APAL with Memory is Better

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Abstract. We introduce Arbitrary Public Announcement Logic with Memory (APALM), obtained by adding to the models a 'memory' of the initial states, representing the information before any communication took place ("the prior"), and adding to the syntax operators that can access this memory. We show that APALM is recursively axiomatizable (in contrast to the original Arbitrary Public Announcement Logic, for which the corresponding question is still open). We present a complete recursive axiomatization, that uses a natural finitary rule, and study this logic's expressivity and the appropriate notion of bisimulation. We then examine Group Announcement Logic with Memory, the extension of APALM obtained by adding to its syntax group announcement operators, and provide a complete finitary axiomatization (again in contrast to the original Group Announcement Logic, for which the only known axiomatization is infinitary).

1 Introduction

Arbitrary Public Announcement Logic (APAL) and its relatives are natural extensions of Public Announcements Logic (PAL), involving the addition of operators $\Box\varphi$ and $\Diamond\varphi$, quantifying over public announcements $[\theta]\varphi$ of some given type. These logics are of great interest both philosophically and from the point of view of applications. Motivations range from supporting an analysis of Fitch's paradox [19] by modeling notions of 'knowability' (expressible as $\Diamond K\varphi$), to determining the existence of communication protocols that achieve certain goals (cf. the famous Russian Card problem, given at a mathematical Olympiad [20]), and more generally to epistemic planning [12], and to inductive learnability in empirical science [6]. Many such extensions have been investigated, starting with the original APAL [2], and continuing with its variants GAL (Group Announcement Logic) [1], Future Event Logic [23], FAPAL (Fully Arbitrary Public Announcement Logic) [27], $APAL^+$ (Positive Arbitrary Announcement Logic) [22], BAPAL (Boolean Arbitrary Public Announcement Logic) [21], etc.

One problem with the above formalisms, with the exception of BAPAL[1], is that they all use *infinitary* axiomatizations. It is therefore not guaranteed that the validities of these logics are recursively enumerable.[2] The seminal paper on APAL [2] proved completeness using an infinitary rule, and then went on to claim that in theorem-proving[3] this rule can be replaced by the following finitary rule: from $\chi \to [\theta][p]\varphi$, infer $\chi \to [\theta]\Box\varphi$, as long as the propositional variable p is "fresh". A similar method is adopted in the completeness proof of GAL in [1] and it was claimed that the infinitary rule used in the completeness proof could be replaced by the finitary rule 'from $\chi \to [\theta][\bigwedge_{i\in G} K_i p_i]\psi$ infer $\chi \to [\theta][G]\psi$', where p_i's are "fresh" and $[G]$ is the group

[1] BAPAL is a very weak version, allowing $\Box\varphi$ to quantify over only purely propositional announcements - no epistemic formulas.

[2] $APAL^+$ is known to be decidable, hence its validities must be r.e., but no recursive axiomatization is known. Also, note that $APAL^+$ is still very weak, in that it quantifies only over positive epistemic announcements, thus not allowing public announcements of ignorance, which are precisely the ones driving the solution process in puzzles such as the Muddy Children.

[3] This means that from any proof of a theorem from the axioms that uses the infinitary rule we can obtain a finitary proof of the same theorem, by using the finitary rule instead.

announcement operator. These are natural $\Box$ and $[G]$-introduction rules, similar to the introduction rule for universal quantifier in First Order Logic (FOL), and they are based on the intuition that variables that do not occur in a formula are irrelevant for its truth value, and thus can be taken to stand for any "arbitrary" formula (via some appropriate change of valuation). However, the soundness of the $\Box$-introduction rule was later disproved via a counterexample by Kuijer [14]. Thus, a long-standing open question concerns *finding a 'strong' version of APAL for which there exists a recursive axiomatization*.[4] Moreover, as we have observed, a slightly modified version of Kuijer's counterexample also proves that the aforementioned $[G]$-introduction rule is unsound, which renders the same question for GAL open.

In this paper, we solve these open questions and focus primarily on the problem concerning APAL. The framework for the 'strong' version of GAL will be developed analogously, as an extension of the one for APAL. Due to the similar syntactic and semantic behaviours of the group announcement ($[G]$) and arbitrary announcement ($\Box$) operators, most of our analysis of the latter also applies to the former.

Our diagnosis of Kuijer's counterexample is that it makes an essential use of a known undesirable feature of PAL and APAL, namely their *lack of memory*: the updated models "forget" the initial states. As a consequence, the expressivity of the APAL $\Box$-modality reduces after any update. This is what invalidates the above rule. We fix this problem by adding to the models a memory of the *initial* epistemic situation W^0, representing the information *before any non-trivial communication took place* ("the prior"). Since communication – gaining more information – deletes possibilities, the set W of currently possible states is a (possibly proper) subset of the set W^0 of initial states. On the syntactic side, we add an operator φ^0 saying that "φ was initially the case" (before all communication). To mark the initial states, we also need a constant 0, stating that "no non-trivial communication has taken place yet". Therefore, 0 will be true only in the initial epistemic situation. It is convenient, though maybe not absolutely necessary, to add a universal modality $U\varphi$ that quantifies over all currently possible states.[5] In the resulting Arbitrary Public Announcement Logic with Memory (APALM), the arbitrary announcement operator $\Box$ quantifies over updates (not only of epistemic formulas but) of arbitrary formulas that do not contain the operator $\Box$ itself.[6] As a result, the range of $\Box$ is wider than in standard APAL, covering announcements that may refer to the initial situation (by the use of the operators 0 and φ^0) or to all currently possible states (by the use of $U\varphi$).

We show that the original finitary rule proposed in [2] *is* sound for APALM and, moreover, it forms the basis of a *complete recursive axiomatization*.[7] Besides its technical advantages, APALM is valuable in its own respect. Maintaining a record of the initial situation in our models helps us to formalize updates that refer to the 'epistemic past' such as "what you said, I knew already" [18]. This may be useful in treating certain epistemic puzzles involving reference to past information states, e.g. "What you said did not surprise me" [15]. The more recent *Cheryl's Birthday* problem also contains an announcement of the form "At first I didn't know when Cheryl's birthday is, but now I know" (although in this particular puzzle the past-tense announcement is redundant and plays no role in the solution).[8] See [18] for more examples.

[4] Here, by 'strong' version we mean one that allows quantification over a sufficiently wide range of announcements (sufficiently wide to avoid Liar-like circles) as intended by a similar restriction in the original APAL.

[5] From an epistemic point of view, it would be more natural to replace U by an operator Ck that describes current common knowledge and quantifies only over currently possible states that are accessible by epistemic chains from the actual state. We chose to stick with U for simplicity and leave the addition of Ck to APAL for future work.

[6] This restriction is necessary to produce a well-defined semantics that avoids Liar-like vicious circles. In standard APAL, the restriction is with respect to inductive construct $\Diamond\varphi$. Thus, formulas of the form $\langle\Diamond p\rangle\varphi$ are allowed in original APAL. APAL and APALM expressivities seem to be incomparable, and that would still be the case if we dropped the above restriction.

[7] We use a slightly different version of this rule, which is easily seen to be equivalent to the original version in the presence of the usual PAL reduction axioms.

[8] Cheryl's Birthday problem was part of the 2015 Singapore and Asian Schools Math Olympiad, and became viral after it was posted on Facebook by Singapore TV presenter Kenneth Kong.

Note though that the 'memory' of APALM is *very limited*: our models do not remember the whole history of communication, but only the initial epistemic situation (before any communication). Correspondingly, in the syntax we do *not* have a 'yesterday' operator $Y\varphi$, referring to the previous state just before the last announcement as in [17], but only the operator φ^0 referring to the initial state. We think of this 'economy' of memory as *a (positive) "feature, not a bug"* of our logic: a detailed record of *all* history is simply not necessary for solving the problem at hand. In fact, keeping all the history and adding a $Y\varphi$ operator would greatly complicate our task by invalidating some of the standard nice properties of PAL and APAL.[9]

So we opt for simplicity, enriching the models and language with just enough memory to recover the full expressivity of $\Box$ after updates, and thus establish the soundness of the $\Box$-introduction rule. Such a limited-memory semantics is sufficient for our purposes, but it also has an intrinsic naturality and simplicity, similar to the one encountered in some Bayesian models, with their distinction between 'prior' and 'posterior' (aka current) probabilities.[10]

Having established the desired results for APALM, we also study a version of GAL with the same memory mechanism – Group Announcement Logic with Memory (GALM) – obtained by extending APALM with group announcement operators. In this logic, the group announcement operators $[G]\varphi$ quantify over updates with formulas of the form $\bigwedge_{i\in G} K_i\varphi_i$, thus, represents *what a group of agents can bring about via simultaneous public announcements*. These updates can have occurrences of every component of the language but $\Box$ and $[G]$ for the same reason explained in footnote 6. We then show, following the same steps as for APALM, that the original finitary $[G]$-introduction rule proposed in [1] is sound for GALM.[11] By using this rule, we provide a complete *finitary* axiomatization for GALM, thus, prove that it is recursively axiomatizable.

On the technical side, our completeness proof involves an essential detour into an alternative semantics for APALM and GALM ('pseudo-models'), in the style of Subset Space Logics (SSL) [16,13]. This reveals deep connections between apparently very different formalisms. Moreover, this alternative semantics is of independent interest, giving us a more general setting for modeling knowability and learnability (see, e.g., [9,10,6]). Various combinations of PAL or APAL with subset space semantics have been investigated in the literature [5,29,28,24,25,9,7,6], including a version of SSL with backward looking public announcement operators that refer to what was true *before* a public announcement [4]. Following the SSL-style, our pseudo-models come with a given family of *admissible sets* of worlds, which in our context represent "publicly announceable" (or communicable) propositions.[12] We interpret $\Box$ in pseudo-models as the so-called 'effort' modality of SSL, which quantifies over updates with announceable propositions (regardless of whether they are syntactically definable or not). The modality $[G]$ on the other hand quantifies over updates with those announceable propositions that are known by some agents in G. The operator $[G]$ is thus modelled as a restricted version of the effort modality. The finitary $\Box$-introduction rule is obviously *sound for the effort modality*, because of its more 'semantic' character. Similarly, the finitary $[G]$-introduction rule is also sound

[9] E.g. the standard Composition Axiom (stating that any sequence of announcements is equivalent to a single announcement) fails in the presence of the Y operator. As a consequence, a logic with full memory of all history would lose some of the appealing features of the APAL operator (e.g. its $S4$ character: $\Box\varphi \to \Box\Box\varphi$). Moreover, this would force us to distinguish between "knowability via one communication step" $\Diamond K$ versus "knowability via a finite communication sequence" $\Diamond^* K$, leading to an unnecessarily complex logic.

[10] In such models, only the 'prior' and the 'posterior' information states are taken to be relevant, while all the intermediary steps are forgotten. As a consequence, all the evidence gathered in between the initial and the current state can be compressed into one set E, called "the evidence" (rather than keeping a growing tail-sequence of all past evidence sets). Similarly, in our logic, all the past communication is compressed in its end-result, namely in the set W of current possibilities, which plays the same role as the evidence set E in Bayesian models.

[11] We again use a slightly different version of this rule, which can easily be proven to be equivalent to the original version in the presence of the PAL reduction axioms. This choice is clearly cosmetic and made in order to simplify the soundness and completeness proofs.

[12] In SSL, the set of admissible sets is sometimes, but not always, taken to be a topology. Here, it will be a Boolean algebra with epistemic operators.

for this effort-like group announcement operator. These observations, together with the important fact that our models for APALM and GALM (unlike original APAL models) can be seen as a special case of pseudo-models, lie at the core of our soundness and completeness proofs.

The paper is organized as follows. In Section 2, we introduce the syntax and semantics of APALM (Section 2.1); then discuss Kuijer's counterexample for the soundness of the finitary $\Box$-introduction rule of the original APAL (Section 2.2); prove expressivity results comparing fragments of the language of APALM (Section 2.3); and present a complete finitary axiomatization (Section 2.4). Section 3 presents the syntax, semantics, and axiomatization of our group announcement logic with memory GALM. In Section 4 we prove soundness, and in Section 5 we prove completeness, for both APALM and GALM. Section 6 contains some concluding comments and ideas for future work.

For readability, most of the rather technical proofs are omitted from the main text and presented in the appendix.

2 Arbitrary Public Announcement Logic with Memory

We start by introducing APALM, obtained by enriching the models of APAL with a record of the initial information states (representing the informational situation before any communication took place) and the language of APAL with operators that can refer to this memory.

2.1 Syntax and Semantics of APALM

Let $Prop$ be a countable set of propositional variables and $\mathcal{AG} = \{1, \ldots, n\}$ be a finite set of agents. The language $\mathcal{L}$ of APALM (Arbitrary Public Announcement Logic with Memory) is recursively defined by the grammar:

$$\varphi \ ::= \ \top \mid p \mid 0 \mid \varphi^0 \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid U\varphi \mid \langle\theta\rangle\varphi \mid \Diamond\varphi,$$

where $p \in Prop$, $i \in \mathcal{AG}$, and $\theta \in \mathcal{L}_{-\Diamond}$ is a formula in the sublanguage $\mathcal{L}_{-\Diamond}$ obtained from $\mathcal{L}$ by removing the $\Diamond$ operator. Given a formula $\varphi \in \mathcal{L}$, we denote by P_φ the set of all propositional variables occurring in φ. We employ the usual abbreviations for $\bot$ and the propositional connectives $\vee, \rightarrow, \leftrightarrow$. The dual modalities are defined as $\hat{K}_i\varphi := \neg K_i\neg\varphi$, $E\varphi := \neg U\neg\varphi$, $\Box\varphi := \neg\Diamond\neg\varphi$, and $[\theta]\varphi := \neg\langle\theta\rangle\neg\varphi$.[13]

We read $K_i\varphi$ as "φ is *known* by agent i"; $\langle\theta\rangle\varphi$ as "θ can be *truthfully announced*, and *after* this public announcement φ is true". U and E are, respectively, the universal and existential modalities quantifying over all *current* possibilities: $U\varphi$ says that "φ is true in *all current alternatives* of the actual state". $\Diamond\varphi$ and $\Box\varphi$ are the (existential and universal) arbitrary announcement operators, quantifying over updates with formulas in $\mathcal{L}_{-\Diamond}$. We can read $\Box\varphi$ as "φ is *stably true* (under public announcements)", i.e., φ stays true no matter what (true) announcements are made. The constant 0, meaning that "*no (non-trivial) announcements* took place yet", holds only at the *initial time*. Similarly, the formula φ^0 means that "*initially* (prior to all communication), φ *was* true".

Definition 1 (Model, Initial Model, and Relativized Model).

- *A* model *is a tuple $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, where $W \subseteq W^0$ are non-empty sets of states, $\sim_i \subseteq W^0 \times W^0$ are equivalence relations labeled by 'agents' $i \in \mathcal{AG}$, and $\|\cdot\| : Prop \rightarrow \mathcal{P}(W^0)$ is a valuation function that maps every propositional variable $p \in Prop$ to a set of states $\|p\| \subseteq W^0$. W^0 is the* initial domain, *representing the initial informational situation before any communication took place; its elements are called* initial states. *In contrast, W is the* current domain, *representing the current informational situation, and its elements are called* current states.

[13] The update operator $\langle\theta\rangle\varphi$ is often denoted by $\langle!\theta\rangle\varphi$ in Public Announcement Logic literature; we skip the exclamation sign, but we will use the notation $\langle!\rangle$ for this modality and $[!]$ for its dual when we do not want to specify the announcement formula θ (so that ! functions as a placeholder for the content of the announcement).

– *For every model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$*, we define its* initial model $\mathcal{M}^0 = (W^0, W^0, \sim_1, \ldots, \sim_n, ||\cdot||)$*, whose both current and initial domains are the initial domain of the original model* $\mathcal{M}$*.*
– *Given a model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$ *and a set* $A \subseteq W$*, we define the* relativized model $\mathcal{M}|A = (W^0, A, \sim_1, \ldots, \sim_n, ||\cdot||)$.

For states $w \in W$ and agents i, we will use the notation $w_i := \{s \in W : w \sim_i s\}$ to denote the restriction to W of w's equivalence class modulo $\sim_i$.

Definition 2 (Semantics). *Given a model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$*, we recursively define a* truth set $[\![\varphi]\!]_{\mathcal{M}}$ *for every formula* $\varphi \in \mathcal{L}$ *as follows (we skip the subscript and simply write* $[\![\varphi]\!]$ *when the current model* $\mathcal{M}$ *is understood):*

$$[\![\top]\!] = W \qquad [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$

$$[\![p]\!] = ||p|| \cap W \qquad [\![K_i\varphi]\!] = \{w \in W : w_i \subseteq [\![\varphi]\!]\}$$

$$[\![0]\!] = \begin{cases} W^0 & \text{if } W = W^0 \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\varphi]\!] = \begin{cases} W & \text{if } [\![\varphi]\!] = W \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![\varphi^0]\!] = [\![\varphi]\!]_{\mathcal{M}^0} \cap W \qquad [\![\langle\theta\rangle\varphi]\!] = [\![\varphi]\!]_{\mathcal{M}|[\![\theta]\!]}$$

$$[\![\neg\varphi]\!] = W - [\![\varphi]\!] \qquad [\![\Diamond\varphi]\!] = \bigcup\{[\![\langle\theta\rangle\varphi]\!] : \theta \in \mathcal{L}_{-\Diamond}\}$$

Observation 1 *Note that we have*

$$w \in [\![\Box\varphi]\!] \quad \textit{iff} \quad w \in [\![[\theta]\varphi]\!] \textit{ for every } \theta \in \mathcal{L}_{-\Diamond}.$$

What we study in this paper is information update via public announcements. But the models given in Definition 1 are too general for this purpose: their current domain W can be *any* subset of the initial domain W^0. Our intended models (which we call "announcement models") will thus be a subclass of these models, in which the current domain comes from updating the initial domain with some public announcement.

Definition 3 (Announcement Models and Validity). *An* announcement model *(or* a-model*, for short) is a model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$ *such that* $W = [\![\theta]\!]_{\mathcal{M}^0}$ *for some* $\theta \in \mathcal{L}_{-\Diamond}$*; i.e.,* $\mathcal{M}$ *can be obtained by updating its initial model* $\mathcal{M}^0$ *with some formula in* $\mathcal{L}_{-\Diamond}$*. A formula* φ *is* APALM valid *(or* valid*, for short) if it is true in every current state* $s \in W$ *(i.e.* $[\![\varphi]\!]_{\mathcal{M}} = W$*) of every announcement model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$.

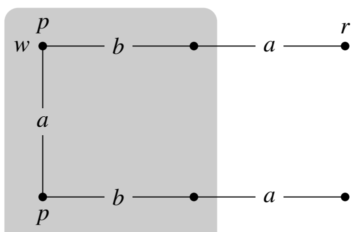


Fig. 1: An a-model $\mathcal{M}$. Initial states are represented by all nodes in the graph, current states are the nodes in shaded areas. Valuation is given by labeling each node with the true atoms, and epistemic relations are represented by arrows with agent names.

Example 1. *Consider the a-model $\mathcal{M} = (W^0, W, \sim_a, \sim_b, \| \cdot \|)$ given in Figure 1, where the initial states include all the nodes of the graph and the current states are the nodes in the shaded area. It is easy to see that the current domain W is obtained by updating the initial domain by $\hat{K}_b p$: the shaded area corresponds to $[\![\hat{K}_b p]\!]_{\mathcal{M}^0}$. The representation in Figure 1 makes it clear that the a-model does not lose the initial domain and specifies the current domain as a subset of the initial one. Since $W^0 \neq W$ (the shaded area does not cover the whole initial domain),* 0 *is false everywhere in the model, that is,* $[\![0]\!] = \emptyset$. *Moreover, while $\hat{K}_b \hat{K}_a K_b r$ was initially true at w, it currently is not: $w \in [\![(\hat{K}_b \hat{K}_a K_b r)^0]\!]$ but $w \notin [\![\hat{K}_b \hat{K}_a K_b r]\!]$ (as $[\![r]\!] = \emptyset$).*

2.2 An Analysis of Kuijer's counterexample

To understand Kuijer's counterexample [14] to the soundness of the finitary $\Box$-introduction rule for the original APAL, recall that, in APAL, $\Box$ quantifies only over updates with *epistemic* formulas. More precisely, the APAL semantics of $\Box$ is given by

$$w \in [\![\Box\varphi]\!] \quad \text{iff} \quad w \in [\![[\theta]\varphi]\!] \text{ for every } \theta \in \mathcal{L}_{epi},$$

where $\mathcal{L}_{epi}$ is the sublanguage generated from propositional atoms $p \in Prop$ using only the Boolean connectives $\neg$ and $\wedge$, and the epistemic operators K_i.

Kuijer takes the formula $\gamma := p \wedge \hat{K}_b \neg p \wedge \hat{K}_a K_b p$, and shows that

$$[\hat{K}_b p]\Box\neg\gamma \to [q]\neg\gamma.$$

is valid in APAL models, i.e., in multi-agent epistemic models with equivalence relations. (In fact, it is also valid in our *a*-models.) But then, by the $[!]\Box$-intro rule (or rather, by its weaker consequence in Proposition 5.5), the formula

$$[\hat{K}_b p]\Box\neg\gamma \to \Box\neg\gamma$$

should also be valid. But this is contradicted by the model $\mathcal{M}$ in Figure 2. The premise $[\hat{K}_b p]\Box\neg\gamma$ is true at w in $\mathcal{M}$, since $\Box\neg\gamma$ holds at w in the updated model $\mathcal{M}|[\![\hat{K}_b p]\!]$ in Figure 3a: indeed, the only way to falsify $\Box\neg\gamma$ would be by deleting u_2 from Figure 3a while keeping (all other nodes, and in particular) node u_1. But in $\mathcal{M}|[\![\hat{K}_b p]\!]$, u_1 and u_2 can not be separated by epistemic sentences: they are bisimilar.

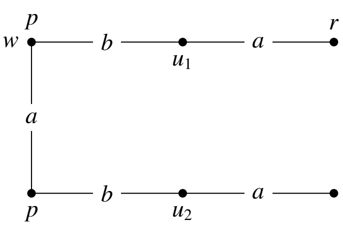
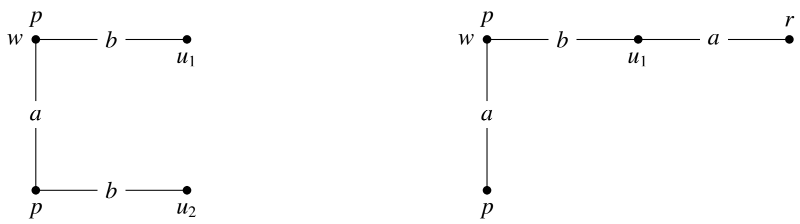


Fig. 2: An epistemic model $\mathcal{M}$. Worlds are nodes in the graph, valuation is given by labeling the nodes with the true atoms, and epistemic relations are given by labeled arrows.

