LEIBNIZ'S PRINCIPLE AND THE PROBLEM OF NONINDIVIDUALITY

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written by

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Abstract

With the emergence of contemporary formal logic, it has become customary to formalise Leibniz's Principle of the Identity of Indiscernibles (PII) by means of the following second order formula: '$\forall x \forall y (\forall p (p(x) \leftrightarrow p(y)) \rightarrow x = y)$'. Under the assumption that this formula captures the ontological meaning of Leibniz's Principle, the relation between a second order formalisation of the scenario presented in Black (1952) as a counterexample to PII and Leibniz's Principle becomes interesting to explore. Furthermore, once shown that the objects described in Black's scenario are nonindividuals, the question arises if and how it is possible to build a theory of collections that does not restrict (as ZFC does) the possibility of being a member of a collection only to individuals. We will present a first attempt to formulate such theory of collections, and some interesting facts will be proved as theorems about collections containing nonindividuals as elements.

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Contents

“We are struggling with language.
We are engaged in a struggle with language.”

Ludwig Wittgenstein, *Culture and Value*

Introduction

Since its first formulation in 1720, Leibniz's Principle of the Identity of Indiscernibles has challenged the minds of numerous philosophers, leading to one of the most impassioned debates within modern and contemporary philosophy. In 1952, Max Black famously presented a counterexample to Leibniz's Principle by imagining an otherwise empty radially symmetrical universe containing only two objects resembling each other in any respect. The origin of this Thesis lies in the attempt to understand Black's (1952) scenario, both metaphysically and logically. Within this work, we will try to answer the questions concerning the nature of the objects inhabiting Black's Universe, and we will analyse the reasons why our standard formal language seems to fail whenever the objects described by Black are considered. This Thesis consists of three chapters, each one broadening a single perspective of the analysis of indiscernible objects. In Chapter 1, a set theoretical formalisation of Black's (1952) scenario will be attempted, with the purpose of showing that no formulation of Black's counterexample in a second order language can be proved correct, if a fragment of ZFC is taken as metatheory and only standard models are considered. Two other issues will be addressed within the same chapter, namely (1) the problem of naming the objects in Black's (1952) scenario, and (2) the problem of finding a correct logical formulation of a defence of Leibniz's Principle from Black's counterexample by means of stating the existence of some property or relation discerning the objects described by Black. In Chapter 2, we will deal with indiscernible objects *tout court*, and we will investigate the features of collections containing such objects. It will be shown that these collections do not belong to the class of collections satisfying the axioms of Zermelo-Fraenkel's Set Theory (ZFC), and that, given the classical characterization of individuality provided by Lowe (2016), indiscernible objects cannot be consistently thought of as individuals. In Chapter 3, we will present the first draft of an axiomatic theory of collections expanding the set theoretical universe of ZFC and allowing indiscernible objects to be members of some collections. We will make a first step towards the development of a suitable formal language for such theory, trying to find a way to solve the problem of how to refer to nonindividual objects. Finally, the first results regarding the structure of collections of nonindividuals will be presented. We hope this work can help to enlighten the understanding of Black's counterexample to Leibniz's Principle, and draw attention to a category of objects that is too often left outside standard ontology, by providing the appropriate tools to talk about those objects and to understand how collections of these kind of objects work.

The outcomes of this project are not only theoretical. Many philosophers discussing the relations between the quantum mechanical description of some isolated systems of particles and Leibniz's Principle have in fact maintained that elementary particles lack individuality. We hope this work can be of interest to those who want to model collections of indiscernible objects, providing a simple formal framework to accomplish such task.

Chapter One

1.1 Introduction

In 1720 Gottfried Wilhelm Leibniz claimed "[...] there are never in nature two beings which are exactly alike, and in which it is not possible to find a difference either internal or based on an intrinsic property" (*Monadology*, G 6:608). Known in analytic ontology with the name of 'Principle of the Identity of Indiscernibles' (henceforth: PII), Leibniz's thesis has been at the heart of an impassioned and still unsolved philosophical debate, and over the years it has been significantly challenged by many philosophers. Kant (1787) famously argued for the possibility of imagining two droplets of water in all respects similar to one another. Hermann Weyl (1928) maintained that the quantum mechanical description of n-many identical particles in an isolated physical system could be consistently thought of in contravention to PII. Max Black (1952) proposed an otherwise empty radially symmetrical universe inhabited only by two indiscernible spheres as a possible scenario breaching the necessary truth of Leibniz's Principle. Two years later, A. J. Ayer (1954) claimed an infinite sequence of four sound tokens to be a counterexample to PII. Other challenges to Leibniz's Principle can be found in the works of Peter Frederick Strawson (1959) and Christian Wütrich (2009)[1]. With the emergence of contemporary formal logic, it has become usual to formalise PII

[1]For a philosophical analysis of the main challenges to the Principle of Identity of Indiscernibles, see Muller (2015).

in a first order language by letting metavariables stand for first order formulas, and in a second order language by means of second order quantification over predicates. First order metavariables and second order quantification are used to formally represent the notion of 'indiscernibility'. Intuitively, two objects a and b are indiscernible when it is not possible to find something distinguishing one from the other. This is formally expressed by saying that two objects a and b are indiscernible whenever everything that is true of a is also true of b, and vice versa, or equivalently that a and b are indiscernible whenever there is no n-ary relation which holds for one and not for the other. The Principle of the Identity of Indiscernibles is then expressed in first order logic by means of the formula:

$$\forall x \forall y((\varphi(x) \leftrightarrow \varphi(y)) \rightarrow x = y),$$

and in second order logic by means of the formula:

$$\forall x \forall y(\forall p(p(x) \leftrightarrow p(y)) \rightarrow x = y),$$

where the variables x and y range over first order objects. The metavariable φ in the first order formulation of PII ranges over first order formulas, while the variable p in the second order formulation ranges over predicates.

In what follows, we will be interested in the first order quantification figuring in both first order and second order PII. Chapter 1 will be completely devoted to the formal analysis of the scenario presented in Black (1952). Although widely challenged, Black's (1952) counterexample to PII is universally acknowledged as a meaningful ontological counterargument to a meaningful ontological thesis. The current disagreement concerning Black's (1952) counterexample is not about whether it is meaningful or not, nor whether it is comparable to Leibniz's Principle at all. The disagreement is about *whether it is true or not.* No doubt has been cast on the fact that if Black's (1952) scenario is a genuine possibility, then PII is infringed. That is, everyone would be willing to admit that the validity of Black's argument would determine the fate of Leibniz's Principle. In this work, we will never challenge the intuition that Black (1952) presents a meaningful argument for the failure of PII. We will challenge, instead, the possibility of finding a second order formalisation of Black's counterexample being in contrast with the second order formalisation of PII. We will challenge this possibility on the ground that, the standard set theoretical metatheory of second order logic being a fragment of Zermelo-Fraenkel's Set Theory with the Axiom of Choice (henceforth: ZFC), no objects of the same kind of the ones postulated by Black (1952) can be consistently thought of as elements of any domain of quantification of any standard model for second order logic. As a result, it will be shown that no standard first order quantification is possible over these objects, and any attribution of properties to them will be shown to be unobtainable. It might be argued that the entire discussion about Black's (1952) argument and Leibniz's Principle of the Identity of Indiscernibles is an informal discussion.[2] Still, a significant part of the current literature about PII and about Black (1952) makes explicit use of the second order formulation of PII.[3] We would like to be clear about the following point: that the matter at issue here is not the correctness of the second order formulation of Leibniz's Principle. The matter at issue here is the formal relation between second order PII and Black's (1952) counterexample. We will show that Black's (1952) scenario does not admit a second order formalisation whenever a fragment of ZFC is taken as metatheory, and standard models are considered. As a consequence, under the assumption of the correctness of the second order formulation of Leibniz's Principle, Black's (1952) argument will turn out to be formally incomparable to PII.

[2]We would like to thank Prof. Luca Incurvati (University of Amsterdam) for having brought this consideration to our attention.

[3]As examples, see Boolos & Jeffrey (1974), Cortes (1976), French (1989), van Dalen (1994), Caulton & Butterfield (2012), and Muller (2015).

1.2 Black's Indiscernible Objects

In 1952 Max Black came up with a counterexample to PII. In his *The Identity of Indiscernibles*[4], written in the form of a dialogue, the philosopher presented the following scenario: an otherwise empty universe inhabited only by two exactly similar spheres, two miles apart from each other. The universe is claimed to be radially symmetrical, its unique centre of symmetry located exactly one mile apart from each sphere. Black (1952) maintained such universe to be a genuine possibility, from which it follows that PII is not necessarily true —that is, it is not a metaphysical truth. In the years that followed, Black's counterexample has been called into question many times as illegitimate. Many philosophers have argued against it in an attempt to secure PII from the alleged threat posed by Black's (1952) scenario. Famously, Ian Hacking (1975) claimed Black's universe to be equivalent to a universe containing only one sphere in a cylindrical spacetime framework, and Della Rocca (2005) warned against the possibility of consistently accepting Black's counterexample to PII and at the same time maintaining an intuitive stance with respect to our everyday answers to cardinality questions[5]. Caulton & Butterfield (2012) argued that Black's spheres are *weakly discernible*[6], meaning that it exists an irreflexive and symmetric relation holding between the two.

Hawley (2009) defines three strategies to defend PII from alleged counterexamples. The first strategy is called 'identity defence', and it consists in identifying the objects figuring in the scenarios maintained to infringe PII, whatever these objects might be. The second strategy, the 'discerning defence', consists in finding a suitable property or relation to discern the objects claimed to be qualitatively identical. Finally, the third strategy, called the

[4]Black, Max (1952). The identity of indiscernibles. *Mind* 61 (242):153-164.

[5]A 'cardinality question' is a question of the form: 'how many —?'.

[6]The notion of 'weak discernibility' (or 'weak discriminability') has been originally presented in Quine (1976).

'summing defence', consists in denying the existence of the objects from time to time described in the scenarios developed against PII, and in claiming instead the existence of a unique object, "with the qualitative features which would have been possessed by the sum of the two indiscernibles, had they both existed and had a sum" (Hawley, 2009). In the following lines, we will be mostly interested in the possibility of defending PII from Black's (1952) counterexample by means of a 'discerning defence'. Showing that Black's counterexample is illegitimate amounts then to finding a way to distinguish the two spheres. In other words, assuming that Black's universe contains exactly two spheres, to defend PII from Black's (1952) argument corresponds to proving that one of the following sentences is true of Black's spheres:

- There is a *property* instantiated by only one of the two spheres: hence, they are not indiscernible.
- There is a *relation* holding for only one of the two spheres: hence, they are not indiscernible.

From the assumptions that (1) the meaning of Leibniz's Principle is entirely captured by the logical formalisations we have presented, that (2) Black's Universe is a genuine possibility, and that (3) the adopted metatheory in the formalisations of PII is a fragment of ZFC, it follows that, in order for Black's counterargument and Leibniz's Principle to be comparable, Black's counterargument needs to be formalisable in the same logic with the same semantics as PII. Only in this way, in fact, it is possible to derive from Black's counterexample formal statements of the form:

- $\exists p(p(a) \wedge \neg p(b))$, and
- $\forall p(p(a) \leftrightarrow p(b))$,

the first confirming and the second breaching PII, where the constants a and b refer to the two spheres. An analysis of the conditions of possibility of such a derivation will keep us occupied for the entire Chapter 1 and for the first part of Chapter 2. One further point needs to be addressed before our argument can be presented: in his dialogue, Black contemplates the possibility of a stranger entering his universe, and naming the spheres 'Castor' and 'Pollux'. Although this is taken as a genuine possibility, it is also considered devastating. The entrance of such a stranger would in fact irreversibly change Black's scenario:

> "All I have conceded is that if something were to happen to introduce a change into my universe, so that an observer entered and could see the two spheres, one of them could then have a name. But this would be a different supposition from the one I wanted to consider. My spheres don't yet have names" (Black, 1952).

Black's conclusion is that, as names, 'Castor' and 'Pollux', as well as the expressions 'the one' and 'the other', and the constants 'a' and 'b', cannot be used to legitimately talk about the spheres, unless having changed the situation under discussion. In the following sections, we will attempt a formalisation of Black's scenario. In our talking, we will use the expressions 'the one' and 'the other', as well as the variables 'x' and 'y'. Every sentence in which these expressions will appear will be replaceable by a formula making no use of the alleged names, and no one of the results we will accomplish will depend on the fact that, for the sake of simplicity and clarity of the language, we have pretended to be able to talk about the spheres in the way we are going to talk about them.

1.3 A Formal Set Theoretical Framework

We begin our analysis of Black's (1952) scenario by asking if it is possible to provide a set theoretical formalisation of Black's Universe. Intuitively, whenever there are a positive number of objects, it seems possible to consider the collection of such objects. The ontological problem whether such collection should be considered a further object in our ontology will be discussed in Chapter 2. For now, we work under the assumption that a collection of objects is something that can be theoretically defined whenever we have a positive number of objects, and that this definition does not give rise to any problem within our ontology. Black (1952) introduces his universe as inhabited by two objects, and in this sense we will consider it as the collection of the objects it contains. In what follows we will not consider the particular symmetry shaping Black's Universe, since no one of the conclusions we want to reach depends on this characteristic, nor can be undermined by it —that is, we will talk about Black's Universe only insofar as it contains the two Black's spheres.

First, we define the notion of 'ZFC-collection' as follows: a ZFC-collection is a collection hereditarily respecting ZFC's axioms. The question we then need to answer becomes: can Black's Universe be considered a ZFC-collection? Let us assume it can. Let U denote Black's Universe, and let the variables u and v range over all the non trivial subsets of U, where a subset $u \subseteq U$ is trivial if and only if $u = \varnothing$ or $u = U$. In Black's case, U containing only two objects, the only non trivial definable subsets $u, v \subseteq U$ are the two singletons containing, respectively, 'one' and 'the other' element of U. By the *Axiom of Extensionality* (henceforth: EXTENSIONALITY), we know that two sets are identical if and only if they contain the same elements. Formally:

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)).$$

By contraposition, we obtain a condition of diversity for two subsets u and v of U, namely: that u and v are different if and only if there is an element $x \in U$ such that x is not an element of the intersection of u and v:

$$\forall u \forall v (u \neq v \leftrightarrow \exists x ((x \in u \wedge x \notin v) \vee (x \notin u \wedge x \in v))).$$

Consider U: provided that u and v exist and that they are non empty, the condition of the existence of some element of U which is not an element of the intersection of u and v is equivalent to the condition of the existence of two elements x and y such that:

$$(x \neq y) \wedge (x \in u \leftrightarrow y \notin u) \wedge (x \in v \leftrightarrow y \notin v) \ ,$$

where x and y are the only objects in U[7]. It follows then from our assumptions that $u = \{x\}$ and $v = \{y\}$, or vice versa, since we assumed both u and v as (1) existing, (2) being non trivial subsets of the domain, and (3) being different.

This argument has been presented to clarify how exactly EXTENSIONALITY depends on the identity conditions we impose on elements. As Krause & Coelho (2005) claims: "[...] for the Axiom of Extensionality to hold, it is necessary to have a criterion for two elements being the same object"[8]. Now, the question follows *which* conditions of identity (if any) we should accept for the two objects x and y in Black's (1952) scenario, pretending, as we have done until now, to be able to name them by the two symbols we have chosen. Continuing with our assumption that Black's environment enables a set theoretical formalisation, in what follows we will show that the criterion of identity we are committed to, when considering Black's objects as elements of a ZFC-collection, can be stated as follows: x is the same object as y if and only if, for any set u, x is an element of u if and only if y is an element of u. Formally: $x = y \leftrightarrow \forall u(x \in u \leftrightarrow y \in u)$.

Proof. For the left to right direction, assume $x = y$ and consider an arbitrary set u such that $x \in u$. As it is usually interpreted, the formula '$x = y$' is

[7]The definition of U as containing only two elements and the definition of the subsets $u \subseteq U$ and $v \subseteq U$ as non empty renders the existence of an element $x \in U$ such that both $x \notin u$ and $x \notin v$ impossible.

[8]The statement of Krause & Coelho (2005) is not equivalent to our conclusion: it is, in fact, a generalisation of it. We have obtained a conclusion for a very particular setting, and to show that all the assumptions we used relative to the particular scenario we were considering can be generalised is far beyond the scope of this work.

true whenever there is a unique object in the domain of quantification such that it is the referent of both the symbols 'x' and 'y' (the equality symbols '$=$' being the formal representation of the concept of 'numerical identity'). It directly follows that $y \in u$. To show that if $y \in u$ then also $x \in u$ is similar.

For the right to left direction, assume that for all u, $x \in u$ if and only if $y \in u$. The conclusion $x = y$ can be then derived following two independent lines of reasoning:

1. By the *Pairing Axiom* (henceforth: PAIRING), consider the set $u = \{x\}$. By construction $x \in u$. Then, by assumption, also $y \in u$, and this leads us to the conclusion that $x = y$.

2. Let φ be the following condition (with parameter x): $\varphi = (w = x)$. By the *Axiom of Separation* (henceforth: SEPARATION), there is a set u such that $u = \{w \in U : w = x\}$. It follows that $u = \{x\}$. By construction $x \in u$. Then, by assumption, also $y \in u$, and this leads us to the conclusion that $x = y$.[9]

We conclude that some restrictions need to be defined whenever we want to consider Black's Universe as a ZFC-collection. Without any restriction, in fact, we are committed to the following condition of identity for two elements x and y, namely: that x is the same object as y if and only if, for any set u, x is an element of u if and only if y is an element of u. We have shown that, independently, the *Axiom of Separation* and the *Pairing Axiom* in conjunction with the *Axiom of Extensionality* commit us to the acceptance of

[9]It might be wondered whether SEPARATION is allowed to play a role in a proof of this kind: admittedly, we are looking for the conditions of identity of two elements, and in order to define them, we use the symbol of equality in the construction of the cornerstone of our proof: the singleton of one of them. But we want to remark that, although this procedure might seem somehow circular, actually it is not: in fact, the identity symbol used in the definition of the condition φ is not the identity symbol we want to define for two elements: it is instead the first order logical constant of equality.

the right to left direction of the biconditional, namely: that if, for all sets u, x is an element of u if and only if y is an element of u, then x and y are one and the same object —in other words, we are implicitly accepting PII at the ground of our formalism. As a consequence, we must doubt the possibility of using both PAIRING and SEPARATION, in conjunction with EXTENSIONALITY. This does not mean that, for any two elements x, y of any given set u there exists no set $v = \{x, y\}$. It means that *not for any* two elements x, y of any given set u there exists a set $v = \{x, y\}$. In particular, the possibility of constructing singletons from elements by means of PAIRING and SEPARATION is no longer guaranteed.

