (i): By (i) and (ii) of Lemma 1.

(ii): This is easily verified by induction on $h(\sigma)$, using (i) of Lemma 1.

■

Lemma 7 (cf. [14, Lemma 12]). *Let* A *be a formula in* $\overline{\mathbf{HAH}}$ *, let* Γ *be a finite, possibly empty set of formulas in* $\overline{\mathbf{HAH}}$ *, and let* a *be a parameter of type* $\mathbf{0}$ *not occurring in* A *or the formulas in* Γ. *If* A *is derivable in* $\overline{\mathbf{HAH}}$ *from* Γ *, then* A^+_a *is derivable in* $\mathbf{NAH_1}$ *from* Γ^+_a.

Proof. By induction on derivations in $\overline{\mathbf{HAH}}$. We consider only the induction steps corresponding to applications of EXT_σ and CR_σ (cf. the proof of Lemma 12 in [14]). Throughout the proof "$\vdash$" means the derivability relation in $\mathbf{NAH_1}$.

Let Σ be a derivation in $\overline{\mathbf{HAH}}$.

Case 1: Σ ends with an application of EXT_σ. Clearly it suffices to show that

$$\vdash [(x^\sigma = y^\sigma)^+_a \,\&\, (x \in z^{[\sigma]})^+_a] \to_{x,y,z} (y \in z)^+_a.$$

This is easily proved by applying the rule of induction IND w.r.t. a, using Lemmas 2(ii) and 6.

Case 2: Σ ends with an application of CR_σ.

Suppose $\Gamma^+_a \vdash \exists y^\sigma (A(y))^+_a$. We have to show that

$$\exists x^{[\sigma]} \forall y (y \in x \leftrightarrow A(y))^+_a \tag{1}$$

is derivable in $\mathbf{NAH_1}$ from Γ^+_a.

First observe that, in view of Definition 6, the formula (1) has the form

$$\exists x^{[\sigma]} \forall y (B_1(x, y) \,\&\, B_2(x, y)),$$

where

$B_1(x, y) \equiv ((y \in x)^+_a)_c(c/z) \to_z (A^+_a(y))_c(c/z)$

and

$B_2(x, y) \equiv (A^+_a(y))_c(c/z) \to_z ((y \in x)^+_a)_c(c/z).$

Now, since $\exists y^{\sigma}(A(y))^{+}_{a}$ is derivable in $\mathbf{NAH_1}$ from Γ^{+}_{a}, we get, by the induction hypothesis and $\mathrm{CR}^{\mathrm{N}}_{\sigma}$,

$$\Gamma^{+}_{a} \vdash \exists z^{[\sigma]}(y \in z \leftrightarrow_{y} A^{+}_{a}(y)).$$

Next assume Γ^{+}_{a} and suppose that for some $b^{[\sigma]}$, not occurring in $A^{+}_{a}(y)$ or the formulas in Γ^{+}_{a},

$$y \in b \leftrightarrow_{y} A^{+}_{a}(y). \tag{2}$$

Let $B(b)$ denote the formula (2). Then it suffices to prove that:

(A). $B(b) \vdash \forall y B_1(b, y)$;

(B). $B(b) \vdash \forall y B_2(b, y)$.

Proof of (A). We first show that (A) is a consequence of

$$\{B(b), (d \in b)^{+}_{a}\} \vdash A^{+}_{a}(d), \tag{3}$$

where it is assumed that d does not occur in $B(b)$ or $A^{+}_{a}(y)$. Indeed, since $B(b) \vdash \exists y(y \in b)$, we get, by (iv) of Lemma 1, $B(b) \vdash \exists y \exists x((x = y)_a \,\&\, x \in b)$, that is, $B(b) \vdash \exists y(y \in b)^{+}_{a}$. Thus (3) implies $B(b) \vdash (y \in b)^{+}_{a} \rightarrow_{y} A^{+}_{a}(y)$; hence, by (iv) of Lemma 1, $B(b) \vdash \forall x[((y \in b)^{+}_{a})_x \rightarrow_{y} (A^{+}_{a}(y))_x]$. This yields

$$B(b) \vdash \forall y B_1(b, y),$$

because, by (iii) of Lemma 1, $\vdash \forall y \exists z((y \in b)^{+}_{a})_z$.

Further, to prove (3), it suffices to show that

$$\{B(b), (e = d)_a \,\&\, e \in b\} \vdash A^{+}_{a}(d)$$

[provided e does not occur in $B(b)$ or $A^{+}_{a}(d)$] and thus, in view of the definition of $B(b)$, that

$$\{A^{+}_{a}(e), (e = d)_a\} \vdash A^{+}_{a}(d). \tag{4}$$

Before continuing, we let $[t, r] := (= \mathrm{sg}(t), S0) \cdot (= t, r)$, for any terms t and r of type $\mathbf{0}$. Note that, by Lemmas 5 and 6 of [14], we have:

(a) $\vdash \forall x \forall y([Sy, x] = 0)$;

(b) $\vdash \forall x([x, x] = 0)$;

(c) $\vdash [0, x] = 0 \leftrightarrow_{x} (x = 0)$.

To prove (4), it is now sufficient to show that

$$\vdash [(y = d)_z \,\&\, [z, x] = 0 \,\&\, b = z \,\&\, A_x^+(y)] \rightarrow_{x,y,z} A_x^+(d),$$

which is done by applying the rule of induction IND w.r.t. b. Basis: By (iii) of Lemma 1, (ii) of Lemma 2, (b), and (c). Induction step: By (i) and (ii) of Lemma 1, (ii) of Lemma 2, and (a).

■

Proof of (B). By (iv) of Lemma 1 and the definition of $B(b)$ we have

$$\{B(b), A_a^+(d)\} \vdash (d \in b)_a^+.$$

Since $B(b) \vdash \exists y A_a^+(y)$, we also have

$$B(b) \vdash A_a^+(y) \rightarrow_y (y \in b)_a^+,$$

so that, by (iv) of Lemma 1,

$$B(b) \vdash \forall x[(A_a^+(y))_x \rightarrow_y ((y \in b)_a^+)_x].$$

Hence

$$B(b) \vdash \forall y[(A_a^+(y))_c(c/z) \rightarrow_z ((y \in b)_a^+)_c(c/z)],$$

because from (iii) of Lemma 1 it follows that $B(b) \vdash \forall y \exists z (A_a^+(y))_z$. Thus $B(b) \vdash \forall y B_2(b, y)$.

■

This completes the proof of Lemma 7.

■ ■

Lemma 8 (*the relationship between* **NAH** *and* $\overline{\mathbf{HAH}}$). (I). *For any formula A in* $\overline{\mathbf{HAH}}$:

(1) $\overline{\mathbf{HAH}} \vdash A \leftrightarrow (A^+)^*$;
(2) $\overline{\mathbf{HAH}} \vdash A$ *if and only if* $\mathbf{NAH} \vdash A^+$.

(2). *For any formula A in* **NAH**:

(3) *A and $(A^*)^+$ are* $\mathbf{NAH_1}$*-equivalent* ;
(4) $\mathbf{NAH} \vdash A$ *if and only if* $\overline{\mathbf{HAH}} \vdash A^*$.

Proof.

(1). Lemma 4.

(3). Lemma 5.

(4). From left to right: Lemma 3. From right to left: By Lemma 7 and (3).

(2). From left to right: By Lemma 7. From right to left: By (4) and (1).

■

3 The relationship between $\overline{\mathbf{HAH}}$ and HAH

To clear up the relationship between $\overline{\mathbf{HAH}}$ and **HAH**, we will consider two systems $\overline{\mathbf{H}}$ and **H** which are obtained from $\overline{\mathbf{HAH}}$ and **HAH** respectively by standard procedures of the addition of new symbols for definable functions (in particular, individual constants) and defined sorts of variables, see [19, §2].

