TRUTH MAKER SEMANTICS AND MODAL INFORMATION LOGIC

Draft, Johan van Benthem, autumn 2017

This is a short note written after attending a spring seminar taught by Kit Fine at Stanford, looking at some of the material presented there from the standpoint of a modal logician. There are no extensive references yet, Fine's relevant work can be found on his webpage.

1 A brief survey of truth making

Models ***M*** are tuples *(S, ≤, V)* with objects *s* in *S* viewed as parts of the world or 'states' of a more abstract sort. Of particular interest is the relation of supremum *s = sup(t, u)* (lowest upper bound) which can be interpreted as saying that the object *s* is a 'sum' or 'merge' of the objects *t* and *u*. It is assumed in Fine's semantics that all suprema exist, something that is related to the introduction of 'impossible worlds'.

The simplest relevant language is a propositional logic with connectives ¬, ∧, ∨. For atomic *p*, the valuation *V(p)* records which states in *S* make *p* true (call this set $V^+(p)$) or false ($V^-(p)$). This can be subject to constraints: for instance, that no state makes a proposition both true and false – though we can also leave things open, with all four possible combinations. The truth definition now works as follows:

$\boldsymbol{M}, s \models p$	iff	$s \in V^+(p)$
$\boldsymbol{M}, s =\mid p$	iff	$s \in V^-(p)$
$\boldsymbol{M}, s \models \neg\varphi$	iff	$\boldsymbol{M}, s =\mid \varphi$
$\boldsymbol{M}, s =\mid \neg\varphi$	iff	$\boldsymbol{M}, s \models \varphi$
$\boldsymbol{M}, s \models \varphi \wedge \psi$	iff	there exist t, u with $s = sup(t, u)$, $\boldsymbol{M}, t \models \varphi$ and $\boldsymbol{M}, u \models \psi$
$\boldsymbol{M}, s =\mid \varphi \wedge \psi$	iff	$\boldsymbol{M}, s =\mid \varphi$ or $\boldsymbol{M}, s =\mid \psi$
$\boldsymbol{M}, s \models \varphi \vee \psi$	iff	$\boldsymbol{M}, s \models \varphi$ or $\boldsymbol{M}, s \models \psi$
$\boldsymbol{M}, s =\mid \varphi \vee \psi$	iff	there exist t, u with $s = sup(t, u)$, $\boldsymbol{M}, t =\mid \varphi$ and $\boldsymbol{M}, u =\mid \psi$

With this in place, one can define various notions of truth and false making. For instance, exact truth making of a formula φ by a state *s* means that no proper part or proper extension of *s* along the ordering ≤ makes φ true. But there are also other versions of truth making where a state has to contain an exact truth maker, or can be extended to one, and so on. Using these notions, once can then define various notions of consequence from premises $\boldsymbol{\varphi}$ to conclusion ψ. Here are a few samples:

- Each truth maker of all premises is a truth maker for the conclusion,
- Each state that is a merge of states making the premises true (one for each premise) is a truth maker for the conclusion,
- Each truth maker of the premises can be extended to one for the conclusion,
- Each truth maker of all premises is a truth maker for the conclusion, and each false maker for the conclusion is a false maker for at least one premise.

These different notions support different valid consequences as is easy to see. Many of them can be reduced to the first-mentioned one using obvious definitions.

First observations. This framework uses well-known notions. The above models are partial orders, and in case we want all suprema to exist, they are complete partial orders (cpo's). The latter structures have been studied extensively, for instance, in Scott's *domain semantics* for information and computation in the 1970s. As for the semantics, the tandem of truth making and false making also occurs in Veltman's *data semantics* from the early 1980s, and in fact, it has long been known that this is equivalent to using a three- or four-valued logic. The above proliferation of notions of valid consequence is well-known from the area of *partial logic* (cf. Stephen Blamey's survey chapter in the 1985 "Handbook of Philosophical Logic").

2 Modal information logic

A more concrete and fruitful comparison along the preceding lines can be made with the modal logic of information proposed in van Benthem 1988, 1996, 2017A. This system had an entirely different motivation, viz. developing an abstract theory of information states, but it shows striking analogies with truth maker semantics.

Models and language of MIL We work with partial orders viewed as universes of information states. The standard universal modality talks about upward structure from a given point, which we highlight for present purposes in the notation $[\uparrow]\varphi$. The downward direction makes sense in discussing weaker information, and so we add a converse modality $[\downarrow]\varphi$. The resulting system is temporal S4.

