Majorizability Types, Assmeblies, and the Fan Theorem

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Abstract

In this thesis we introduce the notion of majorizability types, and the category MMAsm of majorizability modified assemblies. The MMAsm has the vdBB model as the interpretation of finite types. The internal logic of MMAsm captures certain version of modified realizability. Besides, both the Special Fan Functional and the Fan Recursor provably exist in the internal logic of MMAsm. Therefore the Fan Theorem holds in MMAsm. And this provides a computational and categorical justification of the Fan Theorem.

1 Introduction

The constructivism in mathematics is nowadays perhaps no longer an alien to logicians, mathematicians, and computer scientists. The basic observation behind the constructivism is that, in classical mathematical practices, the proofs can be divided into two classes: constructive and non-constructive proofs. The difference is most explicit in those disjunctive and existential statements. For example, to prove

$$\exists x(x \geq 2^{10} \wedge \mathsf{Prime}(x)) \tag{\dagger}$$

one can use his computer and use a few lines of code to find the smallest prime number n such that $n \geq 2^{10}$. In the constructivism terminology, such n is often called a witness of the statement (†). Next we take a well-known example in analysis:

$$\text{Every open cover } \mathcal{U} \text{ of } [0,1] \text{ has a finite subcover}$$

This is normally proved by contradiction: suppose not, then one can construct an infinite sequence of $I_n \subseteq [0,1]$ that cannot be finitely covered. By Cantor Intersection Theorem, $\bigcap_{n\to\infty} I_n$ is a singleton $\{x\}$, so there is some $U \in \mathcal{U}$ that covers $\{x\}$, and also covers slim enough I_n, contradiction to the assumption of I_n's.

One may already feel some slight difference in these two examples. In the first example when we prove the existence of certain x, we do find such natural number x. In the second however, we don't know what a finite subcover is: though denying is existence bring us into problem. In this sense, we say that the first proof is constructive and the second is non-constructive. In general, constructivism in mathematics denies unrestricted use of the Excluded Middle Axiom, and use the intuitionsitic logic as underlying logic for mathematical systems. And in such logic, to prove a disjunctive statement one needs to prove one of its disjuncts; and to claim an existential statement one has to find such an element that one claims to exist.

The intuitionistic logic, however, requires a new interpretation of the logical connectives to make good sense. One of the approach is the **BHK** interpretation. Basically every formula is interpreted as a proof of it. For example, a proof for the atomic formulas are trivial; a proof for $\varphi \to \psi$ requires a function F mapping a proof of φ to a proof of ψ; and a proof for $\exists x\varphi(x)$ consists of some n and a proof of $\varphi(n)$.

Kleene's numerical realizability and its variations can be seen as a reminiscent of the **BHK** interpretation [2]. The numerical realizability defines, for each formula φ, a notion of "number n realizes φ". The n is called a realizer of φ. In the numerical realizability, all the realizers are natural numbers, intuitively thought of Gödel numbering of Turing machines. And it's unsurprising that the Gödel numbering of Turing machines is a model of the numerical realizability. This already incorporates the idea of applying computation concepts into constructivism mathematics. Such idea is more explicit in some later variations. For example, in the modified realizability introduced by Kreisel, every formula is assigned a unique type of its potential realizers, depending on its logical form.

From a categorical logic point of view, there are various categories whose internal logic captures certain notion of realizability. Define assemblies to be (X, E) where X is a set and $E : X \to \mathsf{Pow}_i(\mathbb{N})$ assigns to each $x \in X$ an inhabited subset of $\mathbb{N}$. Here $\mathbb{N}$ can be understood as the Gödel's numbering of Turing machines. A morphism $(X, E) \to (Y, F)$ is a function $f : X \to Y$ that is witnessed: $\exists r \in \mathbb{N}$ s.t. $\forall x \in X$ and $\forall a \in E(x)$, $r \cdot a$ is defined and $ra \in F(fx)$. Thus we get a category of assemblies Asm, where the morphisms are recursive functions. It's a good category in that its internal logic captures Kleene's numerical realizability [15].

Intuitionism, one school of constructivism mathematics, was founded by L.E.J. Brouwer. His philosophy of intuitionistic mathematics can be dated back to the early 1900's [16], which later generated two of Brouwer's most influential idea in intuitionism. Firstly one has the skepticism and ban of the Excluded Middle Principle. Secondly one has the Brouwer's Thesis, consisting of the axioms he proposed after a close examination of (the construction of) the continuum. Among them are the Bar Theorem and the Fan Theorem, and the latter is a corollary of the former. One interesting observation is that, the Fan Theorem itself suffices for many of Brouwer's results. In particular, from the Fan Theorem one can derive the Uniform Continuity Theorem, claiming that every continuous function $[0,1] \to \mathbb{R}$ is uniformly continuous [13]. Therefore the Fan Theorem and its related mathematical principles are of special interest to us. And the thesis has to be seen in the context of Brouwer's philosophy of mathematics.

In this thesis, we have constructed a category of assemblies MMAsm that is expected to capture certain version of realizability (conjectured to be the monotone modified realizability).

Besides, MMAsm matches well with the vdBB structure. In [14], van den Berg and Briseid introduced vdBB as a model for Gödel's T, in which the Fan Theorem holds. In our category MMAsm, the interpretation of the finite types coincide (up to isomorphism) with the vdBB model. This means that vdBB are the finite types in some cartesian closed category with natural number object. And we conjecture that, MMAsm characterizes the monotone modified realizability w.r.t. the vdBB terms.

Finally MMAsm provides a constructive categorical model for the Fan Theorem. The Fan Theorem fails in the famous effective topos Eff [15]. Relying the strong majorizability relation in the MType, MMAsm validates the Fan Theorem, the Special Fan Functional Principle, and the Fan Recursor Principle. And since MMAsm is a category of assemblies, it provides a computable justification of the Fan Theorem.

2 Preliminaries

2.1 Gödel's T and HA^ω

Recall that Gödel's T is a term rewriting system which admits lambda abstraction and application. However, in order to fit our application better, let's regard the Gödel's T as a logical system where the axioms are those of intuitionistic logic and the correspondence of the term rewriting rules. Regarding T as a quantifier-free claculus of functionals also works; we do it just for convenience. Then HA^ω is the many-sorted intuitionisitic logic based on T.

First, we construct the set of finite types $\mathcal{T}$ inductively from the base type 0 and type constructor $\times$ and $\to$. And $\mathcal{T}$ will be the set of types for the language of T and HA^ω. To be succinct, we shall sometimes write $\sigma\tau$ as abbreviation of $\sigma \to \tau$, with the convention that operators associate to the right; also we adopt the convention that the type n denote the type $(n-1)0$, so for example type 2 is $(0 \to 0) \to 0$. We will use : or superscript to indicate the type. Next, the language $\mathcal{L}$ consists of:

- For each type $\sigma \in \mathcal{T}$, infinitely many variables for that type $x^\sigma, y^\sigma, z^\sigma, \ldots$;
- The zero element 0 of type 0, and the successor element S of type $0(0)$. Write $\bar{n}$ for $\mathsf{S}^n 0$;
- For any type $\sigma, \tau, \rho \in \mathcal{T}$, constants $\mathbf{k}^{\sigma,\tau}$, $\bar{\mathbf{k}}^{\sigma,\tau}$, $\mathbf{s}^{\sigma,\tau,\rho}$, $\mathbf{p}^{\sigma,\tau}$, $\mathbf{p}_0^{\sigma,\tau}$, $\mathbf{p}_1^{\sigma,\tau}$;
- For each type σ, a combinator (recursor) $\mathbf{R}^\sigma : \sigma \to (0 \to (\sigma \to \sigma)) \to (0 \to \sigma)$;
- For any type $\sigma \in \mathcal{T}$, a binary relation symbol $=_\sigma$;
- For any type $\sigma, \tau \in \mathcal{T}$, a function symbol $\mathsf{app}_{\sigma,\tau} : (\sigma \to \tau) \times \sigma \to \tau$.

For simplicity, we abbreviate $\mathsf{app}(t, t')$ as tt', and $\mathbf{p}ab$ as $\langle a, b\rangle$ or even (a, b), when no confusion arises. n-ary tuples $\langle a_1, \ldots, a_k\rangle$ are consequently abbreviations of nested pairings, where the arity is explicitly encoded. Now the $\mathcal{L}$ terms are constructed from the variables and constants by applying suitable (w.r.t. the types) terms to function symbols. And axioms for T are as follows:

1. Axioms for intuitionsitic propositional logic, and the rule of universal substitution.
2. Axioms for equality of all types, saying that $=_\sigma$ is an equivalence relation:
   (a) $\forall x^\sigma (x =_\sigma x)$
   (b) $\forall x^\sigma y^\sigma (x =_\sigma y \to y =_\sigma x)$
   (c) $\forall x^\sigma y^\sigma z^\sigma (x =_\sigma y \wedge y =_\sigma z \to x =_\sigma z)$
   (d) $\forall x^{\sigma\tau} y^{\sigma\tau} \forall u^\sigma v^\sigma (x =_{\sigma\tau} y \wedge u =_\sigma v \to xu =_\tau yv)$
3. Axioms for combinators:
   (a) $\forall x^\sigma y^\tau (\mathbf{k}^{\sigma,\tau} xy) = x$, $\forall x^\sigma y^\tau (\bar{\mathbf{k}}^{\sigma,\tau} xy) = y$
   (b) $\forall x^{\sigma(\tau\rho)} y^{\sigma(\tau)} z^\sigma \mathbf{s}^{\sigma,\tau,\rho} xyz = xz(yz)$
   (c) $\forall x_0^\sigma x_1^\tau (\mathbf{p}_i^{\sigma,\tau}(\mathbf{p}^{\sigma,\tau} x_0 x_1)) = x_i$, for $i = 0, 1$
   (d) $\forall x^\sigma f^{0(\sigma\sigma)} (\mathbf{R}^\sigma x f 0 = x)$; $\forall\forall x^\sigma f^{0(\sigma\sigma)} (\mathbf{R}^\sigma x f(\mathsf{S}n) = f(Rxfn))$

To achieve the system HA^ω, we add further:

4. Axioms for the natural number type 0:

(a) $\forall x^0(\neg 0 = \mathsf{S}x)$

(b) $\forall x^0 y^0(\mathsf{S}x = \mathsf{S}y \to x = y)$

(c) Induction schema:

$$\varphi(0) \to (\forall x^0(\varphi(x) \to \varphi(\mathsf{S}x)) \to \forall x^0 \varphi(x))$$

where $\varphi(x)$ is any formula in the language.

In HA^ω, all the prime formulas with only $=_0$ are decidable. However, one cannot prove that everything of type 0 is of the form $\bar{n}$ for some $n \in \mathbb{N}$ (in fact this cannot be expressed in our language). Still, one can prove by induction on the formula structure that the substitution holds for any $\sigma \in \mathcal{T}$: $\mathsf{HA}^\omega \vdash \forall x^\sigma y^\sigma(x =_\sigma y \wedge \varphi(x) \to \varphi(y))$.

If we take as primitive only equality symbol $=_0$ for type 0 and regard the rest $=_\sigma$'s as abbreviations:

$$\forall x^{\sigma\times\tau} y^{\sigma\times\tau}(x =_{\sigma\times\tau} y \leftrightarrow \mathbf{p}_0 x = \mathbf{p}_0 y \wedge \mathbf{p}_1 x = \mathbf{p}_1 y)$$
$$\forall x^{\sigma\tau} y^{\sigma\tau}(x =_{\sigma\tau} y \leftrightarrow \forall u^\sigma xu =_\tau yu)$$

Then every $=_\sigma$ is extensional, and we call the resulting system $\mathsf{E\text{-}HA}^\omega$. By the above discussion, all atomic formulas in $\mathsf{E\text{-}HA}^\omega$ are decidable.

2.2 Realizability

The notion of (numerical) realizability was first raised by Kleene [6], under his intuition that there should exist some connection between *Intuitionism* and the theory of recursive functions. By defining the notion of "number n realizes formula φ", Kleene intended to "capture some essential aspects of the intuitionistic meaning of φ" [2], namely φ is constructively true. And the result is the numerical realizability as a constructive semantics for intuitionistic arithmetic.

One well-known and intuitive variation of realizability is the modified realizability introduced by Kreisel [9], where the realizers are terms in typed lambda calculus. In brief, given a formula φ, its realizers share a specific type given by the logical form of formula φ. Given the system HA^ω, we define HA^ω formula $(x \mathrel{\mathsf{mr}} \varphi)$ inductively on the structure of formula φ:

$$\begin{aligned}
x^0 \mathrel{\mathsf{mr}} \varphi &:= \varphi, \text{ where } \varphi \text{ is atomic} \\
x \mathrel{\mathsf{mr}} (\varphi \wedge \psi) &:= (\mathbf{p}_0 x \mathrel{\mathsf{mr}} \varphi) \wedge (\mathbf{p}_1 x \mathrel{\mathsf{mr}} \psi) \\
x \mathrel{\mathsf{mr}} (\varphi \vee \psi) &:= (\mathbf{p}_0 x = 0 \to \mathbf{p}_1 x \mathrel{\mathsf{mr}} \varphi) \wedge (\mathbf{p}_0 x = 1 \to \mathbf{p}_1 x \mathrel{\mathsf{mr}} \psi) \\
x \mathrel{\mathsf{mr}} (\varphi \to \psi) &:= \forall y(y \mathrel{\mathsf{mr}} \varphi \to xy \mathrel{\mathsf{mr}} \psi) \\
x \mathrel{\mathsf{mr}} (\exists z \varphi) &:= \mathbf{p}_1 x \mathrel{\mathsf{mr}} \varphi[\mathbf{p}_0 x/z] \\
x \mathrel{\mathsf{mr}} (\forall z \varphi) &:= \forall z(xz \mathrel{\mathsf{mr}} \varphi)
\end{aligned}$$

where in the $\to$ case $\forall y$ is typed quantification with the type given by the structure of φ, and in the $\exists$ case $\varphi[\mathbf{p}_0 x/z]$ is the result of substituting all the free occurrence of z in φ with $\mathbf{p}_0 x$. For our purpose, it's sometimes more convenient to consider the realizability notions in the metatheory. One first observation is that, $x \mathrel{\mathsf{mr}} \varphi$ is $\exists$-free. And for any $\exists$-free formula φ, $x \mathrel{\mathsf{mr}} \varphi$ (thus $\exists x(x \mathrel{\mathsf{mr}} \varphi)$) is equivalent to φ itself. Though in this paper we shall focus on the system HA^ω, there is no reason to be restricted to it. One can make similar definitions in other typed systems with Gödel's T.

The formula $\exists x(x \mathrel{\mathsf{mr}} \varphi)$ is also referred to (e.g. [12]) as the modified realizability interpretation of φ, and $x \mathrel{\mathsf{mr}} \varphi$ as the modified realizability predicate. Indeed, from the very beginning, Kleene [6] introduced numerical realizability as "a semantics for 'intuitionistic arithmetic" [12]. The following is a basic result saying that the mr interpretation is a sound semantics for HA^ω:

Theorem 2.1 (Soundness)**.** *For any sentence* φ*,* $\mathsf{HA}^\omega \vdash \varphi \Rightarrow \mathsf{HA}^\omega \vdash \exists x(x \;\mathsf{mr}\; \varphi)$*.*

However, the inverse is not true: not every realized sentence is provable. Consider the following schemata (instances):

$(\mathsf{AC}_{\sigma,\tau})$ $\quad \forall x^\sigma \exists y^\tau \varphi(x,y) \to \exists f^{\sigma(\tau)} \forall x^\sigma \varphi(x, fx)$
$(\mathsf{IP}^{\mathrm{ef}}_\sigma)$ $\quad (\varphi \to \exists y^\sigma \psi(y)) \to \exists y^\sigma(\varphi \to \psi(y))$, where $y \notin \mathsf{FV}(\varphi)$ and φ is $\exists$-free.

where AC stands for "Axiom of Choice" and $\mathsf{IP}^{\mathrm{ef}}$ for "Independence of Premise (for **e**xistence-**f**ree formulas)"[1]. And the following is a well-known characterization of modified realizability [11]:

Lemma 2.2. *Both* AC *and* $\mathsf{IP}^{\mathrm{ef}}$ *are modified realized in* HA^ω*.*

Proof. It's straightforward to verify that $t' := \lambda r.\mathbf{p}(\lambda m.\mathbf{p}_0(rm))(\lambda m.\mathbf{p}_1(rm))$ realizes AC.

For a term t to realize $\mathsf{IP}^{\mathrm{ef}}$, one should have:

$$\forall r[\forall a(\varphi \to \mathbf{p}_1(ra) \;\mathsf{mr}\; \psi(\mathbf{p}_0(ra))) \to \forall a(\varphi \to \mathbf{p}_1(tr)a \;\mathsf{mr}\; \psi(\mathbf{p}_0(tr)))]$$

Then we can pick 0^σ of the type σ of a, and $t := \lambda r.\mathbf{p}(\mathbf{p}_0(r0^\sigma))(\lambda a.\mathbf{p}_1(r0\sigma))$ does the trick. ☐

However, $\mathsf{E\text{-}HA}^\omega \nvdash \mathsf{AC}, \mathsf{IP}^{\mathrm{ef}}$ [2]. And this somehow suggests that, $\mathsf{E\text{-}HA}^\omega$ thinks that modified realizability is as strong as $\mathsf{E\text{-}HA}^\omega$ plus AC, $\mathsf{IP}^{\mathrm{ef}}$ and possibly something else, and fortunately it turns out that these two suffice:

Theorem 2.3 (Characterization)**.** *The following hold for any* HA^ω *formula* φ*:*

1. $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^{\mathrm{ef}} \vdash \varphi \leftrightarrow \exists x(x \;\mathsf{mr}\; \varphi)$
2. $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^{\mathrm{ef}} \vdash \varphi \iff \mathsf{HA}^\omega \vdash \exists x(x \;\mathsf{mr}\; \varphi)$

Proof. Proof for both can be found in [11] (*Theorem 3.4.8*). For (1), consider a nontrivial case $\varphi \to \psi$ as an example. In $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^{\mathrm{ef}}$, one can prove:

$$\begin{aligned} (\psi \to \chi) &\leftrightarrow \exists x(x \;\mathsf{mr}\; \psi) \to \exists y(y \;\mathsf{mr}\; \chi) \\ &\leftrightarrow \forall x(x \;\mathsf{mr}\; \psi \to \exists y(y \;\mathsf{mr}\; \chi)) \\ &\leftrightarrow \forall x \exists y(x \;\mathsf{mr}\; \psi \to y \;\mathsf{mr}\; \chi) && (\mathsf{IP}^{\mathrm{ef}}) \\ &\leftrightarrow \exists Y \forall x(x \;\mathsf{mr}\; \varphi \to Yx \;\mathsf{mr}\; \chi) && (\mathsf{AC}) \\ &\leftrightarrow \exists Y(Y \;\mathsf{mr}\; (\psi \to \chi)) \end{aligned}$$

where for the use of $(\mathsf{IP}^{\mathrm{ef}})$, we use the fact that $x \;\mathsf{mr}\; \varphi$ is always in the negative fragment (thus $\exists$-free). As for (2), simply note that any instance of AC and $\mathsf{IP}^{\mathrm{ef}}$ is realized (though not provable) in HA^ω. ☐

If we work in a variation of HA^ω where the atomic propositions are decidable, for example $\mathsf{E\text{-}HA}^\omega$, then the $\mathsf{IP}^{\mathrm{ef}}$ can be replaced by the following *Independence Principle for negated formulas* as characterization of $\mathsf{E\text{-}HA}^\omega$:

$(\mathsf{IP}^{\neg}_\sigma)$ $\quad (\neg\varphi \to \exists y^\sigma \psi(y)) \to \exists y^\sigma(\neg\varphi \to \psi(y))$, where $y \notin \mathsf{FV}(\varphi)$ and φ is $\exists$-free.

Basically this is because that if the atomic formulas are decidable, then they admit the *Excluded Middle*; or in other words, $\mathsf{E\text{-}HA}^\omega$ treat them and their $\neg\neg$ correspondence as equivalent. Then an easy induction shows that all φ in the negative fragment satisfies $\mathsf{E\text{-}HA}^\omega \vdash \varphi \leftrightarrow \neg\neg\varphi$.

[1] In some references, for example [4] the schemata are written as AC^ω and IP^ω. The superscript ω, as that in HA^ω, stands for all finite types. But to avoid cluster we will omit them throughout the thesis, and simply use AC, IP, etc.m

One variation based on the modified realizability is the monotone realizability [7]. Basically it is Kreisel's modified realizability equipped with Bezem's strong majorizability. To be more formal, first of all one can define the strong majorizability relation $\leq^*_\sigma$ in HA^ω [1] as follows (we will discuss more on the strong majorizability relation $\leq^*$ in **Chapter** 2.4):

$$\begin{aligned}
&\forall x^0 y^0 (x \leq^*_0 y \leftrightarrow x \leq_0 y) \\
&\forall x^{\sigma\times\tau} y^{\sigma\times\tau} (x \leq^*_{\sigma\times\tau} y \leftrightarrow \mathbf{p}_0 x \leq^*_\sigma \mathbf{p}_0 y \wedge \mathbf{p}_1 x \leq^*_\tau \mathbf{p}_1 y) \\
&\forall x^{\sigma\tau} y^{\sigma\tau} (x \leq^*_{\sigma\tau} y \leftrightarrow \forall u^\sigma v^\sigma (u \leq^*_\sigma v) \rightarrow xu \leq^*_\tau yv \wedge yu \leq^*_\tau yv)
\end{aligned}$$

Then $t^\rho \mathsf{\ mmr\ } \varphi$ (reads "t monotone modified realizes formula φ") is an abbreviation of

$$\exists x \leq^*_\rho t (x \mathsf{\ mr\ } \varphi)$$

where $\exists x \leq^* t\varphi(x)$ is the standard abbreviation for $\exists x(x \leq^* t \wedge \varphi(x))$. For soundness, we have the following theorem from [8]:

Theorem 2.4. *Let* $\mathsf{H}^\omega = \mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP}^{\mathrm{ef}}$*, and* φ *be an* $\mathsf{E\text{-}HA}^\omega$ *sentence. Then*

$$\mathsf{H}^\omega \vdash \varphi \Rightarrow \mathsf{E\text{-}HA}^\omega \vdash \exists \underline{x} \leq^* t^*(x \mathsf{\ mr\ } \varphi)$$

2.3 Models for HA^ω

Before interpreting HA^ω, we first think of how to interpret the terms. The idea is to interpret Gödel's T by Turing machines. Further, recall from recursion theory that one can encode Turing machines with natural numbers (called their Gödel numbering), in such a way that there is a universal machine U, and it can simulate any other TM given their encoding (and inputs). In this interpretation, however, the combinators are interpreted regardless of their types. For example we have a TM $\mathbf{p}$ which does the job for any σ and τ, and similar for the others. Then the application is interpreted as taking the second code as inputs for the first code. Besides, we can and do fix one such encoding so that 0 plays a special role: $0a = 0$ and $\mathbf{p}00 = 0$, for any $a \in \mathbb{N}$. More generally, if we consider the Turing machines as Kleene's first model $\mathcal{K}_1$ for partial combinatory algebra, than this possibility is guaranteed by the fixed-point theorem [15].