We first denote by $\overline{\mathcal{H}}$ [resp. $\mathcal{H}$] the system which is the result of successive addition to $\overline{\mathbf{HAH}}$ [resp. **HAH**] of function symbols S_σ, individual constants C_σ, and function symbols G_σ (each of them for any type σ), together with the following axioms:

(1a) $\forall x^{\mathbf{0}}(S_{\mathbf{0}}(x) = Sx)$,

(1b) $\forall x^{\sigma_1 \times \ldots \times \sigma_n}[S_{\sigma_1 \times \ldots \times \sigma_n}(x) = (S_{\sigma_1}(p_1^n x), \ldots, S_{\sigma_n}(p_n^n x))]$,

(1c) $\forall x^{[\sigma]} \forall y^\sigma [y \in S_{[\sigma]}(x) \leftrightarrow \exists z^\sigma (y = S_\sigma(z) \,\&\, z \in x)]$;

(2a) $C_{\mathbf{0}} = 0$,

(2b) $C_{\sigma_1 \times \ldots \times \sigma_n} = (C_{\sigma_1}, \ldots, C_{\sigma_n})$,

(2c) $\forall x^\sigma (x \in C_{[\sigma]} \leftrightarrow x = S_\sigma(C_\sigma))$;

(3a) $\forall x^{\mathbf{0}}(G_{\mathbf{0}}(x) = Sx)$,

(3b) $\forall x^{\sigma_1 \times \ldots \times \sigma_n}[G_{\sigma_1 \times \ldots \times \sigma_n}(x) = (G_{\sigma_1}(p_1^n x), \ldots, G_{\sigma_n}(p_n^n x))]$,

(3c) $\forall x^{[\sigma]} \forall y^\sigma [y \in G_{[\sigma]}(x) \leftrightarrow (y = C_\sigma \vee \exists z (y = G_\sigma(z) \,\&\, z \in x))]$.

We now define, for each type σ, the formulas D_1^σ [resp. D_2^σ] in the language of $\overline{\mathcal{H}}$ [resp. $\mathcal{H}$] by letting:

$$D_1^{\mathbf{0}}(a^{\mathbf{0}}) := \exists y (a = Sy),$$

$$D_1^{\sigma_1\times\ldots\times\sigma_n}(a^{\sigma_1\times\ldots\times\sigma_n}) := D_1^{\sigma_1}(p_1^n a)\,\&\ldots\&\,D_1^{\sigma_n}(p_n^n a),$$

$$D_1^{[\sigma]}(a^{[\sigma]}) := C_\sigma \in a\,\&\,\forall x(x \in a \to (x = C_\sigma \vee D_1^\sigma(x))).$$

$$D_2^{\mathbf{0}}(a^{\mathbf{0}}) := a = a,$$

$$D_2^{\sigma_1\times\ldots\times\sigma_n}(a^{\sigma_1\times\ldots\times\sigma_n}) := D_2^{\sigma_1}(p_1^n a)\,\&\ldots\&D_2^{\sigma_n}(p_n^n a),$$

$$D_2^{[\sigma]}(a^{[\sigma]}) := \exists x(x \in a)\,\&\,\forall z(z \in a \to D_2^\sigma(z)).$$

It is easy to show (by induction on $h(\sigma)$) that $\overline{\mathcal{H}} \vdash \exists x^\sigma D_1^\sigma(x)$ and $\mathcal{H} \vdash \exists x^\sigma D_2^\sigma(x)$, for each type σ.

Let $\overline{\mathbf{H}}$ be obtained from $\overline{\mathcal{H}}$ by adding, for each type $\sigma \neq \mathbf{0}$, new variables $\tilde{x}_0^\sigma, \tilde{x}_1^\sigma, \tilde{x}_2^\sigma, \ldots$, parameters $\tilde{a}_0^\sigma, \tilde{a}_1^\sigma, \tilde{a}_2^\sigma, \ldots$, and the following axioms and inference rules:

(a) $D_1^\sigma(\tilde{a})$, (d) $\forall \tilde{x} D_1^\sigma(\tilde{x})$,

(b) $\dfrac{A(\tilde{a})}{\forall \tilde{x} A(\tilde{x})}$, (e) $\dfrac{D_1^\sigma(t) \quad \forall \tilde{x} A(\tilde{x})}{A(t)}$,

(c) $\dfrac{D_1^\sigma(t) \quad A(t)}{\exists \tilde{x} A(\tilde{x})}$, (f) $\dfrac{\exists \tilde{x} A(\tilde{x}) \quad \begin{matrix}[A(\tilde{a})]\\ C\end{matrix}}{C}$.

It is understood that the standard conventions about the proper parameter $\tilde{a}$ in (b) and (f) are observed.

Finally, we let $\mathbf{H}$ be the system obtained from $\mathcal{H}$ by the addition, for each type $\sigma \neq \mathbf{0}$, of new variables $\dot{x}_0^\sigma, \dot{x}_1^\sigma, \dot{x}_2^\sigma, \ldots$, parameters $\dot{a}_0^\sigma, \dot{a}_1^\sigma, \dot{a}_2^\sigma, \ldots$, and the axioms and inference rules (a') – (f') which are the result of replacing $D_1, \tilde{x}$, and $\tilde{a}$ in (a) – (f) by $D_2, \dot{x}$, and $\dot{a}$, respectively.
This completes the description of $\overline{\mathbf{H}}$ and $\mathbf{H}$.

¿From now on we use $\tilde{x}^\sigma, \tilde{y}^\sigma$, and $\tilde{z}^\sigma$ (resp. $\tilde{a}^\sigma, \tilde{b}^\sigma$, and $\tilde{c}^\sigma$) as metavariables for the variables $\tilde{x}_0^\sigma, \tilde{x}_1^\sigma, \tilde{x}_2^\sigma, \ldots$ (resp. the parameters $\tilde{a}_0^\sigma, \tilde{a}_1^\sigma, \tilde{a}_2^\sigma, \ldots$) and $\dot{x}^\sigma, \dot{y}^\sigma$, and $\dot{z}^\sigma$ (resp. $\dot{a}^\sigma, \dot{b}^\sigma$, and $\dot{c}^\sigma$) for the variables $\dot{x}_0^\sigma, \dot{x}_1^\sigma, \dot{x}_2^\sigma, \ldots$ (resp. the parameters $\dot{a}_0^\sigma, \dot{a}_1^\sigma, \dot{a}_2^\sigma, \ldots$).

Note that $\mathcal{H}$ is a definitional extension of **HAH**, and **H** is an expansion of $\mathcal{H}$. Thus there exists a standard mapping ϕ_1 from the formulas in $Fm(\mathbf{H})$, not containing the parameters $\dot{a}_0^\sigma, \dot{a}_1^\sigma, \dot{a}_2^\sigma, \ldots$, to $Fm(\mathbf{HAH})$ such that the following hold:

(i) $\mathbf{H} \vdash \phi_1(A) \leftrightarrow A$,

(ii) $\mathbf{H} \vdash A$ iff $\mathbf{HAH} \vdash \phi_1(A)$,

(iii) $\phi_1(A) \equiv A$ for $A \in Fm(\mathbf{HAH})$.

Similarly, since $\overline{\mathcal{H}}$ is a definitional extension of $\overline{\mathbf{HAH}}$, and $\overline{\mathbf{H}}$ is an expansion of $\overline{\mathbf{HAH}}$, there exists also a standard mapping ϕ_2 to $Fm(\overline{\mathbf{HAH}})$ of the formulas in $Fm(\overline{\mathbf{H}})$, not containing the parameters $\tilde{a}_0^\sigma, \tilde{a}_1^\sigma, \tilde{a}_2^\sigma, \ldots$, such that (i)–(iii) hold with ϕ_2, $\overline{\mathbf{H}}$, and $\overline{\mathbf{HAH}}$ taking place of ϕ_1, $\mathbf{H}$, and $\mathbf{HAH}$, respectively.

