On the stability of flexible permission structures

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Games with a permission structure are a type of cooperative games with transferable utility in which cooperation is restricted. In these games it is assumed that players can have veto power over other players. Two approaches are distinguished. In the *conjunctive approach*, a player needs permission from all his direct superiors to be able to cooperate. In the *disjunctive approach*, a players needs permission from only one of his direct superiors.

In this thesis we study the stability properties of these permission networks. In order to do so, we first create a new model that allows for superior-successor links to be created and severed and in which links have a cost to them. For the conjunctive approach we find that only trees and forests can be stable, as an agent can only receive less value when the amount of his direct superiors increases. In the disjunctive approach, having more direct superiors can increase the value allocated to a player. Whether trees can be stable or not depends in this case on the size of the link cost.

Contents

1 Introduction

Game theory studies multi-agent decisions, both in situations where agents act individually (strategic game theory) and situations in which cooperation is possible (cooperative game theory). Agents are assumed to be rational, in the sense that they want to maximize their utility. A cooperative game now consists of a set of agents N and a value function v which determines how much value can collectively be created by each coalition. This gives rise to two key questions:

-Which coalitions will form?
-How do we distribute the value generated by the coalitions among the agents?

These two questions can however not be treated in complete isolation from each other. The way in which utility is distributed will influence which set of coalitions is stable (if any) and the coalitions that have formed will determine how much utility there is to be distributed. We usually fix one of the two in order to study the other.

In this thesis we consider a *cooperative game with transferable utility* (a TU-game). In this setting it is assumed that utility can be transferred from one individual to another. This is only possible in situations where players have a common *currency* that is valued the same by every player. In the classical setting of a TU-game, players only differ with respect to the contributions they can make to certain coalitions. Players are assumed to be able to cooperate with any other player and every subset of the set N is thus a feasible coalition. In the literature, several models of games have been developed in which the cooperation is restricted in some way.

In many papers, network structures are used to illustrate the cooperation restrictions. Various assumptions can be made in this context. The links in the network can be directed or undirected. Networks with undirected links are for example used to model communication structures [15,19,20], but can also be used for trade situations between fully informed corporations. [11] Directed links usually relate to relations between unequal agents. Such situations include unidirectional communication of information, domination such as in sports competitions and authority relations. [11,22] In both cases a distinction can be made between approaches for which only the connectedness of a group of agents matters [19] and approaches in which the exact architecture of the network matter. [15, 20, 22]

A last difference that can be found is whether links are assumed to have a cost or not. This is mostly related to whether the paper discusses an allocation rule or whether it discusses the stability of a network. In the last case extra assumptions are necessary with respect to when a link will be formed or severed. In case of directed links it is then often assumed that one agent is the initiator of the link and the other agent can accept or reject. [11, 22]

An important subclass of authority networks are permission structures. These are a type of directed network that explicitly model situations in which agents have veto power over other agents. This means that in games with a permission structure there are players that need permission from other players, their direct superiors, to be able to cooperate. The permission structure thus determines the possible coalitions. Different assumptions can be made in the context of permission structures. In the *conjunctive approach*, as developed in Gilles, Owen & Van den Brink (1992) [13], it is assumed that a player needs permission from all his direct superiors to be able to cooperate. In the *disjunctive approach* on the other hand, as discussed in Gilles & Owen (1999) [12], it is assumed that a player needs permission from at least one of his direct superiors in order to cooperate. The disjunctive and conjunctive approach are further discussed in Van den Brink (1997, 2003, 1999) [2–4], Van den Brink & Gilles (1996) [7], Van den Brink et al. (2015) [8] and Gilles (2010) [11]

The motivation behind analyzing permission structures is that many economic organizations adopt a hierarchical authority structure. Constraints imposed by an organizational structure can influence payoffs considerably as they influence the possible coalitions. To illustrate this we consider a simple example. Consider a production situation with 3 players, $N = \{i, j, h\}$ and suppose i is the only productive player and creates an output of 1. This can be described in terms of a TU-game as follows: $v(E) = 1$ if $i \in E$ and $v(E) = 0$ otherwise for all $E \subseteq N$. If we now assume both j and h are superiors of i we get a different situation. Under the conjunctive approach this game can now be described as follows: $v_c(E) = 1$ if $E = N$ and $v_c(E) = 0$ otherwise. The game v_c is referred to as the conjunctive restriction of the game v. Under the disjunctive approach the game will look somewhat different as in this situation i does not need both j and h in order to be able to be productive, but only one of the two. The disjunctive restriction is now as follows: $v_d(E) = 1$ if $\{i, j\} \subseteq E$ or $\{i, h\} \subseteq E$ and $v_d(E) = 0$ otherwise.

A game with a permission structure consists of a set of players, a value function describing the potential outcomes of the possible coalitions and a permission structure which has the form of a mapping from the set of players N to the powerset of N. This mapping assigns to every player in N a set which determines this player's successors. A modified game is then defined in which the worth of coalitions is restricted according to the permission structure. The relevant coalitions in the modified game are only those that are *autonomous*. This is the case in the disjunctive approach if for every agent in the coalition it holds that at least one of his direct superiors (if he has any) is also part of the coalition. In the conjunctive approach a coalition is autonomous if for every agent in the coalition *all* of his direct superiors are part of the coalition as well. A coalition that is not autonomous will have the value of its largest autonomous subset.

A solution or allocation rule for these games is a function that assigns to every game with a permission structure a distribution of payoffs over the individual players. Allocation rules for TU-games can be applied to the modified games to give rise to a solution for games with a permission structure. For example, the Banzhaf value applied to games with a permission structure has been studied in Van den Brink (2003, 2010). [4, 5] In this thesis we use the Shapley value, which has already been studied in several papers for both the disjunctive and the conjunctive approach and has been referred to as the *disjunctive permission value* and the *conjunctive permission value*.

The Shapley value, as introduced in Shapley (1953) [21], considers all the possible orders in which agents can join a coalition and assigns to every player his average marginal contribution over all these possible orders. The aim of the Shapley value is to determine how important each player is to the value generated by the grand coalition and distribute the total value accordingly. The Shapley value thus focusses on the *fairness* of the distribution and not on whether this distribution creates *stability* or not. Apart from as an allocation function, Shapley viewed his value also as a measure of the power of each player within a game. [25]

The disjunctive and conjunctive permission values have been axiomatized in different ways and it has been found that these two values can be characterized by almost the same set of axioms [2–4, 7]; they only differ with respect to the fairness axiom. The fairness axiom for the disjunctive permission value states that deleting a link between a player (who has more than 1 direct superior) and his direct superior changes the value of these two players by the same amount. For the conjunctive permission value however, this axiom states that deleting such a link will change the payoff of this player and the other direct superiors by the same amount.

The research thus far on permission structures has focused on allocation rules and their axiomatization. However, another interesting topic of investigation is to consider the stability properties of the links that make up a permission structure under the assumption of one specific allocation rule. This is the topic of this thesis. As mentioned before, we will consider the Shapley permission value.

In order to study stability properties, we introduce *games with a flexible permission structure*. These are games in which there is a *permission basis* which determines the direction of possible links that may or may not be formed. We restrict ourselves to permission bases that are cycle-free. A permission basis can be such that certain players cannot form a link with each other in either direction. A game now consists of a triple (N, v, g) where g is one of the possible graphs based on the permission basis.

We assume that links have a certain cost and that all links cost the same for both players involved. The payoff of a player is now the value assigned to him by the Shapley permission value minus the costs of the links he has. We assume that a link will be formed whenever for both players involved the benefits of this link are at least as much as the cost of forming a link. We assume that a link will be broken if at least one of the players involved receives a higher payoff without this link than with.

We find that these assumptions lead to very different stability results for the conjunctive and the disjunctive approach. For the conjunctive approach we find that adding a direct link with a player that is already an indirect superior will not change the set of feasible coalitions and is therefore never beneficial. We also find that for transparent graphs, any link that is broken does change the set of feasible coalitions. Lastly, we show that a player with more than one direct superior will never be worse off by breaking the link with one of those direct superiors. We end by concluding that only trees and forests can be stable in the conjunctive approach.

For the disjunctive approach, unlike the conjunctive approach, we find that a player is never worse off by having more direct superiors. Furthermore, we examine the conditions under which a link adds value to the players involved and find this to be the case, depending on v, as long as there is at least one coalition that becomes feasible after forming the link. We conclude that non-transparent graphs are not necessarily unstable under the disjunctive approach. If the cost of a link is small enough, a tree may thus not be stable in the disjunctive approach, unlike the conjunctive approach. Lastly, we take a closer look at the effect a new link has on the disjunctive permission value of the players involved in the context of the existing graph, in order to get a better idea of which links are more or less likely to form.

The rest of this thesis is organized as follows. In chapter 2 we give an introduction to games with a permission structure and we introduce the conjunctive and the disjunctive approach. We will also briefly discuss some notions from network structures. In chapter 3 we introduce games with a flexible permission structure as an extension to games with a permission structure. In chapter 4 we discuss the stability of flexible permission structures both under the conjunctive approach and under the disjunctive approach. Chapter 5 then presents some examples to illustrate the results of chapter 4. Finally, chapter 6 will give concluding remarks as well as some recommendations for future work.

2 Preliminaries

2.1 Games with permission structures

A situation in which a finite set of players N can generate a certain amount of payoff depending on which coalitions they form is called a *cooperative game with transferable utilities* (a TU-game). Such a game consists of a pair (N, v), where v is a characteristic function $v : 2^N \to \mathcal{R}$ and $v(\emptyset) = 0$. In this thesis, as in most papers, we take the set of players to be fixed and the set of all TU-games on N is then denoted with $\mathcal{G}^N$. An allocation rule is a function that assigns to every TU-game a payoff distribution over the players in N.

In a TU-game, the players only differ in terms of how much they contribute to the payoff a coalition can obtain. Players are assumed to be able to cooperate with any other player. In games with permission structure, however, it is assumed that players are part of a structure which limits the coalitions that can be formed. In this type of games the players are part of a structure in which some players will need permission from certain other players to be allowed to cooperate.

For a finite set of players $N \in \mathbb{N}$ such a structure, called a *permission structure*, is represented by a mapping $S : N \to 2^N$. $j \in S(i)$ denotes that j is a *successor* of i in the permission structure S and that i is a *direct superior* of j. The set of all direct superiors of j is given by $S^{-1}(j) := \{i \in N | j \in S(i)\}$. Furthermore, $\widehat{S} : N \to 2^N$ is a mapping that gives the transitive closure of S. We say that $j \in \widehat{S}(i)$ is true if and only if there is a finite sequence of players $j_1, ..., j_k$ in N such that $j_1 = i, j_k = j$ and $j_{h+1} \in S(j_h)$ for all $1 \le h \le k-1$. The players in $\widehat{S}(i)$ are called the *subordinates* of i and the players in $\widehat{S}^{-1}(i) := \{j \in N | i \in \widehat{S}(j)\}$ are called the *superiors* of i in S. The collection of all permission structures on N is denoted by $\mathcal{S}^N$.

We say that a permission structure S is *acyclic* if it holds for every player $i \in N$ that $i \notin \widehat{S}(i)$. A permission structure S is *quasi-strongly connected* if there exists an $i \in N$ such that $\widehat{S}(i) = N \backslash \{i\}$. The set $\beta_S := \{i \in N | S^{-1}(i) = \emptyset\}$ denotes those players that do not have any (direct) superiors. We call those players *boss players.* A permission structure S is hierarchical if it is both acyclic and quasi-strongly connected. A permission structure S is *weakly connected* if for every bipartition of N into N_1 and

N_2 it holds that there is a player $i \in N_1$ and a player $j \in N_2$ such that $i \in S(j)$ or $j \in S(i)$. Every quasi-strongly connected permission structure is also weakly connected. The opposite, however, does not hold. We denote the set of hierarchical permission structures with $\mathcal{S}_H^N$. As shown by Van den Brink and Gilles (1994) [6], for hierarchical permission structures it holds that there exists a unique boss player, $|\beta_S| = 1$. We denote the set of wealy connected, acyclic permission structures with $\mathcal{S}_W^N$. A triple (N, v, S) with $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}^N$ is called a *game with a permission structure.*

Several assumptions can be made about the way in which the permission structures affects the possibilities for cooperation. In the conjunctive approach, as developed by Gilles, Owen and Van den Brink (1992) [13], it is assumed that every player needs permission from all his direct superiors to cooperate with other players. The result of this restriction is that a coalition E can only form if for every player $i \in E$, all the (direct) superiors are also in E. In particular, E must contain all boss players j in S that have one or more subordinates in E. Thus a coalition E is *feasible* if and only if $\widehat{S}^{-1}(E) \subset E$, where $\widehat{S}^{-1}(E) := \bigcup_{i \in E} \widehat{S}^{-1}(i)$. These sets are called *conjunctive autonomous* coalitions. The set of all conjunctive autonomous coalitions for a permission structure $S \in \mathcal{S}^N$ is given by:

$$\Phi_S^c := \{E \subset N | \forall i \in E, \widehat{S}^{-1}(i) \subset E\}.$$

We call F the *authorizing set* of E if F is the smallest autonomous superset of E:

$$\alpha^c(E) := \bigcap \{F \in \Phi_S^c | E \subset F\}$$

We call F the *conjunctive souvereign part* of E in S if F is the largest autonomous subset of E:

$$\sigma^c(E) := \bigcup \{F \in \Phi_S^c | F \subset E\}.$$

The souvereign part of E consists of all players in E whose superiors are all part of E as well. Using this concept we can now transform a game v into a game that takes into account the restricted possibilities of cooperation enforced by the permission structure S. The resulting game is called a *conjuctive restriction* of v:

$$\mathcal{R}^c_S(v)(E) := v(\sigma^c(E)) \qquad \text{for all } E \subseteq \text{N}.$$

An *allocation rule* for games with a permission structure is a function that assigns to every game with a permission structure (N, v, S) a payoff distribution while taking into account the restricted cooperation possibilities. We will look at the following allocation rule:

$$\phi^c(v, S) := Sh(\mathcal{R}^c_S(v)) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } S \in \mathcal{S}^N.$$

$Sh : \mathcal{S}^N \rightarrow \mathbb{R}$ denotes the Shapley value given, for all $i \in N$, by:

$$Sh_i(v) = \sum_{E \ni i} \frac{\Delta_v(E)}{|E|},$$

where $|E|$ denotes the size of coalition E and the *dividends* $\Delta_v(E)$ are given by:

$$\Delta_v(E) := \sum_{F \subseteq E} (-1)^{|E|-|F|} v(F).$$

The *conjunctive permission value* of a game (N, v, S) is now defined as follows:

$$\phi^c(v, S) := \sum_{E \ni i} \frac{\Delta_{\mathcal{R}^c_S(v)}(E)}{|E|} = \sum_{\substack{F \ni i \\ F = \alpha(F)}} \sum_{\substack{F = \alpha(E) \\ E \subseteq N}} \frac{\Delta_v(E)}{|F|}.$$

Alternatively, in the disjunctive approach, as discussed in Gilles and Owen (1999) [12], a player needs permission from only one of his superiors to be allowed to cooperate with other players. Consequently, a coalition E can be formed only if for every i in E there is a path in the graph from a boss player to i. The coalitions that are feasible are called *disjunctive autonomous* coalitions. The set of all disjunctive autonomous coalitions for a permission structure $S \in \mathcal{S}^N$ is given by:

$$\Phi^d_S := \{E \subseteq N | \forall i \in E \backslash \beta_S, S^{-1} \cap E \neq \emptyset\}.$$

We call F an *authorizing set* of E if F is a smallest autonomous superset of E. The set of authorizing sets of E in the disjunctive approach is given by:

$$\mathcal{A}(E) := \{F \in \Phi^d_S | E \subset F \wedge \neg\exists G \in \Phi^d_S \text{ s.t. } E \subseteq G \subset F\}.$$

Note the difference with the conjunctive approach. In the conjunctive approach every set $E \subseteq N$ has exactly one authorizing set, because a player needs permission from *all* his superiors. In the disjunctive approach however, a coalition can have multiple authorizing sets, because it is enough for a player $i \in N$ to get permission from only *one* of his superiors. We use $F \sim \alpha^d(E)$ to denote that F an authorizing set is for E in the disjunctive approach, or in other words that F is in $\mathcal{A}(E)$. We define $\mathcal{A}^*(E)$ as the set of all finite unions of authorizing sets for coalition E. Thus, $F \in \mathcal{A}^*$ if and only if there are $F_i \in \mathcal{A}(E), 1 \leq i \leq I$ such that $F = \bigcup_{i=1}^{I} F_i$.

