IMPLICIT AND EXPLICIT STANCES IN LOGIC

Johan van Benthem

j.vbenthem@uva.nl, http://staff.fnwi.uva.nl/j.vanbenthem

Institute for Logic, Language and Computation, University of Amsterdam
P.O. Box 94242, 1090 GE AMSTERDAM, The Netherlands

Abstract We identify a pervasive contrast in logic between what we call implicit and explicit design stances. Implicit systems change the meaning of logical constants and sometimes also the definition of consequence, while explicit systems conservatively extend classical systems with new vocabulary. We illustrate the contrast in the traditional setting of intuitionistic and epistemic logic, then take it further to information dynamics, default reasoning, and logics of games and other areas, to show the wide scope of these complementary styles of logical analysis and system design. Throughout we show how awareness of the implicit–explicit contrast leads to new logical questions, from straight-forward technical issues to when implicit and explicit systems can be translated into each other, raising new foundational issues about identity of logical systems. But we also show how a practical facility with these complementary working styles has philosophical consequences, as it throws doubt on strong philosophical claims made by just taking one stance and ignoring the alternative one. We will illustrate the latter benefit for the case of logical pluralism and hyper-intensional semantics.

Keywords Logic, Modality, Implicit, Explicit, Translation.

1 Explicit and implicit stances in logical analysis

The history of logic has themes running from description of ontological structures in the world to elucidating patterns in inferential or communicative human behavior. For both strands, the mathematical foundational era added a methodology of formal systems with semantic notions of truth and validity and matching proof calculi. This modus operandi is standard fare, enshrined in the major systems of the field.

Live disciplines are not finished fields but advancing quests. Logic has a growing agenda, including the study of information, knowledge, belief, action, agency, and other key topics in philosophical logic or computational logic. How are such topics to be brought into the scope of the established mathematical methodology? There are both modifications and extensions of classical logic for these purposes, and the aim of this paper is to point at two main lines, representing a significant contrast.

One line of enriching classical logic adds new operators for new notions to be analyzed, leaving old explanations of existing logical notions untouched. A typical case is modal logic, adding operators for modalities, while nothing changes in the propositional base logic. Let us call this the *explicit* style of analysis, though the label 'conservative' makes sense, too: we do not touch notions already in place.

But there is also another line, where we use new concepts to modify or enrich our understanding of what the old logical constants meant, or what the old notion of valid consequence was meant to do. This leads to non-standard semantics, perhaps rethinking truth as 'support' or 'forcing', and to alternative logics whose laws differ from those of classical logic on the original vocabulary of connectives and quantifiers. Here the richer setting is reflected primarily, not in new laws for new vocabulary, but in deviations on reasoning patterns stated in the old language – and in particular, failures of old laws may then be significant and informative. A paradigmatic case for this approach is intuitionistic logic, but further instances keep emerging all the time. Let us call this the *implicit* style of analysis, without any pejorative connotation. Implicitness is a hall-mark of civilized intercourse. [1]

We will discuss a sequence of illustrations displaying the contrast, and analyze what makes it tick. We set the scene by recalling some key facts about two well-known systems: epistemic logic and intuitionistic logic, presented with a focus on information and knowledge. After that we discuss less standard cases such as logics of information update, default reasoning, games, quantum mechanics, and truth making. Throughout, we take explicit and implicit approaches seriously as equally natural stances, and we discuss new logical questions suggested by their co-existence. Our final conclusion from all this will be that the interplay of the two stances needs to be grasped and appreciated, as it raises many new points and open problems concerning the architecture of logic, while it also has philosophical repercussions. [2]

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[1] The terms 'explicit'/'implicit' may not be optimal, and another way of labeling the contrast would be 'extensionist'/ 'revisionist'. Even so, I have opted for existing words rather then neologisms.

[2] The intended contrast is primarily one in design procedures, but often, it can also be applied to the systems produced. Still, there are some issues in classifying systems to be discussed later on.

A caveat We claim that the contrast elaborated in this paper makes sense widely, but we do not claim that it is clear-cut in all concrete cases. For instance, some `implicit' logics, too, introduce new vocabulary – and more generally, some systems, such as intuitionistic modal logic, have both explicit and implicit features. Also, our contrast seems more compelling in a semantic setting than in a proof-theoretic one. We will discuss some presuppositions, challenges and limitations concerning the explicit/implicit divide in Section 15. Of course, a broad methodological distinction can be fruitful and illuminating even if there are contentious borderline cases.

This may not be an easy paper to classify qua style or results, but we hope that the reader will benefit from looking at logical system design in our broad manner.

2 Information, knowledge, and epistemic logic

A natural addition to the heartland of logic are notions of knowledge and information for agents, that have been part of the discipline from ancient times until today, [34], [7]. In what follows we do not need intricate contemporary logics for epistemology, [32], interesting and innovative though these are. The contrast in modus operandi we are after can be seen at much simpler level, dating back to the 1960s.

Here is a major explicit way of taking knowledge and information seriously. We add modal operators for knowledge to propositional logic, and study the laws of the resulting *epistemic logics* on top of classical logic. These conservative operator extensions of classical logical systems have interesting structure and modeling power, also for notions beyond knowledge, such as belief.

In more detail, the classic [31] proposes an analysis of knowledge that involves an intuitive conception of information as a range of candidates for the real situation ('world', 'state'). This range may be large, and we know little, or small (perhaps as a result of prior information updates eliminating possibilities) and then we know a lot. In this setting, an agent *knows that* φ at a current world s if φ is true in all worlds in the current range of s, the epistemically accessible worlds from s via a binary relation $s \sim t$. To express reasoning in a matching syntax, we take standard propositional logic as a base, and add a clause for formulas of the form $K\varphi$ – subscripted to $K_i\varphi$ for different indices i in case we want to distinguish between different agents i. Then the preceding intuition becomes the following truth definition:

$$\begin{array}{lll} \boldsymbol{M}, s \models p & \text{iff} & s \in V(p) \\ \boldsymbol{M}, s \models \neg\varphi & \text{iff} & \textit{not } \boldsymbol{M}, s \models \varphi \\ \boldsymbol{M}, s \models \varphi\wedge\psi & \text{iff} & \boldsymbol{M}, s \models \varphi \textit{ and } \boldsymbol{M}, s \models \psi \\ \boldsymbol{M}, s \models K\varphi & \text{iff} & \boldsymbol{M}, t \models \varphi \textit{ for all } t \textit{ with } s \sim t. \end{array}$$

This extends classical propositional logic: the base clauses are standard, with one operator clause added. Epistemic accessibility ~ is often taken to be an equivalence relation – but we can vary this if needed. The resulting logic is *S5* for each single agent, without non-trivial bridge axioms relating knowledge of different agents. Thus, basic epistemic logic is a conservative extension of classical logic, and the same holds for its variations like *S4* or *S4.2* that encode other intuitions concerning knowledge, [48]. More intricate laws hold for modalities of common or distributed knowledge in groups, but again these will not be not needed here.

Few people today see the epistemic modality as a conclusive analysis for the full philosophical notion of knowledge. But even so, this system is a perfect fit for another basic notion, the 'semantic information' that an agent has at her disposal, cf. the classic source [6]. And, the simple perspicuous explicit syntax of epistemic logic is still in wide use as a lingua franca for framing epistemological debates, for instance, for or against such basic principles of reasoning about knowledge as

omniscience, or closure	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
introspection	$K\varphi \to KK\varphi$

Significantly, these are debates about intuitively acceptable reasoning principles for knowledge, not about the laws of the underlying propositional logic.

More sophisticated philosophical accounts define knowledge as a notion involving structure beyond mere semantic ranges, such as relevance order, plausibility order of worlds for belief (which we discuss later on), or similarity order for conditionals. Even so, logics for these extended settings tend to be multi-modal systems, that is, they still fall under what we have called the explicit approach. All this is typical for many areas of philosophical logic, such as temporal, deontic, or conditional logic. [3]

3 Intuitionistic logic

Next, consider a second way of taking knowledge and information seriously, which is sometimes presented as a revolt against classical logic. We no longer take the old notions for granted, but *redefine* the meanings of the logical constants, perhaps also the notion of consequence, to get at crucial aspects of knowledge.

A typical instance of this second approach is *intuitionistic logic* that does not add knowledge syntax, but encodes behavior of knowledge in its deviations from the laws of classical consequence. [4] This approach seems more radical, as breaking the classical laws has an iconoclastic appeal, and more subtly, the absence of explicit expressions for epistemic notions makes the behavior of knowledge now show, not in new laws, but implicitly, in *absence* of old laws, or in modifications of such laws. For instance, the well-known intuitionistic failure of Excluded Middle $\varphi \lor \neg\varphi$ tells us something essential about the incompleteness, in general, of our knowledge. But on the positive side, the continued intuitionistic validity of $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$ reveals a more delicate form of introspection for knowledge than the simple *S4* law we had above – where negation now talks about knowledge in an implicit manner.

