Intermediate logics admitting a structural hypersequent calculus

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Abstract

We characterise the intermediate logics which admit a cut-free hypersequent calculus of the form $\mathbf{HLJ} + \mathscr{R}$, where **HLJ** is the hypersequent counterpart of the sequent calculus **LJ** for propositional intuitionistic logic, and $\mathscr{R}$ is a set of so-called structural hypersequent rules, i.e., rules not involving any logical connectives. The characterisation of this class of intermediate logics is presented both in terms of the algebraic and the relational semantics for intermediate logics. We discuss various—positive as well as negative—consequences of this characterisation.

Keywords: Intermediate logics, hypersequent calculi, algebraic proof theory, Heyting algebras.

1 Introduction

Constructing cut-free proof calculi for intermediate logics can be notoriously difficult. In fact, we know only of very few intermediate logics having a cut-free Gentzen-style sequent calculus obtained by adding a finite number of sequent rules to the single-succedent[1] sequent calculus **LJ** for **IPC**.[2] On the other hand few decisively negative results have been obtained in this respect. The few such results in the literature are of a rather general nature ruling out sequent systems with rules of a particular syntactic shape for certain (classes of) logics, see, e.g., [22, Cor. 7.2] and [40].[3] However, by moving to the framework of hypersequent calculi [45, 47, 3] it is possible to construct cut-free hypersequent calculi for many well-known intermediate logics, see, e.g., [4, 21, 20, 26]. Hypersequents are nothing but finite multisets of sequents. In general adding so-called *structural* hypersequent rules, viz., rules not involving the logical connectives, usually behaves well with respect to the cut-elimination procedure. In fact, a systematic approach to the problem of constructing cut-free proof calculi has been developed and a class of formulas, called $\mathcal{P}_3$, has been singled out for which corresponding cut-free structural single-succedent hypersequent calculi may be obtained in a uniform manner [22, 24]. However, negative results demarcating the class of intermediate logics admitting cut-free structural hypersequent calculi are still to some extent lacking. Considering substructural logics proper some meaningful necessary conditions for admitting a cut-free structural hypersequent calculus have been provided [22, Cor. 7.3], [25, Thm. 6.8].

Our contribution consists in singling out a purely semantic criterion determining when an intermediate logic can be captured by a cut-free structural single-succedent hypersequent calculus extending the basic single-succedent hypersequent calculus **HLJ** for **IPC**. This is done by considering a subclass of the so-called $(0, \wedge, \vee, 1)$-stable logics studied in [7, 10, 13]. More precisely, we introduce a class of intermediate logics, which we call $(0, \wedge, 1)$-stable, determined by classes of Heyting algebras closed under taking $(0, \wedge, 1)$-subalgebras of subdirectly irreducible Heyting algebras. We are also able to show that all such intermediate logics are elementarily determined and we obtain a characterisation of the first-order frame conditions determining intermediate logics with a cut-free structural hypersequent calculus. These frame conditions are analogous to the frame conditions introduced by Lahav [42] for constructing analytic hypersequent calculi for modal logics. Finally, we compare the class of $(0, \wedge, 1)$-stable intermediate

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[1] Recall that a sequent is a *single-succedent sequent* if at most one formula occurs on the right-hand side of the sequent arrow.

[2] In [29, 50] a sequent calculus for the intermediate logic **LC** is obtain by adding infinitely many rules to the multi-succedent calculus **LJ′** for **IPC** and [38] gives a Gentzen-like calculus for **KC** in terms of finitely many rules which are, however, non-local. Finally, [2] gives examples of tableau calculi for the seven interpolable intermediate logics from which corresponding sequent calculi may be obtained.

[3] See also [44, 39] for negative result about cut-free sequent systems for modal logics.

logics to the class of $(0, \wedge, \vee, 1)$-stable intermediate logics. We show that there are $(0, \wedge, \vee, 1)$-stable intermediate logics which are not $(0, \wedge, 1)$-stable. Furthermore, we show that the $(0, \wedge, \vee, 1)$-stable logics given by the so-called $(0, \wedge, \vee, 1)$-stable rules [10, Sec. 5] determined by finite well-connected Heyting algebras which are projective as objects in the category of distributive lattices will necessarily be $(0, \wedge, 1)$-stable. Lastly, we show that the $(0, \wedge, 1)$-stable logics are precisely the cofinal subframe logics which are also $(0, \wedge, \vee, 1)$-stable.

The paper is structured as follows. Section 2 contains a short introduction to hypersequent calculi and their algebraic interpretation. Section 3 contains the algebraic characterisation of intermediate logics with a (cut-free) structural hypersequent calculus. In Section 4 the first-order frame conditions associated with the class of intermediate logics admitting a cut-free structural hypersequent calculi are determined and in Section 5 this class of intermediate logics is compared with the class of $(0, \wedge, \vee, 1)$-stable logics.

2 Preliminaries on algebraic proof theory

In this section we will briefly review the necessary background on algebraic proof theory [22, 24, 25] on which the findings of the present paper heavily relies.

2.1 Hypersequents

Let P be a set of propositional letters and let Form be the set of propositional formulas in the language of intuitionistic logic given by the following grammar

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi, \quad p \in \mathsf{P}.$$

Note that in this language both the connective $\leftrightarrow$ and the constant $\top$ is definable, e.g., as $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\bot \to \bot$, respectively.

By a single-succedent sequent (in the language of propositional intuitionistic logic) we shall understand a pair (Γ, Π), written $\Gamma \Rightarrow \Pi$, where Γ is a finite multiset of formulas from Form and Π is a stoup, i.e., either empty or a single formula in Form. The sequent system **LJ**, see, e.g., [32, Chap. 1.3], provides a sequent calculus which is sound and complete with respect to propositional intuitionistic logic **IPC**. However, when adding additional axioms to **LJ** the resulting system is no longer guaranteed to enjoy the same proof-theoretic properties as **LJ** such as cut-elimination.

Nevertheless, many logics for which no cut-free Gentzen-style sequent calculus is available may be captured nicely by the so-called hypersequent calculus formalism. A *hypersequent* is simply a finite multiset of sequents H written as $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$, where the sequents $\Gamma_i \Rightarrow \Pi_i$ are called the components of the hypersequent H. One may think of a hypersequent as a "meta-disjunction" of sequents. The hypersequent formalism can therefore be thought of as a proof-theoretic framework that allows for the manipulation of sequents in parallel. In addition to the usual (internal) structural rules such as contraction and weakening the hypersequent framework allows us to consider a wide variety of so-called *structural hypersequent rule* [22, Sec. 3.1], i.e., hypersequent rules not involving any of the logical connectives, which operates on multiple components at once. For example the structural hypersequent rules

$$\frac{H \mid \Gamma_1, \Gamma_2' \Rightarrow \Pi_1 \qquad H \mid \Gamma_2, \Gamma_1' \Rightarrow \Pi_2}{H \mid \Gamma_1, \Gamma_1' \Rightarrow \Pi_1 \mid \Gamma_2, \Gamma_2' \Rightarrow \Pi_2}\,(com) \qquad \frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow}{H \mid \Gamma_1 \Rightarrow \ \mid \Gamma_2 \Rightarrow}\,(lq)$$

determines hypersequent calculi for the intermediate logics $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$ and $\mathbf{KC} \coloneqq \mathbf{IPC} + \neg p \vee \neg\neg p$, respectively, when added to **HLJ**, viz., the hypersequent version of **LJ**, defined below.

Definition 2.1. Let **HLJ** denote the hypersequent calculus consisting of the following rules.

Logical rules:

$$\frac{}{H \mid \varphi \Rightarrow \varphi}\,(inti) \qquad \frac{}{H \mid \bot \Rightarrow}\,(\bot)$$

$$\frac{H \mid \Gamma \Rightarrow \varphi \qquad H \mid \Gamma, \psi \Rightarrow \Pi}{H \mid \Gamma, \varphi \to \psi \Rightarrow \Pi}\,(L \to) \qquad \frac{H \mid \Gamma, \varphi \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \to \psi}\,(R \to)$$

$$\frac{H \mid \Gamma, \varphi, \psi \Rightarrow \Pi}{H \mid \Gamma, \varphi \wedge \psi \Rightarrow \Pi}\,(L\wedge) \qquad \frac{H \mid \Gamma \Rightarrow \varphi \qquad H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \wedge \psi}\,(R\wedge)$$

$$\frac{H \mid \Gamma, \varphi \Rightarrow \Pi \qquad H \mid \Gamma, \psi \Rightarrow \Pi}{H \mid \Gamma, \varphi \vee \psi \Rightarrow \Pi}\,(L\vee) \qquad \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow \varphi \vee \psi}\,(R\vee_1) \qquad \frac{H \mid \Gamma \Rightarrow \psi}{H \mid \Gamma \Rightarrow \varphi \vee \psi}\,(R\vee_2)$$

The internal structural rules

$$\frac{H \mid \Gamma \Rightarrow \Pi}{H \mid \Gamma, \varphi \Rightarrow \Pi}\,(LW) \qquad \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \varphi}\,(RW) \qquad \frac{H \mid \Gamma, \varphi, \varphi \Rightarrow \Pi}{H \mid \Gamma, \varphi \Rightarrow \Pi}\,(LC)$$

The external structural rules

$$\frac{H}{H \mid G}\,(EW) \qquad \frac{H \mid G \mid G}{H \mid G}\,(EC)$$

The cut-rule

$$\frac{H \mid \Gamma \Rightarrow \varphi \qquad H \mid \Sigma, \varphi \Rightarrow \Pi}{H \mid \Gamma, \Sigma \Rightarrow \Pi}\,(cut)$$

For what follows it will be convenient to have fixed a notion of hypersequent calculus.

Definition 2.2. An *intermediate hypersequent calculus* is a calculus of the form $\mathbf{HLJ} + \mathscr{R}$, for a set $\mathscr{R}$ of hypersequent rules in the language of intuitionistic logic. Furthermore, if $\mathscr{R}$ is a set of structural hypersequent rules in the language of intuitionistic logic we say that $\mathbf{HLJ}+\mathscr{R}$ is a *structural intermediate hypersequent calculus.* A hypersequent H is derivable in $\mathbf{HLJ}+\mathscr{R}$ from a set of hypersequents $\mathcal{H}$, written $\mathcal{H} \vdash_{\mathbf{HLJ}+\mathscr{R}} H$, if H can be obtained using the inference rules from $\mathbf{HLJ}+\mathscr{R}$ possibly using hypersequents in $\mathcal{H}$ as initial assumptions. In the case where $\mathcal{H}$ is empty we simply write $\vdash_{\mathbf{HLJ}+\mathscr{R}} H$. A hypersequent rule (r) is derivable in a calculus $\mathbf{HLJ} + \mathscr{R}$ if the conclusion of (r) is derivable in $\mathbf{HLJ} + \mathscr{R}$ from the premises of (r). Finally, we say that an intermediate hypersequent calculus $\mathbf{HLJ} + \mathscr{R}$ is *cut-free* if any hypersequent derivable in $\mathbf{HLJ} + \mathscr{R}$ can be derived without using the cut-rule.

Remark 2.3. Note that **HLJ** consists of all the rules of the sequent calculus **LJ** with a hypersequent context together with the two external structural rules. Consequently, all of the notions from Definition 2.2 also apply mutatis mutandis to sequent calculi. In particular, it is not difficult to see that **HLJ** and **LJ** derive exactly the same sequents.

Remark 2.4. We are here identifying hypersequent calculi with extensions of the calculus **HLJ**, but we could of course equally well have consider extensions of other cut-free hypersequent calculi for **IPC**. However, for what follows it is essential that we consider single-succedent calculi as we will be relying on results from [22, 25] which only consider the single-succedent case. In fact, it is not immediately clear if the approach of [22, 25] can be successfully adapted to the multi-succedent setting.

Each consistent hypersequent calculus $\mathbf{HLJ} + \mathscr{R}$ determines an intermediate logic, namely,

$$\Lambda(\mathbf{HLJ} + \mathscr{R}) \coloneqq \{\varphi \in \text{Form}\colon \vdash_{\mathbf{HLJ}+\mathscr{R}} \Rightarrow \varphi\}.$$

Definition 2.5. Given an intermediate logic L we say that a hypersequent calculus $\mathbf{HLJ}+\mathscr{R}$ *determines* L if $\Lambda(\mathbf{HLJ} + \mathscr{R}) = L$. Furthermore, given a property P we say that L *admits a hypersequent calculus with property* P if L is determined by a hypersequent calculus $\mathbf{HLJ} + \mathscr{R}$ with the property P.

Remark 2.6. Given a sequent S, say, $\Gamma \Rightarrow \Pi$, there exists a formula φ_S, namely, $\bigwedge \Gamma \to \bigvee \Pi$, such that S and φ_S determine the same super-intuitionistic logic. We make use of the convention that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$. Similarly, given a hypersequent rule (r) there exists a finite set of multi-conclusion rules $\mathscr{M}_r$ such that (r) and $\mathscr{M}_r$ determine the same super-intuitionistic logic, for details see, e.g., [16, Sec. 2]. Thus for the purpose of axiomatising intermediate logics the two formalisms are equally good. However, when considering properties of formal derivations the hypersequent formalism is arguably more natural. We will come back to multi-conclusion rules, in the form of stable universal clauses, in Section 3.