In contrast, the conclusion $\Box\neg\gamma$ is *false* at w in $\mathcal{M}$, since in that original model u_1 and u_2 *could be separated*. Indeed, we could perform an alternative update with the formula $p \vee \hat{K}_a r$, yielding the epistemic model $\mathcal{M}|[\![p \vee \hat{K}_a r]\!]$ shown in Figure 3b, in which γ *is true* at w (contrary to the assertion that $\Box\neg\gamma$ was true in $\mathcal{M}$).



(a) $\mathcal{M}|[\![\hat{K}_b p]\!]$ (b) $\mathcal{M}|[\![p \vee \hat{K}_a r]\!]$

Fig. 3: Two updates of $\mathcal{M}$.

To see that the counterexample does not apply to APALM, notice that a-models keep track of the initial states. When we take $\mathcal{M}$ as an a-model as drawn in Figure 4 - where the initial states and current states collapse - the updated model $\mathcal{M}|[\![\hat{K}_b p]\!]$ consists now of the initial structure together with current set of worlds W in Figure 3a. This structure is given in Figure 5a, where the nodes in the shaded area are the current states. But in this model, $\Box\neg\gamma$ is *no longer true* at w (and so the premise $[\hat{K}_b p]\Box\neg\gamma$ was *not* true in $\mathcal{M}$ when considered as an a-model!). Indeed, we *can* perform a new update of the a-model $\mathcal{M}|[\![\hat{K}_b p]\!]$ with the formula $(p \vee \hat{K}_a r)^0$, which yields the updated model given in Figure 5b:

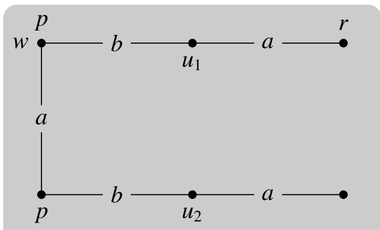


Fig. 4: $\mathcal{M}$ as an a-model. Initial states are nodes in the graphs and current states are represented by the nodes in shaded areas.

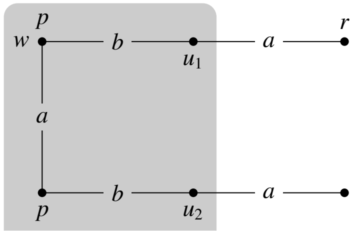
Note that, in this new model, γ *is* the case at w (- thus showing that $\Box\neg\gamma$ was *not* true at w in $\mathcal{M}|[\![\hat{K}_b p]\!]$). So the counterexample simply does not work for APALM.

Moreover, we can see that the unsoundness of $[!]\Box$-intro rule for APAL has to do with its lack of memory, which leads to information loss after updates: while initially (in $\mathcal{M}$) there were epistemic sentences (e.g. $p \vee \hat{K}_a r$) that could separate u_1 and u_2 , there are no such sentences after the update.

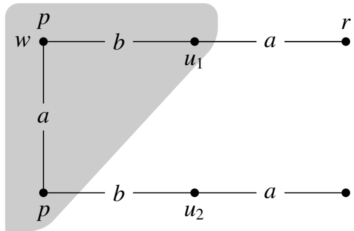
APALM solves this by keeping track of the initial states, and referring back to them, as in $(p \vee \hat{K}_a r)^0$.

2.3 Expressivity

To compare APALM and its fragments with basic epistemic logic (and its extension with the universal modality), consider the *static fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle}$, obtained from $\mathcal{L}$ by removing both the $\Diamond$ operator and the dynamic modality $\langle\varphi\rangle\psi$; as well as the *present-only fragment* $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, obtained by removing the operators 0 and φ^0 from $\mathcal{L}_{-\Diamond,\langle!\rangle}$; and finally the *epistemic fragment* $\mathcal{L}_{epi}$, obtained by further removing the universal



(a) $\mathcal{M}|[\![\hat{K}_b p]\!]$, when $\mathcal{M}$ is an a-model



(b) $(\mathcal{M}|[\![\hat{K}_b p]\!])|[\![(p \vee \hat{K}_a r)^0)]\!]_{\mathcal{M}|[\![\hat{K}_b p]\!]}$

Fig. 5: Two updates of $\mathcal{M}$, when $\mathcal{M}$ is an a-model. Initial states are nodes in the graphs and current states are represented by the nodes in shaded areas.

modality U from $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$. For every a-model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$, consider its *initial epistemic model* $\mathcal{M}^{initial} = (W^0, \sim_1, \ldots, \sim_n, ||\cdot||)$ and its *current epistemic model* $\mathcal{M}^{current} = (W, \sim_1 \cap W \times W, \ldots, \sim_n \cap W \times W, ||\cdot|| \cap W)$.

We need the following auxiliary lemma for some of our expressivity results.

Lemma 1. *There exists a well-founded strict partial order $<$ on $\mathcal{L}$ such that:*

1. *if φ is a subformula of ψ, then $\varphi < \psi$,*
2. $(\theta \to p) < [\theta]p$,
3. $(\theta \to \neg[\theta]\psi) < [\theta]\neg\psi$,
4. $(\theta \to K_i[\theta]\psi) < [\theta]K_i\psi$,
5. $[\langle\theta\rangle\rho]\chi < [\theta][\rho]\chi$,
6. $(\theta \to \varphi^0) < [\theta]\varphi^0$,
7. $(\theta \to U[\theta]\varphi) < [\theta]U\varphi$,
8. $(\theta \to (U\theta \wedge 0)) < [\theta]0$,
9. $\langle\theta\rangle\varphi < \Diamond\varphi$, *for all* $\theta \in \mathcal{L}_{-\Diamond}$.

Proof. The proof is via easy arithmetic calculations following the definitions in Appendix A.1, restricted to language $\mathcal{L}$. Note that, Definition 14 is redundant for the cases restricted to language $\mathcal{L}_{-\Diamond}$.

Proposition 1. *The fragment $\mathcal{L}_{-\Diamond}$ is co-expressive with the static fragment $\mathcal{L}_{-\Diamond,\langle!\rangle}$. In fact, every formula $\varphi \in \mathcal{L}_{-\Diamond}$ is provably equivalent to some formula $\psi \in \mathcal{L}_{-\Diamond,\langle!\rangle}$ (by using* **APALM** *reduction laws, given in Table 1) to eliminate dynamic modalities, as in standard PAL).*

Proof. Use step-by-step the reduction axioms (given in Table 1, Section 2.4), as a rewriting process, and prove termination by $<$-induction on φ by using Lemma 1.

Proposition 2. *The static fragment $\mathcal{L}_{-\Diamond,\langle!\rangle}$ (and hence, also $\mathcal{L}_{-\Diamond}$) is strictly more expressive than the present-only fragment $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, which in turn is more expressive than the epistemic fragment $\mathcal{L}_{epi}$. In fact, each of the operators* 0 *and* φ^0 *independently increase the expressivity of $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$.*

Proof. Consider the a-model in Figure 5a: while u_1 and u_2 are indistinguishable for $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, the sentence $(p \vee \hat{K}_a r)^0$ distinguishes the two. This shows that $\mathcal{L}_{-\Diamond,\langle!\rangle,0}$ is strictly more expressive than $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$. To see that $\mathcal{L}_{-\Diamond,\langle!\rangle,\varphi^0}$ is strictly more expressive than $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$, we just need to consider two a-models $\mathcal{M}_1 = (W^0, W, \sim_1, \ldots, \sim_n, ||\cdot||)$ and $\mathcal{M}_2 = (W, W, \sim_1 \cap W \times W, \ldots, \sim_n \cap W \times W, ||\cdot||)$ such that $W \subset W^0$. As both models have the same underlying current models, they make the same formulas of $\mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0}$ true at the same states in W. However, only the latter makes 0 true (at every state) since it is an initial model. Moreover, it is well-known that $\mathcal{L}_{epi}$ is strictly less expressive than its extension with the universal modality (see, e.g., [11, Chapter 7.1]).

$$
\begin{tikzcd}
 & & \mathcal{L}_{-\Diamond,\langle!\rangle,0} \arrow[r, two heads] & \mathcal{L}_{-\Diamond} \arrow[dr, two heads] \arrow[dd, two heads] & \\
\mathcal{L}_{epi} \arrow[r, two heads] & \mathcal{L}_{-\Diamond,\langle!\rangle,0,\varphi^0} \arrow[ur, two heads] \arrow[dr, two heads] & & & \mathcal{L} \\
 & & \mathcal{L}_{-\Diamond,\langle!\rangle,\varphi^0} \arrow[r, two heads] & \mathcal{L}_{-\Diamond,\langle!\rangle} \arrow[ur, two heads] \arrow[uu, two heads] &
\end{tikzcd}
$$

Fig. 6: Expressivity diagram (Arrows point to the more expressive languages, and reflexive and transitive arrows are omitted.)

The expressivity diagram in Figure 6 summarizes Propositions 1 and 2.

Kuijer's counterexample shows that the standard epistemic bisimulation is not appropriate for APALM, so we now define a new such notion:

Definition 4 (APALM Bisimulation). *An* APALM bisimulation *between a-models* $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ *and* $\mathcal{M}_2 = (W_2^0, W_2, \sim'_1, \ldots, \sim'_n, \|\cdot\|_2)$ *is a total bisimulation B (in the usual sense)*[14] *between the corresponding initial epistemic models* $\mathcal{M}_1^{initial}$ *and* $\mathcal{M}_2^{initial}$*, with the property that:* if $s_1 B s_2$, then $s_1 \in W_1$ iff $s_2 \in W_2$. *Two current states $s_1 \in W_1$ and $s_2 \in W_2$ are* APALM-bisimilar *if there exists an APALM bisimulation B between the underlying a-models such that $s_1 B s_2$.*

Since a-models are always of the form $\mathcal{M} = \mathcal{M}^0|[\![\theta]\!]$ for some $\theta \in \mathcal{L}_{-\Diamond}$, we have a characterization of APALM-bisimulation only in terms of the initial models as stated in Proposition 3. To prove Propositions 3 and 4, we need the following auxiliary Lemmas 2 and 3.

Lemma 2. *Let B be a total epistemic bisimulation between initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM-bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and let $s_1 \in W_1^0$, $s_2 \in W_2^0$ be two initial states such that $s_1 B s_2$. Then we have*

$$s_1 \in [\![\alpha]\!]_{\mathcal{M}_1^0} \text{ iff } s_2 \in [\![\alpha]\!]_{\mathcal{M}_2^0}$$

for all formulas $\alpha \in \mathcal{L}_{-\Diamond}$.

Proof. See Appendix A.2.

Lemma 3. *Let B be a total epistemic bisimulation between initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM-bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and let $s_1 \in W_1^0$, $s_2 \in W_2^0$ be two initial states such that $s_1 B s_2$. Then, for all $\varphi \in \mathcal{L}$, we have*

$$s_1 \in [\![\langle\alpha\rangle\varphi]\!]_{\mathcal{M}_1^0} \text{ iff } s_2 \in [\![\langle\alpha\rangle\varphi]\!]_{\mathcal{M}_2^0}$$

for all formulas $\alpha \in \mathcal{L}_{-\Diamond}$.

[14] A *total bisimulation* between epistemic models $(W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $(W', \sim'_1, \ldots, \sim'_n, \|\cdot\|')$ is an epistemic bisimulation relation (satisfying the usual valuation and back-and-forth conditions from Modal Logic) $B \subseteq W \times W'$, such that: for every $s \in W$ there exists some $s' \in W'$ with sBs'; and dually, for every $s' \in W'$ there exists some $s \in W$ with sBs'.

Proof. Let B be an APALM bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$. The proof goes by $<$-induction on φ, using Lemma 1). We assume the following induction hypothesis: for all $\psi < \varphi$ in $\mathcal{L}$ and all states $s_1 \in W_1^0$, $s_2 \in W_2^0$ with $s_1 B s_2$, we have: $s_1 \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_2^0}$, for all $\alpha \in \mathcal{L}_{-\Diamond}$.

Base cases $\varphi := \top$, $\varphi := p$, and $\varphi := 0$ follow directly from Lemma 2 and the fact that the formulas $\langle\alpha\rangle\top$, $\langle\alpha\rangle p$, and $\langle\alpha\rangle 0$ are in $\mathcal{L}_{-\Diamond}$.

In the following sequence of equivalencies, we make repeated use of the semantic clauses in Defn. 2.

Case $\varphi := \psi^0$
$s_1 \in [\![\langle\alpha\rangle\psi^0]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\psi^0]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $s_1 \in [\![\psi]\!]_{\mathcal{M}_1^0} \cap [\![\alpha]\!]_{\mathcal{M}_1^0}$ (since $\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0} = (W_1^0, [\![\alpha]\!]_{\mathcal{M}_1^0}, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$) iff $s_2 \in [\![\psi]\!]_{\mathcal{M}_2^0} \cap [\![\alpha]\!]_{\mathcal{M}_2^0}$ (by IH, Lemma 2: $\alpha \in \mathcal{L}_{-\Diamond}$) iff $s_2 \in [\![\psi^0]\!]_{\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0}}$ (since $\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0} = (W_2^0, [\![\alpha]\!]_{\mathcal{M}_2^0}, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$) iff $s_2 \in [\![\langle\alpha\rangle\psi^0]\!]_{\mathcal{M}_2^0}$.

Cases $\varphi := K_i\psi$ and $\varphi := U\psi$ follow similarly as in Lemma 2. We spell out here only the case $\varphi := U\psi$. First observe that,
$s_1 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![U\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $\forall s \in [\![\alpha]\!]_{\mathcal{M}_1^0}, s \in [\![\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $\forall s \in [\![\alpha]\!]_{\mathcal{M}_1^0}, s \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_1^0}$.
Suppose $s_1 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_1^0}$ and let $s' \in [\![\alpha]\!]_{\mathcal{M}_2^0}$. Since B is a total bisimulation, there is $s_1' \in W_1^0$ such that $s_1' B s'$. Since $\alpha \in \mathcal{L}_{-\Diamond}$, by Lemma 2, we have $s_1' \in [\![\alpha]\!]_{\mathcal{M}_1^0}$. Then, by the above observation, we have $s_1' \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_1^0}$. Thus, by IH, we obtain that $s' \in [\![\langle\alpha\rangle\psi]\!]_{\mathcal{M}_2^0}$. As $s_2 \in W_2^0$, we then conclude, via similar steps as in the above observation, that $s_2 \in [\![\langle\alpha\rangle U\psi]\!]_{\mathcal{M}_2^0}$. The other direction is similar. For the case $\varphi := K_i\psi$, we also use the back and forth conditions of B.

Case $\varphi := \langle\theta\rangle\psi$ uses the validity of the formula $\langle\alpha\rangle\langle\theta\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\theta\rangle\psi$ which can be easily verified.
$s_1 \in [\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\langle\langle\alpha\rangle\theta\rangle\psi]\!]_{\mathcal{M}_1^0}$ (by $\models \langle\alpha\rangle\langle\theta\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\theta\rangle\psi$) iff $s_2 \in [\![\langle\langle\alpha\rangle\theta\rangle\psi]\!]_{\mathcal{M}_2^0}$ (IH, using $\psi < \langle\theta\rangle\psi$) iff $s_2 \in [\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_2^0}$.

Case $\varphi := \Diamond\psi$
$s_1 \in [\![\langle\alpha\rangle\Diamond\psi]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\Diamond\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $s_1 \in \bigcup_{\theta\in\mathcal{L}_{-\Diamond}}[\![\langle\theta\rangle\psi]\!]_{\mathcal{M}_1^0|[\![\alpha]\!]_{\mathcal{M}_1^0}}$ iff $s_1 \in \bigcup_{\theta\in\mathcal{L}_{-\Diamond}}[\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in \bigcup_{\theta\in\mathcal{L}_{-\Diamond}}[\![\langle\alpha\rangle\langle\theta\rangle\psi]\!]_{\mathcal{M}_2^0}$ (IH, $\langle\theta\rangle\psi < \Diamond\psi$) iff $s_2 \in \bigcup_{\theta\in\mathcal{L}_{-\Diamond}}[\![\langle\theta\rangle\psi]\!]_{\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0}}$ iff $s_2 \in [\![\Diamond\psi]\!]_{\mathcal{M}_2^0|[\![\alpha]\!]_{\mathcal{M}_2^0}}$ iff $s_2 \in [\![\langle\alpha\rangle\Diamond\psi]\!]_{\mathcal{M}_2^0}$.

Proposition 3. *Let $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$ be a-models, and let $B \subseteq W_1^0 \times W_2^0$. The following are equivalent:*

1. *B is an APALM bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$;*
2. *B is a total epistemic bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$ (or equivalently, an APALM bisimulation between $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$), and $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\Diamond}$.*

Proof. (1) $\to$ (2): Let B be an APALM bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$. Then it is obvious (from the definition) that B is also a total bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are a-models, there must exist $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$ s.t. $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta_2]\!]_{\mathcal{M}_2^0}$. Hence, $W_1 = [\![\theta_1]\!]_{\mathcal{M}_1^0}$ and $W_2 = [\![\theta_2]\!]_{\mathcal{M}_2^0}$. To show that $[\![\theta_1]\!]_{\mathcal{M}_1^0} = [\![\theta_2]\!]_{\mathcal{M}_1^0}$, let first $s_1 \in [\![\theta_1]\!]_{\mathcal{M}_1^0} = W_1$. By the definition of APALM bisimulation, there must exist $s_2 \in W_2^0$ such that $s_1 B s_2$. Again by the definition, $s_1 \in W_1$ implies that $s_2 \in W_2 = [\![\theta_2]\!]_{\mathcal{M}_2^0}$. This, together with $s_1 B s_2$, gives us by Lemma 2 that $s_1 \in [\![\theta_2]\!]_{\mathcal{M}_1^0}$. For the converse, let $s_1 \in [\![\theta_2]\!]_{\mathcal{M}_1^0}$; by the definition of APALM bisimulation, there must exist $s_2 \in W_2^0$ such that $s_1 B s_2$. By Lemma 2, we have $s_2 \in [\![\theta_2]\!]_{\mathcal{M}_2^0} = W_2$, and again by the definition of APALM bisimulation (and the fact that $s_1 B s_2$), this implies that $s_1 \in W_1 = [\![\theta_1]\!]_{\mathcal{M}_1^0}$. Given that $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0}$ and $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta_2]\!]_{\mathcal{M}_2^0}$ such that $[\![\theta_1]\!]_{\mathcal{M}_1^0} = [\![\theta_2]\!]_{\mathcal{M}_1^0}$, we can take $\theta := \theta_2$. Then $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta_1]\!]_{\mathcal{M}_1^0} = \mathcal{M}_1^0|[\![\theta_2]\!]_{\mathcal{M}_1^0}$.

(2) → (1): Suppose that B is a total bisimulation between $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$, and $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$ for some common formula $\theta \in \mathcal{L}_{-\Diamond}$. Hence, $W_1 = [\![\theta]\!]_{\mathcal{M}_1^0}$ and $W_2 = [\![\theta]\!]_{\mathcal{M}_2^0}$. We need to verify that $\mathcal{M}_1$ and $\mathcal{M}_2$ are APALM-bisimilar. For this we just need to verify the property that if $s_1 B s_2$, then $s_1 \in W_1$ holds iff $s_2 \in W_2$ holds. Suppose $s_1 B s_2$ and let $s_1 \in W_1 = [\![\theta]\!]_{\mathcal{M}_1^0} \subseteq W_1^0$. By the totality of the bisimulation B, there must exist some $s_2 \in W_2^0$ with $s_1 B s_2$. By Lemma 2, $s_1 \in [\![\theta]\!]_{\mathcal{M}_1^0}$ implies that $s_2 \in [\![\theta]\!]_{\mathcal{M}_2^0} = W_2$. The converse is analogous.

So, to check for APALM-bisimilarity, it is enough to check for total bisimilarity between the initial models and for both models being updates with the same formula.

Next, we verify that this is indeed the appropriate notion of bisimulation.

Corollary 1. APALM formulas are invariant under APALM-bisimulation*: if $s_1 B s_2$ for some APALM-bisimulation relation B between a-models $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$, then: $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2}$ for all $\varphi \in \mathcal{L}$.*

Proof. Let B be some APALM-bisimulation relation between a-models $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$. By Proposition 3, there exists some formula $\theta \in \mathcal{L}_{-\Diamond}$ such that $\mathcal{M}_1 = \mathcal{M}_1^0|[\![\theta]\!]_{\mathcal{M}_1^0}$, $\mathcal{M}_2 = \mathcal{M}_2^0|[\![\theta]\!]_{\mathcal{M}_2^0}$. By the same proposition, B is a total epistemic bisimulation between the initial epistemic models $\mathcal{M}_1^{initial}$ and $\mathcal{M}_2^{initial}$. Thus, for every formula φ, we have the sequence of equivalences: $s_1 \in [\![\varphi]\!]_{\mathcal{M}_1}$ iff $s_1 \in [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}_1^0}$ iff (by Lemma 3) $s_2 \in [\![\langle\theta\rangle\varphi]\!]_{\mathcal{M}_2^0}$ iff $s_2 \in [\![\varphi]\!]_{\mathcal{M}_2}$.