Below are listed three basic models for HA^ω introduced in [12]. To demonstrate the main idea, we focus on the structures $\langle \mathcal{M}_\sigma \rangle_{\sigma\in\mathcal{T}}$ for the finite types.

1. The first structure is the naive model FTS (*Full Type Structure*). Every $\mathcal{M}_\sigma$ consists of functionals of type σ. This is not constructive at all.

2. HRO, the model of *Hereditarily Recursive Operations* is based on the Gödel numbering

$$\begin{aligned}
\mathrm{HRO}_0 &:= \mathbb{N} \\
\mathrm{HRO}_{\sigma\times\tau} &:= \{z \in \mathbb{N} | \mathbf{p}_0 z \in \mathrm{HRO}_\sigma \text{ and } \mathbf{p}_1 z \in \mathrm{HRO}_\tau\} \\
\mathrm{HRO}_{\sigma\tau} &:= \{z \in \mathbb{N} | \forall x \in \mathrm{HRO}_\sigma, z \bullet x \in \mathrm{HRO}_\tau\}
\end{aligned}$$

   where $\bullet$ is the application in pca. To be precise, for HRO_0 we take the Church numerals (namely a canonical encoding of the natural numbers) rather than the natural numbers. As for the constants, app is interpreted as $\bullet$, and $=_\sigma$ as equalities between numbers, while the rest have their natural interpretations.

3. HEO, the model of *Hereditarily Effective Operations*, is also based on Gödel's numbering of TM. Define HEO_σ and partial equivalence relations $\sim_\sigma$ on HEO_σ as follows.

(a) $\text{HEO}_0 := \mathbb{N}$;

(b) $\sim_0$:= equality on natural numbers;

(c) $\text{HEO}_{\sigma\times\tau} := \{z \in \mathbb{N} | \mathbf{p}_0 z \in \text{HEO}_\sigma \text{ and } \mathbf{p}_1 z \in \text{HEO}_\tau\}$;

(d) For any $x, y \in \text{HEO}_{\sigma\times\tau}$, $x \sim_{\sigma\times\tau} y := \mathbf{p}_0 x \sim_\sigma \mathbf{p}_0 y \wedge \mathbf{p}_1 x \sim_\tau \mathbf{p}_1 y$;

(e) For any $x, y : \text{HEO}_\sigma \to \text{HEO}_\tau$, $x \sim_{\sigma(\tau)} y := (\forall uu' \in \text{HEO}_\sigma)(u \sim_\sigma u' \to xu \sim_\tau xu' \wedge yu \sim_\tau yu' \wedge xu \sim_\tau yu)$;

(f) $\text{HEO}_{\sigma(\tau)} := \{x \in \text{HEO}_\sigma \to \text{HEO}_\tau | x \sim_{\sigma\tau} x\}$.

The conditions (e) and (f) can also be replaced by:

(e') For any $x, y : \text{HEO}_\sigma \to \text{HEO}_\tau$, $x \sim_{\sigma(\rho)} y := (\forall u \in \text{HEO}_\sigma)(xu \sim yu)$

(f') $\text{HEO}_{\sigma\tau} := \{x \in \text{HEO}_\sigma \to \text{HEO}_\tau | (\forall uu' \in \text{HEO}_\sigma)(u \sim_\sigma u' \to xu \sim_\tau xu')\ \}$

In words, $\sim_{\sigma(\tau)}$ is defined extensionally, and the type structure $\text{HEO}_{\sigma(\tau)}$ consists of those functionals $x : \text{HEO}_\sigma \to \text{HEO}_\tau$ preserving the partial equivalence relation $\sim$. To be precise, the two $\sim_{\sigma(\tau)}$'s are not equivalent in the two definitions: in the latter, not every x with $x \sim y$ for some y is reflexive. But the two $\text{HEO}_{\sigma(\tau)}$'s are equivalent, since in the latter case we precisely restrict ourselves to those reflexive functionals.

To finalize the definition for typed structures, let equality $=_\sigma$ be interpreted as $\sim_\sigma$, for any finite type σ. The rest are interpreted as that in HRO.

One can easily see that, in HRO, $=_\sigma$ for higher order type σ is decidable (simply the equality of natural numbers); however, in HEO, $=_\sigma$ for higher order type is no longer decidable: this would entail the decidability of whether two numbers encode the same partial recursive functions. Though undecidable, equality in HEO is extensional:

$$x =_{\sigma\tau} y \iff \forall u^\sigma (xu \sim_\tau yu)$$
$$x =_{\sigma\times\tau} y \iff \mathbf{p}_0 x = \mathbf{p}_0 y \wedge \mathbf{p}_1 x = \mathbf{p}_1 y$$

for any $\sigma, \tau \in \mathcal{T}$, as observed in the alternative definition (e'). Recall adding the above extensionality axioms to HA^ω results in $\mathsf{E\text{-}HA}^\omega$. Therefore HEO is also a model for $\mathsf{E\text{-}HA}^\omega$. We are interested in the HEO model. In **Chapter** we present the vdBB structure for $\mathsf{E\text{-}HA}^\omega$, which incorporates the idea of HEO and Bezem's strong majorizability $\leq^*$.

2.4 Strong Majorizability

The notion of *strong majorizability* was introduced by Bezem [1] to provide a model $\mathfrak{M}$ for bar-recursion which admits discontinuous functions. The type structure $\mathfrak{M} = \bigcup \mathfrak{M}_\sigma$ is defined by simultaneous induction on both $\mathfrak{M}_\sigma$ and the strong majorizability relation $\leq^*_\sigma \in \mathsf{Pow}(\mathfrak{M}_\sigma \times \mathfrak{M}_\sigma)$ as follows:

1. $\mathfrak{M}_0 := \mathbb{N}$, and $\leq^*_0$ is simply the less or equal to relation $\leq_0$ on natural numbers.

2. For x, x' in $\mathfrak{M}_\sigma \to \mathfrak{M}\tau$, $x \leq^*_{(\sigma)\tau} x'$ *iff* for any $u, u' \in \mathfrak{M}_\sigma$ with $y \leq^*_\sigma y'$, $xu \leq^*_\tau x'u'$ and $x'u \leq^*_\tau x'u'$. Then the structure

$$\mathfrak{M}_{\sigma(\tau)} := \{x \in \mathfrak{M}_\tau^{\mathfrak{M}_\sigma} | \exists y \in \mathfrak{M}_\tau^{\mathfrak{M}_\sigma} x \leq^*_{(\sigma)\tau} y\}$$

Each $\mathfrak{M}_{(\sigma)\tau}$ consists of those strongly majorized functionals of type $\sigma \to \tau$. For any finite type σ, one can naturally construct a maximal functional $\max_\sigma : \sigma \to \sigma \to \sigma$ as follows:

1. $\max_0(x, y) = \max(x, y)$;
2. $\max_{\sigma(\tau)}(x, y) = \lambda u. \max_\tau(xu, yu)$.

For convenience, we say y is a majorant of x if $x \leq^* y$, and simply say y is a majorant, or that y is monotone if such x exists. Indeed, if $x \leq^*_{\sigma\tau} y$, then $y \leq^*_{\sigma\tau}$, and by definition we have that for any $u \leq^*_\sigma v$, $yu \leq^*_\tau yv$. This explains why we call a majorant "monotone". One can prove some handy properties for the strong majorizability:

Proposition 2.5. *Consider arbitrary finite type σ, and $x \leq^*_\sigma x'$, $y \leq^*_\sigma y'$, the following holds:*

1. $\max(x, y) \leq^*_\sigma \max(x', y')$;
2. $x' \leq^*_\sigma \max(x', y')$;
3. $\max \leq^*_{\sigma(\sigma\sigma)} \max$

Unsurprisingly, one can induce from $\max_\sigma$ the n-ary $\lambda x_1 \dots x_n. \max_\sigma(x_1, \dots, x_n)$ by finite iteration. Besides, one can prove that the combinators $\mathbf{k}$, $\mathbf{s}$, $\mathbf{R}$ (thus $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{i}$) for any finite types are self-majorizing, thus in $\mathfrak{M}$ (see [1] for more details). So we can do λ-abstraction in $\mathfrak{M}$, and $\mathfrak{M}$ contains all primitive recursive functions. Note that the essential difference between the weak majorizabillity $\leq_\sigma$ and the strong majorizability $\leq^*_\sigma$ is that, in the former we don't require a majorant to be monotone, but only:

$$x \leq_{\sigma(\tau)} y \leftrightarrow \forall u^\sigma v^\sigma (u \leq_\sigma v \to xu \leq yv)$$

In general one cannot compute a strong majorant from a weak majorant. But for types of the form $0 \to \sigma$ this is possible, by "taking the maximum of the first n values". Given arbitrary $F : \mathfrak{M}_0 \to \mathfrak{M}_\sigma$, define $F^+ := \lambda n^0. \max_{i \leq n} F(i)$. And this is indeed the construction of strong majorant one wants [1]:

Lemma 2.6. *Suppose functionals $G, F : \mathfrak{M}_0 \to \mathfrak{M}_\sigma$ satisfy that $Gn \leq^*_\sigma Fn$ for all $n \in \mathbb{N}$, then $G, G^+ \leq^* F^+$.*

Proof. First note that Fn is self-majorized, which immediately entails that $F^+(n)$ is self-majorized, for any n^0. Consider any $m \leq_0 n$,

$$G(m) \leq^*_\sigma F(m) \leq^*_\sigma \max_{i \leq n} F(i) = F^+(n) \tag{†}$$

So $G \leq^*_{0(\sigma)} F^+$. And since (†) holds for any $m \leq n$, $\max_{i \leq n} G(i) \leq^*_\sigma F^+(n)$ as well. Therefore $G^+ \leq^*_{0(\sigma)} F^+$. □

Moreover, the type space $\mathfrak{M}_{0(\sigma)}$ for types of the form $0(\sigma)$ are all-included:

Corollary 2.7. $\mathfrak{M}_\sigma^{\mathfrak{M}_0} = \mathfrak{M}_{0(\sigma)}$. *That is, every functional $F : \mathfrak{M}_0 \to \mathfrak{M}_\sigma$ is strongly majorized.*

Proof. For any $n \in \mathbb{N}$, $Fn \in \mathfrak{M}_\sigma$, so there is some $a_n \in \mathfrak{M}_\sigma$ s.t. $Fn \leq^* a_n$. Then $\lambda n^0.a_n$ satisfies that, for any $n \in \mathbb{N}$, $Fn \leq^*_\sigma (\lambda n^0.a_n)(n)$. And by **Lemma** 2.6 we have $F \leq^*_{0(\sigma)} (\lambda n^0.a_n)^+$. □

Theorem 2.8. *Bar induction holds in $\mathfrak{M}$.*

Not on bar induction but also bar recursion holds in $\mathfrak{M}$. Both rely crucially on the majorizability structure: one shrinks each $\mathfrak{M}_\sigma$ by throwing away those unmajorized; so the proof always requires showing that the constructed "good" functionals are majorized, thus inhabits in the structure $\mathfrak{M}$.

2.5 Model vdBB

One of the motivating idea for our majoriazability type and majorizability modified realizability assembly is the following van den Berg & Briseid's structure (called vdBB model) $\langle \mathcal{M}^s_\sigma \rangle_{\sigma \in \mathcal{T}}$ for finite types [14]. We define, simultaneously, two typed structures $\mathcal{M}^t, \mathcal{M}^s$ for finite types. Both are constructed in the HEO flavor, where we take the Gödel numbering of the Turing machines, thus of partial reccursive functions. The idea is to assign an explicit majorant for each functional, which cannot be applied directly in extensional operations. For example, we cannot compute a majorant by simply taking this explicit majorant, since pairs of the form (r, r^*) and r, r^{**} where $r^* \neq r^{**}$ are extensionally equal, but taking the explicit majorant result in different output. Still, they turned out to be useful in contexts where existence of a majorant rather than the precise majorant itself is demanded.

The inductive construction of $\mathcal{M}^s$ and $\mathcal{M}^t$ depend on each other. $\mathcal{M}^t_0$ is simply the (Gödel numbering of) natural numbers, where $\leq^t_0$ and $=^t_0$ are the $\leq_0$ and $=_0$ on natural numbers. For any finite type σ, define $\mathcal{M}^s_\sigma$ based on $\mathcal{M}^t_\sigma$ as:

$$\mathcal{M}^s_\sigma = \{(x, x^*) | x, x^* \in \mathcal{M}^s_\sigma,\ x \leq^t_\sigma x^*\}$$

with $=$ and $\leq$ defined pointwisely on the first component

- $(x, x^*) =^s_\sigma (y, y^*) \iff x =^t_\sigma y$
- $(x, x^*) \leq^s_\sigma (y, y^*) \iff x \leq^t_\sigma y$

For the functional type, define $\mathcal{M}^t_{\sigma(\tau)} = \{\text{codes for functions } f : \mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau\}$. And for $f, g \in \mathcal{M}^t_\tau$, say:

- $f =^t_{\sigma(\tau)} g \iff \forall (x, x^*) \in \mathcal{M}^s_\sigma (f(x, x^*) =^t_\tau g(x, x^*))$
- $f \leq^t_{\sigma(\tau)} g \iff \forall (x, x^*) \in \mathcal{M}^s_\sigma (f(x, x^*) \leq g(x^*, x^*) \wedge g(x, x^*) \leq g(x^*, x^*))$

An immediate consequence is that $f =^t_{\sigma(\tau)} g \iff \forall (x, x^*)(y, y^*),\ (x, x^*) = (y, y^*) \to f(x, x^*) = g(y, y^*)$, since equivalence in $\mathcal{M}^s$ only considers the first component. In another word, $=^t_{\sigma(\tau)}$ is extensional equality.

In $\mathcal{M}^s$ the application app is naturally interpreted as the following operation $\otimes$:

$$(f, f^*) \otimes (x, x^*) := (f(x, x^*), f^*(x^*, x^*))$$

and $\otimes$ can be easily verified to be an extensional function $\mathcal{M}^s_{\sigma(\tau)} \times \mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$. For simplicity we will often omit $\otimes$, when it's clear from the context that that we are talking about the application in $\mathcal{M}^s$. We can then express the $=$ and $\leq$ conditions on $\mathcal{M}^s_{\sigma\tau}$ totally in terms of $\mathcal{M}^s$ structure as follows:

(s1) $(x, x^*) =^s_{\sigma\tau} (y, y^*) \iff (\forall (u, u^*) \in \mathcal{M}^s_\sigma)(x, x^*) \otimes (u, u^*) = (y, y^*)(u, u^*)$

(s2) $(x, x^*) \leq^s_{\sigma\tau} (y, y^*) \iff (\forall (u, u^*) \in \mathcal{M}^s_\sigma)(x, x^*) \otimes (u, u^*) \leq^s_\tau (y, y^*) \otimes (u^*, u^*) \wedge (y, y^*) \otimes (u, u^*) \leq^s_\tau (y, y^*) \otimes (u^*, u^*)$

In short, $=^s_\sigma$ is extensional equality, and $\leq^s_\sigma$ is the strong majorizablity relation.

Now that we have two typed structures $\langle \mathcal{M}^t_\sigma \rangle_{\sigma \in \mathcal{T}}$ and $\langle \mathcal{M}^s_\sigma \rangle_{\sigma \in \mathcal{T}}$. The latter is of main interest for us, while the former is more for the convenience of construction. In fact, careful readers will notice that we even haven't defined application operation in $\mathcal{M}^t$, which is a necessity for interpretation of the terms in Gödel's T. To have a more intuitive picture of $\mathcal{M}^s$, let's conclude how $\mathcal{M}^s_{\sigma(\tau)}$'s look like. Every element in $\mathcal{M}^s_{\sigma(\tau)}$ is a pair (f, f^*) of $\mathcal{M}^t_{\sigma(\tau)}$ elements (thus functions $\mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau$), such that f^* is a $\mathcal{M}^t_{\sigma(\tau)}$-majorant of f. In particular, $\mathcal{M}^s_{\sigma\to\tau}$ is not the function space $\mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$ under ordinary application; rather it's the function space w.r.t. $\otimes$. The relation between $\mathcal{M}^s_\sigma \to \mathcal{M}^S_\tau$ and $\mathcal{M}^s_{\sigma\tau}$ is as follows. Given a function $f : \mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$, $\tilde{f} := (\lambda(x, x^*).(fx)_0, \lambda(x, x^*).(fx)_1) \in \mathcal{M}^s_{\sigma(\rho)}$ satisfies that for any $(x, x^*) \in \mathcal{M}^s_\sigma$, $\tilde{f} \otimes (x, x^*) = f(x, x^*)$.

Definition 2.9 (Combinators)**.** Define for any finite types σ, τ, ρ:

$$\begin{aligned}
\hat{\mathbf{k}}_{\sigma,\tau} &:= (\lambda(x, x^*)^\sigma(y, y^*)^\tau.x, \lambda(x, x^*)^\sigma(y, y^*)^\tau.x^*)\\
\hat{\mathbf{s}}_{\sigma,\tau,\rho} &:= (\lambda(x, x^*)^{\sigma(\tau\rho)}(y, y^*)^{\sigma(\tau)}(z, z^*)^\sigma.x(z, z^*)((y, y^*)(z, z^*)),\\
&\qquad\qquad \lambda(x, x^*)^{\sigma(\tau\rho)}(y, y^*)^{\sigma(\tau)}(z, z^*)^\sigma.x^*(z, z^*)((y, y^*)(z, z^*)))
\end{aligned}$$

Proposition 2.10. $\hat{\mathbf{k}}_{\sigma,\tau}, \hat{\mathbf{s}}_{\sigma,\tau,\rho} \in \mathcal{M}^s$.

Proof. Consider $\hat{\mathbf{s}}$, and the case for $\hat{\mathbf{k}}$ is easier. It suffices to show that $\hat{\mathbf{s}}_0 \le \hat{\mathbf{s}}_1$ in $\mathcal{M}^t_{(\sigma\tau\rho)(\sigma\tau)\sigma\rho}$. By definition, this is equivalent to saying that, given any $(x, x^*), (y, y^*), (z, z^*)$ of appropriate types,

$$\begin{aligned}
&x(z, z^*)((y, y^*)(z, z^*)) \le x^*(z^*, z^*)((y^*, y^*)(z^*, z^*))\\
\iff &x(z, z^*)((y, y^*)(z, z^*)) \le x^*(z^*, z^*)((y^*, y^*)(z^*, z^*))\\
\iff &x(z, z^*)(y(z, z^*), y^*(z^*, z^*)) \le x^*(z^*, z^*)(y^*(z^*, z^*), y^*(z^*, z^*))
\end{aligned}$$

which can be now easily reduced to the fact that $x \le x^*$, $t \le y^*$, $z \le z^*$. □

As for the recursor, one cannot trivially use the recursor R; there are mainly two difficulties. The first problem lies in the monotonicity of the second component. Essentially, if $f, f^* : 0 \to X \to X$ satisfies that $f \le f^*$ (where $\le$ is strong majorizability relation), then the monotonicity of $\mathbf{R}$ requires that $\mathbf{R}x_0f(n+1) = f(n+1, Rx_0fn)$, which is not guaranteed by that f^* is a majorant. The second trouble is that the recursor $\mathbf{R}$ is only for natural number but not for objects (n, n^*) in $\mathcal{M}^s_0$. We solve the two difficulties in order, by introducing $L_\sigma : \mathcal{M}^t_0 \to \mathcal{M}^s_\sigma \to \mathcal{M}^s_{0(\sigma\sigma)} \to \mathcal{M}^s_\sigma$ which does recursion on natural numbers and realizes the monotonicity; then we construct a recursor in $\langle \mathcal{M}^s_\sigma \rangle_{\sigma\in\tau}$ based on L.

Definition 2.11. Define $L_\sigma : \mathcal{M}^t_0 \to \mathcal{M}^s_\sigma \to \mathcal{M}^s_{0(\sigma\sigma)} \to \mathcal{M}^s_\sigma$ to be that, for any $(y, y^*) \in \mathcal{M}^s_\sigma$ and $(z, z^*) \in \mathcal{M}^s_{0(\sigma\sigma)}$,

- $L0(y, y^*)(z, z^*) := (y, y^*)$;
- $L(n+1)(y, y^*)(z, z^*) := (z(n, n)(Ln(y, y^*)(z, z^*)), t)$,

  where $t = \max\{(Ln(y, y^*)(z, z^*))_1, z^*(n, n)([L(n)(y, y^*)(z, z^*)]_1, [L(n)(y, y^*)(z, z^*)]_1)\}$, and the max function is defined inductively on $\mathcal{M}^t_\sigma$: $\max_{\sigma\to\tau}\{x, y\} = \lambda(u, u^*) \max_\tau\{x(u, u^*), y(u, u^*)\}$.

That such L_σ exists is guaranteed by the recursion schema on natural numbers. What's more, we claim that for any $n \in \mathcal{M}^t_0$, and $(y, y^*), (z, z^*)$ of appropriate types, $Ln(y, y^*)(z, z^*) \in \mathcal{M}^s_\sigma$. The main issue is showing that the second component majorizes the first, with a simply proof by induction of n. For $n = 0$, $y \le^t_0 y^*$ is assumed. For $n+1$, the IH tells us that $[Ln(y, y^*)(z, z^*)]_0 \le^t [Ln(y, y^*)(z, z^*)]_1$, so we have the required majorizability conditions:

- $z \le z^*$

- $Ln(y,y^*)(z,z^*) \leq ([Ln(y,y^*)(z,z^*)]_1, [Ln(y,y^*)(z,z^*)]_1)$
- $z(n,n)Ln(y,y^*)(z,z^*) \leq z^*(n,n)([Ln(y,y^*)(z,z^*)]_1, [Ln(y,y^*)(z,z^*)]_1)$

Besides, $Ln(y,y^*)(z,z^*)$ is monotone. So the maximum of $Ln(y,y^*)(z,z^*)$ and $z^*(n,n)([Ln(y,y^*)(z,z^*)]_1, [Ln(y$ is a majorant of $z(n,n)(Ln(y,y^*),(z,z^*))$.