1.4 Singletons

In the following lines, we will show that within the set theoretical formalisation of Black's Non Leibnizian Universe we cannot define and prove the existence of any singleton for elements falling under the category of what we will from now on call u-objects, intuitively defined as the family of (unordered) clusters of entities that could figure in a Non Leibnizian Universe *à la* Black, without altering its symmetry[10].

Within Zermelo-Fraenkel's Set Theory, we can define and prove the existence of the singleton $\{x\}$ of a given element x in four different ways:

(1) Given some element x, by PAIRING we get the existence of the set $\{x, x\}$ which is equivalent, by EXTENSIONALITY, to $\{x\}$;

(2) Given a set u and an element $x \in u$, we define the condition $\varphi = (y = x)$. By SEPARATION, there is a set v such that $v = \{y \in u : y = x\}$, this set being $\{x\}$;

[10]This restriction is needed if we want to be able to generalise our results and to take into account different Black's Universes.

(3) Given two sets u and v and an element $x \in v$, we define the constant function $f : u \to v$ such that for all $y \in u$, $f(y) = x$. By the *Axiom of Replacement* (henceforth: REPLACEMENT), we get the existence of $\{x\}$, which is the image of f;

(4) Given the empty set $\varnothing$, we use the *Power Set Axiom* (henceforth: POWER SET) to obtain the power set of the empty set: $\{\varnothing\}$. Given an element $x \in U$ we define the constant function $f : \wp(\varnothing) \to U$ such that: $f(\varnothing) = x$. By REPLACEMENT we obtain the set $\{x\}$ as the image of $\wp(\varnothing)$ under f.

Can we consistently use one of these ways to define singletons within Black's Universe U? The answer is negative. Clearly, the first two approaches require unrestricted SEPARATION and unrestricted PAIRING in conjunction with EXTENSIONALITY, and we have shown that both independently commit us to a Leibnizian condition of identity for the elements of U.

We then consider the third way. Given $U = \{x, y\}$, we need to define a class function $f : U \to U$ to apply REPLACEMENT and get $\{x\}$ (alternatively, $\{y\}$) as a result. Then, we define the constant function $k : U \to U$ such that, for all $w \in U$, $k(w) = x$[11]. By definition, the image of U under k is $\{x\}$. Hence, by REPLACEMENT, we conclude that $\{x\}$ exists and it is a set. Can we then conclude that the third approach is consistent and sleep tight? No, we cannot. In fact, by definition of function, $k : U \to U$ is a subclass $k \subseteq U \times U$. It follows that k is a collection of ordered n-tuples. In particular, $k = \{(x, x), (y, x)\}$. By definition, $(x, x) = \{\{x\}, \{x, x\}\}$. It follows that, to be properly defined, the function k needs the singleton $\{x\}$: in other words,

[11] k is the only function that can be defined to obtain $\{x\}$, if we require a function to be total for REPLACEMENT to be consistently applied. In the event that the function is allowed to be partial, then two other possible definitions have to be taken into consideration: $f : \{x\} \to U$, which is clearly circular, and $f : \{y\} \to U$, which renders the existence of the singleton of one of the two elements of U depending on the existence of the singleton of the other —the argument being symmetric for $\{y\}$.

the existence of the set $\{x\}$ is a necessary condition for the definition of k. We tacitly assumed the existence of the set we wanted to prove the existence of. As a consequence, doubt must be cast even on REPLACEMENT, under the consideration of Black's (1952) scenario[12].

We end the present section by considering (4). Although the *Null Set Axiom*, stating the existence of the empty set, figures in some axiomatisations of ZFC, nonetheless the existence of the empty set is derivable in ZFC as a theorem. Furthermore, by EXTENSIONALITY, the empty set is unique. As a consequence, the empty set is not a u-object. Until now, we have characterised the notion of 'u-object' only at an informal level, and we have not formally shown how the lack of uniqueness is necessary for an element of a (un-ordered) cluster of objects to be considered a u-object. Nevertheless, it should be informally clear how uniqueness implies some sort of antisymmetry. In the same way, it should be now informally clear that no ordering can be properly induced on a set whose only elements are u-objects without that ordering being symmetric. This consideration (which for now needs to remain informal, being its formalisation only possible when given the results we aim at showing in this chapter) can be found in Black (1952) as the description of the geometry of the universe the two objects inhabit. A complete description of those objects would then need to embed such symmetry, in a way that renders somehow necessary the presence of such a symmetric space whenever u-object are assumed to be spatio-temporally located.

[12]The failure of REPLACEMENT entirely depends on the definition of k as an endomorphism (i.e. the image $i(U)$ of U under k is $i(U) \subseteq U$), which is necessary if we do not allow elements outside Black's Universe to be used in defining its internal structure. In case we thought this condition to be too restrictive and we wanted to allow external collections of objects, the failing condition for REPLACEMENT would be the following: the function f would need to be defined as having a domain containing only u-objects. This is the reason why we cannot apply the present line of reasoning to (4). In fact, the function to define in (4) is clearly not an endomorphism, and the Zermelo-Fraenkel's definition of the empty set has as a consequence the fact that the empty set is not a u-object.

We then consider the fourth way of defining singletons within ZFC. Given the empty set $\varnothing$, we can obtain its power set $\wp(\varnothing)$, whose existence is guaranteed by the *Power Set Axiom*[13]. Let $f : \wp(\varnothing) \to U$ be defined as follows: for all $w \in \wp(\varnothing)$, $f(w) = x$. By REPLACEMENT, the image of $\wp(\varnothing)$ under f, namely $\{x\}$, exists and it is a set. As in considering (3), no problem seems to emerge from this line of reasoning —still, exactly as before, a subtle crack is undermining our possibility to define such a function from the start. By definition, in fact, for any function $f : X \to Y$ (X and Y being sets) there exists (1) an image of the function defined as $f[W] = \{y \in Y : y = f(x)$ for some $x \in W\}$, W being any subset of X, and (2) an inverse image of the function, defined as $f_{-1}(Z) = \{x \in X : f(x) \in Z\}$, Z being any subset of Y. Now, the image h of a function $f : X \to Y$ is itself a function whose domain is the power set of X and whose codomain is the power set of Y ($h : \wp(X) \to \wp(Y)$), and the inverse image h^{-1} of f is also a function, whose domain is the power set of Y and whose codomain is the power set of X ($h^{-1} : \wp(Y) \to \wp(X)$). Although no restriction has to be made on the power set of the empty set[14], some restrictions have to be made on the power set of U, since the existence of its non-trivial subsets $\{x\}$ and $\{y\}$ is internally legitimate only by the possibility of applying either PAIRING, or SUBSET SELECTION, or alternatively REPLACEMENT with the function being an endomorphism (every function defined as not being an endomorphism would fall into the problem we are now dealing with of the existence of its image), and we have shown that the unrestricted validity of these axioms is no longer guaranteed when a set containing u-objects is considered. It follows that no proper image or inverse image can be defined for f, hence f cannot be consistently thought of as a function.

[13]As we have said above, the empty set is not a u-object. It follows from this consideration that we are excused from considering any restriction on the output of the power set operation on it. Hence, the power set of the empty set is defined as usual: $\wp(\varnothing) = \{\varnothing\}$.

[14]See above.

We have shown that all the approaches for the construction of singletons fail when considering Black's Universe (or any other collection containing only u-objects). We can conclude that the proper set theoretical formalisation of Black's (1952) scenario is one in which no singleton for any element of the Universe can be consistently defined, or shown to exist. In analysing the problems of identity and individuality in Quantum Mechanics (henceforth: QM), and the alleged threat posed to PII by isolated physical systems composed of identical particles, French & Krause (2006) connects the problem of the discernibility of two non identical objects by means of a property with the possibility of constructing the singleton of one of the objects in the relevant theory of collections used to represent the scenario under consideration. In what follows, we will see how the impossibility of defining singletons will influence our results about naming and discerning objects.

1.5 Names

We now turn to the question: can we name the objects in Black's Universe? Black (1952) is obscurely clear on this point. On one hand, he allows a stranger to enter his universe, and once entered, to name the spheres Castor and Pollux. At the same time, Black (1952) argues that, as names, 'Castor' and 'Pollux' cannot be consistently used to talk about the spheres in his scenario. Furthermore, we cannot use the expressions 'the one', and 'the other', we cannot use constants to refer to the spheres, and the like. We are given the possibility of naming the objects, but this possibility becomes pointless. We cannot use the names, Black (1952) argues, because we have no means to single out any of his two spheres. But then, how can we name them? Black (1952) connects the possibility of naming the spheres to the possibility of a stranger (an "observer") entering his otherwise closed universe. In other words, a third object entering the universe is a necessary condition for the possibility to name the two original inhabitants. Let us consider again, then, Black's scenario, *before* and *after* the asymptotical entrance of the stranger. If naming is possible only in the 'after universe', something has changed when the stranger has entered. Black (1952) maintains that the change induced by the stranger is so important that the 'after universe' cannot be considered a counterargument to PII anymore. What has changed then? The spheres have suddenly become identifiable. What we have allowed to enter the universe is, in the words of Lowe (2003), an 'individuant'. Now, the spheres are located one to the right and one to the left of the stranger, and he can name them at will. What is it, then, to give a name to something?

In the last two hundreds years, numerous theories have been proposed to explain how exactly proper names (henceforth: names) succeed, within the use of language, in referring to the entities they are names of, and to find the necessary conditions for the reference to take place, and for a string of symbol to be a name. John Stuart Mill (1843) claimed the meaning of a name to be the object referred to by that name. Gottlob Frege (1892) famously introduced the difference between the sense (sinn) of a name and its reference (bedeutung), and argued that both sense and reference contribute to the meaning of a name. Bertrand Russell (1910) maintained that ordinary names have the semantic role of definite descriptions, and his theory was challenged a few years later by Ludwig Wittgenstein in his *Tractatus Logico-Philosophicus* (1922). In his second groundbreaking work, Wittgenstein (1953) developed his theory of 'meaning as use' and of linguistic games. Rudolf Carnap (1947) and Alonzo Church (1951) identified the sense of a name to be an intention, defined as a function from possible worlds to extensions. Michael Devitt (1981) presented the hypothesis of causal chains linking names (as tokens) to the named entities, and Saul Kripke (1980) introduced the concept of 'rigid designator' as a name whose reference is fixed across possible worlds, and maintained the meaning of a name to be established by an initial baptism. John McDowell (1977) used Tarskian clauses to explain how names refer to objects, and Frank Jackson (1998) and David

Chalmers (2006) contributed to ‘two-dimensional semantics’, which provides “finer-grained semantic values than those available within standard possible world semantics, while using the same basic model-theoretic resources” (Schroeter, 2017). These are only some (even if the most influential) of the accounts developed until now within analytic philosophy of language. Sam Cumming (2016) points out that, with the solely exception of Josh Dever’s (1998) theory of names, all the aforementioned theories are committed in a certain degree to some sort of functionalism, according to which names determine functions to entities.

Sam Cumming (2016) reports an example given by Josh Dever, claiming it to be an instance of a use of language that could militate in favor of an anti-functionalist theory of names. The example is the following: hired by Scotland Yard, Sherlock Holmes says: “the murder was committed by two individuals, call them X and Y. First note that, since there is no sign of a struggle, both X and Y were known to the victim”. The question that need to be answered is whether ‘X’ and ‘Y’ are names. In fact they are not introduced by means of definite descriptions or ostensive acts, and this fact seems to threaten the *definiteness* of their reference. We are interested in answering the question whether there is a way to name the objects in Black’s Universe, and to refer to them by means of names without changing the original scenario. We assume then that ‘X’ and ‘Y’, in Dever’s (1998) case, are names: that is, that ‘X’ and ‘Y’ refer to some objects (in this case, the two murderers). The indefiniteness of their reference can be considered either ‘ontological’ or ‘epistemological’. In the case it is epistemological, then ‘X’ and ‘Y’ definitely refer to two unique human beings, without us being able to know whom they actually refer to. Is Dever’s (1998) scenario under this interpretation of ‘X’ and ‘Y’ similar to Black’s? Could we refer to Black’s spheres by using Dever’s names ‘X’ and ‘Y’? Our answer is negative. The impossibility in Black (1952) is that of isolating the objects inside the universe, not being them in principle identifiable. Were Dever’s case similar to Black's, the two spheres would have been in principle identifiable (as the human beings in Dever (1998) are). But once they are identifiable, Black's scenario is turned into what we have called the 'after universe'. Consider now the case of an 'ontological indefiniteness'. In this case 'X' and 'Y' do not uniquely refer to two individuals. Are they still names? Consider 'X'. Can 'X' be a suitable name and at the same time not be referring to a unique object? Our answer is negative. If Holmes is right, there are only two unique individuals who have murdered the man in front of him. Imagine Holmes saying: "now one of the two, say X, has to be seven feet tall". To make sense of this statement, we should have fixed a reference for X. In fact, Holmes' sentence makes sense only if the X he is talking about is the same X as before. Could it make sense otherwise? Even if we could be in the position of not knowing the exact reference of a name, an exact reference is required for some string of symbols to be a name.

The problem of the (im)possibility of naming the objects in Black's counterexample is presented along the lines of Black (1952) as follows (being B the philosopher himself, and A his alleged opponent):

> A: A minute ago, you were willing to allow that somebody might give your spheres different names. Will you let me suppose that some traveller has visited your monotonous "universe" and has named one sphere "Castor" and the other "Pollux"?
> B: All right —provided you don't try to use those names yourself.

In this passage, Black is not listing the rules of the argument, nor he is unjustifiably warning his opponent. He is hinting at a *de facto* impossibility, embedded in his setting: his objects cannot be named from inside the universe. They can be named from the outside, but no name can be used to talk about them. In his words:

> A: Consider one of the spheres, a, ...
> B: How can I, since there is no way of telling them apart? Which one do you want me to consider?

> *A*: This is very foolish. I mean either of the two spheres, leaving you to decide which one you wished to consider. If I were to say to you "Take any book off the shelf" it would be foolish on your part to reply "Which?"
> *B*: It's a poor analogy. I know how to take a book off a shelf, but I don't know how to identify one of two spheres supposed to be alone in space and so symmetrically placed with respect to each other that neither has any quality or character the other does not also have.
> *A*: All of which goes to show as I said before, the unverifiability of your supposition. Can't you imagine that one sphere has been designated as 'a'?
> *B*: I can imagine only what is logically possible. Now it is logically possible that somebody should enter the universe I have described, see one of the spheres on his left hand and proceed to call it 'a'. I can imagine that all right, if that's enough to satisfy you.
> *A*: Very well, now let me try to finish what I began to say about a ...
> *B*. I still can't let you, because you, in your present situation, have no right to talk about a. All I have conceded is that if something were to happen to introduce a change into my universe, so that an observer entered and could see the two spheres, one of them could then have a name. But this would be a different supposition from the one I wanted to consider. My spheres don't yet have names. If an observer were to enter the scene, he could perhaps put a red mark on one of the spheres. You might just as well say "By 'a' I mean the sphere which would be the first to be marked by a red mark if anyone were to arrive and were to proceed to make a red mark!" You might just as well ask me to consider the first daisy in my lawn that would be picked by a child, if a child were to come along and do the picking. This doesn't now distinguish any daisy from the others. You are just pretending to use a name.

The formalisation of Black's (1952) scenario we have attempted and the facts we have proved about it unveil the impossibility Black is pointing at, clarifying its place into the set theoretical framework we have drawn. The impossibility of naming the objects in the universe lies in the metaphysics of those objects: that no singleton can be defined in our environment is the reason of the impossibility of consistently naming the objects. Every theory of names assumes a connection between a name and the object it refers to. If we model this connections by having a fragment of ZFC as metatheory and we use standard models for first or second order logic, then the process of assigning constants to objects is entrusted to a function, the so called 'Interpretation'. We have shown that no function can be properly defined if singletons are not allowed as elements of the power set of its range, since it will have no image for some of its inputs. We can then state the following fact (a theorem): *that no object can in principle be named if it is not possible to define a set containing it as its only element*. And the fact that the objects in Black's scenario cannot be named lies at the ground of the formalisation of his Universe.

One last consideration has to be made about what is called *arbitrary reference*. Famously defended in Breckenridge & Magidor (2012), the thesis of *arbitrary reference* holds that "it is possible to fix the reference of an expression arbitrarily. When we do so, the expression receives its ordinary kind of semantic-value, though we do not and cannot know which value in particular it receives". A clear example of an arbitrary reference can be the one of a mathematical proof concerning natural numbers. It is not uncommon that this kind of proofs begin with the mathematician letting "n be a natural number", and then generalising his result to any natural number. Breckenridge & Magidor (2012) maintains that, in doing the initial stipulation in his proof, the mathematician is actually referring to one of the natural numbers, without knowing the number he is referring to —that is, the constant n is sent to just one element of $\mathbb{N}$ by the function that determines its referent, but the mathematician is in a state of epistemic ignorance about the actual output of the function[15]. It might be thought that Breckenridge

[15]The work of Wylie Breckenridge and Ofra Magidor is not the only attempt to find an

and Magidor's account can be used, in the case of Black's Universe, to arbitrarily talk about one (*just* one) of the two objects, in a way that avoids the problem of the symmetry of the Universe and gives one the possibility to actually say something about *one*, or *the other* object. If this is the case, then some change is needed either in how we construct the semantic for first and second order logic, or in the metatheory with which we interpret our formal languages. If we remain attached to the standard semantics for first order quantification and to the standard set theoretical metatheory of first and second order logic, then arbitrary reference is only possible when the domain of quantification of a certain model only contains individual objects. The example offered in Breckenridge & Magidor (2012) is easily dealt with in the formal context under consideration. A suitable model will have $\mathbb{N}$ as Universe, and the Interpretation $\mathcal{I}$ will assign a particular number to the constant n, without us being able to figure out which one. Unlike numbers, however, Black's objects are not in principle discernible, and they don't obey number's identity conditions. Here we are not dealing with some *a posteriori* impossibility, namely, the impossibility to come to know the exact output of a function —instead, we are dealing with an *a priori* impossibility, namely the impossibility to define a function at all. Our objects are in principle unnameable, and they are in principle unspeakable in the sense that nothing can be said of one of them in isolation, given the impossibility of internally defining its singleton - i.e. given the metaphysics it embeds.

answer to the question of what we are doing when we stipulate a constant being the name of an arbitrary object. Other answers have been proposed in the past years, among which the most interesting have been presented in Rescher (1957), Price (1962), Fine & Tennant (1983), and Shapiro (2004).