We call F the *disjunctive souvereign part* of E in S if F is the largest autonomous subset of E:

$$\sigma^d(E) := \bigcup\{F \in \Phi^d_S | F \subset E\}.$$

The souvereign part of E consists of all players i in E for which there exists a path from a boss player and all players in that path (including the boss player) are in E . Using this concept we can now transform a game v to take into account the restricted possibilities of cooperation enforced by the permission sructure S according to the disjunctive approach. The resulting game is called a *disjuctive restriction* of v:

$$\mathcal{R}^d_S(v)(E) := v(\sigma^d(E)) \qquad \text{for all } E \subseteq \text{N}.$$

The *disjunctive permission value* is defined in the same way as the conjunctive permission value, as the Shapley value of the restricted game v:

$$\phi^d(v,S) := Sh(\mathcal{R}^d_S(v)) \qquad \text{for all } v \in \mathcal{G}^N \text{ and } S \in \mathcal{S}^N.$$

The disjunctive permission value of a game (N, v, S) is thus defined as:

$$\phi^d(v,S) := \sum_{E \ni i} \frac{\Delta_{\mathcal{R}^d_S(v)}(E)}{|E|}.$$

The following example illustrates the difference between the conjunctive and the disjunctive approach(see Figure 1):

Let $N = \{i,j,k,l\}$ and $v \in \mathcal{G}^N$ be given by $v(E) = 1$ for all $E \ni l$ and $v(E) = 0$ otherwise. Let $S \in \mathcal{S}^N$ be given by

$$S(i) = \{j,k\}, S(k) = S(j) = \{l\}, S(l) = \emptyset$$

The conjunctive and disjunctive restrictions of v on S are now given by:

$$\mathcal{R}^c_S(v)(E) = \begin{cases} 1, & \text{if } E = N \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathcal{R}^d_S(v)(E) = \begin{cases} 1, & \text{if } E \supseteq \{i,j,l\} \text{ or } E \supseteq \{i,k,l\} \\ 0, & \text{otherwise} \end{cases}$$

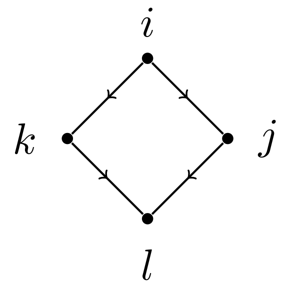


Figure 1

The conjunctive and disjunctive permission values are now respectively $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$ and $\{\frac{5}{12}, \frac{1}{12}, \frac{1}{12}, \frac{5}{12}\}$.

Axiomatizations for the disjunctive and conjunctive permission value for hierarchical permission structures have been given by Van den Brink (1997, 2003, 2015) [2–4] and Van den Brink & Gilles (1996) [7]. It is shown that both can be characterized by 6 axioms of which 5 are the same. [4] The first two axioms are generalizations of *additivity* and *efficiency* of solutions for TU-games.

Axiom 2.1 (Efficiency) *For every $N \subset \mathbb{N}, v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$ it holds that*
$\Sigma_{i \in N} \phi(N, v, S) = v(N)$.

Axiom 2.2 (Additivity) *For every $N \subset \mathbb{N}$, $v, w \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$ it holds that*
$\phi(N, v + w, S) = \phi(N, v, S) + \phi(N, w, S)$,
where $(v + w) \in \mathcal{G}^N$ is defined by $(v + w)(E) = v(E) + w(E)$ for all $E \subset N$.

We call a player $i \in N$ a *null player* if for every coalition $E \subset N$ $v(E) = v(E \backslash \{i\})$. The null player axiom of the Shapley value states that the payoff for a null player is equal to zero. However, in a game with permission structure it might be that, although i is a null player, there are subordinates of i that are not null players. In the case where a non-null player need permission from player i it seems reasonable to give player i a non-null payoff. However, if all subordinates of null player i are also null player we would think it reasonable that i gets a payoff equal to zero. We say that such a player $i \in N$ is an *inessential player* in the game with permission structure (N, v, S).

Axiom 2.3 (Inessential player property) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ and $S \in \mathcal{S}_H^N$ it holds that if $i \in N$ is an inessential pleyer in (N,v,S) then $\phi_i(N, v, S) = 0$.*

Axioms 2.4 and 2.5 are stated for monotone characteristic functions. A characteristic function v is monotone if $v(E) \leq v(F)$ for all $E \subset F \subset N$. We denote the class of all games with monotone v by $\mathcal{G}_M^N$. We call a player i *necessary* in a game (N, v) if $v(E) = 0$

for all $E \subset N\backslash\{i\}$. Thus, a necessary player can always guarantee that other players earn nothing by refusing to cooperate. It seems reasonable that a necessary player gets as least as much as any other player in a monotone game.

Axiom 2.4 (Necessary player property) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}_M^N$ and $S \in \mathcal{S}_H^N$, if $i \in N$ is a necessary player in (N, v) then $\phi_i(N, v, S) \geq \phi_j(N, v, S)$ for all $j \in N$.*

We say that a player i dominates player j completely when all paths from the boss player to j include player i. The set of all players that are completely dominated by player i is denoted by:

$$\overline{S}(i) = \{j \in \widehat{S}(i) | E \in \Phi_S^d \text{ and } j \in E \text{ implies } i \in E\}.$$

Note that in the conjunctive approach $\overline{S}(i) = \widehat{S}(i)$. The next axiom states that if player i completely dominates player j then player i gets at least as much payoff as j.

Axiom 2.5 (Weak structural monotonicity) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}_M^N$, $S \in \mathcal{S}_H^N$ and $i \in N$ it holds that if $j \in \overline{S}(i)$ then $\phi_i(N, v, S) \geq \phi_j(N, v, S)$.*

The five axioms defined thus far are satisfied by both the disjunctive and the conjunctive permission value. These two values differ in the last axiom; *fairness*. For the disjunctive permission value, fairness states that deleting a link between two players i and $j \in S(i)$ changes the payoff of these two players by the same amount. Moreover, also the payoff of all players h that completely dominate i change with the same amount. With $S_{-ij}(i)$ we denote $S(i)\backslash\{j\}$ and $S_{-ij}(h)$= $S(h)$ for all $h \in N\backslash\{i\}$.

Axiom 2.6 (Disjunctive fairness) *For every $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$, $S \in \mathcal{S}_H^N$ and $i \in N$, if $j \in S(i)$ and $|S^{-1}(j)| \geq 2$ then*

$\phi_h(N, v, S) - \phi_h(N, v, S_{-ij}) = \phi_j(N, v, S) - \phi_j(N, v, S_{-ij})$ for all $h \in \{i\} \cup \overline{S}^{-1}(i)$. Where $\overline{S}^{-1}(i) = \{h \in \widehat{S}^{-1}(i) | i \in \overline{S}(h)\}$.

The axiom of disjunctive fairness is not satisfied by the conjunctive permission value, but it does satisfy an alternative fairness axiom. The conjunctive fairness axiom states that deleting a link between a player i and $j \in S(i)$, where $|S^{-1}(j)| \geq 2$, changes the payoff of j and any $h \in S^{-1}(j) \backslash \{i\}$ by the same amount. Moreover, the payoffs of all players that completely dominate one of these direct superiors h also change with the same amount.

Axiom 2.7 (Conjunctive fairness) *For every* $N \subset \mathbb{N}$, $v \in \mathcal{G}^N$ *and* $S \in \mathcal{S}_H^N$, *if* $j \in N$ *and* $i, h \in S^{-1}(j)$ *then*

$$\phi_g(N, v, S) - \phi_g(N, v, S_{-ij}) = \phi_j(N, v, S) - \phi_j(N, v, S_{-ij}) \text{ for all } g \in \{h\} \cup \overline{S}^{-1}(h).$$

The conjunctive and disjunctive permission value can now be characterized by the above axioms.

Theorem 2.8 (Van den Brink (2003)) *An allocation rule* ϕ *is equal to the disjunctive permission value if and only if* ϕ *satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.*

Theorem 2.9 (Van den Brink (2003)) *An allocation rule* ϕ *is equal to the conjunctive permission value if and only if* ϕ *satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.*

We note that all the axioms are stated for hierarchical permission structures. However, only one direction of the proof requires this restriction. For both the disjunctive and the conjunctive permission value, the proof that this value satisfies all axioms stated does not make use of the assumption that the permission structure is hierarchical, but only of the assumption that it is acyclic.

2.2 Games with network structures

Network structures, like permission structures, also restrict the coalitions that can be formed. This restriction is represented by a graph, where the nodes are the players and the links represent pairwise relations. The complete graph on a set of players $N \in \mathbb{N}$ is the set of all subsets of N with size 2 and is denoted by g^N. The set of all possible graphs of N consists of all those graphs g for which it holds that $g \subseteq g^N$. A graph is thus defined by the links it has. With ij we denote a *link* between players i and j. If $ij \in g$ then i and j are directly connected in graph g, while if $ij \notin g$ then this is not the case. With g_{+ij} we denote the graph that results from adding the link ij to graph g. Thus $g_{+ij} = g \cup \{ij\}$. With g_{-ij} we denote the graph that results from deleting the link ij from the graph g (i.e. $g_{-ij} = g \backslash \{ij\}$).

We use $N(g) = \{i | \exists j \text{ s.t. } ij \in g\}$ to denote the set of players in N that are connected to another player by a link. A *path* between i and j in g exists if and only if there is a set of distinct players $h_1, h_2 ... h_k \in N$ such that $\{ih_1, h_1h_2, ..., h_kj\} \subseteq g$. A graph $D \subseteq g$ is a *component* of g if D is a maximal connected subset of g. In other words, D is a component of g if for all $i, j \in N(D), i \neq j$, there exists a path in D connecting i and j and for all $i \in N(D), j \in N(g)$ it holds that if $ij \in g$ then $ij \in D$. We use $C(g)$ to denote the set of all the components in g. Note that a single player is not considered a component. A component always has at least one link. We say that a characteristic function v is *component additive* if $v(g) = \sum_{D \in C(g)} v(D)$

In network structures, the graphs are not fixed. Several assumptions can be made about the formation and severance of links. In this thesis we follow Jackson & Wolinsky (1995) [15] in assuming that the formation of a link requires the consent of both players, while the severance can be done unilaterally. A graph g is said to be *pairwise stable* with respect to a characteristic function v and some payoff function ϕ if:

- for all $ij \in g, \phi_i(g, v) \geq \phi_i(g_{-ij}, v)$ and $\phi_j(g, v) \geq \phi_j(g_{-ij}, v)$

  and
- for all $ij \notin g$, if $\phi_i(g, v) < \phi_i(g_{+ij}, v)$ then $\phi_j(g, v) > \phi_j(g_{+ij}, v)$

The first condition says that there is no player that would get a higher payoff after breaking a link. The second condition states that for each link that can be formed one of the two players involved will have a better payoff without this link than with. This condition contains the assumption that a player will always accept the formation of a link, initiated by another player, if it does not affect his payoff negatively. Furthermore, note that this form of stability assumes that the formation and severance of links happen one at a time. This is a relatively weak notion of stability. Other notions of stability, for example one that allows for group decisions, are of course possible, but will not be considered for now.

Furthermore, we also follow Jackson & Wolinsky (1995) [15] in assuming that a link has a certain cost, which can be interpreted as the cost of maintaining a connection with a different player. We take the cost of each link to be the same. Thus $c_{ij} = c_{gk}$ for any $i, j, g, k \in N$. We will therefore simply use c. The total cost that a player i has to pay is simply the sum of the costs of all the links this player has:

$$c_i(g) := \sum_{\substack{j \in N \\ ij \in g}} c.$$

With $c(g) := \sum_{i \in g} c_i(g)$ we denote the sum over all the players of their cost in graph g. The total value of the graph is now defined as follows:

$$v^*(g) = v(g) - c(g)$$

A graph g is considered *efficient* if for all $g' \subset g^N$ it holds that $v^*(g) \geq v^*(g')$. Efficiency defined in this way indicates maximal total value.

In studying the properties of cooperative games, *unanimity games* often prove useful, as each cooperative game can be written in terms of its unanimity basis. The *unanimity basis* of a game (N, v) is the set of unanimity games $\{u_E | E \subseteq N, E \neq \emptyset\}$ that are defined as follows:

$$u_E(F) = \begin{cases} 1, & \text{if } E \subset F \\ 0, & \text{otherwise} \end{cases}$$

As shown by Harsanyi (1959) [10], the game (N, v) can now be expressed by:

$$v = \sum_{\substack{E \subset N \\ E \neq \emptyset}} \Delta_v(E) \cdot u_E$$

Where Δ_v is the dividend as defined above.

