Undecidable theories of Lyndon algebras

Vera Stebletsova* Yde Venema†

1 Introduction

Lyndon algebras form an interesting connection between the fields of projective geometry and algebraic logic. Their origin lies in the paper [Jónsson 1959] where the author associates a relation algebra with a projective plane. Jónsson's construction was generalized by R. Lyndon to projective geometries of arbitrary dimension, in order to provide "a method for deriving consequences in the algebraic theory of binary relations from certain familiar facts of projective geometry", cf. [Lyndon 1961].

We need to go into some detail concerning this construction. We formalize a projective geometry as a two-sorted structure $\mathcal{G} = (P, L, I)$ where P is a set of points, L a set of lines, and $I \subseteq P \times L$ the binary incidence relation (precise definitions will be given in the next section). Now the *Lyndon algebra* associated with a projective geometry $\mathcal{G} = (P, L, I)$ is the algebraic structure $\mathcal{L}_\mathcal{G} = (\wp(P_e), \cup, (\cdot)^c_{P_e}, \varnothing, \circ, \iota, \{e\})$ defined as follows. The carrier of $\mathcal{L}_\mathcal{G}$ is the power set of the set $P_e = P \cup \{e\}$ — here e is some new, 'useless' point; the operations $\cup$ and $(\cdot)^c_{P_e}$ denote the ordinary set-theoretic operations of union and complement, respectively, and $\varnothing$ denotes the empty set. The interesting operation of Lyndon algebras is $\circ$ which interprets the binary operator symbol ; (in relation algebras corresponding to relational composition). We first define this operation $\circ$ for the atoms of the algebras; that is, singleton subsets of P_e. Here $\overline{pq}$ denotes the line through p and q.

$$\{p\} \circ \{q\} = \begin{cases} \{e, p\} & \text{if } p = q, \\ \overline{pq} \setminus \{p, q\} & \text{if } p \neq q \text{ and } e \notin \{p, q\}, \\ \{p\}, & \text{if } q = e, \\ \{q\}, & \text{if } p = e. \end{cases}$$

This definition extends to arbitrary subsets of P_e by requiring complete additivity for $\circ$; that is, we define $X \circ Y = \bigcup_{x \in X, y \in Y} \{x\} \circ \{y\}$. The operation ι is there only to give Lyndon algebras the right similarity type, namely that of Tarski's relation algebras; it interprets the unary function symbol $\smile$ (corresponding to taking the converse of a relation) and is trivially defined as the identity map: $\iota(X) = X$ for any subset X of P_e. Finally, the singleton $\{e\}$ interprets the constant 1' which denotes the identity relation in a relation algebraic context.

*Department of Philosophy, Utrecht University, Heidelberglaan 8, 3584 CS Utrecht, the Netherlands. E-mail: Vera.Stebletsova@phil.uu.nl. The research of this author is sponsored by the Netherlands Organization for the Advancement of Research (NWO).

†Institute for Logic, Language and Computation, University of Amsterdam, Plantage Muidergracht 24, NL-1018 TV Amsterdam, The Netherlands. E-mail: yde@wins.uva.nl. The research of the author has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

Given a class of G of projective geometries, we denote by $\mathsf{L}(\mathsf{G})$ the class of associated Lyndon algebras. The class of *all* Lyndon algebras is denoted as L. In a number of occasions we will be interested in the Lyndon algebras associated with the class G_d of all d-dimensional projective geometries (d some fixed cardinal); the class of these *d-dimensional Lyndon algebras* is denoted as L_d.

Lyndon algebras have played an important role in the representation theory of relation algebras. First of all, it is straightforward to show that every Lyndon algebra is indeed a relation algebra; that is, it satisfies the axioms that Tarski proposed as a first approximation for an axiomatization of the 'true' relational equations. In [Jónsson 1959] it was proved that the algebra $\mathcal{L}_{\mathcal{G}}$ of a *plane* $\mathcal{G}$ is representable unless the plane satisfies Desargues' Theorem; this result provided natural examples of non-representable relation algebras. Although Lyndon generalized Jónsson's work to geometries of arbitrary dimension, his paper [Lyndon 1961] is usually cited for his work on algebras associated with lines (one-dimensional projective geometries); in fact, some people use the term 'Lyndon algebras' only for this smaller class of algebras. An important result of Lyndon's paper states that for a line $\mathcal{G}$ of order $n \geq 3$, the algebra $\mathcal{L}_{\mathcal{G}}$ is representable iff there is a projective plane of order n. This result formed the basis of Monk's seminal theorem that the equational theory of the class RRA is not finitely axiomatizable, cf. [Monk 1964].

Some further results are known about Lyndon algebras. For instance, Andréka, Givant and Németi proved interesting results on the decidability of various classes of one-dimensional Lyndon algebras, cf. [Andréka, Givant & Németi]. In particular, it is proved that the class L_1 itself has a decidable equational theory. (The cited paper forms an excellent overview of the state of the art concerning decidability and undecidability of various classes of relation algebras.) For some natural classes of geometries, [Givant 199?] contains results connecting the associated Lyndon algebras to the universal class of simple algebras in the variety generated by $\mathsf{L}(\mathsf{G})$. Finally, the paper [Stebletsova 1998] is concerned with the axiomatizability problem of various classes of Lyndon algebras; the main result of that paper is a finite axiomatization for the equational theory of L_d for fixed, finite d.

In this paper we investigate the decidability of the equational theories of Lyndon algebras of higher dimension. Our main result is the following theorem.

Theorem 1 *If G is a class of projective geometries which contains an infinite projective geometry of dimension $d \geq 3$, then the equational theory of $\mathsf{L}(\mathsf{G})$ is undecidable.*

In particular, the class L of all Lyndon algebras has an undecidable equational theory, and so have the classes L_d for $d \geq 3$.