1.6 Relations and Properties

It remains to be shown why the defence of Leibniz's Principle from Black's counterexample claiming that there is a relation or a property distinguishing the two objects (from now on, we will call them the "relation argument" and the "property argument", respectively) in Black's scenario cannot be formalised as correct second order formulas when the metatheory is assumed to be a fragment of ZFC, and in what follows we will use the facts we have established so far in order to make our point clear. What does claiming a relation (or a property) existing between the two objects in Black's Universe amount to? The *relation argument* can be formulated in three ways:

(1) There is an object outside the Universe being related to only one of the two Black's objects, formally expressed as:

$$\exists w \exists r(r(w,x) \wedge \neg r(w,y)),$$

(2) There is an antisymmetric relation between the two objects in the Universe, formally expressed as:

$$\exists r(r(x,y) \wedge \neg r(y,x)),$$

(3) There is an object outside the Universe such that one of the two Black's objects is related to it while the other is not, formally expressed as:

$$\exists w \exists r(r(x,w) \wedge \neg r(y,w))^{16}.$$

In our formalisation we have pretended to be able to actually name Black's objects, which is in fact, as we have shown, impossible. Still, we are analysing the situation of what could and could not be said of them if, in some way,

[16] It is just for simplicity that we have chosen only one of the two possible ways to formalise (1), (2) and (3), since there is no difference in choosing one or the other.

they came up to be nameable after all. By definition, a n-ary relation is a set of n-tuples. In (1), (2) and (3) the relation r is a set of ordered tuples. In (1), the relation is a subset of $\overline{U} \times U$ ($\overline{U}$ being the complement of U): then, $r = \{(w, x)\}$. By definition of ordered tuple, $r = \{\{\{w\}, \{w, x\}\}\}$. But then, either the set $\{w, x\}$ is defined by taking the union $\{w\} \cup \{x\}$, which is clearly inconsistent (having shown that $\{x\}$ cannot be defined), or it is defined as the image of a function by means of REPLACEMENT. But then, there will be no counter-image for the element x under that function, which makes it not well-defined. Hence, (1) has to be abandoned. In (2), $r \subseteq U \times U$, and in particular $r = \{(x, y)\} = \{\{\{x\}, \{x, y\}\}\}$. Then, given the impossibility of the definition of $\{x\}$, also r cannot be defined, and (2) also has to be abandoned. Case (3) is similar, being $r \subseteq U \times \overline{U}$, and in particular $r = \{(x, w)\} = \{\{\{x\}, \{x, w\}\}\}$. We conclude that there is no possibility of defining a relation distinguishing the two spheres in Black's Universe as the *relation argument* would require, and that any second order formalisation of the *relation argument*, under the standard set theoretical metatheory of second order logic, is illegitimate.

The discussion of the *property argument* is a limiting case of the discussion of the *relation argument*: a property being defined as a unary relation, then, given p the property we want to define and being x the only object having that property (the case for y is similar), it follows that $p \subseteq U$, and in particular $p = \{x\}$, which cannot be defined in our framework. We conclude that also any second order formalisation of the *property argument* is illegitimate.

Chapter Two

2.1 Introduction

In Chapter 1 we have attempted a set theoretical formalisation of Black's Universe, and we have shown that a second order formulation of a 'discerning defence' of PII from Black's (1952) counterexample, when a fragment of ZFC represents the standard metatheory interpreting our second order language, is not available. In this Chapter, we will be interested in the nature of i-objects, intuitively defined by inducing an indiscernibility restriction on those objects, that we have called u-objects, that could figure in a Non Leibnizian Universe *à la* Black, without altering its symmetry. We will also be interested in the possibility of developing a suitable formal language to talk about collections of i-objects. The present Chapter will be divided in three sections. In the first section, we will show that any collection containing i-objects does not respect EXTENSIONALITY, and we will analyse the conclusions that can be drawn from this fact. In the second section, we will answer the question whether i-objects are *individuals*. Our answer to the individuality question, in conjunction with the results obtained in the first section, will enlighten the relation between collections of i-objects and ZFC-collections. Finally, in the third section, we will start developing the intuitive ground on which to construct a formal language to talk about collections of i-objects.

It is important to understand that what we have shown so far does not have a direct impact on the possibility to answer the question whether PII has to be considered a metaphysical truth or not. We have in fact shown that, under the assumption that the second order formulation of PII completely captures its ontological significance, and under the assumption that its usual interpretation is achieved by means of the standard models of a second order language when a fragment of ZFC is taken as a metatheory, a large number of the objections raised toward the counterexample presented in Black (1952) must be considered mistaken, for they rest on a misunderstanding of Black's (1952) setting. No conclusion can be drawn from our argument about the fate of PII. The possibility of interpreting the second order formulation of PII by means of a metatheory that could allow for a suitable formalisation of Black's (1952) scenario is still open. As we will show in the present chapter, such metatheory should be chosen as to solve the problem of quantification over nonindividuals. Furthermore, in Chapter 1 we have considered only one of the possible strategies to undermine Black's (1952) counterexample and secure PII from its alleged threat. It is thence possible that Black's counterexample could not be defended from other kinds of objections[17], and it is possible that in the end Leibniz's Principle will walk away a winner from the philosophical battle about its validity. Thence now, the battle still raging outside, all we can do is working towards a clarification.

[17]As we have already mentioned in Chapter 1, Della Rocca (2005) claims that, once accepted Black's counterexample as an effective argument against PII, there is no means to decide how many co-located spheres are there in any region of space of Black's Universe occupied by one of the two spheres: the conclusion being absurd, one must, according to Della Rocca, reject Black's counterexample. On the other hand, Hacking (1975) suggests, with respect to the counterexample to PII formulated by Kant, that PII can be defended by claiming there being not two objects, but only one, which is multilocated. This line of reasoning can be consistently applied to Black's counterexample, and an answer to these objections cannot be found in the argument presented in Chapter 1.

In Chapter 1 we have assumed that there is some sense in which a number n of u-objects can be consistently thought of as a collection, and the results we have obtained in considering the objections falling under the category of ‘Discerning Defences’ that have been raised towards Black’s counterexample to PII enable the conclusion that the number n of u-objects a collection of which can be considered without Leibnizian Identity to be implicitly assumed must be greater than 1. Clearly, this result can be generalised to i-objects. Although an empty collection can be defined of n i-objects when $n = 0$, two observations must be considered with regard to such collection. The first has it that the ontological meaning of the definition of a collection of no objects is questionable, since if we assume there being no object in a universe U and if we maintain a collection not to be an object in its own right, it is not clear how an empty collection can contribute to our ontology. The second consideration has it that, if we consider such an empty collection as an object in its own right (as it is common in pure set theory) and we define it as the collection containing no i-objects, it could still in principle contain individual objects, in which case it will not substantially contribute to our ontology, for it already includes such collections. It follows that it must be defined as a collection containing no objects whatsoever, and it will be shown that, defined in this way, it will turn out to be identical to the empty set, as defined within ZFC. Our ontology already containing all the objects postulated by ZFC, an empty collection so defined would not even minimally enlarge our domain of discourse. It follows that such a collection (1) does not substantially contribute to our ontology, and (2) it is not an i-object, for it can be uniquely defined, and it can be always singled out within any collection containing it as an element.

Under the assumption that whenever there are n i-objects, for $n > 1$, it is consistent to talk about the collection u having those objects as elements, in what follows will rise the questions *which kind of objects* i-objects are, and *which kind of collections* can be built upon them. In the following section we will lay the foundation for an answer to the second question, by contrasting collections of i-objects, henceforth called 'i-collections', and ZFC-collections, with respect to the *Axiom of Extensionality.*

2.2 The Problem of Extensionality

Within ZFC, the *Axiom of Extensionality* gives the conditions under which two sets are identical, defining the identity relation between sets in terms of the membership relation '$\in$' between sets and their elements. This is the reason why EXTENSIONALITY is so important for the characterisation of the concept of 'set'[18]. In fact, as McBride (2003) points out, were we introducing, by means of a definition, a concept under which some objects fall, we would need to provide two distinct criteria: (1) a *criterion of application*, able to distinguish between the objects to which that concept applies and the objects to which it does not, and (2) a *criterion of identity*, able to provide identity conditions for the objects to which the concept applies, giving the possibility of understanding when two objects are the same object, or when they differ from one another. The role of EXTENSIONALITY as a criterion of identity for sets will not be called into question, and the conclusion will be derived that any collection of objects disobeying EXTENSIONALITY cannot be consistenly applied the notion of 'set'. Within ZFC, the *Axiom of Extensionality* is usually formulated as follows:

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x = y),$$

where the variables x, y and z range over sets. Informally, the axiom states that two sets are identical if and only if they have the same elements. In what follows we will show that i-collections do not satisfy EXTENSIONALITY. The argument goes as follows: let a be an i-collection containing only two

[18]For the remainder of this work, we will use 'set' and 'ZFC-collection' interchangeably.

i-objects, and let b be an i-collection containing three i-objects, and such that $a \prec b$, where the formula '$a \prec b$' intuitively means that a is a 'subcollection' of b —a given collection x being a subcollection of a collection y if and only if all elements of x are also elements of y[19]. We assume that our intuition about the difference between a and b is consistent, in the sense that it is legitimate to consider two collections of different cardinality as *different* from one another. By definition of 'subcollection' all elements of a are also elements of b. Let x be an element of b.[20] By assumption, the elements of a and b are pairwise indiscernible. It follows that x is indiscernible from any element of a, and this means that for any property p that can be defined such that the elements of a instantiate p, also x must instantiate it. Let p be the property of 'being a member of a'. By definition, every element of a instantiate p. It follows that also x instantiates it. We then conclude that two i-collections a and b can be consistently said to be different even if all the elements in a are also in b, and vice versa. Therefore, i-collections violate EXTENSIONALITY. This gives us the means to understand why a second order formalisation of Black's (1952) scenario is illegitimate, given a fragment of ZFC as metatheory. A standard model for a second order language allows for the standard first order quantification figuring in a second order formula by means of an interpretation function $\mathcal{I}$, which respond to the ZFC's definition of function, from the language to a domain of quantification which is defined, given a

[19]The notion of 'subcollection' will be formally defined in Chapter 3, once the language $\mathcal{L}^{\text{i}}$ for i-collections will be developed.

[20]It can be argued that any proof regarding i-collections which makes use of arbitrary *singular* reference to one of their elements is to be regarded as formally inadequate, given the difficulties pointed at in Chapter 1 with respect to the possibility of such reference. In this proof, we have chosen the language we consider the simplest and at the same time the most formal. It is unquestionable that the practice of logic has its own language, and even if the objects we are dealing with in the present work can require an adaptation of that language, our position with regard to this point is that a more formal and conservative language has to be preferred, when it can be shown to be paraphrasable by means of sentences not involving singular reference. In the case of our present proof, every sentence can be easily shown to be so paraphrasable.

fragment of ZFC as metatheory, as a ZFC-collection. We can generalise our result to the extent that no collection containing i-objects can be regarded as a ZFC-collection, for it violates EXTENSIONALITY. In fact, as soon as a collection a is allowed to contain some positive number $n > 1$ of i-objects, it is possible to consider another collection b of greater cardinality such that all the elements of a are also elements of b, and vice versa. This allows the conclusion that no standard model can account for i-objects as elements of its domain of quantification, if the chosen metatheory is a fragment of ZFC.

Given the present result, the question imposes itself whether and how identity conditions for collections of indiscernible objects can and have to be defined. We will begin by considering the sub-question whether identity conditions for i-collections have to be defined at all. The question admits two interpretations, and we will deal with them separately. Under the first interpretation, the question asks if any theoretical analysis of i-collections which does not involve considerations about their identity conditions can be considered complete, or at least satisfactory. Our answer is negative: as we have pointed out above, the specification of the identity conditions for objects pertaining to a given kind is of the outmost importance for the theoretical understanding of their nature. This is not supposed to mean that if we do not provide identity conditions in the definition of a certain kind of objects our definition is incomplete —for it could be the case that a kind of objects can find a place in our ontology for which it is meaningless to talk about identity conditions, since the objects falling into that kind are not the entities that could meaningfully figure at the left and at the right side of an equality symbol. In that case, once shown that the objects under discussion are reluctant to any application of the equality symbol, the theoretical analysis can be considered complete, at least with respect to the criterion of identity for such objects. But if this reluctance cannot be shown to obtain or if there is no compelling reason to consider it as a possibility, we believe that identity conditions play a major role in any complete and satisfactory definition. In particular, in the present case two considerations are worth being mentioned: (1) that it seems plausible and intuitively harmless to be in principle able to distinguish between two i-collections, or that two i-collections can be consistently assumed as different: for example, it seems intuitively plausible to distinguish between two different Black's Universes, one containing spheres and the other hemispheres, or alternatively, one containing two and the other three indiscernible objects. Furthermore, (2) given an intuitive meaning of 'collection', it seems unproblematic to single out *one* i-collection, as well as to admit, given an i-collection u, the singleton $\{u\}$[21]. Under the second interpretation, the question whether identity conditions for i-collections have to be defined at all asks if it is necessary to reduce the identity relation for i-collections to other more primitive relations. Again, the answer is negative: if the identity relation can be shown to be definable in terms of some other relations (as in the case of ZFC), such definition must be pursued, at least from a metatheoretical point of view —a theory with a lower number of primitive notions able to take into account notions of greater complexity is in general preferred over a theory able to take into account notions of the same grade of complexity but making use of a greater number of primitives (Schaffer, 2009). On the other hand, it is not new to the philosophical debate that there might exist some facts concerning objects, that have no possible reduction or definition: they should be considered as 'brute' facts, and it should be recognised that nothing can be said of informative and at the same time general about them (Markosian, 1998)[22]: that is, they can at most be listed.

[21]This fact suggests that any i-collection is an individual object, and therefore that any collection of i-collections can be considered as a ZFU-collection, with the i-collections as urelemente.

[22]Markosian (1998) suggests the possibility of 'brute' facts within the debate about which conditions have to be met for a mereological composition of objects to form a whole. We do not want to enter in the debate about mereological composition, even if a mereological discussion of how the concept of 'part' and 'sum' can be applied to indiscernible object, as well as a discussion of how it can be the case that a composition of indiscernible parts can constitute a discernible whole (if it can at all) would be of great interest.

In what follows, we will try to come up with a possible definition of identity for i-collections in terms of more primitive notions. We now answer the sub-question of how to define identity conditions for i-collections. We have two alternatives: either we work with a primitive notion of membership, that intuitively corresponds to the notion of membership as introduced in ZFC, or we work with a primitive notion of subcollection (which we have intuitively characterized above). Let us consider the first alternative: it could be argued that any first order language would require some adjustments in order to accommodate for meaningful formulas containing the membership relation '$\in$'. In fact, we have taken this relation to hold (1) between i-objects and i-collections (i.e. we have assumed that there is some sense in which some indiscernible objects can be said to be elements of some i-collections), and (2) between i-collections and collections of i-collections. With respect to (2), there is no difference in behaviour and in principle between the membership relation we would adopt as primitive and the one defined within ZFC. Instead, with respect to (1) we might be asked in which sense it can be consistently maintained that some indiscernible object is a member of some collection. In fact, given the results obtained in Chapter 1, it might be objected that an indiscernible object a can be said to be a member of a collection x if and only if a binary relation $\in$ can be defined such that $a \in x$, and we have seen that no such relation can be defined, for it would require the singleton $\{a\}$ to be definable in our language. In Chapter 3, we will discuss in detail how to deal with such problem. An answer to this objection is that we can employ as a primitive the relation of 'containment' '$\ni$', which will allow us to avoid any formal problem, since any formula containing the expression '$x \ni a$', for x some collection and a some indiscernible object, does not require any undefinable singleton to be defined —we have shown that it is consistent to define the singleton $\{x\}$ of any i-collection x. In this sense, any expression of the form '$a \in x$' can be seen as informally equivalent to the formally correct expression '$x \ni a$' (given the assumption that for any object and any collection, an object is a member of a given collection if and only if the given collection contains it), and consequently it can be used for the language of our proofs to be kept simple and near to the use of the current mathematics and logic. Still, it can be objected that no indiscernible object can be considered in isolation, and that if we consider a number n of indiscernible objects we are already considering the collections of such objects. Two answers can be given to this objection. The first one is that the expression of the conditions for containment do not need a constant to be attached to an indistinguishable object taken in isolation, since (1) there is no i-collection containing only one object, and (2) we can refer to a ZFC-collection containing the same number of elements and allow only symmetrical and permutation-invariant formulas in our language (the containment condition would then be expressed as a schema in our formal language). The second one maintains that it is possible to have as a primitive the relation of subcollection '$\prec$'. In the third section of Chapter 2, we will argue for the need of having the membership relation as primitive, and in Chapter 3 we will discuss in detail how to deal with the problems that such choice inevitably takes.

Our proposal for a principle of extensionality for i-collections is the following:

$$\forall x \forall y(((x \prec y \lor y \prec x) \land (\kappa(x) = \kappa(y))) \leftrightarrow x = y),$$

where the variables x and y range over i-collections, and, given an i-collection w, $\kappa(w)$ is the *cardinality* of w. From now on, we will call this principle I-EXTENSIONALITY. The subcollection relation is informally defined as follows: x is a subcollection of y ($x \prec y$) if and only if x is included in / is a part of / does not exceed y, and the concept of 'cardinality', which will be taken as a primitive, is informally defined as follows: for a given i-collection x, the cardinality $\kappa(x)$ of x is the number of elements x contains. Furthermore, cardinality facts will be taken as *brute facts*. It will be considered as a brute fact that any i-collection has a definite cardinality, and it will be considered as a brute fact which cardinality any collection is assigned, and that any i-collection is assigned one and only one cardinality. The advantage of having such concept as primitive is twofold: on the one hand, it assures that the identity of i-collections can be defined non-circularly (being the equality relation between cardinalities defined within ZFC), and it makes the objection of Della Rocca (2005), considered before, harmless. In fact, being a brute fact the number of objects in any i-collection, the argument of Della Rocca is no more compelling, for any collection is always in principle assigned one and only one cardinality, and collections of different cardinality are defined as different and they can always be in principle discerned.

2.3 Nonindividuality

Are i-objects *individuals*? With this question, we conclude our philosophical investigation on the metaphysics of indiscernible objects. In the remaining of this work, in fact, we will be primarily interested in examining the structure of collections containing i-objects and in defining a formal language to talk about them. How we will define the syntax and the semantics for such language will largely depend on our answer to the question of individuality. The width of this dependence will be straightened out in the following section.