We would like to know which intermediate logics can be determined by structural intermediate hypersequent calculi as defined in Definition 2.2. That is, we would like to know which intermediate logics are of the form $\Lambda(\mathbf{HLJ}+\mathscr{R})$, for $\mathscr{R}$ a set of structural hypersequent rules. This is interesting to know since such intermediate logics can also be captured by a structural hypersequent calculus without using the cut-rule [22].[4] Thus answering this question will help us better understand—from a semantical point of view—which intermediate logics can be captured by cut-free proof calculi.

2.2 Structural hypersequent calculi and universal clauses

Let $\mathbb{HA}$ denote the variety of Heyting algebras. Then for any intermediate logic $L \supseteq \mathbf{IPC}$ there is a variety $\mathbb{V}(L) \subseteq \mathbb{HA}$ such that $\varphi \leftrightarrow \psi \in L$ if and only if $\mathbb{V}(L) \models \varphi \approx \psi$, for any pair of formulas φ and ψ in the language of propositional intuitionistic logic, see, e.g., [18, Thm. 7.73(iv)]. Thus any intermediate logic is sound and complete with respect to a variety, i.e., an equationally definable class, of Heyting algebras.

Importantly we also have an analogous algebraic completeness theorem for hypersequent calculi.

Theorem 2.7. *Let $\mathscr{R}$ be a set of hypersequent rules and let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. Then the following are equivalent:*

1. $\mathcal{H} \vdash_{\mathbf{HLJ}+\mathscr{R}} H$;
2. $\mathcal{H} \models_{\mathcal{K}(\mathscr{R})} H$,

where $\mathcal{K}(\mathscr{R})$ denotes the class of Heyting algebras validating all the rules belonging to $\mathscr{R}$.

Proof. This is nothing but a modified version of the Lindenbaum-Tarski construction[5], see, e.g., [41, 15, 16]. □

Thus in order to use the algebraic semantics to study structural hypersequent calculi we must identify the classes of Heyting algebras of the form $\mathcal{K}(\mathscr{R})$ for $\mathscr{R}$ a collection of structural hypersequent rules.

Recall, e.g., [17, Def. V.2.19] that a first-order formula (in a language without relational symbols) in prenex-normal form with all quantifiers universal is called a *universal formula* or *universal clause.* Thus, any universal clause may be written[6] as

$$\forall\vec{x}(t_1(\vec{x}) \approx u_1(\vec{x}) \text{ and } \ldots \text{ and } t_m(\vec{x}) \approx u_m(\vec{x}) \implies t_{m+1}(\vec{x}) \approx u_{m+1}(\vec{x}) \text{ or } \ldots \text{ or } t_n(\vec{x}) \approx u_n(\vec{x})),$$

for terms $t_k(\vec{x})$ and $u_k(\vec{x})$, $k \in \{1, \ldots, n\}$. In the presence of the lattice operation $\wedge$ we will use $t \leq u$ as an abbreviation of the equation $t \approx t \wedge u$. Finally, if it is clear from the context we will drop the universal quantifier, leaving it to be understood that the variables, which may or may not be displayed, are all universally quantified. A class of models of a collection of universal formulas is called a *universal class.*

The following two propositions show that the structural intermediate hypersequent calculi correspond to certain kinds of universal classes of Heyting algebras. For details, see, [25, Sec. 3.3, Sec. 4.4].

Proposition 2.8 (cf. [25, Sec. 3.3])**.** *For each structural hypersequent rule (r) there exists a universal clause q_r in the $\{0, \wedge, 1\}$-reduct of the language of Heyting algebras such that*

$$\mathbf{A} \models (r) \iff \mathbf{A} \models q_r,$$

for all Heyting algebras $\mathbf{A}$.

Proof. Given a structural rule (r), say

$$\frac{H \mid S_1 \qquad \ldots \qquad H \mid S_m}{H \mid S_{m+1} \mid \ldots \mid S_n}\,(r)$$

[4] In fact this holds for a much wider class of substructural logics.

[5] Note, however, that unlike the original Lindenbaum-Tarski construction, this construction does not produce free algebras for the universal class of Heyting algebras validating the corresponding rules.

[6] Following [23, 24, 25] we write and for conjunction and or for disjunction to avoid confusion with the lattice operations $\wedge$ and $\vee$. For arbitrary finite conjunctions and disjunctions we use AND and OR, respectively. Similarly, we write $\implies$ for classical implication to avoid confusion with the Heyting algebra operation $\to$.

we associate a unique first-order variable x to each multiset variable Γ occurring in (r) and similarly we associate a unique first-order variable y to each stoup variable Π occurring in (r). Then if S_i is $\Gamma_{i1}, \ldots, \Gamma_{ik} \Rightarrow \Pi_i$ we define t_i to be the term $x_{i1} \wedge \ldots \wedge x_{ik}$ and u_i to be the term y_i. If S_i is $\Gamma_{i1}, \ldots, \Gamma_{ik} \Rightarrow$ we define t_i to be the term $x_{i1} \wedge \ldots \wedge x_{ik}$ and u_i to be the constant 0. Letting q_r be the universal clause

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \ldots \text{ or } t_n \leq u_n,$$

it is then straightforward to verify that (r) and q_r are equivalent. □

Example 2.9. For example the structural hypersequent rules (*com*) and (*lq*) correspond to the universal clauses

$$x_1 \wedge x_2' \leq y_1 \text{ and } x_2 \wedge x_1' \leq y_2 \implies x_1 \wedge x_1' \leq y_1 \text{ or } x_2 \wedge x_2' \leq y_2$$

and

$$x_1 \wedge x_2 \leq 0 \implies x_1 \leq 0 \text{ or } x_2 \leq 0,$$

respectively.

A converse to Proposition 2.8 may be given.

Proposition 2.10 (cf. [25, Sec. 4.4]). *For any universal clause q in the $\{0, \wedge, 1\}$-reduct of the language of Heyting algebras there is a structural hypersequent rule (r_q) such that*

$$\mathbf{A} \models q \iff \mathbf{A} \models (r_q),$$

for any Heyting algebra $\mathbf{A}$.

Proof. Let q be a universal clause in the $\{0, \wedge, 1\}$-reduct of the language of Heyting algebras. Any such universal clause will be equivalent to a finite set of clauses of the form

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_m \leq u_m \implies t_{m+1} \leq u_{m+1} \text{ or } \ldots \text{ or } t_n \leq u_n,$$

where for each $i \in \{1, \ldots, n\}$ the term t_i is a meet of variables, say $x_{i1} \wedge \ldots \wedge x_{ik}$, or the constant 1 and the term u_i is either a variable, say y_i, or the constant 0. Thus without loss of generality we may assume that q is of this form. In fact we may assume that the variables occurring in the terms $\{t_i\}_{i=1}^n$ are disjoint from the variable occurring in the terms $\{u_i\}_{i=1}^n$, [25, Thm. 4.15]. For each variable x_i occurring in the terms $\{t_i\}_{i=1}^n$ we associate a multiset variable Γ_i and for each variable y_i occurring among the terms $\{u_i\}_{i=1}^n$ we associate a stoup variable Π_i. Finally, we let (r_q) be the rule

$$\frac{H \mid S_1 \qquad \ldots \qquad H \mid S_m}{H \mid S_{m+1} \mid \ldots \mid S_n} \, (r_q)$$

where S_i is $\Gamma_{i1}, \ldots, \Gamma_{ik} \Rightarrow \Pi_i$, with the left-hand (resp. right-hand) side empty if t_i (resp. u_i) is a constant. Again, it is easy to verify that (r_q) and (q) are indeed equivalent. □

Thus Propositions 2.8 and 2.10 above establish that for $\mathscr{R}$ a collection of structural hypersequent rules the class $\mathcal{K}(\mathscr{R})$ of Heyting algebras validating all the rules in $\mathscr{R}$ is a universal class of Heyting algebras determined by universal clauses in the $\{0, \wedge, 1\}$-reduct of the language of Heyting algebras and in fact any such class arises in this way.

This together with Theorem 2.13 below allows us to turn the proof-theoretic question of which intermediate logics admit a cut-free structural intermediate hypersequent calculus into a purely model-theoretic question regarding the first-order theory of Heyting algebras.

Given the correspondence between structural intermediate hypersequent calculi and universal clauses in the $\{0, \wedge, 1\}$-reduct of the language of Heyting algebras we may provide the first algebraic characterisation of the class of intermediate logics admitting structural intermediate hypersequent calculi. This characterisation is, however, not very informative and in the following section we shall provide a characterisation which we believe to be more enlightening.

Corollary 2.11. *Let L be an intermediate logic. Then the following are equivalent:*

1. *The logic L admits a structural intermediate hypersequent calculus;*

2. *The variety $\mathbb{V}(L)$ is generated by a universal class of Heyting algebras axiomatised by universal clauses in the $\{0, \wedge, 1\}$-reduct of the language of Heyting algebras.*

Proof. Suppose that L admits a structural intermediate hypersequent calculus, say $\mathbf{HLJ} + \mathscr{R}$ with $\mathscr{R}$ a set of structural hypersequent rules. Then we have that φ is a theorem of L iff the sequent $\Rightarrow \varphi$ is derivable in the hypersequent calculus $\mathbf{HLJ} + \mathscr{R}$. By Theorem 2.7 this is the case iff $\models_{\mathcal{K}(\mathscr{R})} \Rightarrow \varphi$, which in turn happens precisely when the equation $1 \approx \varphi$ is valid on every algebra in the class $\mathcal{K}(\mathscr{R})$. From this we may deduce that the variety $\mathbb{V}(L)$ is indeed generated by the class $\mathcal{K}(\mathscr{R})$ which by Proposition 2.8 is a universal class of Heyting algebras axiomatised by universal clauses in the $\{0, \wedge, 1\}$-reduct as $\mathscr{R}$ is a collection of structural hypersequent rules.

Conversely, if the variety $\mathbb{V}(L)$ is generated by a universal class of Heyting algebras axiomtised by universal clauses in the $\{0, \wedge, 1\}$-reduct, say $\mathcal{U}$, then by Proposition 2.10 there exists a set $\mathscr{R}_{\mathcal{U}}$ of structural hypersequent rules such that the class $\mathcal{K}(\mathscr{R}_{\mathcal{U}})$ of Heyting algebras validating $\mathscr{R}_{\mathcal{U}}$ coincides with the class $\mathcal{U}$. Since by assumption $\mathcal{U}$ generates $\mathbb{V}(L)$ we have that φ is a theorem of L iff $\models_{\mathcal{U}} 1 \approx \varphi$. Therefore, we have that $\varphi \in L$ precisely when $\models_{\mathcal{K}(\mathscr{R}_{\mathcal{U}})} 1 \approx \varphi$, which by Theorem 2.7 is the case exactly when $\vdash_{\mathbf{HLJ}+\mathscr{R}_{\mathcal{U}}} \Rightarrow \varphi$. Thus we may conclude that $\mathbf{HLJ} + \mathscr{R}_{\mathcal{U}}$ is a structural hypersequent calculus for L. $\square$

We finish this section with presenting a syntactic characterisation of the intermediate logics admitting a structural intermediate hypersequent calculus [22].

Definition 2.12 (cf. [26])**.** Let $\mathcal{P}_0 = \mathcal{N}_0$ be a (countable) set of propositional variables and define sets of formulas $\mathcal{P}_n, \mathcal{N}_n$ in the language of intuitionistic logic by the following grammar

$$\begin{aligned} \mathcal{P}_{n+1} &::= \top \mid \bot \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \\ \mathcal{N}_{n+1} &::= \top \mid \bot \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \end{aligned}$$

The key insight is that using the invertible rules of **HLJ**, i.e., rules the premises of which are derivable whenever the conclusion is, any $\mathcal{P}_3$-formula can be transformed into a structural hypersequent rule which preserves the redundancy of the cut-rule when added to **HLJ**. Thus any intermediate logic axiomatisable by $\mathcal{P}_3$-formulas, i.e., any logic of the form $\mathbf{IPC} + \{\varphi_i\}_{i \in I}$ with $\varphi_i \in \mathcal{P}_3$ for all $i \in I$, admits a structural hypersequent calculus in which the cut-rule is redundant.