Two remarks are in order concerning our proof of Theorem 1. First, it turned out that in order to reach undecidability, we do not need the full expressive power of the extra-boolean operators of Lyndon algebras. We define the *simplified Lyndon algebra* $\mathcal{L}^s_{\mathcal{G}}$ of a projective geometry $\mathcal{G} = (P, L, I)$ as the structure $\mathcal{L}^s_{\mathcal{G}} = (\wp(P), \cup, (\cdot)^c_P, \varnothing, \bullet)$. Here the binary operation $\bullet$ on $\wp(P)$ is defined for singletons by

$$\{p\} \bullet \{q\} = \begin{cases} \overline{pq} & \text{if } p \neq q \\ \{p\} & \text{if } p = q, \end{cases}$$

while for arbitrary subsets of P the definition is given by complete additivity. In the next section we will see that when X and Y are suspaces of the geometry, then $X \bullet Y$ is in fact the subspace generated by X and Y. For a class G of projective geometries, we let $\mathsf{L}^s(\mathsf{G})$ denote the class of simplified Lyndon algebras of geometries in G; we will use the notation L^s and L^s_d with obvious meanings. Our main technical result (Theorem 3.1 below) states that for any class G containing an infinite projective geometry of dimension at least three, the equational theory of the class $\mathsf{L}^s(\mathsf{G})$ is undecidable. As we will show in section 3, it is straightforward to derive Theorem 1 from this.

Second, the main idea underlying our proof involves an interesting connection with another kind of algebras that are well-known from algebraic logic, namely the diagonal-free cylindric algebras (cf. the next section for definitions). Roughly speaking, we will show that inside every simplified Lyndon algebra associated with a projective geometry of dimension at least three, we can find a term-definable representable diagonal-free cylindric algebra of dimension three (an RDf_3, for short). And conversely, given a 'small', simple RDf_3 $\mathcal{A}$ and an infinite geometry $\mathcal{G}$ of dimension at least three, we can embed $\mathcal{A}$ in one of these RDf_3's living inside the simplified Lyndon algebra associated with $\mathcal{G}$. These connections, which modify and generalize an idea of [Venema 1998], allow us to use the undecidability of the equational theory of the class RDf_3 in order to prove our main technical result.

Finally, we want to mention that our results leave open an interesting problem, namely whether the class L_2 of Lyndon algebras associated with projective planes has a decidable equational theory.

2 Preliminaries

We will assume a certain familiarity with the basics of geometry and universal algebra. We now fix our notation and terminology, and we gather some specific facts from the fields of projective geometry and diagonal-free cylindric algebras that we need in our proofs. Since Lyndon algebras are the only relation algebras that we will encounter here, we do not need further information on this branch of algebraic logic.

Universal algebra

Given a term $\tau = \tau(x_1, \ldots, x_n)$ in an algebraic similarity type and an algebra $\mathcal{A}$ of this similarity type, τ induces a map: $A^n \to A$. This map is denoted by $\tau^{\mathcal{A}}$.

In this paper we will be dealing with three different similarity types, namely those of relation algebras/Lyndon algebras, simplified Lyndon algebras and three-dimensional diagonal-free cylindric algebras. For brevity, we will denote the second similarity type by SL. All these similarity types extend that of Boolean algebras; for the boolean reducts of the similarity types we will use uniform and familiar notation like $+$, $-$, 0, $\cdot$ and 1. Of these, we consider the first three as primitive and the other two as abbreviations. The relation algebraic similarity type has a binary function symbol ;, a unary $\smile$ and a constant 1'; as abbreviation we use 0' for -1'. The similarity type SL has only one non-boolean function symbol; it is binary and we also denote it by ;. Three-dimensional diagonal-free cylindric algebras will be discussed below.

Projective geometries

As we mentioned before, we model projective geometries as structures $\mathcal{G} = (P, L, I)$, where P is a non-empty set of points, L is a non-empty set of lines, and $I \subseteq P \times L$ is a binary *incidence* relation between points and lines. We use standard geometrical terminology like 'two lines *meet in* the point s'; we may also follow standard practice in identifying a line with the set of points that are incident with it. Now such a two-sorted structure $\mathcal{G}$ is a *projective geometry* or *projective space* if it satisfies the following axioms:

(A1) There exists at least one line, and each line has at least four points;
(A2) Each pair of distinct points (x, y) lies on a unique line, $\overline{xy}$;
(A3) If x, y and z are distinct points, and a line meets lines $\overline{xy}$ and $\overline{xz}$ in distinct points, then it meets $\overline{yz}$.

Now let $\mathcal{G} = (P, L, I)$ be a projective geometry. A subset S of P is a *subspace* of $\mathcal{G}$ if for all distinct points p, q in S, we have $\overline{pq} \subseteq S$. For any $X \subseteq \mathcal{G}$, define $\langle X \rangle$, the *span of* X or *subspace generated by* X, to be the smallest subspace containing X. It is easy to see that $\langle X \rangle$ is the intersection of all subspaces containing X. A *hyperplane* H of a geometry $\mathcal{G}$ is a maximal proper subspace of $\mathcal{G}$. If $\langle X \rangle = S$ we say that X is a *spanning set* of S. We write $\langle X_1, \ldots, X_n \rangle$ for $\langle X_1 \cup \ldots \cup X_n \rangle$, and $\langle p, q, r \rangle$ for $\langle \{p\}, \{q\}, \{r\} \rangle$. A subset S of $\mathcal{G}$ is *independent* if for all $p \in S$ we have $\langle S \setminus \{p\} \rangle \neq \langle S \rangle$. An independent spanning set of $\mathcal{G}$ is called a *basis* of $\mathcal{G}$. Every projective geometry has a basis and all bases of a geometry have the same cardinality; this cardinal is called the *rank* r of $\mathcal{G}$. Finally, we can define the *dimension* d of $\mathcal{G}$ as $d = r - 1$ if r is finite, and $d = \infty$ if r is infinite. One-dimensional projective spaces are called lines, and two-dimensional ones, planes.

One can prove that all lines of a projective space have the same cardinality; a projective geometry is said to have *order* α if each of its lines contains at $\alpha + 1$ points.

For explicit reference we list the following geometric facts. These results can be found in any textbook on projective geometry; for instance, the reader might consult Chapter 2 and 6 of [Garner 1981].