In his long research on individuality E. J. Lowe has come up with two definitions of the notion of ‘individual objects’. Lowe (2003) maintains that (1) an individual object is something that can be consistently thought of as independent from any other individual object, and that (2) whenever an object is an individual, there is something that individuates it (Lowe names this something: an *individuant*). The notion of ‘independence’ is defined in terms of possible scenarios: an object is an individual if and only if it can be consistently thought of as the only inhabitant of an otherwise empty universe. This definition can be challenged by maintaining that some individuals cannot be thought of in a *strictu sensu* otherwise empty universe, for they must be located, if not in time, at least in space. If space is conceded individuality, however, Lowe's definition would return as result that space is the only individual substance —all the other objects being individuated only insofar as they occupy some bounded region of space. In Lowe's (2003) words: "However, this seems to presuppose that space itself has substantival status, in which case material spheres and other material bodies are not, after all, individual substances —rather, space itself is an individual substance and material bodies are non-substantial individuals which exist in virtue of the successive occupancy by matter of contiguous regions of space. (Thus, on this view, what we ordinarily think of as the continuous movement through space of an individual material sphere really just amounts to some matter's successively occupying the members of a continuous series of spherical regions of space.) But, in that case, we have not really addressed the question of what individuates individual substances by saying what individuates such things as material spheres, these turning out not to be individual substances after all. If space itself is the only individual substance, it would seem that it would again have to be self-individuating." An answer to this challenge consists in denying individuality to the space-time framework: if space-time points are not individuals, in fact, the alleged necessity of a space-time framework witnessing the existence of some individuals does not represent a problem for Lowe's definition. The debate over the status of space-time points and over the dependence of the existence of some objects on an underlying space-time framework in which they, if exist, must be located will not be considered here. An answer to these questions is far beyond the scope of the present work, and it would not help in answering the individuality question: Lowe (2016) provides, in fact, a characterisation of what is an individual which is independent from any assumption about the status of space-time, as well as it is independent from the possible dependence of some objects on an underlying spatio-temporal framework.

Lowe's (2003) definition of individuality can then be formulated as follows: something is an individual object if and only if its existence can be thought independently from any other individual, meaning that the universe in which its existence is maintained as possible does not contain other individuals apart from the object the individuality of which is considered. We then consider a general version of Black's (1952) counterexample to PII: a closed universe inhabited by two indistinguishable icosahedrons. Following Lowe's definition, we ask ourselves: is it consistent to think of the existence of one of the two indiscernible icosahedrons in an otherwise empty universe independently from the existence of any other object? We maintain that, if the question is meaningful, the answer is negative. The meaningfulness of the question depends on the possibility to uniquely individuate one of the two icosahedrons —that is, the one that we should or should not be able to think in isolation from everything else. In other words, the question is meaningful if we can find a reference for the expression 'one of the two icosahedrons'. This expression can be understood in two different ways. Under the first interpretation, the expression singles out a particular icosahedron, so to draw a clear distinction between the singled out icosahedron and the remaining one. Under the second, either no particular icosahedron is singled out or, if some icosahedron is, it is singled out in a way as to render the expression 'the remaining icosahedron' meaningless, or illegitimate. In other words, the second interpretation maintains that either the expression 'one of the two icosahedrons' is meaningful without any reference to one of the objects, or that, if a reference is required, there is a sense in which to be the referent of such expression is not something that can in principle distinguish the two icosahedrons. In Chapter 1 we have shown that no reference function can in principle be defined as to range over Black's Universe. It being a limiting case of the example we are considering, it follows that the question whether it is possible to consider the existence of one of the two icosahedrons independently from the existence of any other object is meaningless under the first reading of the expression 'one of the two icosahedrons'. For what concerns the second interpretation: we may ask how it can be possible for the relevant expression to have a meaning without referring to some object, as well as how the property of being the (unique) referent of an expression should be thought of as not being in principle usable in discerning the two icosahedrons. Anyway, let us assume it is possible for that expression to have a meaning without having a reference, or that it is possible to construct a reference function in a suitable way not to make any in principle distinction. Then, the question of whether it is possible to imagine one of the two icosahedrons as the only inhabitant of an otherwise empty universe has to be regarded as meaningful. Further, let us suppose we can in fact imagine one of the two icosahedrons as we are asked to. Now we have two universes, the first one containing two indiscernible icosahedrons, and the second one containing only one of them. Is it now legitimate to ask (1) *which icosahedron* is the one in the new universe? It seems that, under the assumption that the two original icosahedrons were indiscernible, and under the assumption that no distinction has been introduced when one of the two was deemed to be thought of in isolation, the solitary icosahedron in the new universe is still indiscernible from the two in the original universe. But then, both of them are the only icosahedron in the new universe. The conclusion follows that, if (1) is legitimate, then the objects we are talking about cannot be individuals, since whenever one from a positive number of them is thought of as existing in some universe, also the other must be thought of in the exact same universe. If, instead, it is claimed that no answer can be consistently found for (1), then the ground for stating the nonindividuality of such objects must be found somewhere else. It follows that, if the question whether one of n-many indiscernible objects in an otherwise empty universe can be consistently thought of as inhabiting in absolute isolation an otherwise empty universe is meaningful, its answer must be negative.

The existence of an individuant for an object is not an additional condition to the possibility of thinking *that* object in isolation: to think an object in isolation requires the in principle possibility to individuate *that* very object, and the other way around. In Black's (1952) scenario, the impenetrability of the Universe and its symmetry guarantee the impossibility of any internal individuant. Imagine we wanted to distinguish the two spheres by the fact that, being immersed in a space, they occupy different regions of space. If we don't want to grant individuality to space, we cannot claim any difference between the two regions of space. We need something else individuating everything: we need a third object. In Black (1952) the possibility is considered of a stranger entering the Universe, but the reader is warned of the fact that considering Black's Universe *after* the admission of the stranger is not equivalent to considering it *before* such admission. A stranger entering Black's Universe would result in an irredeemable change of the scenario. The stranger, in fact, would represent the only asymmetric element in an otherwise symmetric universe, and the postulate of symmetry in Black's scenario is indispensable. Black's Universe is ultimately inaccessible. It follows that the spheres cannot be identified by means of non-relational spatial properties. Such a possibility would in fact depend on the existence of a point of exact reference, a zero-point of some cartesian system of coordinates, which can be established only by a third object entering the universe. As soon as the stranger enters, however, the universe is not the same anymore.

The second definition of 'individuality' is stated in Lowe (2016), where an object is said to be an individual whenever it meets two conditions: (1) it "determinately counts as *one* entity[23]" and (2) it "has a determinate *identity*". What does it mean to 'count as *one* entity'? Lowe negatively characterises this notion by providing the conditions under which some entity may fail to count as *one*. In his view, this can happen in two different situations: when an entity is a *plurality*, and when an entity lacks countability. An example of a 'plurality' is, according to Lowe, 'the planets of the solar system'. Lowe maintains that 'the planets of the solar system' is an entity on

[23]Lowe (2016) uses the term 'entity' with the same meaning we use the term 'object': "I use 'entity' as the term of broadest possible scope in ontology: everything whatever that does or could exist qualify as an 'entity' in my sense".

its own right, and still it cannot be considered as *one*. As a second possible failure condition, Lowe offers the example of the water in his bathtub. According to Lowe, such entity cannot be consistently counted, for there is no meaningful question of the form 'how many water — ?'. If Lowe's notion of "oneness or unity, and hence of cardinality quite generally" (Lowe, 2016) does admit only *pluralities* and *number lacking objects* as possible infringements, we must conclude that the objects we are considering do respect the first condition of Lowe's definition of individuality. Still, there is room for doubt that 'oneness' can be assigned to our objects. This would require the definition of a third kind of infringement of Lowe's concept. If the concept of 'oneness' or 'unity' is thought of as the concept of cardinality, in fact, it is not unproblematic to say that i-objects can count as *one*. It seems in fact that to count as *one* is equivalent to the possibility of being the element of a set of cardinality one. If the notion of cardinality Lowe is thinking of is (and we think it is) informally described as to indicate the number of objects in a given collection, then we must recognise that i-objects cannot count as *one*, since no singleton can be defined as containing i-objects. We would then conclude that i-objects contravene the first condition for individuality, and therefore they should be considered nonindividuals. Nonetheless, it might be argued that deriving such conclusion from some non *ab origine* contemplated failure condition is grounded on a misunderstanding of Lowe's conception of 'unity', given the assumption that Lowe's definition is accepted as correct. There being no internet connection in heaven (as far as we know), we must then consider our argument still incomplete. The present discussion should have made clear something that was hidden in the conclusions we provided in Chapter 1, namely that it is not unproblematic that, for two objects to count as *two*, each one of them has to count as *one*. This fact represents the departure of the concept of cardinality from the one of countability, which is necessary when i-objects enter the discourse. The second individuality condition is maintained to be infringed whenever, given two objects, "*there is no determinate fact of the matter as to which is which*" (Lowe, 2016). Under this interpretation, any two objects lack a determinate identity when no property or relation can be found to discern one another. Our analysis has shown that no property or relation is formally definable involving Black's spheres if a fragment of ZFC is used as metatheory. Still, it might be argued that to find such property or relation is in the first place a metaphysical task, and if no logical formalisation is allowed for it within the standard metatheory, this should not be a problem for metaphysics. We suspend our judgement on such statement, still holding that a complete metaphysical solution of the problem should at least hinting at a possible formalisation of the relevant claims (unless such formalisation is *shown* to be impossible). In the recent literature, Black's objects have been claimed to be *weakly discernible*, where two objects are weakly discernible whenever there exists a symmetric and irreflexive relation holding between them. As an example, Caulton & Butterfield (2012) claims that if we consider the relation of "being a mile away from — " then the two Black's spheres are related to one another, and no one is related to itself. The hypothesis of weak discernibility and the paradigmatic example given by Caulton & Butterfield (2012) is examined in Lowe (2016) and it is shown to be inapplicable to Black's objects. Furthermore, Lowe himself considers the example of a Black-like scenario: two electrons in an isolated physical system. Under the assumption that from the application of Pauli's Exclusion Principle we can maintain that one electron is spin-down while the other is spin-up, Lowe concludes that electrons are nonindividuals, being there no fact of the matter as to *which* is spin-down and *which* is spin-up. Overlooking on the impossibility of such derivation from the postulates of Quantum Mechanics, Lowe's analysis shows that nonindividuality does not entirely depend on absolute indiscernibility. Even if there was a property or a relation such that it would have been impossible for both the objects in Black's Universe to instantiate, there would still be no fact of the matter as to *which* would have and *which* would have not instantiate it. In conclusion, our answer to the identity question turned out to be negative: i-objects are nonindividuals.

2.4 Towards a Formal Language for i-collections.

In the first section of the present chapter we have shown that i-collections do not satisfy EXTENSIONALITY, and we provided them with a new criterion of identity, which we have called I-EXTENSIONALITY. It maintains that two i-collections are identical whenever (1) one is a subcollection of the other and (2) they have the same cardinality, which can be formulated as follows:

$$\forall x \forall y(((x \prec y \vee y \prec x) \wedge (\kappa(x) = \kappa(y))) \leftrightarrow x = y),$$

where the variables x and y range over i-collections, and given a collection w, $\kappa(w)$ is the cardinality of w. In Chapter 3, we will be interested in defining a formal language that allows us to talk about collections of indiscernible objects, and in describing suitable models for its interpretation. The development of a brand new full-developed theory of collection that could replace ZFC as a metatheory of a second order language is beyond the scope of this work. We will nonetheless present a first draft of our theory of collection, and we will test it in proving as theorems some important facts about the structure of i-collections.

In the previous section, we have shown that i-objects cannot be consistently thought of as *individuals*. As a result, two important consequences of this fact need to be addressed. The first has it that such objects cannot be the referents of individual constants, and that quantification by means of the standard (universal and particular) quantifiers becomes meaningless whenever i-objects are considered, if a fragment of ZFC is chosen as metatheory and a function $\mathcal{I}$ in the sense of ZFC is our interpretation. This follows from the fact that no i-object can be consistently thought of in isolation from all the other objects in a certain domain. As a consequence, no standard reference function can be defined as to have a *single* i-object as input or output, if it is thought of according to the ZFC's definition of function. This impossibility undermines not only the use of individual constants —it being impossible to assign them single witnesses, but also the use of bound variables, for the lack of any definable arbitrary reference. However, we have shown that collections of i-objects are individuals, and from this it follows that we can use the standard logical apparatus to refer to and quantify over them. The second consideration has it that, once i-objects have been shown to be nonindividuals, any formal definition in the standard apparatus of formal logic characterising i-collections in terms of their elements is impossible, since any definition of this kind would require the possibility of ZFC functions to have i-collections as possible ranges, which would in turn require the possibility of defining singletons within i-collections, which we have shown to be impossible. As a consequence, it is possible to talk about i-collections in a first order language. In the following lines we will present the four notions of 'type', 'membership', 'cardinality', and 'ZFC-collection', which we will consider as primitive.

We begin by considering the notion of 'type'. When we intuitively think about i-collections we might wonder whether the only difference between two i-collections can be stated in terms of cardinality, or if we should allow some other qualitative differences between two i-collections. In particular, we may want two i-collections containing elements which are indiscernible inside each collection, but of different kind. We do not see any valid reason for not allowing the existence of nonindividual objects of different kinds. No contradiction seems to follow from considering two otherwise empty universes respectively inhabited by n-many indiscernible tetrahedrons and n-many indiscernible icosahedrons, and no argument from reason seems able to cast doubt on the sensible intuition that this state of affairs represents a genuine possibility. Given the enormous philosophical history of the word 'kind', we will use the word 'types' to denote this different kinds of objects. Collections of nonindividual objects will then differ from one another on the grounds of the number and the types of objects they contain. We may ask if there is a way of giving conditions for the difference between 'types'. We answer negatively: any definition of this kind would in fact be grounded on the possibility of distinguishing some of the objects of one type from some of the objects of another (all the objects of the same type being *in principle* indiscernible), and the only ground of such difference is their belonging to different types, for otherwise there would be no difference at all to talk about. In other words, any conditions for the difference of types must be stated in terms of the differences between nonindividual objects, since the only fact able to ground a difference between types is a difference between the objects pertaining to them. Being nonindividuals, however, i-objects cannot singularly figure at the left and right side of an equality symbol. The only meaningful difference between nonindividual objects can be stated in terms of a difference in their type. We thence must take as primitive to have different types of i-objects. Intuitively, the notion of 'type' is exhausted by the following three principles: (1) if a plurality of i-objects belongs to a single type, then they are mutually indiscernible, (2) no irreducible plurality of i-objects belongs to more than one type, and (3) any irreducible plurality of i-objects belongs to some type.

Given the intuitive formulation of the notion of 'type', let x be some collection containing nonindividual objects. Clearly, there is no intuitive limit to the number of i-objects that x can contain, and it is possible for x to contain different types of objects. This suggests the following definition:

For x some i-collection:

- x is HOMOGENEOUS if and only if x contains only i-objects, and all the elements of x are of the same type;
- x is QUASI-HOMOGENEOUS if and only if x contains only i-objects, and not all the elements of x are of the same type;
- x is NON-HOMOGENEOUS if and only if x contains both i-objects and

individual objects.

We further define the class $\mathcal{H}$ of homogeneous i-collections, the class $\mathcal{Q}$ of quasi-homogeneous i-collections, and the class $\mathcal{N}$ of non-homogeneous i-collections. In the following discussion we will be mostly focused on *homogeneous* i-collections. A formal definition of these classes will only be possible once the language for i-collections will be drawn. At the end of Chapter 3, we aim to give some results pertaining homogeneous i-collections, and the investigation of the differences between $\mathcal{H}$, $\mathcal{Q}$, and $\mathcal{N}$, as well as the study of their interactions will be left for further development.

The second notion we have chosen to consider as primitive is 'membership', characterised as a binary relation holding between elements (i-objects, individual objects, i-collections and ZFC-collections) and collections (respectively: i-collections, collections of i-collections, and ZFC-collections). We denote the membership relation by means of the usual symbol '$\in$', for the intuition behind the meaning of any sentence maintaining some i-objects being elements of some given i-collection is not different from the intuition behind the meaning of any sentence maintaining some individual object being an element of some set. There is no difference in principle between our notion of 'membership' and the intuitive notion of membership underlying ZFC, and our notion coincides with that of ZFC whenever the elements taken into consideration are individuals. The reasons to take the membership relation as a primitive might require further discussion. One of the most important relations in play when it comes to i-collections and to the structure of $\mathcal{H}$ is the relation of subcollection. The membership relation alone does not allow any relevant intuition about i-collections, and nothing more is rendered available to our expression if we extend the application of the notion of 'type' to collections, except for the triviality that if some non empty homogeneous i-collection is given in our domain of type i, then there are some i-objects of type i which we can consistently consider as members of that very i-collection. As it should be clear at this point, the question about which i-objects of type

i are members of some collection x of type i is meaningless. There is a sense in which all of them are members of x, but this must not be confused with an answer to the question of what is the cardinality of x. The intuition behind this distinction can be stated as follows: whenever x is an homogeneous non empty i-collection of cardinality $\kappa(x) = n$ and type i, both the following sentences must be true of x: (1) only n-many i-objects of type i are members of x, and (2) every i-object of type i is a member of x.[24] The meaning of (1) is quantitative, while the meaning of (2) is qualitative. It follows that the truth of (2) cannot ground any cardinality fact about x. Now, while the relation of subcollection can be defined by means of the relation of membership, the converse does not hold. Furthermore, to substitute the notion of 'membership' with the notion of 'subcollection' in the list of primitive notions largely restricts our expressive power and the possibility to prove interesting facts about the structure of i-collections. As an example, it will be shown that, given an i-collection x of cardinality n and type i, then for all $m < n \in \mathbb{N}$ there exists a unique collection y of type i such that y is a subcollection of x (formally: $y \prec x$). As we will see, this fact is of the outmost importance to understand the structure of $\mathcal{H}$. However, the proof of this statement is impossible without a definition of the notion of 'subcollection', since we can in no way prove that two collections x and y such that both $x \prec z$ and $y \prec z$ for some collection z are the same collection, if we cannot prove that either $x \prec y$ or $y \prec x$. We must thence choose in favour of the 'membership' relation as fundamental, for otherwise either we need to state any fact about the

[24]The sentences (1) and (2) are the intuitive non well formed renderings of the only well-formed sentence 'x has cardinality n'. Both (1) and (2), in fact, require quantification over nonindividuals to be stated. If the quantification over first order objects is thought of in the standard way, then, (1) and (2) must be recognised as non well formed sentences. We believe, however, (1) and (2) not to be meaningless. We have used non well formed sentences many times to render the intuitive significance of our formal definitions and reasoning, and this is one of those cases. We believe that the incorrect intuitive renderings of our results could help in understanding, or, at least, in having an intuition about what is going on in these pages.

structure of $\mathcal{H}$ as an axiom, or we are forced to be silent about the validity of such facts.