A precise description of the mappings ϕ_1 and ϕ_2 can easily be obtained from the proofs of Theorems 1.2.7. and 1.2.9. in [19].

Remark 2. When using symbols C_σ, S_σ, and G_σ, we will not always indicate to which of the system, $\mathbf{H}$ or $\overline{\mathbf{H}}$, they belong; which of the two systems is understood will be clear from the context.

Definition 8. For any type σ and any term t of type σ in $\overline{\mathbf{H}}$ or $\mathbf{H}$:

(a) $S_\sigma^0(t) := t$,

(b) $S_\sigma^{n+1}(t) := S_\sigma(S_\sigma^n(t))$.

Note that for each type σ, $\forall x^\sigma(S_\sigma^1(x) = S_\sigma(x))$ is derivable both in $\overline{\mathbf{H}}$ and $\mathbf{H}$.

Lemmas 9 to 18 below are preliminaries to the proof of Theorem 2 in the next section.

Lemma 9. *For any types $\sigma, \sigma_1, \ldots, \sigma_n$ and any $m, n \in \omega$, the following formulas are derivable both in $\overline{\mathbf{H}}$ and $\mathbf{H}$:*

(a) $\forall x^{\sigma_1 \times \ldots \times \sigma_n}[S^m_{\sigma_1 \times \ldots \times \sigma_n}(x) = (S^m_{\sigma_1}(p^n_1 x), \ldots, S^m_{\sigma_n}(p^n_n x))]$,

(b) $\forall x^\sigma[x \in S^n_{[\sigma]}(C_{[\sigma]}) \leftrightarrow x = S^{n+1}_\sigma(C_\sigma)]$,

(c) $\neg(S^n_\sigma(C_\sigma) = S^{n+1}_\sigma(C_\sigma))$,

(d) $\forall x^\sigma \neg(G_\sigma(x) = C_\sigma)$,

(e) $\forall x^\sigma \forall y^\sigma(G_\sigma(x) = G_\sigma(y) \to x = y)$.

Proof. (a): Directly by induction on m.

(b): Directly by induction on n.

(c): By induction on $h(\sigma)$, using (a) and (b).

(d): By induction on $h(\sigma)$, using (c).

(e): By induction on $h(\sigma)$, using (d).
■

Lemma 10. *For any type σ, $\forall\tilde{x}^\sigma\neg(\tilde{x}=C_\sigma)$ is derivable in $\overline{\mathbf{H}}$, where, in case $\sigma=\mathbf{0}$, $\forall\tilde{x}^\sigma\neg(\tilde{x}=C_\sigma)$ denotes $\forall x\neg(Sx=C_\sigma)$.*

Proof. By induction on $h(\sigma)$, using (c) of Lemma 9.
■

Lemma 11. *For any type σ and any $n\in\omega$, the following formulas are derivable in $\mathbf{H}$:*

(a) $\exists\dot{x}\,(\dot{x}=S_\sigma^n(C_\sigma))$,

(b) $\forall y\exists\dot{x}\,(\dot{x}=G_\sigma(y))$.

Here, in case $\sigma=\mathbf{0}$, $\dot{x}$ denotes x.

Proof. By induction on $h(\sigma)$, using (a) and (b) of Lemma 9.
■

Lemma 12. *For any type σ, $\forall y\exists\tilde{x}\,(\tilde{x}=G_\sigma(y))$ is derivable in $\overline{\mathbf{H}}$, where, in case $\sigma=\mathbf{0}$, $\exists\tilde{x}\,(\tilde{x}=G_\sigma(y))$ denotes $\exists x(Sx=G_\sigma(y))$.*

Proof. By induction on $h(\sigma)$.
■

Remark 3. Without loss of generality we may and shall then assume every atomic subformula of any formula A in **HAH** or $\overline{\mathbf{HAH}}$ is of the form either $t^\mathbf{0}=r^\mathbf{0}$, $b^\sigma\in a^{[\sigma]}$, or $b^\sigma\in p_i^n a^\tau$. More precisely, it is easy to show that any formula in **HAH**, resp. $\overline{\mathbf{HAH}}$ is equivalent (in **HAH**, resp. $\overline{\mathbf{HAH}}$) to a formula constructed from the atomic formulas of this kind by means of &, $\rightarrow$, $\forall$, and $\exists$.

Definition 9. Let Δ_1 be the set of formulas in $\mathbf{H}$ such that $A\in\Delta_1$ iff any atomic subformula of A has one of the following forms:

(a) $t^\mathbf{0}=r^\mathbf{0}$,

(b) $b^\sigma\in a^{[\sigma]}$,

(c) $b^\sigma\in\dot{a}^{[\sigma]}$,

(d) $b^\sigma\in p_k^n a^\tau$,

(e) $b^\sigma\in p_k^n\dot{a}^\tau$.

We define a translation $[\ \ ]_1$ that associates with every formula $A \in \Delta_1$ a formula $[A]_1$ in $\overline{\mathbf{H}}$ by letting:

(i) $[t^{\mathbf{0}} = r^{\mathbf{0}}]_1 := t = r$;
(ii) $[a_j^{\mathbf{0}} \in a_i^{[\mathbf{0}]}]_1 := \exists z(z = S(a_j^{\mathbf{0}}) \,\&\, z \in \tilde{a}_i^{[\mathbf{0}]})$;
(iii) $[a_j^{\mathbf{0}} \in \dot{a}_i^{[\mathbf{0}]}]_1 := S(a_j^{\mathbf{0}}) \in G_{[\mathbf{0}]}(a_i^{[\mathbf{0}]})$;
(iv) $[a_j^{\mathbf{0}} \in p_k^n a_i^{\tau}]_1 := \exists z(z = S(a_j^{\mathbf{0}}) \,\&\, z \in p_k^n \tilde{a}_i^{\tau})$;
(v) $[a_j^{\mathbf{0}} \in p_k^n \dot{a}_i^{\tau}]_1 := S(a_j^{\mathbf{0}}) \in p_k^n G_{\tau}(a_i^{\tau})$;
(vi) if $\sigma \neq \mathbf{0}$, then $[a_j^{\sigma} \in a_i^{[\sigma]}]_1 := \exists z(z = \tilde{a}_j^{\sigma} \,\&\, z \in \tilde{a}_i^{[\sigma]})$;
(vii) if $\sigma \neq \mathbf{0}$, then $[a_j^{\sigma} \in \dot{a}_i^{[\sigma]}]_1 := \tilde{a}_j^{\sigma} \in G_{[\sigma]}(a_i^{[\sigma]})$;
(viii) if $\sigma \neq \mathbf{0}$, then $[a_j^{\sigma} \in p_k^n a_i^{\tau}]_1 := \exists z(z = \tilde{a}_j^{\sigma} \,\&\, z \in p_k^n \tilde{a}_i^{\tau})$;
(ix) if $\sigma \neq \mathbf{0}$, then $[a_j^{\sigma} \in p_k^n \dot{a}_i^{\tau}]_1 := \tilde{a}_j^{\sigma} \in p_k^n G_{\tau}(a_i^{\tau})$;
(x) $[\ \ ]_1$ commutes with & and $\rightarrow$;
(xi) $[\mathrm{Q}x_i^{\mathbf{0}} A(a_i^{\mathbf{0}}/x_i^{\mathbf{0}})]_1 := \mathrm{Q}x_i^{\mathbf{0}}[A(a_i^{\mathbf{0}})]_1(a_i^{\mathbf{0}}/x_i^{\mathbf{0}})$;
(xii) if $\sigma \neq \mathbf{0}$, then $[\mathrm{Q}x_i^{\sigma} A(a_i^{\sigma}/x_i^{\sigma})]_1 := \mathrm{Q}\tilde{x}_i^{\sigma}[A(a_i^{\sigma})]_1(\tilde{a}_i^{\sigma}/\tilde{x}_i^{\sigma})$;
(xiii) $[\mathrm{Q}\dot{x}_i^{\sigma} A(\dot{a}_i^{\sigma}/\dot{x}_i^{\sigma})]_1 := \mathrm{Q}x_i^{\sigma}[A(\dot{a}_i^{\sigma})]_1(a_i^{\sigma}/x_i^{\sigma})$;
(xiv) if $\Gamma \subseteq \Delta_1$, then $[\Gamma]_1 := \{[A]_1 : A \in \Gamma\}$.