But there is more. Where suprema exist in the partial order the logic encodes their behavior – and likewise for informational infima, using two binary modalities:

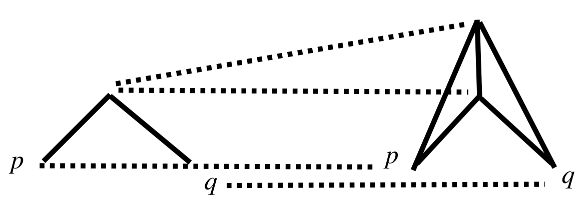
$$\boldsymbol{M}, s \models \langle sup\rangle\varphi\psi \quad \text{iff} \quad \text{there exist } t, u \text{ with } s = sup(t, u), \boldsymbol{M}, t \models \varphi \text{ and } \boldsymbol{M}, u \models \psi$$

$$\boldsymbol{M}, s \models \langle inf\rangle\varphi\psi \quad \text{iff} \quad \text{there exist } t, u \text{ with } s = inf(t, u), \boldsymbol{M}, t \models \varphi \text{ and } \boldsymbol{M}, u \models \psi$$

These are truly new operators.

Fact $\langle sup\rangle pq$ is not definable in the temporal modal language.

With the following two models, the dotted lines are a bisimulation for the temporal language, but $\langle sup\rangle pq$ holds only in the top node on the left, not that on the right.



Further model theory of the system can be developed as for standard modal logic, including a more discerning notion of bisimulation that respects sups and infs.

Valid laws of information Here are a few validities of the modal logic of information:

- $\langle\uparrow\rangle\varphi \leftrightarrow \langle inf\rangle\varphi T$, $\quad \langle\downarrow\rangle\varphi \leftrightarrow \langle sup\rangle\varphi T$
- $\langle sup\rangle\varphi\psi$ and $\langle inf\rangle\varphi\psi$ distribute over disjunction, in both arguments.
- $\langle sup\rangle\varphi\psi \rightarrow \langle sup\rangle\psi\varphi$, $\quad \varphi \rightarrow \langle sup\rangle\varphi\varphi$, and likewise for $\langle inf\rangle$
- $\langle sup\rangle\varphi\psi \rightarrow (\langle\downarrow\rangle\varphi \wedge \langle\downarrow\rangle\psi)$, $\quad \langle inf\rangle\varphi\psi \rightarrow (\langle\uparrow\rangle\varphi \wedge \langle\uparrow\rangle\psi)$

However, one prima facie attractive principle that fails in general is associativity:

$$\langle sup\rangle\langle sup\rangle\varphi\psi\alpha \rightarrow \langle sup\rangle\varphi\langle sup\rangle\psi\alpha$$

plus its right-to-left converse. The reason is that we do not demand existence of all suprema in our partial orders, and such failures readily yield counterexamples.

Remark More abstractly, we can interpret $\langle sup\rangle$ as a standard existential binary modality interpreted via a ternary accessibility relation C:

$$\boldsymbol{M}, s \models \langle sup\rangle\varphi\psi \quad \text{iff} \quad \text{there exist } t, u \text{ such that } Cstu, \boldsymbol{M}, t \models \varphi \text{ and } \boldsymbol{M}, u \models \psi$$

Modal frame correspondence methods then apply to the combined modal logic of C and $\leq$. In particular, imposing an associativity axiom matches first-order associativity conditions on partial orders that can be computed by standard algorithms:

$$\forall yzuv\ ((Sxyz \wedge Syuv) \rightarrow \exists w\ (Sxuw \wedge Swvz))$$

Question Are there interesting general axioms that link $\langle inf\rangle\varphi\psi$ to $\langle sup\rangle\varphi\psi$?

Fact The modal logic of information structures is recursively axiomatizable.

This is since the modal truth conditions translate into the first-order logic of partial orders. However, the less obvious and more interesting issue is the following:

Open problem Is the modal logic of information *decidable*?

Our dropping of associativity seems essential to a positive answer here, since basic modal logics with associative binary modalities are known to encode undecidable word problems. Thus, the use of cpo's in Section 1, which seem just a nice and harmless device for smoothening our models, may actually endanger decidability.

Modal logic of information seems a natural base logic of information structures, and in fact MLI was proposed at the time as an analysis of common patterns behind modal logic, belief revision, resource-sensitive logics, and the like. But alternatively, one can view it is as just an obvious technical expressive completion of S4 whose omission in the literature seems surprising (if we want logics over partial orders, why not take on board some further natural structure found in partial orders). In that sense, finding out the properties of MLI is urgent just "because it's there" – as Mallory famously said when asked why he wanted to climb Mount Everest.