Fact 2.1 Let $\mathcal{G} = (P, L, I)$ be a projective geometry. Then
(1) If S is a subspace of dimension 2 of $\mathcal{G}$, i.e., a plane, then any two of its lines meet.
(2) Every line in $\mathcal{G}$ meets every hyperplane H of $\mathcal{G}$.
(3) Let S_0, S_1 be non-empty subspaces of $\mathcal{G}$. Then

$$\langle S_0, S_1 \rangle \;=\; (S_0 \cap S_1) \cup \bigcup \{\overline{s_0 s_1} \mid s_0 \in S_0, s_1 \in S_1 \text{ and } s_0 \neq s_1\}.$$

(4) If a line intersects a subspace, while it does not form part of the subspace, then the intersection is a singleton point.
(5) If S_i, S_j are subspaces of $\mathcal{G}$, then $S_i \bullet S_j = \langle S_i, S_j \rangle$.
(6) For subspaces S_0, S_1 and S_2 of $\mathcal{G}$ we have $\langle S_0, S_1, S_2 \rangle = \langle S_0, \langle S_1, S_2 \rangle \rangle$.
(7) Let M be an independent set, and $\{M_i : i < 3\}$ a partition of M. Then there are no collinear points $s_0 \in \langle M_0 \rangle$, $s_1 \in \langle M_1 \rangle$ and $s_2 \in \langle M_2 \rangle$.

Boolean algebras with operators

All algebras that we encounter in this paper are Boolean algebras with operators; that is, Boolean algebras augmented with operations that are additive in each of their arguments. The prime examples of such algebras are the so-called complex algebras.

Given a set W and a relation $R \subseteq W^{n+1}$, define the operation $m_R : \wp(W)^n \to \wp(W)$ by $m_R(X_1, \ldots, X_n) = \{y \in W \mid Ryx_1 \ldots x_n \text{ for some } x_1 \in X_1, \ldots, x_n \in X_n\}$. The *complex algebra* of a relational structure $\mathcal{F} = (W, R_i)_{i \in I}$ is defined as the algebra $\mathcal{F}^+ = (\wp(W), \cup, (\cdot)^c_W, \varnothing, m_{R_i})_{i \in I}$.

Of the theory of Boolean algebras with operators we only use the following. A class K of Boolean algebras with operators is a *discriminator class* if there is some unary term $\Diamond(x)$ such that $\mathsf{K} \models x \not\approx 0 \to \Diamond(x) \approx 1$ and $\mathsf{K} \models \Diamond(0) = 0$.

Fact 2.2 *(Cf. [Tarski & Givant 1988]) Let K be a class of Boolean algebras with operators. If K is a discriminator class, then there is some recursive translation which turns any universal formula ϕ to some equation e in such a way that $\mathsf{K} \models \phi$ if and only if $\mathsf{K} \models e$.*

Diagonal-free cylindric algebras

Diagonal-free cylindric algebras, and in particular, the representable ones, form the algebraic counterpart of first-order logic without equality. For a survey of the theory of these algebras the reader is referred to Chapter 5 of [Henkin et alii 1985]; here we confine ourselves to the definitions and the facts that we need later on. Let α be an arbitrary but fixed ordinal.

A *diagonal-free cylindric algebra of dimension* α, for brevity a Df_α, is an algebra $\mathcal{A} = (A, +, \cdot, -, 0, \mathsf{c}_i)_{i\in\alpha}$ such that $(A, +, \cdot, -, 0)$ is a boolean algebra, and each c_i is a unary operation on A such that the following identities hold in $\mathcal{A}$, for all $i, j \in \alpha$:

(C1) $\mathsf{c}_i 0 = 0$

(C2) $x \leq \mathsf{c}_i x$,(i.e., $x + \mathsf{c}_i x = \mathsf{c}_i x$)

(C3) $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$

(C4) $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$.

The variety of all diagonal-free cylindric algebras of dimension α is denoted by Df_α.

In this article we are mainly interested in the representable Df_α's; elements of these algebras represent α-ary relations. For an arbitrary family $U = \{U_i : i < \alpha\}$ of non-empty sets, we define, for any $i < \alpha$, the relation $\equiv_i$ on the Cartesian product $\prod_{i<\alpha} U_i$ by

$$u \equiv_i v \text{ iff } u_j = v_j \text{ for all } j \neq i.$$

Any subalgebra of the structure

$$\mathcal{C}_\alpha(U) \;=\; \big(\prod_{i<\alpha} U_i, \equiv_i\big)^+_{i\in I}$$

is called an *α-dimensional diagonal-free cylindric set algebra with base system U*. The class of such algebras is denoted as Csdf_α. Csdf_α's of the kind $\mathcal{C}_\alpha(U)$ are called *full.* Finally, a Df_α is called *representable* if it is isomorphic to a subalgebra of a product of full Csdf_α's. The class of representable Df_αs, which happens to be a variety, is denoted by RDf_α.

From now we concentrate on the case $\alpha = 3$. An element of a Df_3 $\mathcal{A}$ is called *rectangular* if it satisfies $a = \mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{c}_0\mathsf{c}_2 a \cdot \mathsf{c}_1\mathsf{c}_2 a$; $\mathcal{A}$ is called *rectangularly atomic* if it is atomic and each of its atoms is rectangular.

Proofs of the following facts can be found in Chapter 5 of [Henkin et alii 1985].

Fact 2.3 The following facts hold for diagonal-free cylindric algebras:

(1) A Df_3 is simple if and only if it satisfies $\forall x(x \neq 0 \to \mathsf{c}_0\mathsf{c}_1\mathsf{c}_2 x = 1)$.

(2) A representable Df_3 is simple if and only if it is isomorphic to a Csdf_3.

(3) Any rectangularly atomic Df_3 is representable. If such an algebra is in addition complete and simple, it is isomorphic to a full Csdf_3.

(4) Suppose that $\mathcal{A}$ is a Csdf_3 with base system U, and V is a larger base system in the sense that $|U_i| \leq |V_i|$ for every $i < 3$. Then $\mathcal{A}$ can be embedded in $\mathcal{C}_3(V)$.

(5) RDf_3 is generated by the class of complex algebras $\mathcal{C}_3(U)$ such that each U_i is countable.

3 Proofs

As we mentioned in our introduction, the main technical result of our paper is the following.

Theorem 3.1 *If* G *is a class of projective geometries which contains an infinite projective geometry of dimension* $d \geq 3$*, then the equational theory of* $\mathsf{L}^s(\mathsf{G})$ *is undecidable.*

It is the main aim of this section to prove Theorem 3.1, but let us first see how to derive Theorem 1 from it.