Here Q stands for $\forall$ or $\exists$.

Lemma 13. *The following formulas are derivable in* $\overline{\mathbf{H}}$:

(a) $[a_i^{\sigma} \in b_j^{[\sigma]}]_1 \leftrightarrow \tilde{a}_i^{\sigma} \in \tilde{b}_j^{[\sigma]}$;

(b) $[a_i^{\sigma} = b_j^{\sigma}]_1 \leftrightarrow \tilde{a}_i^{\sigma} = \tilde{b}_j^{\sigma}$;

(c) $[a_i^{\sigma} = \dot{b}_j^{\sigma}]_1 \leftrightarrow \tilde{a}_i^{\sigma} = G_{\sigma}(b_j^{\sigma})$;

(d) $[a_i^{\sigma} \in p_k^n b_j^{\tau}]_1 \leftrightarrow \tilde{a}_i^{\sigma} \in p_k^n \tilde{b}_j^{\tau}$;

Here, in case $\sigma = \mathbf{0}$, $\tilde{a}_i^{\sigma}$, $\tilde{b}_j^{\sigma}$, *and* $\dot{b}_j^{\sigma}$ *denote* $S(a_i)$, $S(b_j)$, *and* b_j , *respectively.*

Proof. (a) and (d): By Definition 9.

(b) and (c): These are easily verified by induction on $h(\sigma)$, using (a), (d), and Lemma 12.

■

Definition 10. Let Δ_2 be the set of formulas in $\overline{\mathbf{H}}$ such that $A \in \Delta_2$ iff any atomic subformula of A has one of the following forms:

(a) $t^{\mathbf{0}} = r^{\mathbf{0}}$,

(b) $b^{\sigma} \in a^{[\sigma]}$,

(c) $b^\sigma \in \tilde{a}^{[\sigma]}$,

(d) $b^\sigma \in p^n_k a^\tau$,

(e) $b^\sigma \in p^n_k \tilde{a}^\tau$.

We define a translation $[\ \ ]_2$ which associates with every formula $A \in \Delta_2$ a formula $[A]_2$ in $\mathbf{H}$ in such a way that:

(i) $[t^{\mathbf{0}} = r^{\mathbf{0}}]_2 := (t = r)$;
(ii) $[a^{\mathbf{0}}_j \in a^{[\mathbf{0}]}_i]_2 := a^{\mathbf{0}}_j \in \dot{a}^{[\mathbf{0}]}_i$;
(iii) $[a^{\mathbf{0}}_j \in \tilde{a}^{[\mathbf{0}]}_i]_2 := a^{\mathbf{0}}_j \in G_{[\mathbf{0}]}(a^{[\mathbf{0}]}_i)$;
(iv) $[a^{\mathbf{0}}_j \in p^n_k a^\tau_i]_2 := a^{\mathbf{0}}_j \in p^n_k \dot{a}^\tau_i$;
(v) $[a^{\mathbf{0}}_j \in p^n_k \tilde{a}^\tau_i]_2 := a^{\mathbf{0}} \in p^n_k G_\tau(a^\tau_i)$;
(vi) if $\sigma \neq \mathbf{0}$, then $[a^\sigma_j \in a^{[\sigma]}_i]_2 := \exists z((z = \dot{a}^\sigma_j) \,\&\, z \in \dot{a}^{[\sigma]}_i))$;
(vii) if $\sigma \neq \mathbf{0}$, then $[a^\sigma_j \in \tilde{a}^{[\sigma]}_i]_2 := \dot{a}^\sigma_j \in G_{[\sigma]}(a^{[\sigma]}_i)$;
(viii) if $\sigma \neq \mathbf{0}$, then $[a^\sigma_j \in p^n_k a^\tau_i]_2 := \exists z(z = \dot{a}^\sigma_j \,\&\, z \in p^n_k \dot{a}^\tau_i)$;
(ix) if $\sigma \neq \mathbf{0}$, then $[a^\sigma_j \in p^n_k \tilde{a}^\tau_i]_2 := \dot{a}^\sigma_j \in p^n_k G_\tau(a^\tau_i)$;
(x) $[\ \ ]_2$ commutes with & and $\rightarrow$;
(xi) $[\mathrm{Q}x^{\mathbf{0}}_i A(a^{\mathbf{0}}_i/x^{\mathbf{0}}_i)]_2 := \mathrm{Q}x^{\mathbf{0}}_i[A(a^0_i)]_2(a^{\mathbf{0}}_i/x^{\mathbf{0}}_i)$;
(xii) if $\sigma \neq \mathbf{0}$, then $[\mathrm{Q}x^\sigma_i A(a^\sigma_i/x^\sigma_i)]_2 := \mathrm{Q}\dot{x}^\sigma_i[A(a^\sigma_i)]_2(\dot{a}^\sigma_i/\dot{x}^\sigma_i)$;
(xiii) $[\mathrm{Q}\tilde{x}^\sigma_i A(\tilde{a}^\sigma_i/\tilde{x}^\sigma_i)]_2 := \mathrm{Q}x^\sigma_i[A(\tilde{a}^\sigma_i)]_2(a^\sigma_i/x^\sigma_i)$;
(xiv) if $\Gamma \subseteq \Delta_2$, them $[\Gamma]_2 := \{[A]_2 : A \in \Gamma\}$.

Here Q is $\forall$ or $\exists$.

Lemma 14. *The following formulas are derivable in* $\mathbf{H}$:

(a) $[a^\sigma_i \in b^{[\sigma]}_j]_2 \leftrightarrow \dot{a}^\sigma_i \in \dot{b}^{[\sigma]}_j$;

(b) $[a^\sigma_i = b^\sigma_j]_2 \leftrightarrow \dot{a}^\sigma_i = \dot{b}^\sigma_j$;

(c) $[a^\sigma_i = \tilde{b}^\sigma_j]_2 \leftrightarrow \dot{a}^\sigma_i = G_\sigma(b^\sigma_j)$;

(d) $[a^\sigma_i \in p^n_k b^\tau_j]_2 \leftrightarrow \dot{a}^\sigma_i \in p^n_k \dot{b}^\tau_j$;

Here, in case $\sigma = \mathbf{0}$, $\dot{a}^\sigma_i$, $\dot{b}^\sigma_j$, *and* $\tilde{b}^\sigma_j$ *denote* a_i, b_j, *and* $S(b_j)$, *respectively.*

Proof. (a) and (d): By Definition 10.

(b) and (c): These are easily verified by induction on $h(\sigma)$, using (a), (d), and Lemma 11.

■

Note that if $A \in Fm(\mathbf{HAH})$, then $[A]_1 \in \Delta_2$; thus $[[A]_1]_2 \in Fm(\mathbf{H})$. Similarly, for any formula A in $\overline{\mathbf{HAH}}$, $[A]_2 \in \Delta_1$; hence $[[A]_2]_1 \in Fm(\overline{\mathbf{H}})$. Moreover, the following two lemmas hold.

Lemma 15. *For any formula A in* **HAH**,

$$[[A]_1]_2 \leftrightarrow A$$

is derivable in **H**.

Proof. By induction on A. Here, as well as in the proof of the next Lemma 16, we consider only the less obvious case where A is of the form $a^\sigma \in p^n_k b^\tau$ and $\sigma \neq \mathbf{0}$. Throughout the proof "$\vdash$"is used for the derivability relation in **H**.