Theorem 2.13 ([22])**.** *Let L be an intermediate logic. Then the following are equivalent:*

1. *The logic L admits a structural intermediate hypersequent calculus;*
2. *The logic L admits a cut-free structural intermediate hypersequent calculus;*
3. *The logic L is axiomatisable by $\mathcal{P}_3$-formulas.*

Proof. That items (1) and (2) are equivalent is established in [22], just as the fact that item (3) entails item (1). That item (1) entails item (3) may be seen via an argument analogous to the one used to prove [24, Prop. 7.5]. We supply the details. Given a structural hypersequent rule (r), as in the proof of Proposition 2.8, there is a finite set of equivalent universal clauses

$$t_1(\vec{x}) \leq u_1(\vec{y}) \text{ and } \dots \text{ and } t_m(\vec{x}) \leq u_m(\vec{y}) \implies t_{m+1}(\vec{x}) \leq u_{m+1}(\vec{y}) \text{ or } \dots \text{ or } t_n(\vec{x}) \leq u_n(\vec{y}), \qquad (q_r)$$

such that the variables $\vec{x}$ and $\vec{y}$ are disjoint and the terms t are (possible empty) meets of variables (i.e., the constant 1) and the terms u are either a single variable or the constant 0. In fact, we may assume that none of the terms t_i are the constant 1 for $i \in \{1, \dots, m\}$. Let φ_r be the formula

$$\bigvee_{j=m+1}^{n} \left(\left(\bigwedge_{i=1}^{m} (t_i(\vec{x}) \to u_i(\vec{y})) \right) \to (t_j(\vec{x}) \to u_j(\vec{y})) \right).$$

It is straightforward to verify that φ_r belongs to $\mathcal{P}_3$.

We claim that q_r and φ_r are equivalent on any Heyting algebra with a second greatest element. Therefore, suppose that $\mathbf{A}$ is such a Heyting algebra. If $\mathbf{A} \models \varphi_r$ then for any valuation ν on $\mathbf{A}$ we have that $\mathbf{A}, \nu \models (\bigwedge_{i=1}^{m} (t_i(\vec{x}) \to u_i(\vec{y}))) \to (t_{j_0}(\vec{x}) \to u_{j_0}(\vec{y}))$ for some $j_0 \in \{m+1, \dots, n\}$, since $\mathbf{A}$ has a second greatest element. Consequently, if $\mathbf{A}, \nu \models t_i \leq u_i$ for all $i \in \{1, \dots, m\}$ then we must have that

$\mathbf{A}, \nu \models \bigwedge_{i=1}^{m}(t_i(\vec{x}) \to u_i(\vec{y}))$ and so $\mathbf{A}, \nu \models t_{j_0}(\vec{x}) \to u_{j_0}(\vec{y})$, i.e., $\mathbf{A}, \nu \models t_{j_0}(\vec{x}) \leq u_{j_0}(\vec{y})$, showing that $\mathbf{A}, \nu \models q_r$.

Conversely, if $\mathbf{A} \models q_r$ then $\mathbf{A}$ also validates any substitution instance of q_r. We construct a substitution σ such that $\mathbf{A} \models \sigma(t_i) \leq \sigma(u_i)$ for all $i \in \{1, \ldots, m\}$ and such that $\sigma(t_j) \geq t_j \wedge \bigwedge_{i=1}^{m}(t_i \to u_i)$ and $\sigma(u_j) = u_j$, for all $j \in \{m+1, \ldots, n\}$. From the existence of such a substitution we may conclude that for any valuation ν on $\mathbf{A}$ there exists $j_0 \in \{m+1, \ldots, n\}$ such that $\mathbf{A}, \nu \models t_{j_0} \wedge \bigwedge_{i=1}^{m}(t_i \to u_i) \leq u_{j_0}$, and hence that $\mathbf{A} \models \bigwedge_{i=1}^{m}(t_i \to u_i) \leq t_{j_0} \to u_{j_0}$, showing that $\mathbf{A}, \nu \models \varphi_r$.

To construct the substitution σ we let $\sigma(x) \coloneqq x \wedge \bigwedge_{i=1}^{m}(t_i \to u_i)$ for each variable $x \in \vec{x}$ and $\sigma(y) \coloneqq y$, for each variable $y \in \vec{y}$. Note that since $\vec{x}$ and $\vec{y}$ are disjoint this is well-defined. It is now easy to verify that this substitution has the desired properties and therefore that $\mathbf{A} \models \varphi_r$. This shows that in fact any Heyting algebra validating the clause q_r also validates the formula φ_r.

Thus, being equivalent on Heyting algebras with a second greatest element, viz., subdirectly irreducible Heyting algebras [5, Thm. IX.4.5], it follows that (r) and φ_r determine the same intermediate logic. □

Remark 2.14. Note that the proof of Theorem 2.13 presented in [22] is semantic in nature and so does not directly yield an explicit procedure for transforming a derivation using the cut-rule into a cut-free derivation. However, in concrete cases an explicit cut-elimination procedure may be given, see, e.g., [20, 21]. We also want to emphasise that it is not the case that the cut-rule is redundant in every structural intermediate hypersequent calculus but only that any such calculus is effectively equivalent a structural intermediate hypersequent calculus in which the cut-rule is redundant.

Example 2.15. For $n \geq 1$ let $\mathbf{BTW}_n, \mathbf{BW}_n$ and $\mathbf{BC}_n$, be the intermediate logics determined by intuitionistic Kripke frames of top width at most n, of width at most n, and of cardinality at most n, respectively. All of these intermediate logics have axiomatisations given by formulas which are ostensibly $\mathcal{P}_3$, see, e.g., [18, Chap. 2], and so by Theorem 2.13 all of these logics admit a hypersequent calculus of the form $\mathbf{HLJ} + \mathscr{R}$, with $\mathscr{R}$ a set of structural hypersequent rules, for which the cut-rule is redundant. Concretely, the rules, (com) and (lq) yields cut-free structural intermediate hypersequent calculi for the intermediate logics $\mathbf{LC}$ and $\mathbf{KC}$, respectively, when added to $\mathbf{HLJ}$.

For more examples of structural hypersequent rules see [22] and [20].

Theorem 2.13 thus gives a very nice syntactic description of the class of intermediate logics which admit cut-free structural intermediate hypersequent calculi. Our aim is then to supply criteria describing this class of intermediate logics in terms of the algebraic and the relational semantics for intermediate logics. Among other things this will allow us to derive negative results showing that certain well-known intermediate logics do not admit such calculi.

3 Algebraic characterisation

In this section we provide a semantic characterisation of the intermediate logics admitting a structural—and therefore also a cut-free—intermediate hypersequent calculus in terms of the algebraic semantics. This section builds on the theory of $(0, \wedge, \vee, 1)$-stable intermediate logics as developed in [7, 10] where these logics are simply called stable intermediate logics.

Notation 3.1. Given $\Sigma \subseteq \{0, \wedge, \vee, \to, 1\}$ we will let Σ_c denote the set $\Sigma \cap \{0, 1\}$ and Σ_o denote the set $\Sigma \cap \{\wedge, \vee, \to\}$. Moreover, if $\mathcal{K}$ is a class of Heyting algebras we let $\mathcal{K}_{si}$ denote the class of subdirectly irreducible Heyting algebras belonging to $\mathcal{K}$.

Definition 3.2. Let $\Sigma \subseteq \{0, \wedge, \vee, \to, 1\}$ and let $\mathbf{A}$ and $\mathbf{B}$ be Heyting algebras. We say that a function $h\colon A \to B$ is a Σ-*homomorphism* if h commutes with the operations in Σ. If $h\colon A \to B$ is a Σ-homomorphism we write $\mathbf{A} \to_\Sigma \mathbf{B}$. A Σ-homomorphism $h\colon \mathbf{A} \to_\Sigma \mathbf{B}$ is called a Σ-*embedding* if the the function $h\colon A \to B$ is injective. In this case we write $\mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$, and say that the algebra $\mathbf{A}$ is a Σ-*subalgebra* of the algebra $\mathbf{B}$.

Definition 3.3. Let $\Sigma \subseteq \{0, \wedge, \vee, \to, 1\}$.

1. We say that a class $\mathcal{K}$ of Heyting algebras is *(finitely)* Σ-*stable* provided that whenever $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$ then $\mathbf{A} \in \mathcal{K}$ for all (finite) Heyting algebras $\mathbf{A}$;

2. We say that an intermediate logic L is *(finitely)* Σ*-stable* provided that whenever $\mathbf{B} \in \mathbb{V}(L)_{si}$ and $\mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$ then $\mathbf{A} \in \mathbb{V}(L)$ for all (finite) Heyting algebras $\mathbf{A}$.

Remark 3.4. As will become evident below, in item (2) of Definition 3.3 we could just as well have chosen the larger class $\mathbb{V}(L)_{wc}$ of well-connected $\mathbb{V}(L)$-algebras, viz., Heyting algebras with a join-irreducible top element, instead of the class $\mathbb{V}(L)_{si}$ of subdirectly irreducible $\mathbb{V}(L)$-algebras. However, we have chosen the definition which aligns best with [7, 10].

Remark 3.5. Note that if $\Sigma \subseteq \Sigma' \subseteq \{0, \wedge, \vee, \to, 1\}$ then any Σ-stable intermediate logic L must necessarily also be Σ'-stable. In particular, for $\Sigma \subseteq \{0, \wedge, \vee, 1\}$, any Σ-stable logic will be $(0, \wedge, \vee, 1)$-stable and so by [7, Thm. 6.8] must enjoy the finite model property. Furthermore, by similar reasoning any $(0, \wedge, 1)$-stable intermediate logic will also be a cofinal stable logic [10].

Finally, it follows from the characterisation of $(0, \wedge, 1)$-stable logics given in Section 4 that any $(0, \wedge, 1)$-stable logic will be characterised by a class of intuitionistic Kripke frames closed under taking (locally) cofinal subframes. Thus any $(0, \wedge, 1)$-stable logic must also be a cofinal subframe logic [52, 53] and as such these logics will be both canonical and elementary [53, Thm. 6.8].

Definition 3.6. Let $\Sigma \subseteq \{0, \wedge, \vee, \to, 1\}$, let $\mathbf{A}$ be a finite Heyting algebra and introduce for each element $a \in A$ a distinct first-order variable x_a. By the *Σ-stable (universal) clause* $q_\Sigma(\mathbf{A})$ *associated with* $\mathbf{A}$ we shall understand the universal clause $\forall \vec{x}\ (P(\vec{x}) \implies C(\vec{x}))$ where

$$P(\vec{x}) = \mathsf{AND}\{x_a \approx a \colon a \in \Sigma_c\} \text{ and } \mathsf{AND}\{x_a \bullet x_{a'} \approx x_{a \bullet a'} \colon a, a' \in A, \bullet \in \Sigma_o\}$$
$$C(\vec{x}) = \mathsf{OR}\{x_a \approx x_{a'} \colon a, a' \in A, a \neq a'\}.$$

Remark 3.7. Stable universal clauses may be seen as a propositional version of diagrams as known from classic Robinson-style model theory, see, e.g., [36, Chap. 1.4]. Variants of the Σ-stable clauses defined above have been studied before under the names stable and canonical multi-conclusion rules [41, 8, 10, 9].

The following lemma shows that the Σ-stable clause associated with a finite algebra $\mathbf{A}$ encodes the property of not containing $\mathbf{A}$ as a Σ-subalgebra.

Lemma 3.8 (cf. [10, Prop. 4.2])**.** *Let* $\Sigma \subseteq \{0, \wedge, \vee, \to, 1\}$ *and let* $\mathbf{A}, \mathbf{B}$ *be Heyting algebras with* $\mathbf{A}$ *finite. Then the following are equivalent:*

1. $\mathbf{B} \not\models q_\Sigma(\mathbf{A})$*;*
2. *There exists a* Σ*-embedding* $h \colon \mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$.

Proof. Given a valuation ν on $\mathbf{B}$ such that $(\mathbf{B}, \nu) \not\models q_\Sigma(\mathbf{A})$ then we obtain a Σ-embedding $h_\nu \colon \mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$ by letting $h_\nu(a) := \nu(x_a)$. Conversely, given a Σ-embedding $h \colon \mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$ we obtain a valuation ν_h on $\mathbf{B}$ such that $(\mathbf{B}, \nu_h) \not\models q_\Sigma(\mathbf{A})$ by letting $\nu_h(x_a) := h(a)$. ☐

We then show that a universal class of Heyting algebras is Σ-stable precisely if it is axiomatisable by Σ-stable clauses.

Lemma 3.9 (cf. [10, Prop. 4.5])**.** *Let* $\Sigma \subseteq \{0, \wedge, \vee, 1\}$ *be given and let* $\mathcal{U}$ *be a universal class of Heyting algebras. Then the following are equivalent:*

1. *The universal class* $\mathcal{U}$ *is* Σ*-stable;*
2. *The universal class* $\mathcal{U}$ *is finitely* Σ*-stable;*
3. *The universal class* $\mathcal{U}$ *is axiomatised by* Σ*-stable clauses.*

Proof. If $\mathcal{U}$ is a (universal) class axiomatised by Σ-stable clauses then $\mathcal{U}$ must be Σ-stable, since universal clauses in the Σ-reduct of the language of Heyting algebras are preserved by Σ-subalgebras. Moreover, any Σ-stable universal class is evidently finitely Σ-stable.