Proposition 4 (Hennessy-Milner). *Let $\mathcal{M}_1 = (W_1^0, W_1, \sim_1, \ldots, \sim_n, \|\cdot\|_1)$ and $\mathcal{M}_2 = (W_2^0, W_2, \sim_1', \ldots, \sim_n', \|\cdot\|_2)$ be a-models with W_1^0 and W_2^0 finite. Then, $s_1 \in W_1$ and $s_2 \in W_2$ satisfy the same APALM formulas iff they are APALM-bisimilar.*

Proof. We only need to prove the left-to-right direction. Let $s_1 \in W_1$ and $s_2 \in W_2$ such that for all $\varphi \in \mathcal{L}$, $\mathcal{M}_1, s_1 \models \varphi$ iff $\mathcal{M}_2, s_2 \models \varphi$. This implies that for all $\varphi \in \mathcal{L}$, $\mathcal{M}_1^0, s_1 \models \varphi$ iff $\mathcal{M}_2^0, s_2 \models \varphi$. To see this, let $\varphi \in \mathcal{L}$ such that $\mathcal{M}_1^0, s_1 \models \varphi$. This means, by the semantics, that $\mathcal{M}_1, s_1 \models \varphi^0$. As $\mathcal{M}_1, s_1$ and $\mathcal{M}_2, s_2$ satisfy the same APALM formulas, we obtain that $\mathcal{M}_2, s_2 \models \varphi^0$, thus, $\mathcal{M}_2^0, s_2 \models \varphi$. The opposite direction is analogous. We then show that the modal equivalence relation in $W_1^0 \times W_2^0$ between the models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ is an APALM bisimulation. We thus need to show the following:

- (Totality) for all $s \in W_1^0$, there exists $s' \in W_2^0$ such that $\mathcal{M}_1^0, s$ and $\mathcal{M}_2^0, s'$ satisfy the same APALM formulas, and for all $s' \in W_2^0$, there exists $s \in W_1^0$ such that $\mathcal{M}_1^0, s$ and $\mathcal{M}_2^0, s'$ satisfy the same APALM formulas:
  Let $s \in W_1^0$ and suppose, toward contradiction, that for no element s' of W_2^0 we have $\mathcal{M}_1^0, s$ and $\mathcal{M}_2^0, s'$ satisfy the same APALM formulas. Since W_2^0 is finite, we can list its elements $W_2^0 = \{w_1, w_2, \ldots, w_n\}$. The first assumption then implies that for all $w_i \in W_2^0$, there exists $\psi_i \in \mathcal{L}$ such that $\mathcal{M}_1^0, s \models \psi_i$ but $\mathcal{M}_2^0, w_i \not\models \psi_i$. Thus, $\mathcal{M}_1^0, s_1 \models E(\psi_1 \wedge \cdots \wedge \psi_n)$ but $\mathcal{M}_2^0, s_2 \not\models E(\psi_1 \wedge \cdots \wedge \psi_n)$, contradicting the assumption that $\mathcal{M}_1^0, s_1$ and $\mathcal{M}_2^0, s_2$ satisfy the same APALM formulas. The second clause follows similarly.
- (Valuation) This follows immediately from modal equivalence.
- (Forth for $\sim_i$) Let $w_1, w_1' \in W_1^0$ and $w_2 \in W_2^0$ such that $\mathcal{M}_1^0, w_1$ and $\mathcal{M}_2^0, w_2$ satisfy the same APALM formulas and $w_1 \sim_i w_1'$. Suppose, toward contradiction, that for no element $w_2' \in W_2^0$ with $w_2 \sim_i' w_2'$, $\mathcal{M}_1^0, w_1'$ and $\mathcal{M}_2^0, w_2'$ satisfy the same APALM formulas. Since W_2^0 is finite, the set $\sim_i'(w_2) = \{t \in W_2^0 : w_2 \sim_i' t\}$ is finite, thus, we can write $\sim_i'(w_2) = \{t_1, \ldots, t_k\}$. As in the proof of (Totality), the assumption implies that for all t_j with $w_2 \sim_i' t_j$, there exists $\psi_j \in \mathcal{L}$ such that $\mathcal{M}_1^0, w_1' \models \psi_j$ but $\mathcal{M}_2^0, t_j \not\models \psi_j$. Therefore, $\mathcal{M}_1^0, w_1 \models \hat{K}_i(\psi_1 \wedge \cdots \wedge \psi_k)$ but $\mathcal{M}_2^0, w_2 \not\models \hat{K}_i(\psi_1 \wedge \cdots \wedge \psi_k)$, contradicting the assumption that $\mathcal{M}_1^0, w_1$ and $\mathcal{M}_2^0, w_2$ satisfy the same APALM formulas. Back condition for $\sim_i$ follows analogously.

We have therefore proven that the modal equivalence relation in $W^0_1 \times W^0_2$ between the models $\mathcal{M}^0_1$ and $\mathcal{M}^0_2$ is an APALM bisimulation between $\mathcal{M}^0_1$ and $\mathcal{M}^0_2$. By Proposition 3, it suffices to further prove that $\mathcal{M}_1 = \mathcal{M}^0_1 | [\![\theta]\!]_{\mathcal{M}^0_1}$, $\mathcal{M}_2 = \mathcal{M}^0_2 | [\![\theta]\!]_{\mathcal{M}^0_2}$ for some common formula $\theta \in \mathcal{L}_{-\Diamond}$. It then suffices to show that $[\![\theta_1]\!]_{\mathcal{M}^0_2} = [\![\theta_2]\!]_{\mathcal{M}^0_2}$, where $W_1 = [\![\theta_1]\!]_{\mathcal{M}^0_1}$ and $W_2 = [\![\theta_2]\!]_{\mathcal{M}^0_2}$.

$[\![\theta_2]\!]_{\mathcal{M}^0_2} \subseteq [\![\theta_1]\!]_{\mathcal{M}^0_2}$: Observe that $\mathcal{M}_1, s_1 \models U(\theta^0_1)$, since $W_1 = [\![\theta_1]\!]_{\mathcal{M}^0_1}$. Moreover, as $\mathcal{M}_1, s_1$ and $\mathcal{M}_2, s_2$ satisfy the same APALM formulas, we obtain that $\mathcal{M}_2, s_2 \models U(\theta^0_1)$. Therefore, for all $y \in [\![\theta_2]\!]_{\mathcal{M}^0_2}$, we have $\mathcal{M}_2, y \models \theta^0_1$, implying that $\mathcal{M}^0_2, y \models \theta_1$. Hence, $[\![\theta_2]\!]_{\mathcal{M}^0_2} \subseteq [\![\theta_1]\!]_{\mathcal{M}^0_2}$.

$[\![\theta_1]\!]_{\mathcal{M}^0_2} \subseteq [\![\theta_2]\!]_{\mathcal{M}^0_2}$: Observe that $\mathcal{M}_2, s_2 \models U(\theta^0_2)$, since $W_2 = [\![\theta_2]\!]_{\mathcal{M}^0_2}$. Moreover, as $\mathcal{M}_1, s_1$ and $\mathcal{M}_2, s_2$ satisfy the same APALM formulas, we obtain that $\mathcal{M}_1, s_1 \models U(\theta^0_2)$. Now suppose, toward contradiction, that $[\![\theta_1]\!]_{\mathcal{M}^0_2} \not\subseteq [\![\theta_2]\!]_{\mathcal{M}^0_2}$, i.e., there is $y \in W^0_2$ such that $\mathcal{M}^0_2, y \models \theta_1$ but $\mathcal{M}^0_2, y \not\models \theta_2$. By the totality of the modal equivalence relation, there exists $x \in W^0_1$ such that $\mathcal{M}^0_1, x \models \theta_1$ but $\mathcal{M}^0_1, x \not\models \theta_2$. The former implies that $x \in W_1$. Therefore, by the latter, we have that $\mathcal{M}_1, x \not\models \theta^0_2$. This implies, since $s_1, x \in W_1$, that $\mathcal{M}_1, s_1 \not\models U(\theta^0_2)$, contradicting $\mathcal{M}_1, s_1$ and $\mathcal{M}_2, s_2$ satisfying the same APALM formulas.

Therefore, we obtain that $[\![\theta_1]\!]_{\mathcal{M}^0_2} = [\![\theta_2]\!]_{\mathcal{M}^0_2}$. Given that $\mathcal{M}_1 = \mathcal{M}^0_1 | [\![\theta_1]\!]_{\mathcal{M}^0_1}$ and $\mathcal{M}_2 = \mathcal{M}^0_2 | [\![\theta_2]\!]_{\mathcal{M}^0_2}$ such that $[\![\theta_1]\!]_{\mathcal{M}^0_2} = [\![\theta_2]\!]_{\mathcal{M}^0_2}$, we can take $\theta := \theta_1$. Then $\mathcal{M}_2 = \mathcal{M}^0_2 | [\![\theta_2]\!]_{\mathcal{M}^0_2} = \mathcal{M}^0_2 | [\![\theta_1]\!]_{\mathcal{M}^0_2}$.

2.4 Axiomatization

Table 1 presents a complete proof system **APALM** for our logic APALM (where recall that P_φ is the set of propositional variables in φ).

Intuitive Reading of the Axioms. Parts (I) and (II) should be obvious. The axiom R$[\top]$ says that *updating with tautologies is redundant*. The reduction laws that do not contain ^{0}, U or 0 are well-known PAL axioms. R_U is the natural reduction law for the universal modality. The axiom R^0 says that the truth value of φ^0 formulas stays the same in time (because the superscript ^{0} serves as a time stamp), so they can be treated similarly to atoms. Ax_0 says that 0 was initially the case, and R_0 says that at any later stage (after any update) 0 can only be true if it was already true before the update and the update was trivial (universally true). Together, these two say that the constant 0 characterizes states where no non-trivial communication has occurred. Axiom 0-U is a synchronicity constraint: *if no non-trivial communication has taken place yet, then this is the case in all the currently possible states*. Axiom 0-*eq* says that *initially, φ is equivalent to its initial correspondent* φ^0. The Equivalences with ^{0} express that ^{0} distributes over negation and over conjunction. $\text{Imp}^0_\Box$ says that *if initially φ was stably true (under any further announcements), then φ is the case now*. Taken together, the elimination axiom $[!]\Box$-elim and introduction rule $[!]\Box$-intro say that *φ is a stable truth after an announcement θ iff φ stays true after any more informative announcement* (of the form $\theta \wedge \rho$). [15]

Proposition 5. *The following schemas and inference rules are derivable in* **APALM**, *where $\varphi, \psi, \chi \in \mathcal{L}$ and $\theta \in \mathcal{L}_{-\Diamond}$:*

1. *from* $\vdash \varphi \leftrightarrow \psi$, *infer* $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$
2. $\vdash \langle\theta\rangle 0 \leftrightarrow (0 \wedge U\theta)$
3. $\vdash \langle\theta\rangle\psi \leftrightarrow (\theta \wedge [\theta]\psi)$
4. $\vdash \Box\varphi \to [\theta]\varphi$
5. *from* $\vdash \chi \to [p]\varphi$, *infer* $\vdash \chi \to \Box\varphi$ $(p \notin P_\chi \cup P_\varphi)$
6. $\vdash (\varphi \to \psi)^0 \leftrightarrow (\varphi^0 \to \psi^0)$
7. $\vdash \varphi^{00} \leftrightarrow \varphi^0$
8. $\vdash \Box\varphi^0 \leftrightarrow \varphi^0$, *and* $\vdash \Diamond\varphi^0 \leftrightarrow \varphi^0$
9. $\vdash (\Box\varphi)^0 \to \Box\varphi^0$
10. $\vdash (0 \wedge \Diamond\varphi^0) \to \varphi$
11. $\vdash \varphi \to (0 \wedge \Diamond\varphi)^0$
12. $\vdash \varphi \to \psi^0$ *if and only if* $\vdash (0 \wedge \Diamond\varphi) \to \psi$

[15] The "freshness" of the variable $p \in P$ in the rule $[!]\Box$-intro ensures that it represents any generic announcement.

	(I) Basic Axioms of system APALM:
(CPL)	all classical propositional tautologies and Modus Ponens
($S5_{K_i}$)	all $S5$ axioms and rules for knowledge operator K_i
($S5_U$)	all $S5$ axioms and rules for U operator
(U-K_i)	$U\varphi \to K_i\varphi$
	(II) Axioms and rules for dynamic modalities $[!]$:
($K_!$)	*Kripke's axiom for* $[!]$: $[\theta](\psi \to \varphi) \to ([\theta]\psi \to [\theta]\varphi)$
($Nec_!$)	*Necessitation for* $[!]$: from φ, infer $[\theta]\varphi$.
(RE)	*Replacement of Equivalents* $[!]$: from $\theta \leftrightarrow \rho$, infer $[\theta]\varphi \leftrightarrow [\rho]\varphi$.
Reduction laws:	
($R[\top]$)	$[\top]\varphi \leftrightarrow \varphi$
(R_p)	$[\theta]p \leftrightarrow (\theta \to p)$
($R_\neg$)	$[\theta]\neg\psi \leftrightarrow (\theta \to \neg[\theta]\psi)$
(R_{K_i})	$[\theta]K_i\psi \leftrightarrow (\theta \to K_i[\theta]\psi)$
($R_{[!]}$)	$[\theta][\rho]\chi \leftrightarrow [\langle\theta\rangle\rho]\chi$
(R^0)	$[\theta]\varphi^0 \leftrightarrow (\theta \to \varphi^0)$
(R_U)	$[\theta]U\varphi \leftrightarrow (\theta \to U[\theta]\varphi)$
(R_0)	$[\theta]0 \leftrightarrow (\theta \to (U\theta \wedge 0))$
	(III) Axioms and rules for 0 and *initial* operator ^{0}:
(Ax_0)	0^0
(0-U)	$0 \to U0$
(0-eq)	$0 \to (\varphi \leftrightarrow \varphi^0)$
(Nec^0)	*Necessitation for* ^{0}: from φ, infer φ^0
Equivalences with ^{0}:	
(Eq^0_p)	$p^0 \leftrightarrow p$
($Eq^0_\neg$)	$(\neg\varphi)^0 \leftrightarrow \neg\varphi^0$
($Eq^0_\wedge$)	$(\varphi \wedge \psi)^0 \leftrightarrow (\varphi^0 \wedge \psi^0)$
Implications with ^{0}:	
(Imp^0_U)	$(U\varphi)^0 \to U\varphi^0$
(Imp^0_i)	$(K_i\varphi)^0 \to K_i\varphi^0$
($Imp^0_\Box$)	$(\Box\varphi)^0 \to \varphi$
	(III) Elim-axiom and Intro-rule for $\Box$:
($[!]\Box$-elim)	$[\theta]\Box\varphi \to [\theta \wedge \rho]\varphi$
($[!]\Box$-intro)	from $\chi \to [\theta \wedge p]\varphi$, infer $\chi \to [\theta]\Box\varphi$ (for $p \notin P_\chi \cup P_\theta \cup P_\varphi$).

Table 1: The axiomatization **APALM**. (Here, $\varphi, \psi, \chi \in \mathcal{L}$, while $\theta, \rho \in \mathcal{L}_{-\Diamond}$.)

13. $\vdash [\theta](\psi \wedge \varphi) \leftrightarrow ([\theta]\psi \wedge [\theta]\varphi)$

14. $\vdash [\theta][p]\psi \leftrightarrow [\theta \wedge p]\psi$

15. $\vdash [\theta]\bot \leftrightarrow \neg\theta$

Proof. See Appendix A.3.

Proposition 6. *All $S4$ axioms and inference rules for $\Box$ are derivable in* **APALM**.

Proof. See Appendix A.3.

We arrive now at one of the main results of our paper.

Theorem 1 (Soundness and Completeness of APALM). *APALM validities are recursively enumerable. Indeed, the axiom system* **APALM** *in Table 1 is sound and complete wrt a-models.*

Both soundness and completeness proofs are rather involved, thus, given in separate Sections 4 and 5, respectively.

3 Group Announcement Logic with Memory: GALM

In this section we turn our focus on the Group Announcement Logic (GAL) introduced in [1]. As briefly mentioned in Introduction, GAL is also an extension of PAL, involving group announcement operators $[G]\varphi$ and $\langle G\rangle\varphi$ (instead of the arbitrary announcement operators $\Box\varphi$ and $\Diamond\psi$). The group announcement operator can be seen as a restricted version of the arbitrary public announcement operator in the sense that it quantifies only over updates with formulas of the form $\bigwedge_{i\in G} K_i\theta_i$, where $\theta_i \in \mathcal{L}_{epi}$ and $i \in G \subseteq \mathcal{AG}$. More precisely, [1] interprets the operator $[G]\varphi$ on epistemic models $M = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ as

$$w \in [\![[G]\varphi]\!] \quad \text{iff} \quad \text{for every set } \{\psi_i : i \in G\} \subseteq \mathcal{L}_{epi}, w \in [\![[\bigwedge_{i\in G} K_i\psi_i]\varphi]\!]. \tag{1}$$

This operator intends to capture communication among a group of agents and what a coalition can bring about via public announcements. While GAL seems to provide more adequate tools than APAL to treat puzzles involving epistemic dialogues, the axiomatization of GAL presented in [1] has a similar shape as the one for APAL in [2]. To recall, [1] proves completeness of GAL also by using an infinitary rule and claims that it is replaceable in theorem-proving by the finitary rule

$$\text{from } \varphi \to [\theta][\bigwedge_{i\in G} K_i p_i]\psi \text{ infer } \varphi \to [\theta][G]\psi, \tag{$R[G]$}$$

where $p_i \notin P_\varphi \cup P_\psi \cup P_\theta$. However, Kuijer's counterexample presented in Section 2.2 constitutes a counterexample also for the soundness of this rule. Consider again the formula $\gamma := p \wedge \hat{K}_b\neg p \wedge \hat{K}_a K_b p$ and let $G = \{a\}$. We show that while

$$[\hat{K}_b p][G]\neg\gamma \to [K_a q]\neg\gamma$$

is valid on epistemic models, its $R[G]$-conclusion

$$[\hat{K}_b p][G]\neg\gamma \to [G]\neg\gamma$$

is not. For the former, suppose that $[\hat{K}_b p][G]\neg\gamma \to [K_a q]\neg\gamma$ is not valid on epistemic models, i.e., that there is an epistemic model $\mathcal{N} = (W, \sim_1, \dots, \sim_n, \|\cdot\|)$ and $w \in W$ such that $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]$ but $w \notin [\![[K_a q]\neg\gamma]\!]$. The latter means that $w \in [\![\langle K_a q\rangle\gamma]\!]$. Therefore, $w \in [\![K_a q]\!]$ and $w \in [\![\gamma]\!]_{\mathcal{N}|[\![K_a q]\!]}$. The latter implies that $w \in \|p\|$ and there are two states w_1, w_2 in $\mathcal{N}|[\![K_a q]\!]$ such that (1) w_1 is $\sim_b$-connected to w and $w_1 \notin \|p\|$, and (2) w_2 is $\sim_a$-connected to w and $w_2 \in [\![K_b p]\!]_{\mathcal{N}|[\![K_a q]\!]}$. In other words, the model in Figure 7 is guaranteed to be a submodel of $\mathcal{N}|[\![K_a q]\!]$.

Moreover, since $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]$ and $w \in [\![\hat{K}_b p]\!]$, we also have that $w \in [\![[G]\neg\gamma]\!]_{\mathcal{N}|[\![\hat{K}_b p]\!]}$. Recall that $w \in [\![K_a q]\!]$. Therefore, neither w nor w_2 have $\sim_a$-access to a states in $\mathcal{N}$ that makes q false. Furthermore, since $K_a q$ is a positive knowledge formula, we have $w \in [\![K_a q]\!]_{\mathcal{N}|\theta}$ for any θ. Then, $w \in [\![[G]\neg\gamma]\!]_{\mathcal{N}|[\![\hat{K}_b p]\!]}$ implies that $w \in [\![\neg\gamma]\!]_{(\mathcal{N}|[\![\hat{K}_b p]\!])|[\![K_a q]\!]_{\mathcal{N}|[\![\hat{K}_b p]\!]}} = [\![\neg\gamma]\!]_{(\mathcal{N}|[\![\langle\hat{K}_b p\rangle K_a q]\!])}$. It is not difficult to see that the model in Figure 7 is also a submodel of $\mathcal{N}|[\![\langle\hat{K}_b p\rangle K_a q]\!]$ (recall that w_1 is in $\mathcal{N}|[\![K_a q]\!]$), thus, $w \in [\![\gamma]\!]_{(\mathcal{N}|[\![\langle\hat{K}_b p\rangle K_a q]\!])}$. This contradicts the assumption that $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]$. Therefore, $[\hat{K}_b p][G]\neg\gamma \to [K_a q]\neg\gamma$ is valid on epistemic models. However, model $\mathcal{M}$ in Figure 2 constitutes a counterexample for $[\hat{K}_b p][G]\neg\gamma \to [G]\neg\gamma$, as $w \in [\![[\hat{K}_b p][G]\neg\gamma]\!]_{\mathcal{M}}$ and $w \in [\![\langle K_a\hat{K}_a(p \vee r)\rangle\gamma]\!]_{\mathcal{M}}$, thus, $w \notin [\![[G]\neg\gamma]\!]$.

To the best of our knowledge, there had been no known recursive axiomatization for GAL or a stronger version of it. In this section, we provide a recursive axiomatization for Group Announcement Logic with

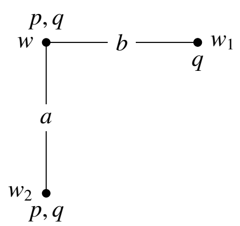


Fig. 7: Submodel of $\mathcal{N}|[\![K_a q]\!]$

Memory (GALM), obtained by extending the syntax of APALM with group announcement operators interpreted on a-models.[16] The language $\mathcal{L}_G$ of GALM is defined recursively, for each group of agents $G \subseteq \mathcal{AG}$, as:

$$\varphi \ ::= \ \top \mid p \mid 0 \mid \varphi^0 \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid U\varphi \mid \langle\theta\rangle\varphi \mid \Diamond\varphi \mid \langle G\rangle\varphi,$$

where $p \in Prop$, $i \in \mathcal{AG}$, and $\theta \in \mathcal{L}_{-\Diamond}$. The dual modality for this new operator is defined as $[G]\varphi := \neg\langle G\rangle\neg\varphi$. $\langle G\rangle\varphi$ and $[G]\varphi$ are the (existential and universal) group announcement operators, quantifying over updates with formulas of the form $\bigwedge_{i\in G} K_i\theta_i$, where $\theta_i \in \mathcal{L}_{-\Diamond}$ and $i \in G$. This restricted quantification over $\mathcal{L}_{-\Diamond}$ captures the assumption that each agent can announce only the ($\Diamond$ and G-free) propositions she knows and nothing else. Analogous to the reading of $\Box$, we read $[G]\varphi$ as "φ is *stably true* under group G's public announcements", i.e., "φ stays true no matter what group G truthfully announces".

We introduce the following abbreviation of *relativized knowledge* for notational convenience:

$$K_i^{\varphi}\psi := K_i(\varphi \to \psi),$$

where $\varphi, \psi \in \mathcal{L}_G$ and $i \in \mathcal{AG}$. The language $\mathcal{L}_G$ is also interpreted on models introduced in Definition 1.

Definition 5. *Given a model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, the semantics for $\mathcal{L}_G$ is defined recursively as in Definition 2 with the following additional clause for $\langle G\rangle$:*

$$[\![\langle G\rangle\varphi]\!] = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!] \ : \ \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\}.$$

Lemma 4. *There exists a well-founded strict partial order $<$ on $\mathcal{L}_G$, such that:*

1. *if φ is a subformula of ψ, then $\varphi < \psi$,*
2. *$\langle\theta\rangle\varphi < \Diamond\varphi$, for all $\theta \in \mathcal{L}_{-\Diamond}$,*
3. *$\langle\theta\rangle\varphi < \langle G\rangle\varphi$, for all $\theta \in \mathcal{L}_{-\Diamond}$.*

Proof. Similar to Lemma 1.

Observation 2 *Note that we have*

$$w \in [\![[G]\varphi]\!] \quad \textit{iff} \quad w \in [\![[\bigwedge_{i\in G} K_i\theta_i]\varphi]\!] \ \textit{ for every } \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}.$$

Proposition 7. *We have $[\![\varphi]\!] \subseteq W$, for all formulas $\varphi \in \mathcal{L}_G$.*

[16] We note that the language of the original GAL in [1] does not include the arbitrary announcement operator $\Box$. The fragment of GALM without the arbitrary announcement operators can be studied in a similar way. We prefer to work with this large language in order be able to present the soundness and completeness proofs for APALM and GALM in a unified way.

Proof. See Appendix A.4.

Our a-models given in Definition 3 are also the *intended* models for GALM, so GALM validities are defined with respect to a-models as in Definition 3. We can now state the main result of this section.

Theorem 2 (Soundness and Completeness of GALM). *GALM validities are recursively enumerable. In fact, the sound and complete axiomatization* **GALM** *wrt a-models is obtained by extending* **APALM** *with the axiom and rule given in Table 2.*

	Elim-axiom and Intro-rule for $[G]$:
($[!][G]$-elim)	$[\theta][G]\varphi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\varphi$
($[!][G]$-intro)	from $\chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$, infer $\chi \to [\theta][G]\varphi$ (for $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$).