We then consider the notion of 'cardinality'. Within ZFC, the notion of cardinality is introduced by stating that two sets have the same cardinality if and only if there is a one-to-one mapping between them, under the assumption of the possibility of assigning a cardinal number to any set. Cardinal numbers, in their turn, are defined either by means of equivalence classes, in presence of the Axiom of Regularity, or as ordinals, in presence of the Axiom of Choice. The idea behind the second definition is that, the Axiom of Choice being equivalent to the Well Ordering Theorem, the consequence follows that for any collection $x \in V$ there exists an ordinal $\lambda \in Ord$ such that an isomorphism can be defined between x and λ. As we will see, there is no sense in which an i-collection can be well ordered. Furthermore, it is not clear what sense can be given to the Axiom of Regularity in the present context. Still, it seems intuitive to talk about the size of i-collections, as well as it seems intuitive to compare them, given a difference in the number of objects they contain. However, the intuition that establishing the cardinality of a collection is not different from counting its elements has to be abandoned as meaningless. What is it then, for a collection in our theory, to have a cardinality? Analysing the concept of 'quasi-cardinal' in Dalla Chiara & Toraldo di Francia's Quaset Theory (Dalla Chiara & Toraldo di Francia, 1993), which we will confront with our theory of collections in the last section of Chapter 3, French & Krause (2006) claims that, being a quasi-cardinal a cardinal, then, "[...] if the concept of cardinal is defined in the usual way, that is, as a particular ordinal, then a quasi-cardinal is an ordinal and hence it may appear that there is no sense in saying that a quaset may have a cardinal but not an ordinal, as we have suggested above. This should be understood in the sense that the concept of cardinal (or of counting) should be taken differently from standard mathematics (Zermelo-Fraenkel set theory), perhaps in the sense of Frege-Russell". According to McMichael (1982), a Frege-Russell cardinal is an equivalence class under the relation of equinumerosity. Insofar as our i-collections can be assigned a cardinal without being isomorphic to any ordinal, and insofar as we recognise this similarity between our concept of 'cardinal' and Dalla Chiara & Toraldo di Francia's (1993) concept of 'quasi-cardinal', we might be asked if, like a 'quasi-cardinal', our cardinal can be said to be a Frege-Russell cardinal. Our answer is negative. In fact, Dalla Chiara & Toraldo di Francia's interpretation of their concept of 'quasi-cardinal' as a Frege-Russell cardinal is possible only because quasets are collections of individual objects. Being an equivalence class under the relation of equinumerosity, in fact, a Frege-Russell cardinal can be defined only on the ground of the possibility of the consistent definition of a one-to-one mapping between two collections. However, if the concept of function is as defined in ZFC, we conclude that no mapping is possible between two i-collections. The philosophical difference between our concept of cardinality and both Frege-Russell and Zermelo-Fraenkel's ones is that a cardinal in our sense is not something that follows from the definition of a given collection. *Au contraire*, the assignment of a cardinal to an i-collection is part of its definition. As a result, cardinals will be assigned, in our theory, brutally, and not on the basis of other operations performed on collections. Elements of i-collections, in fact, cannot be counted, as well as they cannot be ordered or separated —nonetheless, i-collections have a precise number of elements, and this number is unique. For these reasons, we have decided to consider the notion of 'cardinality' as fundamental. In contrast to what happens inside ZFC's universe, however, there is a further distinction that needs to be drawn whenever we want to talk about the *number* of elements in some i-collection. This distinction points towards two notions of 'cardinality', which we call 'partial cardinality' and 'total cardinality'. When considering collections of nonindividual objects, it is intuitively meaningful to broaden the concept of '(total) cardinality' to the concept of 'partial cardinality', which answers the question, given some i-collection x, of how many elements of a given type i are collected in x. Formally, for any type i, any i-collection is by definition assigned a partial cardinality by means of a function $\kappa_i : \mathcal{U} \to Card$ (where $\mathcal{U}$ is the universe of collections we will consider through our investigation), which corresponds to the number of objects of type i in x. Clearly, the notion of '(total) cardinality' is definable in terms of the notion of 'partial cardinality', in the sense that the (total) cardinality of an i-collection can be defined as the sum of all the partial cardinalities for any given type of objects. Therefore, the concept of 'partial cardinality' will be the one we will consider as primitive. For what concerns the derived concept of 'total cardinality' (henceforth: 'cardinality'), any i-collection is by definition assigned a cardinality by means of a function $\kappa : \mathcal{U} \to Card$. Intuitively, the concept of cardinality is characterised as indicating the size of a collection. By definition, to any collection is assigned only one cardinal, and when some cardinal n is assigned to some collection x, we say that x has cardinality n (formally: $\kappa(x) = n$), or that x has n elements.

Finally, we will consider as a primitive concept the one of 'ZFC-collection'. Although this notion will not play an important role when i-collections are considered, still we need it as a primitive in order to be able to state the axioms of ZFC, with a suitable restriction, within the axioms of our theory of collections.

Chapter Three

3.1 The Language

In the following lines we will define the first order formal language $\mathcal{L}^{\mathrm{i}}$, with the intent of developing a suitable formal framework to talk about i-collections and ZFC-collections. In the last section of Chapter 2, we have defined the class $\mathcal{H}$ of homogeneous collections of nonindividual objects, the class $\mathcal{Q}$ of quasi-homogeneous collections of nonindividual objects, and the class $\mathcal{N}$ of non-homogeneous collections, containing both individual and nonindividual objects. We have used, and we will continue using, the symbol 'V' to name the set theoretical universe of ZFC. It will be shown that $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{N}$ and V are not disjoint, for all of them contain the empty set as an element.

The alphabet of the first order language $\mathcal{L}^{\mathrm{i}}$ consists of:

- denumerably infinitely many individual variables: $x, y, z, ...$ ranging over all the collections in $\mathcal{U}$, where $\mathcal{U}$ contains all the i-collections, all the ZFC-collections, all the collections of i-collections, all the collections of collections of i-collections, and so on;
- denumerably infinitely many individual constants: $a, b, c, ...$ interpreted as names for collections in $\mathcal{U}$;

- finitely many[25] nonindividual constants: $\tau_1, \tau_2, ..., \tau_n$ (for some $n \in \mathbb{N}$). These terms are assigned some ZFC-collections in a figurehead domain of quantification by the interpretation function $\mathcal{I}$, and their informal meaning is that of the indefinite description: "something of type i" (for some finite $i \leq n \in \mathbb{N}$);

- the logical constant '$=$' for equality, interpreted as usual;

- a unary predicate 'ς' interpreted as follows: for any collection x, the expression '$\varsigma(x)$' is informally interpreted as: "x is a ZFC-collection";

- the binary predicate '$\in$' of membership, holding between (1) i-objects and i-collections, (2) i-collections and i-collections, (3) ZFC-collections and i-collections, (4) ZFC-collections and ZFC-collections, and (5) i-collections and collections of i-collections, collections of i-collections and collections of collections of i-collections, and so on;[26]

[25]The restriction on the number of nonindividual constants in the alphabet has been made to respect the constraints on the lenght of formulas within a non-infinitary first order logic. Allowing infinitely many nonindividual constants would have turned the majority of our definitions into non well formed formulas (as we shall see in the third section of the present chapter). The possibility of adopting an infinitary logic to accommodate for infinitely many constants and the problems that might arise from such compromise will not be discussed here.

[26]To consider the possibility that i-collections can be elements of i-collections might raise the question about the well-foundedness of i-collections (see Aczel (1988), Barwise & Moss (1991), and Moss(2018)). In the fifth section of the present chapter we will not present a postulate of well-foundedness. Our theory of collections is still under development, and what we are presenting here is a first draft, a blueprint of it that might require a wide revision before being considered completed. At this point of its development, we do not have a clear intuition about the possible shortcomings that allowing non-wellfounded collections in our universe might yield. Within the present work we will be mostly focused on homogeneous i-collections, for which the problem of well-foundedness does not arise. We will then avoid to impose a constraint of well-foundedness on our collections, at least at this stage of development of our theory. Any discussion on the possibility of accepting such constraint will be left as further developments.

- finitely many functional symbols: κ_1, κ_2, ..., κ_n, interpreted as functions $\kappa_i : \mathcal{U} \to Card$. For any input class x, the cardinal number $\kappa_i(x)$ is informally interpreted as the number of i-objects of type i in x;

- the functional symbol 'κ' of cardinality, interpreted as a function $\kappa : \mathcal{U} \to Card$ assigning a cardinal number to any collection in $\mathcal{U}$;

- finitely many functional symbols: π_1, π_2, ..., π_n, interpreted as functions $\pi_i : \mathcal{U} \to \mathcal{U}$. For any input class x, the object $\pi_i(x)$ is informally interpreted as the biggest homogeneous i-collection of i-objects of type i whose cardinality does not exceed the number of i-objects of type i in x. It will be shown that, given a collection $x \in \mathcal{U}$, $\pi_i(x)$ is the biggest homogeneous subcollection y of x of type i;

- logical connectives: $\neg$, $\wedge$, $\vee$, $\to$;

- quantifiers: $\forall$, $\exists$;

- auxiliary symbols: (,).

The terms of the language $\mathcal{L}^{\mathsf{i}}$ are (1) the individual variables, (2) the individual constants, (3) the nonindividual constants, (4) the expressions of the form '$\kappa(\alpha)$', '$\pi_i(\alpha)$' (for some $i \leq n \in \mathbb{N}$), and '$\kappa_i(\alpha)$' (for some $i \leq n \in \mathbb{N}$), where α can stand for an individual variable or for an individual constant. An atomic formula is an expression of $\mathcal{L}^{\mathsf{i}}$ consisting of a n-ary predicate followed by n terms. The atomic formulas of $\mathcal{L}^{\mathsf{i}}$ are:

$$\varsigma(x), \quad x = y, \quad \tau_i \in x, \quad x \in y,$$

and formulas of $\mathcal{L}^{\mathsf{i}}$ are defined by recursion from atomic formulas φ, ψ, by means of the classical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \to \psi$, and quantifiers: $\forall x\varphi$, $\exists x\varphi$. Given a formula $\varphi(v_1, ..., v_n)$, all the variables occurring in φ are among $v_1, ..., v_n$. A sentence of $\mathcal{L}^{\mathsf{i}}$ is a formula with no free variables.

Within ZFC, given some elements $a, b \in V$, the collection containing a and b is usually indicated by means of the sequence of symbols: $\{a, b\}$. Furthermore, given a formula φ in the language of ZFC and some ZFC-collection c, it is possible to define a collection d as follows: $d = \{x \in c : \varphi(x)\}$. We will maintain this convention, for the differences between ZFC-collections and i-collections will be embedded in the formula φ used to define the collections (the language of ZFC being a fragment of the language of our theory of collections), and for the same notation is already in use in other non-standard approaches to set theory (i.e. hyperset theory and multiset theory), giving evidence that such notation is weak enough to be employed in defining objects outside the universe of ZFC. Collections of indiscernible objects will then be defined by means of formulas of $\mathcal{L}^{\mathrm{i}}$ in the usual manner. As we have already mentioned, any term of the form τ_i, for some $i \leq n \in \mathbb{N}$, is informally interpreted as the indefinite description: "something of type i"[27]. In this sense, these terms can be used to describe the internal structure of i-collections. It is important to understand that given some i-collection x, the only relevant elementary facts pertaining to x are: the number of objects it contains (its cardinality), and the kind and type of objects it contains.[28] Within the definition of x by means of a formula, the kind and the type of its elements are given by the chosen formula, and, it being a brute fact, its cardinality is just assigned to x depending on the features of the chosen model. Whenever some i-collection needed to be defined by indirect reference to its elements, we will allow the following construction $x = \{\tau_i, \tau_i, ...\tau_j, \tau_j...\}$ to indicate an i-collection x containing i-objects of types i and j, the number of which is indicated by the number of occurrences of the relevant symbols in the sequence. Although a similar construction is allowed in multiset theory —making use of repeating elements in the definition of a collection, the present construction differs from

[27]The meaning and the function of these symbols within $\mathcal{L}^{\mathrm{i}}$ will be clarified in the next sections, where a semantic approach will be sketched in order to make sense of reference to nonindividuals.

[28]For the present purposes, we consider two kinds of objects: individual and nonindividual objects.

the one of multisets, in that when a multiset is described by means of n-many repetitions of some constant α the interpretation is that it contains the *same* element n-many times, and this has not to be confused with the present notation, since in the present framework the possible multiple occurrence of some symbol τ_i in the definition of a collection cannot be interpreted as the multiple presence of the *same object* within a given collection (for otherwise that object would be an individual).

3.2 The problem of Indirect Reference

Two points need to be addressed before defining the intended models for the first order language $\mathcal{L}^{\mathrm{i}}$. The first point concerns the composition of the class $\mathcal{H}$ of homogeneous i-collections. In the last section of Chapter 2, we have defined the notion of 'homogeneity' as follows: a given i-collection x is *homogeneous* if and only if it only contains indiscernible objects, and all the objects it contains are of the same type. This informal definition can now be stated formally:

$$\mathcal{H} := \{x : \forall y(y \notin x) \wedge \forall i((\tau_i \in x) \to \forall j(\tau_j \in x \leftrightarrow j = i))\},$$

where the variables x and y range over the entire universe $\mathcal{U}$, and the formula '$\forall i((\tau_i \in x) \to \forall j(\tau_j \in x \leftrightarrow j = i))$' is an abbreviation of the well-formed formula:

$$((\tau_1 \in x) \to (\tau_2, ..., \tau_n \notin x)) \wedge ((\tau_2 \in x) \to (\tau_1, \tau_3, ..., \tau_n \notin x)) \wedge ...$$
$$... \wedge ((\tau_n \in x) \to (\tau_1, ..., \tau_{n-1} \notin x)).$$

Clearly, the empty set is an element of $\mathcal{H}$. From the formal definition of the class $\mathcal{H}$ of homogeneous i-collections, and from the assumption that there are only finitely many types of i-objects, it follows that it is possible to define a sequence of collections: $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n$ as follows:

$$\text{for all } i \leq n \in \mathbb{N}\text{: } \mathcal{T}_i := \{x \in \mathcal{H} : \tau_i \in x\}.^{29}$$

Clearly, $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n$ are subcollections of $\mathcal{H}$, and they are mutually disjoint. Furthermore, together with the empty set, they exhaust $\mathcal{H}$. In fact, by definition of $\mathcal{H}$, given an homogeneous i-collection x, either $x = \varnothing$ or $\tau_1 \in x$ or $\tau_2 \in x$ or ... or $\tau_n \in x$: that is, either $x = \varnothing$ or $x \in \mathcal{T}_1$ or $x \in \mathcal{T}_2$ or ... or $x \in \mathcal{T}_n$. It follows that the class of homogeneous i-collection can be thought of as the union of the classes :

$$\mathcal{H} = \{\varnothing\} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup ... \cup \mathcal{T}_n,$$

under the assumption that n is the number of types of i-objects allowed in our domain of quantification. In constructing a model for $\mathcal{L}^{\mathrm{i}}$, we will make use of the following fact: that for any $i \leq n, m \in \mathbb{N}$ there exists only one homogeneous i-collection x such that $x \in \mathcal{T}_i$ and $\kappa(x) = m$. This fact, which is of the outmost importance to understand the structure of the class $\mathcal{H}$ of homogeneous collections, will be shown as a theorem at the end of the chapter.

The second point concerns the nature of i-objects, and the nature of any formal language that can be consistently used to talk about them. In Chapter 2 we have shown that, under the assumption that the definition of individuality provided by Lowe (2016) is considered correct, i-objects are *nonindividuals*. This conclusion undermines the possibility to individuate such objects: it is not possible to single out any i-object, and as a consequence it is not possible to say anything meaningful concerning *one* i-object, that has not already been said of any other of them. As a consequence, no i-object can be consistently defined as to be the output of any ZFC-function, for it is meaningless to ask of two i-objects of the same type if they are the same object.

[29]The assumption of the finiteness of the number of types of i-objects is equivalent to the assumption that, given a model $\Im$ for the language $\mathcal{L}^{\mathrm{i}}$ there exists a finite natural number $n \in \mathbb{N}$ such that, for any $m > n$ there is no collection $x \in \mathcal{U}$ such that $\tau_m \in x$. We remind the reader that the notion of 'type' is taken as primitive.

It follows that no i-object can be assigned to an individual constant, and that no individual variable can be thought of as ranging over nonindividual objects. The problem that we need to solve to be able to talk about nonindividuals is thence twofold: on one side, we need to provide symbols which are neither individual constants nor variables, and on the other side we need to find a way to interpret them. In an attempt to solve the syntactic part of the problem, we have introduced in the alphabet of the language $\mathcal{L}^{\mathrm{i}}$ the symbols $\tau_1, \tau_2, ..., \tau_n$ as nonindividual constants, and we have informally stipulated the meaning of these symbols to be, respectively 'something of type 1', 'something of type 2', ..., 'something of type n', where the use of the word 'something' is such that it does not admit as meaningful any question of the form: "which one —?". These symbols are taken to be indefinite descriptions, and still no arbitrary reference can serve as their interpretation, since no i-object can be in principle referred to by any imaginable function, as defined within ZFC. The problem of how to assign objects to these symbols will then be solved by referring to them indirectly. We will construct a model with two different domains of quantification. The first domain will be the class $\mathcal{U}^{+}$, containing all i-objects, all the i-collections, all the ZFC-collections, all the collections of i-collections, all the collections of collections of i-collections, and so on.[30] Both individual variables and individual constants, as well as all the predicates of any degrees having collections as their argument will be assigned objects in $\mathcal{U} \subseteq \mathcal{U}^{+}$. The second domain of quantification, $V^{\mathfrak{I}}$, will be used as a figurehead domain, and the nonindividual constants $\tau_1, \tau_2, ..., \tau_n$ will be assigned individual objects in $V^{\mathfrak{I}}$. The truth conditions for statements of the form '$\tau_m \in x$' (for some number $m \leq n \in \mathbb{N}$ and some $x \in \mathcal{U}$) will be defined with respect to collections in $\mathcal{H}$, since it will be shown that any i-collection in $\mathcal{U}$ can be defined as the union of homogeneous i-collections and ZFU-collections.

[30]The difference between the classes $\mathcal{U}$ and $\mathcal{U}^{+}$ is that $\mathcal{U}$ only contains collections, while $\mathcal{U}^{+}$ contains all the collections in $\mathcal{U}$ plus the i-objects, according to the structure of the Model.

3.3 Models

Let $\approx_\kappa$ be a binary relation on V defined as follows: for all $x, y \in V$: $x \approx_\kappa y$ iff $|x| = |y|$. Informally, the relation $\approx_\kappa$ obtains between the ZFC-collections x and y when they have the same cardinality. Clearly, $\approx_\kappa$ is an equivalence relation, for it is reflexive, symmetric and transitive.