Thus it remains to be shown that if $\mathcal{U}$ is finitely Σ-stable then $\mathcal{U}$ is axiomatised by Σ-stable clauses. Therefore, let $\mathcal{U}$ be a finitely Σ-stable universal class and let $\mathcal{Q} = \{q_\Sigma(\mathbf{A}) \colon |\mathbf{A}| < \aleph_0, \mathbf{A} \notin \mathcal{U}\}$. We claim that for any Heyting algebra $\mathbf{B}$ we have that $\mathbf{B} \in \mathcal{U}$ iff $\mathbf{B} \models \mathcal{Q}$. To see this let $\mathrm{Th}^{\forall}_{\mathsf{HA}}(\mathcal{U})$ be the universal theory, in the language of Heyting algebras, of $\mathcal{U}$. If $\mathbf{B} \notin \mathcal{U}$ then, by the assumption that $\mathcal{U}$ is a universal class of Heyting algebras, there exist a universal clause $q \in \mathrm{Th}^{\forall}_{\mathsf{HA}}(\mathcal{U})$ such that $\mathbf{B} \not\models q$. Thus, by [10, Lem. 4.3] we must have a finite $(0, \wedge, \vee, 1)$-subalgebra, in particular a Σ-subalgebra, $\mathbf{C}$ of $\mathbf{B}$ such that $\mathbf{C} \not\models q$, i.e., $\mathbf{C} \notin \mathcal{U}$ whence $q_\Sigma(\mathbf{C}) \in \mathcal{Q}$. By Lemma 3.8 we must have that $\mathbf{B} \not\models q_\Sigma(\mathbf{C})$ and so $\mathbf{B} \not\models \mathcal{Q}$.

Conversely, if $\mathbf{B} \not\models \mathcal{Q}$ then for some finite Heyting algebra $\mathbf{A} \notin \mathcal{U}$ we have $\mathbf{B} \not\models q_\Sigma(\mathbf{A})$. By Lemma 3.8 it follows that $\mathbf{A}$ is a Σ-subalgebra of $\mathbf{B}$. Since $\mathcal{U}$ is assumed to be finitely Σ-stable we must conclude that $\mathbf{B} \notin \mathcal{U}$ since otherwise $\mathbf{A} \in \mathcal{U}$. □

We then obtain the first necessary and sufficient conditions in terms of $\mathbb{V}(L)$ for an intermediate logic L to admit a structural intermediate hypersequent calculus.

Proposition 3.10. *Let L be an intermediate logic. Then the following are equivalent:*

1. *The logic L admits a cut-free structural intermediate hypersequent calculus;*
2. *The logic L admits a structural intermediate hypersequent calculus;*
3. *The variety $\mathbb{V}(L)$ is generated by a $(0, \wedge, 1)$-stable universal class of Heyting algebras.*

Proof. The equivalence between items (1) and (2) is contained in Theorem 2.13. That items (2) and (3) are equivalent follows from Corollary 2.11 and Lemma 3.9. □

In principle Proposition 3.10 gives an algebraic characterisation of the intermediate logics L which admit structural intermediate hypersequent calculi in the sense of Definition 2.2. However, we wish to obtain a characterisation which is local in the sense that it pertains to properties of—individual—algebras in the variety $\mathbb{V}(L)$ and not the variety $\mathbb{V}(L)$ taken as a whole. We will obtain such a characterisation by showing that the varieties of Heyting algebras generated by $(0, \wedge, 1)$-stable universal classes are precisely the varieties corresponding to $(0, \wedge, 1)$-stable logics.

3.1 $(0, \wedge, 1)$-stable logics

In this subsection we characterise the $(0, \wedge, 1)$-stable logics in terms of properties of the subdirectly-irreducible algebras in the corresponding variety $\mathbb{V}(L)$. As discussed above this will yield a characterisation of the intermediate logics admitting cut-free structural intermediate hypersequent calculi.

Definition 3.11. Let $\Sigma \subseteq \{0, \wedge, \vee, \to, 1\}$, let $\mathbf{A}$ be a finite Heyting algebra and introduce for each element $a \in A$ a distinct variable x_a. By the *Σ-stable equation $\varepsilon_\Sigma(\mathbf{A})$ associated with* $\mathbf{A}$ we shall understand the equation $1 \approx \bigwedge \Gamma \to \bigvee \Delta$ where

$$\Gamma = \{x_a \leftrightarrow a \colon a \in \Sigma_c\} \cup \{x_a \bullet x_{a'} \leftrightarrow x_{a \bullet a'} \colon a, a' \in A, \bullet \in \Sigma_o\}$$
$$\Delta = \{x_a \to x_{a'} \colon a, a' \in A, a \not\leq a'\}.$$

The Σ-stable equations encode information about finite Heyting algebras in almost the same way as the Σ-stable clauses. However, a version of Lemma 3.8 only obtains for so-called *well-connected* Heyting algebras, that is, Heyting algebras validating the universal clause $\forall x \forall y (1 \leq x \vee y \implies 1 \leq x \text{ or } 1 \leq y)$. Note that every subdirectly irreducible Heyting algebra will be well-connected and that every finite well-connected Heyting algebra will be subdirectly irreducible, see, e.g., [12, Thm. 2.3.14] and the references therein.

We will need the following lemma showing that homomorphic images of a finite Heyting algebra $\mathbf{A}$ must also be $(0, \wedge, 1)$-subalgebras of $\mathbf{A}$.

Lemma 3.12. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite Heyting algebras. If $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, then $\mathbf{B}$ is a $(0, \wedge, 1)$-subalgebra of $\mathbf{A}$.*

Proof. If $h \colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ is a surjective Heyting algebra homomorphism then $\mathbf{B}$ is isomorphic to $\mathbf{A}/F$, as a Heyting algebra, for some filter F on $\mathbf{A}$. As $\mathbf{A}$ is finite the filter F must be a principal filter, say $F = {\uparrow} a$ for some $a \in A$, and therefore $\mathbf{B} \cong [0, a]$. Evidently, we have a $(0, \wedge, 1)$-embedding f from $[0, a]$ into $\mathbf{A}$, given by

$$f(x) = \begin{cases} x & x < a, \\ 1 & x = a, \end{cases}$$

showing that $\mathbf{B} \hookrightarrow_{0,\wedge,1} \mathbf{A}$. □

We may then establish a version of Lemma 3.8 for $(0,\wedge,1)$-stable equations.

Lemma 3.13 (cf. [7, Thm. 6.3])**.** *Let* $\mathbf{A}$, $\mathbf{B}$ *be Heyting algebras with* $\mathbf{A}$ *finite.*

1. *If* $\mathbf{B} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A})$ *then* $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$*;*
2. *If* $\mathbf{B}$ *is well-connected and* $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ *then* $\mathbf{B} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A})$.

Proof. If $\mathbf{B} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A})$ then by [7, Lem. 3.6] we must have a finite Heyting algebra $\mathbf{C}$ which is a $(0,\wedge,\vee,1)$-subalgebra of $\mathbf{B}$, and so in particular a $(0,\wedge,1)$-subalgebra of $\mathbf{B}$, such that $\mathbf{C} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A})$. This means that there is a valuation ν on $\mathbf{C}$ such that $\nu(\bigwedge\Gamma \to \bigvee\Delta) < 1$, where Γ and Δ are as in Definition 3.11. By Wronski's Lemma [51, Lem. 1] there exists a subdirectly irreducible Heyting algebra $\mathbf{D}$ together with a Heyting algebra homomorphism $\pi\colon \mathbf{C} \twoheadrightarrow \mathbf{D}$ such that $\pi(\nu(\bigwedge\Gamma \to \bigvee\Delta)) = c_{\mathbf{D}}$, where $c_{\mathbf{D}}$ denotes the unique co-atom of $\mathbf{D}$. By Lemma 3.12 we have that $\mathbf{D}$ is a $(0,\wedge,1)$-subalgebra of $\mathbf{C}$ and therefore also a $(0,\wedge,1)$-subalgebra of $\mathbf{B}$. We claim that $\mathbf{A}$ is a $(0,\wedge,1)$-subalgebra of $\mathbf{D}$ and therefore also a $(0,\wedge,1)$-subalgebra of $\mathbf{B}$. We obtain a valuation μ on $\mathbf{D}$ such that $\mu(\bigwedge\Gamma \to \bigvee\Delta) = c_{\mathbf{D}}$ by letting $\mu(x_a) = \pi(\nu(x_a))$. From this it follows that $\mu(\bigwedge\Gamma) = 1$ and $\mu(\bigvee\Delta) = c_{\mathbf{D}}$ and hence we may conclude that $h_\mu\colon \mathbf{A} \to \mathbf{D}$ given by $h_\mu(a) = \mu(x_a)$ is an $(0,\wedge,1)$-embedding of $\mathbf{A}$ into $\mathbf{D}$.

Conversely, if there is a $(0,\wedge,1)$-embedding $h\colon \mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ then defining a valuation ν_h on $\mathbf{B}$ by $\nu_h(x_a) = h(a)$ we obtain that $\nu_h(\bigwedge\Gamma) = 1$ by the fact that h is a $(0,\wedge,1)$-homomorphism. Moreover, by the fact that h is also a $(0,\wedge,1)$-embedding we must have that $1 \not\leq \nu_h(x_a \to x_{a'})$ for all $x_a \to x_{a'}$ in Δ. Thus, assuming $\mathbf{B}$ to be well-connected we may conclude that $1 \not\leq \nu_h(\bigvee\Delta)$ and therefore that $1 \not\leq \nu(\bigwedge\Gamma \to \bigvee\Delta)$. Thus, ν_h witnesses that $\mathbf{B} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A})$. □

The following lemma shows that the varieties of Heyting algebras generated by $(0,\wedge,1)$-stable universal classes are in fact axiomatised by $(0,\wedge,1)$-stable equations. Thus a variety can be axiomatised by $(0,\wedge,1)$-stable equations precisely when it can be axiomatised by $(0,\wedge,1)$-stable universal clauses.

Lemma 3.14. *Let* $\mathcal{U}$ *be a universal class axiomatised by a collection* $\{q_{0,\wedge,1}(\mathbf{A}_i)\}_{i\in I}$ *of* $(0,\wedge,1)$*-stable universal clauses. Then the variety generated by the class* $\mathcal{U}$ *is axiomatised by the* $(0,\wedge,1)$*-stable equations* $\{\varepsilon_{0,\wedge,1}(\mathbf{A}_i)\}_{i\in I}$.

Proof. Let $\mathbb{V}$ be the variety determined by the $(0,\wedge,1)$-stable equations $\{\varepsilon_{0,\wedge,1}(\mathbf{A}_i)\}_{i\in I}$. Lemma 3.8 together with item (1) of Lemma 3.13 implies that $\mathcal{U} \subseteq \mathbb{V}$. Furthermore, from item (2) of Lemma 3.13 we may conclude that any subdirectly irreducible $\mathbb{V}$-algebra belongs to $\mathcal{U}$. Consequently, being a subclass of the variety $\mathbb{V}$ containing all subdirectly irreducible $\mathbb{V}$-algebras, the class $\mathcal{U}$ must necessarily generate the variety $\mathbb{V}$. □

Lemma 3.15. *Let* $\Sigma \subseteq \{0,\wedge,\vee,1\}$ *be given. If* $\mathcal{K}$ *is a* Σ*-stable class then so is the universal class generated by* $\mathcal{K}$.

Proof. By [17, Thm. V.2.20] we know that the universal class generated by the class $\mathcal{K}$ is given by $\mathbf{ISP}_U(\mathcal{K})$. Therefore, let $\{\mathbf{B}_i\}_{i\in I}$ be a collection of $\mathcal{K}$-algebras, U an ultrafilter on I and $\mathbf{A}$ a finite Heyting algebra. If $\mathbf{A} \not\hookrightarrow_\Sigma \mathbf{B}_i$ for all $i \in I$ then by Lemma 3.8 we have that $\mathbf{B}_i \models q_\Sigma(\mathbf{A})$ for all $i \in I$ and hence by Łos' Theorem we obtain that $\prod_{i\in I} \mathbf{B}_i/U \models q_\Sigma(\mathbf{A})$ and so $\mathbf{A} \not\hookrightarrow_\Sigma \prod_{i\in I} \mathbf{B}_i/U$. Consequently, if $\mathbf{A} \hookrightarrow_\Sigma \prod_{i\in I} \mathbf{B}_i/U$ then, again by Lemma 3.8, $\mathbf{A} \hookrightarrow_\Sigma \mathbf{B}_i$ for some $i \in I$. Moreover, if $\mathbf{B} \in \mathbf{ISP}_U(\mathcal{K})$ and $\mathbf{A}$ is a finite algebra such that $\mathbf{A} \hookrightarrow_\Sigma \mathbf{B}$ then necessarily $\mathbf{A} \hookrightarrow_\Sigma \mathbf{B}'$ for some $\mathbf{B}' \in \mathbf{P}_U(\mathcal{K})$ whence by the above we have that $\mathbf{A} \in \mathcal{K}$. We have thus shown that $\mathbf{ISP}_U(\mathcal{K})$ is a finitely Σ-stable universal class and as such it must be Σ-stable by Lemma 3.9. □

Theorem 3.16. *Let* L *be an intermediate logic. Then the following are equivalent:*

1. *The logic* L *is* $(0,\wedge,1)$*-stable;*
2. *The variety* $\mathbb{V}(L)$ *is generated by a* $(0,\wedge,1)$*-stable class of finite Heyting algebras;*
3. *The variety* $\mathbb{V}(L)$ *is generated by a* $(0,\wedge,1)$*-stable universal class of Heyting algebras.*

Proof. It follows from Lemma 3.15 that item (2) entails item (3). Furthermore, it follows from [7, Lem. 3.6] that if $\mathbb{V}(L)$ is generated by a $(0, \wedge, 1)$-stable universal class of Heyting algebras, say $\mathcal{U}$, then $\mathbb{V}(L)$ is also generated by the $(0, \wedge, 1)$-stable class of the finite Heyting algebras belonging to $\mathcal{U}$.