3 Games with a flexible permission structure

In this chapter we set out a framework for games with a flexible permission structure. In our setting we assume there is a fixed *permission basis.* The permission basis determines which links can potentially form and thereby defines for each player who his possible superiors and successors are. The permission basis represents the order between players that exists prior to the formation of links, for example an order of proximity to a gas source. Furthermore, the permission basis indicates the boss players and will assure that no cycles form. A permission basis for a player set $N \in \mathbb{N}$ is a mapping $O : N \to 2^N$ which is transitive and assymetric. Thus for any two $i, j \in N$

$$j \in O(i) \text{ implies } i \notin O(j)$$

With $j \in O(i)$ we denote that i is a *potential (direct) superior* of j and that j is a *potential subordinate* of i. The set of all potential superiors of j is given by $O^{-1}(j)$. Note that a permission basis, by definition, is always acyclic. We say that a permission basis is *complete* if for all $i, j \in N, i \neq j$ it holds that either $i \in O(j)$ or $j \in O(i)$. We call a player i for which it holds that $O^{-1}(i) = \emptyset$ a *boss player.* The set of all boss players in a permission basis O is denoted with β_O. The collection of all permission bases on a set of players N is denoted with $\mathcal{O}^N$. We say that a permission basis is hierarchical when, aside from being acyclic, it is also quasi-strongly connected. The set of all hierarchical permission bases is denoted with $\mathcal{O}^N_H$.

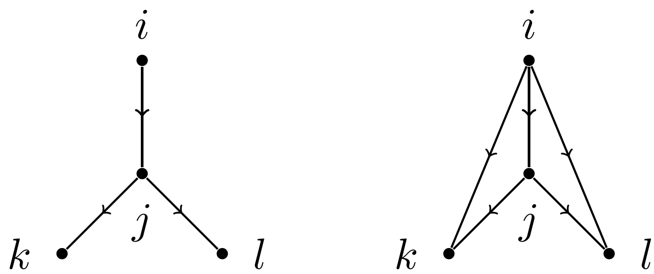


Fig 2a Fig 2b

Figure 2

Figure 2 shows two directed graphs, where the arrows denote the mapping O. The arrow from i to j for example, denotes $j \in O(i)$. The graph in Figure 2a is not a permission basis as it is not transitive. Figure 2b, on the contrary, is an example of a permission basis. However, it is not a *complete* permission basis as $l \notin O(k)$ and $k \notin O(l)$.

Throughout this thesis we will study graphs as a representation for the flexible permission structures. The possible graphs, given a permission basis O, are those graphs g for which it holds that:

for all $ij \in g$, $i \in O(j)$ or $j \in O(i)$

Note that we use undirected links and that the interpretation of these links, who is who's successor, is given by the permission basis. Although ij and ji are thus the same link, we will for clarity use ij when $j \in O(i)$. The maximal graph is now equal to the permission basis O and is denoted by G_O. The possible graphs given a permission basis O consists thus of all those graphs g for which it holds that $g \subseteq G_O$. These graphs will always be acyclic, since they are restricted by a permission basis O.

With $j \in S_{g,O}(i)$ we denote that j is a *successor* of i in the graph $g \subseteq G_O$. This is the case if and only if $ij \in g$ and $j \in O(i)$. i is called a *direct superior* of j. The set of all direct superiors of j is given by $S^{-1}_{g,O}(j) := \{i \in N | j \in S_{g,O}(i)\}$. Furthermore, $\widehat{S}_{g,O} : N \rightarrow 2^N$ is a mapping that gives a transitive closure of $S_{g,O}$. We say that $j \in \widehat{S}_{g,O}(i)$ holds if and only if there is a finite sequence of players $j_1, ..., j_k$ in N such that $j_1 = i, j_k = j$ and $j_{h+1} \in S_{g,O}(j_h)$ for all $1 \leq h \leq k-1$. Thus, $j \in \widehat{S}_{g,O}(i)$ if and only if there is a path from i to j in g. We call j a *subordinate* of i. The players in $\widehat{S}^{-1}_{g,O}(i) := \{j \in N | i \in \widehat{S}_{g,O}(j)\}$ are called a *superiors* of i in g. Note the difference between O and $S_{g,O}$. O describes a *potential* successor relation and it determines which connections *can* form, but does not tell us which links *did* form in the graph g . We use $S_{g,O}$ on the contrary to denote which relations have actually formed in the graph under consideration. We say that a graph is *transparent* if for all $i, j \in N$ such that $i \in S_{g,O}(j)$ it holds that $i \notin \widehat{S}_{g-ji,O}(j)$.

The graph g, based on a permission basis O, now describes an authority structure on the players, but it does not yet specify the effect of this authority structure on the possible outcomes of cooperative behavior as described by a game v. As outlined above, we will look at two different approaches; the conjunctive approach and the disjunctive approach. We recall that in the conjunctive approach a coalition E is formable if and only if for every player $i \in E$ it holds that all the direct superiors of i are also in E. In a flexible permission structure g with permission basis O, a coalition is thus formable only if $\widehat{S}^{-1}_{g,O}(E) \subset E$. As in flexible permission structures there is no guarantee that there is a path between a player $i \in N$ and any of the boss players in the permission basis, we introduce as an extra requirement that a coalition is only feasible if for all $i \in E$, $\widehat{S}^{-1}_{g,O}(i) \cap \beta_O \neq \emptyset$. The set of all conjunctive autonomous coalitions for a graph $g \subseteq G_O$ is now given by:

$$\Phi^c_g := \{E \subseteq N | \forall i \in E \backslash \beta_O,\ \widehat{S}^{-1}_{g,O}(i) \cap \beta_O \neq \emptyset \text{ and } \widehat{S}^{-1}_{g,O}(i) \subset E\}.$$

We call F the authorizing set of E if F is the smallest autonomous superset of E:

$$\alpha^c_g(E) := \bigcap \{F \in \Phi^c_g | E \subset F\}$$

We call F the *conjunctive souvereign part* of E in $g \subseteq G_O$ if F is the largest autonomous subset of E:

$$\sigma^c_g(E) := \bigcup \{F \in \Phi^c_g | F \subset E\}.$$

We can now transform a game v to take into account the restricted possibilities of cooperation enforced by the graph $g \subseteq G_O$. The resulting game is called a *conjuctive restriction* of v:

$$\mathcal{R}^c_g(v)(E) := v(\sigma^c(E)) \qquad \text{for all } E \subseteq \text{N}.$$

The *allocation rule* we will look at for games with a flexible permission structure (N, v, g), is the Shapley value:

$$\phi^c(v,g) := Sh(\mathcal{R}^c_g(v)) \qquad \text{for all } v \in \mathcal{G}^N, O \in \mathcal{O}^N \text{ and } g \subseteq G_O.$$

The *conjunctive permission value*, is now defined in the same way as for non-flexible permission structures:

$$Sh_i(\mathcal{R}^c_g(v)) = \sum_{E \ni i} \frac{\Delta_{\mathcal{R}^c_g(v)}(E)}{|E|} = \sum_{\substack{F \ni i \\ F=\alpha(F)}} \sum_{\substack{F=\alpha(E) \\ E \subseteq N}} \frac{\Delta_v(E)}{|F|}.$$

Finally, to get the payoff of a player i in a conjunctive game (N,v,g) we also take into consideration the costs of all the links i has in the graph g. The final payoff x is now defined as follows:

$$x^c_i(v,g) := \phi^c_i(v,g) - c_i(g).$$

In the disjunctive approach, a player needs permission from only one of his superiors. A coalition E can thus be formed only if for every i in E $S^{-1}_{g,O}(i) \cap E \neq \emptyset$. As in the conjunctive approach we add the requirement that for every player $i \in E$ there is a path in the graph g from a boss player to i. The coalitions that are formable are called *disjunctive autonomous* coalitions. The set of all disjunctive autonomous coalitions for a graph $g \subseteq G_O$ is given by:

$$\Phi^d_g := \{E \subseteq N | \forall i \in E \backslash \beta_O,\ \widehat{S}^{-1}_{g,O}(i) \cap \beta_O \neq \emptyset \text{ and } S^{-1}_{g,O} \cap E \neq \emptyset\}.$$

We call F an authorizing set of E if F is a smallest autonomous superset of E. The set of authorizing sets of E in the disjunctive approach is given by:

$$\mathcal{A}_g(E) := \{F \in \Phi^d_g | E \subset F \wedge \neg\exists G \in \Phi^d_g \text{ s.t. } E \subseteq G \subset F\}.$$

We use F$\sim$ $\alpha^d_g(E)$ to denote that F an authorizing set for E is in the disjunctive approach. We define $\mathcal{A}^*(E)$ as the set of all finite unions of authorizing sets for coalition E. Thus, $F \in \mathcal{A}^*$ if and only if there are $F_i \in \mathcal{A}(E), 1 \leq i \leq I$ such that $F = \bigcup_{i=1}^I F_i$.

We call F the *disjunctive souvereign part* of E in $g \subseteq G_O$ if F is the largest autonomous subset of E:

$$\sigma_g^d(E) := \bigcup\{F \in \Phi_g^d | F \subset E\}.$$

The souvereign part of E consists of all those players i in E such that there exists a path from a boss player to i and all players in this path are in E. We can now transform a game v to get the *disjuctive restriction* enforced by the graph $g \subseteq G_O$:

$$\mathcal{R}_g^d(v)(E) := v(\sigma^d(E)) \qquad \text{for all } E \subseteq \text{N}.$$

The *disjunctive permission value* is again defined as the Shapley value of the restricted game v:

$$\phi^d(v,g) := Sh(\mathcal{R}_g^d(v)) \qquad \text{for all } v \in \mathcal{G}^N, O \in \mathcal{O}^N \text{ and } g \subseteq G_O.$$

The disjunctive permission value of a game (N, v, g) is thus:

$$Sh_i(\mathcal{R}_g^d(v)) = \sum_{E \ni i} \frac{\Delta_{\mathcal{R}_g^d(v)}(E)}{|E|}.$$

Finally, to get the payoff of a player i in a disjunctive game (N, v, g) we also take into consideration the costs of all the links i has in the graph g. The final payoff x is now defined as follows:

$$x_i^d(v,g) := \phi_i^d(v,g) - c_i(g).$$

In accordance with the approach of Jackson & Wolinsky (1995) [15], as discussed in the previous chapter, we assume that a link can be formed whenever it does not decrease the payoff of either of the players involved. We assume that a link is unstable when one of the two players will receive a higher payoff after the link has been broken.

4 Results on link formation

4.1 The conjunctive approach

We start by analyzing the effects of flexible links on the conjunctive approach. In the conjunctive approach a player i needs permission from all his superiors. One consequence of this assumption is that a coalition E is only feasible if for any player $i \in E$ it holds that all his superiors are also in E, whether these are direct superiors or not. We would thus expect that adding a link between i and j when $i \in \widehat{S}_{g,O}(j)$ is already a subordinate of j will not change the set of feasible coalitions. Moreover, as adding this link does add an extra cost, graphs in which such links exist would not be stable.

Theorem 4.1 *For any $v \in \mathcal{G}^N$, $g \subset G_O$ and $O \in \mathcal{O}^N$, and for any $i, j \in N$ such that $i \in \widehat{S}_{g,O}(j)$, and $i \notin S_{g,O}(j)$ it holds that $\phi_i^c(v, g) = \phi_i^c(v, g_{+ji})$.*

Proof:
Let $i \in \widehat{S}_{g,O}(j)$, but $i \notin S_{g,O}(j)$. Let g be an acyclic graph and $g' = g_{+ji}$. We will prove Theorem 4.1 by proving something stronger, namely that $\phi^c(v, g) = \phi^c(v, g')$.

We start by showing that the set of conjunctively autonomous coalitions is the same in g and g' ,$\Phi_g^c = \Phi_{g'}^c$. First note, that since $g \subset g'$ it follows that $\Phi_g^c \supseteq \Phi_{g'}^c$. This means that for any coalition $F \subseteq N$, $F \neq \sigma_g(F)$ implies that $F \neq \sigma_{g'}(F)$ (and $F = \sigma_{g'}(F)$ implies $F = \sigma_g(F)$).

Next we want to show that for all $F = \sigma_g(F)$ it holds that $F = \sigma'_g(F)$. We know that $F = \sigma_g(F)$ if and only if $\widehat{S}^{-1}(F) \subset F$ and thus that for all $F \in \Phi_g^c$ such that $i \in F$ it must hold that $j \in F$. Since the only difference between g and g' the link ji is, it now follows for all $F = \sigma_g(F)$ that $F = \sigma'_g(F)$. Since we now have that $F \neq \sigma_g(F)$ implies $F \neq \sigma_{g'}(F)$ and $F = \sigma_g(F)$ implies $F = \sigma'_g(F)$, we can conclude that $\Phi_g^c = \Phi_{g'}^c$.

As $\Phi_g^c = \Phi_{g'}^c$, we know that for any coalition $E \subseteq N$, $\sigma_g^c(E) = \sigma_{g'}^c(E)$. Therefore, $\mathcal{R}_g^c(v)(E) = \mathcal{R}_{g'}^c(v)(E)$ and thus $\phi^c(v, g) = \phi^c(v, g')$.
□

As we proved the stronger claim that $\phi^c(v, g) = \phi^c(v, g')$, we know that the formation of this link does not change the conjunctive permission value for any player in N. Since i and j have more direct links in g' than in g we have that $c_i(g) < c_i(g')$ and $c_j(g) < c_j(g')$. It then follows that both $x_i^c(v, g') < x_i^c(v, g)$ and $x_j^c(v, g') < x_j^c(v, g)$ and thus the link ji will not be formed.

The proof above does not only show that no link will form between two players $i, j \in N$ such that $i \in \widehat{S}_{g,O}(j)$, but also that a graph in which $i \in S_{g,O}(j)$ and the link ji is not the only path from j to i, is unstable. In other words, it follows from Theorem 4.1 that non-transparent graphs are not stable in the conjunctive approach, as for any non-transparent graph g there is a transparent graph g' such that $\Phi_g^c = \Phi_{g'}^c$ and $c(g) > c(g')$. Another thing that the proof of Theorem 4.1 shows is that for any graph g the set of the conjunctive autonomous coalitions in g is the same as for the transitive closure of g. This conclusion follows directly from our observation that $\Phi_g^c = \Phi_{g+ji}^c$ for any $i \in \widehat{S}_{g,O}(j) \backslash S_{g,O}(j)$.