Intuitionistic logic arose in the analysis of constructive mathematical proof, with logical constants acquiring their meanings in proof rules via the Brouwer-Heyting-Kolmogorov interpretation. In the 1950s, Beth and Kripke proposed models over trees of finite or infinite sequences, and in line with the idea of proof as establishing a conclusion, intuitionistic formulas are true at a node of such a tree when 'verified' in some intuitive sense. A general topological framework for placing all these ideas uniformly is presented in [15]. A standard version that suffices for our purposes here uses partially ordered models $\boldsymbol{M} = (W, \leq, V)$ with a valuation V, setting:

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[3] This brief exposition may be misleading about the agenda of the field. Epistemic logic has come into wider use in computer science, game theory and linguistics because of its potential for describing multi-agent interactions in communication or games. See the Handbook [19] for the state of the art.

[4] This is only one view of intuitionistic logic, though compatible with the epistemic strands in its genesis. On a prominent alternative view, intuitionistic logic is about a non-classical notion of truth.

$\boldsymbol{M}, s \models p$ iff $s \in V(p)$

$\boldsymbol{M}, s \models \varphi \wedge \psi$ iff $\boldsymbol{M}, s \models \varphi$ *and* $\boldsymbol{M}, s \models \psi$

$\boldsymbol{M}, s \models \varphi \vee \psi$ iff $\boldsymbol{M}, s \models \varphi$ *or* $\boldsymbol{M}, s \models \psi$

$\boldsymbol{M}, s \models \neg\varphi$ iff *for no* $t \geq s$, $\boldsymbol{M}, t \models \varphi$

$\boldsymbol{M}, s \models \varphi \rightarrow \psi$ iff *for all* $t \geq s$, *if* $\boldsymbol{M}, t \models \varphi$, *then* $\boldsymbol{M}, t \models \psi$

In such partial orders, we can think of the objects s as information stages or information pieces, while models unraveled to trees give a temporal picture of a record of possible investigations. Next, in line with the idea of accumulating certainty in the process of inquiry, the valuation V in these models is *persistent*, i.e.,

if $\boldsymbol{M}, s \models p$ and $s \leq t$, then also $\boldsymbol{M}, t \models p$.

The truth definition as stated here lifts this persistence property to all formulas φ.

In this modus operandi, in contrast with epistemic logic, there is no separate syntax for knowledge or information – but old logical constants are *re-interpreted*, making negation and implication sensitive to the information structure of new models with an inclusion order that is absent in models for classical logic. In particular, an intuitionistic negation $\neg\varphi$ says that the formula φ is not just 'not true', but that it will never become true at any further stage along the inclusion ordering. Also, failure of classical definability equivalences leads to fine-structure for classical notions like implication, which can now be viewed in several non-equivalent ways.

This 'meaning loading' of the classical operators makes the intuitionistic laws for negation and implication deviate from classical logic. Now earlier points become precise. Famously, this semantics invalidates the law of Excluded Middle $\varphi \vee \neg\varphi$, as this disjunction fails at states where φ is not yet verified though it will later become so. These deviations from classical logic are informative in telling us implicitly about properties of knowledge. Failure of Excluded Middle says that agents cannot decide everything a priori. Thus meaning loading makes the remaining validities informative (they now say something new), and more mysteriously, it packs information in the absence of classical laws – like dogs that do not bark in the night-time.

At the same time, even though the classical language is not extended in intuitionistic logic, there is an increase in expressive power, precisely because classical laws fail.

For instance, $\varphi \rightarrow \psi$ is not equivalent to its classical equivalents $\neg\varphi \vee \psi$ or $\neg(\varphi \wedge \neg\psi)$: intuitionistic logic has at least three candidates for plausible notions of implication. This `splitting' may be seen as an implicit counterpart to the language extensions found in explicit approaches – an issue to which we will return in Section 15. [5]

4 The explicit/implicit contrast: epistemic logic versus intuitionistic logic

So, now we have encountered two major research paradigms in the field of logic, both meant to take information and knowledge seriously – but doing so in very different ways. Let us highlight the major differences that showed in the above:

epistemic logic	explicit, conservative language extension of classical logic
intuitionistic logic	implicit, meaning change old language, non-classical logic

Highlighting the distinction, consider the fact that we do not know the answer to every question, and maybe never will. This showed as follows in intuitionistic logic. Excluded Middle $\varphi \vee_{int} \neg_{int} \varphi$ was not valid – where indices highlight the fact that the failure occurs on the intuitionistic understanding of negation and disjunction – though special cases of this principle may, and do, remain valid. In contrast with this, the law of Excluded Middle is unrestrictedly valid in epistemic logic, but it should not be confused with the invalid epistemic formula $K\varphi \vee_{class} K\neg_{class} \varphi$.

Much more can be said about these two approaches to knowledge and information. But for the purposes of this paper, we will just stipulate that both are based on stable interesting sets of intuitions, both have generated a rich mathematical theory, and both seem bona fide instances of a logical modus operandi in system design.

With this single illustration, we hope the reader has grasped the methodological contrast we are after – and in later sections, we will now explore the 'implicit' versus 'explicit' divide in other cases, adding more depth to what it involves.

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[5] This brief exposition of intuitionistic logic does not do justice to its deep connections with proof theory, universal algebra, and category theory, or to the surprising effects of working in mathematical theories on top of a weaker base logic. See the encyclopedic source [49] for a richer story.

5 Choice or co-existence: translations and merges

But first it may seem time for a choice. Is intuitionistic logic or epistemic logic better or deeper as an analysis of information and knowledge? Should we prefer one over the other? Many philosophers think in this style, but we feel that this adversarial attitude is not very productive, and it also runs counter to known mathematical facts about system connections (for a similar, but more general criticism, cf. [29]).

Already in Gödel's seminal [26], there is a faithful translation from intuitionistic logic into the modal logic *S4* whose underlying intuition follows the present knowledge perspective. We now look at this connection to see what it achieves.

Translating IL into EL The *Gödel translation* ***t*** turns the intuitionistic truth definition into a syntactic recipe, according to the following recursive clauses:

$$\mathbf{t}(p) \quad = \quad \Box p$$

where the modal formulas $\Box p$ are upward persistent on pre-orders,

$$\mathbf{t}(\varphi\wedge\psi) \quad = \quad \mathbf{t}(\varphi) \wedge \mathbf{t}(\psi)$$

$$\mathbf{t}(\varphi\vee\psi) \quad = \quad \mathbf{t}(\varphi) \vee \mathbf{t}(\psi)$$

$$\mathbf{t}(\neg\varphi) \quad = \quad \Box\neg\mathbf{t}(\varphi)$$

$$\mathbf{t}(\varphi \to \psi) \quad = \quad \Box(\mathbf{t}(\varphi) \to \mathbf{t}(\psi))$$

For the standard proof system *IL* of intuitionistic propositional logic, we then have:

Fact $IL \vdash \varphi$ iff $S4 \vdash \mathbf{t}(\varphi)$, for all propositional formulas φ.

This explains key features of intuitionistic logic in modal terms. E.g., varieties of implication place different demands on knowledge: intuitionistic $\varphi \to \psi$ is $\Box(\varphi \to \psi)$, the earlier $\neg\varphi \vee \psi$ is the stronger $\Box\neg\varphi \vee \psi$, and $\neg(\varphi \wedge \neg\psi)$ the weaker $\Box(\varphi \to \Diamond\psi)$. Also, intuitionistic laws like $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$ are special cases of the fact that *S4* has 14 non-equivalent iterations of modalities. But intuitively, the modal setting is richer, as it also supports reasoning about non-persistent formulas that can become false at later stages. Thus, its view of inquiry allows for revision, not just cumulative update.

Uses of translations Some people view translations like this as mere tricks, especially, those who see different logics as separate religions. But the translation facilitates a resounding transfer: everything an intuitionist says or infers can be understood by a classical modal logician. This facilitates traffic of ideas between intuitionistic and epistemic logic, and meaningful contacts between their agendas. For instance, key properties of *S4* such as decidability carry over automatically to intuitionistic logic, and applications keep emerging, such as uses of modal bisimulation in intuitionistic logic, [40]. But also conceptually, ideas from epistemic logic can now enter intuitionistic logic, such as the study of multi-agent scenarios.