We proceed to show that the items (1) and (3) are equivalent.

For that purpose, assume that the logic L is $(0, \wedge, 1)$-stable and define

$$\mathcal{K}' = \{\mathbf{A} \in \mathbb{HA} \colon \exists \mathbf{B} \in \mathbb{V}(L)_{si}(\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B})\},$$

and let $\mathcal{K}$ be the collection of finite Heyting algebras not belonging to $\mathcal{K}'$. Then for any subdirectly irreducible Heyting algebra $\mathbf{B}$ we may observe the following: If $\mathbf{B} \in \mathbb{V}(L)$ and $\mathbf{A} \in \mathcal{K}$ then we must have that $\mathbf{B} \models \varepsilon_{0,\wedge,1}(\mathbf{A})$, since otherwise $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ by Lemma 3.13 entailing that $\mathbf{A} \in \mathcal{K}'$, in direct contradiction with the assumption that $\mathbf{A} \in \mathcal{K}$. Conversely, if $\mathbf{B} \notin \mathbb{V}(L)$ then by [7, Lem. 3.6] we have a finite $(0, \wedge, \vee, 1)$-subalgebra $\mathbf{A}$ of $\mathbf{B}$ such that $\mathbf{A} \notin \mathbb{V}(L)$. In particular $\mathbf{A}$ is a $(0, \wedge, 1)$-subalgebra of $\mathbf{B}$ and hence by Lemma 3.13 $\mathbf{B} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A})$, as $\mathbf{B}$, being subdirectly irreducible, is well-connected. Moreover, we must have that $\mathbf{A} \in \mathcal{K}$ since otherwise, as $\mathbf{A}$ is finite, we would have $\mathbf{A} \in \mathcal{K}'$ and hence $\mathbf{A} \in \mathbb{V}(L)$ by the assumption that L is $(0, \wedge, 1)$-stable. We have thus shown that for any subdirectly irreducible Heyting algebra $\mathbf{B}$ we have

$$\mathbf{B} \in \mathbb{V}(L) \quad \Longleftrightarrow \quad \mathbf{B} \models \{\varepsilon_{0,\wedge,1}(\mathbf{A}) \colon \mathbf{A} \in \mathcal{K}\}.$$

Consequently, the variety $\mathbb{V}(L)$ is axiomatised by the collection of $(0, \wedge, 1)$-stable equations $\{\varepsilon_{0,\wedge,1}(\mathbf{A})\}_{\mathbf{A} \in \mathcal{K}}$. From Lemma 3.14 we may then conclude that $\mathbb{V}(L)$ is identical to the variety generated by the universal class of Heyting algebras determined by the collection of universal $(0, \wedge, 1)$-stable clauses $\{q_{0,\wedge,1}(\mathbf{A})\}_{\mathbf{A} \in \mathcal{K}}$.

Lastly, assume that $\mathbb{V}(L)$ is generated by a $(0, \wedge, 1)$-stable universal class say $\mathcal{U}$. By Lemma 3.9 we have that $\mathcal{U}$ is determined by a collection of $(0, \wedge, 1)$-stable universal clauses, say $\{q_{0,\wedge,1}(\mathbf{A}_i)\}_{i \in I}$. By Lemma 3.14 it then follows that $\mathbb{V}(L)$ is determined by the collection of $(0, \wedge, 1)$-stable equations $\{\varepsilon_{0,\wedge,1}(\mathbf{A}_i)\}_{i \in I}$. Consequently, if $\mathbf{B}$ is a subdirectly irreducible $\mathbb{V}(L)$-algebra and $\mathbf{A}$ is a $(0, \wedge, 1)$-subalgebra of $\mathbf{B}$, then from $\mathbf{A} \notin \mathbb{V}(L)$ we can conclude that $\mathbf{A} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A}_i)$ for some $i \in I$ and so by item (1) of Lemma 3.13 it follows that $\mathbf{A}_i \hookrightarrow_{0,\wedge,1} \mathbf{A}$ and hence that $\mathbf{A}_i \hookrightarrow_{0,\wedge,1} \mathbf{B}$ whence from item (2) of Lemma 3.13 we obtain that $\mathbf{B} \not\models \varepsilon_{0,\wedge,1}(\mathbf{A}_i)$ in direct contradiction with the assumption that $\mathbf{B}$ is a $\mathbb{V}(L)$-algebra. □

We may then obtain the following algebraic characterisation of the intermediate logics admitting a structural intermediate hypersequent calculus and therefore by Theorem 2.13 also a cut-free structural intermediate hypersequent calculus.

Corollary 3.17. *Let L be an intermediate logic. Then the following are equivalent:*

1. *The logic L admits a structural intermediate hypersequent calculus;*
2. *The logic L is $(0, \wedge, 1)$-stable.*

Proof. This follows directly from Theorem 3.16 and Proposition 3.10. □

Remark 3.18. Note that if L is a finitely axiomatisable $(0, \wedge, 1)$-stable intermediate logic then L admits a structural intermediate hypersequent calculus given by only finitely many structural hypersequent rules. To see this simply note that being $(0, \wedge, 1)$-stable $\mathbb{V}(L)$ is axiomatised by a collection of $(0, \wedge, 1)$-equations and and since L is finitely axiomatisable we may conclude that only finitely many of the $(0, \wedge, 1)$-stable equations are needed to axiomatise $\mathbb{V}(L)$. Hence by Lemma 3.14 $\mathbb{V}(L)$ is determined by a finite number of $(0, \wedge, 1)$-stable clauses. Thus from the correspondence between $(0, \wedge, 1)$-stable clauses and structural hypersequent rules we obtain that L indeed admits a structural intermediate hypersequent calculus given by only finitely many structural hypersequent rules.

Remark 3.19. We observe that by Theorem 3.16 we have that in order to check whether or not an intermediate logic is $(0, \wedge, 1)$-stable it suffices to consider the collection of finite subdirectly irreducible $\mathbb{V}(L)$-algebras. In particular, it is possible to use duality to translate the questions of whether or not an intermediate logic L is $(0, \wedge, 1)$-stable into a question about the finite rooted intuitionistic Kripke frames for L, see Section 4. Compare this with the necessary condition for admitting a cut-free structural hypersequent calculus [25, Thm. 6.8] which requires checking closure under certain type of completions of algebras. Of course [25, Thm. 6.8] applies in a much more general setting than Corollary 3.17.

3.2 Applications

We here present some consequences of Corollary 3.17.

Proposition 3.20. *Any intermediate logic admitting a structural intermediate hypersequent calculus has the finite model property.*

Proof. By Corollary 3.17 any intermediate logic admitting a structural intermediate hypersequent calculus will be $(0, \wedge, 1)$-stable, hence also $(0, \wedge, \vee, 1)$-stable, and as such enjoys the finite model property [7, Thm. 6.8]. □

Remark 3.21. Note that Proposition 3.20 gives an alternative way of seeing that every finitely axiomatisable intermediate logic admitting a structural hypersequent calculus is decidable. This of course also already follows from the fact that such logics admit a cut-free intermediate hypersequent calculus given by finitely many rules.

Proposition 3.22. *Any* $(0, \wedge, 1)$*-stable logic is canonical. Thus admitting a structural intermediate hypersequent calculus entails canonicity.*

Proof. By [23, Thm. 4.1] we know that if (r) is a structural hypersequent rule then the class of Heyting algebras validating (r) is closed under (upper) MacNeille completion. Thus if L is a $(0, \wedge, 1)$-stable logic then by Corollary 3.17 there is a collection of structural hypersequent rules $\mathscr{R}$ such that the variety $\mathbb{V}(L)$ is generated by the class $\mathcal{K}(\mathscr{R})$ of Heyting algebras validating all the rules in $\mathscr{R}$. Evidently, $\mathcal{K}(\mathscr{R})$ is a universal class and being closed under (upper) MacNeille completions we may conclude from [34, Thm. 3.6] that the class $\mathcal{K}(\mathscr{R})$ is also closed under (upper) canonical extensions. Finally, since any variety generated by a universal class of Heyting algebras closed under canonical extensions must be canonical [33, Thm. 6.8] we may conclude that $\mathbb{V}(L)$ is indeed canonical. □

Remark 3.23. As we will see in Section 4 any $(0, \wedge, 1)$-stable logic is in fact elementary from which the canonicity of such logics may also be inferred by Fine's Theorem, see, e.g., [18, Thm. 10.22]. Furthermore, every $(0, \wedge, 1)$-stable logic must also be a (locally) cofinal subframe logic. This can be seen either by considering the frame characterisation presented in Section 4 or by an argument similar to the one presented in Section 3.1. From this fact it can also be inferred that $(0, \wedge, 1)$-stable logics must be both canonical and elementary [53].

We conclude this section by drawing attention to some negative consequence of Corollary 3.17.

Proposition 3.24. *Let* $n \geq 2$ *be given. The logic* $\mathbf{BD}_n$*, of intuitionistic Kripke frames of depth at most* n*, does not admit a structural intermediate hypersequent calculus.*

Proof. We know that for $n \geq 2$ the intermediate logic $\mathbf{BD}_n$ is not $(0, \wedge, \vee, 1)$-stable [7, Thm. 7.4(2)] and so in particular it cannot be $(0, \wedge, 1)$-stable. Knowing this the proposition is an immediate consequence of Corollary 3.17. □

Remark 3.25. That this was the case had been expected in the literature, see., e.g., [26, 27]. However, we have not been able to find any proof of this fact before. The logic $\mathbf{BD}_2$ does, however, admit an analytic hypersequent calculus obtained by adding an additional logical hypersequent rule for the introduction of the implication to the multi-succedent hypersequent calculus $\mathbf{HLJ'}$ [26]. Furthermore, the logics $\mathbf{BD}_n$, for $n \geq 2$, do admit analytic display calculi [27], analytic labelled sequent calculi [31] as well as so-called path-hypertableau and path-hypersequent calculi [19].

As a final application of the algebraic characterisation of the intermediate logics admitting structural intermediate hypersequent calculi we give a semantic proof of [22, Cor. 7.2].

Proposition 3.26. *Let* (r) *be a structural sequent rule. Then either the calculus* $\mathbf{LJ} + (r)$ *is inconsistent or the rule* (r) *is derivable in* $\mathbf{LJ}$*.*

Proof. Let (r) be a structural sequent rule, i.e., a structural hypersequent rule the premises and conclusion of which only consists of single component hypersequents. Let $\mathcal{K}(r)$ be the class of Heyting algebras validating (r). If $\mathbf{LJ} + (r)$ is consistent then the class $\mathcal{K}(r)$ is non-trivial. In particular the two element Boolean algebra $\mathbf{2}$ will belong to $\mathcal{K}(r)$, as $\mathcal{K}(r)$ is closed under subalgebras and $\mathbf{2}$ is a subalgebra of every non-trivial Heyting algebra. Because (r) is a sequent rule the class $\mathcal{K}(r)$ will not only be a universal class but in fact a quasi-variety and as such closed under the formation of direct products. Since any bounded distributive lattice can be realised as a subdirect product of the lattice $\mathbf{2}$ [5, Thm. II.10.1] it follows that any Heyting algebra $\mathbf{A}$ will be a $(0, \wedge, \vee, 1)$-subalgebra of some member of $\mathcal{K}(r)$. Therefore, since (r) is structural and so the class $\mathcal{K}(r)$ is $(0, \wedge, 1)$-stable, we may conclude that $\mathbf{A} \in \mathcal{K}(r)$, showing that every Heyting algebra validates the rule (r). Given this the proposition then follows from Theorem 2.7. $\square$

Remark 3.27. Note that the proof of Proposition 3.26 shows the stronger claim that any structural multi-succedent sequent rule is either derivable in $\mathbf{LJ}'$ or derives every formula in $\mathbf{LJ}'$, where $\mathbf{LJ}'$ is the multi-succedent version of $\mathbf{LJ}$.

4 Frame based characterisation

In this section we identify the first-order frame conditions which determine $(0, \wedge, 1)$-stable logics. This is done using the duality theory for $(0, \wedge, 1)$-homomorphism between Heyting algebras developed in [11].