Proof of Theorem 1. Recall that SL denotes the similarity type of simplified Lyndon algebras. We will define a translation $\widehat{\cdot}$ of SL-terms to relation algebraic terms such that for any SL-equation $\sigma \approx \tau$, and any projective geometry $\mathcal{G}$ we have that

(1)
$$\mathcal{L}^s_{\mathcal{G}} \models \sigma \approx \tau \text{ iff } \mathcal{L}_{\mathcal{G}} \models \widehat{\sigma} \approx \widehat{\tau}.$$

From this, Theorem 1 is immediate since $\widehat{\cdot}$ is recursive; namely, it is inductively defined as follows:

$$\begin{array}{rcl}
\widehat{x} & = & x \cdot 0' \\
\widehat{-\tau} & = & -\widehat{\tau} \cdot 0' \\
\widehat{\sigma \cdot \tau} & = & \widehat{\sigma} \cdot \widehat{\tau} \\
\widehat{\sigma ; \tau} & = & ((\widehat{\sigma} + \widehat{\tau} + \widehat{\sigma};\widehat{\tau}) \ \cdot \ \widehat{\sigma};1 \ \cdot \ \widehat{\tau};1) \cdot 0'
\end{array}$$

Now let $\mathcal{G}$ be some projective geometry, and τ an arbitrary SL-term. We claim that

(2)
$$\widehat{\tau}^{\mathcal{L}_{\mathcal{G}}}(Q_1, \ldots, Q_n) = \tau^{\mathcal{L}^s_{\mathcal{G}}}(Q_1 \cap P, \ldots, Q_n \cap P)$$

for arbitrary subsets $Q_1, \ldots, Q_n$ of P_e. This suffices to prove (1) because of the following. Consider an SL-equation $\sigma \approx \tau$, and an arbitrary projective geometry $\mathcal{G}$. First assume that $\mathcal{L}_{\mathcal{G}} \models \widehat{\sigma} \approx \widehat{\tau}$; we will prove that $\mathcal{L}^s_{\mathcal{G}} \models \sigma \approx \tau$. Hence, consider arbitrary subsets $Q_1, \ldots, Q_n$ of P. It follows from $\mathcal{L}_{\mathcal{G}} \models \widehat{\sigma} \approx \widehat{\tau}$ that $\widehat{\sigma}^{\mathcal{L}_{\mathcal{G}}}(Q_1, \ldots, Q_n) = \widehat{\tau}^{\mathcal{L}_{\mathcal{G}}}(Q_1, \ldots, Q_n)$. Hence, (2) gives that $\sigma^{\mathcal{L}^s_{\mathcal{G}}}(Q_1, \ldots, Q_n) = \tau^{\mathcal{L}^s_{\mathcal{G}}}(Q_1, \ldots, Q_n)$. But since $Q_1, \ldots, Q_n$ were arbitrary, this gives that $\mathcal{L}^s_{\mathcal{G}} \models \sigma \approx \tau$, which is what we were after. The other direction is proved in a similar way.

We prove (2) by a straightforward term induction. The only non-trivial part of this proof concerns the case of the induction step when τ is of the form $\tau_0;\tau_1$. In order to cover this case it is sufficient to show that for any $X, Y \subseteq P$ we have that

(3)
$$X \bullet Y \;=\; ((X \cup Y \cup X \circ Y) \cap X \circ P_e \cap Y \circ P_e) \cap P.$$

For a better understanding of the right hand side of this equation is will help to realize that for any $X \subset P_e$,

(4)
$$X \circ P_e = \begin{cases} \varnothing & \text{if } X = \varnothing \\ P_e & \text{otherwise.} \end{cases}$$

In other words, $x;1$ is the unary discriminator term of the class of Lyndon algebras.

Now suppose that p belongs to $X \bullet Y$. Then neither X nor Y is empty, and $p \in P$ by definition of $\bullet$, so by (4) it suffices to show that p belongs to the set $X \cup Y \cup X \circ Y$. It follows from the definition of $\bullet$ that there are $x \in X$ and $y \in Y$ such that either x and y are distinct and $p \in \overline{xy}$, ot otherwise $p = x = y$. In the first case we have either $p = x$ (and thus, $p \in X$), or $p = y$ (and thus, $p \in Y$), or else $p \in \overline{xy} \setminus \{x, y\}$, whence p belongs to $\{x\} \circ \{y\}$ and thus to $X \circ Y$. In the second case we immediately obtain that $p \in X$. In either case we find that $p \in X \cup Y \cup X \circ Y$.

For the other inclusion, suppose that p is an element of the right hand side of (3). It immediately follows that $X \circ P_e$ and $Y \circ P_e$ are non-empty, so by (4) X and Y themselves are not empty. Since $p \in X \cup Y \cup X \circ Y$, we may distinguish the following four cases: $p \in X \setminus Y$, $p \in Y \setminus X$, $p \in X \cap Y$ or $p \in X \circ Y$. In the last two cases it follows immediately that $p \in X \bullet Y$, and since the first two cases are completely symmetric, we only treat the first. That is, assume that p belongs to X but not to Y. Since Y is not empty it must contain some element y which is thus distinct from p. But then from $p \in \overline{py} = \{p\} \bullet \{y\}$ it follows that $p \in X \bullet Y$. This proves (3) and hence (2). QED

Let us now turn to the proof of Theorem 3.1, which is based on three auxiliary results. The first two of these, Lemma 3.3 and Lemma 3.4, provide the details of the earlier-mentioned connection between Lyndon algebras and diagonal-free cylindric set algebras. We then put this connection to work by defining a translation from Df_3-equations to universal formulas in the algebraic *SL*-language, and proving that an equation is valid in the class RDf_3 iff its translation is valid in the class $\mathsf{L}^s(\mathsf{G})$, for any class G of projective geometries that meets the conditions of Theorem 3.1. From this, Theorem 3.1 follows almost immediately.

The following auxiliary notion is used throughout the remainder of the paper.

Definition 3.2 *Let $\mathcal{G} = (P, L, I)$ be a projective geometry. A quadruple $T = (H, S_0, S_1, S_2)$ is called a* tripod *in $\mathcal{G}$ if H is a non-empty proper subspace of $\mathcal{G}$, and S_0, S_1, S_2 are proper, mutually disjoint subspaces of H such that there is no triple of collinear points $s_0 \in S_0$, $s_1 \in S_1$ and $s_2 \in S_2$.*

For a tripod $T = (H, S_0, S_1, S_2)$ in $\mathcal{G}$, we put $A_T = P \setminus H$, and define the following operations on $\wp(A_T)$, for $i < 3$:

$$C_i^l Q = (S_i \bullet Q) \cap A_T. \tag{5}$$

Furthermore, we define the structure $\mathcal{D}_{\mathcal{G},T}$ as:

$$\mathcal{D}_{\mathcal{G},T} = (\wp(A_T), \cup, (\cdot)^c_{A_T}, \varnothing, C_i^l)_{i<3}.$$

We usually omit the first subscript, writing $\mathcal{D}_T$ instead of $\mathcal{D}_{\mathcal{G},T}$.