Let B denote $[[a^\sigma \in p^n_k b^\tau]_1]_2$. Then by Definitions 9 and 10 we have

$$\vdash B \leftrightarrow \exists \dot{z}([z = \tilde{a}]_2 \,\&\, [z \in p^n_k \tilde{b}]_2).$$

Hence by (c) of Lemma 14 and Definition 10,

$$\vdash B \leftrightarrow \exists \dot{z}(\dot{z} = G_\sigma(a) \,\&\, \dot{z} \in p^n_k G_\tau(b)).$$

Further, from (b) of Lemma 11 and the definition of G_σ it follows that

$$\vdash B \leftrightarrow G_\sigma(a) \in p^n_k G_\tau(b),$$

which implies, by (d) of Lemma 9 and the definition of G_σ:

$$\vdash B \leftrightarrow \exists z(a = z \,\&\, z \in p^n_k b),$$

so that $\vdash B \leftrightarrow a \in p^n_k b$.
■

Lemma 16. *For any formula A in* $\overline{\mathbf{HAH}}$

$$[[A]_2]_1 \leftrightarrow A$$

is derivable in $\overline{\mathbf{H}}$.

Proof. By induction on A. Throughout the proof "$\vdash$" refers to the derivability relation in $\overline{\mathbf{H}}$.

Let A be $a^\sigma \in p^n_k b^\tau$ with $\sigma \neq \mathbf{0}$, and let B denote $[[a^\sigma \in p^n_k b^\tau]_2]_1$.

By Definitions 9 and 10 we have

$$\vdash B \leftrightarrow \exists \tilde{z}([z = \dot{a}]_1 \,\&\, [z \in p^n_k \dot{b}]_1).$$

Hence by (c) of Lemma 13 and Definition 9,

$$\vdash B \leftrightarrow \exists \tilde{z}(\tilde{z} = G_\sigma(a) \,\&\, \tilde{z} \in p_k^n G_\tau(b)),$$

which implies, in view of Lemma 12 and the definition of G_σ,

$$\vdash B \leftrightarrow G_\sigma(a) \in p_k^n G_\tau(b).$$

Finally, using the same argument as that in the proof of Lemma 15, we get $\vdash B \leftrightarrow a \in p_k^n b$.
■

Lemma 17. *For any formula A and any set of formulas Γ in* **HAH**: *If A is derivable in* **HAH** *from Γ , then $[A]_1$ is derivable in* $\overline{\mathbf{H}}$ *from $[\Gamma]_1$.*

Proof. Directly by induction on derivations in **HAH**.
■

Lemma 18. *For any formula A and any set of formulas Γ in* $\overline{\mathbf{HAH}}$: *If A is derivable in* $\overline{\mathbf{HAH}}$ *from Γ , then $[A]_2$ is derivable in* **H** *from $[\Gamma]_2$.*

Proof. Directly by induction on derivations in $\overline{\mathbf{HAH}}$.
■

4 The relationship between NAH and HAH

Definition 11. (i) For any formula A in **NAH** with all its parameters of type **0**:

$$\mathcal{F}_1(A) := \phi_1([A^*]_2).$$

(ii) For any formula A in **HAH** with all its parameters of type **0**:

$$\mathcal{F}_2(A) := (\phi_2([A]_1))^+.$$

Observe that $\mathcal{F}_1$ is a mapping of $Fm(\mathbf{NAH})$ to $Fm(\mathbf{HAH})$, while $\mathcal{F}_2$ is a mapping of $Fm(\mathbf{HAH})$ to $Fm(\mathbf{NAH})$. The following theorem shows in what sense $\mathcal{F}_1$ and $\mathcal{F}_2$ realize the translatability of **NAH** and **HAH** into each other.

Theorem 2 (*the relationship between* **HAH** *and* **NAH**). (I). *For any formula A in* **HAH** *such that all its parameters are of type* **0**:

(1) $\mathbf{HAH} \vdash A \leftrightarrow \mathcal{F}_1(\mathcal{F}_2(A))$;
(2) $\mathbf{HAH} \vdash A$ *if and only if* $\mathbf{NAH} \vdash \mathcal{F}_2(A)$.

(II). *For any formula A in* **NAH** *such that all its parameters are of type* **0**:

(3) A *and* $\mathcal{F}_2(\mathcal{F}_1(A))$ *are* $\mathbf{NAH_1}$*-equivalent*;
(4) $\mathbf{NAH} \vdash A$ *if and only if* $\mathbf{HAH} \vdash \mathcal{F}_1(A)$.

In order to proof the theorem we first derive some lemmas.

Lemma 19. *For any type σ ,*

$$\exists x(\tilde{a}^\sigma = G_\sigma(x)) \leftrightarrow [D_2^\sigma(a^\sigma)]_1$$

is derivable in $\overline{\mathbf{H}}$ *, where, in case $\sigma = \mathbf{0}$, $\tilde{a}^\sigma$ denotes Sa.*

Proof. By induction on $h(\sigma)$. (Here, as well as in the proofs of Lemmas 20 to 22 below, we consider only the case where $\sigma = [\tau]$ and $\tau \neq \mathbf{0}$.)

We argue in $\overline{\mathbf{H}}$.

Assume $\tilde{a}^\sigma = G_\sigma(b)$. We must show that this implies $[D_2^\sigma(a^\sigma)]_1$, that is, in view of Lemma 13(a), the conjunction of the following two formulas:

$$\exists \tilde{x}(\tilde{x} \in \tilde{a}^\sigma), \tag{5}$$

$$\forall \tilde{z}^\tau(\tilde{z}^\tau \in \tilde{a}^\sigma \to [D_2^\tau(z)]_1). \tag{6}$$

First observe that $\tilde{a}^\sigma = G_\sigma(b)$ implies (5). by the definition of $G_\sigma(b)$, axiom G_τ, and Lemma 12. Now if $\tilde{c} \in \tilde{a}^\sigma$, then $\tilde{c} \in G_\sigma(b)$; thus, by Lemma 10, we get $\exists z(\tilde{c} = G_\tau(z) \,\&\, z \in b)$; hence, by the induction hypothesis, $[D_2^\tau(c)]_1$.

Conversely, assume (5) and (6). We must show that these imply $\exists x(\tilde{a}^\sigma = G_\sigma(x))$. Observe that, by (6) and the induction hypothesis, we have

$$\forall \tilde{z}[\tilde{z} \in \tilde{a}^\sigma \to \exists x^\tau(\tilde{z} = G_\tau(x)] \tag{7}$$

and hence, by (5), $\exists x(G_\tau(x) \in \tilde{a}^\sigma)$. We thus get, by CR_τ,

$$\exists y^\sigma \forall x(x \in y \leftrightarrow G_\tau(x) \in \tilde{a}^\sigma).$$

Assume $\forall x(x \in b^\sigma \leftrightarrow G_\tau(x) \in \tilde{a}^\sigma)$. It remains to show that $\tilde{a}^\sigma = G_\sigma(b)$ and thus, in view of Lemma 12, that

$$\forall \tilde{x}(\tilde{x} \in \tilde{a}^\sigma \leftrightarrow \tilde{x} \in G_\sigma(b)).$$

But if $\tilde{c} \in \tilde{a}^\sigma$, then, by (6) and the induction hypothesis, we get $\exists x(x \in b \,\&\, \tilde{c} = G_\tau(x))$; hence $\tilde{c} \in G_\sigma(b)$. On the other hand, if $\tilde{c} \in G_\sigma(b)$, then, by Lemma 10, $\exists x(x \in b \,\&\, \tilde{c} = G_\tau(x))$; thus $\tilde{c} \in \tilde{a}^\sigma$.