Given a Heyting algebra $\mathbf{A}$ we let $\mathbf{A}_+$ denote the underlying intuitionistic Kripke frame of the Esakia space $\mathbf{A}_*$ dual to $\mathbf{A}$, i.e., the Kripke frames consisting of the set of prime filters of $\mathbf{A}$ ordered by set-theoretic inclusion. Similarly, given an intuitionistic Kripke frame $\mathfrak{F}$ (Esakia space $\mathfrak{X}$) we let $\mathfrak{F}^+$ ($\mathfrak{X}^*$) denote the Heyting algebra of (clopen) upsets of $\mathfrak{F}$ ($\mathfrak{X}^*$). In the following we will use that $\mathbf{A} \cong (\mathbf{A}_+)^+$ for every finite Heyting algebra $\mathbf{A}$, see, e.g., [12, Thm. 2.2.21].

Definition 4.1 (cf. [11, Def. 6.2])**.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Priestley spaces. We say that a relation $R \subseteq X \times Y$ is a *generalised Priestley morphism* iff

1. If $\neg(xRy)$ then there exists $U \in \mathsf{ClpUp}(\mathfrak{Y})$ such that $y \notin U$ and $R[x] \subseteq U$;
2. If $U \in \mathsf{ClpUp}(\mathfrak{Y})$ then $\Box_R(U) \in \mathsf{ClpUp}(\mathfrak{X})$,

where $\mathsf{ClpUp}(\mathfrak{X})$ denotes the clopen upset of $\mathfrak{X}$ and $\Box_R(U) \coloneqq \{x \in X \colon R[x] \subseteq U\}$.

Moreover if $R^{-1}[Y] = X$ we say that R is *total* and if for every $y \in Y$ there is $x \in X$ such that $R[x] = {\uparrow} y$ then we say that R is *onto*.

Remark 4.2. Finite intuitionistic Kripke frames may be identified with finite Priestley spaces and so, forgetting the topology, we will also speak about generalised Priestley morphisms between finite intuitionistic Kripke frames.

Remark 4.3. Note that if $R \subseteq X \times Y$ is a generalised Priestley morphism between Priestley spaces $\mathfrak{X} \coloneqq (X, \leq_X, \tau_X)$ and $\mathfrak{Y} \coloneqq (Y, \leq_Y, \tau_Y)$ then it is straightforward to verify that $R[x'] \subseteq R[x]$ for all $x, x' \in X$ such $x \leq_X x'$. This observation will be useful when proving Proposition 4.8 and Lemmas 5.5 and 5.6 below.

We are interested in total generalised Priestley morphisms because they are the duals of $(0, \wedge, 1)$-homomorphisms. To be precise we have the following theorem.

Theorem 4.4 ([11])**.** *The category of Heyting algebras and* $(0, \wedge, 1)$*-homomorphisms is dually equivalent to the category of Esakia spaces and total generalised Priestley morphisms. Moreover, under this duality* $(0, \wedge, 1)$*-embeddings corresponds to total onto generalised Priestley morphisms.*

This allows us to translate questions about $(0, \wedge, 1)$-homomorphisms between Heyting algebras into questions about total generalised Priestley morphism between their dual spaces.

Recall [46] that a *geometric axiom* is a first-order sentence of the form

$$\forall \vec{w}\ (\varphi(\vec{w}) \implies \exists v\ \mathsf{OR}_{j=1}^{m} \psi_j(\vec{w}, v)),$$

with $\varphi, \psi_1, \ldots, \psi_m$ conjunctions of atomic formulas and the variable v not occurring free in φ. A *geometric implication* is then taken to be a finite conjunction of geometric axioms.

Definition 4.5 (cf. [42])**.** We say that a geometric axiom $\forall \vec{w}\ (\varphi(\vec{w}) \implies \exists v\ \mathsf{OR}_{j=1}^{m} \psi_j(\vec{w}, v))$ is *simple* if

1. There exists $w_0 \in \vec{w}$ such that $\varphi(\vec{w})$ is the conjunction of the atomic formulas $\{w_0 \leq w\}_{w \in \vec{w}}$;

2. Every atomic subformula of $\psi_j(\vec{w}, v)$ is of the form $w \leq w'$ or $w \leq v$ for $w, w' \in \vec{w}$.

A *simple geometric implication* is then a conjunction of simple geometric axioms.

Example 4.6. The intermediate logics $\mathbf{BTW}_n, \mathbf{BW}_n, \mathbf{BC}_n$, for $n \geq 1$, are all complete with respect to an elementary class of intuitionistic Kripke frames determined by simple geometric implications. Furthermore the logics $\mathbf{BD}_n$, for $n \geq 2$, are all complete with respect to an elementary class of intuitionistic Kripke frames determined by geometric implications, namely $\forall w_1 \dots w_{n+1}(\mathsf{AND}_{i=1}^{n}(w_i \leq w_{i+1}) \implies \mathsf{OR}_{i \neq j}(w_i = w_j))$, which are ostensibly not simple.

Remark 4.7. Intermediate (and modal) logics determined by a class of Kripke frame defined by geometric implications have been shown to admit so-called labelled sequent calculi [49, 46, 31]. Thus as a consequence of Proposition 4.13 below we obtain that any $(0, \wedge, 1)$-stable logic admits a cut-free labelled sequent calculus. This is consistent with the existence of a translation of hypersequents into labelled sequents, see, e.g., [48] for an overview.

Finally, a variant of the simple geometric implications appears in the work of Lahav [42] where these are used to construct analytic hypersequent calculi for modal logics which are sound and complete with respect to a class of Kripke frames determined by such simple geometric implications.

Proposition 4.8. *Let θ be a simple geometric implication. Then for any pair of Priestley spaces $\mathfrak{X} := (X, \leq_X, \tau_X)$ and $\mathfrak{Y} := (Y, \leq_Y, \tau_Y)$, with $\mathfrak{X}$ rooted, and any total and onto generalised Priestley morphism $R \subseteq X \times Y$ we have that*

$$(X, \leq_X) \models \theta \implies (Y, \leq_Y) \models \theta. \tag{†}$$

Proof. It suffices to show that (†) holds for an arbitrary simple geometric axiom θ, say, $\forall \vec{w}\ (\varphi(\vec{w}) \implies \exists \vec{v}\ \mathsf{OR}_{j=1}^{m} \psi_j(\vec{w}, \vec{v}))$.

Therefore, assume that $(X, \leq_X) \models \theta$. Suppose that $y_0, \dots, y_{k-1} \in Y$ are such that $\varphi(y_0, \dots, y_{k-1})$ holds in $(Y, \leq_Y)$, then by the assumption that R is an onto generalised Priestley morphism there are $x_0, \dots, x_{k-1} \in X$ such that $R[x_i] = {\uparrow} y_i$ for each $i \in \{1, \dots, k-1\}$. Because $(X, \leq_X)$ is rooted there is $x_0 \in X$ such that $\varphi(x_0, x_1, \dots, x_{k-1})$ and so since $(X, \leq_X) \models \theta$ there is $z \in X$ such that $\psi_l(\vec{x}, z)$ holds in $(X, \leq_X)$ for some $l \in \{1, \dots, m\}$. Furthermore, since R is total we have $z' \in Y$ such that zRz'. We claim that $\psi_l(\vec{y}, z')$ holds in $(Y, \leq_Y)$. If $x_t \leq_X x_{t'}$ in $(X, \leq_X)$ for some $t, t' \in \{0, \dots, k-1\}$ then we have that ${\uparrow} y_{t'} = R[x_{t'}] \subseteq R[x_t] = {\uparrow} y_t$ and thus $y_t \leq_Y y_{t'}$. Similarly if $x_t \leq_X z$ for some $t \in \{0, \dots, k-1\}$, then we have that $z' \in R[z] \subseteq R[x_t] = {\uparrow} y_t$ and hence that $y_t \leq z'$. This shows that $(Y, \leq_Y)$ satisfies θ. □

Definition 4.9. Recall that a variety $\mathbb{V}$ of Heyting algebras is *elementarily determined* if there exists an elementary class of intuitionistic Kripke frames $\mathcal{F}$ such that the variety is generated by the class of complex algebras $\mathcal{F}^+ := \{\mathfrak{F}^+ : \mathfrak{F} \in \mathcal{F}\}$, with $\mathfrak{F}^+$ denoting the Heyting algebra of upsets of $\mathfrak{F}$.

Remark 4.10. Note that for a given intermediate logic L the corresponding variety $\mathbb{V}(L)$ is elementarily determined iff the logic L is elementary, i.e., sound and complete with respect to an elementary class of intuitionistic Kripke frames.

Corollary 4.11. *Any intermediate logic characterised by a class of intuitionistic Kripke frames defined by simple geometric implications is $(0, \wedge, 1)$-stable.*

Proof. Let L be an intermediate logic characterised by a class of intuitionistic Kripke frames, say $\mathcal{F}$, defined by simple geometric implications. Note that if φ is a formula in the language of propositional intuitionistic logic such that $\mathfrak{F}^+ \not\models \varphi$ then there exists some point-generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ such that $\mathfrak{G}^+ \not\models \varphi$. Therefore, since simple geometric implications are evidently preserved by taking generated subframes, we obtain that $\mathbb{V}(L)$ is in fact determined by the class of rooted intuitionistic Kripke frames belonging to $\mathcal{F}$. Furthermore, any filtration $\mathfrak{F}'$ of an intuitionistic Kripke frame $\mathfrak{F}$ induces an order preserving surjection $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$. Therefore, since on rooted frames any (simple) geometric implication is equivalent to a positive first-order formulas and such formulas are preserved by order-preserving surjections we see that simple geometric implications will be preserved under taking filtrations of rooted intuitionistic Kripke frames. Consequently, we obtain that $\mathbb{V}(L)$ is in fact determined by the finite rooted members of $\mathcal{F}$. In particular $\mathbb{V}(L)$ will be generated by the class of complex algebras obtain from the set $\mathcal{G} := \{\mathfrak{F} \in \mathcal{F} : |\mathfrak{F}| < \aleph_0, \mathfrak{F} \text{ rooted}\}$. Finally, letting $\mathcal{K} := \{\mathbf{A} : \exists \mathbf{B} \in \mathcal{G}^+(\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B})\}$ it follows from Proposition 4.8 together with Theorem 4.4 that $\mathcal{K}$ is a $(0, \wedge, 1)$-stable class of Heyting algebras generating $\mathbb{V}(L)$ and therefore by Theorem 3.16 that the logic L is $(0, \wedge, 1)$-stable. □

To establish the converse of Corollary 4.11 we need the following lemma.

Lemma 4.12. *For any universal* $(0, \wedge, 1)$*-clause* q *there exists a simple geometric implication* θ_q *such that*

$$\mathfrak{F} \models \theta_q \iff \mathfrak{F}^+ \models q,$$

for every rooted intuitionistic Kripke frame $\mathfrak{F}$*.*

Proof. Let a universal $(0, \wedge, 1)$-clause q be given. As before q is equivalent to a finite conjunction of universal clauses of the form

$$t_1(\vec{x}) \leq u_1(\vec{y}) \text{ and } \dots \text{ and } t_m(\vec{x}) \leq u_m(\vec{y}) \implies t_{m+1}(\vec{x}) \leq u_{m+1}(\vec{y}) \text{ or } \dots \text{ or } t_n(\vec{x}) \leq u_n(\vec{y}), \qquad (q')$$

such that every term t_k is a $\{\wedge, 1\}$-term and every term u_k is either 0 or a single variable. Thus, by [25, Thm. 4.15] we may without loss of generality assume that (i) $\vec{x}$ and $\vec{y}$ are disjoint, (ii) every variable in q' occurs exactly once on the right-hand side of q'.

First we will show that for every such clause q' there exists a simple geometric axiom $\theta_{q'}$ such that for every rooted intuitionistic Kripke frame $\mathfrak{F} = (W, \leq)$ we have that $\mathfrak{F} \models \theta_{q'}$ iff $\mathfrak{F}^+ \models q'$. From which we obtain a simple geometric implication θ_q such that for every rooted intuitionistic Kripke frame $\mathfrak{F}$ we have that $\mathfrak{F} \models \theta_q$ iff $\mathfrak{F}^+ \models q$.