Translating EL into IL Our discussion so far may have given the edge to epistemic logic, as it embeds intuitionistic logic. What about the other way around? Intuitively, as we noted, the semantics of *S4* seems richer, allowing non-persistent notions, but the two logics have the same computational complexity (their SAT problems are *Pspace*-complete), so there is no a priori obstacle to mutual translation. In fact, surprisingly, [22] gave a converse translation (with a correction in [27]), which is much less known. It works quite differently from Gödel's ***t***, by mimicking evaluation of modal formulas in finite models inside the intuitionistic language.

Thus, translations between stances occur, and they are significant as manuals for communication and interaction. So, are intuitionistic logic and epistemic logic then really just the same system in different guises because of their faithful mutual embeddings? This further question raises delicate issues of system identity.

Translation and system identity Despite the clear benefits of translations, they need not reduce one logic to another in every relevant aspect. The Gödel translation encodes one particular modal take on the logical constants, which may not be what an intuitionist considers their essence. And there is more. To let the Gödel translation be faithful, deductive power must be restricted to *S4* or logics close to it. This is relevant, since so far, we used *S5* as an epistemic logic, and the Gödel embedding does not work there: *IL* is *Pspace*-complete, and hence more complex than *S5*, which is merely *NP*-complete. And also conversely, studying the syntactical details of the encoding from *EL* into *IL*, one does not get a feeling of strong resemblance between the two systems: it seems more like a case of intuitionistic logic hatching *S4* eggs. [6]

[6] One way of seeing finer differences is in terms of *computational complexity*. Theories that are equivalent under translation, perhaps an inefficient one, may have different computational properties. We will not pursue this angle here, but complexity is a natural way of driving a finer wedge.

Thus, mutual translation, though a strong bond, need not imply system equivalence in all relevant aspects. It is good to search for such connections, but in what follows, we will keep an open eye for intensional differences between translated systems. [7]

From translating to merging Finally, moving away from an emphasis on reduction, there is a weaker but still significant contact between explicit and implicit logics, namely that of compatibility. Can such systems be *merged* in meaningful ways? Intuitionistic modal logics have long existed (for a recent epistemic modal logic, see [2]), and hybrids of explicit and implicit logics keep emerging, as we will see later on. Often this juxtaposition seems routine, but hybrids can also be natural.

6 Dynamic logic of information change

Having introduced our explicit/implicit contrast for two well-known logics, we now move to more recent developments and see where it leads. We start by noting that *inquiry* lies at the heart of both epistemic and intuitionistic logic. Clearly, knowledge and information do not function in isolation, but in an ongoing dynamic process of informational action, or in a social setting, interaction between agents.

Statics and dynamics Key informational actions that guide agents come in three kinds that work together in many natural scenarios. Acts of inference matter – but equally important are acts of observation, and of communication. Such actions, or other events that embody them, are studied in current dynamic-epistemic logics, by adding an explicit vocabulary for the core actions to existing logical systems, and then analyzing the major laws of knowledge change, [11].

Model update Here is a system making the dynamic actions behind basic epistemic logic explicit by representing informational action as model change. The simplest

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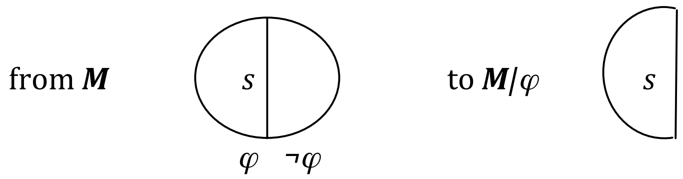
[7] *Digression: proof strength*. Translations only work proof-theoretically with the right deductive power on both sides. The choice of axioms in systems like *S5* or *S4* fine-tunes deductive power – but it does more. Moving from one modal logic to another can lead to a switch in conceptual framework. E.g., the reflexive transitive accessibility relation of *S4* does not just encode an *S5*-style epistemic range: its lack of symmetry also suggests moving forward in inquiry. Like with intuitionistic logic, an *S4*-model can be seen as a process where an agent learns progressively about the actual world, but its non-persistent atoms can also model non-intuitionistic information retraction or real world change. Even so, in translating intuitionistic logic into the modal world of *S4*, rather than the earlier system *S5*, we have gone some way toward adopting the intuitionistic mind set of inquiry.

case of such a change occurs with a *public announcement* or a similar public event $!\varphi$ that produces hard information, where one learns with total reliability that φ is the case. This eliminates all worlds in the current model where φ is false:



As we said when motivating epistemic models, getting information by shrinking a range of options is a common idea in many disciplines, that works for information flow by being told or through observation. We can call this *hard information* because of its irrevocable character: the update step eliminates all counter-examples.

Public announcement logic Public announcements are studied in *PAL*, a system that extends epistemic logic with a dynamic modality for truthful announcements:

$$\boldsymbol{M}, s \models [!\varphi]\psi \quad \text{iff} \quad \textit{if } \boldsymbol{M}, s \models \varphi, \textit{ then } \boldsymbol{M}|\varphi, s \models \psi$$

This dynamic modality has a complete logic that can analyze delicate phenomena, such as complex epistemic assertions, say of current ignorance, changing truth value under update. This typically shows in order dependence: a sequence $!\neg Kp\ ;\ !p$ makes sense, but $!p\ ;\ !\neg Kp$ is contradictory. Here we only display the 'recursion law' for knowledge after update, which is the basic dynamic equation of hard information:

$$[!\varphi]K\psi \leftrightarrow (\varphi \rightarrow K(\varphi \rightarrow [!\varphi]\psi))$$

Together with the *S5*-laws for epistemic logic plus simple axioms for Boolean compounds after update this gives a complete axiomatization for *PAL*. Another interesting law demonstrating the dynamics of *PAL* governs iterated updates:

$$[!\varphi][!\psi]\chi \leftrightarrow [!(\varphi \wedge [!\varphi]\psi)]\chi$$

Recursion axioms reduce formulas with dynamic operators to static base formulas, so the extension of our classical base logic is conservative in the usual explicit style.

General dynamics There is a method here. One 'dynamifies' a given static logic, making its underlying actions explicit and defining them as model transformations. The heart of the dynamic logic is then a compositional analysis of post-conditions for the key actions via recursion laws. This leads to conservative extensions of the base logic, though some systems force redesign of their base, while some recent semantics no longer support all-out reduction. Many further notions can be treated in this style, such as changes in beliefs, inferences, issues, or preferences – by changing the ordering of worlds rather than eliminating them. Dynamic-epistemic logics also deal with public and private events in multi-agent scenarios such as games. [8]

7 Implicit dynamics in intuitionistic logic

We have now extended epistemic logic, an explicit approach to knowledge, to a dynamic logic with explicit informational actions. Is there an implicit counterpart? Given our earlier discussion, it makes sense to search in the realm of intuitionism. We could just add the actions of *PAL* to intuitionistic logic, [3]. But can we be more implicit about informational actions, without putting them into explicit syntax?

Locating the hidden actions Intuitionistic models represent a process of inquiry, with endpoints as final stages where we know the truth about all proposition letters. What are the implicit steps in the background of such a process taking us from node to node? Moves from one state to a successor come in two kinds.

Example The hidden dynamics of intuitionistic models.
Consider two models $\boldsymbol{M_1}$, $\boldsymbol{M_2}$, where the first refutes the classical double negation law $\neg\neg p \rightarrow p$, and the second the law of 'weak excluded middle' $\neg p \vee \neg\neg p$:

The annotations say that the two branches of $\boldsymbol{M_2}$ may be viewed as public announcements of which endpoints, viewed as classical valuations, the process can get to.

[8] Even where a dynamic logic conservatively extends a base logic it may affect our view of the statics. Consider the variety of modal logics with axioms matching conditions on accessibility relations. One can ask *why* such conditions hold in dynamic terms. Say, transitive relations arise from an act ***tc*** of transitive closure of arbitrary models: `reflection' or `exploration'. But then a *K4*-modality $K\varphi$ is an ordinary modality $\Box$ over models resulting from this action, making it a compound $[\boldsymbol{tc}]\Box\varphi$. This faithfully embeds *K4* over transitive models into propositional dynamic logic over arbitrary models. In this dynamics-inspired way, variety of modal logics dissolves in favor of one base logic plus modalities for model change, explaining instead of postulating special relational properties.

This is like *PAL*-style learning by elimination of worlds. But in other intuitionistic steps, like the one in $\boldsymbol{M_1}$, there is no elimination, and we just get more proposition letters true at the next stage. One might view this implicit move as a new type of informational action, namely, 'awareness raising' $\#\varphi$ that some fact φ is the case, where awareness involves *syntactic* in addition to semantic information. [9]

Factual and procedural information But there is more than mere transposing of concerns from dynamic-epistemic logic. The tree structure of intuitionistic models registers *two* notions of semantic information on a par, a distinction also found in epistemic inquiry with long-term scenarios in learning theory, [33]:

(a) *factual information* about how the world is,

(b) *procedural information* about our current investigative process.