The name *tripod* stems from the observation that within the lattice of subspace of $\mathcal{G}$, the subspaces H 'stands' on the three legs S_0, S_1, S_2. Observe that if $T = (H, S_0, S_1, S_2)$ is a tripod, then the dimension of H cannot be smaller than two; hence, only geometries of dimension at least three can contain tripods.

The algebras $\mathcal{D}_{\mathcal{G},T}$ in the definition above form the link between simplified Lyndon algebras and representable diagonal-free cylindric algebras. To start with, observe that for any tripod T in a geometry $\mathcal{G}$, the algebra $\mathcal{D}_{\mathcal{G},T}$ is of the similarity type of diagonal-free algebras. The next lemma establishes that in fact, such a $\mathcal{D}_{\mathcal{G},T}$ is always representable.

Lemma 3.3 *Suppose T is a tripod in the projective geometry $\mathcal{G}$. Then $\mathcal{D}_T$ is isomorphic to a representable diagonal-free cylindric algebra of dimension 3.*

Proof. Let $\mathcal{G} = (P, L, I)$ be a projective geometry and suppose that T is a tripod of $\mathcal{G}$. Define the binary relations $\sim_i$ on A_T for $i < 3$ as follows:

$$p \sim_i q \text{ iff for some } s \in S_i, \text{ the points } p, q \text{ and } s \text{ are collinear.} \tag{6}$$

It then follows immediately from the definitions that for any $Q \subseteq A_T$ we have

$$C_i^l Q = \{p \in A_T : p \sim_i q \text{ for some } q \in Q\}. \tag{7}$$

In other words, $\mathcal{D}_T$ is the complex algebra of the structure $(A_T, \sim_i)_{i<3}$.

The proof of Lemma 3.3 consists in showing that $\mathcal{D}_T$ is an atomic Df_3 such that each of its atoms is rectangular — this suffices by Fact 2.3(3).

It is easy to see that $\mathcal{D}_T$ is complete and atomic, since its boolean reduct is the power set algebra of A_T. In order to verify that $\mathcal{D}_T$ is a diagonal-free cylindric algebra of dimension 3, we will check that the axioms C1–C4 hold. To this end, it suffices to prove that each $\sim_i$ is an equivalence relation and that for each pair of distinct i and j, the relations $\sim_i$ and $\sim_j$ commute, that is, if $p \sim_i q \sim_j r$ for some q then there is a w such that $p \sim_j w \sim_i r$. For, it is easy to see (and in fact, well-known) that, given (7), the fact that $\sim_i$ is an equivalence relation implies that C_i^l satisfies C1–C3, while the commutativity condition on $\sim_i$ and $\sim_j$ ensures the validity of C4.

Hence, let us first establish that $\sim_i$ is an equivalence relation for each $i < 3$. Reflexivity and symmetry are easily proved, so we concentrate on transitivity. Suppose that $p \sim_i q \sim_i r$; that is, there are points s and s' in S_i such that p, q and s and q, r and s' are collinear, respectively. If p, q and r are collinear then by Fact 2.1(4) the points s and s' must be identical; it is then immediate that $p \sim_i r$. Hence, assume that p, q and r are not collinear; consider the projective space $\langle p, q, r\rangle$ which then must be a plane. Since both s and s' are elements of this plane, the line $\overline{ss'}$ is a subset of $\langle p, q, r\rangle$ by the definition of a subspace. In a similar way, the line $\overline{pr}$ forms part of $\langle p, q, r\rangle$. By Fact 2.1(1), the lines $\overline{ss'}$ and $\overline{pr}$ meet at some point s'' which is an element of the subspace S_i since $\overline{ss'} \subseteq S_i$. Since p, r and s'' are collinear, this gives $p \sim_i r$.

Let us now check the commutativity condition. Suppose that for some $i \neq j$, $p \sim_i q$ and $q \sim_j r$; that is, there is an s_i in S_i such that the points p, q and s_i are collinear, and there is an s_j in S_j such that the points q, r and s_j are collinear. It follows easily from the definition of a tripod that the points s_i and s_j are distinct. Now obviously, p, q and r cannot be collinear, for if they were, by Fact 2.1(4) the intersection of this line with H would be a single point; but this would imply that $s_i = s_j$, which gives the desired contradiction. But if p, q and r are not collinear, then the projective space $\langle p, q, r\rangle$ is a plane. The lines $\overline{rs_i}$ and $\overline{ps_j}$ are part of $\langle p, q, r\rangle$, by the definition of a subspace. Hence, by Fact 2.1(1) $\overline{rs_i}$ and $\overline{ps_j}$ must meet in some point, say, w. By definition of $\sim_i$, the collinearity of r, w and s_i implies that $r \sim_i w$, and likewise, the collinearity of p, w and s_j implies that $p \sim_j w$. This proves that $\sim_i$ and $\sim_j$ are commuting equivalence relations indeed.