Table 2: The additional axioms of **GALM**

The axiom and rule in Table 2 are very similar in spirit (and in what they express) to the $[!]\Box$-elim axiom and $[!]\Box$-intro rule, respectively. Together, the elimination axiom $[!][G]$-elim and rule $[!][G]$-intro say that *φ is a stable truth under group G's announcements after an announcement θ iff φ stays true after any more informative announcement from the group G* (of the form $\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i$).

4 Soundness via Pseudo-model Semantics

As GALM is an extension of APALM, we present the soundness and completeness proofs directly for the former. The same results for APALM are obtained following similar steps.

To start with, note that even the soundness of our axiomatic systems is not a trivial matter. As we saw from Kuijer's counterexample, the analogues of our finitary $\Box$ and $[G]$-introduction rules were *not* sound for APAL and GAL, respectively. To prove their soundness on a-models, we need a detour into an equivalent semantics, in the style of Subset Space Logics (SSL) [16,13]: *pseudo-models*.[17]

We first introduce an auxiliary notion: 'pre-models' are just SSL models, coming with a given family $\mathcal{A}$ of "admissible sets" of worlds (which can be thought of as the *communicable* propositions). We interpret $\Box$ in these structures as the so-called "effort modality" of SSL, which quantifies over updates with admissible propositions in $\mathcal{A}$. Analogously, $\langle G\rangle$ quantifies over updates with conjunctions of those admissible propositions in the scope of an epistemic operator labeled by an agent in G. Our 'pseudo-models' are pre-models with additional closure conditions (saying that the family of admissible sets includes the valuations and is closed under complement, intersection, and epistemic operators). These conditions imply that *every set definable by a $\Diamond, \langle G\rangle$-free formula*[18] *is admissible*, and this ensures the soundness of our $\Box$-elimination and $[G]$-elimination axioms on pseudo-models. As for the soundness of the long-problematic $\Box$ and $[G]$-introduction

[17] A more direct soundness proof on a-models is in principle possible, but would require at least as much work as our detour. Unlike in standard EL, PAL or DEL, the meaning of an APALM formula (and therefore of a GALM formula) depends, not only on the valuation of the atoms occurring in it, but also on the family $\mathcal{A}$ of all sets definable by $\mathcal{L}_{-\Diamond}$-formulas. The move from models to pseudo-models makes explicit this dependence on the family $\mathcal{A}$, while also relaxing the demands on $\mathcal{A}$ (which is no longer required to be exactly the family of $\mathcal{L}_{-\Diamond}$-definable sets), and thus makes the soundness proof both simpler and more transparent. Since we will need pseudo-models for our completeness proof anyway, we see no added value in trying to give a more direct soundness proof.

[18] $\Diamond, \langle G\rangle$-free formulas are the sentences in $\mathcal{L}_{-\Diamond}$.

rules on (both pre- and) pseudo-models, this is due to the fact that both the effort modality and $[G]$ operator interpreted on pseudo-models have a more 'robust' range than the arbitrary announcement versions of them: they quantify over admissible sets, regardless of whether these sets are syntactically definable or not. Soundness with respect to our a-models then follows from the observation that they (in contrast to the original APAL models) are in fact equivalent to a *special case of pseudo-models*: the "standard" ones (in which the admissible sets in $\mathcal{A}$ are *exactly* the sets definable by $\Diamond, \langle G\rangle$-free formulas).

Definition 6 (Pre-model). *A* pre-model *is a tuple $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$, where W^0 is the initial domain, $\sim_i$ are equivalence relations on W^0, $\|\cdot\| : Prop \to \mathcal{P}(W^0)$ is a valuation map, and $\mathcal{A} \subseteq \mathcal{P}(W^0)$ is a family of subsets of the initial domain, called* admissible sets *(representing the propositions that can be publicly announced).*

Given a set $A \subseteq W^0$ and a state $w \in A$, we use the notation $w_i^A := \{s \in A : w \sim_i s\}$ to denote the restriction to A of w's equivalence class modulo $\sim_i$. We also introduce the following abbreviation for the semantic counterpart of relativized knowledge: $K_i^A B = \{w \in W^0 : w_i \cap A \subseteq B\}$.

Definition 7 (Pre-model Semantics for $\mathcal{L}_G$). *Given a pre-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we recursively define a* truth set $[\![\varphi]\!]_A$ *for every formula φ and subset $A \subseteq W^0$:*

$$
\begin{aligned}
[\![\top]\!]_A &= A \\
[\![p]\!]_A &= \|p\| \cap A \\
[\![0]\!]_A &= \begin{cases} A & \text{if } A = W^0 \\ \emptyset & \text{otherwise} \end{cases} \\
[\![\varphi^0]\!]_A &= [\![\varphi]\!]_{W^0} \cap A \\
[\![\neg\varphi]\!]_A &= A - [\![\varphi]\!]_A \\
[\![\varphi \wedge \psi]\!]_A &= [\![\varphi]\!]_A \cap [\![\psi]\!]_A
\end{aligned}
\qquad
\begin{aligned}
[\![K_i\varphi]\!]_A &= \{w \in A : w_i^A \subseteq [\![\varphi]\!]_A\} \\
[\![U\varphi]\!]_A &= \begin{cases} A & \text{if } [\![\varphi]\!]_A = A \\ \emptyset & \text{otherwise} \end{cases} \\
[\![\langle\theta\rangle\varphi]\!]_A &= [\![\varphi]\!]_{[\![\theta]\!]_A} \\
[\![\Diamond\varphi]\!]_A &= \bigcup\{[\![\varphi]\!]_B \,:\, B \in \mathcal{A}, B \subseteq A\} \\
[\![\langle G\rangle\psi]\!]_A &= \bigcup\{[\![\psi]\!]_{A\cap\bigcap_{i\in G} K_i^A B_i} : \{B_i : i \in G\} \subseteq \mathcal{A}\}
\end{aligned}
$$

Observation 3 *Note that, for all $w \in A$, we have*

1. *$w \in [\![\Box\varphi]\!]_A$ iff $\forall B \in \mathcal{A}(w \in B \subseteq A \Rightarrow w \in [\![\varphi]\!]_B)$;*
2. *$w \in [\![[G]\varphi]\!]_A$ iff $w \in [\![\varphi]\!]_{A\cap\bigcap_{i\in G} K_i^A B_i}$ for every $\{B_i : i \in G\} \subseteq \mathcal{A}$;*
3. *$[\![\varphi]\!]_A \subseteq A$ for all $A \in \mathcal{A}$ and $\varphi \in \mathcal{L}_G$.*

Observation 3.1 shows that our proposed semantics of $\Box$ on pre-models fits with the semantics of the effort modality in SSL [16,13]. The proof of Observation 3.3 is similar to that of Proposition 7.

Definition 8 (Pseudo-models and Validity). *A* pseudo-model *is a pre-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$, satisfying the following closure conditions:*

1. *$\|p\| \in \mathcal{A}$, for all $p \in Prop$,*
2. *$W^0 \in \mathcal{A}$,*
3. *if $A \in \mathcal{A}$ then $(W^0 - A) \in \mathcal{A}$,*
4. *if $A, B \in \mathcal{A}$ then $(A \cap B) \in \mathcal{A}$,*
5. *if $A \in \mathcal{A}$ then $K_iA \in \mathcal{A}$, where $K_iA := \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in A)\}$.*

A formula $\varphi \in \mathcal{L}_G$ is valid in pseudo-models *if it is true in all admissible sets $A \in \mathcal{A}$ of every pseudo-model $\mathcal{M}$, i.e, $[\![\varphi]\!]_A = A$ for all $A \in \mathcal{A}$ and all $\mathcal{M}$.*

Lemma 5. *Given a pseudo-model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $A, B \in \mathcal{A}$, we have $K_i^A B \in \mathcal{A}$.*

Proof. See Appendix A.5.

Proposition 8. *Given a pseudo-model* $(W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$*,* $A \in \mathcal{A}$*, and* $\theta \in \mathcal{L}_{-\Diamond}$*, we have* $[\![\theta]\!]_A \in \mathcal{A}$.

Proof. See Appendix A.5.

To prove the soundness of our axioms, we need the following lemmas:

Lemma 6. *Given a pseudo-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$*,* $A \in \mathcal{A}$ *and* $\theta \in \mathcal{L}_{-\Diamond}$ *such that* $w \in [\![\theta]\!]_A$*,* $w \in [\![K_i^\theta \rho]\!]_A$ *iff* $w \in K_i^{[\![\theta]\!]_A} [\![\rho]\!]_A$ *for all* $\rho \in \mathcal{L}_{-\Diamond}$.

Proof. See Appendix A.5.

Lemma 7. *Let* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ *and* $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ *be two pseudo-models and* $\varphi \in \mathcal{L}_G$ *such that* $\mathcal{M}$ *and* $\mathcal{M}'$ *differ only in the valuation of some* $p \notin P_\varphi$*. Then, for all* $A \in \mathcal{A}$*, we have* $[\![\varphi]\!]_A^{\mathcal{M}} = [\![\varphi]\!]_A^{\mathcal{M}'}$.

Proof. The proof follows by subformula induction on φ. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ and $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ be two pseudo-models such that $\mathcal{M}$ and $\mathcal{M}'$ differ only in the valuation of some $p \notin P_\varphi$ and let $A \in \mathcal{A}$. We want to show that $[\![\varphi]\!]_A^{\mathcal{M}} = [\![\varphi]\!]_A^{\mathcal{M}'}$. The base cases $\varphi := q (\neq p)$, $\varphi := \top$, $\varphi := 0$, and the inductive cases for Booleans are standard.

Case $\varphi := \psi^0$. Note that $P_{\psi^0} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$, in particular for $W^0 \in \mathcal{A}$. Thus $[\![\psi]\!]_{W^0}^{\mathcal{M}'} = [\![\psi]\!]_{W^0}^{\mathcal{M}}$. Then, $[\![\psi]\!]_{W^0}^{\mathcal{M}'} \cap A = [\![\psi]\!]_{W^0}^{\mathcal{M}} \cap A$ for all $A \in \mathcal{A}$. By the semantics of the initial operator on pseudo-models, we obtain $[\![\psi^0]\!]_A^{\mathcal{M}'} = [\![\psi^0]\!]_A^{\mathcal{M}}$.

Case $\varphi := K_i\psi$. Note that $P_{K_i\psi} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}} = [\![\psi]\!]_A^{\mathcal{M}'}$. Observe that $[\![K_i\psi]\!]_A^{\mathcal{M}} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{\mathcal{M}}\}$ and, similarly, $[\![K_i\psi]\!]_A^{\mathcal{M}'} = \{w \in A : w_i^A \subseteq [\![\psi]\!]_A^{\mathcal{M}'}\}$. Then, since $[\![\psi]\!]_A^{\mathcal{M}} = [\![\psi]\!]_A^{\mathcal{M}'}$, we obtain $[\![K_i\psi]\!]_A^{\mathcal{M}} = [\![K_i\psi]\!]_A^{\mathcal{M}'}$.

Case $\varphi := U\psi$. Note that $P_{U\psi} = P_\psi$. Then, by IH, we have that $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. We have two case: (1) If $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}} = A$, then $[\![U\psi]\!]_A^{\mathcal{M}'} = A = [\![U\psi]\!]_A^{\mathcal{M}}$. (2) If $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}} \neq A$, then $[\![U\psi]\!]_A^{\mathcal{M}'} = [\![U\psi]\!]_A^{\mathcal{M}} = \emptyset$.

Case $\varphi := \langle\theta\rangle\psi$. Note that $P_{\langle\theta\rangle\psi} = P_\theta \cup P_\psi$. By IH, we have $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$ and $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for every $A \in \mathcal{A}$. By Proposition 8, we know that $[\![\theta]\!]_A^{\mathcal{M}} = [\![\theta]\!]_A^{\mathcal{M}'} \in \mathcal{A}$. Therefore, in particular, we have $[\![\psi]\!]_{[\![\theta]\!]_A^{\mathcal{M}'}}^{\mathcal{M}'} = [\![\psi]\!]_{[\![\theta]\!]_A^{\mathcal{M}}}^{\mathcal{M}}$. Therefore, by the semantics of $\langle ! \rangle$ on pseudo-models, we obtain $[\![\langle\theta\rangle\psi]\!]_A^{\mathcal{M}'} = [\![\langle\theta\rangle\psi]\!]_A^{\mathcal{M}}$.

Case $\varphi := \Diamond\psi$. Note that $P_{\Diamond\psi} = P_\psi$. Since *the same* family of sets $\mathcal{A}$ is carried by both models $\mathcal{M}$ and $\mathcal{M}'$ and since (by IH) $[\![\psi]\!]_A^{\mathcal{M}'} = [\![\psi]\!]_A^{\mathcal{M}}$ for all $A \in \mathcal{A}$, we get:

$$[\![\Diamond\psi]\!]_A^{\mathcal{M}'} = \bigcup \{[\![\psi]\!]_B^{\mathcal{M}'} : B \in \mathcal{A}, B \subseteq A\} = \bigcup \{[\![\psi]\!]_B^{\mathcal{M}} : B \in \mathcal{A}, B \subseteq A\} = [\![\Diamond\psi]\!]_A^{\mathcal{M}}.$$

Case $\varphi := [G]\psi$. Note that $P_{[G]\psi} = P_\psi$. Then, by (IH), we have that $[\![\psi]\!]_B^{\mathcal{M}'} = [\![\psi]\!]_B^{\mathcal{M}}$ for every $B \in \mathcal{A}$. In particular, $[\![\psi]\!]_B^{\mathcal{M}'} = [\![\psi]\!]_B^{\mathcal{M}}$ for the B's of the form $A \cap K_i^A C$ with $A, C \in \mathcal{A}$ (recall that pseudo-models are closed under K_i^A operation and conjunction, see Definition 8 and Lemma 5). Since the same family of sets $\mathcal{A}$ is carried by both models $\mathcal{M}$ and $\mathcal{M}'$, we obtain:

$$[\![[G]\psi]\!]_A^{\mathcal{M}'} = \bigcup \{[\![\psi]\!]_{A \cap \bigcap_{i \in G} K_i^A B_i}^{\mathcal{M}'} : \{B_i : i \in G\} \subseteq \mathcal{A}\} = \bigcup \{[\![\psi]\!]_{A \cap \bigcap_{i \in G} K_i^A B_i}^{\mathcal{M}} : \{B_i : i \in G\} \subseteq \mathcal{A}\} = [\![[G]\psi]\!]_A^{\mathcal{M}}.$$

Proposition 9. *The system* **GALM** *is sound wrt pseudo-models. Thereofore, the system* **APALM** *is also sound wrt pseudo-models.*

Proof. The soundness of most of the axioms follows simply by spelling out the semantics. We present here only the soundness of the axioms $[!]\Box$-elim, $[!][G]$-elim and rules $[!]\Box$-intro, $[!][G]$-intro:

For the elimination axioms, let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be a pseudo-model, $A \in \mathcal{A}$, and $w \in A$ arbitrarily chosen:

($[!]\Box$-elim): Let $\rho \in \mathcal{L}_{-\Diamond}$ and suppose (1) $w \in [\![[\theta]\Box\varphi]\!]_A$ and (2) $w \in [\![\theta \wedge \rho]\!]_A$. We need to show that $w \in [\![\varphi]\!]_{[\![\theta\wedge\rho]\!]_A}$. Assumption (1) means that if $w \in [\![\theta]\!]_A$ then $w \in [\![\Box\varphi]\!]_{[\![\theta]\!]_A}$. By assumption (2) and since $w \in [\![\theta \wedge \rho]\!]_A \subseteq [\![\theta]\!]_A$, we have $w \in [\![\Box\varphi]\!]_{[\![\theta]\!]_A}$. Thus, by the semantics of $\Box$, we have $w \in \{u \in [\![\theta]\!]_A : \text{for all } B \in \mathcal{A}(u \in B \subseteq [\![\theta]\!]_A \Rightarrow u \in [\![\Box\varphi]\!]_B\}$. Therefore, for $B := [\![\theta \wedge \rho]\!]_A \subseteq [\![\theta]\!]_A$ (since by Proposition 8, $[\![\theta \wedge \rho]\!]_A \in \mathcal{A}$) we have $w \in [\![\varphi]\!]_{[\![\theta\wedge\rho]\!]_A}$.

($[!][G]$-elim): Let $\{\rho_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}$ and suppose (1) $w \in [\![[\theta][G]\varphi]\!]_A$ and (2) $w \in [\![\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A$. Assumption (1) means that if $w \in [\![\theta]\!]_A$ then $w \in [\![[G]\varphi]\!]_{[\![\theta]\!]_A}$. I.e., by the semantic clause for $[G]$, we have that if $w \in [\![\theta]\!]_A$ then for all $\{B_i : i \in G\} \subseteq \mathcal{A}$, $w \in [\![\varphi]\!]_{[\![\theta]\!]_A \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A} B_i}$. By (2) we have that $w \in [\![\theta]\!]_A$ and $w \in [\![\bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A$. Thus, by (1), we obtain $w \in [\![\varphi]\!]_{[\![\theta]\!]_A \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A} [\![\rho_i]\!]_A} = [\![\varphi]\!]_{[\![\theta]\!]_A \cap \bigcap_{i\in G} [\![K_i^\theta \rho_i]\!]_A} = [\![\varphi]\!]_{[\![\theta\wedge\bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A}$ (by Proposition 8 and Lemma 6). Thus if $w \in [\![\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A$ then $w \in [\![\varphi]\!]_{[\![\theta\wedge\bigwedge_{i\in G} K_i^\theta \rho_i]\!]_A}$, i.e., $w \in [\![[\theta \wedge \bigwedge_{i\in G} K_i^\theta \rho_i]\varphi]\!]_A$.

($[!]\Box$-intro): Suppose $\models \chi \to [\theta \wedge p]\varphi$ and $\not\models \chi \to [\theta]\Box\varphi$, where $p \notin P_\chi \cup P_\theta \cup P_\varphi$. The latter means that there exists a pseudo model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin [\![\chi \to [\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Therefore $w \in [\![\chi \wedge \neg[\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Thus we have (1) $w \in [\![\chi]\!]_A^{\mathcal{M}}$ and (2) $w \in [\![\neg[\theta]\Box\varphi]\!]_A^{\mathcal{M}}$. Because of (2), $w \in [\![\langle\theta\rangle\Diamond\neg\varphi]\!]_A^{\mathcal{M}}$, and, by the semantics, $w \in [\![\Diamond\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}}}$. Therefore, applying the semantics of $\Box$, we obtain (3) there exists $B \in \mathcal{A}$ s.t. $w \in B \subseteq [\![\theta]\!]_A^{\mathcal{M}} \subseteq A$ and $w \in [\![\neg\varphi]\!]_B^{\mathcal{M}}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ such that $\|p\|' := B$ and $\|q\|' = \|q\|$ for any $q \neq p \in \mathit{Prop}$. In order to use Lemma 7 we must show that $\mathcal{M}'$ is a pseudo-model. For this we only need to verify that $\mathcal{M}'$ satisfies the closure conditions given in Definition 8. First note that $\|p\|' := B \in \mathcal{A}$ by the construction of $\mathcal{M}'$, so $\|p\|' \in \mathcal{A}$. For every $q \neq p$, since $\|q\|' = \|q\|$ and $\|q\| \in \mathcal{A}$ we have $\|q\|' \in \mathcal{A}$. Since $\mathcal{A}$ is the same for both $\mathcal{M}$ and $\mathcal{M}'$, and $\mathcal{M}$ is a pseudo-model, the rest of the closure conditions are already satisfied for $\mathcal{M}'$. Therefore $\mathcal{M}'$ is a pseudo-model. Now continuing with our soundness proof, since $p \notin P_\chi \cup P_\theta \cup P_\varphi$, by Lemma 7, we obtain $[\![\chi]\!]_A^{\mathcal{M}'} = [\![\chi]\!]_A^{\mathcal{M}}$, $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$ and $[\![\neg\varphi]\!]_A^{\mathcal{M}'} = [\![\neg\varphi]\!]_A^{\mathcal{M}}$. Since $\|p\|' = B \subseteq [\![\theta]\!]_A^{\mathcal{M}'} \subseteq A$ we have $\|p\|' = [\![p]\!]_A^{\mathcal{M}'}$. Because of (3) we have that $w \in [\![\theta]\!]_A^{\mathcal{M}'}$ and $w \in [\![\neg\varphi]\!]_B^{\mathcal{M}'} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![p]\!]_A^{\mathcal{M}'}} = [\![\langle p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$. Thus, $w \in [\![p]\!]_A^{\mathcal{M}'}$, so $w \in [\![\theta \wedge p]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}'} \cap [\![p]\!]_A^{\mathcal{M}'} = [\![p]\!]_A^{\mathcal{M}'}$ simply because $[\![p]\!]_A^{\mathcal{M}'} \subseteq [\![\theta]\!]_A^{\mathcal{M}'}$. Since $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![p]\!]_A^{\mathcal{M}'}}$ we obtain $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta\wedge p]\!]_A^{\mathcal{M}'}}$. Putting everything together, $w \in [\![\theta \wedge p]\!]_A^{\mathcal{M}'}$ and $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta\wedge p]\!]_A^{\mathcal{M}'}}$, we obtain that $w \in [\![\langle\theta \wedge p\rangle\neg\varphi]\!]_A^{\mathcal{M}'}$ and $w \in [\![\chi]\!]_A^{\mathcal{M}'}$. Therefore $\mathcal{M}', w \models \chi \wedge \langle\theta \wedge p\rangle\neg\varphi$, which contradicts the validity of $\chi \to [\theta \wedge p]\varphi$.

($[!][G]$-intro): Suppose $\models \chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$ and $\not\models \chi \to [\theta][G]\varphi$ where $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$. The latter means that there exists a pseudo model $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ such that for some $A \in \mathcal{A}$ and some $w \in A$, $w \notin [\![\chi \to [\theta][G]\varphi]\!]_A^{\mathcal{M}}$. Therefore $w \in [\![\chi \wedge \neg[\theta][G]\varphi]\!]_A^{\mathcal{M}}$. Thus we have (1) $w \in [\![\chi]\!]_A^{\mathcal{M}}$, and (2) $w \in [\![\neg[\theta][G]\varphi]\!]_A^{\mathcal{M}}$. Item (2) means $w \in [\![\langle\theta\rangle\langle G\rangle\neg\varphi]\!]_A^{\mathcal{M}}$. Then, by the semantics of $\langle!\rangle$, we have $w \in [\![\langle G\rangle\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}}}$. Therefore by the semantics of $\langle G\rangle$ we obtain: (3) there exists $\{B_i : i \in G\} \subseteq \mathcal{A}$ s.t. $w \in [\![\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}}} B_i}$.