How we can get knowledge matters, not just what is the case. While endpoints record eventual factual information states, the branching tree structure of intuitionistic models, both its available and its missing intermediate stages, encodes further non-trivial information: viz. agents' knowledge about the process of inquiry.

This challenges simple views of how intuitionistic and epistemic logic connect. The epistemic logic for semantic information is *S5*, and the fact that the Gödel translation takes us into *S4* reflects a view of intuitionistic models as temporal processes of inquiry. Thus, an explicit counterpart to intuitionistic logic needs a temporal version of dynamic epistemic logic. Indeed, temporal `protocol models' with designated admissible histories satisfying constraints on inquiry, [11], model procedural information in long-term processes of inquiry or learning beyond local dynamic steps.

Thus, both epistemic logic and intuitionistic logic have dynamic extensions having to do with inquiry, and these can be developed in both explicit and implicit styles. Moreover, this process is not routine and interesting new issues come to the fore.

8 Dynamic semantics, meaning as information change potential

Intuitionistic logic is not the only vehicle for a meaningful comparison with *PAL*. Explicit logics need not have unique implicit companions, there may be more mat-

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[9] With such a new operator, one could also make more general distinctions between `aware' and `unaware' versions of logical constants, say, implications – but we will not pursue this line here.

chings. Indeed, the more striking implicit counterpart to dynamic epistemic logics may well be another logical paradigm, that we will discuss now, raising new issues.

Here is a fundamental idea from the area of *dynamic semantics* of natural language, going back to classical sources like [28], [48]. The guiding intuition of this approach to language involves communication-oriented 'information change potential':

> The meaning of an expression is its potential for changing information states of someone who accepts the information conveyed by the expression.

This sounds like a plea for taking informational actions seriously, as we did in the preceding section. But this time, they are treated, not by adding new operators, but implicitly, by loading the meanings of classical vocabulary with dynamic features.

Dynamic semantics comes in many forms. In what follows, we use Veltman's update semantics *US* for a modal propositional language and its novel account of consequence, [50], for a comparison with the explicit *PAL* approach. In this semantics, on a universe of information states (in the simplest version, sets of atomic valuations representing ways the actual world might be), each modal propositional formula φ denotes a *state transformation* $[[\varphi]]$ by the following recursion:

$$
\begin{aligned}
[[p]](S) &= S \cap [[p]] \\
[[\neg\varphi]](S) &= S - [[\varphi]](S) \\
[[\varphi \vee \psi]](S) &= [[\varphi]](S) \cup [[\psi]](S) \\
[[\varphi \wedge \psi]](S) &= [[\psi]]([[\varphi]](S)) \\
[[\Diamond\varphi]](S) &= S, \textit{ if } [[\varphi]](S) \neq \emptyset, \text{ and } \emptyset, \textit{ otherwise.}
\end{aligned}
$$

Conjunction now stands for a dynamic notion: sequential composition of actions, while an existential modality becomes a 'test' on the current information state.

As with intuitionistic logic, these new meanings for old operators result in deviations from classical logic. In particular, conjunction is no longer commutative, reflecting the typical order dependence of dynamic acts. Facts about the informational process are now encoded in the logic of the logical operators in this system.

This encoding becomes even more pronounced with the introduction of a new notion of dynamic consequence saying that, after processing the information in the successive premises, the conclusion has no further effect:

$\varphi_1, \ldots, \varphi_n \models \psi$ iff for every information state X in any model, $\varphi_n(\ldots (\varphi_1(X)))$ is a fixed point for $[[\psi]]$: i.e., this set stays the same under an update $[[\psi]]$.

Dynamic consequence differs from classical consequence, and its deviations encode typical features of the update process, like sensitivity to order or multiplicity of premises. But typically for the implicit style, what changes here are the classical laws of logic, not its methodology. Completeness theorems exist for dynamic consequence.

There are many highly sophisticated systems of dynamic semantics for other classes of expressions, with different notions of meaning, state change and dynamic consequence, cf. [28] – but the above seems a fair introduction to one basic mechanics.

9 A new contrast: dynamic semantics versus dynamic logic of information

PAL and dynamic semantics Dynamic epistemic logics like *PAL* and update semantics for propositional logic both take information change seriously, with analogous scenarios and intuitions. And both systems have a precise account for the dynamics of informational actions. But one does so explicitly, and the other implicitly:

> *Dynamic semantics* keeps the actions implicit, while giving the old language of propositional logic richer dynamic meanings supporting a new notion of consequence, with a technical theory that differs from standard propositional logic.
>
> *Dynamic epistemic logic* makes the actions explicit, provides them with explicit recursion laws, extends the old base language while retaining the old meanings for it, and in all this, it still works with standard consequence.

As before, this is not just a matter of attaching two labels `implicit' and `explicit'. Seeing things in terms of our contrast leads to new questions and open problems. One straightforward illustration concerns new system design.

Inquiry and questions A current innovation in dynamic semantics is inquisitive semantics for natural language, [18], where formulas get richer 'inquisitive meanings' reflecting their role in, not just conveying information, but also in directing discourse. The resulting logic is a non-classical intermediate logic related to Medvedev's logic of problems from the intuitionistic tradition. Our analysis then suggests the design of an explicit counterpart. Such dynamic-epistemic logics of inquiry – not tied to natural language, but closer to epistemology and learning theory – exist, and they involve explicit acts of 'issue management', where questions and related actions modify current issue structures on top of epistemic models, and the logical language has explicit modalities for such model-changing actions, [14], [30].

In the remainder of this section, we go into more depth on the foundational issue of how the two views of dynamics are related, and show new issues that emerge.

Translations between US and S5 As with epistemic and intuitionistic logic, there are translations between dynamic semantics and dynamic-epistemic logic, but they involve new issues. Our first observation comes from [9]:

Fact There is a faithful translation from update-validity into the modal logic *S5*.

The following is a recursive map *tr* from propositional formulas φ to modal formulas $tr(\varphi)(q)$, where q is a fresh proposition letter (note the clause for conjunction):

$$tr(p) = q \wedge p$$
$$tr(\neg\varphi) = q \wedge \neg tr(\varphi)$$
$$tr(\varphi\vee\psi) = tr(\varphi) \vee tr(\psi)$$
$$tr(\Diamond\varphi) = q \wedge \Diamond(q \wedge tr(\varphi))$$
$$tr(\varphi\wedge\psi) = [tr(\varphi)/q]tr(\psi)$$ [10]

Now consider *S5*-models $\boldsymbol{M}(q:=S)$ marked to show that q denotes the set of worlds S. The above *US* semantics works here on sets of worlds X to produce values $[[\varphi]]^{\boldsymbol{M}}(S)$. Then the following equivalence holds for all *US*-formulas φ and subsets S of $\boldsymbol{M}$:

$$\boldsymbol{M}(q:=S), s \models tr(\varphi) \text{ iff } s \in [[\varphi]](S)$$ [11]

As a corollary, for update validity, we have that

$$\varphi_1, \ldots, \varphi_n \models_{US} \psi \text{ iff } \models_{S5} tr(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \psi) \leftrightarrow tr(\varphi_1 \wedge \ldots \wedge \varphi_n)$$

In fact, connections run both ways. There is also a converse embedding:

Fact There is a faithful translation from *S5*-validity into update validity.

---

[10] We may have overdosed on occurrences of '$q \wedge$' here, but this makes proofs more perspicuous.

[11] Here is the crucial inductive step. $\boldsymbol{M}(q:=S), s \models tr(\varphi\wedge\psi)$ iff $\boldsymbol{M}(q:=S), s \models [tr(\varphi)/q]tr(\psi)$ iff (using an obvious substitution lemma, with *[ ]* ordinary denotation brackets) $\boldsymbol{M}(q:=[tr(\varphi)]^{\boldsymbol{M}(q:=S)}), s \models tr(\psi)$ iff (by the inductive hypothesis) $\boldsymbol{M}(q:= [[\varphi]](S)), s \models tr(\psi)$ iff $s \in [[\psi]]([[\varphi]](S))$ iff $s \in [[\varphi\wedge\psi]](S)$.

To see this, transform *S5*-formulas φ into their normal form $nf(\varphi)$ of modal depth 1. Then, for *S5*-validities, the update function $[[nf(\varphi)]]$ is the identity on all sets, while for non-validities, on any counter-model, $[[nf(\varphi)]]$ returns the empty set.

System identity Now an earlier issue returns. Do the preceding results say that *US* is the same system as *S5*? Our translations reduce valid consequence both ways, which is enough for the standard notion of system equivalence. But the intuitive novelty of *US* is that it does something more: it can express the dynamics of model change. However, the details of our first translation give information about this aspect, too: *S5* can define model changes in ambient sets q using the formulas $tr(\varphi)$ as indicated, and this process even simulates the working of *US* in a recursive manner.