3 Translating truth maker logic into modal information logic

There is more here than just analogy. We now provide a complete and faithful translation from the truth maker logic of Section 1 into the MLI of Section 2. Our translation will use technical ideas that have been around since at least the 1980s (cf. van Benthem 1986) , but that may not be as well-known as they should be.

Two-component translation Take new proposition letters p^+ and p^- for each atomic proposition letter p. For each propositional formula φ, we now recursively extend this as follows, closely following the truth definition in Section 1:

$(\neg\varphi)^+ = (\varphi)^-$	$(\neg\varphi)^- = (\varphi)^+$
$(\varphi \wedge \psi)^+ = \langle sup \rangle(\varphi)^+(\psi)^+$	$(\varphi \wedge \psi)^- = (\varphi)^- \vee (\psi)^-$
$(\varphi \vee \psi)^+ = (\varphi)^+ \vee (\psi)^+$	$(\varphi \vee \psi)^- = \langle sup \rangle(\varphi)^-(\psi)^-$

Theorem $\varphi_1, \ldots, \varphi_n \models \psi$ is valid in truth maker semantics
iff $(\varphi_1)^+, \ldots, (\varphi_n |)^+ \models (\psi)^+$ in modal information logic.

The reason is that each model $\boldsymbol{M}$ for our modal language can be transformed in an obvious way into a truth maker model $Tr(\boldsymbol{M})$ such that

$Tr(\boldsymbol{M}), s \models \varphi$ iff $\boldsymbol{M}, s \models (\varphi)^+$
$Tr(\boldsymbol{M}), s =| \varphi$ iff $\boldsymbol{M}, s \models (\varphi)^-$

This translation can be enhanced using further vocabulary in our modal language that is needed when we give a standard modal analysis of further notions in truth maker logic (a priori, these further notions need not have been modal, but perhaps surprisingly, they all are). For instance, using the strict versions $[\uparrow^s]$, $[\downarrow^s]$ of our order modalities, referring to the strict variant $x < y := x \leq y \ \&\ \neg\, y \leq x$) of the partial order, whose complete theory is easy to add in a standard modal style,

strict truth making can be defined as	$[\downarrow^s]\neg\varphi \wedge \varphi \wedge [\uparrow^s]\neg\varphi$
partial truth making as	$\langle\uparrow\rangle\varphi$
loose truth making as	$\langle\downarrow\rangle\varphi$

As we said earlier, further varieties of consequence are also easily definable.

Our modal language also deals straightforwardly with various special conditions that have been considered for denotations of propositions in truth maker semantics. For instance, here are two important cases:

closure under merges	imposes	$\langle sup \rangle\varphi\varphi \rightarrow \varphi$
convexity	imposes	$\varphi \rightarrow [\uparrow](\langle\uparrow\rangle\varphi \rightarrow \varphi)$

Finally, various additional assumptions relating truth making and false making can be put into our language in the form of requirements on the p^+ and p^- predicates.

4 Discussion

The preceding results are not completely conclusive, since models for Fine's truth maker semantics have some special features. We discuss a few of these.

Complexity One assumption is that all suprema exist. One way of thinking about this requirement is as arising from a harmless standard completion of arbitrary partial orders, say, by means of Fine's algebraic 'quasi-filters'. And there may also be metaphysical reasons for thinking that the universe contains all recombinations of parts, or of states. (By contrast, this assumption is much less appealing when thinking about states in modal information logic, as incompatibility is a key notion of interest there – that we do not want to sweep under the rug.) In any case, as we noted before, this smooth 'filling out' of the models may come with a price in terms of undecidability of the resulting logic. Moreover, since completeness of cpo's is a second-order condition, even recursive axiomatizability may not be immediate.

Aside It was remarked at the seminar that Fine's insistence on decidability may be a red herring, since he motivates this by means of the logical omniscience problem. But that seems more of an epistemological than a metaphysical issue, unless one thinks that the metaphysical universe should be especially easy for us to grasp. Moreover, it is well-known in logic that decreasing a set of validities has no obvious connection with lowering computational complexity: things can go either way.

Maximal objects Another special assumption considered in some metaphysics (as reported by Fine) requires models to have maximal states in the ordering, where each proposition is either made true or made false. This amounts to another sort of completion for models, whose effects we have not yet considered in modal information models. Incidentally, a milder intermediate form would focus on so-called `generic branches' – a notion familiar from set-theoretic forcing – in a model, where each proposition is eventually made true or false at some stage on such a branch. For further uses and theory of generic branches, cf. van Benthem 2016.