Now consider the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|')$ such that $\|p_i\|' = B_i$ and $\|q\|' = \|q\|$ for any $q \neq p_i \in \mathit{Prop}$ for all $i \in G$. Observe that since $[\![\theta]\!]_A \subseteq A$, by Boolean operations of sets we obtain that $K_i^{[\![\theta]\!]_A}(A \cap B_i) = K_i^{[\![\theta]\!]_A} B_i$. In order to use Lemma 7 we must show that $\mathcal{M}'$ is a pseudo-model as in the soundness proof of $[!]\Box$-intro. First note that for every $q \neq p_i$, since $||q||' = ||q||$ and $||q|| \in \mathcal{A}$, we have $||q||' \in \mathcal{A}$. Moreover, since for every $i \in G$, $||p_i||' = B_i \in \mathcal{A}$, we conclude that $\mathcal{M}'$ satisfies Definition 8.1. Since $\mathcal{A}$ is the same for both $\mathcal{M}$ and $\mathcal{M}'$, and $\mathcal{M}$ is a pseudo model, the rest of the closure conditions are satisfied already. Therefore $\mathcal{M}'$ is a pseudo model. Now continuing with our soundness proof, given $p_i \notin P_\chi \cup P_\theta \cup P_\varphi$ for all $i \in G$, by Lemma 7, we obtain $[\![\chi]\!]_A^{\mathcal{M}'} = [\![\chi]\!]_A^{\mathcal{M}}$ and $[\![\theta]\!]_A^{\mathcal{M}'} = [\![\theta]\!]_A^{\mathcal{M}}$. We moreover have that

$$[\![\neg\varphi]\!]^{\mathcal{M}}_{[\![\theta]\!]_A^{\mathcal{M}} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}}} B_i} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} B_i} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} ||p_i||'} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} (A \cap ||p_i||')}$$

by the above observation. And by Lemma 6, we obtain

$$[\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} (A \cap ||p_i||')} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta]\!]_A^{\mathcal{M}'} \cap \bigcap_{i\in G} K_i^{[\![\theta]\!]_A^{\mathcal{M}'}} ([\![p_i]\!]_A^{\mathcal{M}'})} = [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\!]_A^{\mathcal{M}'}}.$$

Therefore, $w \in [\![\neg\varphi]\!]^{\mathcal{M}'}_{[\![\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\!]_A^{\mathcal{M}'}}$. I.e., $w \in [\![\langle \theta \wedge \bigwedge_{i\in G} K_i^\theta p_i \rangle \neg\varphi]\!]_A^{\mathcal{M}'}$. Since we also have that $w \in [\![\chi]\!]_A^{\mathcal{M}'}$, we conclude that $w \in [\![\chi \wedge \langle \theta \wedge \bigwedge_{i\in G} K_i^\theta p_i \rangle \neg\varphi]\!]_A^{\mathcal{M}'}$, contradicting the validity of $\chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$. Therefore, $\models \chi \to [\theta][G]\varphi$.

Definition 9 (Standard Pre-model). *A pre-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, || \cdot ||)$ *is* standard *if and only if* $\mathcal{A} = \{[\![\theta]\!]_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$.

Proposition 10. *Every standard pre-model is a pseudo-model.*

Proof. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, || \cdot ||)$ be a standard pre-model. This implies that $\mathcal{A} = \{[\![\theta]\!]_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$. We need to show that $\mathcal{M}$ satisfies the closure conditions given in Definition 8. Conditions (1) and (2) are immediate.

For (3): let $A \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, we know that $A = [\![\theta]\!]_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Since $\theta \in \mathcal{L}_{-\Diamond}$, we have $\neg\theta \in \mathcal{L}_{-\Diamond}$, thus, $[\![\neg\theta]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![\neg\theta]\!]_{W^0} = W^0 - [\![\theta]\!]_{W^0}$, thus, we obtain $W^0 - A \in \mathcal{A}$.

For (4): let $A, B \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, $A = [\![\theta_1]\!]_{W^0}$ and $B = [\![\theta_2]\!]_{W^0}$ for some $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$. Since $\theta_1, \theta_2 \in \mathcal{L}_{-\Diamond}$, we have $\theta_1 \wedge \theta_2 \in \mathcal{L}_{-\Diamond}$, thus, $[\![\theta_1 \wedge \theta_2]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![\theta_1 \wedge \theta_2]\!]_{W^0} = [\![\theta_1]\!]_{W^0} \cap [\![\theta_2]\!]_{W^0} = A \cap B$, thus, we obtain $A \cap B \in \mathcal{A}$.

For (5): let $A \in \mathcal{A}$. Since $\mathcal{M}$ is a standard pre-model, $A = [\![\theta]\!]_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Since $\theta \in \mathcal{L}_{-\Diamond}$, we have $K_i\theta \in \mathcal{L}_{-\Diamond}$, thus, $[\![K_i\theta]\!]_{W^0} \in \mathcal{A}$. Observe that $[\![K_i\theta]\!]_{W^0} = \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in [\![\theta]\!]_{W^0})\} = K_i[\![\theta]\!]_{W^0}$, thus, we obtain $K_iA \in \mathcal{A}$.

Equivalence between the standard pseudo-models and announcement models. For Proposition 11 only, we use the notation $[\![\varphi]\!]_A^{PS}$ to refer to pseudo-model semantics (as in Definition 7) and use $[\![\varphi]\!]_{\mathcal{M}}$ to refer to the semantics on *a*-models (as in Definition 5).

Proposition 11.

1. *For every standard pseudo-model* $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, || \cdot ||)$ *and every set* $A \in \mathcal{A}$*, we denote by* $\mathcal{M}_A$ *the model* $\mathcal{M}_A = (W^0, A, \sim_1, \ldots, \sim_n, || \cdot ||)$*. Then:*
   (a) *For every* $\varphi \in \mathcal{L}_G$*, we have* $[\![\varphi]\!]_{\mathcal{M}_A} = [\![\varphi]\!]_A^{PS}$ *for all* $A \in \mathcal{A}$*.*
   (b) $\mathcal{M}_A$ *is an a-model, for all* $A \in \mathcal{A}$*.*

2. *For every a-model* $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, || \cdot ||)$*, we denote by* $\mathcal{M}'$ *the pre-model* $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, || \cdot ||)$*, where* $\mathcal{A} = \{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\Diamond}\}$*. Then*

(a) $\mathcal{M}'$ is a standard pseudo-model.
(b) For every $\varphi \in \mathcal{L}_G$, we have $[\![\varphi]\!]_{\mathcal{M}} = [\![\varphi]\!]^{PS}_{W}$.

Proof. See Appendix A.5.

Corollary 2. *Validity on standard pseudo-models coincides with validity on the a-models.*

Proof. This is a straightforward consequence of Proposition 11.

Corollary 3. *The system* **GALM** *is sound wrt a-models. Moreover, the system* **APALM** *is sound wrt a-models.*

Proof. Follows immediately from Proposition 9 and Corollary 2.

It is important to note that the equivalence between standard pseudo-models and *a*-models (given by Proposition 11 above, and underlying our soundness result) is *not* trivial (the proof is in Appendix A.5). It relies in particular on the equivalence between the effort modality and the arbitrary announcement operator $\Box$ (see Lemma 22.1 in Appendix A.5) and on the equivalence between the *purely syntactic* and *purely semantic* descriptions of the group announcement operator $[G]$ on standard pseudo models (see Lemma 22.2 in Appendix A.5). These equivalences hold *only because our models and language retain the memory of the initial situation*. Hence, a similar equivalence fails for the original APAL and GAL.

5 Completeness

In this section we prove the completeness of **GALM** and **APALM**. First, we show completeness with respect to pseudo-models, via an innovative modification of the standard canonical model construction. This is based on a method previously used in [7], that makes an essential use of the finitary $\Box$ and $[G]$-introduction rules, by requiring our canonical theories T to be (not only maximally consistent, but also) "witnessed". Roughly speaking, a theory T is witnessed if: every $\Diamond\varphi$ occurring in every "existential context" in T is witnessed by some atomic formula p, meaning that $\langle p\rangle\varphi$ occurs in the same existential context in T, and if for every $\langle G\rangle\varphi$ occurring in every "existential context" in T is witnessed by some formula $\wedge_{i\in G}K_i p_i$, meaning that $\langle\wedge_{i\in G}K_i p_i\rangle\varphi$ occurs in the same existential context in T. Our canonical pre-model will consist of all *initial, maximally consistent, witnessed theories* (where a theory is 'initial' if it contains the formula 0). A Truth Lemma is proved, as usual. Completeness for (both pseudo-models and) *a*-models follows from the observation that our canonical pre-model *is standard*, hence it is (a standard pseudo-model, and thus) equivalent to a genuine *a*-model.

We now proceed with the details. The appropriate notion of "existential context" is represented by *possibility forms*, in the following sense.

Definition 10 (Necessity forms and possibility forms). *For any finite string $s \in (\{\bullet^0\}\cup\{\varphi\to \mid \varphi \in \mathcal{L}_G\}\cup\{K_i : i \in \mathcal{A}\} \cup \{U\} \cup \{\rho \mid \rho \in \mathcal{L}_{-\Diamond}\})^* = NF$, we define pseudo-modalities $[s]$ and $\langle s\rangle$. These pseudo-modalities are functions mapping any formula $\varphi \in \mathcal{L}_G$ to another formula $[s]\varphi \in \mathcal{L}_G$ (*necessity form*), respectively $\langle s\rangle\varphi \in \mathcal{L}_G$ (*possibility form*). The necessity forms are defined recursively as $[\epsilon]\varphi = \varphi$, $[s,\bullet^0]\varphi = [s]\varphi^0$, $[s,\varphi\to]\varphi = [s](\varphi \to \varphi)$, $[s,K_i]\varphi = [s]K_i\varphi$, $[s,U]\varphi = [s]U\varphi$, $[s,\rho]\varphi = [s][\rho]\varphi$, where ϵ is the empty string. For possibility forms, we set $\langle s\rangle\varphi := \neg[s]\neg\varphi$.*

Example: $[K_i,\bullet^0,\Diamond p\to,0,U]$ is a necessity form such that $[K_i,\bullet^0,\Diamond p\to,0,U]\varphi = K_i(\Diamond p \to [0]U\varphi)^0$.

Definition 11 (Theories: witnessed, initial, maximal). *: Let $\mathcal{L}_G^P$ be the language of GALM based only on some countable set P of propositional variables. Similarly, let NF^P denote the corresponding set of strings defined based on $\mathcal{L}_G^P$ (necessity and possibility forms are as given in Definition 10). A* P-theory *is a consistent set of formulas in $\mathcal{L}_G^P$ (where "consistent" means consistent with respect to the axiomatization of* **GALM** *formulated for $\mathcal{L}_G^P$). A* maximal P-theory *is a P-theory Γ that is maximal with respect to $\subseteq$ among all P-theories; in other words, Γ cannot be extended to another P-theory. A* P-witnessed theory *is a P-theory Γ such that, for every $s \in NF^P$ and $\varphi \in \mathcal{L}_G^P$, (1) if $\langle s\rangle\Diamond\varphi$ is consistent with Γ then there is $p \in P$ such that $\langle s\rangle\langle p\rangle\varphi$ is consistent with Γ (or equivalently: if $\Gamma \vdash [s][p]\neg\varphi$ for all $p \in P$, then $\Gamma \vdash [s]\Box\neg\varphi$), and (2) for every $G \subseteq \mathcal{AG}$, if $\langle s\rangle\langle G\rangle\varphi$ is consistent with Γ then there is $\{p_i : i \in G\} \subseteq P$ such that $\langle s\rangle\langle \wedge_{i\in G} K_i p_i\rangle\varphi$ is consistent with Γ. A P-theory Γ is called* initial *if* $0 \in \Gamma$. *A* maximal P-witnessed theory *Γ is a P-witnessed theory that is not a proper subset of any P-witnessed theory. A* maximal P-witnessed initial theory *Γ is a maximal P-witnessed theory such that* $0 \in \Gamma$.

Lemma 8. *For every necessity form $[s]$, there exist formulas $\theta \in \mathcal{L}_{-\Diamond}$ and $\psi \in \mathcal{L}_G$, with $P_\psi \cup P_\theta \subseteq P_s$, such that for all $\varphi \in \mathcal{L}_G$, we have*

$$\vdash [s]\varphi \textit{ iff } \vdash \psi \to [\theta]\varphi.$$

Proof. See Appendix A.6.

Lemma 9. *The following rules are admissible in* **GALM***:*

1. *if $\vdash [s][p]\varphi$ then $\vdash [s]\Box\varphi$, where $p \notin P_s \cup P_\varphi$,*
2. *if $\vdash [s][\bigwedge_{i\in G} K_i p_i]\varphi$ then $\vdash [s][G]\varphi$, where $p_i \notin P_s \cup P_\varphi$.*

Proof. For (1), suppose $\vdash [s][p]\varphi$. Then, by Lemma 8, there exist $\theta \in \mathcal{L}_{-\Diamond}$ and $\psi \in \mathcal{L}_G$ such that $\vdash \psi \to [\theta][p]\varphi$. By the auxiliary reduction in Proposition 5.14, we get $\vdash \psi \to [\theta \wedge p]\varphi$. By the construction of the formulas ψ and θ, we know that $P_\psi \cup P_\theta \subseteq P_s$, and so $p \notin P_\psi \cup P_\theta \cup P_\varphi$. Therefore, by ([!]$\Box$-intro), we have $\vdash \psi \to [\theta]\Box\varphi$. Applying again Lemma 8, we obtain $\vdash [s]\Box\varphi$. The proof of (2) goes in a similar way as the one before given that ($\star$) $[\theta][\bigwedge_{i\in G} K_i p_i]\varphi \leftrightarrow [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$ is derivable in **GALM** (by using the appropriate reduction axioms and RE). Let $s \in NF^P$ such that $\vdash [s][\bigwedge_{i\in G} K_i p_i]\varphi$ where $p_i \notin P_s \cup P_\varphi$. Then, by Lemma 8, we obtain that $\vdash \chi \to [\theta][\bigwedge_{i\in G} K_i p_i]\varphi$. Therefore, by ($\star$), we have that $\vdash \chi \to [\theta \wedge \bigwedge_{i\in G} K_i^\theta p_i]\varphi$. By the [!][G]-intro rule we then obtain $\vdash \chi \to [\theta][G]\varphi$. Again by Lemma 8, we get $\vdash [s][G]\varphi$.

Lemma 10. *For every maximal P-witnessed theory Γ, and every formula $\varphi, \psi \in \mathcal{L}_G^P$,*

1. *$\Gamma \vdash \varphi$ iff $\varphi \in \Gamma$*
2. *$\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$,*
3. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,*
4. *$\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ implies $\psi \in \Gamma$.*
5. **GALM**$_P$ *$\subseteq \Gamma$, where* **GALM**$_P$ *is* **GALM** *formulated for $\mathcal{L}_G^P$.*

Proof. The proof is standard. We prove only item (5): suppose **GALM**$_P \not\subseteq \Gamma$. This means that there is a sentence $\psi \in \mathcal{L}_G^P$ such that $\psi \in$ **GALM**$_P$ but $\psi \notin \Gamma$. The former means that $\vdash \psi$, thus, $\Gamma \vdash \psi$. Items (2) and (1) implies that if $\psi \notin \Gamma$ then $\Gamma \vdash \neg\psi$, contradicting consistency of Γ.

Lemma 11. *For every $\Gamma \subseteq \mathcal{L}_G^P$, if Γ is a P-theory and $\Gamma \nvdash \neg\varphi$ for some $\varphi \in \mathcal{L}_G^P$, then $\Gamma \cup \{\varphi\}$ is a P-theory. Moreover, if Γ is P-witnessed, then $\Gamma \cup \{\varphi\}$ is also P-witnessed.*

Proof. The proof of the first claim is standard. We only prove the second claim. Suppose that Γ is P-witnessed but $\Gamma \cup \{\varphi\}$ is not P-witnessed. By the previous statement, we know that $\Gamma \cup \{\varphi\}$ is consistent. Since $\Gamma \cup \{\varphi\}$ is not P-witnessed, it violates either (1) or (2) in Definition 11. First suppose $\Gamma \cup \{\varphi\}$ does not satisfy (1), that is, there is $s \in NF^P$ and $\psi \in \mathcal{L}_G^P$ such that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$ but $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle p\rangle\psi$ for all $p \in P$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][p]\neg\psi$ for all $p \in P$. Therefore, $\Gamma \vdash \varphi \to [s][p]\neg\psi$ for all $p \in P$.

Note that $\varphi \to [s][p]\neg\psi := [\varphi \to, s][p]\neg\psi$, and $[\varphi\to, s] \in NF^P$. We thus have $\Gamma \vdash [\varphi \to, s][p]\neg\psi$ for all $p \in P$. Since Γ is P-witnessed, we obtain $\Gamma \vdash [\varphi \to, s]\Box\neg\psi$. By unraveling the necessity form $[\varphi\to, s]$, we get $\Gamma \vdash \varphi \to [s]\Box\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s]\Box\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\Diamond\psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\Diamond\psi$. Now suppose $\Gamma\cup\{\varphi\}$ does not satisfy (2). This means that there is $s \in NF^P$ and $\psi \in \mathcal{L}_G^P$ such that for some group $G \subseteq \mathcal{A}$, the set $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\langle G\rangle\psi$ but $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\psi$ for all $\{p_i : i \in G\} \subseteq P$. This implies that $\Gamma \cup \{\varphi\} \vdash [s][\bigwedge_{i\in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Therefore, $\Gamma \vdash \varphi \to [s][\bigwedge_{i\in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Note that $\varphi \to [s][\bigwedge_{i\in G} K_i p_i]\neg\psi := [\varphi \to, s][\bigwedge_{i\in G} K_i p_i]\neg\psi$, and $[\varphi\to, s] \in NF^P$. We thus have $\Gamma \vdash [\varphi \to, s][\bigwedge_{i\in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Since Γ is P-witnessed, we obtain $\Gamma \vdash [\varphi \to, s][G]\neg\psi$. By unraveling the necessity form $[\varphi\to, s]$, we get $\Gamma \vdash \varphi \to [s][G]\neg\psi$, thus, $\Gamma \cup \{\varphi\} \vdash [s][G]\neg\psi$, i.e., $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle G\rangle\psi$, contradicting the assumption that $\Gamma \cup \{\varphi\}$ is consistent with $\langle s\rangle\langle G\rangle\psi$. All together we obtain that $\Gamma \cup \{\varphi\}$ is P-witnessed.

Lemma 12. *If $\{\Gamma_i\}_{i\in\mathbb{N}}$ is an increasing chain of P-theories such that $\Gamma_i \subseteq \Gamma_{i+1}$, then $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is a P-theory.*

Proof. Let $\{\Gamma_i\}_{i\in\mathbb{N}}$ be an increasing chain of P-theories with $\Gamma_i \subseteq \Gamma_{i+1}$ and suppose, toward contradiction, that $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is not a P-theory, i.e., suppose that $\bigcup_{n\in\mathbb{N}} \Gamma_n \vdash \bot$. This means that there exists a finite $\Delta \subseteq \bigcup_{n\in\mathbb{N}} \Gamma_n$ such that $\Delta \vdash \bot$. Then, since $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is a union of an increasing chain of P-theories, there is some $m \in \mathbb{N}$ such that $\Delta \subseteq \Gamma_m$. Therefore, $\Gamma_m \vdash \bot$ contradicting the fact that Γ_m is a P-theory. Hence, $\bigcup_{n\in\mathbb{N}} \Gamma_n$ is a P-theory.

Lemma 13. *For every maximal P-witnessed theory T, both $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\}$ and $\{\theta \in \mathcal{L}_G^P : U\theta \in T\}$ are P-witnessed theories.*

Proof. Observe that, by axiom (T_{K_i}), $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\} \subseteq T$. Therefore, as T is consistent, the set $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\}$ is consistent. Let $s \in NF^P$, $\psi \in \mathcal{L}^P$, and $G \subseteq \mathcal{AG}$ such that $\{\theta \in \mathcal{L}^P : K_i\theta \in T\} \vdash [s][p]\neg\varphi$ for all $p \in P$ and $\{\theta \in \mathcal{L}_G^P : K_i\theta \in T\} \vdash [s][\bigwedge_{i\in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. By normality of K_i, $T \vdash K_i[s][p]\neg\varphi$ for all $p \in P$ and $T \vdash K_i[s][\bigwedge_{i\in G} K_i p_i]\neg\psi$ for all $\{p_i : i \in G\} \subseteq P$. Since $K_i[s][p]\neg\varphi := [K_i, s][p]\neg\varphi$ and $K_i[s][\bigwedge_{i\in G} K_i p_i]\neg\psi := [K_i, s][[\bigwedge_{i\in G} K_i p_i]\neg\psi$ are necessity forms and T is P-witnessed, we obtain $T \vdash [K_i, s]\Box\neg\varphi$ and $T \vdash [K_i, s][G]\neg\varphi$, i.e., $T \vdash K_i[s]\Box\neg\varphi$ and $T \vdash K_i[s][G]\neg\varphi$. As T is maximal, we have $K_i[s]\Box\neg\varphi \in T$ and $K_i[s][G]\neg\varphi \in T$, thus $[s]\Box\neg\varphi \in \{\theta \mid K_i\theta \in T\}$ and $[s][G]\neg\varphi \in \{\theta \mid K_i\theta \in T\}$. The proof for $\{\theta \in \mathcal{L}_G^P : U\theta \in T\}$ follows similarly.

Lemma 14 (Lindenbaum's Lemma). *Every P-witnessed theory Γ can be extended to a maximal P-witnessed theory T_Γ.*

Proof. See Appendix A.6.

Lemma 15 (Extension Lemma). *Let P be a countable set of propositional variables and P' be a countable set of* fresh *propositional variables, i.e., $P \cap P' = \emptyset$. Let $\widetilde{P} = P \cup P'$. Then, every initial P-theory Γ can be extended to an initial $\widetilde{P}$-witnessed theory $\widetilde{\Gamma} \supseteq \Gamma$, and hence to a maximal $\widetilde{P}$-witnessed initial theory $T_\Gamma \supseteq \Gamma$.*

Proof. See Appendix A.6.

To define our canonical pseudo-model, we first put, for all maximal P-witnessed theories T, S:

$$T \sim_U S \quad \text{iff} \quad \forall\varphi \in \mathcal{L}_G^P\,(U\varphi \in T \;\text{ implies }\; \varphi \in S)\,.$$

Definition 12 (Canonical Pre-Model). *Given a maximal P-witnessed initial theory T_0, the* canonical pre-model for T_0 *is a tuple $\mathcal{M}^c = (W^c, \mathcal{A}^c, \sim_1^c, \ldots, \sim_n^c, \|\cdot\|^c)$ such that:*

- $W^c = \{T :$ *T is a maximal P-witnessed theory such that $T_0 \sim_U T$*$\}$,

– $\mathcal{A}^c = \{\widehat{\theta} \ : \ \theta \in \mathcal{L}^P_{-\Diamond}\}$ *where* $\widehat{\varphi} = \{T \in W^c \ : \ \varphi \in T\}$ *for any* $\varphi \in \mathcal{L}^P_G$,
– *for every* $T, S \in W^c$ *and* $i \in \mathcal{AG}$ *we define:*

$$T \sim^c_i S \quad \textit{iff} \quad \forall \varphi \in \mathcal{L}^P_G \, (K_i\varphi \in T \ \textit{ implies } \ \varphi \in S)\,.$$

– $||p||^c = \{T \in W^c \ : \ p \in T\} = \widehat{p}$.