Finally, we turn to the proof that all atoms of $\mathcal{D}_T$ are rectangular. We claim that it is sufficient to check the following condition:

$$\text{if } p \sim_i q \sim_j r \sim_k p \text{ for distinct } i, j, k < 3, \text{ then } p = q = r. \tag{8}$$

For, using (8) we can reason as follows. Consider an arbitrary atom of $\mathcal{D}_T$. By definition of $\mathcal{D}_T$, this atom must be a singleton, say $\{p\}$ for some point p in $A_T = P \setminus H$. First we show that

$$C_0^l\{p\} = C_0^lC_1^l\{p\} \cap C_0^lC_2^l\{p\}. \tag{9}$$

For, suppose that $q \in C_0^lC_1^l\{p\} \cap C_0^lC_2^l\{p\}$. Then by (7), there exist s and t in A_T such that $p \sim_0 s \sim_1 q$ and $p \sim_0 t \sim_2 q$, respectively; it follows that $q \sim_1 s \sim_0 t \sim_2 q$. Therefore, $q = s$ by (8). Hence, $q \in C_0^l\{p\}$, since $s \in C_0^l\{p\}$. The opposite inclusion can be easily derived using the fact that the axioms C2 and C4 hold in $\mathcal{D}_T$. This proves (9), which places us in the right position to prove that $\{p\}$ is a rectangular atom:

$$\{p\} = C_0^lC_1^l\{p\} \cap C_0^lC_2^l\{p\} \cap C_1^lC_2^l\{p\}. \tag{10}$$

First, take an arbitrary point q in $C_0^lC_1^l\{p\} \cap C_0^lC_2^l\{p\} \cap C_1^lC_2^l\{p\}$. Applying (9), we find that $q \in C_0^l\{p\} \cap C_1^lC_2^l\{p\}$. Then by (7), there exists r such that $p \sim_0 q \sim_2 r \sim_1 p$; hence, by (8) p and q must be identical, showing that q belongs to $\{p\}$. The opposite inclusion is easily proved using axiom C2.

Hence, the only condition left to check is (8). Suppose that for some distinct $i, j, k < 3$, we have $p \sim_i q \sim_j r \sim_k p$ while p and q are distinct. Then by Fact 2.1(4), the intersection of the line $\overline{pq}$ with H is some point $s_i \in S_i$, and the intersection of the line $\overline{qr}$ with H is some point $s_j \in S_j$. It follows easily from the definition of a tripod that $S_i \cap S_j = \varnothing$; hence, the points s_i and s_j are distinct. Obviously, p, q and r can not be collinear, otherwise we obtain a contradiction with the fact that s_i and s_j must be distinct (we gave an analogous argument when checking the commutativity condition above). But then the subspace $\langle p, q, r\rangle$ is a plane.

Now consider the intersection S of $\langle p, q, r\rangle$ and H, which must be a subspace. Hence, since S contains the points $s_i = \overline{pq} \cap H$ and $s_j = \overline{qr} \cap H$, it contains the line $\overline{s_is_j}$. However, the plane $\langle p, q, r\rangle$ is not contained in H, for instance, $p \notin H$; therefore, S cannot be bigger than a line. We conclude that S is identical to the line $\overline{s_is_j}$. Now consider the line $\overline{rp}$; this line must intersect H in a point $s_k \in S_k$, since $r \sim_k p$. But since s_k also belongs to the plane $\langle p, q, r\rangle$ we find $s_k \in S$. By our previous observation then, s_k must be incident with the line $\overline{s_is_j}$, contradicting the tripod condition that there are *no* collinear points $s_0 \in S_0$, $s_1 \in S_1$ and $s_2 \in S_2$. QED

The next Lemma works the other way around. It states that for a 'small', simple RDf_3 $\mathcal{A}$ and an infinite geometry $\mathcal{G}$ of dimension at least three, we can find a tripod in $\mathcal{G}$ such that $\mathcal{A}$ can be embedded in $\mathcal{D}_T$.

Lemma 3.4 *Let $\mathcal{A}$ be a diagonal-free cylindric set algebra of dimension 3, with base system $\{U_i : i < 3\}$, where each U_i is countable. Let $\mathcal{G}$ be an infinite projective geometry of dimension $d \geq 3$. Then there is a tripod T in $\mathcal{G}$ such that $\mathcal{A}$ can be embedded in $\mathcal{D}_T$.*

Proof. For the definition of T, we start with taking a basis M of $\mathcal{G}$. Since $\mathcal{G}$ is an infinite projective geometry, there are two (not necessarily exclusive) possibilities: $\mathcal{G}$ may be of infinite order, or of infinite dimension. Observe that in the second case M will be infinite as well. Fix some $t \in M$ and define H as the subspace generated by $M \setminus \{t\}$; it is easy to see that H is a hyperplane. By our assumption concerning the dimension of $\mathcal{G}$, we have that $|M \setminus \{t\}| \geq 3$. Partition $|M \setminus \{t\}|$ into three non-empty sets M_0, M_1, M_2, with the proviso that if M is infinite, then each M_i is infinite as well. We define S_i to be the span of M_i: $S_i = \langle M_i\rangle$.

Clearly, $H = \langle S_0, S_1, S_2\rangle$. Furthermore, it follows from Fact 2.1(7) that there is no triple of collinear points $s_0 \in S_0$, $s_1 \in S_1$ and $s_2 \in S_2$. Hence, it follows that $T = (H, S_0, S_1, S_2)$ is a tripod in $\mathcal{G}$; let A_T, C_i^l and $\mathcal{D}_T$ be as in Definition 3.2.

Two further observations underpin our proof. First, $\mathcal{D}_T$ is simple and complete; and second, each $\sim_i$-equivalence class is infinite.

Let us first see why these two claims suffice to show that $\mathcal{A}$ can be embedded in $\mathcal{D}_T$. To start with, the completeness and simplicity of $\mathcal{D}_T$ make that $\mathcal{D}_T$ is isomorphic to a full diagonal-free cylindric *set* algebra $\mathcal{B}$ (instead of merely being representable); this follows from the second part of Fact 2.3(3). Now suppose that $\{Y_i : i < 3\}$ is the base system of $\mathcal{B}$; it is easy to see that if $\mathcal{D}_T$ and $\mathcal{B}$ are isomorphic, then their underlying relational structures $(A_T, \sim_i)_{i<3}$ and $(\prod_{i<3} Y_i, \equiv_i)$ must be isomorphic as well. But then it follows from the second observation that each $\equiv_i$-equivalence class is infinite; from this it is immediate that each Y_i is infinite too. Now recall that $\mathcal{A}$ is a three-dimensional diagonal-free cylindric set algebra with base system $\{U_i : i < 3\}$, where each U_i is countable. Hence, for each $i < 3$, $|U_i| \leq |Y_i|$ since Y_i is infinite. Then we may apply Fact 2.3(4) and conclude that $\mathcal{A}$ can be embedded in $\mathcal{B}$, and therefore, in $\mathcal{D}_T$, by isomorphism.

Let us now justify these two key observations; first, concerning the simplicity of $\mathcal{D}_T$, we claim that it suffices to show that

(11) for any $p, q \in A_T$, there are $v, u \in A_T$ such that $p \sim_0 v \sim_1 u \sim_2 q$.