A model $\Im$ for the language $\mathcal{L}^{\mathrm{i}}$ is a n-tuple: $\Im = \langle\, \mathcal{U}^{+},\ V^{\Im},\ \mathcal{I},\ f\ \rangle$, where:

- $\mathcal{U}^{+}$ is the first domain of quantification, and it is defined as the class containing all the i-objects, all the i-collections, all the ZFC-collections, all the collections of i-collections, all the collections of collections of i-collections, and so on;

- $V^{\Im}$ is the second 'figurehead' domain of quantification. It is defined as the disjoint union of n-many ZFC's universes indexed by the natural numbers $\mathbb{N}$, where n is the number of admitted types of i-objects:

$$V^{\Im} \;\equiv\; V_1 \uplus V_2 \uplus \ldots \uplus V_n;$$

- $f : \mathcal{H} \setminus \varnothing \to V^{\Im}$ is a function assigning to any i-collection $x \in \mathcal{H} \setminus \varnothing$ a finite subclass $F_i \subseteq V_i$ (for some $i \in \mathbb{N}$) of the equivalence class of collections $y \in V_i$ under the equivalence relation $\approx_\kappa$ such that $|y| = \kappa(x)$. We impose two restrictions on f:

  (1) for any two collections $x, y \in \mathcal{H} \setminus \varnothing$ and any two subclasses $V_i, V_j \subseteq V^{\Im}$: if x and y contain objects of the same type and $f(x) \subseteq V_i$ and $f(y) \subseteq V_j$ then $i = j$;

  (2) for any two collections $x, y \in \mathcal{H} \setminus \varnothing$ containing objects of the same type: if $\kappa(x) \leq \kappa(y)$ then $\bigcup f(x) \subseteq \bigcup f(y)$, and if $\kappa(y) \leq \kappa(x)$ then $\bigcup f(y) \subseteq \bigcup f(x)$;

- $\mathcal{I} : \mathcal{L}^{\mathfrak{i}} \to \mathcal{U}^{+} \uplus V^{\Im}$ is the interpretation function, defined as usual. $\mathcal{I}$ assigns individual objects in $\mathcal{U}^{+}$ to individual constants and variables, and it assigns elements in $V^{\Im}$ to the symbols $\tau_1, \tau_2, ..., \tau_n$ as follows: given a term τ_i for some $i \in \mathbb{N}$, $\mathcal{I}(\tau_i)$ is an arbitrarily chosen object in the class: $\bigcap \{F_i : F_i = f(x) \text{ for some } x \in \mathcal{H}\}$.

As usual, individual variables and constants are interpreted within the real domain of quantification $\mathcal{U}^{+}$. Nonindividual constants are interpreted as objects in a defined restriction of the figurehead domain, $V^{\Im}$, which is meant to allow indirect reference to nonindividual objects. The nonindividual objects in $\mathcal{U}^{+}$ do not figure as outputs of the interpretation function $\mathcal{I}$, and this fact reflects the impossibility we have pointed out of a direct reference to nonindividuals. The indirect reference is obtained by interpreting nonindividual objects as if they were individuals (as objects in a restriction of $V^{\Im}$ that mirrors the structure of $\mathcal{H}$), and the truth conditions for sentences of the form '$\tau_i \in x$', for some $i \leq n \in \mathbb{N}$ and some $x \in \mathcal{H}$ are given, as a consequence, as if the objects said to be elements of x were individuals. The indefiniteness of the informal interpretation of such terms is given by allowing the formal interpretation to be possible only within a figurehead domain of quantification. Given a sentence of the form '$\tau_i \in x$', for some $i \leq n \in \mathbb{N}$ and some $x \in \mathcal{H}$, its truth conditions are given respecting the fact that, having taken the concept of 'type' as primitive and having informally defined the non empty collections in the class $\mathcal{H}$ as collections of objects of the same type, the possibility follows of considering such collections as typed, and to recognise that given some collection of a certain type, it is necessary that some objects of that type belong to it. Therefore, we define the truth condition of the sentence '$\tau_i \in x$', for some $i \in \mathbb{N}$ as follows:

$$\Im \models (\tau_i \in x) \text{ if and only if } \mathcal{I}(\tau_i) \in^{i} \bigcup f(x),$$

where $\Im$ is a model for $\mathcal{L}^{\mathfrak{i}}$, and '$\in^{i}$' is the membership relation within the i-th copy of V in $V^{\Im}$. All other truth conditions for statements of any other form are defined as usual. Furthermore, $\mathcal{I}$ interprets the functional symbols $\kappa_1, \kappa_2, ..., \kappa_n$ as functions $\kappa_i : \mathcal{U} \to Card$, assigning a cardinal number to any collection corresponding to the number of elements of a given type i in that collection, and $\mathcal{I}$ interprets the functional symbol κ as a function $\kappa : \mathcal{U} \to Card$, assigning a cardinal number to any collection. The cardinality of ZFC-collections is assigned as usual, by defining the relevant isomorphism with an ordinal. For collections of i-collections, the cardinality is assigned as it is usually assigned in Zermelo-Fraenkel's Set Theory with Urelemente (ZFU). The cardinality of i-collections is assigned on the basis of the assigment of their partial cardinality. Finally, $\mathcal{I}$ interprets the functional symbols $\pi_1, \pi_2, ..., \pi_n$ as functions $\pi_i : \mathcal{U} \to \mathcal{U}$ assigning homogeneous i-collections of a given cardinality to any i-collection in $\mathcal{U}$, under the following condition: given a collection $x \in \mathcal{U}$ and a number $i \leq n \in \mathbb{N}$, $\pi_i(x)$ is the homogeneous collection containing objects of type i and whose cardinality corresponds to the number of objects of type i in x.

3.4 Subcollections, Unions and Intersections

With the help of the machinery presented in the last three sections, we can define some non primitive notions and objects that will be important in investigating the structure of the universe $\mathcal{U}^+$ of collections and nonindividual objects. In what follows, we will define the cardinality functions, the notion '$\prec$' of subcollection, the objects $\pi_1, \pi_2, ..., \pi_n$, and the operations of union and intersection for i-collections (respectively: '$\sqcup$' and '$\sqcap$'). In the next section, we will submit a non complete list of principles that can be regarded as a first attempt towards an axiomatization of a theory of i-collections.

We begin by defining the (total) cardinality function $\kappa : \mathcal{U} \to Card$ as follows:

$$\text{for all } x \in \mathcal{U}\text{: } \kappa(x) = |c(x)| + \kappa_1(x) + \kappa_2(x) + ... + \kappa_n(x),$$

where the symbols '$|\cdot|$' refer to the ZFC's notion of cardinality and, for any collection x, the collection $c(x)$ is defined as $c(x) = \{y : y \in x\}$. Clearly, for any ZFC-collection x, $\kappa(x) = |x|$. For any $i \leq n \in \mathbb{N}$ and any collection $x \in \mathcal{U}$, we will interpret the output of the function $\kappa_i(x)$ as a measure of how many i-objects of type i are collected in x as elements.

We continue by defining the notion of 'subcollection', which is thought of as a binary relation holding between collections, and it is formally represented by the symbol '$\prec$'. Given two collections a and b, we say that a is a subcollection of b, formally written as: '$a \prec b$' if and only if a is a part / does not exceed / is wholly contained in b. The meaning of the term 'wholly' in our informal characterisation can be formally translated by saying that, when some collection a is a subcollection of a collection b, the cardinality of a is not greater than the cardinality of b. It follows that the relation $\prec$ is a partial order (i.e. it is reflexive, antisymmetric and transitive). At first sight, the subcollection relation might resemble the subset relation $\subseteq$ as defined within ZFC. The intuition behind both definitions is the same: both relations characterise pairs of collections such that the first element of the pair is wholly contained in the second element. It turns out that these two relations differ in some respects, and in what follows we will discuss the relevant dissimilarities between them. Our formal definition for the relation of subcollection is introduced as an abbreviation into $\mathcal{L}^{\mathfrak{i}}$ as follows:

$$\begin{aligned} x \prec y \text{ abbreviates:} \quad & (\forall z(z \in x \rightarrow z \in y) \wedge \\ & (\tau_1 \in x \rightarrow (\tau_1 \in y \wedge \kappa_1(y) \geq \kappa_1(x)) \wedge \\ & (\tau_2 \in x \rightarrow (\tau_2 \in y \wedge \kappa_2(y) \geq \kappa_2(x)) \wedge \\ & \;\vdots \\ & (\tau_n \in x \rightarrow (\tau_n \in y \wedge \kappa_n(y) \geq \kappa_n(x))), \end{aligned}$$

where the variables x, y and z range over collections in $\mathcal{U}$. It is possible to prove as theorems three facts about subcollections, namely that (1) any collection x is a subcollection of itself, that (2) the notion of 'subcollection' coincides with the notion of 'subset' within the universe of ZFC, and that (3) whenever a collection x is a subcollection of a collection y, the cardinality of x is not greater than the cardinality of y.

Theorem 1: For all $x \in \mathcal{U}$: $x \prec x$.

Proof: Let x be some collection in $\mathcal{U}$. Clearly, for all $y \in \mathcal{U}$, if $y \in x$ then $y \in x$. Furthermore, let $i \leq n \in \mathbb{N}$ be some natural number such that $\tau_i \in x$. Clearly, $\tau_i \in x$. By definition of partial cardinality, $\kappa_i(x)$ is unique, hence $\kappa_i(x) \leq \kappa_i(x)$. It follows that $x \prec x$.

Theorem 2: For all $x, y \in V$: $x \prec y$ iff $x \subseteq y$.

Proof: For the left to right direction, let x, y be two collections in V such that $x \prec y$. By definition, for all z, if $z \in x$ then $z \in y$. Therefore, $x \subseteq y$. For the left to right direction, let x, y be two collections in V such that $x \subseteq y$. By definition, for all z, if $z \in x$ then $z \in y$. Furthermore, x and y being ZFC-collections, for all $i \in \mathbb{N}$: $\tau_i \notin x$. It follows that $x \prec y$.

Theorem 3: For all $x, y \in \mathcal{U}$: $x \prec y$ implies $\kappa(x) \leq \kappa(y)$.

Proof: Let x, y be some collections in $\mathcal{U}$, and assume $x \prec y$. By definition of (total) cardinality, the cardinality of x is defined as the sum of all the partial cardinalities of x, in turn summed to the cardinality of the collection (if any) containing all the individual objects in x. The same holds for y. Let z be an individual object such that $z \in x$. By definition of subcollection, $z \in y$. It follows that the cardinality of the collection containing all the individual objects in x is not greater than the cardinality of the collection containing all the individual objects in y. Now, let i be some natural number $i \leq n \in \mathbb{N}$ (n being the number of allowed types of i-objects in $\mathcal{U}^+$), and assume $\tau_i \in x$. By definition of subcollection, $\tau_i \in y$ and the partial cardinality $\kappa_i(x)$ is not greater than the partial cardinality $\kappa_i(y)$. We conclude that the (total) cardinality of x is not greater than the (total) cardinality of y: $\kappa(x) \leq \kappa(y)$.

We look now at the definition of the objects $\pi_1, \pi_2, ..., \pi_n \in \mathcal{U}$. Given some collection $x \in \mathcal{U}$, we define the object $\pi_i(x)$, for some $i \leq n \in \mathbb{N}$, as the unique collection respecting the following condition:

$$(\pi_i(x) \in \mathcal{H}) \wedge (\tau_i \in \pi_i(x) \leftrightarrow \tau_i \in x) \wedge (\kappa(\pi_i(x)) = \kappa_i(x)).$$

Informally, given some $i \leq n \in \mathbb{N}$ and some $x \in \mathcal{U}$, the collection $\pi_i(x)$ is defined as the only homogeneous collection such that $\pi_i(x)$ has as members something of type i in the number of the partial cardinality of x with respect to i: that is, $\pi_i(x)$ contains as many objects of type i as x. The notion of 'as many as' cannot be defined in terms of the notion of *one-to-one correspondence* when applied to i-collections, since no function f can be defined between $\pi_i(x)$ and x, if we want to maintain the standard set theoretical definition of a function within ZFC. It follows that, given two i-collections a and b, we characterise the meaning of the sentence 'a has as many objects as b' as saying that the cardinal number associated by the cardinality function κ to a happens to be the same as the cardinal number associated by κ to b —we remind here that the value of the cardinality of homogeneous collections has been taken as a brute fact. In order to prove the existence of the object $\pi_i(x)$ for any collection $x \in \mathcal{U}$ and any natural number $i \leq n \in \mathbb{N}$ (n being the number of allowed types), we need to rely on two of the principles that will be presented in the next section. The first has it that there exists, for any natural number $n < 1$ or $n > 1$ and any type i of i-objects, an homogeneous collection $x \in \mathcal{H}$ containing n i-objects of type i, and the second states that any collection $x \in \mathcal{U}$ can be defined as the union of a number n of homogeneous i-collections and ZFC-collections. The second principle is needed to prevent the possibility that the cardinality of $\pi_i(x)$ is $\kappa(\pi_i(x)) = 1$. In fact, this principle forces the fact that there is no type k of i-objects such that x contains only *one* object of type k. For the present purpose, it is sufficient to present these two principles informally, since it is easy to see how the existence of $\pi_i(x)$ follows from them. The proof of the uniqueness of $\pi_i(x)$ follows from I-EXTENSIONALITY, which we will regard as an axiom. We now prove that, for any collection x, $\pi_i(x)$ is a subcollection of x.

Theorem 4: For all $x \in \mathcal{U}$ and all $i \leq n \in \mathbb{N}$: $\pi_i(x) \prec x$.

Proof: Let x be some collection in $\mathcal{U}$, and consider the collection $\pi_i(x)$, for some $i \leq n \in \mathbb{N}$. Being $\pi_i(x)$ an homogeneous collection, then by definition no individual object is a member of $\pi_i(x)$. It follows that the first conjunct of the definition of subcollection is vacuously true. Now, by first order logic, either (1) $\tau_i \in x$ or (2) $\tau_i \notin x$. Consider (1): then $\tau_i \in \pi_i(x)$ and $\kappa(\pi_i(x)) = \kappa_i(x) \leq \kappa(x)$. Hence, $\pi_i(x) \prec x$. Consider (2): then $\pi_i(x) = \varnothing$ and by definition $\varnothing \prec x$.

Theorem 5: For all $x \in \mathcal{U}$ and all $i \leq n \in \mathbb{N}$: $\pi_i(x)$ is unique.

Proof: Let x be some collection in $\mathcal{U}$, and consider the collections $y = \pi_i(x)$ and $z = \pi_i(x)$, for some $i \leq n \in \mathbb{N}$. We want to show that $y = z$. By I-EXTENSIONALITY, it amounts to show that (1) $y \prec z$ or $z \prec y$, and (2) $\kappa(y) = \kappa(z)$. We begin with (2): by definition, $\kappa(y) = \kappa_i(x)$ and $\kappa(z) = \kappa_i(x)$. By uniqueness of $\kappa_i(x)$, it follows that $\kappa(y) = \kappa(z)$. Now we consider (1): by first order logic, either (1.1) $\tau_i \in x$ or (1.2) $\tau_i \notin x$. Consider (1.1): then $\tau_i \in y$ and $\tau_i \in z$ and, y and z being by definition homogeneous, it follows that $y \prec z$. Consider (1.2): then $y = \varnothing$ and $z = \varnothing$, and hence $y \prec z$. We can conclude that $y = z$, and hence that $\pi_i(x)$ is unique.

In what follows, we will define the operations of union $\sqcup$ and intersection $\sqcap$ for collections in $\mathcal{U}$. We begin by defining the union $x \sqcup y$ of two collections $x, y \in \mathcal{U}$ as the unique collection that respects the following conditions:

0. $\forall z((z \in x \vee z \in y) \leftrightarrow z \in x \sqcup y)$

1. $\forall u, v((u = \pi_1(x) \wedge v = \pi_1(y)) \rightarrow (((u = \pi_1(x \sqcup y)) \leftrightarrow (\kappa(u) \geq \kappa(v))) \wedge ((v = \pi_1(x \sqcup y)) \leftrightarrow (\kappa(v) \geq \kappa(u)))))$

2. $\forall u, v((u = \pi_2(x) \wedge v = \pi_2(y)) \rightarrow (((u = \pi_2(x \sqcup y)) \leftrightarrow (\kappa(u) \geq \kappa(v))) \wedge ((v = \pi_2(x \sqcup y)) \leftrightarrow (\kappa(v) \geq \kappa(u)))))$

$\vdots$

n. $\forall u, v((u = \pi_n(x) \wedge v = \pi_n(y)) \rightarrow (((u = \pi_n(x \sqcup y)) \leftrightarrow (\kappa(u) \geq \kappa(v))) \wedge ((v = \pi_n(x \sqcup y)) \leftrightarrow (\kappa(v) \geq \kappa(u)))))$.

The definition of the operation of union between two collections $x, y \in \mathcal{U}$ is meant as a generalization of the operation of union as defined within ZFC. The union of two collections is thought of as the collection containing all the objects that were members of either one or the other of the collections that were taken to be united. The conditions 1 - n had to be added to force the partial cardinality with respect to some type i of the union of two collections to be always equal to the partial cardinality with respect to i of the collection having the greater number of objects of type i figuring in the union. Given two collections $x, y \in \mathcal{U}$, the existence of their union $x \sqcup y$ will be taken as a postulate of our system. The uniqueness of such a union will be proved in what follows:

Theorem 6: For all $x, y \in \mathcal{U} : x \sqcup y$ is unique.

Proof: Let x, y be some collections in $\mathcal{U}$, and let $u, v \in \mathcal{U}$ be such that $u = x \sqcup y$ and $v = x \sqcup y$. We want to show that $u = v$, which amounts to show, by I-EXTENSIONALITY, that (1) $\kappa(u) = \kappa(v)$ and that (2) $u \prec v$ or $v \prec u$. We start by considering (1). Assume for *reductio* that $\kappa(x) \neq \kappa(y)$. Then either (1.1) $\kappa(u) > \kappa(v)$ or (1.2) $\kappa(v) > \kappa(u)$. Consider (1.1): by definition, either (1.1.1) there exists some $w \in \mathcal{U}$ such that $w \in u$ and $w \notin v$, or (1.1.2) there exists some natural number $n \in \mathbb{N}$ such that $\kappa(\pi_n(u)) > \kappa(\pi_n(v))$. Consider (1.1.1): By definition, w being a member of u, either $w \in x$ or $w \in y$. In both cases, $w \in v$, by definition of union: *contradiction*. Consider (1.1.2): by definition, either $\pi_n(u) = \pi_n(x)$ or $\pi_n(u) = \pi_n(y)$. If $\pi_n(u) = \pi_n(x)$, then $\kappa(\pi_n(x)) \geq \kappa(\pi_n(y))$. By definition of union, $\pi_n(v) = \pi_n(x)$. It follows that $\pi_n(u) = \pi_n(v)$, hence $\kappa(\pi_n(u)) = \kappa(\pi_n(v))$: *contradiction*. If, instead, $\pi_n(u) = \pi_n(y)$, then $\kappa(\pi_n(y)) \geq \kappa(\pi_n(x))$. By definition of union, $\pi_n(v) = \pi_n(y)$. It follows that $\pi_n(u) = \pi_n(v)$, hence $\kappa(\pi_n(u)) = \kappa(\pi_n(v))$: *contradiction*. Case (1.2) is similar. We conclude that $\kappa(u) = \kappa(v)$. Now consider (2): consider an arbitrary w such that $w \in u$. By definition, w being a member of u, either $w \in x$ or $w \in y$. In both cases, $w \in v$, by definition of union: the first conjunct of the definition of subcollection holds. Now, assume $\tau_n \in u$, for some $n \in \mathbb{N}$. Then, $\pi_n(u) \neq \varnothing$. It follows that $\pi_n(x) \neq \varnothing$ or $\pi_n(y) \neq \varnothing$. By definition of union, $\pi_n(v) \neq \varnothing$: hence, $\tau_n \in v$. By definition, either (2.1) $\kappa(\pi_n(x)) \geq \kappa(\pi_n(y))$, or (2.2) $\kappa(\pi_n(y)) \geq \kappa(\pi_n(x))$. Consider (2.1): then $\pi_n(x) = \pi_n(u)$ and $\pi_n(x) = \pi_n(v)$, which implies that $\pi_n(u) = \pi_n(v)$. It follows that $\kappa(\pi_n(v)) \geq \kappa(\pi_n(u))$. Case (2.2) is similar. Hence: $u \prec v$. We conclude that $u = v$, which amounts to say that $x \sqcup y$ is unique.