Now we are ready to define a real recursor in $\mathcal{M}^s$. Define

$$\hat{\mathbf{R}}_\sigma = (\lambda(n,n^*)(y,y^*)(z,z^*).[Ln(y,y^*)(z,z^*)]_0, \lambda(n,n^*)(y,y^*)(z,z^*).[Ln(y,y^*)(z,z^*)]_1)$$

where $(n,n),(y,y^*),(z,z^*)$ are of types $0, \sigma, 0 \to \sigma \to \sigma$, respectively. And we claim that $\hat{\mathbf{R}}_\sigma$ is a recursor in our typed structure $\mathcal{M}^s$. This boils down to two things: (1) $\hat{\mathbf{R}}_\sigma \in \mathcal{M}^s$, (2) $\hat{\mathbf{R}}_\sigma$ acts as a recursor. The first item is obvious by the construction of $\hat{\mathbf{R}}_\sigma$ from L_σ, and the relation between function space $\mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$ and $\mathcal{M}^s_{\sigma\tau}$. So we verify the second item as follows:

Proposition 2.12. *For arbitrary $(n,m) \in \mathcal{M}^s_0$, $(y,y^*) \in \mathcal{M}^s_\sigma$ and $(z,z^*) \in \mathcal{M}^s_{0(\sigma\sigma)}$,*

- $\hat{\mathbf{R}}(0,m)(y,y^*)(z,z^*) = (y,y^*)$
- $\hat{\mathbf{R}}(\mathsf{S}n,m)(y,y^*)(z,z^*) = (z,z^*)(n,m)[\hat{\mathbf{R}}(n,m)(y,y^*)(z,z^*)]$

Proof. The first item is quite obvious: note that $L0(y,y^*)(z,z^*) = (y,y^*)$. For the second case, note that in $\mathcal{M}^s$ it suffices to show that the first components coincide.

$$\begin{aligned}
(\hat{\mathbf{R}}_\sigma(\mathsf{S}n,m)(y,y^*)(z,z^*))_0 &= (L(\mathsf{S}n)(y,y^*)(z,z^*))_0 \\
&= z(n,n)(Ln(y,y^*)(z,z^*)) \\
((z,z^*)(n,m)[\hat{\mathbf{R}}(n,m)(y,y^*)(z,z^*)])_0 &= z(n,m)[\hat{\mathbf{R}}(n,m)(y,y^*)(z,z^*)] \\
&= z(n,m)(Ln(y,y^*)(z,z^*))
\end{aligned}$$

Since $(n,n) =^s_0 (n,m)$, the above two results are equivalent. Note that the second components may not be equivalent, but that does not matter:

$$\begin{aligned}
(\hat{\mathbf{R}}_\sigma(\mathsf{S}n,m)(y,y^*)(z,z^*))_1 &= (L(\mathsf{S}n)(y,y^*)(z,z^*))_1 \\
&= \max\{(Ln(y,y^*)(z,z^*))_1, \\
&\qquad z^*(n,n)([Ln(y,y^*)(z,z^*)]_1, [Ln(y,y^*)(z,z^*)]_1)\} \\
((z,z^*)(n,m)[\hat{\mathbf{R}}(n,m)(y,y^*)(z,z^*)])_1 &= z^*(m,m)([\hat{\mathbf{R}}(n,m)(y,y^*)(z,z^*)]_1, [\hat{\mathbf{R}}(n,m)(y,y^*)(z,z^*)]_1) \\
&= z^*(m,m)([Ln(y,y^*)(z,z^*)]_1, [Ln(y,y^*)(z,z^*)]_1)
\end{aligned}$$

even though both are majorants of the first component, say $z(n,m)(Ln(y,y^*)(z,z^*)$. □

Remark 2.13. Later we will see that the trick of constructing $\hat{\mathbf{R}}_\sigma$ from L_σ can be seen as a special case of the above process, in proving that MType has a natural number object.

Therefore we can conclude that $\langle \mathcal{M}^s_\sigma \rangle_{\sigma \in \mathcal{T}}$ is a model for Gödel's T. What's more, recall our earlier claim that $\langle \mathcal{M}^s \rangle_{\sigma \in \mathcal{T}}$ is designed to be a combination of Bezem's strong majorizability relation and HEO. This is clear once we observe that the conditions **(s1)**, **(s2)** for $=^s$ and $\leq^s$ can be equivalently stated as follows:

(s1') $(x,x^*) =^s_{\rho(\tau)} (y,y^*) \iff (\forall (u,u^*)(v,v^*) \in \mathcal{M}^s_{\sigma(\tau)})(u,u^*) =^s_\sigma (v,v^*) \to (x,x^*) \otimes (u,u^*) = (y,y^*) \otimes (v,v^*)$

(s2') $(x,x^*) \leq^s_{\rho(\tau)} (y,y^*) \iff (\forall (u,u^*)(v,v^*) \in \mathcal{M}^s_\sigma)(u,u^*) \leq^s_\sigma (v,v^*) \to (x,x^*) \otimes (u,u^*) \leq (y,y^*) \otimes (v,v^*) \wedge (y,y^*) \otimes (u,u^*) \leq (y,y^*) \otimes (v,v^*)$

In other words, $=^s_{\sigma(\tau)}$ is defined extensionally, while $\leq^s_{\sigma(\tau)}$ is defined in Bezem's style.

As introduced in the beginning of this section, one of the basic observation is that, for some propositions to hold in a constructivism setting, one requires only the *existence* of certain majorants rather than the majorants themselves. Indeed, we can prove that the Fan Theorem holds in the vdBB structure $\langle \mathcal{M}^s_\sigma \rangle_{\sigma \in \mathcal{T}}$.

3 Majorizability Types

From a categorical point of view, the HEO model for finite types are exactly the finite types in the category PERS of partial equivalence relations. In the above discussion, we have seen the vdBB model as a computational model for the Fan Theorem. In this chapter, we define a category MType of majorizability types. The finite types in MType are exactly those in vdBB. Therefore, MType preforms the same role to vdBB, as that of PERS to HEO.

Definition 3.1 (WPO)**.** A relation $\leq$ on set A is called a *weak partial order* if it's majorant-reflexive and transitive in the following sense:

1. $x \leq y \Rightarrow y \leq y$
2. $x \leq y \wedge y \leq z \Rightarrow x \leq z$

This notion of WPO is extracted from Bezem's strong majorizability relation [1]. So we adopt the same notion of the majorant and being monotone. One of the motivation is that, for y to be a majorant, we require some extra condition. For example in the strong majorizability relation we require monotonicity. We often uses a^* for majorant of a, in Bezem's flavour.

This definition has some results in some routine concepts for ordered structures. For example, it's unwise to define the least element under a WPO $\leq_A$ totally, since that would imply reflexivity for all elements and turns the WPO into a trivial one (namely a preorder). And it's more natural to have the following definition of the partial least element:

Definition 3.2. Given a WPO $(A, \leq_A)$, and we say $x \in A$ is the *partial smallest element* on A if for any $a \in A$ with $a \leq_A a$, we have $x \leq_A a$.

Definition 3.3. A *majorizability type* (maj-type as abbreviation) is a triple $(A, \leq_A, \sim_A)$, where:

- $0 \in A \subseteq \mathbb{N}$
- $\leq_A$ is a WPO on A with 0 the partial smallest element
- $\sim_A$ is an equivalence relation on A
- there is a computable function $\mu_A : A \to A$ such that $a \leq \mu_A a$ for any $a \in A$
- there is $\max_A \in \mathbb{N}$ such that:
  1. $a \leq a,\ b \leq b \Longrightarrow a, b \leq \max_A(a, b)$
  2. $a \leq a^*, b \leq b^* \Longrightarrow \max_A(a, b) \leq_A \max_A(a^*, b^*)$

such that $\leq_A$ is extensional w.r.t $\sim_A$: if $a \sim a'$, $b \sim b'$ and $a \leq b$, then $a' \leq b'$.

Note that the computable function μ_A guarantees that every element has a majorant, and we can even compute it. However, we need to point out that it's *not* required that μ_A is defined extensional. Sometimes we shall refer to this μ_A as a "computable majorant function", since it's a computable function that computes the majorants.

Remark 3.4. The computable majorant function μ_A is a bit artificial here, and can be some inconvenient. As an alternative, one can define maj-types as above except that the existence of μ_A is replaced by that every element is a pair $(a, a^*) \in A$ such that $a \leq a^*$. Then it's not hard to see that these two definitions are equivalent. Given a maj-type of this form, the computable

μ_A is simply $\lambda(x, x^*).x^*$. On the other hand, given a maj-type with majorant function μ_A, we can rewrite the elements as $(a, \mu_A(a))$, thus of the above form.

In fact, later we shall see that the exponentials and finite types in the category of majorizability types are exactly of the form.

Next we start to construct a category MType of majorizability types. This is not that straightforward: to define the morphisms in MType, we go through the concepts of quasi-morphisms and pre-morphisms. A *quasi-morphism* $(A, \leq_A, \sim_A) \to (B, \leq_B, \sim_B)$ is some $r \in \mathbb{N}$ that encodes a function $A \to B$ that preserves $\sim$ (or extensional). A *pre-morphism* $(A, \leq_A, \sim_A) \to (B, \leq_B, \sim_B)$ is a pair of quasi-morphisms, where r^* majorizes r (denoted as $r \preceq_{A,B} r^*$) in the following sense: for any $a \leq_A a^*$, we have $ra \leq_B r^*a^*$ and $r^*a \leq_B r^*a^*$. And it's easy to verify that $\preceq_{A,B}$ is also a WPO. Two pre-morphisms (r_0, r_0^*) and (r_1, r_1^*) are equivalent: $r_0 a \sim_B r_1 a$ for any $a \in A$. And this is denoted as $(r_0, r_0^*) \approx_{A,B} (r_1, r_1^*)$. When the context is clear, we will always omit the subscripts in $\preceq_{A,B}$ and $\approx_{A,B}$. One remark is that, later one shall see that $\preceq$ and $\approx$ corresponds to the majorizability structure of the exponentials.

A morphism $(A, \leq_A, \sim_A) \to (B, \leq_B, \sim_B)$ is an equivalence class of the pre-morphisms over $\approx$. Since every pre-morphism is at least $\approx$ with itself, it suffices to find a pre-morphism for the existence of a morphism. In particular, for any directed type $(A, \leq_A, \sim_A)$, $\mathbf{i} : A \to A$ is a monotone function that trivially preserves $\sim_A$ and $\leq_A$. So $(\mathbf{i}, \mathbf{i})$ is a pre-morphism $A \to A$. Given two morphisms $[(r, r^*)] : B \to C$ and $[(s, s^*)] : A \to B$, we have a morphism $[(r \circ s, r^* \circ s^*)] : A \to C$ as their composition, where $r \circ s := \lambda x.r(sx)$. So the directed types and the morphisms between them form a category of directed types, called MType.

For the commutativity in MType, it also suffices to check the pre-morphisms. Given $[(s, s^*)] : B \to C$ and $[(r, r^*)] : A \to B$, and $[(t, t^*)] : A \to C$, then the triangle commutes means that the two morphisms fall in the same pre-morphism equivalence class: $(s, s^*) \circ (r, r^*) \approx (t, t^*)$. That is to say, for any $a \in A$, $(s \circ r)a \sim_B ta$ holds.

Now we show some basic properties of the category MType:

Proposition 3.5. MType *is has finite products.*

Proof. The final object is $1 = (\{0\}, \leq_1, =)$, where $\leq_1$ is trivial. That there exists a unique morphism $(A, \leq_A, \sim_A) \to (\{0\}, \leq_1, =)$ is because that the constant function is computable, self-majorized, and the equivalence relation on 1 is trivial.

The product is computed pointwise. Given directed types $(A, \leq_A, \sim_A)$ and $(B, \leq_B, \sim_B)$, their product is $(A \times B, \leq_A \times \leq_B, \sim_A \times \sim_B)$, where $A \times B = \{\mathbf{p}ab | a \in A, b \in B\}$, and the relations taken pointwise. Obviously we can compute the majorants by pairing the respective majorant functions, so this is again computable. The max function on $A \times B$ is also constructed pairwise. As for the projections, $(\mathbf{p}_0, \mathbf{p}_0) : A \times B \to A$ is a pre-morphism: $\mathbf{p}_0$ preserves $\sim$ and $\leq$, and it's monotone since $\sim$ on the product $A \times B$ is defined pairwise. So $[(\mathbf{p}_0, \mathbf{p}_0)]$ is a MType morphism.

In both cases, the UMP is easy to verify. For example, given $(X, \leq_X, \sim_X)$, morphisms $[(r_A, r_A^*)] : X \to A$ and $[(r_B, r_B^*)] : X \to B$, one can take arbitrary (s_A, s_A^*) and (s_B, s_B^*) from the equivalence class and construct a pre-morphism $(\lambda x.\mathbf{p}(s_A x)(s_B x), \lambda x.\mathbf{p}(s_A^* x)(s_B^* x))$. What's more, such pre-morphisms are all equivalent. So one can construct the morphism:

$$[(\lambda x.\mathbf{p}(r_A x)(r_B x), \lambda x.\mathbf{p}(r_A^* x)(r_B^* x))]$$

whose uniqueness is guaranteed by the commutativity condition.

Therefore we can conclude that MType has finite products. $\square$

Proposition 3.6. MType *has binary coproduct.*

Proof. Given $(A, \leq_A, \sim_A)$ and $(B, \leq_B, \sim_B)$, their coproduct is $(A+B, \leq_A + \leq_B, \sim_A + \sim_B)$, where $A + B := \{\mathbf{pk}a, \mathbf{p\bar{k}}b | a \in A, b \in B\}$, and the rest structure follows naturally. The morphisms are $\tau_A := [(\lambda a.\mathbf{pk}a, \lambda a.\mathbf{pk}a)]$ and $\tau_B := [((\lambda b.\mathbf{pk}a, \lambda a.\mathbf{pk}a))]$

Given any $(X, \leq_X, \sim_X)$ and morphisms $[(r_A, r_A^*)] : A \to X$, $[(r_B, r_B^*)] : B \to X$, the unique morphism $A + B \to X$ is represented by $(\lambda m.(\mathbf{p}_0 m) r_A r_B (\mathbf{p}_1 m), \lambda m.(\mathbf{p}_0 m) r_A^* r_B^* (\mathbf{p}_1 m))$. This is because any $m \in A + B$ is either $\mathbf{pk}a$ or $\mathbf{p\bar{k}}b$, and w.l.o.g. if $m = \mathbf{pk}a$ then

$$\begin{aligned}(\lambda m.(\mathbf{p}_0 m) r_A r_B (\mathbf{p}_1 m))(\mathbf{pk}a) &= \mathbf{k} r_A r_B a \\ &= r_A a\end{aligned}$$

and $r_A a \in X$ since (r_A, r_A^*) is a premorphism $A \to X$. Besides it's obvious that

$$\lambda m.(\mathbf{p}_0 m) r_A r_B (\mathbf{p}_1 m) \preceq \lambda m.(\mathbf{p}_0 m) r_A^* r_B^* (\mathbf{p}_1 m)$$

so the pair is a premorphism $A + B \to X$. Finally, for the uniqueness, note that any premorphism $(s, s^*) : A + B \to X$ commuting the diagram requires that for any $a \in A$ and $b \in B$,

$$\begin{cases}(s \circ \tau_A) a = r_A a \\ (s \circ \tau_B) b = r_B b\end{cases}$$

which implies that

$$\begin{cases}s(\mathbf{pk}a) = r_A a \\ s(\mathbf{p\bar{k}}b) = r_B b\end{cases}$$

Note that elements in $A+B$ are exhausted by $\mathbf{pk}a$ and $\mathbf{p\bar{k}}b$, so by definition $s \approx \lambda m.(\mathbf{p}_0 m) r_A r_B (\mathbf{p}_1 m)$. That they lie in the same equivalence class means that the represented morphism is unique. □

Remark 3.7. MType has neither equalizer nor initial object.

Proposition 3.8. MType *has exponentials.*

Proof. The exponential is simply the pre-morphism space. Given MType objects $(A, \leq_A, \sim_A)$ and $(B, \leq_B, \sim_B)$, to keep the morphisms natural numbers (rather than equivalence classes of natural numbers), we cannot take arbitrary representatives in the equivalence classes. Instead we take just the pre-morphisms. Define the exponential B^A to be $(B^A, \leq_{B^A}, \sim_{B^A})$, where:

- $B^A = \mathsf{PreMor}(A, B)$, namely the set of MType pre-morphisms $A \to B$.
- $\leq_{B^A}$ is simply $\preceq$: $(r, r^*) \leq_{B^A} (s, s^*) \iff \forall a \leq_A a^*$ in A, $ra \leq_B sa^*$ and $sa \leq_B sa^*$
- $\sim_{B^A}$ is simply $\approx$; that is, $(r, r^*) \sim (s, s^*) \iff \forall a \in A$, $ra \sim_B sa$

First of all, it's easy to verify that $\leq_{B^A}$ is WPO on B^A, and $\sim_{B^A}$ is an equivalence relation on B^A. To see that $\leq_{B^A}$ is extensional, suppose $(r_0, r_0^*) \sim (r_1, r_1^*)$, $(s_0, s_0^*) \sim (s_1, s_1^*)$, and $(r_0, r_0^*) \leq (s_0, s_0^*)$. Then for any $a, a^* \in A$ with $a \leq_A a^*$, we have $r_1 a \sim_B r_0 a$, $r_0 a \leq_B s_0 a^*$, $s_0 a \sim_B s_1 a$, $s_0 a^* \sim_B s_1 a^*$. By the extensionality of $\leq_B$, we know that $r_1 a \leq s_0 a^* \leq s_1 a^*$; similarly $s_0 a \leq_B s_0 a^*$ implies $s_1 a \leq_B s_1 a^*$.

Besides, we have a computable function μ_{B^A} to compute a majorant for every pre-morphism $(r, r^*) \in B^A$, by simply taking the second component.

What's more, the $\max_{B^A}$ function on B^A can be defined as:

$$\max_{B^A}\{(r, r^*), (s, s^*)\} := (\lambda x. \max_B\{rx, sx\}, \lambda x. \max_B\{r^* x, s^* x\})$$

and it suffices to check three conditions: (1) it's a MType pre-morphism, (2) $\max_{B^A}$ is monotone, (3) $\max_{B^A}$ is a maximal function for monotone elements.

Now let's consider the UMP of exponentials. First of all let the evaluation morphism $ev : B^A \times A \to B$ be the morphism represented by the pairing of

$$(ev_0, ev_1) := (\lambda y.(\mathbf{p}_0(\mathbf{p}_0 y))(\mathbf{p}_1 y), \lambda y.(\mathbf{p}_0(\mathbf{p}_0 y))(\mathbf{p}_1 y))$$

which, in words, is applying the second component to the first one). Consider an arbitrary pre-morpihsm $(r, r^*) : X \times A \to B$. It's not hard to verify that the pairing of $\tilde{r} := \lambda x.\mathbf{p}(\lambda a.r(\mathbf{p}xa))(\lambda a.r^*(\mathbf{p}xa))$ and $\tilde{r}^* := \lambda x.\mathbf{p}(\lambda a.r^*(\mathbf{p}xa))(\lambda a.r^*(\mathbf{p}xa))$ is a pre-morphism $X \to B^A$, thus a representative for some morphism $X \to B^A$. To check that $(\tilde{r}, \tilde{r^*})$ commutes the exponential triangle, we just check the pre-morphism level. For any $\mathbf{p}xa \in X \times A$,

$$\begin{aligned}[ev_0 \circ (\tilde{r} \times \mathsf{id}_A)](\mathbf{p}xa) &= (\lambda y.(\mathbf{p}_0(\mathbf{p}_0 y))(\mathbf{p}_1 y))(\mathbf{p}(\tilde{r}x)a) \\ &= (\lambda a.r(\mathbf{p}xa))a \\ &= r(\mathbf{p}xa)\end{aligned}$$

As for the uniqueness, it suffices to verify that for any pre-morphism $(s, s^*) : X \to B^A$ that commutes the triangle, $(\tilde{r}, \tilde{r}^*) \approx (s, s^*)$ holds, so they represent the same morphism. By the definition of $\approx$,

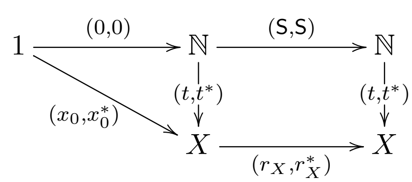
$$\begin{aligned}&(s, s^*) \approx (\tilde{r}, \tilde{r}^*) \\ \iff& s \sim \tilde{r} \text{ as quasi-morphism} \\ \iff& \forall x \in X, sx \sim_{B^A} \tilde{r}x \\ \iff& \forall x \in X, a \in A, ev_0(\mathbf{p}sx)a \sim_B ev_0(\mathbf{p}\tilde{r}x)a = r(\mathbf{p}xa) \\ \iff& \text{commutativity of the exponential diagram}\end{aligned}$$

We can finally conclude that B^A is the exponential. □

Proposition 3.9. MType *has a nno.*

Proof. We claim that $(\mathbb{N}, \leq_{\mathbb{N}}, \sim_{\mathbb{N}})$ where $\leq_{\mathbb{N}}$ is $\leq$ relation on $\mathbb{N}$ and $\sim_{\mathbb{N}}$ is natural number equality, is a natural number object in MType. The successor morphism is represented simply by the pre-morphism (S, S), where S can easily be verified to be monotone.

Consider an arbitrary majorizability type $(X, \leq_X, \sim_X)$, an endomorphism $[(r_X, r_X^*)]$, and a morphism $[(x_0, x_0^*)] : 1 \to X$. One demand the existence of a unique morphism $[(t, t^*)] : \mathbb{N} \to X$:



As one can imagine, we use the recursor $\mathbf{r}$ to form the pre-morphism $(t, t^*) : \mathbb{N} \to X$. However, we cannot apply $\mathbf{r}$ naively: the monotonicity will cause problem. For example, consider the pair:

$$(\lambda n.\mathbf{r}x_0(\lambda kx.r_X x)n, \lambda n.\mathbf{r}x_0^*(\lambda kx.r_X^* x)n)$$

though the first component works perfectly well, we cannot guarantee that the second component is monotone. Intuitively, for this we need the computable majorant function μ_X.