■

Lemma 20. *For any type σ, $[D_2^\sigma(\dot{a}^\sigma)]_1$ is derivable in $\overline{\mathbf{H}}$, where, in case $\sigma = \mathbf{0}$, $\dot{a}^\sigma$ denotes a.*

Proof. By induction on $h(\sigma)$.

Suppose that $\sigma = [\tau]$ with $\tau \neq \mathbf{0}$. It is easy to show, using Definition 9 and Lemma 19, that $[D_2^\sigma(\dot{a}^\sigma)]_1$ is equivalent in $\overline{\mathbf{H}}$ to the formula

$$\exists\tilde{x}(\tilde{x} \in G_\sigma(a^\sigma))\,\&\,\forall\tilde{z}[\tilde{z} \in G_\sigma(a^\sigma) \to \exists y(\tilde{z} = G_\tau(y))].$$

By axiom G_τ and Lemma 12 we get

$$\overline{\mathbf{H}} \vdash \exists\tilde{x}(\tilde{x} \in G_\sigma(a^\sigma)).$$

Moreover, by Lemma 10,

$$\overline{\mathbf{H}} \vdash \forall\tilde{z}[\tilde{z} \in G_\sigma(a^\sigma) \to \exists y(\tilde{z} = G_\tau(y))].$$

■

Let $\Delta_2^{'}$ be the set of formulas in $\overline{\mathbf{H}}$ such that $A \in \Delta_2^{'}$ iff any atomic subformula of A either belongs to Δ_2 or has one of the following forms:

(f) $C_\sigma \in a^{[\sigma]}$,

(g) $b^\sigma \in C_{[\sigma]}$,

(h) $C_\sigma \in \tilde{a}^{[\sigma]}$.

In order to state the next lemma we extend Definition 10 to the formulas in $\Delta_2^{'}$ by adding the clauses:

(xv) $[C_\sigma \in a_i^{[\sigma]}]_2 := C_\sigma \in \dot{a}_i^{[\sigma]}$;
(xvi) $[a_i^{\mathbf{0}} \in C_{[\mathbf{0}]}]_2 := a_i^{\mathbf{0}} \in C_{[\mathbf{0}]}$;
(xvii) if $\sigma \neq \mathbf{0}$, then $[a_i^\sigma \in C_{[\sigma]}]_2 := \dot{a}_i^\sigma \in C_{[\sigma]}$;
(xviii) $[C_\sigma \in \tilde{a}_i^{[\sigma]}]_2 := C_\sigma \in G_{[\sigma]}(a_i^{[\sigma]})$.

Lemma 21. *For any type σ,*

$$\exists x(\dot{a}^\sigma = G_\sigma(x)) \leftrightarrow [D_1^\sigma(a^\sigma)]_2$$

is derivable in $\mathbf{H}$; here, in case $\sigma = \mathbf{0}$, $\dot{a}^\sigma$ denotes a.

Proof. By induction on $h(\sigma)$.

Suppose that $\sigma = [\tau]$ and $\tau \neq \mathbf{0}$. We argue in $\mathbf{H}$.

First, assume $\dot{a}^\sigma = G_\sigma(b)$. We have to show that this implies $[D_1^\sigma(a^\sigma)]_2$, that is, in view of Definition 10 and Lemmas 11 and 14(a), the conjunction of the following two formulas:

$$C_\tau \in \dot{a}^\sigma, \tag{8}$$

$$\forall \dot{z}^\tau(\dot{z} \in \dot{a}^\sigma \to (\dot{z} = C_\tau \vee [D_1^\tau(z)]_2)). \tag{9}$$

Observe that $\dot{a}^\sigma = G_\sigma(b)$ implies $C_\tau \in \dot{a}^\sigma$, by the definition of G_σ. Now if $\dot{c} \in \dot{a}^\sigma$, then, by the definition of G_σ, we get $\dot{c} = C_\tau \vee \exists z(\dot{c} = G_\tau(z))$ and thus, by the induction hypothesis, $\dot{c} = C_\tau \vee [D_1^\tau(c)]_2$.

Conversely, assume (8) and (9). We must show that these imply $\exists x(\dot{a}^\sigma = G_\sigma(x))$. Note that by CA_τ we have

$$\exists x^\sigma \forall y^\tau(y \in x \leftrightarrow G_\tau(y) \in \dot{a}^\sigma).$$

Assume

$$\forall y^\tau(y \in b \leftrightarrow G_\tau(y) \in \dot{a}^\sigma). \tag{10}$$

Clearly it remains to show that $\dot{a}^\sigma = G_\sigma(b)$ and thus, in view of Lemma 11, that

$$\forall \dot{x}(\dot{x} \in \dot{a}^\sigma \leftrightarrow \dot{x} \in G_\sigma(b))$$

But if $\dot{c} \in \dot{a}^\sigma$. then by (9) and the induction hypothesis we have $\dot{c} = C_\tau \vee \exists x(\dot{c} = G_\tau(x))$; hence, by (10), $\dot{c} \in G_\sigma(b)$. On the other hand, if $\dot{c} \in G_\sigma(b)$, then by the definition of G_σ we get $\dot{c} = C_\tau \vee \exists x(x \in b \,\&\, \dot{c} = G_\tau(x))$, which implies, in view of (8) and (10), $\dot{c} \in \dot{a}^\sigma$.

■

Lemma 22. *For any type* σ, $[D_1^\sigma(\tilde{a}^\sigma)]_2$ *is derivable in* $\mathbf{H}$ *, where, in case* $\sigma = \mathbf{0}$, $\tilde{a}^\sigma$ *denotes* Sa.

Proof . By induction on $h(\sigma)$.

Suppose that $\sigma = [\tau]$ with $\tau \neq \mathbf{0}$. It is easily verified, in view of Definition 10 and Lemmas 11 and 21, that $[D_1^\sigma(\tilde{a}^\sigma)]_2$ is equivalent in $\mathbf{H}$ to the conjunction of the two formulas:

$$C_\tau \in G_\sigma(a^\sigma),$$

$$\forall \dot{x}[\dot{x} \in G_\sigma(a^\sigma) \to (\dot{x} = C_\tau \vee \exists y(\dot{x} = G_\tau(y))].$$

Evidently each of these formulas is derivable in $\mathbf{H}$.

■

Lemma 23. (i) *Let A be any formula in Δ_1 with all its parameters of type* $\mathbf{0}$. *Then*

$$\overline{\mathbf{H}} \vdash [\phi_1(A) \leftrightarrow A]_1.$$

(ii) *Let A be any formula in Δ_2 with all its parameters of type* $\mathbf{0}$. *Then*

$$\mathbf{H} \vdash [\phi_2(A) \leftrightarrow A]_2.$$

Proof. First note that, for any formula $A \in \Delta_1$ such that all its parameters are of type $\mathbf{0}$, there is a derivation Σ in $\mathbf{H}$ of $\phi_1(A) \leftrightarrow A$ such that Σ may contain only applications of the rules of many-sorted intuitionistic predicate logic and the axioms and rules (a$'$) – (f$'$). With this in view, the statement (i) is easily verified by induction on Σ, using Lemmas 19 and 20.

The proof of (ii) is similar to that of (i), using Lemmas 21 and 22.