In the following we write $P(\vec{x}, \vec{y})$ for the left-hand side of the clause q', and for $k \in \{1, \dots, n\}$ we let $x_{k_1}, \dots, x_{k_{m_k}}$ denote the variables occurring in term $t_k(\vec{x})$, if any, and let y_{k_0} denote the variable occurring in the term $u_k(\vec{y})$, if any. Thus we then have that

$$\begin{aligned}
\mathfrak{F}^+ \not\models q' &\iff \exists \vec{U}, \vec{V} \in \mathsf{Up}(\mathfrak{F})\ (P(\vec{U}, \vec{V}) \text{ and } \mathsf{AND}_{j=m+1}^{n}(t_j(\vec{U}) \not\subseteq u_j(\vec{V}))) \\
&\iff \exists \vec{U}, \vec{V} \in \mathsf{Up}(\mathfrak{F})\ \exists \vec{w} \in W\ (P(\vec{U}, \vec{V}) \text{ and } \mathsf{AND}_{j=m+1}^{n}(w_j \in t_j(\vec{U})) \text{ and } w_j \notin u_j(\vec{V})) \\
&\iff \exists \vec{U}, \vec{V} \in \mathsf{Up}(\mathfrak{F})\ \exists \vec{w} \in W\ (P(\vec{U}, \vec{V}) \text{ and } \mathsf{AND}_{j=m+1}^{n}(\uparrow w_j \subseteq t_j(\vec{U})) \text{ and } u_j(\vec{V}) \subseteq (\downarrow w_j)^{\mathrm{c}}) \\
&\iff \exists \vec{U}, \vec{V} \in \mathsf{Up}(\mathfrak{F})\ \exists \vec{w} \in W\ (P(\vec{U}, \vec{V}) \text{ and } \mathsf{AND}_{j=m+1}^{n}(\mathsf{AND}_{k=1}^{m_j}(\uparrow w_j \subseteq U_{j_k}) \text{ and } V_{j_0} \subseteq (\downarrow w_j)^{\mathrm{c}}).
\end{aligned}$$

The special syntactic shape of the clause q' ensures that that the second-order variables among $\vec{U}$ only occur negatively in $P(\vec{U}, \vec{V})$ and that the second-order variables among $\vec{V}$ only occur positively in $P(\vec{U}, \vec{V})$. Moreover, every second-order variable among $\vec{U}, \vec{V}$ occurs exactly once somewhere on the right-hand side. This allows us to eliminate all the second-order variables via a standard and straightforward application of the Ackermann Lemma [1], see, e.g., [28, Lem. 0.1], to obtain that

$$\mathfrak{F}^+ \not\models q' \iff \mathfrak{F} \models \exists \vec{w}\ \mathsf{AND}_{i=1}^{m} \left(\bigcap_{k=1}^{m_i} \uparrow w_{i_k} \subseteq (\downarrow w_{i_0})^{\mathrm{c}} \right),$$

for some collection $\vec{w}$ of first-order variables. Thus we see that

$$\mathfrak{F}^+ \models q' \iff \mathfrak{F} \models \forall \vec{w} \exists v\ \mathsf{OR}_{i=1}^{m}(\mathsf{AND}_{k=1}^{m_i}(w_{i_k} \leq v) \text{ and } (v \leq w_{i_0})).$$

This shows that q' is equivalent to a formula in the first-order language of intuitionistic Kripke frames.

To see that q' is equivalent to a simple geometric implication on rooted intuitionistic Kripke frames simply note that if for some $i \leq m$ we have that the variable w_{i_0} occurs as one of the variables w_{i_k}, say $w_{i_{k'}}$, then it must be the case that $\mathsf{AND}_{k=1}^{m_i}(w_{i_k} \leq v)$ and $(v \leq w_{i_0})$ is equivalent to $\mathsf{AND}_{k=1}^{m_i}(w_{i_k} \leq w_{i_0})$ and $(w_{i_0} \leq w_{i_{k'}})$. On the other hand if for some $i \leq m$ we have that the variable w_{i_0} does not occur as one of the variables w_{i_k} then we must have that $\mathsf{AND}_{k=1}^{m_i}(w_{i_k} \leq v)$ and $(v \leq w_{i_0})$ is equivalent to $\mathsf{AND}_{k=1}^{m_i}(w_{i_k} \leq v \text{ and } w_{i_k} \leq w_{i_0})$. Thus we obtain a formula $\psi(\vec{w}, v)$, which is a disjunction of conjunctions of atomic formulas of the form $w \leq w'$ and $w \leq v$, such that

$$\mathfrak{F}^+ \models q' \iff \mathfrak{F} \models \forall \vec{w} \exists v \psi(\vec{w}, v).$$

Finally, letting $\theta_{q'}$ be the formula $\forall w_0 \forall \vec{w}(\mathsf{AND}_{w \in \vec{w}}(w_0 \leq w) \implies \exists v \psi(\vec{w}, v))$, for w_0 some fresh first-order variable, we obtain a simple geometric axiom such that $\theta_{q'}$ is equivalent to q' on rooted intuitionistic Kripke frames. □

Proposition 4.13. *Any variety of Heyting algebras generated by a* $(0, \wedge, 1)$*-stable universal class of Heyting algebras is elementarily determined by a class of intuitionistic Kripke frames defined by simple geometric implications.*

Proof. Given a variety $\mathbb{V}$ of Heyting algebras generated by a $(0, \wedge, 1)$-stable universal class, say $\mathcal{U}$, axiomatised by $(0, \wedge, 1)$-stable clauses, say $\{q_i\}_{i \in I}$, we see, by an argument completely similar to the one found in the proof of Corollary 4.11, that $\mathbb{V}$ will be generated by the class $\mathcal{F}^+ := \{\mathfrak{F}^+ : \forall i \in I\ (\mathfrak{F} \models \theta_{q_i})\}$, where θ_i is the simply geometric implication corresponding to q_i obtain from Lemma 4.12. □

We summarise our findings by amending Theorem 2.13 with two additional items.

Theorem 4.14. *Let L be an intermediate logic. Then the following are equivalent*

1. *The logic L admits a structural intermediate hypersequent calculus;*
2. *The logic L admits a cut-free structural intermediate hypersequent calculus;*
3. *The logic L is axiomatisable by $\mathcal{P}_3$-formulas;*
4. *The logic L is $(0, \wedge, 1)$-stable;*
5. *The logic L is characterised by a class of intuitionistic Kripke frames defined by simple geometric implications.*

5 Comparison with $(0, \wedge, \vee, 1)$-stable logics

The class of $(0, \wedge, \vee, 1)$-stable intermediate logics was first introduced and studied in [7] under the name of stable logics. In [10] a characterisation of $(0, \wedge, \vee, 1)$-stable intermediate logics were given of which Theorem 3.16 may be seen as an analogue. We here compare the class of $(0, \wedge, 1)$-stable intermediate logics to the class of $(0, \wedge, \vee, 1)$-stable intermediate logics.

Proposition 5.1. *The set of* $(0, \wedge, 1)$*-stable logics is a proper subset of the set of* $(0, \wedge, \vee, 1)$*-stable logics.*

Proof. Evidently each $(0, \wedge, 1)$-stable logic is also a $(0, \wedge, \vee, 1)$-stable logic. To show that there exists $(0, \wedge, \vee, 1)$-stable logics which are not $(0, \wedge, 1)$-stable, consider the following pair of Heyting algebras:

A **B**

We easily see that $\mathbf{A}$ is a $(0, \wedge, 1)$-subalgebra of $\mathbf{B}$ but not a $(0, \wedge, \vee, 1)$-subalgebra of $\mathbf{B}$. Let $\mathbb{V}$ be the variety axiomatised by the $(0, \wedge, \vee, 1)$-stable equation $\varepsilon_{0,\wedge,\vee,1}(\mathbf{A})$ associated with $\mathbf{A}$. Then the intermediate logic L corresponding to this variety is $(0, \wedge, \vee, 1)$-stable [10, Prop. 5.3]. Since $\mathbf{B}$ is well-connected and $\mathbf{A} \not\hookrightarrow_{0,\wedge,\vee,1} \mathbf{B}$, we may conclude that $\mathbf{B}$ belongs to $\mathbb{V}$ [10, Prop. 5.1]. Consequently, assuming that L is $(0, \wedge, 1)$-stable $\mathbf{A}$ must also belong to $\mathbb{V}$. But then $\mathbf{A} \models \varepsilon_{0,\wedge,\vee,1}(\mathbf{A})$ which, since any finite well-connected Heyting algebra refutes its own $(0, \wedge, \vee, 1)$-stable equation [10, Prop. 5.1], is absurd. □

Despite the fact that there are $(0,\wedge,\vee,1)$-stable logics which are not $(0,\wedge,1)$-stable all the examples of $(0,\wedge,\vee,1)$-stable logics considered so far [7, Sec. 7] are in fact $(0,\wedge,1)$-stable. The following theorem may be seen as explaining why this indeed the case. Furthermore, this also provide us with examples of $(0,\wedge,\vee,1)$-stable logics which are not $(0,\wedge,1)$-stable.

Theorem 5.2. *For* $\mathbf{A}$ *a finite well-connected Heyting algebra the following are equivalent:*

1. *The* $(0,\wedge,\vee,1)$*-stable clause* $q_{0,\wedge,\vee,1}(\mathbf{A})$ *associated with* $\mathbf{A}$ *is equivalent to a collection of universal* $(0,\wedge,1)$*-clauses;*
2. *The* $(0,\wedge,\vee,1)$*-stable clause* $q_{0,\wedge,\vee,1}(\mathbf{A})$ *associated with* $\mathbf{A}$ *is equivalent to the* $(0,\wedge,1)$*-stable clause* $q_{0,\wedge,1}(\mathbf{A})$ *associated with* $\mathbf{A}$*;*
3. *The Heyting algebra* $\mathbf{A}$ *is weakly projective as an object in the category* DL *of distributive lattices and lattice homomorphism.*

Proof. Evidently item 2 entails item 1. Conversely, to see that item 1 entails item 2, it suffices, due to Lemma 3.8, to show for any Heyting algebra $\mathbf{B}$ that

$$\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B} \iff \mathbf{A} \hookrightarrow_{0,\wedge,\vee,1} \mathbf{B}.$$

Since $\{0,\wedge,1\} \subseteq \{0,\wedge,\vee,1\}$ the implication $\mathbf{A} \hookrightarrow_{0,\wedge,\vee,1} \mathbf{B} \implies \mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ evidently obtains. To establish the converse let $\mathbf{B}$ be given and suppose that $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$, say via $h\colon A \hookrightarrow B$. If $\mathbf{A} \not\hookrightarrow_{0,\wedge,\vee 1} \mathbf{B}$ then $\mathbf{B} \models q_{0,\wedge,\vee,1}(\mathbf{A})$ and so since, by assumption, $q_{0,\wedge,\vee,1}(\mathbf{A})$ is equivalent to collection of universal $(0,\wedge,1)$-clauses and such clauses are preserved by $(0,\wedge,1)$-embeddings we must have that $\mathbf{A} \models q_{0,\wedge,\vee,1}(\mathbf{A})$ which is absurd as every Heyting algebra refutes all of the stable clauses associated with it.

To see that item 3 entails item 2 suppose that $\mathbf{A}$ is weakly projective as an object in the category DL. We claim that the $(0,\wedge,\vee,1)$-stable clause $q_{0,\wedge,\vee,1}(\mathbf{A})$ associated with $\mathbf{A}$ is equivalent to the $(0,\wedge,1)$-stable clause $q_{0,\wedge,1}(\mathbf{A})$ associated with $\mathbf{A}$. As before it suffices to show that $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B} \implies \mathbf{A} \hookrightarrow_{0,\wedge,\vee,1} \mathbf{B}$. Therefore, suppose that $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$, say via $h\colon A \hookrightarrow B$. Since $\mathbf{A}$ is assumed to be weakly projective as an object in the category DL it follows from a well-known result [6] that the poset $J_0(\mathbf{A})$ of join-irreducibles[7] of $\mathbf{A}$ including the element 0 is a $(0,\wedge)$-subalgebra of $\mathbf{A}$. Moreover, by the assumption that $\mathbf{A}$ is well-connected $J_0(\mathbf{A})$ will in fact be a $(0,\wedge,1)$-subalgebra of $\mathbf{A}$. Consequently, restricting h to $J_0(\mathbf{A})$ we obtain a $(0,\wedge,1)$-homomorphism $h_0\colon J_0(\mathbf{A}) \hookrightarrow_{0,\wedge,1} \mathbf{B}$. Because $\mathbf{A}$ is weakly projective we have by [6, Thm. 4] that h_0 has a unique extension to a $(0,\wedge,\vee)$-homomorphism $\widehat{h_0}\colon \mathbf{A} \to_{0,\wedge,\vee} \mathbf{B}$. In fact, since $\widehat{h_0}$ is an extension of h_0 and $1 \in J_0(\mathbf{A})$ we obtain that $\widehat{h_0}(1) = h_0(1) = h(1) = 1$. Thus, $\widehat{h_0}\colon \mathbf{A} \to_{0,\wedge,\vee,1} \mathbf{B}$ leaving us with the task of proving that $\widehat{h_0}$ is injective. By direct inspection of the construction of the map $\widehat{h_0}$ it may easily be verified that $\widehat{h_0}(a) \leq h(a)$ for all $a \in A$. As a consequence of this we see that if $\widehat{h_0}(a_1) = \widehat{h_0}(a_2)$ for some $a_1, a_2 \in A$ then for each $a_1' \in J_0(\mathbf{A}) \cap {\downarrow} a_1$ we must have

$$h(a_1') = h_0(a_1') = \widehat{h_0}(a_1') \leq \widehat{h_0}(a_1) = \widehat{h_0}(a_2) \leq h(a_2).$$

From this and the fact that h is a $(0,\wedge,1)$-embedding we may conclude that $a_1' \leq a_2$ for all $a_1' \in J_0(\mathbf{A}) \cap {\downarrow} a_1$. By a completely analogous argument we may deduce that $a_2' \leq a_1$ for all $a_2' \in J_0(\mathbf{A}) \cap {\downarrow} a_2$. Since $\mathbf{A}$ is finite every element is uniquely determined by the set of join-irreducible elements below it and so we must have that $a_1 = a_2$ and therefore that $h\colon \mathbf{A} \hookrightarrow_{0,\wedge,\vee,1} \mathbf{B}$, as desired.