Let $\ell \in \mathbb{N}$ encodes the function $X \to (0 \to X \to X) \to (0 \to X)$ recursively refined as follows. Given any $y \in X$ and $s : 0 \to X \to X$, define ℓ as:

- $\ell ys0 = y$;
- $\ell ys(n+1) = \max_X\{\mu_X(\ell ysn), s(\ell ysn)n\}$

Then let the second component t^* of the pre-morphism be $\lambda n.\ell x_0^*(\lambda kx.r_X^* x)n$, and it suffices to verify that:

1. t^* is extensional
2. t^* majorizes t

Item (1) is trivial since $\sim_{\mathbb{N}}$ is natural number equality. Note that ℓ is not extensional, but this does not affect the extensionality of t^*. For item (2), given any $(m, m^*) \leq_0^s (n, n^*)$, we show that $t(m, m^*) \leq t^*(n, n^*)$ and $t^*(m, m^*) \leq t^*(n, n^*)$. Both are done by induction on n. For $n = 0$, m must also be 0, and $t(m, m^*) = x_0 \leq_X x_0^* = t^*(n, n^*)$. For $n+1$, we can prove the following two items in sequence: (a) $t^*(n, n^*) \leq t^*(n+1, n^*)$, (b) $t(n+1, n^*) \leq t^*(n+1, n^*)$. Then by IH we can easily derive that $t^*(m, m^*) \leq t^*(n+1, n^*)$ and $t(m, m^*) \leq t^*(n+1, n^*)$, for any $m \leq n+1$. Recall that by definition,

$$t^*(n+1, n^*) = \max\{\mu_X(t^*(n, n^*)), r_X^*(t^*(n, n^*))\}$$

For item (a), note that $\mu_X(t^*(n, n^*)) \geq t^*(n, n^*)$ (by definition of μ_X) and $r_X^*(t^*(n, n^*))$ is monotone (by IH). For item (b), note that $r_X \leq r_X^*$ and $t(n, n^*) \leq t^*(n, n^*)$ implies that $t(n+1, n^*) = r_X(t(n, n^*)) \leq r_X^*(t^*(n, n^*))$.

So far we have constructed a pre-morphism $(t, t^*) : \mathbb{N} \to X$. Now we check the correctness of (t, t^*):

Commutativity For the triangle, simply note that $t0 = x_0$. For the square, we can prove by induction on $n \in \mathbb{N}$ that $(r_X \circ t)n = t(\mathsf{S}n)$.

Uniqueness Suppose pre-morphism $(u, u^*) : \mathbb{N} \to X$ commutes the diagram. Then by the same analysis as above,

$$\begin{cases} u0 = x_0 \\ u(\mathsf{S}n) = (r_X \circ u)n \quad n \in \mathbb{N} \end{cases}$$

So u and $t = \lambda n.\mathbf{r}x_0(\lambda kx.r_X x)n$ are extensionally equivalent, and $(t, t^*) \approx (u, u^*)$.

Therefore we can conclude that $(\mathbb{N}, \leq_{\mathbb{N}}, \sim_{\mathbb{N}})$ is a nno in the category MType. □

Corollary 3.10. MType *is a cartesian closed category with nno.* ***it does not have equalizers***

The monomorphisms and epimorphisms in MType do *not* correspond to (codes of) injective and surjective functions. This is essentially because that two pre-morphisms represent the same MType morphism *iff* they are extensionally equal. So it's natural to guess that monomorphisms and epimorphisms in MType corresponds to (codes of) injective and surjective functions w.r.t. extensionality. Let's first give the formal definition:

Definition 3.11. A MType pre-morphism $(r, r^*) : X \to Y$ is injective w.r.t. extensionality if $\forall x_0, x_1 \in X$, $rx_0 \sim_Y rx_1$ implies $x_0 \sim_X x_1$; is surjective w.r.t. extensionality if $\forall y \in Y$, $\exists x \in X$ such that $rx \sim_Y y$.

And the definition can be naturally generalized to MType morphisms: $[(r,r^*)]$ is injective (surjective) w.r.t extensionality if every pre-morphism $(r_0,r_0^*)\in[(r,r^*)]$ is injective (surjective) w.r.t. extensionality. In fact, suppose (r_0,r_0^*) and (r_1,r_1^*) are both in equivalence class $[(r,r^*)]$. Then:

$$\begin{aligned}(r_0,r_0^*) \text{ is extensionally surjective} &\iff \forall y\in Y,\exists x\in X \text{s.t.} r_0 x\sim_Y y\\ &\iff \forall y\in Y,\exists x\in X \text{s.t.} r_1 x\sim_Y y\\ &\iff (r_1,r_1^*) \text{ is extensionally surjective}\end{aligned}$$

and similarly for injectivity. Now we can formalize the above intuition of characterizing monomorphisms and epimorphisms in MType. However, this intuition is only partly correct, as shown by the following proposition:

Proposition 3.12. *That an* MType *morphism* $[(r,r^*)]$ *is extensionally injective (surjective) implies that it's monic (epic).*

Proof. $[(r,r^*)]$ is epic *iff* for any morphisms $[(u,u^*)]$ and $[(v,v^*)]$ (of appropriate domains and codomains), $[(u,u^*)]\circ[(r,r^*)]=[(v,v^*)]\circ[(r,r^*)]$ implies $[(u,u^*)]=[(v,v^*)]$. In terms of pre-morphisms, *iff* for any pre-morphism (u,u^*) and (v,v^*), $(u,u^*)\circ(r,r^*)\approx(v,v^*)\circ(r,r^*)$ implies $(u,u^*)\approx(v,v^*)$.

Suppose $(r,r^*):X\to Y$ is extensionally surjective, then for any $y\in Y$ there exists $x\in X$ s.t. $rx\sim_Y y$. If pre-morphisms $(u,u^*),(v,v^*):Y\to Z$ satisfy that $(u,u^*)\circ(r,r^*)\approx(v,v^*)\circ(r,r^*)$ while $(u,u^*)\not\approx(v,v^*)$, then there exists some $y\in Y$ s.t. $uy\not\sim_Z vy$. For this y there also exists some $x\in X$ such that $rx\sim y$. Then we have found some x such that $(u\circ r)x\not\sim(v,r)x$, contradictory to the assumption that $(u,u^*)\circ(r,r^*)\approx(v,v^*)\circ(r,r^*)$.

Applying a similar reasoning, $[(r,r^*)]$ is mono *iff* for any pre-morphisms (u,u^*) and (v,v^*), $(r,r^*)\circ(u,u^*)\approx(r,r^*)\circ(v,v^*)$ implies $(u,u^*)\approx(v,v^*)$. Suppose $(r,r^*):X\to Y$ is extensionally injective, namely for any $x_0,x_1\in X$ with $rx_0\sim rx_1$, we have $x_0\sim x_1$. Further suppose that pre-morphisms $(u,u^*),(v,v^*):W\to X$ satisfy that (v,v^*), $(r,r^*)\circ(u,u^*)\approx(r,r^*)\circ(v,v^*)$ but $(u,u^*)\not\approx(v,v^*)$. Then there exists $w\in W$ such that $uw\not\sim_X vw$. However, $(r\circ u)w\sim(r\circ v)w$, which implies $uw\sim vw$, contradiction.

And we can conclude that being extensionally injective (surjective) is sufficient condition of being monic (epic). ∎

However, the reverse is false, namely

We claim that for $[(r,r^*)]:X\to Y$ to be (part of) isomophism, it suffices to show that the extensionally injective and surjective morphism $[(r,r^*)]$ has a computable extensional inverse $[(s,s^*)]:Y\to X$: for any $x\in X$, $(s\circ r)x\sim_X x$. To see this, we prove that $s\circ r$ and $r\circ s$ respectively represent id_X and id_Y.

Given arbitrary $x\in X$, $(s\circ r)x\sim_X x$, so $(s\circ r)\approx\mathsf{id}_X$. Given arbitrary $y\in Y$, the extensional surjectivity entails that there exists $x\in X$ s.t. $y\sim_Y rx$. Then $s(rx)\sim x$ and that r is extensional imply that $(r\circ s)y\sim r(s(rx))\sim rx\sim y$.

Remark 3.13. When talking about natural number objects in MType, one might immediately think of the finite type $\mathcal{M}_0^s$ in the $\langle\mathcal{M}_\sigma^s\rangle_{\sigma\in\mathcal{T}}$ structure: there is a WPO relation $\leq_\sigma^s$ and equivalence relation $=_\sigma^s$ which behave well with each other. So it's unsurprising that from every $\mathcal{M}_\sigma^s$ one can construct a majorizability type; in particular, the majorizability type $(\mathcal{M}_0^s,\leq_0^s,=_0^s)$ is also a nno.

Given arbitrary finite type $\sigma\in\mathcal{T}$, $(\mathcal{M}_\sigma^s,\leq_\sigma^s,=_\sigma^s)$ is a majorizability type. The computable majorant function is simply taking the second component of a pair, and the maximal function $\max_\sigma$ is defined inductively from $\max_0$. Besides, $\mathcal{M}_{\sigma\times\tau}^s$ is the binary product of $\mathcal{M}_\sigma^s$ and $\mathcal{M}_\tau^s$ in

MType. The case for $\mathcal{M}^s_{\sigma(\tau)}$ and $\mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$ is a bit tricky. They are *not* precisely identical: note that $\mathcal{M}^s_{\sigma(\tau)}$ consists of pairs of functions $\mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau$, while the exponential is composed of pairs of quasi-morphisms $\mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$ (as MType objects). However, they are still very alike: in MType, $\mathcal{M}^s_{\sigma(\tau)} \cong \mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$. Recall the underlying sets for each majorizability type:

- $\mathcal{M}^s_{\sigma(\tau)} = \{(m, m^*) | m, m^* : \mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau \text{ and } \forall (x, x^*) \in \mathcal{M}^s_\sigma, m(x, x^*) \leq^t_\tau m^*(x, x^*)\}$
- $\mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau = \{(r, r^*) | r, r^* \in \mathsf{PreMor}(\mathcal{M}^s_\sigma, \mathcal{M}^s_\tau) \text{ and } r \leq r^*\}$

Consider the quasi-morphisms t_0 and t_1, where:

- $t_0 := \lambda(r, r^*).(\lambda(x, x^*).\mathbf{p}_0(r(x, x^*)), \lambda(x, x^*).\mathbf{p}_0(r^*(x, x^*)))$
- $t_1 := \lambda(m, m^*).(\lambda(x, x^*).m(x, x^*), \lambda(x, x^*).m^*(x^*, x^*))$

We have the following claims:

1. t_0 is quasi-morphism $(\mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau) \to \mathcal{M}^s_{\sigma(\tau)}$; t_1 is quasi-morphism $\mathcal{M}^s_{\sigma(\tau)} \to (\mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau)$. One mainly needs to check the extensionality.
2. $t_0 \leq t_0$. For any $(r, r^*) \leq (s, s^*)$ in the exponential $\mathcal{M}^s_\sigma \to \mathcal{M}^t_\tau$,

$$t_0(r, r^*) = (\lambda(x, x^*).\mathbf{p}_0(r(x, x^*)), \lambda(x, x^*).\mathbf{p}_0(r^*(x, x^*)))$$
$$t_0(s, s^*) = (\lambda(x, x^*).\mathbf{p}_0(s(x, x^*)), \lambda(x, x^*).\mathbf{p}_0(s^*(x, x^*)))$$

   and $\lambda(x, x^*).\mathbf{p}_0(r(x, x^*)) \leq \lambda(x, x^*).\mathbf{p}_0(s(x, x^*))$ implies that $t_0(r, r^*) \leq t_0(s, s^*)$
3. $t_1 \leq t_1$. Proof is similar.

   So far we can conclude that (t_0, t_0) and (t_1, t_1) are MType pre-morphisms.
4. Both (t_0, t_0) and (t_1, t_1) are extensionally injective and surjective. We simply prove two cases among the four.
   (a) For any $(r, r^*), (s, s^*) \in \mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$, suppose $t_0(r, r^*) \sim t_0(s, s^*)$ as elements in $\mathcal{M}^s_{\sigma(\tau)}$. Then $\lambda(x, x^*).\mathbf{p}_0(r(x, x^*)) \sim \lambda(x, x^*).\mathbf{p}_0(s(x, x^*))$ in $\mathcal{M}^\tau_t$, so for any (x, x^*), $\mathbf{p}_0(r(x, x^*)) \sim \mathbf{p}_0(s(x, x^*))$, and $r(x, x^*) \sim s(x, x^*)$ by definition of $\sim^s_{\sigma(\tau)}$. So $(r, r^*) \sim (s, s^*)$, and (t_0, t_0) is injective.
   (b) For any $(r, r^*) \in \mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$, we take $t_0(r, r^*)$, then

$$t_1(t_0(r, r^*)) = (\lambda(x, x^*).\mathbf{p}_0(r(x, x^*)), \lambda(x, x^*).\mathbf{p}_0(r^*(x, x^*)))$$

   and since $(t_1(t_0(r, r^*)))(x, x^*) = \mathbf{p}_0(r(x, x^*))$, we know that $t_1(t_0(r, r^*)) \sim (r, r^*)$. So (t_1, t_1) is extensionally surjective.
5. $[(t_0, t_0)]$ and $[(t_1, t_1)]$ form an isomorphism. By the above discussion, it suffices to show that (t_1, t_1) is a computable inverse of (t_0, t_0). But this is already shown above: $(t_1 \circ t_0)(r, r^*) \sim (r, r^*)$ for any $(r, r^*) \in \mathcal{M}^s_\sigma \to \mathcal{M}^s_\tau$.

Therefore we know that the finite type structures $\langle \mathcal{M}^s_\sigma \rangle_{\sigma \in \mathcal{T}}$ are, up to isomorphism, the products and exponentials in MType constructed from the natural number object. Later when we talk about the internal logic of the category of assemblies and interpret the finite types, we shall use this observation to use both the intuition of the $\langle \mathcal{M}^s_\sigma \rangle$ structure and categorical construction of the exponentials.

4 Majorizability Assembly

Definition 4.1. Define assemblies based on the directed types. A *majorizability modified assembly* (sometimes simply called assembly) is a tuple $(X, \mathsf{P}_X, \leq_X, \sim_X, \alpha_X)$, where:

- X is a set
- $(\mathsf{P}_X, \leq_X, \sim_X)$ forms a directed type
- $\alpha_X : X \to \mathsf{Pow}_i(\mathsf{P}_X)$, where $\mathsf{Pow}_i(\mathsf{P}_X)$ is the set of nonempty subsets of P_X

such that every $\alpha_X(x)$ is closed under $\sim_X$. The set $\mathsf{P}_X \subseteq \mathbb{N}$ is called the set of potential realizers, while $\alpha_X(x)$ is called the (set of) actual realizers for $x \in X$.

For simplicity, we shall sometimes refer to an assembly simply by its underlying set, and the assembly structure with corresponding subscripts, when no confusion arises. Defining the morphisms is, as that for majoriability type, a bit tricky. A map $f : X \to Y$ is *witnessed* if there exists some MType pre-morphism $(r, r^*) : \mathsf{P}_X \to \mathsf{P}_Y$ such that for any $x \in X$ and $a \in \alpha_X(x)$s, $ra \in \alpha_X(fx)$. But for a MMAsm morphism f, we require it to be not only witnessed but also majorant: there exist pre-morphism $(r, r^*) : \mathsf{P}_X \to \mathsf{P}_Y$ such that r witnesses f [2]. We shall use $(r, r^*) \Vdash f$ to denote that r witnesses f with r^* majorizing r, or simply $r \Vdash f$ when we don't need the majorant to be explicit. Given any assembly A, the identity function $\mathsf{id}_A : A \to A$ is always witnessed by $(\mathbf{i}, \mathbf{i})$. And given two morphisms f, g, their composition $g \circ f$ is witnessed by the composition of their witnesses $(r_g \circ r_f, r^*_g \circ r^*_f)$, where $r \circ s$ is abbreviation of $\lambda x.r(sx)$.

So the assemblies and morphisms form the category of majorizability modified assembly MMAsm.

The category MType embeds into MMAsm by the inclusion functor $I : \mathsf{MType} \to \mathsf{MMAsm}$ defined as follows. For the objects, let $I((A, \leq_A, \sim_A))$ be $(A/\sim_A, A, \leq_A, \sim_A, \alpha_A)$, with α_A mapping an equivalence class to the set of its members, and it's obviously an assembly. For the morphisms, given a MType morphism $[(r, r^*)] : \mathcal{A} \to \mathcal{B}$, $I([(r, r^*)])$ is the function f_r mapping $[a] \in A/\sim_A$ to $[ra] \in B/\sim_B$. First of all, f_r is well-defined since r is extensional. And for the same reason, f_r is witnessed by (r, r^*) (in fact by any $(s, s^*) \approx (r, r^*)$).

Note that any morphism in MMAsm requires a witness with majorant, which generates a MType pre-morphism. And any pre-morphisms witnessing the same MMAsm morphism should be in the same equivalence class, namely they represent the same MType morphism. Thus the functor I is full and faithful. Besides, I preserves finite products. In particular, the final object in MMAsm is simply $1 = (\{*\}, \{0\}, \leq, =, \alpha_1)$. What's more, we will see that I also preserves nno. But before proving that, let's first list some basic properties of MMAsm. And one shall note that MMAsm has some categorical structures, e.g. equalizer, that MType lacks.

Proposition 4.2. MMAsm *has finite limits.*

Proof. The terminal object 1 is already given. The binary product of $(A, \mathsf{P}_A, \leq_A, \sim_A, \alpha_A)$ and $(B, \mathsf{P}_B, \leq_B, \sim_B, \alpha_B)$ is $(A \times B, \mathsf{P}_A \times \mathsf{P}_B, \leq_\times, \sim_\times, \alpha_\times)$, where $\leq_\times$, $\sim_\times$ and $\alpha_\times$ are all defined pointwise. The projection morphisms are respectively witnessed by $\mathbf{p}_0$ and $\mathbf{p}_1$, and since they are monotone, they are covered by themselves. The UMP is trivial.

As for the equalizers, unlike the above case for binary products, we don't have equalizers in MType. To construct the equalizer of two MMAsm morphisms $f, g : A \to B$, we take the underlying set to be the corresponding equalizer in **Sets**, say $E = \{a \in A | f(a) = g(a)\}$. Then take P_E simply be P_A, $\leq_E$ and $\sim_E$ be respectively the restriction of $\leq_A$ and $\sim_A$ on E. And the morphism

[2] There are other alternatives

$\iota_E : E \to A$ is the inclusion map witnessed by $[(\mathbf{i}, \mathbf{i})]$. It's easy to see that the diagram commutes $f \circ \iota_E = g \circ \iota_E$. Finally let's check the UMP. Given any assembly X and morphism $h : X \to A$ such that $f \circ h = g \circ h$. Then $h' : X \to E$ such that $h'(x) = h(x)$ for any $x \in X$ is a well-defined function. What's more, $(r_{h'}, \hat{r}_{h'}) := (r_h, \hat{r_h})$ witnesses h', so h' is a pre-morphism that commutes the whole equalizer diagram. The uniqueness follows immediately from that for **Sets**. □

Proposition 4.3. MMAsm *has exponentials.*

Proof. The MMAsm exponentials also relies on the MType exponentials. Given two assemblies $(A, \mathsf{P}_A, \leq_A, \sim_A, \alpha_A)$ and $(B, \mathsf{P}_B, \leq_B, \sim_B, \alpha_B)$, their exponential is $(B^A, \mathsf{P}_B^{\mathsf{P}_A}, \leq_{B^A}, \sim_{B^A}, \alpha_{B^A})$, where:

- $B^A = \mathsf{MMAsm}(A, B)$, namely the morphism space
- $(\mathsf{P}_B^{\mathsf{P}_A}, \leq_{B^A}, \sim_{B^A})$ is the exponential $\mathsf{P}_A \to \mathsf{P}_B$ in MType;
- α_{B^A} maps a morphism f to those pre-morphisms (r, r^*) in MType such that $(r, r^*) \vDash f$.

Since every morphism $f : A \to B$ is witnessed with majorant, this is a well-defined MMAsm object. We go on to verify the UMP property of the exponentials.

The evaluation map $ev : B^A \times A \to B$ is simply the application function, witnessed by the monotone element $\lambda z.(\mathbf{p}_0(\mathbf{p}_0 z))(\mathbf{p}_1 z)$ (which is also the evaluation arrow in the category MType). Suppose $f : X \times A \to B$ is an MMAsm morphism, witnessed by $(r_f, \hat{r}_f)$. We claim that $\tilde{f} = \lambda xa.f(x, a)$ is an MMAsm morphism $X \to B^A$. If so, then the commutativity of the exponential triangle and the uniqueness are trivial. For this, one needs to show that $\tilde{f}$ is witnessed with cover. Let

$$s_{\tilde{f}} := \lambda r_x.(\lambda r_a.r_f(\mathbf{p}r_x r_a), \lambda r_a.\hat{r}_f(\mathbf{p}r_x r_a))$$
$$\hat{s}_{\tilde{f}} := \lambda r_x.(\lambda r_a.\hat{r}_f(\mathbf{p}r_x r_a), \lambda r_a.\hat{r}_f(\mathbf{p}r_x r_a))$$

and they are MType quasi-morphisms $\mathsf{P}_X \to \mathsf{P}_{B^A}$. Note that $(s_{\tilde{f}}, \hat{s}_{\tilde{f}}) \vDash \tilde{f}$ means that given any $x \in X$ and $r_x \in \alpha_X(x)$, $s_{\tilde{f}} r_x \in \alpha_{B^A}(\tilde{f}(x))$. And this in turn equals to saying that $(\mathbf{p}_0(s_{\tilde{f}} r_x), \mathbf{p}_1(s_{\tilde{f}} r_x))$ witnesses the MMAsm morphism $\tilde{f}(x) : A \to B$. Since the majorizability relation is quite obvious, it suffices to show that, for any $a \in A$ and $r_a \in \alpha_A(a)$, $r_f(\mathbf{p}r_x r_a) \in \alpha_B(\tilde{f}(x)(a))$. □

Proposition 4.4. MMAsm *has a natural number object.*

Proof. One can simply take the image of the nno in MType along functor I. Recall that $(\mathbb{N}, \leq_\mathbb{N}, \sim_\mathbb{N})$ is a natural number object in MType. And its image under I is $(\mathbb{N}, \mathbb{N}, \leq_\mathbb{N}, \sim_\mathbb{N}, \alpha_\mathbb{N})$, where $\alpha_\mathbb{N}(n) = \{\bar{n}\}$. And the successor morphism $suc : \mathbb{N} \to \mathbb{N}$ is witnessed by MType pre-morphism (S, S).