■

Proof of Theorem 2. (1): First, by Lemma 23(ii), we have

$$\mathbf{H} \vdash [\phi_2([A]_1)]_2 \leftrightarrow [[A]_1]_2.$$

Then by Lemma 15,

$$\mathbf{H} \vdash [\phi_2([A]_1)]_2 \leftrightarrow A.$$

Thus

$$\mathbf{HAH} \vdash \phi_1([\phi_2([A]_1)]_2) \leftrightarrow A. \tag{11}$$

Now by Lemma 8(1),

$$\overline{\mathbf{HAH}} \vdash \phi_2([A]_1) \leftrightarrow (\phi_2([A]_1)^+)^*,$$

that is, by the definition of $\mathcal{F}_2$,

$$\overline{\mathbf{HAH}} \vdash \phi_2([A]_1) \leftrightarrow (\mathcal{F}_2(A))^*.$$

This implies, in view of Lemma 18,

$$\mathbf{H} \vdash [\phi_2([A]_1)]_2 \leftrightarrow [(\mathcal{F}_2(A))^*]_2.$$

Hence

$$\mathbf{HAH} \vdash \phi_1([\phi_2([A]_1)]_2) \leftrightarrow \phi_1([(\mathcal{F}_2(A))^*]_2),$$

from which it follows that

$$\mathbf{HAH} \vdash A \leftrightarrow \mathcal{F}_1(\mathcal{F}_2(A)),$$

by (11) and the definition of $\mathcal{F}_1$.

(2): If $\mathbf{HAH} \vdash A$, then, by Lemma 17, $\overline{\mathbf{H}} \vdash [A]_1$; hence $\overline{\mathbf{HAH}} \vdash \phi_2([A]_1)$; thus $\mathbf{NAH} \vdash \mathcal{F}_2(A)$, by (2) of Lemma 8.

Conversely, if $\mathbf{NAH} \vdash \mathcal{F}_2(A)$, then by (4) of Lemma 8, $\overline{\mathbf{HAH}} \vdash (\mathcal{F}_2(a))^*$; hence, by Lemma 18, $\mathbf{H} \vdash [(\mathcal{F}_2(A))^*]_2$. We thus get $\mathbf{HAH} \vdash \mathcal{F}_1(\mathcal{F}_2(A))$, which implies, in view of (1), $\mathbf{HAH} \vdash A$.

(3): From Lemma 23(i) it follows that

$$\overline{\mathbf{H}} \vdash [\phi_1([A^*]_2)]_1 \leftrightarrow [[A^*]_2]_1.$$

Then by Lemma 16 and the definition of $\mathcal{F}_1$,

$$\overline{\mathbf{H}} \vdash [\mathcal{F}_1(A)]_1 \leftrightarrow A^*;$$

hence

$$\overline{\mathbf{HAH}} \vdash \phi_2([\mathcal{F}_1(A)]_1) \leftrightarrow A^*.$$

This implies, in view of Lemma 7 and the definition of $\mathcal{F}_2$, that $\mathcal{F}_2(\mathcal{F}_1(A))$ and $(A^*)^+$ are $\mathbf{NAH_1}$-equivalent; thus $\mathcal{F}_2(\mathcal{F}_1(A))$ and A are $\mathbf{NAH_1}$-equivalent, by Lemma 8(3).

(4): If $\mathbf{NAH} \vdash A$, then, by (4) of Lemma 8, $\overline{\mathbf{HAH}} \vdash A^*$; hence, by Lemma 18, $\mathbf{H} \vdash [A^*]_2$; thus $\mathbf{HAH} \vdash \mathcal{F}_1(A)$.

Conversely, assume $\mathbf{HAH} \vdash \mathcal{F}_1(A)$; then, by (2), $\mathbf{NAH} \vdash \mathcal{F}_2(\mathcal{F}_1(A))$; hence, by (3), $\mathbf{NAH} \vdash A$.

This completes the proof of Theorem 2.

■

Corollary 1. *Let A be a formula in* **HA** *not containing* $\to$. *Then* $\mathbf{HAH} \vdash A$ *if and only if* $\mathbf{NAH} \vdash A$.

Corollary 2. *The systems* $\mathbf{HAH^c}$ (= **HAH** *plus the Principle of the Excluded Middle*) *and* **NAH** *are equiconsistent.*

Proof. Combining the extension of the Gödel-Gentzen negative translation to the case of $\mathbf{HAH^c}$ and **HAH**, given in [16], with Corollary 1.

■

Addendum: The relationship between NPC and Nelson's sequent calculus $\mathbf{P_1}$

Throughout the following we shall presuppose some familiarity with [17].

Given any finite, possibly empty sequence $\Delta \equiv A_1, \dots, A_n$ of formulas in **NPC**, we use $\bar{\Delta}$ to denote the set $\{B\colon B \text{ is a formula in } \Delta\}$.

The following theorem shows that **NPC**, resp. $\mathbf{NPC_1}$ is equivalent to (a slight modification of) the Σ-free fragment of $\mathbf{P_1}$, resp. $\mathbf{P_2}$ (cf. [14, Remark 4]). [1]

Theorem 3. *Let $\mathbf{P_i^-}$ denote the Σ-free fragment of $\mathbf{P_i}$ ($\mathbf{i} = 1, 2$) such that every sequent in $\mathbf{P_i^-}$ has just one formula in its succedent. Let $\mathbf{P_i^\star}$ be the sequent calculus obtained from $\mathbf{P_i^-}$ by adding all the axioms of* **NPC** *as initial sequents (viz. sequents of the form $\to A$, where A is an axiom of* **NPC**) *and, in case $\mathbf{i} = 1$, the rule*

$$(\star) \quad \frac{\Gamma, \exists x A(x) \to \mathcal{T}}{\Gamma, A(a) \to \mathcal{T}}\ ,$$

where a does not occur in the upper sequent. Then:

(1). *For any sequent $\Delta \to A$ in $\mathbf{P_1^\star}$, $\Delta \to A$ is derivable in $\mathbf{P_1^\star}$ if and only if A is derivable in* **NPC** *from $\bar{\Delta}$;*

(2). *For any sequent $\Delta \to A$ in $\mathbf{P_2^\star}$, $\Delta \to A$ is derivable in $\mathbf{P_2^\star}$ if and only if A is derivable in $\mathbf{NPC_1}$ from $\bar{\Delta}$.*

Previous to proving the theorem we will assemble in Lemma 24 some easily verified facts about the derivability relation in $\mathbf{P_1^\star}$.

Lemma 24. *For any sequents Δ_1 and Δ_2 and any formulas A and B in $\mathbf{P_1^\star}$, the following hold (here "$\vdash$" refers to the derivability relation in $\mathbf{P_1^\star}$):*

(i) *If $\vdash \Delta_1, \Delta_2 \to \mathcal{T}$, then $\vdash \Delta_1 \to \mathcal{T}$ and $\Delta_2 \to \mathcal{T}$;*
(ii) *if $\vdash \Delta_1, \Delta_2 \to \mathcal{T}$ and $\vdash \Delta_i \to A$, then $\vdash \Delta_1, \Delta_2 \to A$ (for $i = 1, 2$);*
(iii) *if $\vdash \Delta_1, \Delta_2 \to \mathcal{T}$, then $\vdash \Delta_1, \Delta_2 \to \&\bar{\Delta}_i$ (for $i = 1, 2$);*
(iv) *if $\vdash \Delta_1, \Delta_2 \to \mathcal{T}, \vdash \Delta_1 \to A$, and $\vdash \Delta_2 \to B$, then*
$\vdash \Delta_1, \Delta_2, A, B \to \mathcal{T}$;
(v) *if $\vdash \Delta_1, \Delta_2 \to \mathcal{T}$ and $\vdash \Delta_i \to A$, then $\vdash \Delta_1, \Delta_2, A \to \mathcal{T}$ (for $i = 1, 2$).*

[1] Both Nelson's sequent calculi $\mathbf{P_1}$ and $\mathbf{P_2}$ contain, in addition to the logical constants of **NPC**, the quatified disjunction Σ.

Remark 4. This lemma holds with any of the systems $\mathbf{P_i}$, $\mathbf{P_i^-}$ $(\mathbf{i} = 1, 2)$, or $\mathbf{P_2^\star}$ taking place of $\mathbf{P_1^\star}$.

Proof of Theorem 3. (Throughout the proof the denotation of the inference rules of $\mathbf{P_1}$ will be the same as in [17]; we will write "$\vdash$", resp. "$\vdash_{\mathbf{NPC}}$" for the derivability relation in $\mathbf{P_1^\star}$, resp. **NPC**.)