For, assume that (11) holds and take an arbitrary non-empty subset Q of $A_T = P \setminus H$, and an arbitrary point $p \in A_T$. Since Q is non-empty it contains some point q. Then (11) yields points v and u in A_T such that $p \sim_0 v \sim_1 u \sim_2 q$; from this and the definitions we may easily infer that $p \in C^l_0 C^l_1 C^l_2 Q$. But since p was arbitrary, this shows that $C^l_0 C^l_1 C^l_2 Q = A_T$ whence $\mathcal{D}_T$ is simple indeed, by Fact 2.3(1).

In order to prove (11), let p and q be arbitrary points in A_T. Leaving the easy case where p and q are identical to the reader, we assume that $p \neq q$; hence, we may speak of the *line* $\overline{pq}$. Since H is a hyperplane, by Fact 2.1, clauses (2) and (4), $\overline{pq}$ intersects H at a single point, say h. Note that $H = \langle S_0, S_1, S_2 \rangle = \langle S_0, \langle S_1, S_2 \rangle \rangle$ (cf. Fact 2.1(6)). Then, by Fact 2.1(3), there are points $t \in \langle S_1, S_2 \rangle$ and $s_0 \in S_0$ such that either $t = s_0 = h$ or t, s_0 and h are collinear. Observe that the first case is impossible by the fact that $S_0 \cap \langle S_1, S_2 \rangle = \varnothing$. Furthermore, since $t \in \langle S_1, S_2 \rangle$, we may assume, again by Fact 2.1(3), the existence of points $s_1 \in S_1$ and $s_2 \in S_2$ such that either $t = s_1 = s_2$ or t, s_1 and s_2 are collinear. Again, the first case can be excluded, now on the ground that $S_1 \cap S_2 = \varnothing$. Hence, we may conclude that t, s_0 and h on the one hand, and t, s_1 and s_2 on the other, are collinear, and also that t, s_0, s_1 and s_2 must all be distinct. In the following we assume that h is also distinct from t, s_0, s_1 and s_2. (The easier cases where h coincides with one of the other points are left as exercises to the reader).

But if h and t are distinct, then t does not lie on the line $\overline{pq}$, and hence, the subspace $\langle p, q, t \rangle$ is a plane. Observe that this plane also goes through h and s_0. By Fact 2.1(1) the lines $\overline{qt}$ and $\overline{s_0 p}$ meet in some point v. By definition of $\sim_i$, we have $p \sim_0 v$. Now consider the subspace $\langle q, s_1, s_2 \rangle$ which is also a plane since s_1 and s_2 are distinct points in H, and q does not belong to H. Note that the points t and v are also in this plane. Let u be the intersection of $\overline{q s_2}$ and $\overline{s_1 v}$. Then we have $v \sim_1 u$ and $u \sim_2 q$ by definition of $\sim_i$. This proves (11).

Finally, it left to show that all $\sim_i$-equivalence classes are infinite. Let E be an $\sim_i$-equivalence class. We now distinguish cases as to the reason why $\mathcal{G}$ is infinite. If $\mathcal{G}$ is of infinite *order*, consider a point p from E and the line $\overline{p s_i}$ for some point $s_i \in S_i$. Note that s_i is the only point on this line which belongs to H, by Fact 2.1(4). All other points of the line $\overline{p s_i}$ belong to the $\sim_i$-equivalence class E, and as there are infinitely many of these points, E is infinite.

If, on the other hand, the *dimension* of $\mathcal{G}$ is infinite, we reason as follows. Take an arbitrary point p of E and consider the set B of all lines connecting p to some point in S_i. Observe that two distinct lines in B cannot intersect in any point *distinct* from p. It is thus obvious that the function mapping a line in B to the (unique!) point where the line intersects S_i, is a bijection from B onto S_i. In particular, this shows B to be infinite, since S_i is infinite by our proviso on M_i. But since any line must contain at least three points, we may choose on each line in B a point which is distinct from p and not in S_i. The collection of these points forms then an infinite subset of E. Hence, E itself must be infinite. QED

In order to put the above two lemmas to work we now specify a translation mapping Df_3-equations to universal formulas in the language simplified Lyndon algebras.

Definition 3.5 *Let* h, s_i, $(i < 3)$, *be distinguished variables in the language of simplified Lyndon algebras. The translation* $\bar{\cdot}$, *mapping* Df_3*-terms to SL-terms, is defined by the following term induction:*

$$\begin{aligned}
\overline{x} &= x \cdot -\mathsf{h} \\
\overline{-\sigma} &= -\overline{\sigma} \cdot -\mathsf{h} \\
\overline{\sigma \cdot \tau} &= \overline{\sigma} \cdot \overline{\tau} \\
\overline{\mathsf{c}_i \sigma} &= (\overline{\sigma}; \mathsf{s}_i) \cdot -\mathsf{h}
\end{aligned}$$

Let $\tau \prec \sigma$ *abbreviate the formula* $((-\sigma + \tau) \approx 1) \wedge \neg(\sigma \approx \tau)$ *and* $\mathsf{subspace}(\sigma)$ *the equation* $\sigma; \sigma \approx \sigma$. *Define*

$$\begin{aligned}
\mathsf{tripod}(\mathsf{h}, \mathsf{s}_0, \mathsf{s}_1, \mathsf{s}_2) \quad = \quad & \mathsf{h} \prec 1 \wedge \bigwedge_{0 \leq i < 3} (\mathsf{s}_i \not\approx 0) \wedge \bigwedge_{0 \leq i < 3} (\mathsf{s}_i \prec \mathsf{h}) \wedge \\
& \wedge \mathsf{subspace}(\mathsf{h}) \wedge \bigwedge_{0 \leq i < 3} \mathsf{subspace}(\mathsf{s}_i) \wedge \mathsf{s}_0 \cdot (\mathsf{s}_1; \mathsf{s}_2) \approx 0.
\end{aligned}$$

Finally, define the translation tr of Df_3*-equations to formulas of simplified Lyndon algebras by:*

$$tr(\sigma \approx \tau) := \mathsf{tripod}(\mathsf{h}, \mathsf{s}_0, \mathsf{s}_1, \mathsf{s}_2) \rightarrow \overline{\sigma} \approx \overline{\tau}.$$

Observe that the translation of a Df_3-equation is a boolean combination of *SL*-equations and thus, it can be seen as a universal formula.