Suppose X is an assembly, $x_0 : 1 \to X$ and $f : X \to X$ are two MMAsm morphisms, witness respectively by (r_0, r_0^*) and (r_f, r_f^*). So $x_0(*) \in X$ is witnessed by $r_0 0$. Then the function $h : \mathbb{N} \to X$ inductively defined by

$$h(0) = x_0$$
$$h(n+1) = f(h(n))$$

is witnessed by $(\mathbf{r}(r_0 0)(\lambda ks.r_f s), \mathbf{r}(r_0^* 0)(\lambda ks.r_f^* s))$. The uniqueness follows from that of h in **Sets**. □

Since MMAsm has finite limits, it has pullbacks. Since the pullbacks will play an important role in interpreting logic in the category, it's worthwhile to have a closer look at the pullbacks in MMAsm. Unsurprisingly, the pullbacks consist of those in **Sets** with extra MType and witness structure. Given MMAsm morphisms $f_0 : X_0 \to Y$ and $f_1 : X_1 \to Y$, witnessed respectively by $(r_0, \hat{r}_0)$ and $(r_1, \hat{r}_1)$. The pullback is $X_0 \times_Y X_1 = (X_0 \times_Y X_1, \mathsf{P}_{\times_Y}, \leq_{\times_Y}, \sim_{\times_Y}, \alpha_{\times_Y})$, where $X_0 \times_Y X_1$, is the pullback of f_0 and f_1 in the category of **Sets**, the potential realizer set $\mathsf{P}_{\times_Y}$ is simply $\mathsf{P}_{X_0} \times \mathsf{P}_{X_1}$, and $\leq_{\times_Y}$, $\sim_{\times_Y}$ and $\alpha_{\times_Y}$ are restriction on those for the product $X_0 \times X_1$. And the projection map $\pi_i : X_0 \times_Y X_1 \to X_i$ is witnessed with cover by the monotone $\mathbf{p}_i$.

There is a forgetful functor $\mathcal{U} : \mathsf{MMAsm} \to \mathbf{Sets}$ which extracts the underlying set of the assembly and the functions for the morphisms. $\mathcal{U}$ is not full as there are uncomputable functions, which cannot be tracked. But $\mathcal{U}$ does preserves some basic categorical notions. $\mathcal{U}$ creates monomorphisms and epimorphisms. Consequently, an MMAsm morphism $X \to A$ is a subobject of A iff its underlying function is injective. And every subobject X of A is isomorphic to one whose underlying set is a subset of A: take the image of X under the subobject morphism, and keep the MType structure. So it suffices to consider only those subobjects whose underlying sets are subsets of A. Also $\mathcal{U}$ creates isomorphism.

Recall that a *cover* is a morphism that cannot factor through a strict subobject of its codomain. And a good factorization is the so-called "image-cover factorization". Before showing that in our category MMAsm there is also such good factorization, we shall first give a characterization of the covers in MMAsm. Then we will show that in MMAsm, there is a standard cover for every morphism, which is achieved by the "image-cover factorization".

Definition 4.5. We say an MMAsm morphism $f : X \to Y$ is *effectively epic* if there exists MType pre-morphism (s, s^*) satisfying:

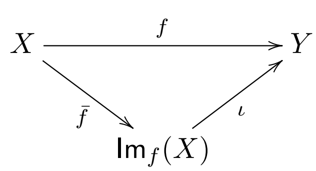
$$\forall y \in Y,\ \forall m_y \in \alpha_Y(y),\ \exists x \in f^{-1}(y) \text{ such that } s m_y \in \alpha_X(x)$$

$$\forall y \in Y,\ \exists x \in f^{-1}(y) \text{ such that } \forall m_y \in \alpha_Y(y),\ s m_y \in \alpha_X(x)$$

If one notes that the definition of an effectively epic morphism is nothing but listing the essential conditions for finding an "inverse" for a factorization, then it's straightforward to prove that, in MMAsm the covers are precisely those effectively epic morphisms. Before that, we first present a standard construction of cover from arbitrary morphism:

Proposition 4.6. MMAsm *admits "image-cover factorization".*

Proof. Let $f : X \to Y$ be an MMAsm morphism, witnessed by a MType pre-morphism $(r_f, \hat{r}_f)$. The image of f is $(\mathsf{Im}_f(X), \mathsf{P}_X, \leq_X, \sim_X, \alpha_{\mathsf{Im}})$, where $\mathsf{Im}_f(X)$ is the set-theoretical image and α_{Im} witnesses the "source". In another word, $\mathsf{Im}_f(X) = f[X]$, and $\alpha_{\mathsf{Im}}(y) = \bigcup_{f(x)} = \alpha_X(x)$. And it's an assembly, since the directed type structure is inherited from that of X. Then we can commute the diagram with the MMAsm morphism $\iota : \mathsf{Im}_f(X) \to Y$: just note that ι is witnessed by $(r_f, \hat{r}_f)$ as well. The diagram is as follow, where $\bar{f}$ is set-theoretically the same as f but categorically different (w.r.t. the codomains).



To see that $\bar{f}$ is a cover, suppose $g_0 : X \to Z$ and $g_1 : Z \rightarrowtail \mathsf{Im}_f(X)$ is a factorization of $\bar{f}$, where $(s_0, s_0^*) \vDash g_0$. We claim that g_1 is a bijection, and the inverse map $g_1^{-1} : \mathsf{Im}_f(X) \to Z$ is witnessed by (s_0, s_0^*) as well: $\forall y \in \mathsf{Im}_f(X)$ and $m_y \in \alpha_{\mathsf{Im}}(y)$, there is some $x \in X$ with $f(x) = y$ such that $m_y \in \alpha_X(x)$, so $s_0 m_y \in \alpha_Z(g_0(x))$. By commutativity, $g_1^{-1}(y) = g_0(x)$, and the above becomes that for any $m_y \in \alpha_{\mathsf{Im}}(y)$, $s_0 m_y \in \alpha_Z(g^{-1}(y))$. So (s_0, s_0^*) witnesses g^{-1}. Therefore g_1 is (part of) an isomorphism. □

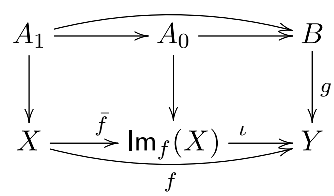
Next we extend the above proof to show that in MMAsm the covers are exactly those effectively epic morphisms.

Proposition 4.7. *In* MMAsm*, the covers are exactly the effectively epic morphisms.*

Proof. Suppose $f : X \to Y$ is a cover. By definition Y is isomorphic to $\mathsf{Im}_f(X)$, and $\iota : \mathsf{Im}_f(X) \to Y$ is oart of the isomorphism. This means that there exists an inverse of ι witnessed by pre-morphism (s, s^*). This means that, for any $y \in Y$, $m_y \in \alpha_Y(y)$, $sm_y \in \alpha_{\mathsf{Im}}(y)$. Since $\alpha_{\mathsf{Im}}(y) = \bigcup_{x \in f^{-1}(y)} \alpha_X(x)$, we know that there exists $x \in X$ such that $sm_y \in \alpha_X(x)$. So f is effectively epic.

Suppose $f : X \to Y$ is effectively epic, with the pre-morphism (s, s^*). Then it suffices to show that Y is isomorphic to the canonical cover $\mathsf{Im}_f(X)$. Note that we already have a morphism $\iota : \mathsf{Im}_f(X) \to Y$, so it remains to find another morphism $Y \to \mathsf{Im}_f(X)$ such that they two form an isomorphism. For this, let $g : Y \to X$ select, for any $y \in Y$, some $x \in X$ that satisfies the effectively epic condition. Then it's immediate that $g : Y \to X$ is a MMAsm morphism (witnessed by (s, s^*)). And the composition $f \circ g : Y \to \mathsf{Im}_f(X)$ is the demanded morphism. □

Recall that a category is *regular* if it's finitely complete and has pullback-stable image factorization. Now, for MMAsm to be regular, it only remains to be tested whether the image factorization above is stable under pullbacks. In other words,



suppose A_0 and A_1 are respectively the pullbacks of ι and f along g, while $A_1 \to A_0$ is the unique morphism induced by the pullback A_0, then $A_1 \to A_0 \to B$ is an image factorization.

For every MMAsm morphism $f : Y \to X$, one can induce a pullback functor $f^* : \mathsf{Sub}(X) \to \mathsf{Sub}(Y)$ by taking the pullback along f. And functor f^* has both left and right adjoints, say $\exists_f$ and $\forall_f$. Recall the image and dual image in **Sets** are defined as:

- $\mathsf{Im}_f(B) := \{x \in X | \exists y \in B, f(y) = x\}$
- $\mathsf{DIm}_f(B) := \{x \in X | \forall y \in B, f(y) = x \Rightarrow y \in B\}$

The left adjoint $\exists_f : \mathsf{Sub}(Y) \to \mathsf{Sub}(X)$ maps a subobject $\iota_B : B \rightarrowtail Y$ to its image through f. That is, $\exists_f(B) = \mathsf{Im}_{f \circ \iota_B}(B)$, and the . And it's action on morphisms is evident. And the adjointness is straightforward to be verified.

The rightadjoint $\forall_f : \mathsf{Sub}(Y) \to \mathsf{Sub}(X)$ maps a subobject $\iota_B : B \rightarrowtail Y$ to its dual image through f. The MType structure is a bit tricky, where for every $x \in \mathsf{DIm}_f(B)$ we require not only an actual realizer of it in X but also a track of the inclusion map $f^{-1}(x) \to B$. That is, $\forall_f(B) = (\mathsf{DIm}_f(B), \mathsf{P}_X \times (\mathsf{P}_Y \to \mathsf{P}_B), \sim_\forall, \leq_\forall, \alpha_\forall)$, where:

- $\sim_\forall$ and $\leq_\forall$ are defined pointwise from that for P_X and the exponential $\mathsf{P}_Y \to \mathsf{P}_B$

- $\alpha_\forall(x) = \alpha_X(x) \times \{\text{actual realizer for } f^{-1}(x) \to B\}$, and $f^{-1}(x) \to B$ is the set of MMAsm morphisms between $(f^{-1}(x), \mathsf{P}_Y, \leq_Y, \sim_Y, \alpha_{Y\restriction})$ and B.

Since in this paper we will focus on interpreting logic in the categories, those projections $\pi_Y : X \times Y \to X$ are of main interest to us. In this case, given a subobject $\iota_B : B \to X \times Y$ we can slightly simplify the potential realizers (consequently the actual realizers) in $\forall_{\pi_Y}(B)$. By the above definition, the potential realizer set is $\mathsf{P}_X \times (\mathsf{P}_{X\times Y} \to \mathsf{P}_B) = \mathsf{P}_X \times (\mathsf{P}_X \times \mathsf{P}_Y \to \mathsf{P}_B)$, and an actual realizer for $x \in \mathsf{DIm}(B)$ consists of a witness of it in X, together with a function $\mathsf{P}_X \times \mathsf{P}_Y \to \mathsf{P}_B$ such that works for any $a \in \mathsf{P}_X$. So we may simplify the potential realizer set to be $\mathsf{P}_X \times (\mathsf{P}_Y \to \mathsf{P}_B)$, and

All the above argument entails that MMAsm is a Heyting category. So every category of subobjects $\mathsf{Sub}(X)$ is a Heyting algebra, where morphisms are interpreted as the $\leq$ relation. Suppose $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ are two subobjects of X. Their meet $X_0 \wedge X_1$ is simply the pullback in MMAsm, or equivalently the product in the subobject category. Their join $X_0 \vee X_1$ is the image of the universal morphism $X_0 + X_1 \to X$. The Heyting implication $X_0 \to X_1$ is defined as $\forall_{f_0}(f_0^*(X_1))$, and can be verified as below:

$$\begin{aligned} & A \leq \forall_{f_0}(f_0^*(X_1)) \\ \iff & f_0^*(A) \leq (f_0^*(X_1) \\ \iff & f_0^*(A) \leq X_1 \end{aligned}$$

where $\leq$ is the relation in the algebra $\mathsf{Sub}(X)$.

To be more explicit, the $X_0 \to X_1$ has the underlying set $\{x \in X, x \notin X_0\} \cup (X_0 \cap X_1)$, the potential realizers $\mathsf{P}_X \times (\mathsf{P}_0 \to (\mathsf{P}_0 \times \mathsf{P}_1))$. The rest are defined accordingly. In fact, the potential realizers set can be simplified as $\mathsf{P}_X \times (\mathsf{P}_0 \to \mathsf{P}_1)$.

In particular, since the negated formulas $x.\neg\varphi(x)$ are defined as $x.\varphi(x) \to \bot$, its interpretation in MMAsm is $[x.\neg\varphi(x)] = [x.\varphi(x) \to \bot]$, whose potential realizer set is $\mathsf{P}_{[\varphi(x)]} \to \mathsf{P}_{[\bot]} = \mathsf{P}_{[\varphi(x)]} \to \{\bar{0}\}$, and an actual realizer for $\bar{x} \in [\neg\varphi(x)]$ is trivial (any potential realizer works, in particular $\bar{0} \in \mathsf{P}_{\neg\varphi(x)}$).

The functor $I : \mathsf{MType} \to \mathsf{MMAsm}$ preserves natural number objects. Suppose $(X, \mathsf{P}_X, \leq_X, \sim_X, \alpha_X)$ is an assembly, $x_0 : 1 \to X$ is a morphism mapping $*$ to $x_0 \in X$ and witnessed by $(t_0, \hat{t}_0)$, and $f : X \to X$ is a morphism witnessed by $(r_f, \hat{r}_f)$. The morphism $h : \mathbb{N} \to X$ is recursively defined as $h(0) = x_0$ and $h(n+1) = f(h(x))$, so $g : \mathcal{M}_0^s \to X$ defined as $g(n, n^*) = h(n)$ is a function witnessed by $(\lambda r.R(t_0 0)(\lambda k.r_f), \lambda r.R(\hat{t}_0 0)(\lambda k.\hat{r}_f))$ **(this witness is rubbish, need to be verified later)**.

Corollary 4.8. MMAsm *is a Heyting category with a natural number object.*

Later we will see that, a category's being regular means that we can interpret many-sorted first-order logic in out category MMAsm. In particular, we can interpret all the finite types. As a result, MMAsm is a model for HA^ω.

4.1 Basic Properties for Finite Types

We have a look at some basic properties of the finite types in MMAsm:

Proposition 4.9. *Let σ be a finite type.*

1. *For any $x, x' \in X$, if $\alpha_\sigma(x) \cap \alpha_\sigma(x') \neq \emptyset$, then $x = x'$.*

2. *Suppose $a, a' \in \mathsf{P}_\sigma$ satisfies that $a \sim_\sigma a'$ and $a \in \alpha_\sigma(x)$ for some x^σ, then $a' \in \alpha_X(x)$.*

3. *For any $a \in P_\sigma$, there is some $x \in \sigma$ such that $a \in \alpha_\sigma(x)$.*

4. *Suppose $a, a' \in \mathsf{P}_\sigma$ satisfies that $a, a' \in \alpha_\sigma(x)$ for some x^σ, then $a \sim_X a'$.*

Proof. 1. Suppose $\alpha_{\sigma\tau}(f) \cap \alpha_{\sigma\tau}(f') \neq \emptyset$, say (r, r^*) lies in the intersection. Then for any x^σ and its witness r_x, rr_x lies in both $\alpha_\tau(fx)$ and $\alpha_\tau(f'x)$. By IH this implies $f(x) = f'(x)$. And since this holds for any x^σ, we have $f =_{\sigma\tau} f'$.

2. Suppose $a, a' \in \mathsf{P}_{\sigma\tau}$, $a \sim_{\sigma\tau} a'$ and $a \in \alpha_\sigma(f)$ for some $f : \sigma \to \tau$. We shall prove that a' witnesses f as well. For any x^σ and $r_x \in \alpha_\sigma(x)$, $a'r_x \sim_\tau ar_x$ by the definition of $a \sim a'$. And IH tells us then that $a'r_x \in \alpha_\tau(fx)$. So a' witnesses f as well.

3. An element in $\mathsf{P}_{\sigma\to\tau}$ is of the form (r, r^*) where both r, r' are extensional functions $\mathsf{P}_\sigma \to \mathsf{P}_\tau$, and $r \leq r^*$. Then the underlying function $f : \sigma \to \tau$ does the following: given arbitrary x^σ and $r_x \in \alpha_\sigma(x)$, $f(x)$ is the y such that $rr_x \in \alpha_\tau(y)$ (whose existence guaranteed by *item* 1). To see that this f is well-defined, note that for other $r'_x \in \alpha_\sigma(x)$, $r_x \sim_\sigma r'_x$ so $rr_x \sim_\tau rr'_x$, and $r'_x \in \alpha_\tau(y)$ for the same y (*item* 2).

4. Suppose $a, a' \in \alpha_{\sigma\tau}(f)$ for some $f : \sigma \to \tau$. To show that $a \sim_{\sigma\tau} a'$, one requires that $ar \sim a'r$ for any $r \in P_\sigma$. Since every $r \in \mathsf{P}_\sigma$ witnesses some x^σ, it suffices to consider those r_x, and the rest is obvious.

□

So the finite type objects in MMAsm behave well: every potential realizer is an actual realizer for exactly one functional of the type; the equivalence relation $\sim_\sigma$ is encoding the same functional; the WPO $\leq_\sigma$ is the strong majorizability relation. Later when dealing with some mathematical principle, we will see that these properties turn out to be crucial. For convenience, we offer the following definitions:

Definition 4.10 (Modest Assembly)**.** An assmebly $(X, \mathsf{P}_X, \leq_X, \sim_X, \alpha_X)$ is *modest* if for any different $x, x' \in X$, $\alpha_X(x) \cap \alpha_X(x') = \emptyset$.

Definition 4.11 (Full Assembly)**.** An assembly $(X, \mathsf{P}_X, \leq_X, \sim_X, \alpha_X)$ is *full* if every $a \in \mathsf{P}_X$ sits in some $\alpha_X(x)$.

5 Logic in MMAsm

From a mathematical logic point of view, categories are interesting in that they provide structures for various logical systems. On one hand one can find suitable categories to interpret a given logic; on the other hand a category $\mathbf{C}$, one has its so-called *internal logic*. But the guideline is the same. The objects A in $\mathbf{C}$ are regarded as (interpretation of) types. The arrows $A \to B$ are taken as function symbols of type $A \to B$, or terms of type B with free variables of type A. If $\mathbf{C}$ has subobjects, then $X \rightarrowtail A$ can be seen as a predicate on type A, or collections of those objects of type A such that the formula (corresponding to) X is true. If $\mathbf{C}$ admits finite products and coproducts, then one can interpret conjunction and disjunction.

For further logical structures, let's start from $\mathbf{C}$ being regular. A regular category is a finitely complete category with pullback-stable image factorization. Then for any morphism $f : Y \to X$, the pullback functor $f^* : \mathsf{Sub}(X) \to \mathsf{Sub}(Y)$ has a left adjoint $\exists_f$ (by taking the image), which serves to interpret the existential quantifier. Since $\mathsf{Sub}(X)$ has the greatest element, and products in $\mathsf{Sub}(X)$ are essentially pullbacks in $\mathbf{C}$, we can interpret logics with truth, conjunction and existential quantifier. In another word, the internal logic of regular categories is the regular logic. Moving a step forward, if in each subobject category $\mathsf{Sub}(X)$ one has finite unions that are stable under pullbacks, then one can interpret disjunction and false (empty union) in $\mathbf{C}$. Such categories are called coherent categories, whose internal logic is coherent logic. So far, we still haven't included $\forall$ quantifiers into our language. And one way to achieve this is by requiring right adjoint for each f^*. The result is that each $\mathsf{Sub}(X)$ is now a Heyting algebra, which gives the name Heyting category, and one can interpret many-sorted first-order logic.

5.1 General framework

First of all, we will present a standard interpretation of many-sorted first order logic in any Heyting category. Next we will prove some handy lemmas for doing logic in MMAsm. In this chapter are listed well-known results, and one can find further details in [5].

Consider a language $\mathcal{L}$ consisting of:

- A set of basic types $X, Y, Z, \ldots$, and type constructors $\times$, $\to$;
- An infinite set of variables $x_1^X, x_2^X, \ldots$ for each type X;
- A set of relation symbols $R, S, \ldots$ each of a unique type. If R is of type $X_1 \times \cdots \times X_k$, we call $\mathsf{sg}(R) = (X_1, \ldots, X_k)$ its signature;
- A set of function symbols $f, g, h, \ldots$ each of a unique type. Given function $f : X_1 \times \cdots \times X_k \to Y$, call $\mathsf{sg}(f) = (X_1, \ldots, X_k)$ its signature, while $\mathsf{tp}(f) = Y$ its type;
- A specific relation $=_X$ for any type X.

The $\mathcal{L}$-terms are defined as routine. And each term t has a unique type denoted as $\mathsf{tp}(t)$. The $\mathcal{L}$-formulas consists of:

- Constants $\top$ and $\bot$;
- Atomic formulas $Rt_1 \cdots t_n$, where $\mathsf{tp}(t_i) = X_i$ and $\mathsf{sg}(R) = (X_1, \ldots, X_k)$;
- Formulas constructed from atomic formulas with logical connectives $\wedge, \vee, \to$ and quantifiers $\forall x^X$ and $\exists x^X$.

As usual, $\neg\varphi$ will be taken as abbreviation of $\varphi \to \bot$, and $\varphi \leftrightarrow \psi$ of $(\varphi \to \psi) \wedge (\psi \to \varphi)$. A sequence of distinct variables $\vec{x} = (x_1, \dots, x_n)$ is called a *context*. Following [3], we say a context is *appropriate* for a term t or a formula φ if all its free variables appear in $\vec{x}$, while no bounded variable does so. Sometimes we want to make the context salient, and following [3] we use $\vec{x}.t$ and $\vec{x}.\varphi$ to denote that term t and the formula φ is in context $\vec{x}$, respectively.

Now we are ready to interpret $\mathcal{L}$-formulas in a regular category $\mathbf{C}$. Every basic type (i.e. not built by type constructor) X is interpreted as an object $[X]$ of $\mathbf{C}$, and composed types are interpreted inductively by taking products and/or exponentials, depending on their structures. A tuple of types $(X_1, \dots, X_k)$ is also interpreted as the product $[X_1] \times \cdots \times [X_k]$. Each relation symbol R is interpreted as a subobject $[R] \rightarrowtail [\mathsf{sg}(R)]$, and each function symbol f as a morphism $[f] : [\mathsf{sg}(f)] \to [\mathsf{tp}(f)]$. In particular, $=_X$ is interpreted as $\Delta_X : X \rightarrowtail X \times X$. Recall that for a $\mathbf{C}$ morphism $f : Y \to X$, f^* is the pullback functor $\mathsf{Sub}(X) \to \mathsf{Sub}(Y)$ given by "pulling along f".

On the term level, the idea is to interpret each term $\vec{x}.t$ of type Y a morphism

$$[\vec{x}.t] : [X_1] \times \cdots \times [X_n] \to [Y]$$

where $X_1 \times \cdots \times X_n$ is the type of the context $\vec{x}$. So for each variable $\vec{x}.x_i$, define $[\vec{x}.x] := \pi_i : [X_1] \times \cdots \times [X_n] \to [X_i]$. A constant a^X is interpreted as $[a] : 1 \to [X]$. Given a function symbol $f : X_1 \times \cdots \times X_n \to Y$ and terms $t_1^{X_1}, \dots, t_n^{X_n}$, then the term $\vec{z}.ft_1 \cdots t_n$ is interpreted as a morphism:

$$[\vec{z}.ft_1 \cdots t_n] := \langle [\vec{z}.t_1], \dots, [\vec{z}.t_n] \rangle^* ([f])$$

in $\mathbf{C}([\mathsf{tp}(z)], [Y])$.