In any graph g in which there is no component without boss player and for any i such that $S_{g,O}^{-1}(i) = j$, breaking the link ji would change the set of autonomous coalitions, as removing such a link would make all autonomous coalitions that contain i nonautonomous. Theorem 4.2 states that for any transparent graph removing a link between an agent i and a superior of i also changes the set of conjunctive autonomous coalitions when i has more than one superior. It then follows that for any such graph there exists a $v \in \mathcal{G}^N$ such that $\mathcal{R}_g^c(v) \neq \mathcal{R}_{g-ji}^c(v)$ for $i \in S_{g,O}(j)$.

Theorem 4.2
Let $g \subseteq G_O$ with $O \in \mathcal{O}^N$ be a transparent graph such that $i, j, h \in N$ and $i \in S_{g,O}(j) \cap S_{g,O}(h)$. It then holds that $\Phi_g^c \neq \Phi_{g-ji}^c$.

Proof:
As g is transparent we know that $i \notin \widehat{S}_{g,O}(j) \backslash S_{g,O}(j)$ and thus $i \notin \widehat{S}_{g-ji,O}(j)$. Now take a coalition $E = \alpha_{g-ji}^c(\{i\})$. Clearly $E \in \Phi_{g-ji}^c$. As $i \notin \widehat{S}_{g-ji,O}(j)$ we know that j cannot be in E. This means that $S_{g,O}^{-1}(i) \not\subset E$ and that E is not autonomous in g. Thus, we can conclude that for a transparent graph g and any $i \in S_{g,O}(j) \cap S_{g,O}(h)$ it holds that $\Phi_g^c \neq \Phi_{g-ji}^c$.

□

We now know which links do and do not change the set of conjunctive autonomous coalitions, but to know which links are stable we need to know more. As links can be broken unilaterally, it follows that any link ji that has a negative effect on the conjunctive permission value of either j or i will be broken. Since in the conjunctive approach a player i needs permission from all his superiors we can expect all those superiors to be able to demand a part of i's value contribution. Thus, for $v \in \mathcal{G}_M^N$ a monotone characteristic function, we would expect a player in the conjunctive approach to be better off, or in other words to receive a higher payoff, when he has less superiors.

It is shown by Van den Brink (1999) [3] that for any monotone v and hierarchical permission structure S with $i, j, h \in N$, $j \neq h$ and $i \in S(j) \cap S(h)$ it holds that $\phi_i^c(v, S) \leq \phi_i^c(v, S_{-ji})$. We will show that this result holds in a more general case. Theorem 4.3 states that in any acyclic graph g, based on a permission basis O with $i, j, h \in N, j \neq h$ and $i \in S_{g,O}(j) \cap S_{g,O}(h)$ it holds that $\phi_i^c(v, g) \leq \phi_i^c(v, g_{-ji})$ The following proof follows the same reasoning as the proof by Van den Brink (1999) [3].

Theorem 4.3

For any $v \in \mathcal{G}_M^N, g \subseteq G_O$ and $O \in \mathcal{O}^N$ with $i, j, h \in N, j \neq h$ and $i \in S_{g,O}(j) \cap S_{g,O}(h)$ it holds that $\phi_i^c(v, g) \leq \phi_i^c(v, g_{-ji})$.

Proof:

Note that for all $E \subseteq N$, $\sigma_g^c(E) \subseteq \sigma_{g_{-ji}}^c(E)$ and $E \not\supseteq \{i, h\}$ and for $E \ni j$ we have that $\sigma_g^c(E) = \sigma_{g_{-ji}}^c(E)$. It then follows for monotone v that:

$\phi_i^c(v, g) - \phi_i^c(v, g_{-ji}) \leq 0$ iff:

$$\sum_{E \ni i} (v(\sigma_g^c(E)) - v(\sigma_g^c(E \backslash \{i\})) - (v(\sigma_{g_{-ji}}^c(E)) + v(\sigma_{g_{-ji}}^c(E \backslash \{i\})))$$
$$= \sum_{E \ni i} (v(\sigma_g^c(E)) - v(\sigma_{g_{-ji}}^c(E))) \leq 0$$

Where the last inequality follows from the monotonicity of v.

□

Note that it is not necessarily the case that an acyclic graph $g \subseteq G_O$ consist of exactly one component. However, even if g consists of more than one component, it still holds that $\sigma_g^c(E) \subseteq \sigma_{g-ji}^c(E)$. Furthermore, if v is a component additive characteristic function, the different components of graph g can be considered seperately and adding or removing a link within a component $C \subseteq g$ will then not affect any players outside component C. From Theorem 4.3 it thus follows that for $v \in \mathcal{G}_M^N, g \subseteq G_O$ and $O \in \mathcal{O}_W^N$ a graph in which some player i has more than one direct superior is not stable. The conjunctive approach thus leads to a situation where the only stable graphs are forests or trees.

It clearly holds that any player that is part of a component $C \subseteq g$ that does not contain any boss player, is a null player. As all players in such a component are null players it follows for all these players i, by the inessential player property of the conjunctive permission value, that $\phi_i^c(v, g) = 0$. The payoff for any $i \in C$ will thus be $-c_i(g)$. Recall that $c_i(g)$ is the sum of the costs of all the links that player i has in graph g. Any player i in such a component would obtain a higher payoff ($x_i^c(v, g') = 0$) if he would break all his links. It thus follows that a graph g which has any inessential players is unstable. When $O \in \mathcal{O}_H^N$ is hierarchical there will be at most one component that contains players that are not inessential players, as there is only one boss player. For hierarchical permission bases it thus follows that a stable graph will have at most one component.

Furthermore, when v is superadditive, any player i such that $v(\{i\}) > 2 \cdot c$ will have at least some possible links that will be beneficial for him. In particular, forming a direct link with a boss player will increase both i's payoff and the boss player's payoff with at least $v(\{i\}) - c$. Thus, in a stable graph, any such player i will be (indirectly) connected to a boss player.

When the cost of forming a link is relatively high, stable graphs will be only those with a flat hierarchy. This is the case since a player loses part of his value to his superiors. The more superiors he has, the less value he is left with. In general it is thus better for a player to be directly connected to a boss player, as this will maximize his payoff.

However, as we will show in the next chapter, a flat hierarchy does not necessarilly yield the best payoff for the boss player.

4.2 The disjunctive approach

In the disjunctive approach, a player needs permission from only one of his direct superiors. This implies that a player with more direct superiors has more leverage towards his superiors and could thus claim a higher payoff. For monotone v and a hierarchical permission structure S it has indeed been shown by Van den Brink (1999) [3] that in the disjunctive approach for $i, j, h \in N$, $j \neq h$ and $i \in S(j) \cap S(h)$ it holds that $\phi_i^d(v, g) \geq \phi_i^d(v, g_{-ji})$ and $\phi_j^d(v, g) \geq \phi_j^d(v, g_{-ji})$. We will show that this result holds in a more general case. Theorem 4.4 states that in the disjunctive approach in any acyclic graph $g \subseteq G_O$, based on a permission basis $O \in \mathcal{O}^N$ it is still true for $i, j, h \in N$ with $j \neq h$ and $i \in S(j) \cap S(h)$ that $\phi_i^d(v, g) \geq \phi_i^d(v, g_{-ji})$. By disjunctive fairness it then follows that $\phi_j^d(v, g) \geq \phi_j^d(v, g_{-ji})$ holds as well. The proof follows the same structure as the proof by Van den Brink (1999) [3].

Theorem 4.4
For any $v \in \mathcal{G}_M^N, g \subseteq G_O$ and $O \in \mathcal{O}^N$ with $i, j, h \in N, j \neq h$ and $i \in S_{g,O}(j) \cap S_{g,O}(h)$ it holds that $\phi_i^d(v, g) \geq \phi_i^d(v, g_{-ji})$.

Proof:
Note that for all $E \subseteq N$, $\sigma_g^d(E) \supseteq \sigma_{g_{-ji}}^d(E)$ and for $E \not\supseteq \{i, j\}$ we have that $\sigma_g^d(E) = \sigma_{g_{-ji}}^d(E)$. It then follows for monotone v that:

$\phi_i^d(v, g) - \phi_i^d(v, g_{-ji}) \geq 0$ iff:

$$\sum_{E \ni i} (v(\sigma_g^d(E)) - v(\sigma_g^d(E \backslash \{i\})) - (v(\sigma_{g_{-ji}}^d(E)) + v(\sigma_{g_{-ji}}^d(E \backslash \{i\})))$$
$$= \sum_{E \ni i} (v(\sigma_g^d(E)) - v(\sigma_{g_{-ji}}^d(E))) \geq 0$$

Where the last inequality follows from the monotonicity of v.

□

In order for the link between i and j to be stable, however, it must hold for both players that $\phi_i^d(v,g) - \phi_i^d(v,g_{-ji}) \geq c$ and $\phi_j^d(v,g) - \phi_j^d(v,g_{-ji}) \geq c$. The benefit of the link must thus at least be strictly greater than 0. This can only be the case if the disjunctive restriction of v is different after the link was made; $\mathcal{R}_g^d(v) \neq \mathcal{R}_{g_{+ji}}^d(v)$. Theorem 4.5 gives the necessary condition for $\mathcal{R}_g^d(v) \neq \mathcal{R}_{g_{+ji}}^d(v)$ to be true.

Theorem 4.5
Let $g \subset G_O$ and $O \in \mathcal{O}^N$ be such that $i \notin S_{g,O}(j)$ and $i \in O(j)$.
Then there exists a $v \in \mathcal{G}^N$ such that $\mathcal{R}_g^d(v) \neq \mathcal{R}_{g_{+ji}}^d(v)$ if and only if there is at least one $E \in \mathcal{A}_g(j)$ such that $E \cap S_{g,O}^{-1}(i) = \emptyset$.

Proof:
Let E be a coalition such that $E \in \mathcal{A}_g(j)$ and $E \cap S_{g,O}^{-1}(i) = \emptyset$ and let F be $E \cup \{i\}$. Note that $\mathcal{A}_g(j) = \mathcal{A}_{g_{+ji}}(j)$. As $E \cap S_{g,O}^{-1}(i) = \emptyset$ it follows that $\sigma_g^d(F) = E$. However, as $E \in \mathcal{A}_g(j)$ and thus $E \in \mathcal{A}_{g_{+ji}}(j)$, F is an autonomous coalition in graph g_{+ji}; $\sigma_{g_{+ji}}^d(F) = F$. Thus, for any $v \in \mathcal{G}^N$ which assigns a different value to E than to F it holds that $\mathcal{R}_g^d(v) \neq \mathcal{R}_{g_{+ji}}^d(v)$.

Now suppose there is no coalition $E \in \mathcal{A}_g(j)$ such that $E \cap S_{g,O}^{-1}(i) = \emptyset$. Note that for all $E \not\supseteq \{i,j\}$ it holds that $\sigma_g^d(E) = \sigma_{g_{-ji}}^d(E)$. Take a random $E \in \mathcal{A}_g(j)$. As before F is an autonomous coalition in g_{+ji}. However, as E is autonomous in g and E contains a direct superior of i, F is autonomous in g as well. As we chose E randomly we can conclude that for any $E \in \mathcal{A}_g(j)$, $\sigma_g^d(E \cup \{i\}) = \sigma_{g_{+ji}}^d(E \cup \{i\}) = E \cup \{i\}$. It now follows that $\Phi_g^d = \Phi_{g_{+ji}}^d$ and thus that for all $v \in \mathcal{G}^N$, $\mathcal{R}_g^d(v) = \mathcal{R}_{g_{+ji}}^d(v)$.
□

Theorem 4.5 shows that in the disjunctive approach a link between i and j will only be formed if adding this link will make some coalitions feasible that were not feasible without the link ji. We point out that this is always true for a direct link with a boss player. Moreover, once a player is directly connected to a boss player, the links he has with other superiors become superfluous and will be broken. However, in order for a link between i and j to be formed, it is not enough for that link to increase the set of feasible coalitions. It must also be the case that the disjunctive permission value of both players involved increases by forming this link. In the case of a monotone v, this will already be the case if the restriction of v assigns a higher value to at least one coalitions after the link ji has been formed. However, even when the disjunctivce permission value of both players increase, it may not increase enough to compensate for the cost of the link.

Theorems 4.4 and 4.5 tell us when a link between i and j can have a positive effect on the disjunctive permission value of both players, but they do not tell us anything about the size of this effect. Theorem 4.6 is a restatement of Theorem 4.4, but with stronger assumptions. Although the proof we give for theoren 4.6 is not applicable to as broad a situation as the proof of Theorem 4.4 is, it has as a merit that it gives a much more specific idea of the size of the effect of forming a link (the proof given earlier only tells us that the effect is not negative).

In the next proof we will distinguish two cases. The first covers the situations where $i \in \widehat{S}_{g,O}(j)$. In this case, j is already a superior of i before the link ji is formed, but forming the link will make j a direct superior of i as well. The second case covers the situation where $i \notin \widehat{S}_{g,O}(j)$. With g' we denote $g+ji$. We will use $\mathcal{A}_g(\{i\})$ to denote the authorizing sets of i in g and $\mathcal{A}_{g'}(\{i\})$ to denote the authorizing sets of i in g'. When these two sets have the same size, this means that there is exactly one shortest path p from i to j in g, such that $p \subset p_i$ for any other path p_i that may exist between i and j in g.

As Gilles & Owen (1999) [12] remark, in some cases it can be shown that the dividend of a coalition is always 1, 0 or -1, but in other cases there is no proof for this conjecture yet. For that reason, we will restrain ourselfs to graphs where for every coalition $F \in \mathcal{A}_g(\{i\})$ there exists a player $k \in F$ such that $k \notin G$ for all $G \in \mathcal{A}_g(\{i\})\backslash F$, as it has been shown that in acyclic permission structures and for $v = u_i$, the unanimity game of i, the dividend of coalitions with this property is either 1 or -1.

Theorem 4.6

Let v be superadditive, $g \subset G_O$ and $O \in \mathcal{O}^N$ be an acyclic graph and let $|\mathcal{A}_g(\{i\})| -$

$|\mathcal{A}_{g'}(\{i\})| = 0$ *when* $i \in \widehat{S}_{g,O}(j)$. *Let* g *be such that for every coalition* $F \in \mathcal{A}_g(\{i\})$ *there exists a player* $k \in F$ *such that* $k \notin G$ *for all* $G \in \mathcal{A}_g(\{i\})\backslash F$. *Furthermore, let it be the case that for any player* $k \neq i$, $k \neq j$ *that is part of the path between* j *and* i *in* g *it holds that* $\forall E \in \mathcal{A}_g(\{i\}), k \in E$ *if and only if* $j \in E$.
Then $v(\{i\}) > 0$ *implies* $\phi_i^d(v, g') > \phi_i^d(v, g)$.