(2). This is easily verified by induction on derivations in the corresponding system.

(1). From left to right: By (2). From right to left: In view of the axiom (1a) and the structural rule (13a) of $\mathbf{P_1}$, it is clearly sufficient to show that if A is derivable in **NPC** from $\bar{\Delta}$, and $\Delta \to \mathcal{T}$ is derivable in $\mathbf{P_1^\star}$, then the sequent $\Delta \to A$ is derivable in $\mathbf{P_1^\star}$. [Indeed, $\bar{\Delta} \vdash_{\mathbf{NPC}} A$ implies $\vdash_{\mathbf{NPC}} \exists(\&\bar{\Delta})$, by the definition of a derivation in **NPC**.] The proof is performed using a routine though somewhat cumbersome induction on derivations in **NPC**.

Let $<\Sigma_1, \Sigma_2>$ be a derivation in **NPC**. We consider only two less obvious cases where Σ_2 ends with an application of $\to$ I and $\to$ E$_1$.

Case (A): The last inference rule in Σ_2 is an application of $\to$ I. Then Σ_2 has the form

$$\dfrac{\begin{matrix} & [A(\bar{a})] \\ \Pi & \vdots \\ \exists \bar{x} A(\bar{x}) & B(\bar{a}) \end{matrix}}{A(\bar{x}) \to_{\bar{x}} B(\bar{x})} \,.$$

Suppose that for some finite, possibly empty sequences Δ_1 and Δ_2 of formulas in **NPC**:

$$\bar{\Delta}_1 \vdash_{\mathbf{NPC}} \exists \bar{x} A(\bar{x}), \tag{a}$$

$$\bar{\Delta}_2 \cup \{A(\bar{a})\} \vdash_{\mathbf{NPC}} B(\bar{a}), \tag{b}$$

$$\vdash \Delta_1, \Delta_2 \to \mathcal{T}, \tag{c}$$

where it is assumed that the parameters in $\bar{a}$ do not occur in the formulas in $\bar{\Delta}_2$.

We must show that

$$\vdash \Delta_1, \Delta_2 \to A(\bar{x}) \to_{\bar{x}} B(\bar{x}).$$

First note that by (c) and Lemma 24(i) we have $\vdash \Delta_1 \to \mathcal{T}$; hence by (a) and the induction hypothsis,

$$\vdash \Delta_1 \to \exists \bar{x} A(\bar{x}). \tag{d}$$

Next, by (c), (d), and Lemma 24,

$$\vdash \Delta_2, \exists \bar{x} A(\bar{x}) \to \mathcal{T};$$

thus, by the above mentioned rule ($\star$),

$$\vdash \Delta_2, A(\bar{a}) \to \mathcal{T}. \tag{e}$$

Now by the induction hypothesis from (e) and (b) it follows that

$$\vdash \Delta_2, A(\bar{a}) \to B(\bar{a}).$$

So, by the structural rule (17) and the rule $(2)^*$,

$$\vdash \exists \bar{x} A(\bar{x}), \Delta_2 \to A(\bar{x}) \to_{\bar{x}} B(\bar{x}). \tag{f}$$

Finally, by the cut rule (18) applied to (c), (d), and (f) we get

$$\vdash \Delta_1, \Delta_2 \to A(\bar{x}) \to_{\bar{x}} B(\bar{x}).$$

Case (B): The last inference rule in Σ_2 is an application of $\to \mathrm{E}_1$. Then Σ_2 has the form

$$\frac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ A(\bar{t}) & A(\bar{x}) \to_{\bar{x}} B(\bar{x}) \end{array}}{B(\bar{t})} .$$

Suppose that for some finite, possibly empty sequences Δ_1 and Δ_2 of formulas in **NPC**:

$$\bar{\Delta}_1 \vdash_{\mathbf{NPC}} A(\bar{t}), \tag{g}$$

$$\bar{\Delta}_2 \vdash_{\mathbf{NPC}} A(\bar{x}) \to_{\bar{x}} B(\bar{x}), \tag{h}$$

$$\vdash \Delta_1, \Delta_2 \to \mathcal{T} \tag{i}$$

We must show that

$$\vdash \Delta_1, \Delta_2 \to B(\bar{t}).$$

First, by (i) and Lemma 24, we have $\vdash \Delta_1 \to \mathcal{T}$ and $\vdash \Delta_2 \to \mathcal{T}$. Hence, by (g), (h), and the induction hypothesis:

$$\vdash \Delta_1 \to A(\bar{t}), \tag{j}$$

$$\vdash \Delta_2 \to A(\bar{x}) \to_{\bar{x}} B(\bar{x}). \tag{k}$$

Next, by (k), the existence rules (20a) and (20b), and Lemma 24, we obtain $\vdash \exists\bar{x}A(\bar{x}) \to \mathcal{T}$ and $\vdash \exists\bar{x}B(\bar{x}) \to \mathcal{T}$, hence, by the rule ($\star$), $\vdash A(\bar{a}) \to \mathcal{T}$ and $\vdash B(\bar{a}) \to \mathcal{T}$, where it is assumed that $\bar{a}$ is a sequence of distinct parameters which occur in neither $\exists\bar{x}A(\bar{x})$ nor $\exists\bar{x}B(\bar{x})$. Thus by the rule (1b), $\vdash A(\bar{a}) \to A(\bar{a})$ and $\vdash B(\bar{a}) \to B(\bar{a})$. Moreover, in view of Lemma 24, from (k) and the existence rule (20a) we conclude that

$$\vdash A(\bar{x}) \to_{\bar{x}} B(\bar{x}), \exists\bar{x}A(\bar{x}) \to \mathcal{T}.$$

This implies, by the rule ($\star$),

$$\vdash A(\bar{x}) \to_{\bar{x}} B(\bar{x}), A(\bar{a}) \to \mathcal{T}. \tag{l}$$

Thus by the rule (3) applied to $A(\bar{a}) \to A(\bar{a})$, $B(\bar{a}) \to B(\bar{a})$, and (l),

$$\vdash A(\bar{x}) \to_{\bar{x}} B(\bar{x}), A(\bar{a}) \to B(\bar{a}). \tag{m}$$

Now from (i), (j), and (k) it follows (by Lemma 24) that

$$\vdash A(\bar{x}) \to_{\bar{x}} B(\bar{x}), A(\bar{t}) \to \mathcal{T}; \tag{n}$$

thus by the specification rule (19) applied to (m) and (n),

$$\vdash A(\bar{x}) \to_{\bar{x}} B(\bar{x}), A(\bar{t}) \to B(\bar{t}); \tag{o}$$

Moreover, by (i), (j), and Lemma 24, we have

$$\vdash \Delta_2, A(\bar{t}) \to \mathcal{T}. \tag{p}$$

Hence by the cut rule (18) applied to (k), (o), and (p),

$$\vdash \Delta_2, A(\bar{t}) \to B(\bar{t});$$

so that by the structural rule (17),

$$\vdash A(\bar{t}), \Delta_2 \to B(\bar{t}). \tag{q}$$

Finally, by the rule (18) applied to (j), (q), and (i) we get

$$\vdash \Delta_1, \Delta_2 \to B(\bar{t}).$$

This completes the proof of Theorem 3.

■

As an immediate consequence of Theorem 3 we have

Corollary 3. *For any sequent $\Delta \to A$ in $\mathbf{P}_1^\star$, $\Delta \to A$ is derivable in $\mathbf{P}_1^\star$ if and only if both the sequents $\Delta \to A$ and $\to \exists(\&\bar{\Delta})$ are derivable in $\mathbf{P}_2^\star$.*

Remark 5. For the sake of completeness, we have included in the reference list below some works of Griss not mentioned in our [14], as well as Franchella's recent papers dealing with philosophical aspects of the Griss viewpoint.

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