On the formula level, each formula is interpreted as a subobject of (the object associated with) its context. In particular, sentences are interpreted as subobjects of the terminal object 1. Recall that in a Heyting category, every $\mathsf{Sub}(A)$ has a HA structure. Assume that $\vec{x}$ is of type X, and y of type Y. The constants $\vec{x}.\top$ and $\vec{x}.\bot$ are interpreted as the top and bottom element of $\mathsf{Sub}([X])$. The boolean connectives are interpreted in accordance with counterparts in the HA structure. As for the quantified formulas, we use the adjunctions. Let $\pi : [X] \times [Y] \to [X]$ be the projection, then:

- $[\vec{x}.\exists y\varphi] := \exists_\pi [(\vec{x}, y).\varphi]$
- $[\vec{x}.\forall y\varphi] := \forall_\pi [(\vec{x}, y).\varphi]$

We say $\varphi \vdash_{\vec{x}} \psi$ is a *sequent* if $\vec{x}$ is appropriate type for both φ and ψ. And $\varphi \vdash_{\vec{x}} \psi$ is satisfied in category $\mathbf{C}$ if $[\varphi] \leq [\psi]$ in $\mathsf{Sub}([X])$. In particular, when φ is $\top$, we simply write $\mathbf{C} \vDash \psi$, and read "ψ is valid in $\mathbf{C}$". Now we fix a certain Heyting category $\mathbf{C}$.

Lemma 5.1 (Substitution). *Suppose $\vec{x}.\varphi$ is a $\mathbf{C}$-typed formula, and $\vec{y}.t_1, \dots, t_n$ is a sequence of terms such that $\vec{x}$ and $\vec{t}$ share the same arity and types: $\mathsf{tp}(t_i) = \mathsf{tp}(x_i)$. Then*

$$[\vec{y}.\varphi(\vec{t}/\vec{x})] = \langle [\vec{y}.t_1], \dots, [\vec{y}.t_n] \rangle^* [\vec{x}.\varphi]$$

Theorem 5.2 (Soundness). *Suppose Σ is a set of $\mathcal{L}$ formulas and φ an $\mathcal{L}$ formula such that $\Sigma \vdash \mathcal{L}$. If $\mathbf{C} \vDash \Sigma$, then $\mathbf{C} \vDash \varphi$. In particular, if Σ is empty, then $\mathbf{C} \vDash \varphi$.*

And there are some tricks in playing around with the internal logic of Heyting categories [5]:

Lemma 5.3. *Given a Heyting category $\mathbf{C}$,*

1. *$\mathbf{C} \vDash \varphi \to \psi$ iff $[\varphi]$ is a subobject of $[\psi]$*

2. $\mathbf{C} \vDash \forall x \varphi$ *iff* $[\varphi] \cong X$ *in* $\mathbf{C}$.

Now let's apply the general framework to our category MAsm. Given an assembly $(X, \mathsf{P}_X, \leq_X, \sim_X, \alpha_X)$, recall that the canonical subobjects are $(A, \mathsf{P}_A, \leq_A, \sim_A, \alpha_A)$ where $A \subseteq X$. So suppose φ and ψ are formulas of type X, then they are interpreted as subobjects $[\varphi]$ and $[\psi]$ of X. So we may assume that their underlying set are $X_0, X_1 \subseteq X$. By the definition, for $\mathbf{C} \vDash \varphi \to \psi$ one needs to show that $[\varphi]$ is a subobject of ψ. In this case, it suffices to show that: (1) $X_0 \subseteq X_1$, namely any $\vec{x} : X$ satisfying φ also satisfies ψ, (2) The inclusion map $\iota : X_0 \to X_1$ is witnessed. This means that there exists a premorphism $(r, r^*) : \mathsf{P}_0 \to \mathsf{P}_1$ such that for any $x \in X_0$ and $m_0 \in \alpha_0(x)$, $rm_0 \in \alpha_1(x)$.

As for the universal sentence $\forall x \varphi(x)$, one requires the isomorphism $[\varphi] \cong X$. Since the inclusion $\iota : [\varphi] \to X$ is trivial, it remains to show that one has the identity map $[\varphi] \to X$ and it's also witnessed.

Finally let's have a look at the existential statements. Suppose $\varphi(x, y)$ is of type (X, Y). Then to prove $\mathbf{C} \vDash \exists y \varphi(x, y)$, one needs to find a MMAsm morphism $X \to [\varphi(x, y)]$.

5.2 MMAsm and Monotone Modified Realizability

First of all we show that MMAsm is a model for extensional modified realizability mr. This means that MMAsm validates E-HA$^\omega$ and the characterization of mr, namely AC and IP$^\neg$.

Proposition 5.4. MMAsm $\Vdash$ E-HA$^\omega$

Proof. In addition to HA$^\omega$, we need to verify functional extensionality. For any MMAsm object X, Y,

$$\mathsf{MMAsm} \vDash \forall f^{Y^X} g^{Y^X} (\forall x^X (fx =_Y gx) \to f =_{Y^X} g)$$

By definition, this requires a computation from $[\forall x(fx = gx)]$ to $[f = g]$, both as subobjects of $Y^X \times Y^X$. This is essentially by **Lemma** 5.5. Note that ()the underlying set of) S consists of those functionals $f, g : X \to Y$ together with $x \in X$ on which they coincide. So $\forall_\pi(S)$ is exactly $[f, g. \forall x(fx = gx)]$. And by **Lemma** 5.5, $[f, g. \forall x(fx = gx)]$ factors through $\Delta : Y^X \to Y^X \times Y^X$, we know that there is morphism $[f, g. \forall x(fx = gx)] \to [f, g. f = g]$, which is demanded. ☐

The following lemma for extensionality can be found in [5], [3]:

Lemma 5.5. *Let* $\mathbf{C}$ *be a Cartesian closed Heyting category, then* $\mathbf{C}$ *internal extensionality. To be precise, let* X, Y *be arbitrary* $\mathbf{C}$ *objects, and*

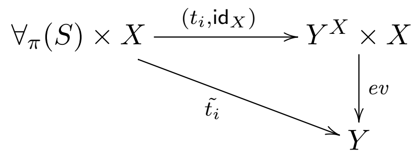
$$\begin{array}{ccc} S & \xrightarrow{k_1} & Y \\ {\scriptstyle k_0}\downarrow & & \downarrow{\scriptstyle \Delta} \\ Y^X \times Y^X \times X & \xrightarrow[ev_{0,2}, ev_{0,1}]{} & Y \times Y \end{array}$$

is a pullback. Then $\forall_\pi(k_0) : \forall_\pi(S) \to Y^X \times Y^X$ *factors through* $\Delta : Y^X \to Y^X \times Y^X$*, where projection* $\pi : Y^X \times Y^X \times X \to Y^X \times Y^X$.

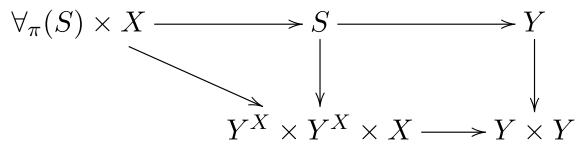
Proof. This is equivalent to showing that the compositions $t_i = \pi_i \circ \forall_\pi(k_0)$

$$\forall_\pi(S) \xrightarrow{\forall_\pi(k_0)} Y^X \times Y^X \xrightarrow{\pi_i} Y^X$$

for $i = 0, 1$, are equal. And this, since $\mathbf{C}$ is c.c., is again the same as the equivalence of their transposes:



Note that the pullback of $(t_0, t_1) : \forall_\pi(S) \to Y^X \times Y^X$ along $\pi : Y^X \times Y^X \times X \to Y^X \times Y^X$ is simply $(t_0, t_1, \mathsf{id}_X) : \forall_\pi(S) \times X \to Y^X \times Y^X \times X$. Then by the adjunction $f^* \dashv \forall_\pi$, we know that $\forall_\pi(S) \times X = \pi^* \circ \forall_\pi(S)$ factors through S. Therefore $\forall_\pi(k_0)$ factors through $\Delta : Y^X \to Y^X \times Y^X$ as in the following diagram:



where the left triangle is by the adjunction UMP, and the right square is the pullback diagram. □

For MMAsm to be a category for monotone modified realizability, one needs the following statement:

Conjecture 5.6. For any HA^ω sentence φ,

$$\mathsf{MMAsm} \vDash \varphi \iff \text{there exists } \mathsf{vdBB} \text{ term } t \text{ s.t. } t \ \mathsf{mmr} \ \varphi$$

Due to the limit of time and energy, we haven't yet formally (dis)proved the conjecture.

5.3 Some Constructivism Principles in MMAsm

Proposition 5.7. $\mathsf{MMAsm} \Vdash \mathsf{IP}^{\neg}_{\sigma,\tau}$

Proof. Recall that

$$\mathsf{IP}^{\neg}_{\sigma,\tau} \equiv \forall x^\sigma((\neg P(x) \to \exists y^\tau R(x, y)) \to \exists y^\tau(\neg P(x) \to R(x, y)))$$

and for simplicity let's denote it as $\forall x(\varphi(x) \to \psi(x))$. For this to hold in MMAsm, we need an algorithm (universal in $r_x \in \alpha_{[\sigma]}(\bar{x})$) that computes from a witness of $\varphi(\bar{x})$ to a witness of $\psi(\bar{x})$.

For the antecedent,

$$\begin{aligned}
\mathsf{P}_{[\varphi(x)]} &= \mathsf{P}_{[\neg P(x) \to \exists y R(x,y)]} \\
&= \mathsf{P}_{[\sigma]} \times (\mathsf{P}_{[\neg P(x)]} \to \mathsf{P}_{[\exists y R(x,y)]}) \\
&= \mathsf{P}_{[\sigma]} \times (\mathsf{P}_{[x.\neg P(x)]} \to \mathsf{P}_{[R(x,y)]}) \\
&= \mathsf{P}_{[\sigma]} \times (\mathsf{P}_{[x.P(x) \to \bot]} \to \mathsf{P}_{R[x,y]}) \\
&= \mathsf{P}_{[\sigma]} \times (\mathsf{P}_{[\sigma]} \times (\mathsf{P}_{[P(x)]} \to \mathsf{P}_{[\bot]}) \to \mathsf{P}_{[R]})
\end{aligned}$$

And the actual realizer for $\bar{x}$ in $[\varphi(x)]$ is a pair $(r_x, (r_\varphi, \hat{r}_\varphi))$, where $r_x \in \alpha_{[\sigma]}(x)$, and $(r_\varphi, \hat{r}_\varphi)$ is a pre-morphism that computes a witness of $R(\bar{x}, \bar{y})$ for some $\bar{y}$ from a witness of $\neg P(\bar{x})$. So we shall first have a look at how the two assemblies, $[\varphi(x)]$ and $[\psi(x)]$ looks like.

For the descendant, we have:

$$\begin{aligned}
\mathsf{P}_{[\psi(x)]} &= \mathsf{P}_{[\exists y(\neg P(x)\to R(x,y))]}\\
&= \mathsf{P}_{[\neg P(x)\to R(x,y)]}\\
&= \mathsf{P}_{[\sigma]}\times\mathsf{P}_{[\tau]}\times(\mathsf{P}_{[x,y.\neg P(x)]}\to\mathsf{P}_{[R]})\\
&= \mathsf{P}_{[\sigma]}\times\mathsf{P}_{[\tau]}\times(\mathsf{P}_{[x,y.P(x)\to\bot]}\to\mathsf{P}_{[R]})\\
&= \mathsf{P}_{[\sigma]}\times\mathsf{P}_{[\tau]}\times(\mathsf{P}_{[\sigma]}\times\mathsf{P}_{[\tau]}\times(\mathsf{P}_{[x,y.P(x)]}\to\mathsf{P}_{[\bot]})\to\mathsf{P}_{[R]})
\end{aligned}$$

The actual realizer for $\psi(\bar{x})$ is a triple $(r_x, r_y, (r_\psi, \hat{r}_\psi))$, where $r_x \in \alpha_{[\sigma]}(\bar{x})$, $r_y \in \alpha_{[\tau]}(\bar{y})$ for some $\bar{y}$, and $(r_\psi, \hat{r_\psi})$ is a pre-morphism such that r_ψ tracks a computation from arbitrary witness of $\neg P(\bar{x})$ to some witness of $R(\bar{x}, \bar{y})$.

Essentially, a witness of $\varphi(\bar{x}) \to \psi(\bar{x})$ should be able to compute a $\bar{y}$ for the descendant. And this is (partly) fulfilled by a witness for the projection $[R] \to [\tau]$. So let's assume that $[R] \to [\tau]$ is witnessed by $(r_y, \hat{r}_y)$. We claim that $(r_{\mathsf{IP}}, \hat{r}_{\mathsf{IP}})$ defined as:

$$\begin{aligned}
r_{\mathsf{IP}} &:= \lambda(r_x,(r_\varphi,\hat{r}_\varphi)).(r_x,(r_y\circ r_\varphi)(\mathbf{p}r_x0),(\lambda(s_x,s_y,t).r_\varphi(\mathbf{p}r_x0),\lambda(s_x,s_y,t).\hat{r}_\varphi(\mathbf{p}r_x0)))\\
\hat{r}_{\mathsf{IP}} &:= \lambda(r_x,(r_\varphi,\hat{r}_\varphi)).(r_x,(\hat{r}_y\circ \hat{r}_\varphi)(\mathbf{p}r_x0),(\lambda(s_x,s_y,t).\hat{r}_\varphi(\mathbf{p}r_x0),\lambda(s_x,s_y,t).\hat{r}_\varphi(\mathbf{p}r_x0)))
\end{aligned}$$

The first observation is that, if everything goes on well, then $r_{\mathsf{IP}} \le \hat{r}_{\mathsf{IP}}$ is trivial. And the verification boils down to the two levels of potential realizers and actual realizers.

Potential Given $(r_x, (r_\varphi, \hat{r}_\varphi)) \in \mathsf{P}_{[\varphi(x)]}$, $r_{\mathsf{IP}}(r_x, (r_\varphi, \hat{r}_\varphi))$ is a triple consisting of r_x, $r_y(r_\varphi(\mathbf{p}r_x0))$, and a pair $(\lambda(s_x, s_y, t).r_\varphi(r_x, \mathbf{p}_0 t), \lambda(s_x, s_y, t).\hat{r}_\varphi(r_x, \mathbf{p}_0 t))$. It's obvious that $r_x \in \mathsf{P}_{[\sigma]}$; note that $0 \in \mathsf{P}_{[P(x)]} \to \mathsf{P}_{[\bot]}$, and one knows that $r_y(r_\varphi(\mathbf{p}r_x0)) \in \mathsf{P}_\sigma$. And to see that

$$(\lambda(s_x,s_y,t).r_\varphi(\mathbf{p}r_x0),\lambda(s_x,s_y,t).\hat{r}_\varphi(\mathbf{p}r_x0))$$

is an element of $\mathsf{P}_{[\sigma]} \times \mathsf{P}_{[\tau]} \times (\mathsf{P}_{[x,y.P(x)]} \to \mathsf{P}_{[\bot]}) \to \mathsf{P}_{[R]}$, it only remains to check the majoriability relation, which is derived from that $r_\varphi \le \hat{r}_\varphi$. That $\hat{r}_{\mathsf{IP}}(r_x, (r_\varphi, \hat{r}_\varphi))$ lies in $\mathsf{P}_{[\psi(x)]}$ is similar. And it's easy to see that $r_{\mathsf{IP}} \le \hat{r}_{\mathsf{IP}}$ as MType quasi-morphisms.

Actual Suppose $(r_x, (r_\varphi, \hat{r}_\varphi))$ is a witness of $\varphi(\bar{x})$ (i.e. in $\alpha_{[\varphi(x)]}(\bar{x})$). $\mathbf{r}_{\mathsf{IP}}$ is still a triple, whose first component r_x is trivial. For the other two, we make a case distinction:

- $\bar{x} \in [x.\neg P(x)]$ Then $\alpha_{[\neg P(x)]}(\bar{x}) = \alpha_{[\sigma]}(\bar{x}) \times (\mathsf{P}_{[P(x)]} \to \mathsf{P}_{[\bot]})$, and in particular, $\mathbf{p}r_x0 \in \alpha_{[\neg P(x)]}(\bar{x})$. So $r_\varphi(\mathbf{p}r_x0) \in \alpha_{[\exists y R(x,y)]}(\bar{x}) = \bigcup_{\hat{y}\in[\tau]} \alpha_{[R]}(\bar{x}, \hat{y})$, and there is some $\bar{y}$ such that $r_\varphi(\mathbf{p}r_x0) \in \alpha_{[R]}(\bar{x}, \bar{y})$. What's more, $r_y(r_\varphi(\mathbf{p}r_x0)) \in \alpha_{[\tau]}(\bar{y})$ for this very same $\bar{y}$.
- $\bar{x} \notin [x.\neg P(x)]$. Then $(\bar{x}, \hat{y}) \notin [x, y].\neg P(x)$ for any $\hat{y} \in [\tau]$, which implies that actual realizers for $[x, y.\neg P(x, y)] \to [R(x, y)]$ are trivially those computing the potential realizers.

$$\begin{aligned}
\alpha_{[\psi(x)]}(\bar{x}) &= \bigcup_{\hat{y}\in[\tau]} \alpha_{[x,y.\neg P(x)\to R(x,y)]}(\bar{x},\hat{y})\\
&= \bigcup_{\hat{y}\in[\tau]} \alpha_{[\sigma]}(\bar{x})\times\alpha_{[\tau]}(\hat{y})\times(\mathsf{P}_{[x,y.\neg P(x)]}\to\mathsf{P}_{R(x,y)})
\end{aligned}$$

$r_\varphi(\mathbf{p}r_x0)$ certainly lies in $\mathsf{P}_{[R(x,y)]}$, and consequently $(r_y \circ r_\varphi)(\mathbf{p}r_x0)$ lies in $\mathsf{P}_{[\tau]}$. By the basic property of finite types in MMAsm, there exists some $\bar{y}$ such that $(r_y \circ r_\varphi)(\mathbf{p}r_x0) \in \alpha_{[\tau]}(\bar{y})$. So $(r_x, (r_y \circ r_\varphi)(\mathbf{p}r_x0), r_\varphi(\mathbf{p}r_x0)) \in \alpha_{[\psi(x)]}(\bar{x})$.

Now that $(r_{\mathsf{IP}}, \hat{r}_{\mathsf{IP}})$ witnesses $[\varphi(x)] \to [\psi(x)]$. So given any $\bar{x} \in [\sigma]$ and $r_x \in \alpha_{[\sigma]}(\bar{x})$, $(r_x, (r_{\mathsf{IP}}, \hat{r}_{\mathsf{IP}}))$ witnesses $[\varphi(\bar{x}) \to \psi(\bar{x})]$. This computation is universal in r_x. Therefore we can conclude that $\mathsf{MMAsm} \vDash \mathsf{IP}^{\neg}_{\sigma,\tau}$. □

Remark 5.8. Note that $\mathsf{IP}^{\neg}$ also holds for more types than the finite types. A closer look at the proof above tells us that we only used the fact that in $[\tau]$, every potential realizer is an actual realizer for some $\bar{y}^{\tau}$. Therefore $\mathsf{MMAsm} \vDash \mathsf{IP}^{\neg}_{X,Y}$, for arbitrary X, and those Y satisfying: $\forall a \in \mathsf{P}_Y$, $\exists y \in Y$ such that $a \in \alpha_Y(y)$.

Proposition 5.9. $\mathsf{MMAsm} \Vdash \mathsf{AC}_{\sigma,\tau}$ *for any* $\sigma, \tau \in \mathcal{T}$.

Proof. Recall that

$$\mathsf{AC}_{\sigma} \equiv \forall a^{\alpha}(\forall x^{\sigma}\exists y^{\tau} R(x,y,a) \to \exists f^{\sigma\tau}\forall x^{\sigma} R(x, fx, a))$$

For simplicity let's denote the above sentence as $\forall a(\varphi(a) \to \psi(a))$.To prove that $\mathsf{AC}_{\sigma\tau}$ holds in MMAsm, we need to find a universal computation from $\bar{a} \in [\alpha]$ and a witness r_a of a to $[\varphi(\bar{a}) \to \psi(\bar{a})]$. And that means a witness of $[\varphi(\bar{a})] \to [\psi(\bar{a})]$. So we first have a look at how $[\varphi(\bar{a})]$ and $[\psi(\bar{a})]$ look like.

First let's deal with the antecedent. $[R]$ is a subobject of $[\sigma] \times [\tau] \times [\alpha]$. By definition,

$$\begin{aligned}\mathsf{P}_{[\forall x \exists y R(x,y,\alpha)]} &= \mathsf{P}_{[\alpha]} \times (\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[\exists y R(x,y,a)]}) \\ &= \mathsf{P}_{[\alpha]} \times (\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[R]})\end{aligned}$$

And an actual realizer of $\bar{a}$ in $[\forall x \exists y R(x,y,a)]$ is a pair $(r_a, (r_\varphi, \hat{r}_\varphi))$, where $r_a \in \alpha_{[\alpha]}(\bar{a})$, and $(r_\varphi, \hat{r}_\varphi)$ is a pre-morphism such that r_φ computes, from every $\bar{x}$ and actual realizer r_x of $\bar{x}$, a witness of $R(\bar{x}, \bar{y}, \bar{a})$ for some $\bar{y} \in \tau$.

Turning to the descendant, note that $[R(x, fx, a)]$ is the subobject of $[\sigma] \times [\sigma \to \tau] \times [\alpha]$, constructed by the pullback of $[R] \rightarrowtail [\sigma] \times [\tau] \times [\alpha]$ along the natural evaluation $[\sigma \to \tau] \times [\sigma] \times [\alpha] \to [\sigma] \times [\tau] \times [\alpha]$.

$$\begin{aligned}\mathsf{P}_{[\exists f \forall x R(x,fx,a)]} &= \mathsf{P}_{[\forall x R(x,fx,a)]} \\ &= \mathsf{P}_{[\alpha]\times[\sigma\to\tau]} \times (\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[R(x,fx,a)]}) \\ &= \mathsf{P}_{[\alpha]\times[\sigma\to\tau]} \times (\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[R]})\end{aligned}$$

An actual realizer of $\bar{a}$ in $[\exists f \forall x R(x,y,a)]$ is a pair $((r_a, r_f), (r_\psi, \hat{r}_\psi))$, where $r_a \in \alpha_{[\alpha]}(a)$, $r_f \in \alpha_{[\sigma\to\tau]}(\bar{f})$ for some $\bar{f} \in [\sigma \to \tau]$, and $(r_\psi, \hat{r}_\psi)$ is a pre-morphism such that r_ψ computes, from $r_x \in \alpha_{[\sigma]}(\bar{x})$, a witness of $R(\bar{x}, \bar{f}\bar{x}, \bar{a})$.