Proof:
First, note that if g is such that for every coalition $F \in \mathcal{A}_g(\{i\})$ there exists a player $k \in F$ such that $k \notin G$ for all $G \in \mathcal{A}_g(\{i\})\backslash F$, then this also holds for g'. Suppose that this is not the case. Consider a coalition $E \in \mathcal{A}_g(\{i\}) \cap \mathcal{A}_{g'}(\{i\})$, with a player k that is not in any coalition $G \in \mathcal{A}_g(\{i\})\backslash E$. These are exactly those coalitions that do not contain j. Since for any $H \in \mathcal{A}_{g'}(\{i\})\backslash\mathcal{A}_g(\{i\})$ it holds that $H \subset G$ for some $G \in \mathcal{A}_g(\{i\})$ it follows that k is also not in any coalition $G' \in \mathcal{A}_{g'}(\{i\})\backslash E$.

Now suppose there is a coalition $E \in \mathcal{A}_{g'}(\{i\})\backslash\mathcal{A}_g(\{i\})$ such that there is no $k \in E$ such that $k \notin H$ for all $H \in \mathcal{A}_{g'}(\{i\})\backslash E$. However, as we assume that $|\mathcal{A}_g(\{i\})| - |\mathcal{A}_{g'}(\{i\})| = 0$, it holds that there is exactly one path from j to i in g. It follows that for any $H \in \mathcal{A}_{g'}(\{i\})\backslash\mathcal{A}_g(\{i\})$ there is exactly one $G \in \mathcal{A}_g(\{i\})\backslash\mathcal{A}_{g'}(\{i\})$ such that $H \subset G$. Therefore if it holds for some $E \in \mathcal{A}_{g'}(\{i\})\backslash\mathcal{A}_g(\{i\})$ that there is no $k \in E$ such that $k \notin H$ for all $H \in \mathcal{A}_{g'}(\{i\})\backslash E$, then it must also hold for the coalition $F \in \mathcal{A}_g(\{i\})$ such that $E \subset F$ that there is no $k \in F$ such that $k \notin G$ for all $G \in \mathcal{A}_g(\{i\})\backslash F$. We can thus conclude it is also true in g' that for every coalition $F \in \mathcal{A}_g(\{i\})$ there exists a player $k \in F$ such that $k \notin G$ for all $G \in \mathcal{A}_g(\{i\})\backslash F$.

Now consider the unanimity game u_i. We define $w_i := \mathcal{R}_g^d(u_i)$ as the disjunctive restricted value of the unanimity game u_i on graph g based on an acyclic permission basis O. Let $E \subseteq N$ be a coalition. The following holds for the dividend of E: [12]

- $E \notin \mathcal{A}_g(E) \Rightarrow \Delta_{w_i}(E) = 0$.
- $E \in \mathcal{A}_g(\{i\}) \Rightarrow \Delta_{w_i}(E) = 1$.
- $E \notin \mathcal{A}^*(\{i\}) \Rightarrow \Delta_{w_i}(E) = 0$.

Furthermore, let $\eta_i(E) := |\{E' \in \mathcal{A}_g(\{i\}) | E' \subset E\}|$ be the amount of coalitions in

$\mathcal{A}_g(\{i\})$ that are a subset of E. As we have that for every coalition $F \in \mathcal{A}_g(\{i\})$ there exists a player $k \in F$ such that $k \notin G$ for all $G \in \mathcal{A}_g(\{i\})\backslash F$, we know that for all $E \in \mathcal{A}^*(\{i\})$ [12]:

$$\Delta_{w_i}(E) = (-1)^{\eta_i(E)-1}.$$

Therefore, to prove Theorem 4.6 we need only look at coalitions E such that $E \in \mathcal{A}_g(\{i\}), E \in \mathcal{A}_{g'}(\{i\}), E \in \mathcal{A}^*_g(\{i\})$ or $E \in \mathcal{A}^*_{g'}(\{i\})$. Now consider a coalition E such that $i \in E$ and $j \notin E$. We know that $\Delta_{w_i}(E) = w_i(E) - \sum_{S \subset E} \Delta_{w_i}(S)$. Since $j \notin E$, E will be autonomous in g' if and only if E is autonomous in g. It thus follows that $w_i(E) = w'_i(E)$. For all $S \subset E$ the same holds and we can thus conclude that $\Delta_{w_i}(E) = \Delta'_{w_i}(E)$ for all $E \not\ni j$.

We can now conclude that to prove Theorem 4.6 we only need to consider those $E \subseteq N$ for which it holds that $j \in E$ and $E \in \mathcal{A}_g(\{i\}), E \in \mathcal{A}_{g'}(\{i\}), E \in \mathcal{A}^*_g(\{i\})$ or $E \in \mathcal{A}^*_{g'}(\{i\})$. There are two different cases we need to distinguish; $i \in \widehat{S}_{g,O}(j)$ or $i \notin \widehat{S}_{g,O}(j)$. In the first case there already exists a path from j to i in g. In the second case, such a path does not exist in g (but clearly it does exist in g'). Let's first consider the case where $i \in \widehat{S}_{g,O}(j)$.

Lemma 4.7

There is no coalition $E \ni i, j$ such that $E \in \mathcal{A}_g(\{i\})$ and $E \in \mathcal{A}_{g'}(\{i\})$.

Proof:

Let E be such that $i, j \in E$ and $E \in \mathcal{A}_{g'}(\{i\})$. By definition of $\mathcal{A}$ it must hold that there is only one path between j and i. As $i \notin S_{g,O}(j)$, E is then not an autonomous coalition in g and therefore $E \notin \mathcal{A}_g(\{i\})$.

Let E be such that $i, j \in E$ and $E \in \mathcal{A}_g(\{i\})$. As $i \notin S_{g,O}(j)$, for any path from j to i there must be some agent k different form j and i, such that k is part of this path. Since $i \in S_{g',O}(j)$ there does exist a path in g' without any such k. This path is clearly shorter than any of the paths between j and i in g. Thus, for any E be such that $i, j \in E$ and $E \in \mathcal{A}_g(\{i\})$ there exists a coalition F in g' such that F is a strict subset of E. Thus E will not be a smallest superset of i in g' and therefore $\notin \mathcal{A}_{g'}(\{i\})$.

$\square$

By definition of $\mathcal{A}^*_g$ and our specifications of graph g it now follows easily from Lemma 4.7 that there is also no $E \ni i, j$ such that $E \in \mathcal{A}^*_g(\{i\})$ and $E \in \mathcal{A}^*_{g'}(\{i\})$.

We define $r := |\mathcal{A}_g(\{i\}) \cap \mathcal{A}_{g'}(\{i\})|$ as the amount of coalitions in $\mathcal{A}_g(\{i\})$ that do not contain j. Note that this amount is the same in g as in g'. We define $\mathscr{L}$ as the size of a coalition containing j in $\mathcal{A}_g(\{i\})$ and $\mathscr{L}'$ as the size of such a coalition in $\mathcal{A}_{g'}(\{i\})$. Let $E, F \in \mathcal{A}_{g'}(\{i\}) \backslash \mathcal{A}_g(\{i\})$ be two coalitions containing j in $\mathcal{A}_{g'}(\{i\})$. We define $L := |E \backslash F|$ as the amount of players in E that are not in F. For simplicity we assume that L is the same irrespective of which two coalitions we pick, but we will argue later that this does not matter for the validity of our proof. We make the same assumption for all coalitions containing j in $\mathcal{A}_g(\{i\})$. Furthermore, note that the assumption that $|\mathcal{A}_g(\{i\})| - |\mathcal{A}_{g'}(\{i\})| = 0$ entails that in g there is exactly one path p from j to i such that $p \subset E$ for some $E \in \mathcal{A}_g(\{i\})$. From this it also follows that for any two coalitions $E, F \in \mathcal{A}_g(\{i\}) \backslash \mathcal{A}_{g'}(\{i\})$ we can find two coalitions $E', F' \in \mathcal{A}_{g'}(\{i\}) \backslash \mathcal{A}_g(\{i\})$ such that $E \backslash F = E' \backslash F'$. We define $k := \mathscr{L} - \mathscr{L}'$ as the difference in size between a coalition in $\mathcal{A}_g(\{i\})$ containing j and a coalition in $\mathcal{A}_{g'}(\{i\})$ containing j. Note that $k = |p| - 2$, as there is exactly one shortest path p from j to i. We define $n := |\mathcal{A}_{g'}(\{i\}) \backslash \mathcal{A}_g(\{i\})|$ as the amount of coalitions containing j in $\mathcal{A}_{g'}(\{i\})$. We note that in the first case this is the same as the amount of coalitions containing j in $\mathcal{A}_g(\{i\})$ (by the assumption that $|\mathcal{A}_g(\{i\})| - |\mathcal{A}_{g'}(\{i\})| = 0$). We define $\mathscr{R}$ as the number of agents in a coalition in $\mathcal{A}_g(\{i\}) \cap \mathcal{A}_{g'}(\{i\})$ that are not in a coalition $\mathcal{A}_{g'}(\{i\}) \backslash \mathcal{A}_g(\{i\})$. Since we know that for all $E \in \mathcal{A}_g(\{i\})$ and $k \in p \backslash \{i, j\}$ (we recall that p is the shortest path between j and i in g) $k \in E$ if and only if $j \in E$,we know that the set of these $\mathscr{R}$ for $\mathcal{A}_g(\{i\}) \backslash \mathcal{A}_{g'}(\{i\})$ is the same as for the coalitions in $\mathcal{A}_{g'}(\{i\}) \backslash \mathcal{A}_g(\{i\})$.We assume this number to be the same for any coalition in $\mathcal{A}_g(\{i\}) \cap \mathcal{A}_{g'}(\{i\})$, but we will show that this assumption does not affect the validity of our proof.

Let n be 1. Now $\phi^d_i(v, g') - \phi^d_i(v, g) = Sh'_i(w_i) - Sh_i(w_i)$. As argued above we can limit ourself to those coalitions $E \in \mathcal{A}_g(\{i\}), E \in \mathcal{A}_{g'}(\{i\}), E \in \mathcal{A}^*_g(\{i\})$ and $E \in \mathcal{A}^*_{g'}(\{i\})$ such that $j \in E$. From this it follows that $Sh_i(w_i) - Sh'_i(w_i) = \sum_{x=0}^{r} \binom{r}{x} (-1)^x \frac{1}{\mathscr{R}x + \mathscr{L}'} -$

$\sum_{x=0}^{r}\binom{r}{x}(-1)^x\frac{1}{\mathscr{R}x+\mathscr{L}}$, where $x = \eta_i(E) - 1$ is equal to the amount of coalitions $F \in \mathcal{A}_g(\{i\}) \cap \mathcal{A}_{g'}(\{i\})$ such that $F \subset E$. Note that $(-1)^x$ is equal to the dividend of a coalition and $\mathscr{R}x + \mathscr{L}$ is equal to the size of a coalition. We now get that:

$$
\begin{aligned}
\sum_{x=0}^{r}\binom{r}{x}(-1)^x\frac{1}{\mathscr{R}x+\mathscr{L}'} - \sum_{x=0}^{r}\binom{r}{x}(-1)^x\frac{1}{\mathscr{R}x+\mathscr{L}} &= \\
\sum_{x=0}^{r}\binom{r}{x}(-1)^x\left(\frac{1}{\mathscr{R}x+\mathscr{L}-k} - \frac{1}{\mathscr{R}x+\mathscr{L}}\right) &= \\
\int_0^1\sum_{x=0}^{r}\binom{r}{x}(-1)^x\left(z^{\mathscr{R}x+\mathscr{L}-k-1} - z^{\mathscr{R}x+\mathscr{L}-1}\right)dz &= \\
\int_0^1\sum_{x=0}^{r}\binom{r}{x}(-1)^x(z^{\mathscr{R}})^x\left(z^{\mathscr{L}-k-1} - z^{\mathscr{L}-1}\right)dz &= \\
\int_0^1(1-z^{\mathscr{R}})^r\left(z^{\mathscr{L}-k-1} - z^{\mathscr{L}-1}\right)dz &= \\
\int_0^1(1-z^{\mathscr{R}})^r(1-z^k)z^{\mathscr{L}-k-1}\,dz &> 0.
\end{aligned}
$$

Where the fourth equality holds by the fact that $(y+z)^r = \sum_{x=0}^{r}\binom{r}{x}y^{r-x}z^x$

Now suppose $\mathscr{R}$ is not the same for every coalition in $\mathcal{A}_g(\{i\}) \cap \mathcal{A}_{g'}(\{i\})$ that is not in $\mathcal{A}_g(\{i\}) \backslash \mathcal{A}_{g'}(\{i\})$. Let $\mathscr{R}_a$ be smallest of these $\mathscr{R}$ and $\mathscr{R}_z$ be the biggest. As the value of $Sh'_i(w_i) - Sh_i(w_i)$ increases with $\mathscr{R}$, the true value of $Sh'_i(w_i) - Sh_i(w_i)$ in the case of differing $\mathscr{R}$ must be somewhere in between the value we obtain when using $\mathscr{R}_a$ everywhere and the value we obtain when using $\mathscr{R}_z$ everywhere. We can therefore conclude that $Sh'_i(w_i) - Sh_i(w_i) > 0$ will still hold.