And yet the two systems feel intensionally different, and *US* seems a new discovery. I must leave the matter of detecting finer intensional differences open here, but will return to the issue of comparing dynamic components by drawing in the logic *PAL*.

PAL and S5 Similar points can be made concerning public announcement logic.

Fact There are faithful translations between *PAL*-validity and modal *S5*.

This time, the reason is that the recursion laws provide an effective truth-preserving translation from all *PAL*-formulas with dynamic modalities into the *S5* base language, while for that static fragment, *PAL* is a conservative extension of *S5*.

Comparing US and PAL directly Composing their mutual translations with *S5* gives faithful embeddings between *US* and *PAL*, our paradigms of implicit and explicit dynamics. But despite what was said before, going via the static logic *S5* does not relate the dynamic character of both approaches directly. Can we do better?

There is an obstacle here. Update semantics typically recurses on the structure of modal propositional formulas interpreted as updates, whereas the characteristic axioms of *PAL* do not recurse on the inner structure of announcements $!\varphi$, but on the postconditions ψ for the dynamic modalities $[!\varphi]\psi$. To overcome this difference, we merge, and enrich *PAL* with conversational programs built from actions $!\varphi$ by standard program operations of union and sequential composition, [11]. The following translation then arises, where T stands for the logical expression 'True':

$$Tr(p) = \,!p$$
$$Tr(\neg\varphi) = \,!\neg\langle Tr(\varphi)\rangle T$$
$$Tr(\varphi\vee\psi) = Tr(\varphi) \cup Tr(\psi)$$
$$Tr(\Diamond\varphi) = \,!\Diamond\langle Tr(\varphi)\rangle T$$
$$Tr(\varphi\wedge\psi) = Tr(\varphi) \,;\, Tr(\psi)$$

Now it is easy to show that, for models ***M*** whose domain is the set *S* that:

$$[[\varphi]](S) = \{s \in S \mid \boldsymbol{M}, s \models \langle Tr(\varphi)\rangle T\}$$

To see how this works, compare the *PAL* program $!\Diamond\neg\langle !p\rangle T \,;\, !p$ for the consistent *US* formula $\Diamond\neg p \wedge p$ with the program $!p \,;\, !\Diamond\neg\langle !p\rangle T$ for the inconsistent $p \wedge \Diamond\neg p$.

Tr does not translate *US* updates into single *PAL* actions, but it comes close. Earlier on, we saw how public announcements are closed under sequential composition, and hence $Tr(\varphi\wedge\psi)$ amounts to announcing just one suitable S5-formula. [12]

Open problem Is there also a direct translation from *PAL* actions into *US* updates?

Discussion These translations again have various uses. Decidability of dynamic consequence follows from that for *S5*. And we can use results about *PAL* as road signs for *US*. E.g., the logic of *PAL* extended with conversational programs that allow finite iterations is non-axiomatizable and not arithmetically definable, [39]. So, dynamic semantics for discourse rather than sentences might run the same complexity risk.

But earlier reservations apply: despite the translations, *US* and *PAL* seem intuitively different. For instance, recall our notion of 'procedural information' in intuitionistic models. Viewing *PAL* as a logic of inquiry, a generalized semantics of 'protocol models' with restricted temporal histories of updates makes sense, [11]. This natural modification changes the laws of *PAL*, and it blocks the translation of *PAL* into *S5*. However, it is unclear if protocol models make sense in dynamic semantics. Also, *PAL* update has a natural extension to dynamic-epistemic logics with much more drastic model changes modeling the dynamics of partly private information, and it is unclear if these richer transformations have any role in a dynamic semantics.

[12] Our result is just a pilot case, and much more can be done. E.g., [21] translates dynamic predicate logic, [28], another well-known dynamic semantics format, into standard dynamic logic of programs. One can also give translations from inquisitive logics into logics of issue management, and so on.

What these two examples suggest is a more demanding criterion of system identity: equality or difference in ‘natural generalizations’. But is there a formal basis to this, or would the criterion merely concern our current powers of imagination?

We found natural translations between dynamic semantics and dynamic-epistemic logics. Still, implicit and explicit approaches do not collapse, and again we might be content with creating merges. Either way, the realm of information dynamics seems a rich source for our explicit/implicit contrast, raising interesting issues of its own.

10 Dynamic logics of soft information

Our discussion so far centered on the statics and dynamics of knowledge. However, the implicit/explicit contrast applies just as well to logics of belief, perhaps the more important attitude in agency. The case of belief shows interesting new features and suggests new comparisons between implicit and explicit logic systems. We start with belief dynamics in explicit style, moving to implicit counterparts later.

Belief and conditional belief Epistemic-doxastic models for belief order the earlier bare epistemic ranges by a relation of ‘relative plausibility’ $\leq xy$ between worlds x, y. These models interpret operators of absolute and conditional belief:

$$\boldsymbol{M}, s \models B\varphi \quad \text{iff} \quad \boldsymbol{M}, t \models \varphi \text{ for all } t \sim s \textit{ maximal} \text{ in the order} \leq \text{on } \{u \mid u \sim s\}$$

$$\boldsymbol{M}, s \models B^{\psi}\varphi \quad \text{iff} \quad \boldsymbol{M}, t \models \varphi \text{ for all} \leq\text{-maximal } t \text{ in } \{u \mid s \sim u \text{ and } \boldsymbol{M}, u \models \psi\}$$

But there is a richer repertoire of epistemic notions available on this models. For instance, on binary world-independent orderings, a good addition is ‘safe belief’, a standard modality intermediate in strength between knowledge and belief:

$$\boldsymbol{M}, s \models [\leq]\varphi \quad \text{iff} \quad \boldsymbol{M}, t \models \varphi \text{ for all } t \text{ with } s \leq t$$

Logics for conditional belief are like those of [37], [17]. For a more general picture of natural modalities that can be defined on these models, see [4].

Belief change under hard information Beliefs guide our decisions and actions, going beyond what we know. But beliefs can be wrong, and new information can lead to correction and learning. One trigger for belief revision are the earlier public announcements. Here is the recursion law governing the matching model changes:

$$[!\varphi]B\psi \leftrightarrow (\varphi \rightarrow B^{\varphi}\, [!\varphi]\psi)$$

A similar principle for updating conditional beliefs axiomatizes the system completely. There is also a recursion law for safe belief under public announcement, which is even simpler. The following equivalence holds on plausibility models:

$$[!\varphi][\leq]\psi \leftrightarrow (\varphi \rightarrow [\leq][!\varphi]\psi)$$

Belief change under soft information But richer belief models also support new transformations. In addition to hard information, there is *soft information*, when we take a signal as serious, but not infallible. Its mechanism is not eliminating worlds, but changing plausibility order. A widely used soft update is 'radical upgrade':

> $\Uparrow\varphi$ changes a current model $\boldsymbol{M}$ to $\boldsymbol{M}\Uparrow\varphi$, where all φ-worlds become better than all $\neg\varphi$-worlds; within these zones, the old order remains.

The dynamic modality for radical upgrade is interpreted as follows:

$$\boldsymbol{M}, s \models [\Uparrow\varphi]\psi \text{ iff } \boldsymbol{M}\Uparrow\varphi, s \models \psi$$

and its dynamic logic can again be axiomatized completely using recursion laws.

Logics of belief change Recursion laws exist for belief changes under a wide variety of soft events representing different levels of trust or acceptance for new information, [4], [25]. An area where this variety makes special sense is Learning Theory, [24]: different update rules induce different policies for reaching true belief in the limit. The Handbook [19] has details on the landscape of modal logics for belief change, plus connections with *AGM*-style postulational approaches.

The systems presented here are explicit in a double sense. Not only do events and acts that usually stay in the background of logical systems become first-class objects of study, but also, dynamic logics for knowledge and belief have explicit syntax and laws for these actions. The new structure is not put into the meanings of the original language, and so we get conservative extensions of earlier static logics, although sometimes there is a need for some redesign of the original static vocabulary.

11 Non-monotonic consequence relations as implicit devices

Next, how can we do belief revision in an alternative implicit style? One line runs via dynamic semantics, with new meanings for linguistic expressions such as epistemic modals, [43], [50], [51]. All our earlier points apply, but we will not pursue them here. Instead, we show how our contrast can also take us, perhaps surprisingly, to an area of implicit logic that seems quite different from those discussed so far.