Now let's turn to the proof of AC in MMAsm. Essentially one needs to be able to compute the choice function $\bar{f}$. The idea is that, note that one has the projection $\pi : [R] \to [\sigma \times \tau \times \alpha] \to [\tau]$ witnessed by some $(r_y, \hat{r}_y)$, and so $(r_y \circ r_\varphi, \hat{r}_y \circ \hat{r}_\varphi)$ witnesses a morphism $[\sigma] \to [\tau]$, and this is indeed what we want. That is, $(r_{\mathsf{AC}}, r\hat{_{\mathsf{AC}}})$ defined as:

$$\begin{aligned}r_{\mathsf{AC}} &:= \lambda(r_a, (r_\varphi, \hat{r}_\varphi)).((r_a, (r_y \circ r_\varphi, \hat{r}_y \circ r_\varphi)), (\lambda r_x.r_\varphi r_x, \lambda r_x.\hat{r}_\varphi r_x,)) \\ \hat{r}_{\mathsf{AC}} &:= \lambda(r_a, (r_\varphi, \hat{r}_\varphi)).((r_a, (\hat{r}_y \circ r_\varphi, \hat{r}_y \circ r_\varphi)), (\lambda r_x.\hat{r}_\varphi r_x, \lambda r_x.\hat{r}_\varphi r_x))\end{aligned}$$

witnesses $[\varphi(\bar{a})] \to [\psi(\bar{a})]$ (it is universal in $r_a \Vdash_{[\alpha]} \bar{a}$, and the majorizability relation is also obvious). And the proof boils down to the potential realizer and actual realizer levels.

Potential Suppose $(r_a, (r_\varphi, \hat{r}_\varphi)) \in \mathsf{P}_{[\varphi(\bar{a})]}$, i.e. $r_a \in \mathsf{P}_{[\alpha]}$, $(r_\varphi, \hat{r}_\varphi) \in \mathsf{P}_{[\sigma]} \to \mathsf{P}_{[R]}$. Then

$$r_{\mathsf{AC}}(r_a, (r_\varphi, \hat{r}_\varphi)) = ((r_a, (r_y \circ r_\varphi, \hat{r}_y \circ \hat{r}_\varphi)), (\lambda r_x.r_\varphi r_x, \lambda r_x.\hat{r}_\varphi r_x))$$

Note that both $r_y \circ r_\varphi$ and $\hat{r}_y \circ \hat{r}_\varphi$ are quasi-morphisms $\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[\tau]}$ and the latter majorizes the former, so their pairing is an element in $\mathsf{P}_{[\sigma\to\tau]}$. Besides, $\lambda r_x.r_\varphi r_x$ and $\lambda r_x.\hat{r}_\varphi r_x$ are both quasi-morphisms $\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[R]}$ with the majorizability relation, so their pairing is an element of $\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[R]}$.

Actual Suppose $(r_a, (r_\varphi, \hat{r}_\varphi))$ is an actual witness of $[\varphi(\bar{a})]$. Then r_a witnesses $\bar{a}$ in $[\alpha]$, and $r_\varphi \leq \hat{r}_\varphi$ (in $\mathsf{P}_{[\sigma\to R]}$) while r_φ computes, from any $\bar{x} \in [\sigma]$ and a witness r_x, a witness of $[R(\bar{x}, \bar{y}, \bar{a})]$ for some $\bar{y} \in [\tau]$. Again,

$$r_{\mathsf{AC}}(r_a, (r_\varphi, \hat{r}_\varphi)) = ((r_a, (r_y \circ r_\varphi, \hat{r}_y \circ \hat{r}_\varphi)), (\lambda r_x.r_\varphi r_x, \lambda r_x.\hat{r}_\varphi r_x))$$

First, given any $\bar{x} \in [\sigma]$ and $r_x \in \alpha_{[\sigma]}(\bar{x})$, both $r_y(r_\varphi \circ r_x)$ and $\hat{r}_y(\hat{r}_\varphi r_x)$ are in $\mathsf{P}_{[\tau]}$, so $(r_y \circ r_\varphi, \hat{r}_y \circ \hat{r}_\varphi)$ is a pre-morphism $\mathsf{P}_{[\sigma]} \to \mathsf{P}_{[\tau]}$. And by the property of the finite types in MMAsm we know that it witnesses some $\hat{f} : [\sigma] \to [\tau]$. Next, given the same r_x, $\lambda r_x.r_\varphi r_x$ computes, for exactly the same $\bar{x}$ and $\hat{f}$, a witness of $[R(\bar{x}, \hat{f}\bar{x}, \bar{a})]$. And the majorizability $(\lambda r_x.r_\varphi r_x) \leq (\lambda r_x.\hat{r}_\varphi r_x)$ is also obvious. So $r_{\mathsf{AC}}(r_a, (r_\varphi, \hat{r}_\varphi))$ is a witness of $[\psi(\bar{a})]$.

Since $r_{\mathsf{AC}} \leq \hat{r}_{\mathsf{AC}}$, the above argument show that pair $(r_{\mathsf{AC}}, \hat{r}_{\mathsf{AC}})$ witnesses $[\varphi(\bar{a})] \to \psi(\bar{a})$ (the inclusion map). So $\mathsf{MMAsm} \Vdash \forall a(\varphi(a) \to \psi(a))$. □

But there is a price to pay. CT and CONT fail in MMAsm. In fact they conflict with $\mathsf{E\text{-}HA}^\omega + \mathsf{AC}$. The Church's thesis CT basically claims that every total function is computable. This is formalized as follows [13]:

(CT) $\forall f \exists e \forall y \exists z (Teyz \wedge Uz = fy)$

Here T is the Klenne's T-predicate, where $Teyz$ basically says that z encodes a terminating computation of inputting y for the program encoded by e. U is the so-called "result-extracting function" that Uz returns the result from the code of a terminating computation z. The detailed proofs can be found in [13].

Proposition 5.10. $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} \vdash \neg\mathsf{CT}, \ \neg\mathsf{CONT}$

Proof. Let's scratch the proof of $\neg\mathsf{CT}$. AC and $\forall f \exists e \forall y \exists z (Teyz \wedge Uz = fy)$ implies that $\exists E \forall f \forall y \exists z (T(Ef)yz \wedge Uz = fy)$. Then extensionality implies that, for any extensionally equivalent functions f and g, E computes their codes Ef and Eg which are identical. Contradiction. □

6 The Fan Theorem and Related Mathematical Principles

Besides the general mathematics principles discussed above, in this section we have a look at some more specific constructivism principles, such as Fan Theorem.

6.1 Notations

Throughout the rest of this paper, we shall use the following notation:

- For a type X, X^* denotes the set of finite sequences of X objects, and $X^{\mathbb{N}}$ the set of infinite sequences of X objects.
- For a finite sequence $s = \langle x_1, \dots, x_n \rangle : X^*$, $|s|$ denotes the length of s, which in this case is n.
- Given finite sequences $s, t : X^*$, we use $s * t$ for their concatenation. For an object $x \in X$, we use $s * x$ as abbreviation of $s * \langle x \rangle$. Also, $s * \alpha$ denotes the concatenation where $\alpha : X^{\mathbb{N}}$ is an infinite sequence.
- For an infinite sequence $\alpha : X^{\mathbb{N}}$, $\alpha(i)$ denotes its i-th object, and $\bar{\alpha}(n)$ denotes the initial sequence $\langle \alpha(0), \dots, \alpha(n-1) \rangle$.
- A set $S \subseteq X^*$ is a *cover* (of $X^{\mathbb{N}}$) if for any $\alpha : X^{\mathbb{N}}$, there exists $s \in S$ such that $s = \bar{\alpha}(|s|)$ (i.e. s is an initial segment of α).
- For an object $a : X$, we shall use $\bar{a}$ to denote the infinite X sequence consisting solely of a.
- The canonical extension $\hat{s} : X^{\mathbb{N}}$ of $s : X^*$ is defined by $\hat{s}(i) = s(i)$ if $i < |s|$, and $\hat{s}(i) = 0_X$ otherwise. Use the notation above, $\hat{s} = s * \bar{0}_X$.
- Given $s, t : \sigma^*$, we say s is an *initial sequence* of t if $s(i) = t(i)$ for any $i < |s|$, and denote this as $s \sqsubseteq t$. And this definition is naturally generalized to where $t : \sigma^{\mathbb{N}}$.
- In this chapter, we will use $\leq$ instead of $\leq^*$ for the majorizability relation. This will not cause confusion since $\leq_0^*$ is just $\leq_0$.

Next we list out a few conditions that we will frequently refer to later. For predicate P on σ^*:

$$\mathbf{Dec}(P) := \forall u^{\sigma^*}(Pu \vee \neg Pu) \qquad \textit{Decidable}$$

$$\mathbf{Bar}(P) := \forall \alpha^{\sigma^{\mathbb{N}}} \exists k^{\mathbb{N}} P(\bar{\alpha}(k)) \qquad \textit{Bar}$$

$$\mathbf{Mono}(P) := \forall u^{\sigma^*} v^{\sigma^*}(u \sqsubseteq v \wedge Pu \to Pv) \qquad \textit{Monotone}$$

$$\mathbf{Back}(P) := \forall u^{\sigma^*}(\forall x^{\sigma}(P(u * x)) \to Pu) \qquad \textit{Backwards Induction}$$

$$\mathbf{Back}'(P) := \forall u^{\sigma^*}(\neg Pu \to \exists x^{\sigma} \neg P(u * x)) \qquad \textit{Backwards Induction'}$$

$$\mathbf{Uni}(P) := \exists m^{\mathbb{N}} \forall \alpha^{\sigma^*} P(\bar{\alpha}(m)) \qquad \textit{Uniform Bar}$$

$$\mathbf{Uni}'(P) := \exists m^{\mathbb{N}} \forall \alpha^{2^{\mathbb{N}}}(\exists k^{\mathbb{N}} \leq m) P(\bar{\alpha}(k)) \qquad textitUniformBar'$$

$$\mathbf{Triv}(P) := \forall u^{\sigma^*} P(u) \qquad \textit{Trivial Bar}$$

To work on these structure in MMAsm, we should be able to interpret all the types. In particular we need to interpret 2^* in MMAsm. For any natural number m, the object m in MMAsm is $(\{0, \dots, m-1\}, \mathsf{P}_m, \leq_m, \sim_m, \alpha_m)$, where $\mathsf{P}_m = \{\bar{0}, \dots \overline{m-1}\}$, and the rest defined as restriction of object $\mathbb{N}$ on P_m. In particular, 2 in MMAsm has the underlying set $\{0,1\}$. Then 2^m is the exponential object in MMAsm. The quasi-morphisms are computable functions $\{0,\ldots,m-1\} \to \{0,1\}$ (since preservation of $\sim$ is trivial); the equivalence of quasi-morphisms are simply extensional equivalence; and two quasi-morphisms $u \leq v$ if $\forall k, k^* \in m$ with $k \leq_2 k^*$, $uk \leq_2 vk^*$ and $vk \leq_2 vk^*$. In particular, the witness $\lambda i.\bar{1}$ of $1^m \in 2^m$ is the greatest quasi-morphism $\mathsf{P}_m \to \mathsf{P}_2$, where $1^m(i) = 1$ for any $i < m$.

Another approach is to build 2^* from scratch. Take the underlying set simply as 2^*, and the coding r_u for each finite sequence $u \in 2^*$ is a pair $r_u = \langle k, m\rangle$, where k is the length, and m the coding of the elements. Then we naturally have $\mathbf{p}_i$ to extract from r_u (a witness of) the i-th element in the sequence.

6.2 Fan Theorem

Let's first discuss the Fan Theorem. And in this thesis we shall focus on the that for binary sequences, namely $\Sigma = 2 = \{0,1\}$ (not to be confused with the coproduct $2 = 1+1$). Let $\mathbf{Tree}_2$ be the set of binary trees, namely $T \subseteq 2^*$ such that $T \neq \emptyset$, $\forall u, v \in 2^*(u \in T \wedge v \sqsubseteq u \to v \in T)$ and $\forall \alpha \in 2^{\mathbb{N}} \forall v \in 2^*(\alpha \in T \wedge v \sqsubseteq u \to v \in T)$. Here by $\alpha \in T$ where $\alpha \in T$, we mean that $\forall w \in 2^*(w \sqsubseteq \alpha \to w \in T)$.

(FT) $(\forall P \in \mathbf{Tree}_2)(\mathbf{Bar}(P) \to \mathbf{Uni'}(P))$

Adding the condition **Dec**, one gets the Decidable Fan Theorem ($\mathsf{FT_D}$). In brief, FT says that every bar has a uniform upper bound. The following is another commonly version of Fan Theorem, which we call the Monotone Fan Theorem:

($\mathsf{FT_{Mon}}$) $(\forall P \in \mathbf{Tree}_2)(\mathbf{Bar}(P) \wedge \mathbf{Mono}(P) \to \mathbf{Uni}(P))$

That $\mathsf{FT} \Rightarrow \mathsf{FT_{Mon}}$ is obvious. And the inverse direction requires $\mathbf{Mono}(P)$. The Fan Theorem holds classically, and since **Dec** is trivial classically, it implies that $\mathsf{FT}_{\mathcal{D}}$ holds as well. And to see this, note that FT is classically equivalent to the Weak König's Lemma which claims that every infinite tree has an infinite path. This is formalized as follows:

(WKL) $(\forall T \in \mathbf{Tree}_2)[\forall n^{\mathbb{N}} \exists u \in 2^*(|u| = n \wedge u \in T) \to \exists \alpha \in 2^{\mathbb{N}} \forall n^{\mathbb{N}} \bar{\alpha}(n) \in T]$

Since WKL holds classically and it is classically equivalent to FT, we know that FT holds classically []. However, in an intuitionistic setting things are different. For example, since CT is incompatible with FT, any system compatible with CT will not force FT [13]. On the other hand, $\mathsf{E\text{-}HA}^\omega$ does not force ¬FT either.

Axiom of Majorizability (MAJ) Everything has a majorant. And it's formalized as follows:

$$\forall^\sigma x \exists y^\sigma (x \leq_\sigma y)$$

Proposition 6.1. $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{MAJ} \vdash \mathsf{FT}$

Proof. By AC, $\forall \alpha \exists k P(\bar{\alpha}(k))$ implies $\exists F^{2^{\mathbb{N}} \to \mathbb{N}} \forall \alpha^{2^{\mathbb{N}}} P(\bar{\alpha}(F\alpha))$. Then by MAJ, $\exists F^*(F \leq F^*)$. Since $\forall \alpha^{2^{\mathbb{N}}}(\alpha \leq \bar{1})$, $\forall \alpha F\alpha \leq F^*\bar{1}$, and this $F^*\bar{1}$ is the upper bound m that we want. □

We claim that our category MMAsm provides a computable justification of the Fan Theorem. And this is not achieved, for example, by the famous effective topos Eff. The bad news is that, we haven't yet shown if MAJ holds in MMAsm, though we conjecture that it doesn't:

Conjecture 6.2. $\mathsf{MMAsm} \vDash \neg\mathsf{MAJ}$

If $\mathsf{MMAsm} \vDash \mathsf{MAJ}$, then FT follows immediately from **Proposition** 6.1. If $\mathsf{MMAsm} \vDash \neg\mathsf{MAJ}$, as we conjecture, then we can still do some trick to prove that FT holds in MMAsm. Basically this is because, even though MAJ is internally false, it is externally true: we can see from the "outside" of the model that every element in an assembly has a majorant. We will introduce some other mathematical principles that implies FT. Then proving them in MMAsm immediately entail that $\mathsf{MMAsm} \vDash \mathsf{FT}$.

6.3 Special Fan Functional

In [10] Norman and Sanders introduced the Special Fan Functional. As for the Fan Theorem, in the thesis we only focus on the binary sequences. First we informally state the following Special Fan Functionals Principles. Note that they do not have the demanded functional explicit in the statement, so they are not the "real" Special Fan Functional Principles, and we use the subscript 0 to distinguish it.

Special Fan Functional Principle(SFF_0) For any $F : 2^{\mathbb{N}} \to \mathbb{N}$, there exists $\alpha_1, \ldots, \alpha_n \in 2^{\mathbb{N}}$ such that

$$\{\bar{\alpha}_1(F\alpha_1), \ldots, \bar{\alpha_n}(F\alpha_n)\}$$

is a cover.

And we have another fan functional as follows:

Special Fan Functional Principle* (SFF_0^*) For any $F : 2^{\mathbb{N}} \to \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$\{\bar{\hat{u}}F(\hat{u}) | u \in 2^*, |u| \leq m\}$$

is a cover, and there is a least m for this. That is,

$$\mathsf{SFF}_0^* := \forall F^{(2^{\mathbb{N}})\mathbb{N}} \exists_{\mathsf{lst}} m^{\mathbb{N}} \forall \alpha \in 2^{\mathbb{N}} \exists u^{2^*} (|u| \leq m \wedge \overline{u * \bar{0}} F(u * \bar{0}) \sqsubseteq \alpha)$$

Note that the real functional requires, for example in SFF_0^*, the existence of $\Theta : (2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$. The above statements can be directed formalized of form $\forall\exists$, and under the Axiom of Choice the existence of such Θ is equivalent to the above statements. We will later give the real "functional" version of SFF, but before that these informal statements already suffices to prove some basic properties of the Special Fan Functional.

It's immediate that SFF_0^* implies SFF_0. With the restriction of F to those continuous functions, one also has the inverse direction:

Proposition 6.3. *In* $\mathsf{E\text{-}HA}^{\omega}$ *we have: for any continuous* $F : 2^{\mathbb{N}} \to \mathbb{N}$*, if there exists* $\alpha_1, \ldots, \alpha_n \in 2^{\mathbb{N}}$ *such that* $\{\bar{\alpha}_1(F\alpha_1), \ldots, \bar{\alpha}_n(F\alpha_n)\}$ *is a cover, then there exists* $m \in \mathbb{N}$ *such that* $\{\bar{\hat{u}}(F\hat{u}) | u \in 2^*, |u| \leq m\}$

Proof. Consider $F^{-1}[\{F\alpha_i\}]$. By the continuity of F, this is an open set, thus there is some $t_i \in 2^*$ such that $\alpha_i \in t_i * 2^{\mathbb{N}}$ and $t_i * 2^{\mathbb{N}} \subseteq F^{-1}[\{F\alpha_i\}]$. Let $k_i = \max\{|t_i|, F(\alpha_i)\}$, and $\beta_i := \bar{\alpha_i}(k_i) * \bar{0}$. Then we claim that the following conditions are satisfied:

1. $F\beta_i = F\alpha_i$
2. $\bar{\alpha_i}(k_i) = \bar{\beta_i}(k_i)$

If so, then note that $F\alpha_i \leq k_i$, and $\bar{\beta}_i(F\beta_i) = \bar{\alpha}_i(F\alpha_i)$. This implies that these $\bar{\beta}_i(F\beta_i)$'s form a cover, where each β_i is of the form $v_i * \bar{0}$. So let $m = \max_i |v_i|$, and $\{u * \bar{0} | u \in 2^*, |u| \leq m\}$ contains all β_i's, so $\{\overline{u * \bar{0}}(Fu * \bar{0}) | u \in 2^*, |u| \leq m\}$ is a cover.

For item (1), note that t_i is an initial sequence of β, so $F\beta_i = F\alpha_i$. For item (2), α_i and β_i are indeed constructed to coincide on the first k_i elements. Finally note that all the above discussion are valid in $\mathsf{E\text{-}HA}^\omega$. □

However, in general SFF_0 and SFF_0^* are not equivalent classically. And this follows immediately from the fact that SFF_0 holds classically, while SFF_0^* does not. For the former, one can basically use the fact that the topology on $2^{\mathbb{N}}$ with the base $\{s * 2^{\mathbb{N}} | s \in 2^*\}$ is compact (it's indeed a Cantor space).

Example 6.4. We given a counter example to show that SFF_0^* does not hold classically. This is done by offering some $F : 2^{\mathbb{N}} \to \mathbb{N}$ such that even

$$\{\overline{u * \bar{0}}(F(u * \bar{0})) | u \in 2^*\}$$

not a cover, not to say a finite subset of it.

Define $F : 2^{\mathbb{N}} \to \mathbb{N}$ to be that, for any sequence of the form $u * \bar{0}$, $F(u * \bar{0}) = |u| + 1$. So for any $u \in 2^*$, $\overline{u * \bar{0}}(F(u * \bar{0}))$ is exactly $u * 0$. But these finite sequences obviously cannot cover $2^{\mathbb{N}}$: consider $\bar{1}$, for example.

Besides, note that such F is not continuous. Suppose F is, then the inverse image $F^{-1}[\{n+1\}]$ should be open in $2^{\mathbb{N}}$. By definition, $F^{-1}[\{n+1\}]$ at least contains those $\alpha \in 2^{\mathbb{N}}$ such that $\alpha(i) = 0$ for all $i \geq n$. But the only open set containing these α is the whole set, which means that $F(\beta) = n + 1$ for any $\beta \in 2^{\mathbb{N}}$, contradiction.

Now let's formalize the real Special Fan Fucntionals SFF, SFF^* as follows:

Special Fan Functional Principle (SFF) $\exists \Theta^{(2^{\mathbb{N}}\mathbb{N})(2^{\mathbb{N}})^*} \forall F^{2^{\mathbb{N}}\mathbb{N}} \forall \alpha^{2^{\mathbb{N}}} \exists \beta^{2^{\mathbb{N}}} [\beta \in \Theta(F) \wedge \bar{\beta}(F\beta) \sqsubseteq \alpha]$

Special Fan Functional Principle* (SFF^*) $\exists \Theta^{(2^{\mathbb{N}}\mathbb{N})\mathbb{N}} \forall F^{2^{\mathbb{N}}\mathbb{N}} \forall \alpha^{2^{\mathbb{N}}} \exists u^{2^*} [|u| = \Theta(F) \wedge \overline{u * \bar{0}} F(u * \bar{0}) \sqsubseteq \alpha]$

And in both statements, the Θ are the Special Fan Functionals. In SFF, Θ selects a finite sequence of infinite binary sequences $\{\beta_1, \ldots, \beta_k\}$ such that $\{\bar{\beta}_i(F\beta_i) | 1 \leq i \leq k\}$ forms a cover of $2^{\mathbb{N}}$. Similar to that of SFF_0 and SFF_0^*, SFF_0 immediately implies SFF_0^* by taking m to be the greatest length of the sequences in $\Theta(F)$. Besides, SFF and SFF^* respectively imply SFF_0 and SFF_0^*. Then the failure of SFF_0^* classically implies that SFF^* fails classically. And AC holds classically implies that SFF holds classically as well.