Now, we prove as theorems that (1) the notion of union $\sqcup$ coincides with the notion of ZFC's union within the universe of ZFC, that (2) whenever a collection u is the union of two collections x and y, then both x and y are subcollections of u, that (3) for any collection $x \in \mathcal{U}$, the union of x with itself is x, and that (4) for any collection $x \in \mathcal{U}$, the union of x with the empty set is x.

Theorem 7: For all $x, y \in V$: $x \sqcup y = x \cup y$.

Proof: Let x, y be two collections in V. By definition, x and y contain only individual objects, i.e. there is no natural number $n \in \mathbb{N}$ such that $\tau_n \in x$ or $\tau_n \in y$. It follows that the definition of $x \sqcup y$ becomes the following: $x \sqcup y$ is the unique collection satisfying the following condition: $\forall z\ ((z \in x \vee z \in y) \leftrightarrow z \in x \sqcup y)$. By definition, $x \cup y = \{z : z \in x \vee z \in y\}$. We conclude that $x \sqcup y = x \cup y$.

Theorem 8: For all $x, y \in \mathcal{U}$: $x \prec x \sqcup y$ and $y \prec x \sqcup y$.

Proof: Let x, y be some collections in $\mathcal{U}$. We want to show that (1) $x \prec x \sqcup y$ and (2) $y \prec x \sqcup y$. We start by proving (1): by definition of union, the following condition: $\forall z((z \in x \vee z \in y) \leftrightarrow z \in x \sqcup y)$ holds, which in turn implies that also: $\forall z(z \in x \rightarrow z \in x \sqcup y)$ holds. Hence, the first conjunct of the definition of subcollection holds. Now, let $i \in \mathbb{N}$ be some natural number such that $\tau_i \in x$. By definition, $\pi_i(x) \neq \varnothing$, from which it follows that $\pi_i(x \sqcup y) \neq \varnothing$. Hence: $\tau_i \in x \sqcup y$. We are left with showing that $\kappa_i(x \sqcup y) \geq \kappa_i(x)$. By definition of π_i, $\kappa_i(x) = \kappa(\pi_i(x))$ and $\kappa_i(x \sqcup y) = \kappa(\pi_i(x \sqcup y))$. By definition of union, either (1.1) $\pi_i(x \sqcup y) = \pi_i(x)$, or (1.2) $\pi_i(x \sqcup y) = \pi_i(y)$. Consider (1.1): then clearly $\kappa(\pi_i(x \sqcup y)) \geq \kappa(\pi_i(x))$. Consider (1.2): then by definition $\kappa(\pi_i(y)) \geq \kappa(\pi_i(x))$, from which it follows that $\kappa(\pi_i(x \sqcup y)) \geq \kappa(\pi_i(x))$. We conclude that $x \prec x \sqcup y$. The proof of (2) is similar. We conclude that $y \prec x \sqcup y$.

Theorem 9: For all $x \in \mathcal{U}$: $x \sqcup x = x$.

Proof: Let x be some collection in $\mathcal{U}$. We want to show that $x \sqcup x = x$. By I-EXTENSIONALITY, this amounts to show that (1) $x \sqcup x \prec x$ or

$x \prec x \sqcup x$, and that (2) $\kappa(x \sqcup x) = \kappa(x)$. By *Theorem 8*, $x \prec x \sqcup x$. Consider (2): by definition, $\kappa(x) = \kappa(\zeta) + \kappa_1(x) + \kappa_2(x) + ... + \kappa_n(x)$ and $\kappa(x \sqcup x) = \kappa(\xi) + \kappa_1(x \sqcup x) + \kappa_2(x \sqcup x) + ... + \kappa_n(x \sqcup x)$, where $\zeta = \{u : u \in x\}$ and $\xi = \{u : u \in x \sqcup x\}$. By definition of union, $\xi = \{u : u \in x\}$. Hence: $\zeta = \xi$, from which it follows that $\kappa(\zeta) = \kappa(\xi)$. By definition of union, for all $i \leq n \in \mathbb{N}$: $\pi_i(x \sqcup x) = \pi_i(x)$. It follows that for all $i \leq n \in \mathbb{N}$: $\kappa_i(x \sqcup x) = \kappa_i(x)$. We conclude that $x \sqcup x = x$.

Theorem 10: For all $x \in \mathcal{U}$: $x \sqcup \varnothing = x$.

Proof: Let x be some collection in $\mathcal{U}$. We want to show that $x \sqcup \varnothing = x$. By I-EXTENSIONALITY, this amounts to show that (1) $x \sqcup \varnothing \prec x$ or $x \prec x \sqcup \varnothing$, and (2) $\kappa(x \sqcup \varnothing) = \kappa(x)$. By *Theorem 8*, $x \prec x \sqcup \varnothing$. Consider (2): by definition of κ, $\kappa(x) = \kappa(\zeta) + \kappa_1(x) + \kappa_2(x) + ... + \kappa_n(x)$ and $\kappa(x \sqcup \varnothing) = \kappa(\xi) + \kappa_1(x \sqcup \varnothing) + \kappa_2(x \sqcup \varnothing) + ... + \kappa_n(x \sqcup \varnothing)$, where $\zeta = \{u : u \in x\}$ and $\xi = \{u : u \in x \sqcup \varnothing\}$. By definition of empty set[31], $\xi = \{u : u \in x\}$. Hence: $\zeta = \xi$, from which it follows that $\kappa(\zeta) = \kappa(\xi)$. Furthermore, for all $i \leq n \in \mathbb{N}$: $\pi_i(\varnothing) = \varnothing$. It follows that for all $i \leq n \in \mathbb{N}$: $\kappa(\pi_i(x)) \geq \kappa(\pi_i(\varnothing))$. Hence, for all $i \leq n \in \mathbb{N}$: $\pi_i(x \sqcup \varnothing) = \pi_i(x)$, which implies that $\kappa_i(x \sqcup \varnothing) = \kappa_i(x)$. We conclude that $x \sqcup \varnothing = x$.

[31]We will include the existence and the definition of the empty set in the list of axioms in the next section. The empty set $\varnothing$ will be defined as the unique collection such that: $\forall x(x \notin \varnothing) \wedge (\tau_1 \notin \varnothing) \wedge (\tau_2 \notin \varnothing) \wedge ... \wedge (\tau_n \notin \varnothing)$.

We conclude this section by defining the operation of intersection, and by proving some theorems about it. We define the intersection $x \sqcap y$ of two collections $x, y \in \mathcal{U}$ as the unique collection respecting the following conditions:

0. $\forall z((z \in x \wedge z \in y) \leftrightarrow z \in x \sqcap y)$
1. $\forall u, v((u = \pi_1(x) \wedge v = \pi_1(y)) \rightarrow (((v = \pi_1(x \sqcap y)) \leftrightarrow (\kappa(u) \geq \kappa(v))) \wedge ((u = \pi_1(x \sqcap y)) \leftrightarrow (\kappa(v) \geq \kappa(u)))))$
2. $\forall u, v((u = \pi_2(x) \wedge v = \pi_2(y)) \rightarrow (((v = \pi_2(x \sqcap y)) \leftrightarrow (\kappa(u) \geq \kappa(v))) \wedge ((u = \pi_2(x \sqcap y)) \leftrightarrow (\kappa(v) \geq \kappa(u)))))$

$\vdots$

n. $\forall u, v((u = \pi_n(x) \wedge v = \pi_n(y)) \rightarrow (((v = \pi_n(x \sqcap y)) \leftrightarrow (\kappa(u) \geq \kappa(v))) \wedge ((u = \pi_n(x \sqcap y)) \leftrightarrow (\kappa(v) \geq \kappa(u)))))$.

The definition of the operation of intersection for collections in $\mathcal{U}$ is meant as a generalisation of the operation of intersection as defined within ZFC, in the same way as the definition of union was. Informally, the intersection of two collections is thought of as the unique collection containing all and only the elements that were contained in both the input collections. Following the definition of union, the conditions 1 - n had to be stated to force the partial cardinality with respect to some type i of the intersection of two collections to be always equal to the partial cardinality with respect to i of the collection having the least number of objects of type i figuring in the intersection. In order to prove the existence and the uniqueness of the intersection $x \sqcap y$ of two collections $x, y \in \mathcal{U}$, we give an equivalent definition of the operation of intersection by means of the operation of union. First, we define a selection function $h : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ as follows:

For all $w, v \in \mathcal{H}$:

$$h(\langle w, v \rangle) = \begin{cases} w, & \text{if } w \prec v \text{ and } \kappa(w) < \kappa(v) \\ y, & \text{if } v \prec w \\ \varnothing, & \text{otherwise} \end{cases}$$

Intuitively, h takes as inputs pairs of homogeneous collections and, when they contain objects of the same type, h chooses the smaller one. Having defined h, we can give an alternative and equivalent definition of the intersection $x \sqcap y$ of two collections $x, y \in \mathcal{U}$ as follows:

$$x \sqcap y = \bigsqcup \{v : v \in \text{ran}(h) \text{ when } \text{dom}(h) = \mathcal{F}_x \times \mathcal{F}_y\} \sqcup \{u : u \in x \wedge u \in y\},$$

where the collections $\mathcal{F}_x$ and $\mathcal{F}_y$ (respectively, the collection of all biggest homogeneous subcollections of x and y) are defined as follows:

$$\mathcal{F}_x = \{w \in \mathcal{H} : w = \pi_i(x) \text{ for some } i \leq n \in \mathbb{N}\}, \text{ and}$$

$$\mathcal{F}_y = \{v \in \mathcal{H} : v = \pi_i(y) \text{ for some } i \leq n \in \mathbb{N}\}.$$

The intersection is then defined as a finite union of homogeneous collections and individual objects. In this way, the *existence* of the intersection $x \sqcap y$ of collections $x, y \in \mathcal{U}$ is a consequence of the existence of the union of any two collections, which will be stated as a postulate. Similarly, the *uniqueness* of the intersection is a consequence of the uniqueness of the union. In the following lines we prove as theorems that (1) the notion of intersection $\sqcap$ coincides with the notion of ZFC's intersection $\cap$ within the universe of ZFC, that (2) whenever a collection u is the intersection of two collections x and y, then u is a subcollection of both x and y, that (3) for any collection $x \in \mathcal{U}$, the intersection of x with itself is x, and that (4) for any collection $x \in \mathcal{U}$, the intersection of x with the empty set is empty.

Theorem 11: For all $x, y \in V$: $x \sqcap y = x \cap y$.

Proof: Let x, y be two collections in V. By definition, x and y contain only individual objects, i.e. there is no natural number $n \in \mathbb{N}$ such that $\tau_n \in x$ or $\tau_n \in y$. It follows that the definition of $x \sqcap y$ becomes the following: $x \sqcap y$ is the unique collection satisfying the following condition: $\forall z((z \in x \wedge z \in y) \leftrightarrow z \in x \sqcap y)$. By definition, $x \cap y = \{z : z \in x \wedge z \in y\}$. We conclude that $x \sqcap y = x \cap y$.

Theorem 12: For all $x, y \in \mathcal{U}$: $x \sqcap y \prec x$ and $x \sqcap y \prec y$.

Proof: Let x, y be some collections in $\mathcal{U}$. We want to show that (1) $x \sqcap y \prec x$ and (2) $x \sqcap y \prec y$. We start by proving (1): by definition of intersection, $\forall z((z \in x \wedge z \in y) \leftrightarrow z \in x \sqcap y)$, which in turn implies that $\forall z(z \in x \sqcap y \rightarrow z \in x)$. Hence, the first conjunct of the definition of subcollection holds. Now, let $i \in \mathbb{N}$ be some natural number such that $\tau_i \in x \sqcap y$. By definition, $\pi_i(x \sqcap y) \neq \varnothing$, from which it follows that $\pi_i(x) \neq \varnothing$. Hence: $\tau_i \in x$. We are left with showing that $\kappa_i(x \sqcap y) \leq \kappa_i(x)$. By definition of π_i, $\kappa_i(x) = \kappa(\pi_i(x))$ and $\kappa_i(x \sqcap y) = \kappa(\pi_i(x \sqcap y))$. By definition of intersection, either (1.1) $\pi_i(x \sqcap y) = \pi_i(x)$, or (1.2) $\pi_i(x \sqcap y) = \pi_i(y)$. Consider (1.1): then clearly $\kappa(\pi_i(x \sqcap y)) \leq \kappa(\pi_i(x))$. Consider (1.2): then by definition $\kappa(\pi_i(y)) \leq \kappa(\pi_i(x))$, from which it follows that $\kappa(\pi_i(x \sqcap y)) \leq \kappa(\pi_i(x))$. We conclude that $x \sqcap y \prec x$. The proof of (2) is similar. We conclude that $x \sqcap y \prec y$.

Theorem 13: For all $x \in \mathcal{U}$: $x \sqcap x = x$.

Proof: Let x be some collection in $\mathcal{U}$. We want to show that $x \sqcap x = x$. By I-EXTENSIONALITY, this amounts to show that (1) $x \sqcap x \prec x$ or $x \prec x \sqcap x$, and (2) $\kappa(x \sqcap x) = \kappa(x)$. By *Theorem 12*, $x \sqcap x \prec x$. Consider (2): by definition, $\kappa(x) = \kappa(\zeta) + \kappa_1(x) + \kappa_2(x) + ... + \kappa_n(x)$

and $\kappa(x \sqcap x) = \kappa(\xi) + \kappa_1(x \sqcap x) + \kappa_2(x \sqcap x) + ... + \kappa_n(x \sqcap x)$, where $\zeta = \{u : u \in x\}$ and $\xi = \{u : u \in x \sqcap x\}$. By definition of conjunction, $\xi = \{u : u \in x\}$. Hence: $\zeta = \xi$, from which it follows that $\kappa(\zeta) = \kappa(\xi)$. By definition of intersection, for all $i \leq n \in \mathbb{N}$: $\pi_i(x \sqcap x) = \pi_i(x)$. It follows that for all $i \leq n \in \mathbb{N}$: $\kappa_i(x \sqcap x) = \kappa_i(x)$. We conclude that $x \sqcap x = x$.

Theorem 14: For all $x \in \mathcal{U}$: $x \sqcap \varnothing = \varnothing$.

Proof: Let x be some collection in $\mathcal{U}$. We want to show that $x \sqcap \varnothing = \varnothing$. By i-extensionality, this amounts to show that (1) $x \sqcap \varnothing \prec \varnothing$ or $\varnothing \prec x \sqcap \varnothing$, and (2) $\kappa(x \sqcap \varnothing) = \kappa(\varnothing)$. By definition of empty set, $\varnothing \prec x \sqcap \varnothing$: hence (1) holds. Consider (2): by definition of κ, $\kappa(x \sqcap \varnothing) = \kappa(\xi) + \kappa_1(x \sqcap \varnothing) + \kappa_2(x \sqcap \varnothing) + ... + \kappa_n(x \sqcap \varnothing)$, where $\xi = \{u : u \in x \sqcap \varnothing\}$. By definition of empty set, $\xi = \varnothing$. Furthermore, for all $i \leq n \in \mathbb{N}$: $\pi_i(\varnothing) = \varnothing$. It follows that for all $i \leq n \in \mathbb{N}$: $\kappa(\pi_i(\varnothing)) \leq \kappa(\pi_i(x))$. Hence, for all $i \leq n \in \mathbb{N}$: $\pi_i(x \sqcap \varnothing) = \pi_i(\varnothing) = \varnothing$, which implies that $\kappa_i(x \sqcap \varnothing) = \kappa_i(\varnothing) = 0$. We conclude that $x \sqcap \varnothing = \varnothing$.

This concludes the present section. In what follows, we will present a list of postulates for our system, and we will prove some facts about the structure of $\mathcal{H}$.

3.5 Postulates

We present a list of postulates that we think should be considered as fundamental truths about i-collections. The following list does not have the pretence to be plenary. A thorough enquiry about the necessary principles governing the structures of i-collections is far beyond the scope of the present work. In Chapter 2 and Chapter 3, we have tried to investigate the nature and the structure of i-collections, and while some of the features of such collections have proved themselves intuitive, others (which we will be dealing with in the end of Chapter 3) have challenged our reasoning and understanding. The language we have proposed is a simplified version of what should be a proper general formal language for i-collections, as well as our models are fragments of what completely general models for i-collections should be. For example, we have assumed a finite number of types of i-objects. This assumption helped us in building a suitable language, and in glimpsing the realm of these objects. This has been a first acquaintance, and as such we will not make any pretence of completeness or generality. The structure of the present work has led us to the point in which some principles need to be stated for collections of nonindividuals, and in what follows we will state the ones we think the most intuitive and fundamental, in order to prove facts about i-collections and their structure. Being them postulates, no justification can be given for their acceptance except for, maybe, their intuitive truth.

As a starting point, we need to postulate that whenever there exist some nonindividual objects of a certain type, there exists a collection containing those objects as elements. This principle cannot be properly formulated, without the assumption that the number of types of objects is finite. As a result, we give as a condition the existence of a natural number $n \in \mathbb{N}$ such that n is the number of allowed types of i-objects in any model. We call this condition FINITENESS.

Now we can formulate the postulate of comprehension:

COMPREHENSION: $\forall i \leq n \in \mathbb{N}\ \forall m > 1 \in \mathbb{N}\ \exists x\ ((\tau_i \in x) \wedge \kappa(x) = m)$.

The following postulate gives the identity conditions for i-collections. We have already formulated it, and now we officially include it in our list of principles:

$$\text{I-EXTENSIONALITY: } \forall x \forall y (((x \prec y \lor y \prec x) \land (\kappa(x) = \kappa(y))) \leftrightarrow x = y).$$

Next, we postulate the existence of the empty collection:

$$\text{NULL: } \exists x(\forall y(y \notin x) \land (\tau_1 \notin x) \land (\tau_2 \notin x) \land \ldots \land (\tau_n \notin x)).$$

Clearly the empty collection is identical to the empty set. The proof of this statement (which is a theorem of our system), lies on the use of the postulate of I-EXTENSIONALITY, together with the definition of the partial and total cardinality of collections.