Varieties of consequence In the 1980s, the study of different consequence relations modeling varieties of common sense-based problem solving started in Artificial Intelligence, and it has since entered other fields. In particular, the consequence notion of *circumscription* [38], [47] says that, in problem solving or related tasks, the following inferences are allowed:

> A conclusion need not be true in all models of the premises,
> but only in the most preferred, or most plausible models.

Thus, problem solving involves only inspection of currently most relevant cases.

This style of reasoning deviates from classical logic. In particular, it is 'non-monotonic': a conclusion C may follow from a premise P in this sense, but it may fail to follow from the extended set of premises P, Q. For, the maximal models within the set of models for the conjunction $P \wedge Q$ need not be maximal among the models of P.

Many forms of defeasible inference have been studied, given the large repertoire of human reasoning styles – and complete structural rules or proof systems have been found, following what Bolzano and Peirce already advocated in the 19th century. These systems, that usually drop some classical rules, while retaining modified variants, are typically taken to encode basic features of such styles of reasoning.

Non-standard consequence as implicit logic This looks like an implicit approach in our sense. What is new about a reasoning practice is not explicitly on the table, but it shows in differences and analogies with classical consequence for a classical logical language. But non-standard consequence relations have concrete motivations, they do not just arise by tinkering with classical structural rules.

Making it explicit Can we provide alternative explicit accounts leaving the notion of consequence standard, while adding vocabulary to bring out the origins of the new consequence notions? Of course, we need a guiding semantic perspective for doing so, and this will depend on the precise motivation for the new consequence relation. In the following case study, we concentrate on the role of belief in circumscription – though explicitizing consequence relations may well involve other notions, too.

From inference to belief change Revisiting the original AI scenarios, one can also construe problem solving differently. We have beliefs about a problem and where a solution might go, based on scenarios that seem most plausible to consider – either deep beliefs based on experience in problem solving, or light beliefs as lacking considerations to the contrary. Then, as we take in new information relevant to the problem, this set of scenarios changes, and beliefs are modified. [13] Now this fits precisely with our models of beliefs. For instance, a circumscriptive consequence

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi$$

translates into our earlier dynamic logic for belief change, using the formula

$$[!\varphi_1]\ldots[!\varphi_n]B\psi$$

This translation explains the deviations of non-monotonic logic from classical consequence, as the structural rules of circumscription now follow from the dynamic logic of belief revision. For instance, it is easy to see how $[!\varphi]B\psi$ does not imply $[!\varphi][!\alpha]B\psi$ for all formulas α – except for special cases. This explanation of the deviant inferential behavior has two sources: the key attitude in practical reasoning is fallible belief rather than knowledge, and also, we have explicit dynamic events. [14]

Remark There are well-known analogies between non-monotonic consequence and conditionals in the style of Lewis [37]. Instead of $[!\varphi]B\psi$, this might favor conditional belief $B^{\psi}\varphi$ pre-encoding effects of learning that ψ. The two versions are not quite the same, as update $!\psi$ restricts a model to its ψ-worlds, while a conditional can still look at $\neg\psi$-worlds when evaluating φ. But these details need not concern us here.

Either way, our general points apply. The juxtaposition of perspectives raises interesting issues. Again we see a trade-off between implicit and explicit approaches:

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[13] An early source for this style of thinking in computer science is Levesque's logic of belief, [36].

[14] This simple analysis serves to make our point, but it is not the only explicit view of belief and non-monotonic logic. E.g., on the richer epistemic view developed in [46], default reasoning submits candidates for belief while further actions sift available evidence that supports one's eventual beliefs.

nonstandard consequence	classical language, deviant rules of reasoning [15]
explicit dynamic reanalysis	new language with belief and action modalities, consequence is just classical consequence.

On the second approach, non-standard reasoning is a mixture of classical reasoning and further features of informational actions, not a family of radical alternatives. Dynamic logics of belief change enrich the original language with informational events and attitude changes, but then work conservatively with a classical consequence relation, explaining deviant features of non-standard consequence by attitude and information change through the recursion laws for the new dynamic operators. In the following section, we evaluate this difference in approaches.

12 Comparisons and switches

We have seen that non-monotonic consequence relations can be translated faithfully into a classical logic with operators for attitudes and informational events. But as before, this does not identify the two perspectives: one can still have advantages over the other. For instance, implicit approaches focus on structural rules, which are a natural high-level vantage point allowing for elegant theory. On the other hand, an explicit dynamic approach provides an emancipation of informational events in problem solving that is of interest per se, as it adds new events beyond inference.

New dynamic logics A neutral two-way view here sees switching perspectives, [11]. In one direction, given an implicit non-standard notion of consequence, one can tease out informational or other events motivating it, and write their explicit dynamic logic. This style of analysis, backed up by mathematical representation theorems, replaces non-standard deviation from classical logic into dynamic extension of classical logic. Explicit events behind non-standard notions of consequence are sometimes easy to find, as in the above analysis of circumscription, but there is no automatic method for this art – and there are unresolved challenges concerning major substructural logics, [42]. In particular, no explicit-style dynamic reanalysis seems to be known so far for linear logic, whose non-classical notion of consequence is primarily based on proof-theoretic resource intuitions.

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[15] It should be noted that non-standard consequence relations, too, can support the introduction of new vocabulary, an issue that will be discussed a bit further in Section 15 below.

New notions of consequence Vice versa, given an explicit dynamic logic of informational events, one can package some basic structure in the form of new consequence relations, and study those per se. The latter move can even add to our fund of styles of reasoning. Here is an illustration. Logics of belief change predict the existence of new styles of inference based on their repertoire of different informational events and attitudes. In particular, problem solving may involve different attitudes, such as both knowledge and belief, and also, it may take some new information as hard, and some in the earlier soft sense, leading to variants of circumscription such as

soft-weak circumscription	$[\Uparrow\varphi_1]\ldots[\Uparrow\varphi_n]B\psi$
soft-strong circumscription	$[\Uparrow\varphi_1]\ldots[\Uparrow\varphi_n]K\psi$

where the premises are just taken as soft upgrades, not as public announcements. Different structural rules will then encode differences in the underlying process of drawing a conclusion. Thus, we generate new notions of consequence, and even more would arise with other mixtures of knowledge, belief and update actions.

Thus, in the study of consequence relations, implicit and explicit approaches live in harmony, and we can often perform a Gestalt switch from one to the other. Such switches also suggest precise mathematical system translations in our earlier sense.

Philosophical repercussions While the preceding analysis may seem just technical, well-known positions based on non-standard logics may come under pressure by an explicit-style reanalysis. In particular, existence of different consequence relations on a par has led to the thesis of Logical Pluralism, a view that logic should acknowledge competing views on the nature of logical consequence, and perhaps also other core notions of the field, [8]. But in our view, this grand conclusion depends on taking the implicit methodology for granted. On a dynamic explicit re-analysis as presented here, the competition between consequence relations disappears, and we get compatible extensions of classical logic without any commitment to competition. The second view need not be superior to the first, but its very existence undermines strong conclusions arising from looking at consequence in only one stance. [16]

[16] As pointed out by a referee, this need not mean that pluralism goes away. On the analysis presented here, pluralism for consequence relations might now morph into pluralism for different natural extensions of the vocabulary of classical logic. But in this version, the thesis is much less contentious.

13 Further examples

We have seen how the implicit/explicit contrast runs through both static and dynamic logics for knowledge and belief, as well as for logics for consequence relations. Further examples in this epistemic line can be found by moving from information flow to design stances in information-driven agency and *games*: for instance, in [12], implicit logic games and explicit game logics are naturally entangled strands throughout. But once one sees the contrast put forward in this paper, it applies to any part of logic whatsoever, not just information and agency. To demonstrate this broader range for our analysis, we discuss two examples, one from the philosophy of physics and one from contemporary metaphysics. [17]

Quantum logic Our first example concerns a stronghold of non-classical logic since the 1930s. Consider the field of *quantum logic*, where the classical distributive law

$$(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$$

fails in the domain of physical quantum phenomena. There are of course reasons for this failure: measurements disturb the current state of a physical system – but this is left implicit in quantum logic. There is a long tradition of research in this area, which has resulted in an extensive algebraic and modal theory of quantum logics.

The first explicit companion to all this seems the dynamic measurement logic of Baltag and Smets, cf. [5]. Their system of 'quantum PDL' has dynamic modalities for measurement actions that satisfy perspicuous laws mirroring physical quantum facts, but it remains squarely based on classical logic. In doing so, it explains all the deviant features of quantum logic in a uniform manner as properties of a small fragment of the explicit language. For instance, failure of distributivity becomes failure of actions to distribute over choice, a well-known phenomenon in logics of computation, which has nothing to do with propositional logic. But an explicit dynamic logic of measurement can also express further significant properties of physical systems, and analyze more types of measurement action on these, making traditional quantum logic a poor projection of what goes on from a physical point of view.