Next, we cancel the assumption that $n = 1$. $Sh'_i(w_i) - Sh_i(w_i)$ is now equal to :

$$\sum_{y=1}^{n}\binom{n}{y}\sum_{x=0}^{r}\binom{r}{x}(-1)^{x+y+1}\left(\frac{1}{\mathscr{R}x+\mathscr{L}-k+L(y-1)}-\frac{1}{\mathscr{R}x+\mathscr{L}+L(y-1)}\right)=$$

$$\int_0^1\sum_{y=1}^{n}\binom{n}{y}\sum_{x=0}^{r}\binom{r}{x}(-1)^{x+y+1}\left(z^{\mathscr{R}x+\mathscr{L}-k+L(y-1)-1}-z^{\mathscr{R}x+\mathscr{L}+L(y-1)-1}\right)dz=$$

$$\int_0^1\sum_{y=1}^{n}\binom{n}{y}\sum_{x=0}^{r}\binom{r}{x}(-1)^{x}(z^{\mathscr{R}})^{x}(-1)^{y+1}\left(z^{\mathscr{L}+L(y-1)-k-1}-z^{\mathscr{L}+L(y-1)-1}\right)dz=$$

$$\int_0^1\sum_{y=1}^{n}\binom{n}{y}(-1)^{y+1}(1-z^{\mathscr{R}})^{r}\left(z^{\mathscr{L}-k+Ly-L-1}-z^{\mathscr{L}+Ly-L-1}\right)dz=$$

$$\int_0^1\sum_{y=0}^{n}\binom{n}{y}(-1)^{y}(-1)(1-z^{\mathscr{R}})^{r}(z^{L})^{y}\left(z^{\mathscr{L}-k-L-1}-z^{\mathscr{L}-L-1}\right)-$$

$$\binom{n}{0}(-1)^{1}(1-z^{\mathscr{R}})^{r}(z^{L})^{0}\left(z^{\mathscr{L}-k-L-1}-z^{\mathscr{L}-L-1}\right)dz=$$

$$\int_0^1(-1)(1-z^{L})^{n}(1-z^{\mathscr{R}})^{r}\left(z^{\mathscr{L}-k-L-1}-z^{\mathscr{L}-L-1}\right)+$$

$$(1-z^{\mathscr{R}})^{r}\left(z^{\mathscr{L}-k-L-1}-z^{\mathscr{L}-L-1}\right)dz=$$

$$\int_0^1(-1)(1-z^{L})^{n}(1-z^{\mathscr{R}})^{r}(1-z^{k})z^{\mathscr{L}-k-L-1}+(1-z^{\mathscr{R}})^{r}(1-z^{k})z^{\mathscr{L}-k-L-1}\,dz>0.$$

The inequality at the end holds because $\int_0^1(1-z^L)^n\,dz$ will always have a value between 0 and 1.

We assumed that L is always the same, but of course this need not be the case. However, since we know that there is only one path between j and i in g, we know that for any coalition in $\mathcal{A}_g(\{i\})\backslash\mathcal{A}_{g'}(\{i\})$ with length $\mathscr{L}_n$ there is a coalition in $\mathcal{A}_{g'}(\{i\})\backslash\mathcal{A}_g(\{i\})$ with size $\mathscr{L}_n-k$. Therefore, if we take the set of the differences L between the coalitions in $\mathcal{A}_g(\{i\})\backslash\mathcal{A}_{g'}(\{i\})$ this is the same set as for the coalitions in $\mathcal{A}_{g'}(\{i\})\backslash\mathcal{A}_g(\{i\})$. Let L_a be smallest of these L and L_z be the biggest. The true value of $Sh_i'(w_i)-Sh_i(w_i)$ in the case of differing L must be somewhere in between the value we obtain when using L_a everywhere and the value we obtain when using L_z everywhere. We can therefore conclude that $Sh_i'(w_i)-Sh_i(w_i)>0$ will still hold.

Now let us continue with the second case, where $i\notin\widehat{S}_{g,O}(j)$. In this case $\mathcal{A}_g(\{i\})\subset$

$\mathcal{A}_{g'}(\{i\})$. As a result of this $r = |\mathcal{A}_g(\{i\})|$, $\mathscr{L} = 0$ and k becomes irrelevant. Thus we get that $Sh'_i(w_i) - Sh_i(w_i)$ is equal to:

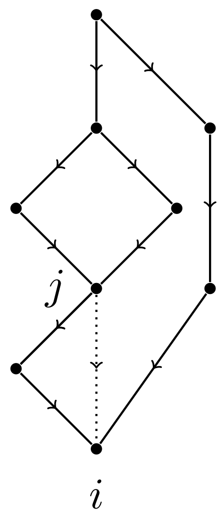
$$
\begin{aligned}
&\sum_{y=1}^{n}\binom{n}{y}\sum_{x=0}^{r}\binom{r}{x}(-1)^{x+y+1}\frac{1}{\mathscr{R}x + \mathscr{L}' + L(y-1)} = \\
&\int_0^1 \sum_{y=1}^{n}\binom{n}{y}\sum_{x=0}^{r}\binom{r}{x}(-1)^{x+y+1} z^{\mathscr{R}x+\mathscr{L}'+L(y-1)-1}\, dz = \\
&\int_0^1 \sum_{y=1}^{n}\binom{n}{y}\sum_{x=0}^{r}\binom{r}{x}(-1)^{x}(z^{\mathscr{R}})^{x}(-1)^{y+1} z^{\mathscr{L}'+L(y-1)-1}\, dz = \\
&\int_0^1 \sum_{y=1}^{n}\binom{n}{y}(-1)^{y+1}(1-z^{\mathscr{R}})^{r} z^{\mathscr{L}'+Ly-L-1}\, dz = \\
&\int_0^1 \sum_{y=0}^{n}\binom{n}{y}(-1)^{y}(-1)(1-z^{\mathscr{R}})^{r}(z^{L})^{y} z^{\mathscr{L}'-L-1} \; - \\
&\binom{n}{0}(-1)^{1}(1-z^{\mathscr{R}})^{r}(z^{L})^{0} z^{\mathscr{L}'-L-1}\, dz = \\
&\int_0^1 (-1)(1-z^{L})^{n}(1-z^{\mathscr{R}})^{r} z^{\mathscr{L}'-L-1} \; + \\
&(1-z^{\mathscr{R}})^{r} z^{\mathscr{L}'-L-1}\, dz > 0.
\end{aligned}
$$

With respect to both L and $\mathscr{R}$ we can reason in the same way as above. We can thus conclude that $Sh'_i(w_i) - Sh_i(w_i) > 0$ in both cases discussed. As v is superadditive it holds that $\Delta_v(E) \geq 0$ for any coalition $E \subseteq N$. As the disjunctive permission value satisfies the additivity axiom the game (N, v) can thus be written as the sum of the unanimity games of all the coalitions $E \subseteq N$ multiplied by positive integers. Therefore, we can conclude that Theorem 4.6 is true.

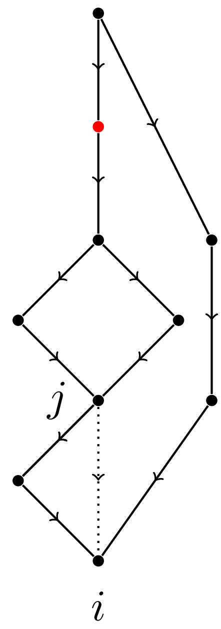
□

It is easy to see that this proof works more generally whenever there is a coalition $C \subseteq \widehat{S}_{g,O}(i) \cup \{i\}$ such that $v(C) > 0$. In that case, to look at the influence of the link ji we would consider all coalitions E such that $j \in E$ and $E \in \mathcal{A}_g(\{C\}), E \in \mathcal{A}_{g'}(\{C\}), E \in \mathcal{A}^*_g(\{C\})$ or $E \in \mathcal{A}^*_{g'}(\{C\})$. The rest of the proof then works exactly the same.

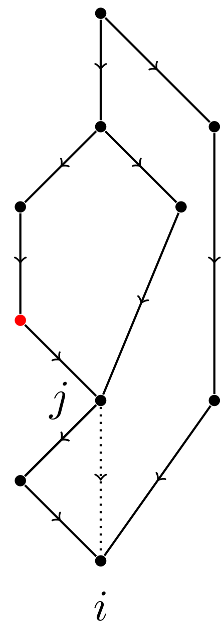
Based on the proof just given we can now investigate when a link is more or less



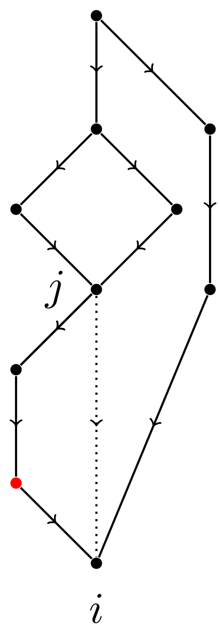
Graph g



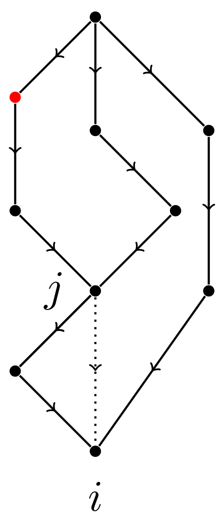
$g_1 : \mathscr{L} + 1$



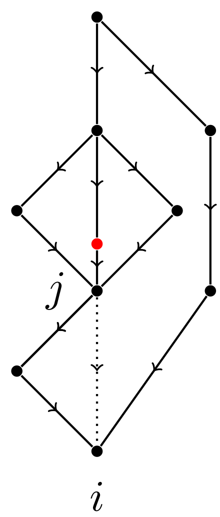
$g_2 : \mathscr{L}$ and L increase



$g_3 : \mathscr{L} + 1$ and $k + 1$



$g_4 : L + 1$



$g_5 : n + 1$

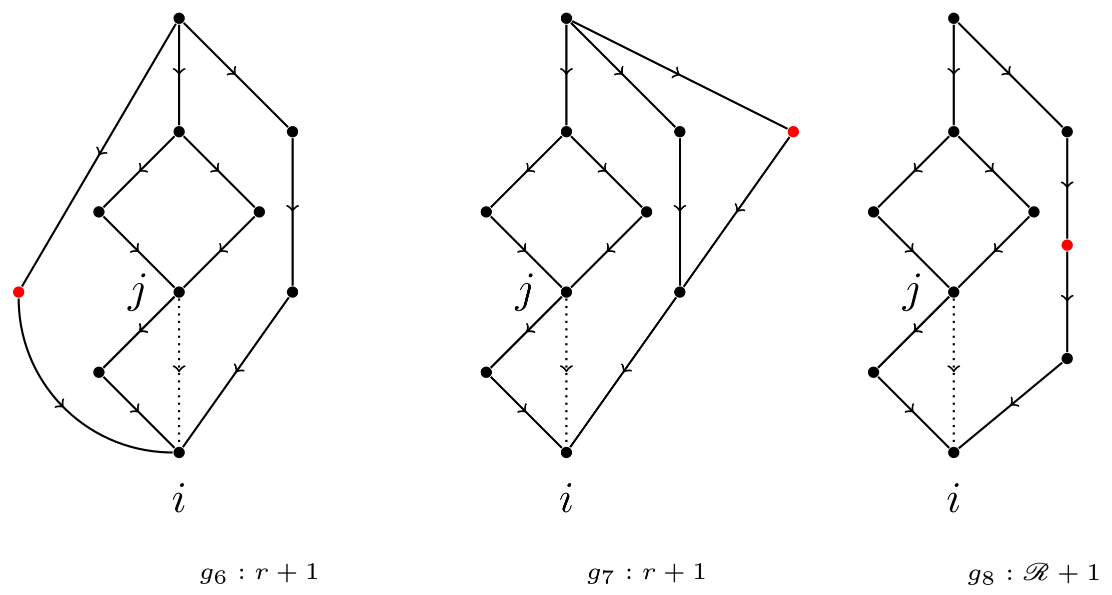


Figure 3

beneficial to make for players i and j. We will investigate this by looking at the effect of adding one superior of i to the graph g and considering whether this would make the effect of link ji on the disjunctive permission value for i and j bigger or smaller. We distinguish 7 cases of which one is not relevant for the situation were $i \notin \widehat{S}_{g,O}(j)$. Figure 3 shows a graph g in which $i \in \widehat{S}_{g,O}(j)$ and 8 different ways in which g can be extended to give i an extra superior. We will treat 2 of these cases as one, as they have the same effect. Unless stated otherwise, our discussion of the effect of the extra superior will hold for both the case where $i \in \widehat{S}_{g,O}(j)$ and the case in which $i \notin \widehat{S}_{g,O}(j)$

Graph g_1 shows a possible way of adding a superior of i to g that increases both $\mathscr{L}$ and $\mathscr{L}'$ with 1. As $z < 1$, we get that with an increasing $\mathscr{L}$ (or $\mathscr{L}'$ if $i \notin \widehat{S}_{g,O}(j)$), $z^{\mathscr{L}-L(-k)-1}$ decreases. The result of this is that the benefit of the link ji with respect to the disjunctive permission value for j and i decreases when i has more superiors of this type.

The added superior in graph g_2 increases $\mathscr{L}, \mathscr{L}'$ and L. If there is only one path from the boss player β to i in g, then this change will increase both $\mathscr{L}$ and L with 1 and the value of $z^{\mathscr{L}-L(-k)-1}$ will stay the same. However, the increase of L will make $(1 - z^L)^n$

increase, which in turn will make the total benefit of the link ji decrease. In case there is more than one path from β to i in g things get more complicated. As L will always be smaller than $\mathscr{L}$ the effect of the added player will be greater on the average value of L than on the average value of $\mathscr{L}$. This means that $\mathscr{L} - L(-k) - 1$ decreases which causes $z^{\mathscr{L}-L(-k)-1}$ to increase. As mentioned before, the increasing L has an decreasing effect on the total sum since it increases $(1 - z^L)^n$. It is thus unclear what the effect is of the addition of this type of superior on the benefit of the link ji when there is more than one path from β to i.

The graph g_3 shows a situation where $\mathscr{L}$ and k both increase with 1. Note that by assumption, there is always only one path between i and j in the case where $i \in \widehat{S}_{g,O}(j)$. In the case where $i \notin \widehat{S}_{g,O}(j)$ this situation does not exist. As $\mathscr{L}$ and k increase with the same amount $z^{\mathscr{L}-L-k-1}$ stays the same. However, $(1-z^k)$ increases as k increases. The result of this is that the benefit of the link ji with respect to the disjunctive permission value for j and i increases. This is very intuitive, as the 'shortcut' created by the link ji surpasses more players in g_3 than in g.

Graph g_4 shows a situation in which L increases with 1 and all other variables stay the same. Note, however, that if $n > 2$, L will still increase but with a smaller number than 1. Although $\mathscr{L}$ does not change in this case, we get the same situation as in g_2. The increase of L has a positive effect on the total sum by increasing the value of $z^{\mathscr{L}-L(-k)-1}$. However, as it also increases the value of $(1 - z^L)^n$ it has a negative effect on the sum at the same time.

In graph g_5 the added superior changes n. When $L = 1$, as is the case in g, adding this superior will only change n. However, if L is bigger than 1 in g the added superior will decrease L, of which, as noted before, the effect on the value of $Sh_i'(w_i) - Sh_i(w_i)$ is unclear. If we look at the effect of increasing n on its own, we see that it decreases the value of $(1 - z^L)^n$, which increases the total value.

Both g_6 and g_7 show a way in which r can be increased by 1. The way depicted in g_6 could have an effect on no other variable in case $\mathscr{R} = 1$ (or $r = 0$) in g. The option of g_7 will always decrease $\mathscr{R}$. As $\mathscr{R}$ decreases, $1 - z^{\mathscr{R}}$ will decrease as well. The increase of n means that $(1 - z^{\mathscr{R}})^n$ decreases even more. The benefit of the link ji with respect to the disjunctive permission value for j and i will thus decrease in this situation.