Conversely, to see that item 2 entails item 3 suppose that $\mathbf{A}$ is not weakly projective as an object in the category DL. We exhibit a (finite) Heyting algebra $\mathbf{B}$ such that $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ but $\mathbf{A} \not\hookrightarrow_{0,\wedge,\vee,1} \mathbf{B}$, showing that the universal clauses $q_{0,\wedge,1}(\mathbf{A})$ and $q_{0,\wedge,\vee,1}(\mathbf{A})$ are not equivalent. To this effect let $P \coloneqq J(\mathbf{A})^{\partial}$, be the order dual of $J(\mathbf{A})$. Note that the Heyting algebra $\mathsf{Up}(P)$ of upsets of P is isomorphic to $\mathbf{A}$, as $\mathbf{A}$ is finite. Again, by the characterisation of finite weakly projective distributive lattices [6], $\mathbf{A}$ not being weakly projective entails the existence of $a_1, a_2 \in J(\mathbf{A})$ such that $a_1 \wedge a_2 \notin J_0(\mathbf{A})$, in particular a_1 and a_2 must be incomparable. Let $b_1, \ldots, b_n$ be the set of join-irreducibles which are below $a_1 \wedge a_2$ in $\mathbf{A}$. Necessarily, $n \geq 2$. Given this, let P' be the poset obtained from P by adding a new element a_0 covering a_1, a_2 and covered by $b_1, \ldots, b_n$. Thus $|P'| = |P|+1$. Evidently there can be no order-preserving surjection from P' onto P, since this would entail that a_1 and a_2 are comparable. Consequently, letting

[7]That is, non-zero elements $a \in A$ such that $a = b \vee c$ entails $a = b$ or $a = c$, for all $b, c \in A$.

$\mathbf{B}$ denote the dual Heyting algebra $\mathsf{Up}(P')$ of P', this shows that $\mathbf{A} \not\hookrightarrow_{0,\wedge,\vee,1} \mathbf{B}$. We claim, however, that $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$. To establish this it suffice by Theorem 4.4 to exhibit a total and onto generalised Priestley morphism $R \subseteq P' \times P$. We claim that letting $R \subseteq P' \times P$ be given by

$$R[a] \coloneqq \begin{cases} \uparrow a & \text{if } a \neq a_0 \\ \uparrow\{a_1, a_2\} & \text{if } a = a_0, \end{cases}$$

is such a generalised Priestley morphism. It may readily be verified that R is a generalised Priestley morphisms. Moreover, that R is total and onto is evident from the definition. □

Remark 5.3. We note that the use of generalised Priestley morphisms in the proof of Theorem 5.2 can be avoided. To see this observe, with the notation of the proof of Theorem 5.2, that P is, in fact, a subposet of P', whence by Priestley duality we have a surjective map $h\colon \mathbf{B} \twoheadrightarrow_{0,\wedge,\vee,1} \mathbf{A}$. Moreover, we may easily verify that $h^{-1}(0) = \{0\}$ and $h^{-1}(1) = \{1\}$. From [37, Cor. 5.4] we know that $\mathbf{A}$, being a finite distributive lattice, is projective as a meet-semilattice and therefore we obtain a map $\overline{h}\colon \mathbf{A} \to_{\wedge} \mathbf{B}$ such that $h \circ \overline{h}$ is the identity on $\mathbf{A}$. In particular, $\overline{h}$ must be injective and as $h(\overline{h}(0)) = 0$ and $h(\overline{h}(1)) = 1$ it follows that $\overline{h}(0) = 0$ and $\overline{h}(1) = 1$. Thus we have $\overline{h}\colon \mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$.

Remark 5.4. Theorem 5.2 can be seen as explaining why all of the examples of $(0, \wedge, \vee, 1)$-stable logics considered in [7, Sec. 7] are in fact $(0, \wedge, 1)$-stable logics, as all of these logics are axiomatised by $(0, \wedge, \vee, 1)$-stable equations associated with finite well-connected Heyting algebras which are weakly projective as objects in the category DL.

We conclude this section by showing that the $(0, \wedge, 1)$-stable logics are precisely the intermediate logics which are both cofinal subframe logics and $(0, \wedge, \vee, 1)$-stable. Recall [18, Chap. 11.3] that an intermediate logic is a cofinal subframe logic if it can be axiomatised by cofinal subframe formulas or alternatively if it is sound and complete with respect to a class of Kripke frames closed under taking cofinal subframes [18, Thm. 11.25].

For this we need two simple lemmas.

Lemma 5.5. *Let $S \subseteq W_1 \times W_2$ be a generalised Priestley morphism between finite intuitionistic Kripke frames $\mathfrak{F}_1 \coloneqq (W_1, \leq_1)$ and $\mathfrak{F}_2 \coloneqq (W_2, \leq_2)$. Then $\mathfrak{F}_2$ is the image under an order-preserving map of a cofinal subframe of $\mathfrak{F}_1$.*

Proof. Let $W_1' = \{w_1 \in W_1 : \exists w_2 \in W_2\ S[w_1] = \uparrow w_2\}$. Then we see that mapping each $w_1 \in W_1'$ to the necessarily unique element $w_2 \in W_2$ such that $S[w_1] = \uparrow w_2$ determines a map $f\colon W_1' \to W_2$ which must be surjective as S is onto. Furthermore, because S is a generalised Priestley morphism we have that $w_1 \leq_1 v_1$ implies $S[v_1] \subseteq S[w_1]$ and consequently that f is order-preserving when considering W_1' as a subframe of W_1.

We then note that for any $w_1 \in \max(W_1)$ since S is total we have $w_2 \in W_2$ such that $w_1 S w_2$. Moreover, if for some $v_1 \in W_1'$ we have $v_1 \leq_1 w_1$ then $S[w_1] \subseteq S[v_1] = \uparrow f(v_1)$ as S is a generalised Priestley morphism. Thus we may define $g\colon W_1' \cup \max(W_1) \to W_2$ by letting $g(w_1) = f(w_1)$ if $w_1 \in W_1'$ and letting $g(w_1)$ be some element of $S[w_1]$ if $w_1 \in \max(W_1) \backslash W_1'$. Since S is total this is a well-defined order-preserving map. Evidently $W_1' \cup \max(W_1)$ is a cofinal subframe of W_1 and so $\mathfrak{F}_2$ is the image of a cofinal subframe of $\mathfrak{F}_1$ under an order-preserving map. □

Lemma 5.6. *Let $S \subseteq W_1 \times W_2$ be a generalised Priestley morphism between finite intuitionistic Kripke frames $\mathfrak{F}_1 \coloneqq (W_1, \leq_1)$ and $\mathfrak{F}_2 \coloneqq (W_2, \leq_2)$, with $\mathfrak{F}_1$ rooted. Then $\mathfrak{F}_2$ is a cofinal subframe of an image of a rooted cofinal subframe of $\mathfrak{F}_1$ under an order-preserving map.*

Proof. From Lemma 5.5 we know that $\mathfrak{F}_2$ is the image of a cofinal subframe $\mathfrak{F}_1' \coloneqq (W_1', \leq_1')$ of $\mathfrak{F}_1$, under an order-preserving map, say $f\colon W_1' \to W_2$. Let w_0 be the root of $\mathfrak{F}_1$. If $w_0 \in W_1'$ then there is nothing to show. If w_0 is not in W_1' then letting $W_1'' \coloneqq W_1' \cup \{w_0\}$ we obtain a rooted cofinal subframe $\mathfrak{F}_1''$ of $\mathfrak{F}_1$. Similarly, by adjoining a new root w_0' to W_2 we obtain a rooted frame $\mathfrak{F}_2'$ of which $\mathfrak{F}_2$ is a cofinal subframe. Finally, the map f extends to a surjective order-preserving map from $\mathfrak{F}_1''$ to $\mathfrak{F}_1'$ by mapping w_0 to w_0'. □

Proposition 5.7. *Let L be an intermediate logic. Then the following are equivalent.*

1. *L is $(0, \wedge, 1)$-stable;*

2. L is a $(0, \wedge, \vee, 1)$*-stable, cofinal subframe logic.*

Proof. Every $(0, \wedge, 1)$-stable logic is evidently $(0, \wedge, \vee, 1)$-stable. Furthermore, by Proposition 4.13 every $(0, \wedge, 1)$-stable logic is sound and complete with respect to a class of intuitionistic Kripke frames determined by simple geometric implications. It is straightforward to verify that such first-order formulas are preserved by taking cofinal subframes. Consequently, being generated by a class of Kripke frames closed under cofinal subframes it follows that any $(0, \wedge, 1)$-stable logic is indeed a cofinal subframe logic.

Conversely, suppose that L is a cofinal subframe logic which is $(0, \wedge, \vee, 1)$-stable. We show that if $\mathbf{A}$ and $\mathbf{B}$ are finite Heyting algebras with $\mathbf{B}$ subdirectly irreducible such that $\mathbf{B} \in \mathbb{V}(L)$ and $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ then $\mathbf{A} \in \mathbb{V}(L)$. Therefore, let $\mathfrak{F} = (W, \leq)$ be the dual intuitionistic Kripke frame of $\mathbf{B}$ and let $\mathfrak{F}' = (W', \leq')$ be the dual intuitionistic Kripke frame of $\mathbf{A}$. By the assumption that $\mathbf{A} \hookrightarrow_{0,\wedge,1} \mathbf{B}$ we have a total and onto generalised Priestley morphism $S \subseteq W \times W'$. Moreover, since $\mathbf{B}$ is subdirectly irreducible we have that $\mathfrak{F}$ is rooted and hence by Lemma 5.6 that $\mathfrak{F}'$ is a cofinal subframe of an image $\mathfrak{G}'$ under an order-preserving map of a rooted cofinal subframe $\mathfrak{G}$ of $\mathfrak{F}$. Since by assumption L is a cofinal subframe logic we must have that $\mathfrak{G}$ is also an L-frame. Moreover, since $\mathfrak{G}$ is rooted and L is $(0, \wedge, \vee, 1)$-stable we obtain that $\mathfrak{G}'$ is an L-frame [7, Thm. 6.7], and so, again using the fact that L is a cofinal subframe logic, we see that $\mathfrak{F}'$ is an L-frame. We may therefore conclude that $\mathbf{A} \in \mathbb{V}(L)$ as desired. It follows that $\mathbb{V}(L)$ is generated by a $(0, \wedge, 1)$-stable class of Heyting algebras and therefore that L is $(0, \wedge, 1)$-stable. □

Remark 5.8. It is known that there are continuum-many $(0, \wedge, \vee, 1)$-stable logics [7, Thm. 6.13] just as it is known that there are continuum-many cofinal subframe logics [18, Thm. 11.19]; however, we have not been able to determine how many $(0, \wedge, 1)$-stable logics there are. The problem is that, using duality, we cannot work with order-preserving surjections which are the duals of $(0, \wedge, \vee, 1)$-embeddings, but we must work with total and onto generalised Priestley morphism which are more complicated. In this context it is no longer clear if an argument similar to the one presented in [7] will work.

6 Future work

We conclude by mentioning a number of open questions arising in the context of the present work.

It is not clear if the property of being $(0, \wedge, 1)$-stable can be effectively verified for finitely axiomatisable intermediate logics. Thus it is left open whether or not it is decidable if a finitely axiomatisable intermediate logic admits a (cut-free) structural intermediate hypersequent calculus.[8]

Much of the original work on algebraic proof theory has been done in the context of substructural logic. Consequently, we find it worth investigating if a similar characterisation of substructural logics admitting structural hypersequent calculi can be given. However, since the subdirectly irreducible residuated lattices are more complicated than their Heyting algebra counterparts it is not immediately clear if all the necessary results transfer to the setting of substructural logic. Here the work of Bezhanishvili et. al. [14], investigating canonical formulas in the context of certain substructural logics might be helpful.

Finally, it is our hope that the findings in Section 4 will help to bridge the gap between the different approaches to systematic proof theory regarding substructural, intermediate and modal logics. Some of these approaches are primarily based on the algebraic semantics [22, 24, 25, 35], while others primarily make use of the relational semantics [31, 42, 43].

7 Acknowledgements

I would like to thank Nick Bezhanishvili, Julia Ilin and Yde Venema from the University of Amsterdam for helpful comments on earlier drafts of this paper. I would also like to thank the two anonymous reviewers for their comments.

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