[17] Again, these topics raise new issues of their own that we will only touch upon in what follows.

Discussion This is not just reformulation into an explicit format, again, there are serious philosophical consequences. Quantum mechanics was famously touted by Quine as a realm where not even the laws of logic are immune to revision in scientific theory construction. What was taken for granted in Quine's analysis was that quantum logic in implicit style is the only game in town. But this claim dissolves when we have a mathematically elegant and conceptually perspicuous classical logic that explicitly puts measurement where it belongs: at center stage.

This brief exposition may not do justice to explicit quantum dynamic logic, but suffice it to say that this new approach placing measurement actions and quantum information flow at center stage is more than just logic-internal system redesign. It fits well with a substantive topic, viz. recent investigations into analogies between the foundations of quantum mechanics and theories of computation.

Truth maker semantics Our second example shows our contrast at work in a very recent development. `Truth maker semantics' has been touted as a hyper-intensional paradigm springing the bounds of standard modal logic, cf. [23]. In our terms, truth making is an implicit approach changing the meanings of the logical constants to reflect metaphysically (or, in other applications, epistemic) important structure, and defining new notions of consequence based on these changed meanings. Thus, it makes sense to look for a translation from truth maker logic into an explicit companion, namely, a standard modal logic over the same class of models.

We give a brief explanation of how this can be done for one simple pilot system.

Models for truth making ***M*** are tuples $(S, \leq, V)$ with objects s in S viewed as parts of the world or abstract states. The binary relation $\leq$ is a partial order between states. The relation of supremum $s = sup(t, u)$ (lowest upper bound) says that object s is a sum or merge of the t and u. It is often assumed in the literature that all suprema exist, often as 'impossible worlds' in case the merged states are incompatible.

The simplest relevant language here is a propositional logic with connectives $\neg, \wedge, \vee$. For atomic p, a valuation V records which states in S make p true, the set $V^+(p)$, or false, $V^-(p)$. This can be made subject to further constraints: for instance, that no state makes a proposition both true and false. The truth definition is as follows:

$$\boldsymbol{M}, s \models p \quad \text{iff} \quad s \in V^+(p)$$

$\boldsymbol{M}, s =\mid p$ iff $s \in V^{-}(p)$

$\boldsymbol{M}, s \models \neg\varphi$ iff $\boldsymbol{M}, s =\mid \varphi$

$\boldsymbol{M}, s =\mid \neg\varphi$ iff $\boldsymbol{M}, s \models \varphi$

$\boldsymbol{M}, s \models \varphi \wedge \psi$ iff there exist t, u with $s = sup(t, u)$, $\boldsymbol{M}, t \models \varphi$ and $\boldsymbol{M}, u \models \psi$

$\boldsymbol{M}, s =\mid \varphi \wedge \psi$ iff $\boldsymbol{M}, s =\mid \varphi$ or $\boldsymbol{M}, s =\mid \psi$

$\boldsymbol{M}, s \models \varphi \vee \psi$ iff $\boldsymbol{M}, s \models \varphi$ or $\boldsymbol{M}, s \models \psi$

$\boldsymbol{M}, s =\mid \varphi \vee \psi$ iff there exist t, u with $s = sup(t, u)$, $\boldsymbol{M}, t =\mid \varphi$ and $\boldsymbol{M}, u =\mid \psi$

One can also define further notions of 'loose' and 'inexact' truth and false making.

Next one can define various notions of consequence, such that each truth maker for all premises being a truth maker for the conclusion, or each truth maker of the premises being extendable to one for the conclusion, as well as versions that add conditions on false making. All support different laws for the propositional base language. Thus propositional logic is the locus where the new conceptual analysis shows.

Modal information logic Now essentially these same structures have been around in modal logic since the 1990s as models of abstract information states, [10]. The universal modality $[\uparrow]\varphi$ describes upward structure from a point, and downward $[\downarrow]\varphi$ describes weaker information. The logic is then temporal *S4*. Where suprema exist in the order, the logic describes them using binary modalities:

$\boldsymbol{M}, s \models \langle sup \rangle \varphi\psi$ iff there exist t, u with $s = sup(t, u)$, $\boldsymbol{M}, t \models \varphi$ and $\boldsymbol{M}, u \models \psi$

$\boldsymbol{M}, s \models \langle inf \rangle \varphi\psi$ iff there exist t, u with $s = inf(t, u)$, $\boldsymbol{M}, t \models \varphi$ and $\boldsymbol{M}, u \models \psi$

It is easy to show that $\langle sup \rangle pq$ is not definable in the temporal modal language, making this a natural extension of the ordinary modal logic *S4*.

As for laws of reasoning, the modal logic of information has interesting validities, but details are not relevant here. One principle that fails though is associativity:

$$\langle sup \rangle \langle sup \rangle \varphi\psi\alpha \rightarrow \langle sup \rangle \varphi \langle sup \rangle \psi\alpha$$

The reason is that, unlike in truth maker semantics, we do not demand existence of all suprema in our partial orders. The modal logic of information structures is axiomatizable, but a major open problem is whether it is decidable, [10].

[18]

Translating truth maker logic into modal information logic The models just described and their modal logic are an explicit companion to truth maker logic. And the connection is very close. Here is a two-component recipe for translating from implicit truthmaker logic into explicit modal logic, where the simultaneous use of variants + and – is a standard trick in reducing three-valued logic to classical logics.

Take new proposition letters p^+ and p^- for each atomic proposition letter p. Now, for each propositional formula φ, we recursively extend this double set-up as follows, closely following the above truth definition:

$(\neg\varphi)^+ \;=\; (\varphi)^-$ $\qquad\qquad$ $(\neg\varphi)^- \;=\; (\varphi)^+$

$(\varphi \wedge \psi)^+ \;=\;$ <sup>$(\varphi)^+(\psi)^+$ $\qquad\qquad$ $(\varphi \wedge \psi)^- \;=\; (\varphi)^- \vee (\psi)^-$

$(\varphi \vee \psi)^+ \;=\; (\varphi)^+ \vee (\psi)^+$ $\qquad\qquad$ $(\varphi \wedge \psi)^- \;=\;$ <sup>$(\varphi)^-(\psi)^-$

Theorem $\quad \varphi_1, \ldots, \varphi_n \models \psi$ is valid in truth maker semantics

iff $(\varphi_1)^+, \ldots, (\varphi_n)^+ \models (\psi)^+$ in modal information logic.

We do not provide a formal proof, but the translation is almost self-explanatory.

The translation can accommodate a notion of inexact truth making as $\langle\downarrow\rangle\varphi$, and other modal combinations can deal with `loose truth making'. Adding strict versions $[\uparrow^s]$, $[\downarrow^s]$ of the order modalities defines strict truth making as $[\downarrow^s]\neg\varphi \wedge \varphi \wedge [\uparrow^s]\neg\varphi$. Also, the earlier-mentioned varieties of consequence are easily seen to be modally definable, and with a little more effort, so are special conditions that have been considered for truth maker denotations such as closure under merge, or convexity.

Discussion What does our translation achieve? First, it enlists methods from modal logic in the study of truth making – though not all questions are settled automatically, such as decidability or explicit axiomatization. But in addition, the translation has a clear philosophical point: truth maker semantics is entirely compatible with classical modal logic, refuting claims about irreducibility of its hyper-intensionality.

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[18] Since logics with associative modalities often encode undecidable word problems, this might be a warning sign for the use of `impossible worlds' as suprema in truth making. The practice of throwing in such worlds looks like harmless smoothening of the universe, but it induces associativity.

Finally, in exploring metaphysical intuitions, an explicit modal logic might even be more appropriate than propositional logic re-interpreted via truth making, as it sets no linguistic constraints on how to describe the structures we find in our universe. If we truly love something, why not give it its own vocabulary?

14 Implicit versus explicit stances at work

After this broad array of different examples, it might seem time for a precise formal definition of the implicit/explicit contrast. But we do not have one to offer, and we doubt that a definition exists covering all cases in a useful manner. Even so, we did identify recognizable general features. Implicit approaches enrich old meanings, and locate important information in deviant notions of consequence – explicit approaches introduce new vocabulary, but conservatively extend classical logic. And with this difference comes plurality of alternative logics for implicit approaches, and compatible extensions of classical logic on explicit approaches. These features should be enough to recognize the two styles when one sees them: they are natural approaches toward any subject in logic. [19] Moreover, our terminology is not just a matter of assigning labels to what already exists. As we have shown by many examples, seeing the contrast raises interesting new issues, both practical and theoretical. We summarize a few strands that occurred in the preceding sections.