Recall that RDf_3 denotes the class of representable diagonal-free cylindric algebras, and $\mathsf{L}^s(\mathsf{G})$ denote the class of simplified Lyndon algebras associated with a class G of projective geometries.

Lemma 3.6 *Let* G *be a class of projective geometries which contains an infinite projective geometry of dimension* $d \geq 3$. *Then for any pair of* Df_3*-terms* σ *and* τ*:*

$$\mathsf{RDf}_3 \models \sigma \approx \tau \textit{ iff } \mathsf{L}^s(\mathsf{G}) \models tr(\sigma \approx \tau).$$

Proof. Two main observations underpin our proof of this Lemma. Let $\mathcal{G} = (P, L, I)$ be a projective geometry and consider subsets H, S_0, S_1 and S_2 of P. Our first observation is that

(12) $\mathsf{tripod}(H, S_0, S_1, S_2)$ holds in $\mathcal{L}^s_{\mathcal{G}}$ iff (H, S_0, S_1, S_2) is a tripod of $\mathcal{G}$.

The proof of (12) is straightforward. For instance, it is easy to show that $\mathsf{subspace}(S)$ holds in $\mathcal{L}^s_{\mathcal{G}}$ if and only if S is a subspace of $\mathcal{G}$ (cf. Fact 2.1(5).

And second, if we assume that $T = (H, S_0, S_1, S_2)$ is a tripod in $\mathcal{G}$, we have for any term σ:

$$(13) \qquad \sigma^{\mathcal{D}_T}(Q_1 \cap A_T, \ldots, Q_n \cap A_T) \;=\; \overline{\sigma}^{\mathcal{L}^s_{\mathcal{G}}}(Q_1, \ldots, Q_n, H, S_0, S_1, S_2),$$

for all $Q_1, \ldots, Q_n \in \wp(P)$. The details of this proof are immediate by the definitions and therefore left to the reader — the crucial case of the inductive step is immediate by the definition (5).

Now suppose that the formula $tr(\sigma \approx \tau)$ is not valid in the class $\mathsf{L}^s(\mathsf{G})$. Then there is some projective geometry $\mathcal{G}$ in G such that $tr(\sigma \approx \tau)$ does not hold in $\mathcal{L}^s_{\mathcal{G}}$; that is, there are $H, S_0, S_1, S_2, \vec{Q}$ such that $\mathsf{tripod}(H, S_0, S_1, S_2)$ holds in $\mathcal{L}^s_{\mathcal{G}}$ while $\overline{\sigma}^{\mathcal{L}^s_{\mathcal{G}}}(\vec{Q}, H, S_0, S_1, S_2) \neq \overline{\tau}^{\mathcal{L}^s_{\mathcal{G}}}(\vec{Q}, H, S_0, S_1, S_2)$. It follows from (12) that $T = (H, S_0, S_1, S_2)$ is a tripod in $\mathcal{G}$; thus, by Lemma 3.3 $\mathcal{D}_T$ is a representable diagonal-free cylindric algebra of dimension 3. But we may also, now using (13), infer that $\sigma^{\mathcal{D}_T}(Q_1 \cap A_T, \ldots, Q_n \cap A_T) \neq \tau^{\mathcal{D}_T}(Q_1 \cap A_T, \ldots, Q_n \cap A_T)$. But then we have shown that the equation $\sigma \approx \tau$ is not valid in the class RDf_3.

For the opposite direction, assume that the equation $\sigma \approx \tau$ is not valid in the class RDf_3. By Fact 2.3(5) we may assume that the equation can be refuted in some diagonal-free cylindric set algebra $\mathcal{A}$ with unit $V = Y_0 \times Y_1 \times Y_2$ for some non-empty, *countable* sets Y_i, $i < 3$. By Lemma 3.4, this algebra $\mathcal{A}$ can be embedded in the structure $\mathcal{D}_{\mathcal{G},T}$, where $\mathcal{G}$ is some infinite geometry in $\mathsf{L}^s(\mathsf{G})$ of dimension at least three, and $T = (H, S_0, S_1, S_2)$ is an appropriate tripod in $\mathcal{G}$. It follows that we can also refute $\sigma \approx \tau$ in $\mathcal{D}_T$, so $\sigma^{\mathcal{D}_T}(\vec{Q}] \neq \tau^{\mathcal{D}_T}[\vec{Q})$ for some $\vec{Q} \subseteq A_T$. By (12), $\mathsf{tripod}(H, S_0, S_1, S_2)$ holds in $\mathcal{L}^s_{\mathcal{G}}$ since (H, S_0, S_1, S_2) is a tripod in $\mathcal{G}$. Applying (13), we get $\overline{\sigma}^{\mathcal{L}^s_{\mathcal{G}}}(\vec{Q}, H, S_0, S_1, S_2) \neq \overline{\tau}^{\mathcal{L}^s_{\mathcal{G}}}(\vec{Q}, H, S_0, S_1, S_2)$. So $tr(\sigma \approx \tau)$ does not hold in $\mathcal{L}^s_{\mathcal{G}}$. QED

We are now in a position to bring together our previous results and prove our main technical result.

Proof of Theorem 3.1. Suppose G is a class of projective geometries which contains an infinite projective geometry of dimension $d \geq 3$. Let $\sigma \approx \tau$ be a Df_3-equation. By the previous Lemma, $\sigma \approx \tau$ is valid in the class RDf_3 iff $tr(\sigma \approx \tau)$ is valid in the class of Lyndon algebras over G. We already say that the translation tr turns Df_3-equations into universal formulas in the language of simplified Lyndon algebras. Now it is relatively simple to show that $\mathsf{L}^s(\mathsf{G})$ is a discriminator class: the unary discriminator term is $x;1$. (All that is required is to show that for every non-empty set Q of points in a projective geometry $\mathcal{G} = (P, L, I)$ we have that $Q \bullet P = P$, and this is striaghtforward.) Hence, by Fact 2.2 this means that there is a recursive procedure of rewriting universal formulas into equations of simplified Lyndon algebras. So the equational theory of RDf_3 reduces to the equational theory of $\mathsf{L}^s\mathsf{G}$. The equational theory of RDf_3 is undecidable; hence, the equational theory of $\mathsf{L}^s(\mathsf{G})$ is undecidable as well. QED

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