As usual, it is easy to see (given the $S5$ axioms for K_i and for U) that $\sim_U$ and $\sim^c_i$ are equivalence relations.

Lemma 16 (Existence Lemma for K_i). *Let T be a maximal P-witnessed theory, $\alpha \in \mathcal{L}^P_{-\Diamond}$, and $\varphi \in \mathcal{L}^P_G$ such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. Then, there is a maximal P-witnessed theory S such that $T \sim_i S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof. Let $\alpha \in \mathcal{L}^P_{-\Diamond}$ and $\varphi \in \mathcal{L}^P_G$ such that $\alpha \in T$ and $K_i[\alpha]\varphi \notin T$. The latter implies that $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \nvdash [\alpha]\varphi$, hence, $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \nvdash \neg\neg[\alpha]\varphi$. Then, by Lemmas 11 and 13, we obtain that $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is a P-witnessed theory. Note that $\vdash \neg[\alpha]\varphi \leftrightarrow (\alpha \wedge [\alpha]\neg\varphi)$ (see Proposition 5.3). We therefore obtain that $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \vdash \alpha$, thus, $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \nvdash \neg\alpha$ (since $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\}$ is consistent). Therefore, by Lemma 11, $\{\psi \in \mathcal{L}^P_G : K_i\psi \in T\} \cup \{\neg[\alpha]\varphi\} \cup \{\alpha\}$ is also a P-witnessed theory. We can then apply Lindenbaum's Lemma (Lemma 14) and extend it to a maximal P-witnessed theory S such that $S \sim_i T$, $\alpha \in S$, and $[\alpha]\varphi \notin S$.

Lemma 17 (Existence Lemma for U). *Let T be a maximal P-witnessed theory, $\alpha \in \mathcal{L}^P_{-\Diamond}$, and $\varphi \in \mathcal{L}^P_G$ such that $\alpha \in T$ and $U[\alpha]\varphi \notin T$. Then, there is a maximal P-witnessed theory S such that $T \sim_U S$, $\alpha \in S$ and $[\alpha]\varphi \notin S$.*

Proof. See the proof of Lemma 16.

Corollary 4. *For $\varphi \in \mathcal{L}_G$, we have $\widehat{U\varphi} = W^c$ if $\widehat{\varphi} = W^c$, and $\widehat{U\varphi} = \emptyset$ otherwise.*

Proof. If $\widehat{\varphi} = W^c$, suppose $\widehat{U\varphi} \neq W^c$. The latter means that there is a $T \in W^c$ such that $U\varphi \notin T$. Then, by Lemma 17 (when $\alpha := \top$), there is a maximal P-witnessed theory S such that $T \sim_U S$ and $\varphi \notin S$. Since $T_0 \sim_U T \sim_U S$ and $\sim_U$ is transitive, we have $T_0 \sim_U S$, thus, $S \in W^c$. Therefore, $\widehat{\varphi} \neq W^c$, contradicting the initial assumption. If $\widehat{\varphi} \neq W^c$, then there is a $T \in W^c$ such that $\varphi \notin T$. Since $T \sim_U S$ for all $S \in W^c$, we obtain by the definition of $\sim_U$ that $U\varphi \notin S$ for all $S \in W^c$. Therefore, $\widehat{U\varphi} = \emptyset$.

Lemma 18. *Every element $T \in W^c$ is an initial theory (i.e. $0 \in T$).*

Proof. Let $T \in W^c$. By the construction of W^c, we have $T_0 \sim_U T$. Since $0 \to U0$ is an axiom and T_0 is maximal, $(0 \to U0) \in T_0$. Thus, since $0 \in T_0$, we obtain $U0 \in T_0$ (by Lemma 10.4). Therefore, by the definition of $\sim_U$ and since $T_0 \sim_U T$, we have that $0 \in T$.

Corollary 5. *For all $\varphi \in \mathcal{L}^P_G$, we have $\widehat{\varphi} = \widehat{\varphi^0}$.*

Proof. Since $0 \in T$ for all $T \in W^c$, we obtain by axiom (0-*eq*) that $\varphi \leftrightarrow \varphi^0 \in T$ for all $T \in W^c$. Therefore, $\widehat{\varphi} = \widehat{\varphi^0}$.

Lemma 19 (Truth Lemma). *Let $\mathcal{N}^c = (W^c, \mathcal{A}^c, \sim^c_1, \ldots, \sim^c_n, V^c)$ be the canonical pre-model for some T_0 (in $\mathcal{L}^P_G$) and $\varphi \in \mathcal{L}^P_G$. Then, for all $\alpha \in \mathcal{L}^P_{-\Diamond}$ we have $[\![\varphi]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\varphi}$.*

Proof. The proof is by $<$-induction on φ, using the following *induction hypothesis* (IH): for all $\psi < \varphi$, we have $[\![\psi]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle\psi}$ for all $\alpha \in \mathcal{L}_{-\Diamond}$. The cases for the Boolean connectives are straightforward. The cases for K_i and U are standard, using $\vdash \langle\alpha\rangle K_i\psi \leftrightarrow \alpha \wedge K_i[\alpha]\psi$ and Lemma 16 for K_i, and $\vdash \langle\alpha\rangle U\psi \leftrightarrow \alpha \wedge U[\alpha]\psi$ and Lemma 17 for U.

Base case $\varphi := \top$. Then $[\![\top]\!]_{\widehat{\alpha}} = \widehat{\alpha} = \widehat{\langle\alpha\rangle\top}$, by Defn.7 and the fact that $\vdash \alpha \leftrightarrow \langle\alpha\rangle\top$.

Base case $\varphi := p$. Then $[\![p]\!]_{\widehat{\alpha}} = \|p\|^c \cap \widehat{\alpha} = \widehat{p} \cap \widehat{\alpha} = \widehat{p \wedge \alpha} = \widehat{\langle\alpha\rangle p}$, by Defn.7, the defn. of $\|\cdot\|^c$, R_p, and Proposition 5.3.

Base case $\varphi := 0$. Then $[\![0]\!]_{\widehat{\alpha}} = W^c$ if $\widehat{\alpha} = W^c$, and $[\![0]\!]_{\widehat{\alpha}} = \emptyset$ otherwise. Also, $\widehat{\langle\alpha\rangle 0} = \widehat{0 \wedge U\alpha} = \widehat{0} \cap \widehat{U\alpha} = \widehat{0} \cap \widehat{U\alpha} = \widehat{U\alpha}$ (by Propositions 5.2 and Lemma 18). By Corollary 4, $\widehat{U\alpha} = W^c$ if $\widehat{\alpha} = W^c$, and $\widehat{U\alpha} = \emptyset$ otherwise. So $[\![0]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle 0}$.

Case $\varphi := \psi^0$. Follows easily from $\widehat{\top} = W^c$ and R[$\top$], Corollary 5, and R^0.

Case $\varphi := \langle\chi\rangle\psi$. Straightforward, using the fact that $\vdash \langle\alpha\rangle\langle\chi\rangle\psi \leftrightarrow \langle\langle\alpha\rangle\chi\rangle\psi$ (by $\mathrm{R}_{[!]}$)

Case $\varphi := \Diamond\psi$.

($\Rightarrow$) Suppose $T \in [\![\Diamond\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7, that $\alpha \in T$ and there exists $B \in \mathcal{A}^c$ such that $T \in B \subseteq \widehat{\alpha}$ and $T \in [\![\psi]\!]_B$ (see Observation 3.1). By the construction of $\mathcal{A}^c$, we know that $B = \widehat{\theta}$ for some $\theta \in \mathcal{L}^P_{-\Diamond}$. Therefore, $T \in [\![\psi]\!]_B$ means that $T \in [\![\psi]\!]_{\widehat{\theta}}$. Moreover, since $\widehat{\theta} \subseteq \widehat{\alpha}$ and, thus, $\widehat{\theta} = \widehat{\alpha} \cap \widehat{\theta} = \widehat{\alpha \wedge \theta}$, we obtain $T \in [\![\psi]\!]_{\widehat{\alpha \wedge \theta}}$. By Lemma 4.1, we have $\psi < \Diamond\psi$. Therefore, by IH, we obtain $T \in \widehat{\langle\alpha \wedge \theta\rangle\psi}$. Then, by axiom ([!]$\Box$-elim) and the fact that T is maximal, we conclude that $T \in \widehat{\langle\alpha\rangle\Diamond\psi}$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\Diamond\psi}$, i.e., $\langle\alpha\rangle\Diamond\psi \in T$. Then, since T is a maximal P-witnessed theory, there is $p \in P$ such that $\langle\alpha\rangle\langle p\rangle\psi \in T$. By Lemma 4.2, we know that $\langle p\rangle\psi < \Diamond\psi$. Thus, by IH on $\langle p\rangle\psi$, we obtain that $T \in [\![\langle p\rangle\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7 and Observation 3.3, that $T \in [\![\psi]\!]_{[\![p]\!]_{\widehat{\alpha}}} \subseteq [\![p]\!]_{\widehat{\alpha}}$. Since $p < \Diamond\psi$, by IH on p, we obtain that $[\![p]\!]_{\widehat{\alpha}} = \widehat{\langle\alpha\rangle p} \subseteq \widehat{\alpha}$. By the construction of $\mathcal{A}^c$, we moreover have $\widehat{\langle\alpha\rangle p} \in \mathcal{A}^c$. Therefore, as $T \in [\![\psi]\!]_{\widehat{\langle\alpha\rangle p}}$ and $\widehat{\langle\alpha\rangle p} \subseteq \widehat{\alpha}$, by Definition 7, we conclude that $T \in [\![\Diamond\psi]\!]_{\widehat{\alpha}}$.

Case $\varphi := \langle G\rangle\psi$.

($\Rightarrow$) Suppose $T \in [\![\langle G\rangle\psi]\!]_{\widehat{\alpha}}$. This means by Definition 7 that $T \in \widehat{\alpha}$ and there exists $\{B_i : i \in G\} \subseteq \mathcal{A}^c$ such that $T \in [\![\psi]\!]_{\widehat{\alpha} \cap \bigcap_{i\in G} K_i^{\widehat{\alpha}} B_i}$. By the construction of $\mathcal{A}^c$ we know that for all $i \in G$, $B_i = \widehat{\theta_i}$ for some $\theta_i \in \mathcal{L}^P_{-\Diamond}$. Therefore $T \in [\![\psi]\!]_{\widehat{\alpha} \cap \bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{\theta_i}}$. It suffices to show that: $\widehat{\alpha} \cap \bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{\theta_i} = \widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha \theta_i}$. First we need to show that $K_i^{\widehat{\alpha}}\widehat{\theta_i} = \widehat{K_i^\alpha\theta_i}$. Note that $K_i^{\widehat{\alpha}}\widehat{\theta_i} = K_i(\widehat{\alpha} \to \widehat{\theta_i}) = K_i(\widehat{\alpha \to \theta_i})$ and $\widehat{K_i^\alpha\theta_i} = \widehat{K_i(\alpha \to \theta_i)}$. For ($\subseteq$): Let $T \in K_i^{\widehat{\alpha}}\widehat{\theta_i}$, then for all $S \sim_i T$, $S \in \widehat{\alpha \to \theta_i}$. Therefore $T \in \widehat{K_i(\alpha \to \theta_i)}$ and so $T \in \widehat{K_i^\alpha\theta_i}$. For ($\supseteq$): Let $T \in \widehat{K_i^\alpha\theta_i}$, this means that $K_i(\alpha \to \theta_i) \in T$. Thus for all $S \sim_i T$, $\alpha \to \theta_i \in S$. Therefore $T \in K_i^{\widehat{\alpha}}\widehat{\theta_i}$. Using this, it is easy to see that $\widehat{\alpha} \cap \bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{\theta_i} = \widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i}$. We then obtain that $T \in [\![\psi]\!]_{\widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i}}$. Since $\psi < \langle G\rangle\psi$, by I.H. we have that $T \in \widehat{\langle\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i\rangle\psi}$. Thus $\langle\alpha \wedge \bigwedge_{i\in G} K_i^\alpha\theta_i\rangle\psi \in T$. By ([!][$G$]-elim) we have $\langle\alpha\rangle\langle G\rangle\psi \in T$.

($\Leftarrow$) Suppose $T \in \widehat{\langle\alpha\rangle\langle G\rangle\psi}$, i.e., $\langle\alpha\rangle\langle G\rangle\psi \in T$. Since T is a maximal P-witnessed theory, there is $\{p_i : i \in G\} \subseteq P$ such that $\langle\alpha\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\psi \in T$. By Lemma 4.3, we know that $\langle\bigwedge_{i\in G} K_i p_i\rangle\psi < \langle G\rangle\psi$. Thus, by IH on $\langle\bigwedge_{i\in G} K_i p_i\rangle\psi$, we obtain that $T \in [\![\langle\bigwedge_{i\in G} K_i p_i\rangle\psi]\!]_{\widehat{\alpha}}$. This means, by Definition 7, that $T \in [\![\psi]\!]_{[\![\bigwedge_{i\in G} K_i p_i]\!]_{\widehat{\alpha}}}$. By IH on $\bigwedge_{i\in G} K_i p_i$, we obtain that $T \in [\![\psi]\!]_{\widehat{\langle\alpha\rangle\bigwedge_{i\in G} K_i p_i}}$. By Proposition 3.2 and the reduction axioms (R_{K_i}) and (R_p), it is easy to see that the formula $\langle\alpha\rangle\bigwedge_{i\in G} K_i p_i \leftrightarrow \alpha \wedge \bigwedge_{i\in G} K_i^\alpha p_i$ is derivable in **GALM**. Therefore,

$$[\![\psi]\!]_{\widehat{\langle\alpha\rangle\bigwedge_{i\in G} K_i p_i}} = [\![\psi]\!]_{\widehat{\alpha \wedge \bigwedge_{i\in G} K_i^\alpha p_i}} = [\![\psi]\!]_{\widehat{\alpha} \cap \bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{p_i}}.$$

Thus $T \in [\![\psi]\!]_{\widehat{\alpha} \cap \bigcap_{i\in G} K_i^{\widehat{\alpha}}\widehat{p_i}}$. Since $B_i := \widehat{p_i} \in \mathcal{A}^c$ for every $i \in G$, we obtain that $T \in [\![\langle G\rangle\psi]\!]_{\widehat{\alpha}}$.

Corollary 6. *The canonical pre-model $\mathcal{M}^c$ is standard (and hence a pseudo-model).*

Proof. $\mathcal{A}^c = \{\widehat{\theta} : \theta \in \mathcal{L}^P_{-\Diamond}\} = \{\widehat{\langle\top\rangle\theta} : \theta \in \mathcal{L}^P_{-\Diamond}\} = \{[\![\theta]\!]_{\widehat{\top}} : \theta \in \mathcal{L}^P_{-\Diamond}\} = \{[\![\theta]\!]_{W^c} : \theta \in \mathcal{L}^P_{-\Diamond}\}$.

Lemma 20. *For every $\varphi \in \mathcal{L}^P_G$, if φ is consistent then $\{0, \Diamond\varphi\}$ is an initial P_φ-theory.*

Proof. Let $\varphi \in \mathcal{L}^P_G$ s.t. $\varphi \nvdash \bot$. By the *Equivalences with* ^{0} in Table 1, we have $\vdash \bot^0 \leftrightarrow (p \wedge \neg p)^0 \leftrightarrow (p^0 \wedge \neg p^0) \leftrightarrow (p \wedge \neg p) \leftrightarrow \bot$. Therefore, $\vdash \psi \to \bot^0$ iff $\vdash \psi \to \bot$ for all $\psi \in \mathcal{L}^P_G$. Then, by Proposition 5.12, we obtain $\vdash \varphi \to \bot$ iff $\vdash (0 \wedge \Diamond\varphi) \to \bot$. Since $\varphi \nvdash \bot$, we have $0 \wedge \Diamond\varphi \nvdash \bot$, i.e., $\{0, \Diamond\varphi\}$ is a P_φ-theory. By definition, it is an initial one.

Corollary 7. **GALM** *is complete with respect to standard pseudo models.*

Proof. Let φ be a consistent formula. By Lemma 20, $\{0, \Diamond\varphi\}$ is an initial P_φ-theory. By Extension and Lindenbaum Lemmas, respectively, we can extend P_φ to some $P \supseteq P_\varphi$ and extend $\{0, \Diamond\varphi\}$ to some maximal P-witnessed theory T_0 such that $(0 \wedge \Diamond\varphi) \in T_0$. So T_0 is initial and we can construct the canonical pseudo-model $\mathcal{M}^c$ for T_0. Since $\Diamond\varphi \in T_0$ and T_0 is P-witnessed, there exists $p \in P$ such that $\langle p\rangle\varphi \in T_0$. By Truth Lemma (applied to $\alpha := p$), we get $T_0 \in [\![\varphi]\!]_{\widehat{p}}$. Hence, φ is satisfied at T_0 in the set $\widehat{p} \in \mathcal{A}^c$.

Theorem 3. **APALM** *is complete with respect to standard pseudo models.*

The completeness proof for **APALM** with respect to standard pseudo models is obtained by following the same steps in the completeness proof of **GALM** without the parts required for the operator $\langle G\rangle$. This involves, for example, defining the witnessed theories only with respect to $\Diamond$ and modifying the auxiliary lemmas accordingly. This proof is presented in the earlier, shorter version [8] of this paper.

Corollary 8. **GALM** *is complete with respect to a-models. Moreover,* **APALM** *is complete with respect to a-models.*

Proof. **GALM** completeness follows immediately from Corollaries 7 and 2. **APALM** completeness follows from Theorem 3 and Corollary 2

6 Conclusions and Future Work

This paper solves the open question of finding a strong variant of APAL and GAL that is recursively axiomatizable. Our system APALM is inspired by our analysis of Kuijer's counterexample [14], which lead us to add to APAL a 'memory' of the initial situation. We then used similar methods to obtain a recursive axiomatization for the memory-enhanced variant GALM of GAL. The soundness and completeness proofs crucially rely on a Subset Space-like semantics and on the equivalence between the effort modality and the arbitrary announcement modality (and on the equivalence between their $[G]$ counterparts), thus revealing the strong link between these two formalisms.

A further comment on the connection with the yesterday operator. The limited form of memory provided by APALM is in fact enough to 'simulate' the yesterday operator $Y\varphi$ on any given model, by using context-dependent formulas. For instance, the dialogue in Cheryl's birthday puzzle (Albert: "I don't know when Cheryl's birthday is, but I know that Bernard doesn't know it either"; Bernard: "At first I didn't know when Cheryl's birthday is, but I know now"; Albert: "Now I also know"), can be simulated by the following sequence of announcements[19]: first, the formula $0 \wedge \neg K_a c \wedge K_a \neg K_b c$ is announced (where 0 marks the fact that this is the first announcement), then $(\neg K_b c)^0 \wedge K_b c$ is announced, and finally $K_a c$ is announced.

For another example: if instead we change the story so that the third announcement (by Albert) is "I knew you knew it (just before you said so)", then the last step of this alternative scenario corresponds to

[19] Here, we use the abbreviation $K_a c = \bigvee\{K_a(d \wedge m) : d \in D, m \in M\}$, where D is the set of possible days and M is the set of possible months, to denote the fact that Albert knows Cheryl's birthday and, similarly, use $K_b c$ for Bernard.

announcing the formula $([0 \wedge \neg K_a c \wedge K_a \neg K_b c]K_a K_b c)^0$ (saying that, just after the first announcement but before the second, Albert knew that Bernard knew the birthday). This shows how the logic can simulate the use of *any* (iterated) Y's in concrete examples, although at the cost of repeating the relevant part of history inside the announcement in order to mark the exact time when the announced formula was meant to be true.

A more systematic treatment of the yesterday operator on (a version of) our announcement models and its connection to arbitrary and group announcements are topics for future research. Yet another line of further work concerns other meta-logical properties, such as decidability and complexity, of APALM and GALM.

The acknowledgements section has been removed for the purpose of blind reviewing.

References

1. Ågotnes, T., Balbiani, P., van Ditmarsch, H., and Seban, P. (2010) Group announcement logic. *Journal of Applied Logic*, **8**, 62–81.
2. Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., and de Lima, T. (2008) 'Knowable' as 'known after an announcement'. *The Review of Symbolic Logic*, **1**, 305–334.
3. Balbiani, P. and van Ditmarsch, H. (2015) A simple proof of the completeness of APAL. *Studies in Logic*, **8 (1)**, 65–78.
4. Balbiani, P., van Ditmarsch, H., and Herzig, A. (2016) Before announcement. *Advances in Modal Logic 11, proceedings of the 11th conference on "Advances in Modal Logic," held in Budapest, Hungary, August 30 - September 2, 2016*, pp. 58–77.
5. Balbiani, P., van Ditmarsch, H., and Kudinov, A. (2013) Subset space logic with arbitrary announcements. *Proc. of the 5th ICLA*, pp. 233–244, Springer.
6. Baltag, A., Gierasimczuk, N., Özgün, A., Vargas-Sandoval, A. L., and Smets, S. (2017) A dynamic logic for learning theory. *Dynamic Logic. New Trends and Applications*, pp. 35–54, Springer International Publishing.
7. Baltag, A., Özgün, A., and Vargas-Sandoval, A. L. (2017) Topo-logic as a dynamic-epistemic logic. *Proceedings of the 6th LORI*, pp. 330–346, Springer Berlin Heidelberg.
8. Baltag, A., Özgün, A., and Vargas-Sandoval, A. L. (2018) Apal with memory is better. Moss, L. S., de Queiroz, R., and Martinez, M. (eds.), *Logic, Language, Information, and Computation*, pp. 106–129, Springer Berlin Heidelberg, Berlin, Heidelberg.
9. Bjorndahl, A. (2018) Topological subset space models for public announcements. van Ditmarsch, H. and Sandu, G. (eds.), *Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*, pp. 165–186, Springer International Publishing, Cham.
10. Bjorndahl, A. and Özgün, A. (2017) Topological subset space models for belief. *Proc. of the 16th TARK*, vol. 251 of *EPTCS*, pp. 88–101, Open Publishing Association.
11. Blackburn, P., de Rijke, M., and Venema, Y. (2001) *Modal Logic*, vol. 53 of *Cambridge Tracts in Theoretical Computer Scie*. Cambridge University Press, Cambridge.
12. Bolander, T. and Andersen, M. B. (2011) Epistemic planning for single-and multi-agent systems. *Journal of Applied Non-Classical Logics*, **21**, 9–34.
13. Dabrowski, A., Moss, L. S., and Parikh, R. (1996) Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, **78**, 73–110.
14. Kuijer, L. B. (2015) Unsoundness of $R(\Box)$, unpublished Manuscript. Available online at `http://personal.us.es/hvd/APAL_counterexample.pdf`.
15. McCarthy, J. (1990) Formalization of two puzzles involving knowledge. *Formalizing Common Sense: Papers by John McCarthy*, Ablex Publishing Corporation, original manuscript dated 1978-1981.
16. Moss, L. S. and Parikh, R. (1992) Topological reasoning and the logic of knowledge. *Proc. of the 4th TARK*, pp. 95–105, Morgan Kaufmann.
17. Renne, B., Sack, J., and Yap, A. (2009) Dynamic epistemic temporal logic. *Proc. of the 2nd LORI*, pp. 263–277, Springer.
18. van Benthem, J. (2002) *One is a lonely number: on the logic of communication*, pp. 96–129. Lecture Notes in Logic, Cambridge University Press.
19. van Benthem, J. (2004) What one may come to know. *Analysis*, **64**, 95–105.