Finding complementary analyses If we see one stance on a topic, we can usually find a dual one. Thus our contrast becomes a force for logical system design. We saw this with dynamic semantics of questions, which suggested an explicit companion logic of issue modifying events. And conversely, explicit logics of belief change suggested new implicit notions of consequence in the area of non-monotonic logic.

Transfer of ideas Different stances on the same thing facilitate creative borrowing, since their agendas may differ. For instance, epistemic logic has a rich tradition of multi-agent and group knowledge. Intuitionistic logic can then profit from the same ideas, creating accounts of mathematics closer to research as a social activity, cf. [1]. But one can also borrow ideas inside one stance. Say, intuitionistic logic started

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[19] Also, recall that our contrast was meant to apply to design stances, i.e., activities, and only in a derived sense to the logical systems produced by these. It may even be impossible to classify logical systems uniquely as implicit or explicit when we disregard their genesis.

from the proof-theoretic BHK interpretation of the logical constants, which met up with semantics only afterwards. Could a similar proof-theoretic analysis apply to dynamic semantics, a major implicit paradigm for information dynamics? Or, for another example inside the implicit realm: can BHK-style proof analysis be taken to non-monotonic logics, and thus to belief rather than knowledge?

Translation and identity criteria for logics The explicit/implicit contrast also suggests new mathematical issues of translation or reduction between logical systems. We have given some new examples, and no doubt much more can be proved about translating between implicit and explicit logics. Even so, there is no automatic algorithm for turning one sort of system into the other. Finding illuminating counterparts as we have done is an art rather than a science, and it may well remain so.

We have also suggested that, even when implicit and explicit logics can be mutually embedded under translation – clearly a telling fact – subtle differences may remain. Here we encountered a general issue in the foundations of logic. There is no generally accepted criterion for when two (presentations of) a logical system should count as the same. Mutual interpretability is a significant notion of equivalence that allows for much transfer of information, so we should always see if it occurs, but it need not be the last word. [20] In fact, one vexing problem that makes it hard to judge the merits of this notion is a scarcity of negative results. There are no general methods showing *non*-translatability between logical systems. Perhaps, in the end, there is too much translatability in the realm of logics, and a finer sieve is needed.

Merging Where we cannot translate different stances into each other, a weaker connection is compatibility in meaningful *merges*. Many systems in the literature combine implicit and explicit features: intuitionistic modal logics, [20], merged logic games and game logics, [1], dynamic-epistemic inquisitive logics, [44], joint linear-temporal logics, and so on. Often, these merges feel natural. A recent case is the intuitionistically flavored possibility semantics for classical logic in [13], [45]. [21]

[20] For a recent extensive discussion of translatability and identity issues, cf. the dissertation [35].

[21] Merges are a case where an over-zealous exclusive use of the implicit/explicit contrast makes no sense. Is intuitionistic modal logic explicit because it has modalities, or implicit because its underlying propositional base packages information that could be brought out by a Gödel translation?

15 General discussion

The above concludes our case for making the explicit/implicit distinction and seeing where it leads. We conclude with a few more critical points, or at least, challenges, having to do with a proper understanding of the contrast, and with its limitations.

First-person and third-person views We have noted the existence of our contrast, but we have not offered an explanation of why it is there, or perhaps more to the point, of the co-existence of explicit and implicit aproaches. It has been suggested by readers of this paper that the background may lie in some well-known distinctions in logic and philosophy. One is that between *reasoning with,* from an internal first-person perspective, and *reasoning about,* from an external third-person perspective. Implicit logics might give the reasoning with, and explicit ones reasoning about. But while this seems appealing, it does not quite fit with the way these systems function in practice. For instance, epistemic logic with operators can also be used in first-person mode by agents reasoning about their own information, and on the other hand, say, dynamic semantics has also been applied to third-person discourse. There may be a correlation between explicit design and a third-person stance, and implicit design with a first-person stance, but it does not seem very tight.

Object language and meta-language Another distinction that seems congenial is that between *object language* and *meta-language*. We can often formalize the meta-language of the semantics for a logical system in some other logic – the 'standard translation' for modal logic is a typical example, [16]. Is the meta-logic then the explicit version of an implicit object logic? In some cases this correlation fits well. One might view modal *S4* as formalizing a small, but significant part of the first-order meta-theory of intuitionistic semantics. [22] But even so, a complete identification of explicit logics with meta-theory inspired extensions of implicit logics does not fit universally. It is not at all clear, for instance, in which sense paradigmatic systems in philosophical logic such as doxastic or conditional logics are meta-logics of implicit systems – or for an example discussed at length in this paper, in which

[22] Further telling illustrations occur at the interface of logic and games. The modal-dynamic game logic of [41] formalizes part of the meta-theory of first-order evaluation games. Vice versa, game logics induce logic games, that is, implicit practices for analyzing their semantics – and this design cycle can even be iterated, cf. the research program developed in [12].

sense dynamic-epistemic logic is a meta-logic of dynamic semantics. Again, the object/meta distinction offers an interesting correlation, but no more than that.

Choosing locally Co-existence means that both implicit and explicit stances have intrinsic value, but even so, particular areas may bring further reasons for using one rather than the other. For instance, are there favored stances in human *cognition*? Indeed, it has been claimed that natural language conveys much information implicitly, perhaps for ease of coding. Implicit logics would then model this reality directly, whereas explicit logics of information and agency are outside theorists' views of language. But this does not fit the empirical facts as I would see them. Natural language is a medium where both stances occur, in the guise of what one might call participating versus observing modes on cognitive activities. A key feature to keep in mind here is the *universality* of natural language. We switch between the two modes all the time, while firmly staying inside the same medium of communication. There may be local cognitive preferences between going explicit or implicit, but we doubt they can be justified in a sweeping manner.

We end with a few more critical points about potential bias in our analysis.

The challenge from proof theory Does the implicit/explicit contrast really make sense all across logic, or does it have presuppositions that limit its applicability? Most of our discussion was couched in semantic terms, focusing on meaning and valid consequence. Does it transfer to a combinatorial proof-theoretic setting?

There are severe challenges here. To mention just one, in a proof-theoretic perspective, given any logical system, it makes sense to look for natural inferential fragments that use less proof power for applications. In this way, we can start with an explicit logic and change its base calculus, without having a primary semantic motivation for doing so. In that setting, our contrast does not readily apply. [23]

But there is more. Proof-theoretically weaker logics often come with an extended vocabulary that has a purely combinatorial motivation of encoding proof patterns, cf. the product or residual operations in the substructural logics of [42]. In the latter

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[23] Thomas Icard (p. c.) points out the case of `natural logics', very small efficient fragments of logical systems, as a form of design where it is hard to contrast implicit and explicit features. For such small fragments, there is often no difference between classically valid and non-classical consequence.

case, logics that we classified as implicit from a semantic point of view do support new `explicit' vocabulary, but now for intrinsically combinatorial reasons, whose semantic rationale may not be apparent. Finding a proof-theoretic companion, if one exists, for the explicit/implicit contrast of this paper remains an open problem.

The ubiquity of new language design Outside of proof theory, our discussion may have suggested that new language design is a preserve of the explicit design stance. But even this is not quite right. It is true that many implicit logics focus on reinterpretation of existing vocabulary and the resulting changes in valid consequence, but in some cases, new vocabulary does get created, such as the earlier-mentioned non-monotonic logics that can have new operators referring to minimization along orderings, [9]. In fact, as a general point of methodology, in all cases of weakening logics by extending model classes through the introduction of new semantic primitives (ordering, admissibility, and the like), there is a potential for introducing new vocabulary reflecting those primitives. Then the linguistic difference between explicit and implicit shifts to different *motivations* for introducing new vocabulary – an intriguing subject whose discussion we leave open in this paper.

16 Conclusion

We have identified what we see as a significant methodological contrast running through logic, between implicit and explicit stances. We use the word 'stance' here, and not 'system', because we do not identify logic with a family of formal systems. Some logical systems can indeed be called implicit or explicit, but the contrast as we have discussed it also applies to broader working habits in analysis and design.

We have not hidden the fact that our contrast also has its limitations, and that it does not apply to all logical systems, or travel well from semantics to proof theory. The jury is still out on whether we can spring those boundaries.

Even so, seeing the contrast as it stands reveals patterns running through the field of logic, and it suggests new questions of a conceptual and technical nature. We have shown this in concrete instances of system design and in translations between systems. So, awareness of our contrast means new interesting work to be done, and more generally, we see it as a force toward a better understanding of the coherence of logic, in systems and in working habits. But also, and perhaps more importantly, we have pointed out in several illustrations (dynamic semantics, logical pluralism, Quinean revision in logic, truth maker semantics) that awareness of the contrast has serious philosophical consequences, since it undercuts sweeping ideological views that are tacitly based on taking one stance while ignoring the other.

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