Graph g_8 lastly, shows the addition of a superior that will only effect $\mathscr{R}$. In this case $\mathscr{R}$ is increased by 1, but this will be less in case $r > 1$. When $\mathscr{R}$ increases $1 - z^{\mathscr{R}}$ increases as well and subsequenly the total benefit of the link ji increases.

If we now compare the stable graphs of the conjunctive approach with those of the disjunctive approach we can see a clear difference. In the conjunctive approach only trees and forests are stable. In the disjunctive approach, however, even non-transparent graphs can be stable. To see this, consider the following example (see Figure 4):

Let $N = \{h, i, j, k, l\}$ and $v \in \mathcal{G}^N$ be given by $v(E) = 1$ for all $E \ni i$, $v(E) = 2$ for all $E \supset \{i, j\}$ and $v(E) = 0$ otherwise. Let $O \in \mathcal{O}^N$ be given by

$$O(l) = \{h, i, j, k\}, O(k) = \{i, j\}, O(h) = \{i, j\}, O(j) = \{i\}, O(i) = \emptyset$$

The disjunctive restriction of v given the graph in Figure 4 is now given by:

$$\mathcal{R}_g^d(v)(E) = \begin{cases} 1, & \text{if } E \supseteq \{i, h, l\} \text{ and } E \not\ni j \\ 2, & \text{if } E \supseteq \{i, j, k, l\} \text{ or } E \supseteq \{i, j, h, l\} \\ 0, & \text{otherwise} \end{cases}$$

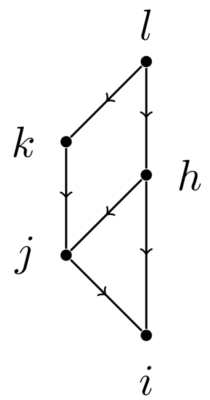


Figure 4

The graph g in Figure 4 is clearly not transparent, due to the link between i and h

and the fact that $i \in \widehat{S}_{g_{-hi},O}(h)$ via j. However, if we compare the disjunctive permission value for i in g ($\frac{23}{60}$) with those in $g_{-ji}(\frac{20}{60})$ and $g_{-hi}(\frac{18}{60})$, we see that as long as the cost of a link is small enough ($c < \frac{1}{20}$) i will not want to break any link. By disjunctive fairness we know that this means that both j and h do not want to break their link with i either.

Due to this difference an inefficient graph is more likely to form in the disjunctive approach than in the conjunctive approach. For any graph g which is not a tree, there exists a graph g' which is a tree such that the total value of these graphs is the same ($v^*(g) = v^*(g')$), but $c(g) > c(g')$. Hence, a graph that is not a tree can never be an efficient graph. In the conjunctive approach, a tree graph will never change into a graph that is not a tree, although as we've shown, in the disjunctive approach it could.

Although a tree graph can be efficient, this is not necessarilly the case. For example, a graph containing null players that are not boss players will not be efficient, as the same total value can be generated by a graph without such players. The next chapter will study some example situations for both the conjunctive and the disjunctive approach, of which several contain such null players.

5 Applications

In this section we discuss some applications for both conjuctive and disjunctive flexible permission structures. First we will use the conjuctive approach to study an organization structure where the only 'productive' players are the ones in the lowest level of the permission basis. We show that the organization structure will end up as a tree. Subsequently we will argue that the existence of an unproductive middleman can increase the payoff of the boss player. Next we use the disjunctive approach to look at a buyer-seller situation and we will show that in most cases, it is beneficial for both buyers and sellers to have more than one connection. We will also use the disjunctive approach to look at a buyer-seller situation in which there is a middleman who pays for the cost of transportation.

5.1 Organizations with unproductive superiors

Consider a hierarchical production organization in which the productive players are all in the lowest level. In other words for all i such that $v(\{i\}) > 0$ it holds that $O(i) = \emptyset$. We will use the conjunctive approach to show that this organization will transition into a tree structure.

Consider the player set $N = \{g, h, i, j, k, l, m\}$ and let the permission basis $O \in \mathcal{O}^N$ be defined by:

$$O(g) = N \backslash \{g\}, O(i) = \{j, k, l, m\}, O(j) = \{l, m\}, O(k) = O(l) = \{m\}, O(h) = O(m) = \emptyset$$

Let $v \in \mathcal{G}^N$ be defined as follows:

$$v(E) = \begin{cases} 1, & \text{if } E \ni h \text{ or } E \ni m \\ 2, & \text{if } E \supseteq \{h, m\} \\ 0, & \text{otherwise} \end{cases}$$

Graph g_1 in Figure 5 shows the maximal transparent graph based on permission basis O. We can start with the maximal transparent graph, since by Theorem 4.1 we know that non-transparent graphs are never stable under the conjunctive permission value.

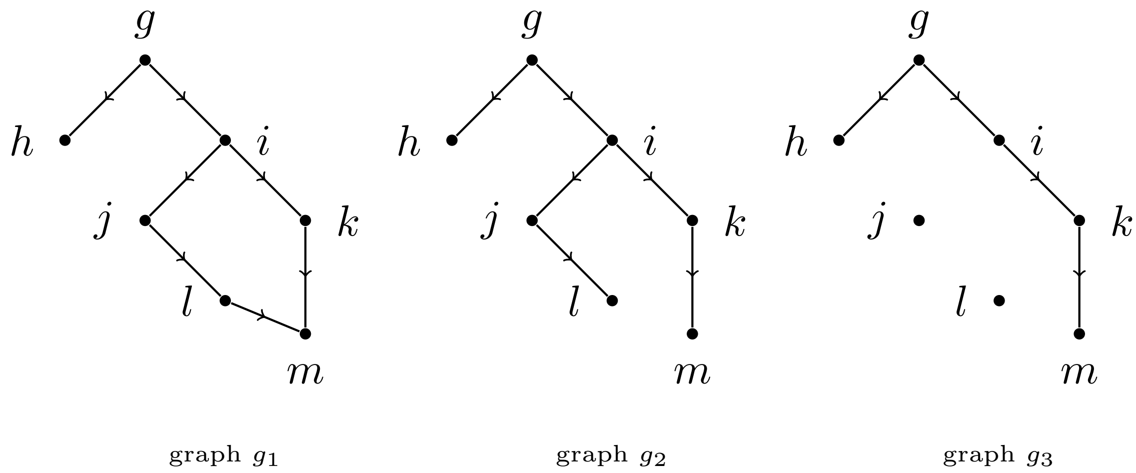


Figure 5

By Theorem 4.3 we know that m will receive a higher payoff when removing either the link with l or the link with k. Both links are thus unstable in g_1. Suppose m breaks the link with l. In the resulting graph g_2, the link km is stable, as without it the conjunctive permission value for both k and m will become 0. However, both j and l have become inessential players by the breaking of link lm. As mentioned in chapter 4.1, inessential players are better off when breaking all the links and the links that j and l are involved in are therefor not stable. We thus end up in graph g_3. It is easy to see that no player wants to break any link in g_3 and by Theorem 4.1 we also know that no player wants to form any link. Graph g_3 is thus stable.

Note that if m would have chosen to break the link with k, this would have had a less beneficial result for him. In g_3 the payoff for m is $\frac{1}{4} - c$. If m would have broken with k her payoff would have been $\frac{1}{5} - c$. Furthermore, note that the graph we start with matters for the end result. Had we started with a completely disconnected graph, we would have ended up with the graph $\{gh, gm\}$. The graph $\{gh, gm\}$ is the only efficient graph based on permission basis O and also gives the highest payoff for g, h and m. Our example thus shows that stable graphs are not necessarilly efficient graphs.

The next example shows that unproductive middlemen aren't always useless. In some cases they can increase the payoff of the boss players, and in that way incentivize unefficient graphs. By taking over the supervision(link cost) of several productive work-

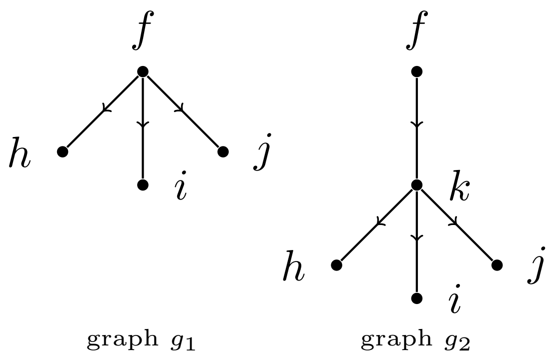


Figure 6

ers from the boss players, these middlemen can save the boss player some costs, thereby increasing the boss's payoff.

Let $N = \{f, h, i, j, k\}$, let $O \in \mathcal{O}^N$ be defined as follows:

$$O(f) = N \backslash \{f\}, O(k) = \{h, i, j\}, O(h) = O(i) = O(j) = \emptyset$$

Let $v \in \mathcal{G}^N$ be additive where $v(\{h\}) = v(\{i\}) = v(\{j\}) = 1$ and $v(\{f\}) = v(\{k\}) = 0$.

Now compare graph g_1 and g_2 in Figure 6. The costs for f in g_1 are three times as high as those in g_2. In return, f loses part of the payoff he can claim from h, i and j to player k. In g_1 the payoff for player f is $1\frac{1}{2} - 3c$ while h, i and j receive $\frac{1}{2} - c$. In g_2 player f receives $1 - c$ and h, i and j receive $\frac{1}{3} - c$. It thus follows that whenever $\frac{1}{4} < c < \frac{1}{3}$, player f will be better off in a structure with an unproductive middleman.

In the case of unproductive superiors and an additive v a structure with a middleman is only beneficial for the boss player when there are at least 3 productive workers under the supervision of a middleman. To see this, note that in the case of 2 productive workers the cost of a link must be at least $\frac{1}{3}$ of the average value of the two workers in order for the boss player to benefit from the existence of a middleman. However, if the link cost is this big, then at least one of the workers will have a negative payoff in the graph with the middleman, making this graph unstable. On the other hand, if the middleman is productive, a graph with middleman can already be beneficial for the boss player when

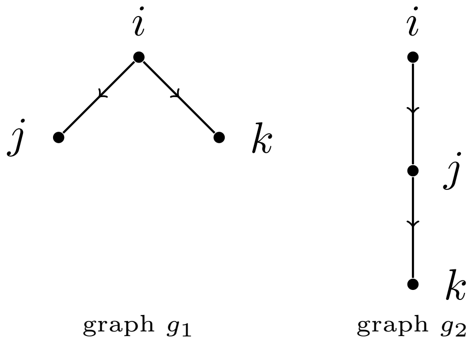


Figure 7

there is only one productive worker in the lowest level. This is shown in the next example.

Let $v \in \mathcal{G}^N$ be additive and let $v, N = \{i, j, k\}$ and $O \in \mathcal{O}^N$ be given by:

$v(\{j\}) = v(\{k\}) = 1$ and $v(\{i\}) = 0$

and $O(i) = \{j, k\}, O(j) = \{k\}, O(k) = \emptyset$

Now consider the two graphs in Figure 7. The conjunctive restrictions of v in graphs g_1 and g_2 respectively are as follows:

$$\mathcal{R}^c_{g_1}(v)(E) = \begin{cases} 1, & \text{if } E = \{i, j\} \text{ or } E = \{i, k\} \\ 2, & \text{if } E = \{i, j, k\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathcal{R}^c_{g_2}(v)(E) = \begin{cases} 1, & \text{if } E = \{i, j\} \\ 2, & \text{if } E = \{i, j, k\} \\ 0, & \text{otherwise} \end{cases}$$

The conjunctive permission values for g_1 and g_2 are respectively $\phi^c(v, g_1) = (1, \frac{1}{2}, \frac{1}{2})$ and

$\phi^c(v, g_2) = (\frac{5}{6}, \frac{5}{6}, \frac{2}{6})$. We can see that in this case player 1 receives a higher payoff in g_2 than in g_1 whenever $\frac{1}{6} < c < \frac{1}{3}$. If we would allow players to deny the formation of a link even when this link increases their payoff, we would expect player i to deny a link initiated by player k, as he can expect player k to then form a link with player j.

These examples show that flat hierarchies are most beneficial for the players at the lowest level of the permission basis, but not necessarilly for the other players. In the last example both graphs are efficient, but in some cases (Figure 6) the boss player will receive a higher payoff in an inefficient graph than in any efficient alternative.

5.2 Additive buyer-seller games

In this subsection we consider buyer-seller relationships as flexible permission structures. A buyer-seller supply chain represents a situation in which a manufacturer (the seller) sells a product to a retailer (the buyer) who in turn will sell it to consumers. Case studies have shown that an abundance of industries is organized as network structures [16]. In these cases it is the buyer who produces value (by selling to a consumer), but only in cooperation with a seller. A permission structure in which the sellers are the boss players and the buyers the subordinates of these sellers, thus seems like a natural way of modelling this type of relationship.

Buyer-seller realtionships have been extensively modeled in the literature in various ways; under the assumption of constant demand, of varying demands, without considering logistic costs and as network structures [9,16,17,24]. In our case we assume constant demand of the product, with as only cost the initial communication cost for forming a link.

We consider a game with 3 sellers and 5 buyers, where v is additive and every buyer has the potential to produce a value of 1. The permission basis $O \in \mathcal{O}^N$ is a graph in which there is a link from every seller to every buyer, but no links between sellers or between buyers. The potential of a buyer is only realised if the buyer has a connection to at least one of the sellers. Suppose now that the cost of a link is $\frac{1}{8}$. The disjunctive permission value of any buyer when connected to 1, 2 or all the sellers is respectively $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$. With c equal to $\frac{1}{8}$, we get a payoff of $\frac{3}{8}$ when a buyer is connected to 1 seller, $\frac{5}{12}$ when connected to 2 and $\frac{3}{8}$ when connected to all sellers (see Figure 8 for an example

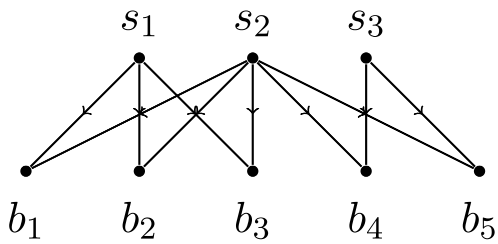


Figure 8

of a stable graph for this case). In a stable graph, every buyer will thus have a link with exactly two sellers.