20. van Ditmarsch, H. (2003) The russian cards problem. *Studia logica*, **75**, 31–62.
21. van Ditmarsch, H. and French, T. (2017) Quantifying over boolean announcements, avaliable online at `https://arxiv.org/abs/1712.05310`.
22. van Ditmarsch, H., French, T., and Hales, J. (2018) Positive announcements, `https://arxiv.org/abs/1803.01696`.
23. van Ditmarsch, H., French, T., and Pinchinat, S. (2010) Future event logic-axioms and complexity. *Advances in Modal Logic*, **8**, 77–99.
24. van Ditmarsch, H., Knight, S., and Özgün, A. (2014) Arbitrary announcements on topological subset spaces. *Proc. of the 12th EUMAS*, pp. 252–266, Springer.
25. van Ditmarsch, H., Knight, S., and Özgün, A. (2015) Announcement as effort on topological spaces. *Proc. of the 15th TARK*, pp. 95–102.
26. van Ditmarsch, H., Knight, S., and Özgün, A. (2015) Announcement as effort on topological spaces–Extended version. *Synthese*, forthcoming.
27. van Ditmarsch, H., van der Hoek, W., and Kuijer, L. B. (2016) Fully arbitrary public announcements. *Advances in Modal Logic*, **11**, 252–267.
28. Wang, Y. N. and Ågotnes, T. (2013) Multi-agent subset space logic. *Proc. of the 23rd IJCAI*, pp. 1155–1161, IJCAI/AAAI.
29. Wáng, Y. N. and Ågotnes, T. (2013) Subset space public announcement logic. *Proc. of 5th ICLA*, pp. 245–257, Springer.

A Appendix

A.1 Definition of the complexity measure for the Proofs of Lemmas 1 and 4

In some of our inductive proofs, we need a complexity measure on formulas different from the standard one based on subformula complexity. The standard notion requires only that formulas are more complex than their subformulas, while we also need that $\Diamond\varphi$ and $\langle G\rangle\varphi$ are more complex than $\langle\theta\rangle\varphi$ for all $\theta \in \mathcal{L}_{\Diamond}$. To the best of our knowledge, such a complexity measure was first introduced in [3] for the original APAL language from [2]. Similar measures have later been introduced for topological versions of APAL in [25,26,7].

Definition 13 (Size of formulas in $\mathcal{L}_G$). *The* size $s(\varphi)$ *of formula $\varphi \in \mathcal{L}_G$ is a natural number recursively defined as:*

$$\begin{aligned} s(\top) = s(p) = s(0) &= 1, \\ s(\neg\varphi) = s(\varphi^0) = s(K_i\varphi) = s(U\varphi) = s(\Diamond\varphi) = s(\langle G\rangle\psi) &= s(\varphi) + 1, \\ s(\varphi \wedge \psi) = s(\langle\varphi\rangle\psi) &= s(\varphi) + s(\psi) + 1. \end{aligned}$$

Definition 14 ($\Diamond, G$-Depth of formulas in $\mathcal{L}_G$). *The* $\Diamond, G$-depth $d(\varphi)$ *of formula $\varphi \in \mathcal{L}_G$ is a natural number recursively defined as:*

$$\begin{aligned} d(\top) = d(p) = d(0) &= 0, \\ d(\neg\varphi) = d(\varphi^0) = d(K_i\varphi) = d(U\varphi) &= d(\varphi) \\ d(\varphi \wedge \psi) = d(\langle\varphi\rangle\psi) &= max\{d(\varphi), d(\psi)\}, \\ d(\Diamond\varphi) = d(\langle G\rangle\varphi) &= d(\varphi) + 1. \end{aligned}$$

Finally, we define our intended complexity relation $<$ as lexicographic merge of $\Diamond, G$-depth and size, exactly as in [3]:

Definition 15. *For any $\varphi, \psi \in \mathcal{L}_G$, we put*

$$\varphi < \psi \text{ iff either } d(\varphi) < d(\psi), \text{ or } d(\varphi) = d(\psi) \text{ and } s(\varphi) < s(\psi).$$

Definition 16 (Subformula). *Given a formula $\varphi \in \mathcal{L}_G$, the set $Sub(\varphi)$ of subformulas of φ is recursively defined as*

$$Sub(\varphi) = \{\varphi\} \quad \text{if } \varphi \text{ is } \top, p \text{ or } 0,$$

$$\begin{aligned} Sub(\neg\varphi) &= Sub(\varphi) \cup \{\neg\varphi\} & Sub(\varphi \wedge \psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\varphi \wedge \psi\} \\ Sub(\varphi^0) &= Sub(\varphi) \cup \{\varphi^0\} & Sub(\langle\varphi\rangle\psi) &= Sub(\varphi) \cup Sub(\psi) \cup \{\langle\varphi\rangle\psi\} \\ Sub(K_i\varphi) &= Sub(\varphi) \cup \{K_i\varphi\} & Sub(\Diamond\varphi) &= Sub(\varphi) \cup \{\Diamond\varphi\} \\ Sub(U\varphi) &= Sub(\varphi) \cup \{U\varphi\} & Sub(\langle G\rangle\varphi) &= Sub(\varphi) \cup \{\langle G\rangle\varphi\}. \end{aligned}$$

A.2 Proofs of results in Section 2.3

Proof of Lemma 2 By Proposition 1, it is enough to prove the claim for all formulas $\alpha \in \mathcal{L}_{-\Diamond,\langle!\rangle}$. Let B be an APALM bisimulation between initial a-models $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$. The proof is by subformula induction on α, using the following *induction hypothesis* (IH): for all $\beta \in Sub(\alpha)$, we have $s_1 \in [\![\beta]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\beta]\!]_{\mathcal{M}_2^0}$ for all $s_1 \in W_1^0$, $s_2 \in W_2^0$ such that $s_1 B s_2$.

Base case $\alpha := \top$: Since $s_1 \in W_1^0 = [\![\top]\!]_{\mathcal{M}_1^0}$ and $s_2 \in W_2^0 = [\![\top]\!]_{\mathcal{M}_2^0}$, we trivially obtain that $s_1 \in [\![\top]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![\top]\!]_{\mathcal{M}_2^0}$.

Base case $\alpha := p$: Since $s_1 B s_2$, $s_1 \in [\![p]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![p]\!]_{\mathcal{M}_2^0}$ follows by Definition 4, valuation condition.

Base case $\alpha := 0$: Since $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ are initial a-models, by the semantics, we have $s_1 \in W_1^0 = [\![0]\!]_{\mathcal{M}_1^0}$ and $s_2 \in W_2^0 = [\![0]\!]_{\mathcal{M}_2^0}$. We therefore trivially obtain that $s_1 \in [\![0]\!]_{\mathcal{M}_1^0}$ iff $s_2 \in [\![0]\!]_{\mathcal{M}_2^0}$.

Case $\alpha := \beta \wedge \gamma$ and $\alpha := \neg\beta$ follow straightforwardly by the semantics and IH.

In the following sequence of equivalencies, we make repeated use of the semantic clauses in Defn. 2.

Case $\alpha := \beta^0$
$s_1 \in [\![\beta^0]\!]_{\mathcal{M}_1^0}$ iff $s_1 \in [\![\beta]\!]_{\mathcal{M}_1^0} \cap W_1^0$ iff $s_2 \in [\![\beta]\!]_{\mathcal{M}_2^0} \cap W_2^0$ (by IH and $s_2 \in W_2^0$) iff $s_2 \in [\![\beta^0]\!]_{\mathcal{M}_2^0}$.

Case $\alpha := U\beta$
$s_1 \in [\![U\beta]\!]_{\mathcal{M}_1^0}$ iff $\forall s \in W_1^0, s \in [\![\beta]\!]_{\mathcal{M}_1^0}$ iff $\forall s' \in W_2^0, s' \in [\![\beta]\!]_{\mathcal{M}_2^0}$ (since B is total and IH) iff $s_2 \in [\![U\beta]\!]_{\mathcal{M}_2^0}$.

Case $\alpha := K_i\beta$
$s_1 \in [\![K_i\beta]\!]_{\mathcal{M}_1^0}$ iff $(\forall s \in W_1^0)(s \sim_i s_1$ implies $s \in [\![\beta]\!]_{\mathcal{M}_1^0})$ iff $(\forall s \in W_2^0)(s \sim_i' s_2$ implies $s \in [\![\beta]\!]_{\mathcal{M}_2^0})$ (back and forth condition, IH) iff $s_2 \in [\![K_i\beta]\!]_{\mathcal{M}_2^0}$.

A.3 Proofs of results in Section 2.4

Proof of Proposition 5

1. from $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [\theta]\varphi \leftrightarrow [\theta]\psi$: Follows directly by ($K_!$) and ($Nec_!$).

2. $\langle\theta\rangle 0 \leftrightarrow (0 \wedge U\theta)$: Follows from the definition of $\langle\theta\rangle 0 := \neg[\theta]\neg 0$ and the axiom ($R_\neg$.)

3. $\langle\theta\rangle\psi \leftrightarrow (\theta \wedge [\theta]\psi)$: Follows from the definition $\langle\theta\rangle\psi := \neg[\theta]\neg\psi$ and the axiom ($R_\neg$.)

4. $\Box\varphi \to [\rho]\varphi$ ($\rho \in \mathcal{L}_{-\Diamond}$ arbitrary):

1. $\vdash \Box\theta \leftrightarrow [\top]\Box\theta$ (R[$\top$])
2. $\vdash [\top]\Box\theta \to [\top \wedge \rho]\theta$, (for arbitrary $\rho \in \mathcal{L}_{-\Diamond}$) ([!]$\Box$-elim)
3. $\vdash [\top \wedge \rho]\Box\theta \to [\rho]\theta$, (for arbitrary $\rho \in \mathcal{L}_{-\Diamond}$) ($\vdash (\top \wedge \rho) \leftrightarrow \rho$ and (RE))
4. $\vdash \Box\theta \to [\rho]\theta$, (for arbitrary $\rho \in \mathcal{L}_{-\Diamond}$) (1-3, CPL)

5. from $\vdash \chi \to [p]\varphi$, infer $\vdash \chi \to \Box\varphi$ ($p \notin P_\chi \cup P_\varphi$): proof follows analogously to the above case by using (RE), ([!]$\Box$-intro) with $\theta := \top$, and (R[$\top$]).

6. $(\varphi \to \psi)^0 \leftrightarrow (\varphi^0 \to \psi^0)$: This is straightforward by the set of axioms called *Equivalences with* ^{0}.

7. $\vdash \varphi^{00} \leftrightarrow \varphi^0$:

1. $\vdash 0 \to (\varphi \leftrightarrow \varphi^0)$ (0-eq)
2. $\vdash (0 \to (\varphi \leftrightarrow \varphi^0))^0$ (Nec^0)
3. $\vdash 0^0 \to (\varphi \leftrightarrow \varphi^0)^0$ (Prop.5.6)
4. $\vdash 0^0 \to (\varphi^0 \leftrightarrow \varphi^{00})$ (Prop.5.6)
5. $\vdash 0^0$ (Ax$_0$)
6. $\vdash \varphi^0 \leftrightarrow \varphi^{00}$ (4, 5, MP)

8. $\Box\varphi^0 \leftrightarrow \varphi^0$ and $\varphi^0 \leftrightarrow \Diamond\varphi^0$: From left-to-right direction of both cases follow from the T-axiom for $\Box$. From right-to-left direction we will only prove $\varphi^0 \to \Box\varphi^0$ since the remaining implication follows simply by definition of the dual for $\Box$. By an instance of the rule in Proposition 5.5 (($\Box$-intro)), it is sufficient to show that $\vdash \varphi^0 \to [p]\varphi^0$ for $p \notin P_\varphi$:

1. $\vdash \varphi^0 \to (p \to \varphi^0)$ (for $p \notin P_\varphi$, CPL)
2. $\vdash \varphi^0 \to [p]\varphi^0$ (R^0)
3. $\vdash \varphi^0 \to \Box\varphi^0$ (($\Box$-intro) rule for $p \notin P_\varphi$)

9. $\vdash (\Box\varphi)^0 \to \Box\varphi^0$

1. $\vdash \Box\varphi \to \varphi$ (S4 for $\Box$)
2. $\vdash (\Box\varphi \to \varphi)^0$ (Nec^0)
3. $\vdash (\Box\varphi)^0 \to \varphi^0$ (Prop.5.6, 2, MP)
4. $\vdash (\Box\varphi)^0 \to \Box\varphi^0$ (Prop.5.8)

10. $\vdash (0 \wedge \Diamond\varphi^0) \to \varphi$

1. $\vdash 0 \to (\varphi^0 \leftrightarrow \varphi)$ (0-*eq*)
2. $\vdash 0 \to (\varphi^0 \to \varphi)$ (CPL)
3. $\vdash 0 \to (\Diamond\varphi^0 \to \varphi)$ (Prop.5.8)
4. $\vdash (0 \wedge \Diamond\varphi^0) \to \varphi$ (CPL)

11. $\vdash \varphi \to (0 \land \Diamond\varphi)^0$

1. $\vdash (\Box\neg\varphi)^0 \to \neg\varphi$ $\quad$ ($\mathrm{Imp}^0_\Box$)
2. $\vdash \neg\neg\varphi \to \neg(\Box\neg\varphi)^0$ $\quad$ (contraposition of 1)
3. $\vdash \neg\neg\varphi \to (\neg\Box\neg\varphi)^0$ $\quad$ ($\mathrm{Eq}^0_\neg$)
4. $\vdash \varphi \to (\Diamond\varphi)^0$ $\quad$ (the defn. of $\Diamond$)
5. $\vdash \varphi \to (0^0 \land (\Diamond\varphi)^0)$ $\quad$ (Ax_0)
6. $\vdash \varphi \to (0 \land \Diamond\varphi)^0$ $\quad$ ($\mathrm{Eq}^0_\land$)

12. $\vdash \varphi \to \psi^0$ if and only if $\vdash (0 \land \Diamond\varphi) \to \psi$
From left-to-right: Suppose $\vdash \varphi \to \psi^0$ and show: $\vdash (0 \land \Diamond\varphi) \to \psi$.

1. $\vdash (0 \land \Diamond\psi^0) \to \psi$ $\quad$ (Prop.5.10)
2. $\vdash \Diamond\varphi \to \Diamond\psi^0$ $\quad$ (by assumption and $\mathrm{Nec}_\Box$)
3. $\vdash (0 \land \Diamond\varphi) \to (0 \land \Diamond\psi^0)$ $\quad$ (2 and CPL)
4. $\vdash (0 \land \Diamond\varphi) \to \psi$ $\quad$ (3, 1, CPL)

From right-to-left: Suppose $\vdash (0 \land \Diamond\varphi) \to \psi$ and show $\vdash \varphi \to \psi^0$.

1. $\vdash \varphi \to (0 \land \Diamond\varphi)^0$ $\quad$ (Prop.5.11)
2. $\vdash (0 \land \Diamond\varphi) \to \psi$ $\quad$ (assumption)
3. $\vdash ((0 \land \Diamond\varphi) \to \psi)^0$ $\quad$ (Nec^0)
4. $\vdash (0 \land \Diamond\varphi)^0 \to \psi^0$ $\quad$ (Prop.5.6)
5. $\vdash \varphi \to \psi^0$ $\quad$ (1, 4, CPL)

13. $[\theta](\varphi \land \psi) \leftrightarrow ([\theta]\varphi \land [\theta]\psi)$: Follows from ($\mathrm{K}_!$) and ($\mathrm{Nec}_!$).

14. $[\theta][p]\varphi \leftrightarrow [\theta \land p]\varphi$

1. $\vdash [\theta][p]\varphi \leftrightarrow [\langle\theta\rangle p]\varphi$ $\quad$ ($\mathrm{R}_{[!]}$)
2. $\vdash [\langle\theta\rangle p]\varphi \leftrightarrow [\theta \land [\theta]p]\varphi$ $\quad$ (Prop.5.3, RE)
3. $\vdash [\theta \land [\theta]p]\varphi \leftrightarrow [\theta \land (\theta \to p)]\varphi$ $\quad$ (R_p, RE)
4. $\vdash [\theta \land (\theta \to p)]\varphi \leftrightarrow [\theta \land p]\varphi$ $\quad$ (CPL, RE)
5. $\vdash [\theta][p]\varphi \leftrightarrow [\theta \land p]\varphi$ $\quad$ (1-4, CPL)

15. $[\theta]\bot \leftrightarrow \neg\theta$: this is an easy consequence of Prop. 5.13, (R_p), and ($\mathrm{R}_\neg$).

Proof of Proposition 6 The derivation of (Nec) rule for $\Box$ easily follows from ($\mathrm{Nec}_!$) and Prop. 5.5. The T-axiom for $\Box$ follows from Prop. 5.4, RE, and $\mathrm{R}[\top]$.

For the K-axiom:

1. $\vdash(\Box(\varphi \to \psi) \wedge \Box\varphi) \to ([p](\varphi \to \psi) \wedge [p]\varphi)$	$(p \notin P_\varphi \cup P_\psi$, Prop. 5.4)
2. $\vdash([p](\varphi \to \psi) \wedge [p]\varphi) \to [p]\psi$	($K_!$)
3. $\vdash(\Box(\varphi \to \psi) \wedge \Box\varphi) \to [p]\psi$	(1, 2, CPL)
4. $\vdash(\Box(\varphi \to \psi) \wedge \Box\varphi) \to \Box\psi$	$(p \notin P_\varphi \cup P_\psi$, Prop. 5.5)

For the 4-axiom:

1. $\vdash\Box\varphi \to [p \wedge q]\varphi$	(for some $p, q \notin P_\varphi$, Prop. 5.4)
2. $\vdash\Box\varphi \to [p]\Box\varphi$	([!]$\Box$-intro)
3. $\vdash\Box\varphi \to \Box\Box\varphi$	$(p \notin P_\varphi$, Prop. 5.5)

A.4 Proofs of results in Section 3

Proof of Proposition 7 The proof is by $<$-induction on φ, using Lemma 4 and the following *induction hypothesis* (IH): for all $\psi < \varphi$ and all models $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we have $[\![\psi]\!] \subseteq W$. The base cases $\varphi := \top$, $\varphi := p$, and $\varphi := 0$ are straightforward by the semantics given in Defn.2. The inductive cases for Booleans are immediate. Similarly, the following cases make use of the corresponding semantic clause in Defn.2.

Case $\varphi := \psi^0$: $[\![\psi^0]\!] = [\![\psi]\!]_{\mathcal{M}^0} \cap W \subseteq W$.

Case $\varphi := K_i\psi$: $[\![K_i\psi]\!] = \{w \in W : w_i \subseteq [\![\varphi]\!]\} \subseteq W$.

Case $\varphi := U\psi$: $[\![U\psi]\!] \in \{\emptyset, W\}$, thus $[\![U\psi]\!] \subseteq W$.

Case $\varphi := \langle\theta\rangle\psi$: Since $\theta < \langle\theta\rangle\psi$ (Lemma 4.1), by the IH on θ, we have that $[\![\theta]\!] \subseteq W$. Moreover, since $\psi < \langle\theta\rangle\psi$ (Lemma 4.1), by the IH on ψ, we also have that $[\![\psi]\!]_{\mathcal{M}|[\![\theta]\!]} \subseteq [\![\theta]\!]$ (recall that $\mathcal{M}|[\![\theta]\!] = (W^0, [\![\theta]\!], \sim_1, \ldots, \sim_n, \|\cdot\|)$). Therefore, by Defn.2, we obtain that $[\![\langle\theta\rangle\psi]\!] = [\![\psi]\!]_{\mathcal{M}|[\![\theta]\!]} \subseteq [\![\theta]\!] \subseteq W$.

Case $\varphi := \Diamond\psi$: By Lemma 4.2, it follows that for each $\theta \in \mathcal{L}_{-\Diamond}$, $\langle\theta\rangle\psi < \Diamond\psi$. Then, by the IH, we have that for all $\theta \in \mathcal{L}_{-\Diamond}$, $[\![\langle\theta\rangle\psi]\!] \subseteq W$. Thus $\bigcup\{[\![\langle\theta\rangle\psi]\!] : \theta \in \mathcal{L}_{-\Diamond}\} \subseteq W$, i.e., $[\![\Diamond\psi]\!] \subseteq W$.

Case $\varphi := \langle G\rangle\psi$: By Lemma 4.3, it follows that for each $\theta \in \mathcal{L}_{-\Diamond}$, $\langle\theta\rangle\psi < \langle G\rangle\psi$. Then, by the IH, we have that for all $\theta_i \in \mathcal{L}_{-\Diamond}$, $[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\psi]\!] \subseteq W$. Thus $\bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\psi]\!] : \theta_i \in \mathcal{L}_{-\Diamond}\} \subseteq W$, i.e., $[\![\langle G\rangle\psi]\!] \subseteq W$.

A.5 Proofs of results in Section 4

Proof of Lemma 5 First note that by Defn.8.5 and Boolean operations of sets we have,
$K_i^A B = \{w \in W^0 : w_i \cap A \subseteq B)\} = \{w \in W^0 : \forall s \in W^0((s \in A \text{ and } w \sim_i s) \Rightarrow s \in B)\}$
$= \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in A \Rightarrow s \in B))\} = \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow (s \in (W^0 - A) \text{ or } s \in B))\}$
$= \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in (W^0 - A) \cup B)\} = K_i((W^0 - A) \cup B)$.
Then, by Definition 8.(3-5) and $A, B \in \mathcal{A}$, we obtain $K_i^A B = K_i((W^0 - A) \cup B) \in \mathcal{A}$.

Proof of Proposition 8 The proof is by subformula induction on θ. The base cases and the inductive cases for the Booleans are immediate (using the conditions in Definition 8).

Case $\theta := \psi^0$: By the semantics, $[\![\psi^0]\!]_A = [\![\psi]\!]_{W^0} \cap A \in \mathcal{A}$, since $[\![\psi]\!]_{W^0} \in \mathcal{A}$ (by the fact that $W^0 \in \mathcal{A}$ and IH), $A \in \mathcal{A}$ (by assumption), and $\mathcal{A}$ is closed under intersection.