The next postulate states the existence of the collection union $x \sqcup y$, defined in the previous section, given any two collections $x, y \in \mathcal{U}$:

$$\text{UNION: } \forall x \forall y \exists z (z = x \sqcup y).$$

Then, we state the existence, given any collection $x \in \mathcal{U}$, of the collection containing all the subcollections of x:

$$\text{S-FAMILY: } \forall x \exists y \forall z (z \in y \leftrightarrow z \prec x).$$

In the previous section, we have shown that the notion of subcollection coincides with the notion of subset inside the universe of ZFC. It follows that, within V, $\mathfrak{S}(x) = \wp(x)$, where $\mathfrak{S}(x)$ is the collection whose existence is postulated in S-FAMILY, and $\wp(x)$ is the ZFC's power set.

The following postulate states the condition for the cardinality of collections in $\mathcal{U}$:

CARDINAL: $\forall i \leq n \in \mathbb{N}\ \forall x((x = \varnothing \leftrightarrow \kappa(x) = 0) \wedge (\tau_i \in x \rightarrow \kappa(x) > 1))$.

Our second to last postulate, which is not independent from the CARDINAL postulate, states that there exist no homogeneous or quasi-homogeneous i-collection containing only one element:

LIMITATION: $\forall x(\forall y(y \notin x) \rightarrow \kappa(x) \neq 1)$.

As a last principle, we postulate that whenever a collection $x \in \mathcal{U}$ exists, then there are $n+1$ collections (where $n \in \mathbb{N}$ is the number of allowed types of i-objects) such that their union is x:

SPAN: $\forall x \exists y_0, y_1, ..., y_n(x = y_0 \sqcup y_1 \sqcup ... \sqcup y_n)$.

The collection y_0 is defined as the biggest collection of individuals which is a subcollection of x: i.e. $y_0 = \{z : z \in x\}$, and the collections $y_1, y_2, ..., y_n$ are referred to by means of the expressions: $\pi_1(x), \pi_2(x), ..., \pi_n(x)$.

Finally, we will have all the axioms of ZFC appropriately relativised to ZFC-collections, by means of the substitution, whenever it is necessary to express the meaning of ZFC's postulates, of the quantified expressions: '$\forall x \varphi(x)$' and '$\exists x \varphi(x)$' with the relativised expressions: '$\forall x(\varsigma(x) \rightarrow \varphi(x))$' and '$\exists x(\varsigma(x) \wedge \varphi(x))$'.

3.6 The structure of i-collections

In what follows, we will prove some theorems about *homogeneous* i-collections, that will be important for understanding the structure of $\mathcal{H}$. In particular, we will prove that for any natural number n and any type i of i-objects, there is at most one homogeneous collection of cardinality n containing i-objects of type i, and that the class $\mathcal{H}$ is not closed under the operation of arbitrary finite union and s-family. We will also state some derived facts, in the form of corollaries.

Theorem 15: For all $i \leq n, m \in \mathbb{N}$ there exist at most one collection $x \in \mathcal{H}$ such that $\tau_i \in x$ and $\kappa(x) = m$.

Proof: By LIMITATION, there is no $x \in \mathcal{H}$ such that $\kappa(x) = 1$. It follows that not for all numbers $m \in \mathbb{N}$ there is a collection x in $\mathcal{H}$ such that $\kappa(x) = m$. When $m = 0$, x is the empty collection. By NULL and I-EXTENSIONALITY, x is unique. Let $m \in \mathbb{N}$ be such that $m > 1$, and consider some arbitrary number $i \leq n \in \mathbb{N}$. Let $x, y \in \mathcal{H}$ be two homogeneous collections such that $\tau_i \in x$ and $\tau_i \in y$, and $\kappa(x) = \kappa(y) = m$. By definition of $\mathcal{H}$ for all $j \in \mathbb{N}$, $\tau_j \in x$ if and only if $j = i$, and the same holds for y. It follows that, for any $k \leq n \in \mathbb{N}$, if $\tau_k \in x$ then $\tau_k \in y$. Hence $x \prec y$. We conclude that $x = y$.

Corollary: For any non empty collection $x \in \mathcal{H}$ of cardinality $\kappa(x) = n$, and for any $m \leq n \in \mathbb{N}, m \neq 1$ there is a unique subcollection $y \prec x$ such that $\kappa(y) = m$.

Corollary: For any two homogeneous collections $x, y \in \mathcal{T}_i$ (for any $i \leq n \in \mathbb{N}$), either $x \prec y$ or $y \prec x$.

Corollary: For all $i \leq n \in \mathbb{N}$, the structures $\langle \mathcal{T}_i, \prec \rangle$ and $\langle \mathbb{N}, \leq \rangle$ are isomorphic.

Theorem 16: $\mathcal{H}$ is not closed under arbitrary finite unions.

Proof: Let x and y be some homogeneous collections such that $x \in \mathcal{T}_i$ and $y \in \mathcal{T}_j$, for some $i, j \leq n \in \mathbb{N}$ and $j \neq i$. By definition of union, both $\tau_i \in x \sqcup y$ and $\tau_j \in x \sqcup y$. It follows that $x \sqcup y \notin \mathcal{H}$.

Theorem 17: For all $x, y \in \mathcal{T}_i$ (for any $i \leq n \in \mathbb{N}$), either $x \sqcup y = x$ or $x \sqcup y = y$.

Proof: Let $i \leq n$ be some natural number $i \in \mathbb{N}$ and let x and y be some collections in $\mathcal{T}_i$. By definition of union, $x \sqcup y \in \mathcal{T}_i$, since both x and y contains only nonindividual objects and only of type i. Furthermore, by definition of $\mathcal{T}_i$, $x = \pi_i(x)$ and $y = \pi_i(y)$, which means that either $x = \pi_i(x \sqcup y)$, in which case, by definition of $\mathcal{T}_i$, $x \sqcup y = x$, or $y = \pi_i(x \sqcup y)$, in which case, by definition of $\mathcal{T}_i$, $x \sqcup y = y$.

Theorem 18: For all $x \in \mathcal{H}$, $\mathfrak{S}(x) \notin \mathcal{H}$.

Proof: Consider an arbitrary collection $x \in \mathcal{H}$. By *Theorem 1*, $x \prec x$, which in turn implies, by S-FAMILY, that $x \in \mathfrak{S}(x)$. Hence, by definition of $\mathcal{H}$, $\mathfrak{S}(x) \notin \mathcal{H}$.

Theorem 19: For all $x \in \mathcal{Q}$, $\mathfrak{S}(x) \notin \mathcal{Q}$.

Proof: The proof is similar to the one of *Theorem 18*.

Theorem 20: For all $x \in \mathcal{H}$, $\kappa(\mathfrak{S}(x)) = \kappa(x)$.

Proof: Let $x \in \mathcal{H}$ be some homogeneous collection such that $\kappa(x) = n$, for some $n \in \mathbb{N}$. By *Theorem 15*, x has n-many subcollections. It follows that $\kappa(\mathfrak{S}(x)) = \kappa(x) = n$.

3.7 Quasi-Set Theories

The possibility of developing a language for collections of indiscernible objects has been considered and widely discussed in Philosophy of Physics. Within the standard interpretation of the theory of Quantum Mechanics (henceforth: QM), in fact, particles of the same kind in an isolated physical system can be considered, under certain circumstances, indiscernible. The problem of 'indiscernibility' has been analysed by numerous philosophers, and three kinds of discernibility have been identified in an attempt to clarify the nature of the reality described by QM: 'absolute discernibility', 'relative discernibility' and 'weak discernibility'. According to Ladyman (2007), "[t]wo objects are 'absolutely discernible' if there exists a formula in one free variable which is true of one object and not the other. [...] [They are] 'relatively discernible' just in case there is a formula in two free variables that applies to them in one order only. [...] Finally, two objects are 'weakly discernible' just in case there is a two-place irreflexive relation that they satisfy".[32] An agreement on the correct kind of discernibility obtaining between particles of the same kind in isolated physical systems is yet to be found. Among the philosophers holding that quantum particles should be regarded as *absolutely indiscernible*, the question has been asked if the current mathematical representation of indiscernibility (which makes use of the concept of 'invariance under isomorphism') is correct, and if it is possible to develop a theory of collections of indiscernible elements to provide a foundation of mathematics

[32] Ladyman (2007) is not alone in drawing a distinction between different kinds of indiscernibility, and in using them to analyse Black-like scenarios. The notion of 'weak indiscernibility' was famously presented in Quine (1976), and since then it has been widely studied among those philosophers interested in the relations between 'identity' and 'indiscernibility'. Saunders (2003, 2006), Ladyman & Bigaj (2010), Linnebo & Muller (2013), and Dorato & Morganti (2013) are only few of the examples of the application of the Quinean distinctions between different grades of discernibility to the problem posed by Quantum Mechanics. Hawley (2006) offers a philosophical argument for the possibility of absolutely indiscernible objects, and Ladyman, Linnebo & Pettigrew (2012) analyses the different kinds of indiscernibility from a logical point of view.

able to represent indiscernibility straightforwardly. In particular, two theories of collections of indiscernible objects have been developed to solve the problem of indiscernibility: the Quaset Theory, presented in Dalla Chiara & Toraldo di Francia (1993, 1995), and the Quasi-Set Theory, presented in French & Krause (2006). A wide comparison with the theory of collections presented in this work would be difficult, since our theory has only been sketched. However an analysis, even if partial, of the similarities and the differences between our theory of collections and both Quaset Theory and Quasi-Set Theory might prove itself important for the understanding of our theory, as well as for guiding its future developments. We begin by considering Dalla Chiara & Toraldo di Francia's Quaset Theory. Dalla Chiara & Toraldo di Francia (1993) introduces the notion of 'quaset' as "a collection of elements which may be *indistinguishable* from one another". Consistently with the intuitions behind our theory of collections, a quaset can be assigned a cardinality without being isomorphic to any ordinal number. Furthermore, as expected, quasets infringe EXTENSIONALITY. The Language of Quaset Theory is a standard first order language with equality ($=$) interpreted as a logical constant. The theory has numerous primitives: identity '$=$', inclusion '$\subseteq$', definite membership '$\in$' and definite non-membership '$\notin$', cardinality '$card^*$', and quasi intersection '$\cap_*$'. The large number of primitives is not the only difference between Quaset Theory and our theory of collections. The main difference is that Dalla Chiara and Toraldo di Francia consider the elements of quasets *individuals*. This allows them to quantify over these objects and to name them by means of standard individual constants. In this sense, whilst them both being expansions of the universe of ZFC, our theory of collections and Quaset Theory differ in a very radical way: the nonindividuality of i-objects, in fact, completely permeates our theory, and any difference in the syntax and in the semantics between our theory and the theory of quasets is a consequence of this very fact. On the contrary, French and Krause's Quasi-Set Theory is a theory of collections of nonindividuals, and in this sense is more similar to our theory than Dalla Chiara and

Toraldo di Francia's one. The Language of Quasi-Set theory is a first order language without identity. French and Krause expand Zermelo-Fraenkel's Set Theory with Urelemente (ZFU) by considering two different kinds of urelemente: M-atoms, which are the standard urelemente of ZFU, and m-atoms, which are nonindividual objects, interpreted as the fundamental particles of quantum physics. The concept of extensional identity '$=_E$' applies only to ZFC-collections and M-atoms. This is obtained by "restricting the concept of formula: expressions like $x = y$ are not well formed if x and y denote m-atoms" (French & Krause, 2006). The primitive notions of Quasi-Set Theory are the notions of indistinguishability '$\equiv$', membership '$\in$', quasi-cardinality 'qc', and set '$Z(x)$'. French and Krause's notion of indistinguishability allows them to take into account different kinds of nonindividual indiscernible objects without having to postulate a primitive notion of 'type'. In fact, our notion of 'type' can be defined as a function from classes of indistinguishable objects to cardinals (given the notion of indistinguishability). So why not choosing an indistinguishability relation instead of the notion of type? Because, we submit, such choice is potentially dangerous and we doubt that a semantics simpler than ours can be provided for a first order theory of collections of nonindividuals. French and Krause's notion of indistinguishability requires, to be of any use at all, the possibility to construct well-formed sentences of the form: '$\exists x \exists y(m(x) \wedge m(y) \wedge x \equiv y)$', stating the existence of two nonindividuals x and y which are claimed to be indistinguishable. Then, it must be the case that some reference function can be defined for nonindividuals, which we have considered to be impossible, if the metatheory is still assumed to be a fragment of ZFC. The logical details at the ground of their language are not specified, and as a result our analysis of the reference in Quasi-Set Theory cannot be carried any further. Following Berto (2017), direct reference to nonindividuals appears everywhere in French and Krause's theory. For example, a quasi-set is defined as something that is not an urelement: $Q(x) := \neg(m(x) \vee M(x))$. Another example is the first conjunct of the axiom $(Q9) : \forall x \forall y(m(x) \wedge x \equiv y \rightarrow m(y))$. If we want these sentences to be meaningful, we need to be able to directly refer to x and y singularly, and we don't know how this is obtained in French and Krause's theory. In Berto's (2017) words: "The quasi-set theory presented in the book[33] and in various papers has variables that range over things allegedly lacking identity, variables which can be bound by quantifiers. To pick one example at random, one who claims: 'either the non-individual y belongs to the quasi-set A or not, as in the case of an atom, where an electron either belongs or does not belong to it, although we cannot name it unambiguously. Here, y does not act as a name for an individual' (French and Krause [2006]: 319) ... while denying that we can name arbitrary particles, is using variables to refer to them and say things about them". Thence, even if our notion of 'type' could be in principle defined in terms of the indistinguishability relation, we prefer to maintain it as a primitive, since we want to be firm on our position that no direct reference is possible to nonindividual objects, at least given the standard semantic of first order logic. We consider two last points about Quasi-Set Theory. The first has it that within Quasi-Set Theory the difference between the notions of 'extensional identity' and 'indistinguishability' is stated in terms of the notion of 'membership': unlike extensional identity, in fact, the indistinguishability relation does not allow substitutivity for membership. In fact, it is not the case that '$(x \in w) \land (y \equiv x)$' entails '$y \in w$'. As a result, French and Krause seem to have much more to say about collections of nonindividuals than we have to, since they can express *which one*, from a pair of i-objects, belongs to some collections and *which one* does not. Formally, in fact, the following sentence is well-formed in Quasi-Set Theory: $\exists x \exists y \exists w((((m(x) \land m(y)) \land (x \in w)) \land (y \equiv x)) \land (y \notin w))$. From our point of view, such expression can be consistently expressed only when the assumption of the nonindividuality of x and y is abandoned. Furthermore, the aforementioned sentence seems inconsistent (although well-formed). In fact, French & Krause (2006) defines the notion of indistinguishability as

[33] French, S. & Krause, D. (2006). *Identity in Physics: A Historical, Philosophical, and Formal Analysis.* Oxford University Press.

"agreement with respect to all the attributes". How can then x and y be indistinguishable, if the formula '— is member of w' is true of x and not true of y? The second point has it that with the notion of 'strong singleton', French & Krause (2006) allows for the possibility of collections containing only *one* nonindividual. According to the definition we have employed in our analysis of individuality, the possibility of a strong singleton w of quasi-cardinality $qc(w) = 1$ is equivalent to the admission that the element of w is an individual object.

As we have remarked many times in this work, our theory is still under construction, and numerous revisions will be needed in order to complete it. Being a theory for collections of i-objects, we have devoted this last part of our work to a limited comparison with the two prominent theories of collections of indiscernible objects presented in the last few years. We hope with this comparison to have given a contribution to a wider understanding of our theory, which will guide our future hiking in the still unexplored territory of i-collections.

Conclusion

This work has revolved around the notions of 'indiscernibility' and 'nonindividuality', within the fields of metaphysics and logic. Our investigation has focused on the nature of indiscernible objects, and on the possibility of developing a formal language that could allow us to talk about collections containing them as elements. We have begun our investigation by considering Leibniz's Principle of the Identity of Indiscernibles (PII) and its relations with one of its most famous counterexamples, presented in 1952 by the philosopher Max Black. Never challenging the assumption that the classical second order formalisation of PII exhausted its ontological meaning, we have shown that maintaining a fragment of ZFC as metatheory and considering standard models for second order logic have resulted in the impossibility of finding a second order formalisation of the scenario presented in Black (1952), if the assumption that Black's Universe contains *two* objects is not abandoned. The reasons for this have been shown and discussed in the beginning of Chapter 2, where we have proved that every collection containing indiscernible objects that could suitably inhabit a universe *à la* Black without altering its symmetry would infringe the ZFC's Axiom of Extensionality, admitting the possibility of two collections being different and still containing the same elements. As a consequence, collections of indiscernible objects have been shown to not be ZFC-collections. Chapter 2 was also devoted to a metaphysical analysis of indiscernible objects. In particular, we have shown that under the conditions for individuality presented in Lowe (2016), such objects cannot be consistently thought of as individuals. As a result, we have studied the possibility of developing a language being expressive enough to allow us to talk about collections of nonindividuals, expanding the set theoretical universe of ZFC. After having discussed the primitive notions that such language would have, we have entirely devoted Chapter 3 to a presentation of a first draft of it, within a new theory of collections. We have described the syntax and the semantics of such language, and we have tried to solve the problem of how to refer to nonindividual objects in order to establish the conditions for the truth of those formulas stating that some i-collections contain nonindividual objects of a given type. We have then defined new concepts and operations in our new language (in particular, the concepts of 'subcollection', 'union' and 'intersection'), and we have shown that, whenever two ZFC-collections are considered, our new concepts coincide, respectively, with the concepts of 'subset', 'union' and 'intersection' as defined within ZFC. We have then presented the first proposal for a list of postulates of our new theory of collections, and, as the British explorer Percy Harrison Fawcett returning from his first journey in South America, we have brought witnesses of interesting facts happening in the still uncharted realm of collections of nonindividuals.

We hope that this work can be considered as an evidence that there is much to gain in accepting nonindividual objects in our ontology and that consequently this will lead to a more profound research on the correct formal framework encompassing them. Throughout this Thesis, our argumentation has been carried out entirely within a purely theoretical paradigm: our interest towards nonindividual objects and a theory of collection being expressive enough to talk about collections having indiscernible elements was purely theoretical. Still, we hope that our work can contribute to an advancement of the research on the possibility to model systems of indiscernible objects. As far as we know, this kind of research has been carried out among those philosophers of Physics advancing the idea that elementary particles have to be regarded as absolutely indiscernible, or even as objects lacking individuality.

Within the foregoing pages, we have presented a first draft of a theory of collections that could accommodate for the existence and the behaviour of collections of nonindividuals. There is still a long way to go before this theory can be considered completed. The research of an exhaustive list of axioms is required, and careful thought needs to be put on the problem of the non well-foundedness of i-collections. Furthermore, the possibility of developing simpler semantics for our language should be taken into consideration. Finally, the majority of the facts concerning i-collections need to be discovered and proved, and a first representation of this still uncharted territory awaits.

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