Now we prove that SFF holds in MMAsm by showing that $\mathsf{MMAsm} \vDash \mathsf{SFF}^*$. The idea is to make use of the fact that for any MMAsm object X and $(x, x^*) \in X$, (x, x^*) has a majorant, namely (x^*, x^*).

Proposition 6.5. $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{MAJ} \vdash \mathsf{SFF}^*$

Proof. By MAJ, every $F : 2^{\mathbb{N}} \to \mathbb{N}$ has a majorant F^*, and AC tells us that there is a selection function for such F^* from F. And one can prove that in $2^{\mathbb{N}}$, $\bar{1}$ is the greatest element, namely:

$$\forall y^{2^{\mathbb{N}}} (\forall i^{\mathbb{N}} (y(i) = 1) \to \forall x^{2^{\mathbb{N}}} (x \leq y))$$

So by the definition of $\leq$, one has $Fx \leq m^* := F^*\bar{1}$ for any $x^{2^{\mathbb{N}}}$.

We claim that $A = \{\overline{u * \bar{0}}(F(u * \bar{0}))|u \in 2^*, |u| = F^*(\bar{1})\}$ is a cover. To see this, given arbitrary $\alpha \in 2^{\mathbb{N}}$, $\bar{\alpha}(m^*)$ has length $F^*(\bar{1})$, and satisfies the condition for A. Besides, since $F(\bar{\alpha}(m^*) * \bar{0}) \leq m^*$, we know that

$$\overline{\bar{\alpha}(m^*) * \bar{0}}F(\bar{\alpha}(m^*) * \bar{0}) \sqsubseteq \overline{\bar{\alpha}(m^*) * \bar{0}}(m^*) = \bar{\alpha}(m^*) \sqsubseteq \alpha$$

which means that every $\alpha \in 2^{\mathbb{N}}$ is covered. Therefore we can define $\Theta : (2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$ as $\Theta(F) = F^*(\bar{1})$. □

However, as mentioned above we don't expect to have MAJ in the internal logic of MMAsm. We can do the following trick. For any $F : (2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$, we know it has a majorant F^*, and the above argument shows that there exists some $M \in \mathbb{N}$ such that $\{\overline{u * \bar{0}}F(u * \bar{0})|u \in 2^*, |u| \leq M\}$ is a cover. Then we can do the following algorithm: starting from $m = 1$, one checks if $\{\overline{u * \bar{0}}F(u * \bar{0})|u \in 2^*, |u| \leq m\}$ is a cover. Such check terminates, since it suffices to check $2^m = \{u \in 2^* ||u| = m\}$. This algorithm terminates no later than $m = M$, and it computes the smallest such m.

Theorem 6.6. $\mathsf{MMAsm} \vDash \mathsf{SFF}^*$

Proof. Let's abbreviate SFF^* as $\forall F \exists n \varphi(F, n)$, and $\varphi(F, n) := \forall \alpha \exists u \psi(F, n, \alpha, u)$, then

$$\begin{aligned} \mathsf{P}_{[\forall F \exists n \varphi(F,n)]} &= \mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \to \mathsf{P}_{[\exists n \varphi(F,n)]} \\ &= \mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \to \mathsf{P}_{[\varphi(F,n)]} \end{aligned}$$

As for the actual realizers, note that $[\forall F \exists n \varphi(F, n)$ is a subobject of 1. And $\alpha_{[\forall F \exists n \varphi(F,n)]}(*)$ consists of those pre-morphisms $(r_\varphi, \hat{r}_\varphi) : \mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \to \mathsf{P}_{[\varphi(F,n)]}$, such that given any $F : 2^{\mathbb{N}} \to \mathbb{N}$ and a witness $(r_F, \hat{r}_F)$ of it, $r_\varphi r_F$ is a witness of $\varphi(F, n)$ for some $n \in \mathbb{N}$. Also note that $\mathsf{P}_{2^{\mathbb{N}}(\mathbb{N})}$ is full in the sense that every $a \in \mathsf{P}_{2^{\mathbb{N}}(\mathbb{N})}$ is an actual witness of some functional $2^{\mathbb{N}} \to \mathbb{N}$ (so the potential realizer is not a matter here).

Again, $\varphi(F, n) = \forall \alpha \exists u \psi(F, n, \alpha, u)$ is interpreted as a subobject of $[2^{\mathbb{N}}(\mathbb{N}) \times \mathbb{N}]$, and

$$\begin{aligned} \mathsf{P}_{[\varphi(F,n)]} &= \mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \times \mathsf{P}_{[\mathbb{N}]} \times (\mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \times \mathsf{P}_{[\mathbb{N}]} \times \mathsf{P}_{[2^{\mathbb{N}}]} \to \mathsf{P}_{[\exists u \psi(F,n,\alpha,u)]}) \\ &= \mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \times \mathsf{P}_{[\mathbb{N}]} \times (\mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \times \mathsf{P}_{[\mathbb{N}]} \times \mathsf{P}_{[2^{\mathbb{N}}]} \to \mathsf{P}_{[\psi(F,n,\alpha,u)]}) \end{aligned}$$

As for the actual realizer, $\alpha_{[\varphi(F,n)]}(\bar{F}, \bar{n})$ are those triples $((r_F, \hat{r}_F), (\bar{n}, \bar{n}^*), (r_\psi, \hat{r}_\psi)$ where $(r_F, \hat{r}_F) \in \alpha_{[2^{\mathbb{N}}(\mathbb{N})]}(\bar{F})$, $(\bar{n}, \bar{n}^*) \in \alpha_{[\mathbb{N}]}(\bar{n})$, and $(r_\psi, \hat{r}_\psi)$ is a pre-morphism $\mathsf{P}_{[2^{\mathbb{N}}(\mathbb{N})]} \times \mathsf{P}_{[\mathbb{N}]} \times \mathsf{P}_{[2^{\mathbb{N}}]} \to \mathsf{P}_{[\psi]}$ such that, given any $(r_F, \hat{r}_F) \in \alpha_{[2^{\mathbb{N}}(\mathbb{N})]}(\bar{F})$, $(\bar{n}, \bar{n}^*) \in \alpha_{[\mathbb{N}]}(\bar{n})$, and an actual realizer $(r_\alpha, \hat{r}_\alpha)$ for an arbitrary $\alpha \in 2^{\mathbb{N}}$, r_ψ computes a realizer of $\psi(\bar{F}, \bar{n}, \bar{\alpha}, \bar{u})$, for some $u \in 2^*$. We let r_ψ encode the following computation: one checks every $u \in 2^n$ in sequence, and outputs the first u such that $\overline{u * \bar{0}}(F(u * \bar{0}))$ covers α. And such r_ψ is majoraized, for example, by $\hat{r}_\psi := \lambda r_F n r_\alpha.1^n$.

Now let's turn to SFF^* itself. By definition, $\hat{r}_F$ is a majorant of r_F, and it's a MType quasi-morphism $\mathsf{P}_{2^{\mathbb{N}}} \to \mathsf{P}_{\mathbb{N}}$. So $\hat{r}_F$ indeed encodes a function, say $\bar{F}^* : 2^{\mathbb{N}} \to \mathbb{N}$. Then that $\bar{F} \leq \bar{F}^*$ implies $\bar{F}(\alpha) \leq \bar{F}^*(\bar{1})$, for any $\alpha \in 2^{\mathbb{N}}$. Let $m = \bar{F}^*(\bar{1})$, and $\{\overline{u * \bar{0}}(F(u * \bar{0}))|u \in 2^*, |u| \leq m\}$ is apparently a cover. Then we simply starts from $k = 0$ and check if $\{\overline{u * \bar{0}}(F(u * \bar{0}))|u \in 2^*, |u| \leq k\}$ is a cover. First, checking whether it's a cover is decidable, since one only needs to check whether the finite set 2^k is covered. Second, this procedure always terminates since we already have an upper bound m for k. In short, we can compute *the least* n such that $\{\overline{u * \bar{0}}(F(u * \bar{0}))|u \in 2^*, |u| \leq k\}$ forms a cover. Note that this procedure is extensional, so we have a computation encoded by some $r_\varphi \in \mathbb{N}$. Now it only remains to find a majorant $\hat{r}_\varphi$ of r_φ. And we claim that $\hat{r}_\varphi := \lambda r_F.r_F\bar{1}$ does the job, where $\bar{1}$ is the encoding for $\bar{1} \in 2^{\mathbb{N}}$. In fact, for any $F \leq F^*$ and witness r_F and r_{F^*} of them (let's omit their majorants here), $r_\varphi r_F \leq \hat{r}_\varphi r_{F^*} = r_{F^*}\bar{1}$ by the definition of r_φ.

Therefore we can conclude that SFF^* holds in MMAsm. □

As the name "special fan functional" suggests, SFF is closely related with FT. One can indeed think of statement **Bar**(P) as "P is a cover": for any $\alpha \in 2^{\mathbb{N}}$, there exists $k \in \mathbb{N}$ such that $\alpha(k) \in P$. Then if one thinks of, for any $\alpha \in 2^{\mathbb{N}}$, $F(\alpha)$ as the value such that $\bar{\alpha}(F\alpha) \in P$, the finite cover given by SFF immediately generates an upper bound for the bar.

Proposition 6.7. $\mathsf{E\text{-}HA}^{\omega} + \mathsf{AC} + \mathsf{SFF} \vdash \mathsf{FT}$

Proof. Assuming **Bar**(P), then we have:

$$\begin{aligned}
\mathbf{Bar}(P) &\to \exists F \in 2^{\mathbb{N}}(\mathbb{N})\forall \alpha \in 2^{\mathbb{N}} P(\bar{\alpha}(F\alpha)) && (\mathsf{AC})\\
&\to \exists \alpha_1 \dots \alpha_n \in 2^{\mathbb{N}}, \{\bar{\alpha_i}(F\alpha_i) | i = 0, \dots, n\} \text{ forms a cover} && (\mathsf{SFF})\\
&\to \forall \alpha \in 2^{\mathbb{N}} \exists k \leq \max\{|\alpha_i|\}, P(\bar{\alpha}(k))\\
&\to \exists n \in \mathbb{N} \forall \alpha \in 2^{\mathbb{N}} P(\bar{\alpha}(k))
\end{aligned}$$

□

Therefore one can immediately derive that FT holds in MMAsm.

6.4 Fan Recursor

In this section we discuss the Fan Recursor. The Fan Recursor can be seen as a "baby" version of the fame Bar Recursor.

Definition 6.8 (Fan Recursor)**.** A *Fan recursor* FR is a functional $(2^{\mathbb{N}} \to \mathbb{N}) \to 2^* \to \mathbb{N}$ defined as follows:

$$\mathsf{FR}(N, u) = \begin{cases} 0 & N(\hat{u}) < |u| \\ 1 + \max\{\mathsf{FR}(N, u * 0), \mathsf{FR}(N, u * 1)\} & \text{otherwise} \end{cases}$$

where $N : 2^{\mathbb{N}} \to \mathbb{N}$, $u \in 2^*$, and $\hat{u}$ denotes the canonical extension $u * \bar{0}$ of u.

Example 6.9. The first thing to point out is that, in the classical setting this definition may not even make sense. Think of the following $N : (\mathbb{N} \to 2) \to \mathbb{N}$:

$$N(\alpha) = \begin{cases} k & \#1\text{'s in } \alpha \text{ is finite, say } k \\ 0 & \text{otherwise} \end{cases}$$

Then consider $u = 1$. $N(\hat{u}) = 1 \geq |u|$, so one goes to the recursive case. In general, one meets $\mathsf{FR}(N, 1^{k+1} * \bar{0})$ at the k-th recursion, and $N(1^{k+1} * \bar{0}) = k + 1 \geq |1^{k+1}|$ means an extra recursion. So $N(u)$ is no well-defined ($\to \infty$).

Note that if $N(\hat{u}) < u$, then $\bar{\hat{u}}N(\hat{u}) \sqsubseteq u$. Intuitively, $\mathsf{FR}(N, u)$ gives a length m of extension of u, such that for all $v \in 2^*$ of length $|v| \leq |u| + m$, their canonical extension under $N : 2^{\mathbb{N}} \to \mathbb{N}$ forms a cover. This shall remind one of the special fan functional SFF. Let $C \subseteq 2^*$, $\alpha_0 \in 2^{\mathbb{N}}$ and $M \subseteq 2^{\mathbb{N}}$, then we say C *covers* α_0 if there exists $u \in C$ such that $u \sqsubseteq \alpha_0$, and C *covers* M if for any $\alpha \in M$, there exists $u \in C$ such that $u \sqsubseteq \alpha$. In particular, for some $u \in 2^*$, we use C covers u as abbreviation of A covers $u * 2^{\mathbb{N}}$. And their counterparts in the formal language are respectively $Cov(C, M)$ and $Cov(C, u)$ Besides, let $Ext(u, n, N) := \{\overline{v * \bar{0}}N(v * \bar{0}) | u \sqsubseteq v \in 2^*, |v| \leq |u| + n\}$.

Lemma 6.10. *Let* $m = \mathsf{FR}(N, u)$*, then* $Ext(u, m, N)$ *forms a cover of* u*.*

Proof. Prove by induction on m.

- $m = 0$. It can only be the case that $N(u * \bar{0}) < |u|$. Then $\overline{u * \bar{0}N(u * \bar{0})} \sqsubseteq u$, and $\mathit{Ext}(u, 0, N)$ covers u.

- $m + 1$. By definition we know that

$$\max\{\mathsf{FR}(N, u * 0), \mathsf{FR}(N, u * 1)\} = m$$

  And IH tells us that $\mathit{Ext}(u*0, \mathsf{FR}(N, u*0), N)$ and $\mathit{Ext}(u*1, \mathsf{FR}(N, u*1), N)$ cover $u*0$ and $u*1$ respectively. Note two things: firstly for any $k_0 \leq k_1$, $\mathit{Ext}(u, k_0, N) \subseteq \mathit{Ext}(u, k_1, N)$; secondly $\mathit{Ext}(u, m+1, N) = \mathit{Ext}(u*0, m, N) \cup \mathit{Ext}(u*1, m, N)$. So $\mathit{Ext}(u*0, m, N)$ and $\mathit{Ext}(u*1, m, N)$ cover $u * 0$ and $u * 1$ respectively, and their union covers $(u * 0) * 2^{\mathbb{N}} \cup (u * 1) * 2^{\mathbb{N}} = u * 2^{\mathbb{N}}$. That is, $\mathit{Ext}(u, m + 1, N)$ covers u.

Therefore $\mathit{Ext}(u, \mathsf{FR}(N, u), N)$ covers u. □

However, as will be shown by the following example, $\mathsf{FR}(N, u)$ does *not* return the least m such that $\mathit{Ext}(u, m, N)$ covers u.

Example 6.11. Let functional $N : 2^{\mathbb{N}} \to \mathbb{N}$ be defined as:

$$N(\alpha) = \begin{cases} 0 & \alpha \equiv \bar{0} \\ 99 & \text{otherwise} \end{cases}$$

Consider the empty sequence $\langle\rangle$, then it's easy to verify that

$$\mathsf{FR}(N, \langle\rangle) = 100 + \max\left\{\mathsf{FR}(N, v) | |v| = 100, v \neq 0^{100}\right\}$$

But obviously $\mathit{Ext}(u, 0, \langle\rangle)$ is a cover of the empty sequence $\langle\rangle$.

However, the existence of $\mathsf{FR}(N, u)$ guarantees the existence of a least such m. More importantly in our context, this least m is computable: fix u and N, one can simply start from $m = 0$ and check if $\mathit{Ext}(u, m, N)$ covers u, by verifying whether every $v \in u * 2^m$ is covered. Define *Fan Recursor Principle* FRP as:

Fan Recursor Principle $\mathsf{FRP} := \exists \mathsf{FR}^{2^{\mathbb{N}}(\mathbb{N}) \to 2^* \to \mathbb{N}} \forall F^{2^{\mathbb{N}}(\mathbb{N})} \forall u^{2^*} [\mathit{Base}(\mathsf{FR}, F, u) \wedge \mathit{Ind}(\mathsf{FR}, F, u)]$, where:

$$\begin{aligned} \mathit{Base}(\mathsf{FR}, F, u) &:= F(u * \bar{0}) < |u| \to \mathsf{FR}(F, u) = 0 \\ \mathit{Ind}(\mathsf{FR}, F, u) &:= F(u * \bar{0}) \geq |u| \to \mathsf{FR}(F, u) = \max\{\mathsf{FR}(F, u * 0), \mathsf{FR}(F, u * 1)\} \end{aligned}$$

Proposition 6.12. $\mathsf{E\text{-}HA}^\omega \vdash \mathsf{FRP} \to \mathsf{SFF}$.

Proof. Given $F : 2^{\mathbb{N}} \to \mathbb{N}$. Consider the empty sequence $\langle\rangle$. By **Lemma** 6.10, the set

$$\mathit{Ext}(\langle\rangle, \mathsf{FR}(F, \langle\rangle), F) = \{\overline{v * \bar{0}}F(v * \bar{0}) | v \in 2^*, |v| \leq \mathsf{FR}(F, \langle\rangle)\}$$

is a cover for $\langle\rangle$, or simply a cover. □

Note that SFF holds classically while FRP does not. While both are compatible with $\mathsf{E\text{-}HA}^\omega$, FRP is strictly stronger than SFF. And such strong FRP also holds in MMAsm:

Proposition 6.13. $\mathsf{MMAsm} \vDash \mathsf{FRP}$

Proof. For simplicity, let's abbreviate FRP as $\exists\mathsf{FR}\varphi(\mathsf{FR})$, and $\varphi(\mathsf{FR}) := \forall N\forall u\psi(\mathsf{FR}, N, u)$ (where we omit the sorts here). Then we have:

$$\begin{aligned}
\mathsf{P}_{[\mathsf{FRP}]} &= \mathsf{P}_{[\exists\mathsf{FR}\varphi(\mathsf{FR})]} \\
&= \mathsf{P}_{[\varphi(\mathsf{FR})]} \\
&= \mathsf{P}_{[\forall N\forall u\psi(\mathsf{FR},N,u)]} \\
&= \mathsf{P}_{[(2^{\mathbb{N}}\to\mathbb{N})\to 2^*\to\mathbb{N}]} \times (\mathsf{P}_{[2^{\mathbb{N}}\to\mathbb{N}]} \times \mathsf{P}_{[2^*]} \to \mathsf{P}_{[\psi(N,u,\mathsf{FR})]})
\end{aligned}$$

And an actual realizer of $\mathsf{P}_{[\mathsf{FRP}]}$ is a pair $((r_{\mathsf{FR}}, \hat{r_{\mathsf{FR}}}), (r_\psi, \hat{r}_\psi))$ such that $(r_{\mathsf{FR}}, \hat{r}_{\mathsf{FR}})$ witnesses some FR, and $(r_\psi, \hat{r}_\psi)$ computes, from a witness of any N and u, a witness of $\psi(N, u, \mathsf{FR})$. So to show that $\mathsf{MMAsm} \vDash \mathsf{FRP}$, one needs to find some $\mathsf{FR} : (2^{\mathbb{N}} \to \mathbb{N}) \to 2^* \to \mathbb{N}$ with the aforementioned witnesses.

Recall that the potential problem of finding such an FR is that $\mathsf{FR}(N, u)$ might diverge for some N, u. And for every $\mathsf{FR}(N, u)$ to converge, it suffices that, for any $u \in 2^*$, there is some "uniform bound" $m \in \mathbb{N}$ such that for any $v \in u{*}2^m$, there exists v' with $u \sqsubseteq v' \sqsubseteq v$ such that $N(v'{*}\bar{0}) \leq |v'|$. Then $\mathsf{FR}(u) \leq m$. Since functional N is majorized, say by N^*, $N(\alpha) \leq N^*(\bar{1})$. And this means that we have a trivial upper bound $N^*(\bar{1})$. Then we know that the computation $\mathsf{FR}(N, u)$ always converges, for any computable N and u. Let r_{FR} be (one of) its codes. Besides, it's easy to see that r_{FR} is majorized by $\hat{r}_{\mathsf{FR}} := \lambda N^* u^*. N^*(\bar{1})$. So $(r_{\mathsf{FR}}, \hat{r}_{\mathsf{FR}})$ majorizes FR.

Careful readers might have noticed that r_{FR} does exactly what we require for a witness of $\forall N\forall u\psi(\mathsf{FR}, N, u)$: given r_N and r_u respectively witnessing N and u, r_{FR} returns 0 if $N(u^*0) < |u|$; otherwise returns the maximum of $r_{\mathsf{FR}}(r_N, r_{u*0})$ and $r_{\mathsf{FR}}(r_N, r_{u*1})$. And this algorithm terminates for the existence of the majorant. □

7 Discussion

7.1 Conclusion

In this thesis we have constructed the category MType of majorizaibility types, whose interpretation of finite types are computably equivalent to the vdBB model for Gödel's T. Based on MType, we introduce the category MMAsm of majorizability modified assemblies.

As a Heyting category with nno, MMAsm provides a model for $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\neg$. We expect it to characterize the monotone modified realizability, but this is not yet verified.

In the last part, we reviewed Brouwer's Fan Theorem, as well as some of its related principles. We have shown that not only the Fan Theorem, but also the Special Fan Functional Principle and the Fan Recursor Principle hold in our category MMAsm.

7.2 Future Work

The first work is, of course, to check the unproven conjectures in this thesis. That consists of:

1. MAJ fails in MMAsm.
2. MMAsm is a category for monotone modified realizability.

Item (2) shouldn't be too hard, and one mainly do induction on the structure of φ. For item (1), the very first idea is try to find a counterexample in type 2.

Next, we would like to follow the "assembly to tripos to topos" procedure, and construct a topos based on the category MMAsm. However, the routine procedure encounters an immediate problem. For the left and right adjoints in the tripos construction, one needs to take union/intersection of the potential realizers. But every potential realizer is of the form (a, a^*), and we are not sure what to take in the union/intersection of $X \ni (a, a^*)$ and $Y \ni (a, a^{**})$.

Thirdly, Ferreira and Nunes introduced the bounded modified realizability in [4]. The basic idea is that one does not care about the precise witnesses for arithmetic statements; rather, one extract majorants of potential witnesses. Its difference with the monotone modified realizability is tiny: in the latter case the precise witness always exists, while in the former case a constructive witness may not exists (but still majorized).

Last but not least, there are lots of other constructivism mathematical principles to be considered in the internal language of MMAsm. For example the Bar Induction and the Bar Recursion.

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