We can see that if the link cost is not too high (in this case if it is less than $\frac{1}{6}$), it is beneficial for a buyer to have a link with more than one seller. This is in accordance with the predictions made in Malone et al. (1987) [18] and the results of model 1 in Bakos & Brynjolfsson (1993) [1]. A buyer that has access to more than one seller has more bargaining power. This stronger competition among the sellers will decrease the price a seller can ask, which improves the situation for the buyer [17, 23]. However, we also see that there is a maximum to the amount of sellers that a buyer wants to be connected to. Every next seller which a buyer connects to provides him with a smaller benefit. At some point, this benefit will not be bigger than the cost of forming a link anymore.

If we now look at the perspective of a seller, we see that for him more connections is simply better. The benefit of a link for a selller does not depend on how many links he himself already has, it depends on how many links the buyer he wants to connect with already has. That is, a buyer that has no connection to any seller yet, is worth $\frac{1}{2} - \frac{1}{8}$(but may become worth less when this buyer forms more links), while a buyer that already has one connection is worth $\frac{1}{6} - \frac{1}{8}$. Thus, while in a stable graph all the buyers are connected to the same amount of sellers, the sellers might have different amounts of buyers. It is even possible for a seller to have no links at all!

The intuition behind this is that a seller will always want to sell more and therefor more buyers to sell to is always better. However, a buyer will only want to buy a certain amount and so for him the benefit of more sellers is only in the greater bargaining power it gives him.

The next example shows a situation in which it is better for buyers to connect to a

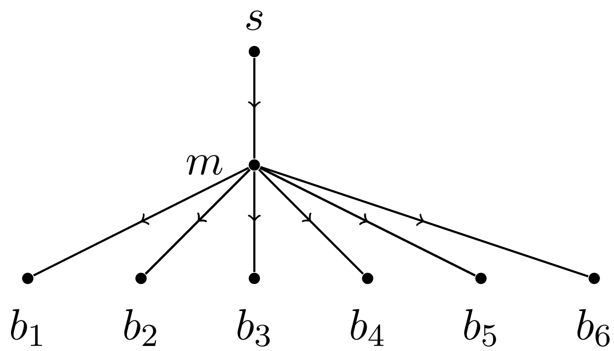


Figure 9

middleman than to directly connect with the seller. Consider a situation in which the manufacturer is located quite far from his potential buyers. In this case, the cost of forming a link with the seller would be expected to be relatively high, as the product needs to be shipped over a significant distance. If instead, we introduce a middleman who takes up the shipping cost for many buyers at once, both the seller and the buyers can be better off.

Consider a situation with one seller s, one middleman m and 6 buyers; $N = \{s, m, b_1, ..., b_6\}$. Let $v \in \mathcal{G}^N$ be additive where $v(\{b_i\}) = 1$ for $1 \leq i \leq 6$ and $v(\{s\}) = v(\{m\}) = 0$. The permission basis $O \in \mathcal{O}^N$ is defined as expected:

$$O(s) = N \backslash \{s\}, O(m) = \{b_1, ..., b_6\} \text{ and } O(b_i) = \emptyset \text{ for } 1 \leq i \leq 6.$$

Figure 9 shows the graph with middleman. Now let the cost for a link with the seller be $\frac{1}{2}$ and the cost for a link between buyer and middleman $\frac{1}{6}$. If a buyer would connect directly to the seller instead of via the middleman, both buyer and seller would get exactly zero, as the benefit of this link for either player ($\frac{1}{2}$) is equal to the cost of forming the link. In the situation as depicted in Figure 9, however, the buyers have a payoff of $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, the seller of $6 \cdot \frac{1}{3} - \frac{1}{2} = 1\frac{1}{2}$ and the middleman gets $6 \cdot \frac{1}{3} - \frac{1}{2} - 6 \cdot \frac{1}{6} = \frac{1}{2}$. The situation in Figure 9 is stable, as forming a direct link with the seller when a buyer is already connected to the middleman will be even less beneficial than forming a link with the seller only. Forming this link would thus decrease the payoff of both the buyer and the seller.

We can see from the previous two examples that in the disjunctive approach having multiple links can significantly increase a players payoff. Graphs with middleman such as in Figure 6 and 7 will generally not be stable as unlike the conjunctive approach, forming a direct with the boss player will in most cases increase the disjunctive permission value of a player. However, as shown in our last example, if not all links are equally expensive it can be the case that the stable graphs of a game under the disjunctive approach coincide with those under the conjunctive approach.

6 Conclusion and future work

6.1 Conclusion

In this thesis we have developed a framework for games with a flexible permission structure. We have based ourselves on games with permission structures as defined by Gilles, Owen & Van den Brink (1992) [13] and Gilles & Owen (1999) [12] in which players need permission from one or more of their direct superiors in order to cooperate. The framework we have built allows for links to be formed and severed in case this has a positive effect on the payoff of the players between which the link exists. We followed Jackson & Wolinsky (1995) [15] in assuming a link will be formed if it does not decrease the payoff of either of the players. A link will be severed if this strictly increases the payoff of one of the two players involved. We also assumed that links have a certain cost c and that this cost is the same for every link and for any player.

In chapter 4 we then discussed the stability of the graphs as defined by the framework set out in chapter 3. We have shown that in the conjunctive approach, where a player needs permission from *all* his direct superiors, only trees and forests are stable. We have further noted that as the cost of the links increases it becomes less likely for a graph with many layers of hierarchy to be stable. This is due to the fact that a player with more superiors loses a bigger percentage of his contribution to his superiors.

For the disjunctive approach it is less easy to determine which links are stable. In the disjunctive approach, a player benefits from having more direct superiors. Because of this, whether a link will be stable or not is a matter of whether the benefits outweigh the cost of a link. We then took a closer look on how a link changes the disjunctive permission value of a player in the proof of theorem 2.6. With the help of this, we considered some example graphs to examine in which situations the formation of a link will increase the disjunctive permission value more or less.

In chapter 5 we have discussed applications both for games with a conjunctive and a disjunctive flexible permission structure. We have considered organization structures with unproductive superiors in several examples for the conjunctive approach and buyer-seller situations as an illustration for the disjunctive approach. These examples have also shown that in games with a flexible permission structure several graphs can be stable. These alternatives do not only differ in structure, but also yield a different payoff distribution.

We have not yet shown that a game with a flexible permission structure will always converge to a stable graph. This question remains for future research.

6.2 Future work

6.2.1 Alternative allocation rules

In this thesis we used the Shapley value to study the stability of links in games with a flexible permission structure. An alternative allocation rule to consider is the Banzhaf value, which has also been axiomatized for permission structures. [4, 5] It has been shown that the Banzhaf permission values and Shapley permission values share a lot of properties. Both satisfy additivity, the inessential player property, the necessary player property, weak structural monotonicity and one-player efficiency. The last axiom is an efficiency axiom that only holds for one-player games. Aside from these axioms, both the disjunctive Banzhaf and Shapley permission value satisfy disjunctive fairness and the conjunctive Banzhaf and Shapley permission value satisfy conjunctive fairness.

An important difference between the two values is that the Banzhaf permission values do not satisfy efficiency. Although this may sound like a major drawback, the Banzhaf permission values do have some desirable properties that the Shapley permission values lack. Van den Brink (2003, 2010) [4, 5] has shown that the Banzhaf permission values differ from the Shapley permission values on three split axioms. The first of these three, *power split neutrality*, states that for $i, j, h \in N$ and $j \in S(i) \cap S(h)$ where $i \neq h$, when the link between j and i is broken, the sum of the payoffs of i and h, the two direct superiors, stays the same. Not only does this axiom not hold for the Shapley permission values, these values do not even satisfy the weaker *opposite change property*, which states that the payoff of the two direct superiors changes in opposite direction. The Shapley permission values do, however, satisfy the opposite change property when v is monotone.

The other two split axioms consider situations where a player is split into two instead of a split of power. *Vertical split neutrality* states that if we introduce a new player h who is a null player and who enters the permission structure as the unique direct superior of some player j while getting all j's direct superiors in the original game as direct superior, then the sum of the payoff of h and j in this new game is equal to the payoff of j in the old game. We can see this change as splitting up player j into two players in the vertical direction.

Horizontal split neutrality is similar to vertical split neutrality, but for situations in which we split j into two on the horizontal level. Horizontal split neutrality states that if a player j has no successors and we let a player h enter the game who has the same direct superiors as j and also no successors and j and h veto each other in the new game, then the sum of the payoffs of h and j in the new game is the same as the payoff of j in the original game.

The Shapley permission values satisfy their own split neutrality axioms, which differ from the ones for the Banzhaf permission values in the sense that they require the total sum of the payoffs of all players to remain the same. The Banzhaf permission values thus differ from the Shapley permission values with respect to how they react to changes in the game in certain situations where there is a split of power or of players. As the Shapley permission values satisfy a global alternative of the split neutrality axioms instead of the pairwise versions satisfied by the Banzhaf permission values, it can be beneficial for a player to pretend to be more than one. Under the Banzhaf permission values this will not increase a players payoff and thus there will be no incentive for a player to cheat in this way. For future research it would thus be interesting to consider which graphs will be stable in the conjunctive and disjunctive approach under the Banzhaf permission values.

Jackson (2005) [14] argues that the Myerson value, which is an extension of the Shapley value for communication and network structures, has a serious deficit in the sense that it takes the graph as fixed when the payoff distribution is calculated. This is equally true for the conjunctive and disjunctive permission value. According to Jackson, it is important to take into account which alternative networks could have formed as this can influence the incentives to form certain networks. Aside from that, it can be argued that it is only fair to take into account whether a player fulfills a role in the graph that could have been fulfilled by other players as well, or whether he's the unique player who can take this specific position.

The first criticism Jackson has is on the insensitivity of the Myerson value to alternative networks. Both the conjunctive and the disjunctive permission value have this same property. To see this for the conjunctive permission value, we look again at the graph in Figure 1. Note that under the conjunctive permission value this graph would turn into either $\{ik, kl\}$ or $\{ij, jl\}$. The payoff in both cases would be $\frac{1}{3}$ for the three players with a link and 0 for the fourth player. The conjunctive permission value thus does not take into account that player k and j can be replaced by each other, while i and l are irreplaceable.

To see the same for the disjunctive permission value, look again at the buyer-seller situation as depicted in Figure 8, but now suppose all the buyers are connected to s_1 and s_2. Note that this situation, in which seller s_3 has no links, would also be stable. Under the disjunctive permission value the payoff of s_1 and s_2 does not depend on the existence of s_3. However, the existence of this third seller does matter for the fact whether there are alternative graphs that could have formed or not. This shows the insensitivity of the disjunctive permission value to alternative networks.

One of the axioms underlying the Myerson value that Jackson considers problematic is the axiom of equal bargaining power. Equal bargaining power states that players benefit equally from the link they form between them. Although the conjunctive permission value does not satisfy this property, as is shown in Van den Brink (1999) [3], the disjunctive permission value does satisfy equal bargaining power, which follows directly from disjunctive fairness.

The problem with equal bargaining power can be seen from the following example: Let $N = \{g, i, j\}$ and $v \in \mathcal{G}^N$ be given by $v(E) = 1$ for all $E \supseteq i$ or $E \supseteq j$ and $v(E) = 0$ otherwise. Let $O \in \mathcal{O}^N$ be given by

$$O(g) = \{i, j\}, O(i) = O(j) = \emptyset$$

If we now consider the two graphs depicted in Figure 10 we can see that $v(g_1) = v(g_2)$. As g_2 has more links than g_1 the total value of this graph is therefore less. However, if we look at the disjunctive permission value for the players in both graphs we see that in graph g_1 player g and i receive $\frac{1}{2}$ and player j receives 0, while in g_2 players i and

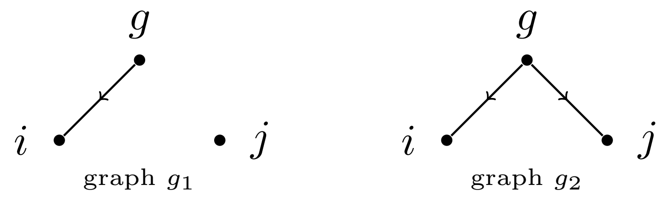


Figure 10

j receive $\frac{1}{6}$ while g gets $\frac{2}{3}$. Due to equal bargaining power, player g and i receive the same in g_1. This does not only provide player g with an incentive to form a link with j and thereby creating an inefficient graph, it is also questionable whether the equal bargaining power axiom is fair when the roles of the players are not comparable.

Jackson argues that in cases of full symmetry of players the most natural distribution is one in which all players receive the same payoff. However, in situations where one can assume that productive players actually put in some effort to produce this value, it seems unreasonable to allocate the null players in such a graph the same as the productive players. Nevertheless, it would be interesting for future work to look into allocation functions that take into account that a graph is not fixed and that alternative networks may form or could have formed.

6.2.2 Alternative assumptions for link formation

In this thesis we assumed that a link can be formed if it does not decrease the payoff of either players involved. We assumed a link will be severed if this increases the payoff of either of the players, irrespective of how it changes the payoff of the other. Of course, other assumptions could also be made and could result in different stability properties for both the conjunctive and the disjunctive approach.

With respect to the making of a link, an alternative would be to assume that only one of the two, either the successor or the direct superior, can initiate the link. The other player can then only accept or deny this link. The result of this change would be that only links will form that strictly increase the payoff of the initiator and do not decrease the payoff of the acceptor. A link that would increase the acceptor's payoff and would not change that of the initiator would not form in this situation, while it could form under the assumptions made in this thesis.

One benefit of this inequality in link formation is that we can also consider strategic moves from players with respect to link formation. In our current approach, any link that is not harming either of the players could form. If only one of the two players can initiate, than one link becomes more likely to form than another. Consider the example in Figure 7. If only the superiors can initiate a link, a smart player i would initiate a link with j, but not with k, as it is better for i if j forms a link with k. We would thus expect g_2 to form. On the other hand, if only the successor is allowed to initiate a link, we would expect g_1 to form as this is the more beneficial situation for player k.

With respect to the severance of links it would also be interesting to look at the effects of assuming that both players need to agree in order for a link to be broken. In that case links that are beneficial to one player, but costly for the other will be stable. Aside from this, we could also look at the consequences of assuming that only one of the two players can break the link. On the Dutch labor market, for example, it's not very easy to fire someone, yet it is almost always possible to resign. Assuming that a link can only be broken by a successor would better match such a situation.

Lastly, it would be interesting to look at alternative assumptions with respect to the link costs. In this thesis we assumed both players share the cost of a link equally. However, in some situations it would be more natural to assume an unequal share or even that only one of the two players pays the cost. Especially in combination with alternative assumptions with respect to link formation and severance, an equal sharing of the link cost might be unnatural.

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