No battle of paradigms What does the above translation achieve? We see its main virtue as conceptual. Truth maker semantics is entirely compatible with classical modal logic, no irreducible hyper-intensionality is involved. Both the models and the languages of truth maker semantics have clear analogues in well-studied areas of modal logic. So, this should suffice for toning down some current propaganda.

But how can this be? Many people think that truth maker semantics extends classical modal semantics, since we now have models containing not just total states, but also partial states. So how can this be translated back into standard modal logic, and even into a very special modal logic? I believe that there is a general point here about abstraction, that is often misunderstood. Modal logic is a mildly expressive and decidable language talking about relational models: i.e., annotated graphs. Now philosophers often give these models very specific interpretations: with points as complete 'possible worlds'. Any change away from possible worlds then looks like an extension: indeed, a conceptual revolution. But in fact, modal logic applies to any graph-like structure, where 'points' can just as well be partial stages, or whatever. That is, it applies to most generalized semantics, since their models have points, too. In fact, as in intuitionistic or truth maker semantics, new models usually come with further order structure between points, which often means: more relations, and so more modalities for these, satisfying special axioms reflecting special properties of the additional relations. Thus, perhaps paradoxically, most generalized semantics I know of are in fact *specialized semantics* from a modal point of view.

This coexistence is a much more productive perspective than the competition of logics and paradigm shifts touted by philosophers: these disregard mathematical reality, and failure to see the connections may even hamper the study of new ideas.

Work remains to be done Technically speaking, the link to modal logic may even be helpful in understanding the workings of truth maker semantics, as in our discussion of associativity. What our translation does not yield, though, is an *explicit axiomatization* of truth maker consequence. This requires further work – as is usual with translations between logics. Also, we do not settle the *complexity* of truth maker validity, since modal information logic might be more complex than its fragment consisting of our translations of formulas in truth maker semantics.

Open problem Is there a converse embedding from modal information logic into propositional truth maker logic?

Aside Our translation takes propositional truth/false maker logic into a very special fragment of modal information logic, as it only uses syntactically *positive* formulas. (It may be thought that we will need classical negation to separate p^+ from p^-, using something like a universal modality $U\neg(p^+ \wedge p^-)$, but this can be circumvented: just translate as in Section 3, and move $U\neg(p^+ \wedge p^-)$ to a positive disjunct $E(p^+ \wedge p^-)$ in the conclusion.) We do not know if this positive syntax is significant for complexity.

Choice of language The modal translation also raises another type of issue, namely, which language is most suitable for describing the original models for truth maker semantics. After all, these models are supposed to stand for an independent metaphysical structure about which we have intuitions, and whose laws of reasoning we want to study. Then why would it be obvious that the standard language of propositional logic, or predicate logic, would be the right vehicle for this?

Implicit versus explicit Studying these laws in terms of a standard propositional language with modified meanings and deviant laws is an 'implicit' approach, while our modal logic MLI is an 'explicit' approach in the sense of van Benthem 2017B, extending the vocabulary for state structure while staying on a classical base logic.

The latter approach suggests further extensions of the truth maker language that would reflect more possibly metaphysically relevant structure. One obvious source for this is the symmetry of partial orders in upward and downward directions, with infima as relevant as suprema. This would lead to a 'splitting' of classical connectives in the spirit of relevant logic (or more radically, linear logic) where the logic of the richer vocabulary may reveal much more of the semantic structure.

Objects, predication, quantification A good test on our analysis and conclusions is how they fare on truth maker semantics for modal logic and for predicate logic. But, at present, it is not clear what a canonical proposal for the latter would be.

5 References

The following sources contain the techniques and models highlighted in the above.

J. van Benthem, 1986, 'Partiality & Non–Monotonicity in Classical Logic', *Logique et Analyse* 29, 225–247. – , 'Semantic Parallels in Natural Language and Computation', in H–D Ebbinghaus et al., eds., *Logic Colloquium. Granada 1987*, North–Holland, Amsterdam, 331–375. –, 1996, 'Modal Logic as a Theory of Information', in J. Copeland, ed., *Logic and Reality. Essays on the Legacy of Arthur Prior*, Clarendon Press, Oxford, 135–168. –, 2016, 'Tales from an Old Manuscript', in J. van Eijck, R. Iemhoff & J. Joosten, eds., *Liber Amicorum Alberti*, College Publications, London, 5–14. –, 2017A, 'Constructive Agents', *Indagationes Mathematicae* online, Brouwer Anniversary issue, to appear, –, 2017B, 'Implicit and Explicit Stances in Logic', manuscript, ILLC, University of Amsterdam.