Case $\theta := K_i\psi$: Note that $[\![K_i\psi]\!]_A = \{w \in A : w_i^A \subseteq [\![\varphi]\!]_A\} = A \cap \{w \in W^0 : w_i^A \subseteq [\![\varphi]\!]_A\}$ (by Definition 8) $= A \cap \{w \in W^0 : \forall s \in W^0((s \in A \text{ and } w \sim_i s) \Rightarrow s \in [\![\varphi]\!]_A)\}$. We then obtain, by CPL and Boolean operations of sets that $[\![K_i\psi]\!]_A = A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - A) \cup [\![\varphi]\!]_A)\}$. Moreover, $A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in ((W^0 - A) \cup [\![\varphi]\!]_A)\} = A \cap K_i((W^0 - A) \cup [\![\varphi]\!]_A)$ by Definition 8.(3-5) (since $A \in \mathcal{A}$ and $[\![\varphi]\!]_A \in \mathcal{A}$ by IH). Therefore, $[\![K_i\psi]\!]_A = A \cap K_i((W^0 - A) \cup [\![\varphi]\!]_A)$ is in $\mathcal{A}$.

Case $\theta := U\psi$: By Definition 8, $[\![U\psi]\!]_A \in \{\emptyset, A\} \subseteq \mathcal{A}$.

Case $\theta := \langle\varphi\rangle\psi$: Since $A \in \mathcal{A}$, we have $[\![\varphi]\!]_A \in \mathcal{A}$ (by IH on φ), and hence $[\![\langle\varphi\rangle\psi]\!]_A = [\![\psi]\!]_{[\![\varphi]\!]_A} \in \mathcal{A}$ (by the semantics and IH on ψ).

Proof of Lemma 6 Observe that $K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A = K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A)$ (as in Lemma 5). Moreover, it's easy to see that,

$[\![K_i^\theta\rho]\!]_A = [\![K_i(\theta \to \rho)]\!]_A = \{w \in A : w_i^A \subseteq [\![\theta \to \rho]\!]_A\} = A \cap \{w \in W^0 : \forall s \in W^0(w \sim_i s \Rightarrow s \in (W^0 \setminus [\![\theta]\!]_A \cup [\![\rho]\!]_A))\}$ (since $[\![\theta]\!]_A \subseteq A$). We therefore obtain that $[\![K_i^\theta\rho]\!]_A = A \cap K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A)$ (by Boolean operations of sets and the defn. of K_i). Thus, $[\![K_i^\theta\rho]\!]_A = A \cap K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A) \subseteq K_i((W^0 - [\![\theta]\!]_A) \cup [\![\rho]\!]_A) = K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A$. Therefore if $w \in [\![\theta]\!]_A \subseteq A$, $w \in [\![K_i^\theta\rho]\!]_A$ iff $w \in K_i^{[\![\theta]\!]_A}[\![\rho]\!]_A$.

The proof of Proposition 11 needs the following lemmas.

Lemma 21. *The sentence $(K_i(\varphi \to \psi))^0 \leftrightarrow K_i(K_i(\varphi \to \psi))^0$ is valid on pseudo-models.*

Proof. It is easy to see that the direction from left-to-right follows from the fact that the semantics for φ^0 is state-independent, and the direction from right-to-left is an instance of the T-axiom for K_i.

Lemma 22. *Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \dots, \sim_n, \|\cdot\|)$ a standard pseudo-model, $A \in \mathcal{A}$ and $\varphi \in \mathcal{L}_G$, then the following holds:*

1. $[\![\Diamond\varphi]\!]_A = \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$,
2. $[\![\langle G\rangle\varphi]\!]_A = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_A : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\}$.

Proof.

1. For ($\subseteq$): Let $w \in [\![\Diamond\varphi]\!]_A$. Then, by the semantics of $\Diamond$ in pseudo-models, there exists some $B \in \mathcal{A}$ such that $w \in B \subseteq A$ and $w \in [\![\varphi]\!]_B$. Since $\mathcal{M}$ is standard, we know that $A = [\![\psi]\!]_{W^0}$ and $B = [\![\chi]\!]_{W^0}$ for some $\psi, \chi \in \mathcal{L}_{-\Diamond}$. Moreover, since $B = [\![\chi]\!]_{W^0} \subseteq A = [\![\psi]\!]_{W^0}$, we have $B = [\![\chi]\!]_{W^0} \cap [\![\psi]\!]_{W^0} = [\![\chi^0]\!]_{[\![\psi]\!]_{W^0}} = [\![\chi^0]\!]_A$, and so $w \in [\![\varphi]\!]_B = [\![\varphi]\!]_{[\![\chi^0]\!]_A} = [\![\langle\chi^0\rangle\varphi]\!]_A \subseteq \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$.
For ($\supseteq$): Let $w \in \bigcup\{[\![\langle\theta\rangle\varphi]\!]_A : \theta \in \mathcal{L}_{-\Diamond}\}$. Then we have $w \in [\![\langle\theta\rangle\varphi]\!]_A = [\![\varphi]\!]_{[\![\theta]\!]_A}$, for some $\theta \in \mathcal{L}_{-\Diamond}$. Moreover, since $[\![\theta]\!]_A \in \mathcal{A}$ (by Proposition 8) and $[\![\theta]\!]_A \subseteq A$ (by Observation 3), it follows that $w \in [\![\Diamond\varphi]\!]_A$ (by the semantics of $\Diamond$ in pseudo-models).
2. For ($\subseteq$): Let $w \in [\![\langle G\rangle\varphi]\!]_A$. Then, by Definition 7, we have $w \in [\![\varphi]\!]_{A\cap\bigcap_{i\in G} K_i^A B_i}$ for some $\{B_i : i \in G\} \subseteq \mathcal{A}$. Since $\mathcal{M}$ is a standard pseudo-model, we know that each $B_i = [\![\rho_i]\!]_{W^0}$ and $A = [\![\psi]\!]_{W^0}$ for some $\rho_i, \psi \in \mathcal{L}_{-\Diamond}$. Thus,

$$w \in [\![\varphi]\!]_{[\![\psi]\!]_{W^0}\cap\bigcap_{i\in G} K_i^{[\![\psi]\!]_{W^0}}[\![\rho_i]\!]_{W^0}} = [\![\varphi]\!]_{[\![\psi]\!]_{W^0}\cap\bigcap_{i\in G}[\![K_i(\psi\to\rho_i)]\!]_{W^0}} = [\![\varphi]\!]_{[\![\psi]\!]_{W^0}\cap[\![\bigwedge_{i\in G} K_i(\psi\to\rho_i)]\!]_{W^0}}$$

by Lemma 6 and the semantics. By the semantics of 0 and Lemma 21, we obtain

$$[\![\varphi]\!]_{[\![\psi]\!]_{W^0}\cap[\![\bigwedge_{i\in G} K_i(\psi\to\rho_i)]\!]_{W^0}} = [\![\varphi]\!]_{[\![\bigwedge_{i\in G}(K_i(\psi\to\rho_i))^0]\!]_A} = [\![\varphi]\!]_{[\![\bigwedge_{i\in G} K_i(K_i(\psi\to\rho_i))^0]\!]_A}.$$

Thus, for $\theta_i := (K_i(\psi \to \rho_i))^0$, $w \in [\![\varphi]\!]_{[\![\bigwedge_{i\in G} K_i\theta_i]\!]_A} = [\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_A$. For ($\supseteq$): Let $\{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}$ such that $w \in [\![\varphi]\!]_{[\![\bigwedge_{i\in G} K_i\theta_i]\!]_A}$. Note that $[\![\bigwedge_{i\in G} K_i\theta_i]\!]_A = \bigcap_{i\in G}[\![K_i\theta_i]\!]_A = A \cap \bigcap_{i\in G} K_i^A[\![\theta_i]\!]_A$. Since $\mathcal{M}$ is a standard pseudo-model, we know that $B_i := [\![\theta_i]\!]_A \in \mathcal{A}$ for every $i \in G$ and by our initial assumption $w \in [\![\varphi]\!]_{A\cap\bigcap_{i\in G} K_i^A[\![\theta_i]\!]_A}$, so we obtain $w \in [\![\langle G\rangle\varphi]\!]_A$.

Proof of Proposition 11

1. Let $\mathcal{M} = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be a standard pseudo-model. Then, $A \in \mathcal{A}$ implies $A = [\![\theta]\!]^{PS}_{W^0} \subseteq W^0$ for some θ, hence $\mathcal{M}_A = (W^0, A, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a model.
   (a) The proof is by <-induction from Lemma 4. The base cases and the inductive cases for Booleans are straightforward.
   Case $\varphi := \psi^0$. We have $[\![\psi]\!]^{PS}_A = [\![\psi]\!]^{PS}_{W^0} \cap A = [\![\psi]\!]_{\mathcal{M}^0_A} \cap A = [\![\psi^0]\!]_{\mathcal{M}_A}$ (by Defn.7, IH, and Defn. 2).
   Case $\varphi := K_i\psi$. We have $[\![K_i\psi]\!]^{PS}_A = \{w \in A : w_i^A \subseteq [\![\psi]\!]^{PS}_A\} = \{w \in A : w_i \subseteq [\![\psi]\!]_{\mathcal{M}_A}\} = [\![K_i\psi]\!]_{\mathcal{M}_A}$ (by Defn.7, IH, and Defn. 2).
   Case $\varphi := U\psi$. By Definitions 2 and 7, we have:
   $$[\![U\psi]\!]_{\mathcal{M}_A} = \begin{cases} A & \text{if } [\![\psi]\!]_{\mathcal{M}_A} = A \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\psi]\!]^{PS}_A = \begin{cases} A & \text{if } [\![\psi]\!]^{PS}_A = A \\ \emptyset & \text{otherwise} \end{cases}$$
   By IH, $[\![\psi]\!]^{PS}_A = [\![\psi]\!]_{\mathcal{M}_A}$, therefore, $[\![U\psi]\!]^{PS}_A = [\![U\psi]\!]_{\mathcal{M}_A}$.
   Case $\varphi := \langle\psi\rangle\chi$. By Defn. 2, we know that $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}_A} = [\![\chi]\!]_{\mathcal{M}_A|[\![\psi]\!]_{\mathcal{M}_A}}$. Now consider the relativized model $\mathcal{M}_A|[\![\psi]\!]_{\mathcal{M}_A} = (W^0, [\![\psi]\!]_{\mathcal{M}_A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$. By Lemma 4.1 and IH, we have $[\![\psi]\!]_{\mathcal{M}_A} = [\![\psi]\!]^{PS}_A$. Moreover, by the definition of standard pseudo-models, we know that $A = [\![\theta]\!]^{PS}_{W^0}$ for some $\theta \in \mathcal{L}_{-\Diamond}$. Therefore, $[\![\psi]\!]_{\mathcal{M}_A} = [\![\psi]\!]^{PS}_A = [\![\psi]\!]^{PS}_{[\![\theta]\!]^{PS}_{W^0}} = [\![\langle\theta\rangle\psi]\!]^{PS}_{W^0}$. Therefore, $[\![\psi]\!]_{\mathcal{M}_A} \in \mathcal{A}$. We then have
   $$[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}_A} = [\![\chi]\!]_{\mathcal{M}_{[\![\psi]\!]_{\mathcal{M}_A}}} = [\![\chi]\!]_{\mathcal{M}_{[\![\psi]\!]^{PS}_A}} = [\![\chi]\!]^{PS}_{[\![\psi]\!]^{PS}_A} = [\![\langle\psi\rangle\chi]\!]^{PS}_A,$$
   by the semantics and IH on ψ and on χ (since $[\![\psi]\!]^{PS}_A \in \mathcal{A}$).
   Case $\varphi := \Diamond\psi$. By Defn.2, Lemma 4.2, IH, the fact that $\mathcal{M}$ is a standard pseudo model, and Lemma 22.1 - applied in this order - we obtain the following equivalences:
   $$[\![\Diamond\psi]\!]_{\mathcal{M}_A} = \bigcup\{[\![\langle\chi\rangle\psi]\!]_{\mathcal{M}_A} : \chi \in \mathcal{L}_{-\Diamond}\} = \bigcup\{[\![\langle\chi\rangle\psi]\!]^{PS}_W : \chi \in \mathcal{L}_{-\Diamond}\} = [\![\Diamond\psi]\!]^{PS}_A.$$
   Case $\varphi := \langle G\rangle\psi$. By Defn.5, Lemma 4.3, IH, the fact that $\mathcal{M}$ is a standard pseudo model, and Lemma 22.2 - applied in this order - we obtain the following equivalences:
   $$[\![\langle G\rangle\psi]\!]_{\mathcal{M}_A} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_{\mathcal{M}_A} : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]^{PS}_A : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\Diamond}\}.$$
   Therefore $[\![\langle G\rangle\psi]\!]_{\mathcal{M}_A} = [\![\langle G\rangle\psi]\!]^{PS}_A$.
   (b) By part (a), $[\![\varphi]\!]_{\mathcal{M}^0_A} = [\![\varphi]\!]_{\mathcal{M}_{W^0}} = [\![\varphi]\!]^{PS}_{W^0}$ for all φ. Since $\mathcal{M}$ is standard, we have $A = [\![\theta]\!]^{PS}_{W^0} = [\![\theta]\!]_{\mathcal{M}^0_A}$ for some $\theta \in \mathcal{L}_{-\Diamond}$, so $\mathcal{M}_A$ is an a-model.
2. Let $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$ be an a-model. Since $\mathcal{A} = \{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\Diamond}\} \subseteq \mathcal{P}(W^0)$, the model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is a pre-model. Therefore, the semantics given in Definition 7 is defined on $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$.
   (a) By Proposition 10, it suffices to prove that the pre-model $\mathcal{M}' = (W^0, \mathcal{A}, \sim_1, \ldots, \sim_n, \|\cdot\|)$ is standard, i.e. that $\{[\![\theta]\!]_{\mathcal{M}^0} : \theta \in \mathcal{L}_{-\Diamond}\} = \{[\![\theta]\!]^{PS}_{W^0} : \theta \in \mathcal{L}_{-\Diamond}\}$. For this, we need to show that for every a-model $\mathcal{M} = (W^0, W, \sim_1, \ldots, \sim_n, \|\cdot\|)$, we have $[\![\theta]\!]_{\mathcal{M}} = [\![\theta]\!]^{PS}_W$ for all $\theta \in \mathcal{L}_{-\Diamond}$.
   We prove this by subformula induction on θ. The base cases and the inductive cases for Booleans are straightforward.
   Case $\theta := \psi^0$. Then $[\![\psi^0]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^0} \cap W = [\![\psi]\!]^{PS}_{W^0} \cap W = [\![\psi^0]\!]^{PS}_W$ (by Defn.2, IH, and Defn.7).
   Case $\theta := K_i\psi$. We have $[\![K_i\psi]\!]_{\mathcal{M}} = \{w \in W : w_i \subseteq [\![\psi]\!]_{\mathcal{M}}\} = \{w \in W : w_i^W \subseteq [\![\psi]\!]^{PS}_W\} = [\![K_i\psi]\!]^{PS}_W$ (by Defn.2, IH, and Defn.7).
   Case $\theta := U\psi$. By Definitions 2 and 7, we have:

$$[\![U\psi]\!]_{\mathcal{M}} = \begin{cases} W & \text{if } [\![\psi]\!]_{\mathcal{M}} = W \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![U\psi]\!]^{PS}_W = \begin{cases} W & \text{if } [\![\psi]\!]^{PS}_W = W \\ \emptyset & \text{otherwise} \end{cases}$$

By IH, $[\![\psi]\!]^{PS}_W = [\![\psi]\!]_{\mathcal{M}}$, therefore, $[\![U\psi]\!]^{PS}_W = [\![U\psi]\!]_{\mathcal{M}}$.
Case $\theta := \langle\psi\rangle\chi$. By Definition 2, we know that $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}} = [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}}$. Now consider the relativized model $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}} = (W^0, [\![\psi]\!]_{\mathcal{M}}, \sim_1, \dots, \sim_n, ||\cdot||)$. By Lemma 4.1 and IH on ψ, we have $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]^{PS}_W$. Moreover, by the definition of a-models, we know that $W = [\![\theta]\!]_{\mathcal{M}^0}$ for some $\theta \in \mathcal{L}_{-\diamond}$. Therefore, $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^0|[\![\theta]\!]_{\mathcal{M}^0}} = [\![\langle\theta\rangle\psi]\!]_{\mathcal{M}^0}$. Hence, since $\langle\theta\rangle\psi \in \mathcal{L}_{-\diamond}$, the model $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}$ is also an a-model obtained by updating the initial model $\mathcal{M}^0$ by $\langle\theta\rangle\psi$. We then have $[\![\langle\psi\rangle\chi]\!]_{\mathcal{M}} = [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}}$ (by Defn.2) $= [\![\chi]\!]_{\mathcal{M}|[\![\psi]\!]^{PS}_W}$ (by IH on ψ) $= [\![\chi]\!]^{PS}_{[\![\psi]\!]^{PS}_W}$ (by IH on χ, $\mathcal{M}|[\![\psi]\!]_{\mathcal{M}}$ is an a-model) $= [\![\langle\psi\rangle\chi]\!]^{PS}_W$ (by Defn.7).

(b) The proof of this part follows by $<$-induction on φ (where $<$ is as in Lemma 4). All the inductive cases are similar to ones in the above proof, except for the cases $\varphi := \diamond\psi$ and $\varphi := \langle G\rangle\psi$, shown below.
Case $\varphi := \diamond\psi$. By Defn.2, Lemma 4.2, IH, the fact that $\mathcal{M}'$ is a standard pseudo model, and Lemma 22.2 - applied in that order - we obtain the following equivalences:

$$[\![\diamond\psi]\!]_{\mathcal{M}} = \bigcup\{[\![\langle\chi\rangle\psi]\!]_{\mathcal{M}} : \chi \in \mathcal{L}_{-\diamond}\} = \bigcup\{[\![\langle\chi\rangle\psi]\!]^{PS}_W : \chi \in \mathcal{L}_{-\diamond}\} = [\![\diamond\psi]\!]^{PS}_W.$$

Case $\varphi := \langle G\rangle\psi$. By Defn.5, Lemma 4.3, IH, the fact that $\mathcal{M}'$ is a standard pseudo model and Lemma 22.2 - applied in that order - we obtain the following: equivalences,

$$[\![\langle G\rangle\psi]\!]_{\mathcal{M}} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]_{\mathcal{M}} : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\} = \bigcup\{[\![\langle\bigwedge_{i\in G} K_i\theta_i\rangle\varphi]\!]^{PS}_W : \{\theta_i : i \in G\} \subseteq \mathcal{L}_{-\diamond}\}.$$

Therefore, $[\![\langle G\rangle\psi]\!]_{\mathcal{M}} = [\![\langle G\rangle\psi]\!]^{PS}_W$.

A.6 Proofs of results in Section 5

Proof of Lemma 8 We proceed by induction on the structure of necessity forms. For $s := \epsilon$, take $\psi := \top$ and $\theta := \top$, then it follows from the axiom R[$\top$]. For the inductive cases we will verify only $s := s', \bullet^0$; $s := s', \eta \rightarrow$; $s := s', U$; and $s := s', \rho$. The case $s := s', K_i$ is analogous to the case $s := s', U$.

Case $s := s', \bullet^0$

$\vdash [s', \bullet^0]\varphi$ iff $\vdash [s']\varphi^0$ (by Defn. 10) iff $\vdash \psi' \rightarrow [\theta']\varphi^0$ (for some $\psi' \in \mathcal{L}$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow (\theta' \rightarrow \varphi^0)$ (by R^0) iff $\vdash (\psi' \wedge \theta') \rightarrow \varphi^0$ iff $\vdash (0 \wedge \diamond(\psi' \wedge \theta')) \rightarrow \varphi$ (by Prop. 5.12) iff $\vdash \psi \rightarrow [\theta]\varphi$ (since $\psi := 0 \wedge \diamond(\psi' \wedge \theta') \in \mathcal{L}$ and $\theta := \top \in \mathcal{L}_{-\diamond}$).

Case $s := s', \eta \rightarrow$

$\vdash [s', \eta \rightarrow]\varphi$ iff $\vdash [s'](\eta \rightarrow \varphi)$ (by Defn. 10) iff $\vdash \psi' \rightarrow [\theta'](\eta \rightarrow \varphi)$ (for some $\psi' \in \mathcal{L}$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow ([\theta']\eta \rightarrow [\theta']\varphi)$ (by K$_!$) iff $\vdash (\psi' \wedge [\theta']\eta) \rightarrow [\theta']\varphi$ iff $\vdash \psi \rightarrow [\theta]\varphi$ (since $\psi := \psi' \wedge [\theta']\eta \in \mathcal{L}$ and $\theta := \theta' \in \mathcal{L}_{-\diamond}$).

Case $s := s', U$

$\vdash [s', U]\varphi$ iff $\vdash [s']U\varphi$ (by Defn. 10) iff $\vdash \psi' \rightarrow [\theta']U\varphi$ (for some $\psi' \in \mathcal{L}$ and $\theta' \in \mathcal{L}_{-\diamond}$, by IH) iff $\vdash \psi' \rightarrow (\theta' \rightarrow U[\theta']\varphi)$ (by R$_U$) iff $\vdash (\psi' \wedge \theta') \rightarrow U[\theta']\varphi$ iff $\vdash E(\psi' \wedge \theta') \rightarrow [\theta']\varphi$ (pushing U back with its dual E, since U is an S5 modality) iff $\vdash \psi \rightarrow [\theta]\varphi$ ($\psi := E(\psi' \wedge \theta') \in \mathcal{L}$ and $\theta := \theta' \in \mathcal{L}_{-\diamond}$).

Case $s := s', \rho$

$\vdash [s', \rho]\varphi$ iff $\vdash [s'][\rho]\varphi$ (by Defn. 10) iff $\vdash \psi' \rightarrow [\theta'][\rho]\varphi$ (by IH) iff $\vdash \psi' \rightarrow [\langle\theta'\rangle\rho]\varphi$ (by $R_{[!]}$) iff $\vdash \psi \rightarrow [\theta]\varphi$ ($\psi := \psi' \in \mathcal{L}$ and $\theta := \langle\theta'\rangle\rho \in \mathcal{L}_{-\Diamond}$)

In each case, it is easy to see that $P_\psi \cup P_\theta \subseteq P_s$.

Proof of Lemma 14 (Lindenbaum's Lemma) The proof proceeds by constructing an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ of P-witnessed theories, where $\Gamma_0 := \Gamma$, and each Γ_i is recursively defined. Since we have to guarantee that each Γ_i is P-witnessed, we follow a two-fold construction, where $\Gamma_0 = \Gamma_0^+ := \Gamma$. Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ be an enumeration of all pairs of the form $\gamma_i = (s_i, \varphi_i)$ consisting of any necessity form $s_i \in NF^P$ and any formula $\varphi_i \in \mathcal{L}_G^P$. Let (s_n, φ_n) be the nth pair in the enumeration. We then set

$$\Gamma_n^+ = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\varphi_n\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \\ \Gamma_n & \text{otherwise} \end{cases}$$

Note that the empty string ϵ is in NF^P, and for every $\psi \in \mathcal{L}_G^P$ we have $\langle\epsilon\rangle\psi := \psi$ by the definition of possibility forms. Therefore, the above enumeration of pairs includes every formula ψ of $\mathcal{L}_G^P$ in the form of its corresponding pair (ϵ, ψ). By Lemma 11, each Γ_n^+ is P-witnessed. Then, if φ_n is of the form $\varphi_n := \Diamond\theta$ for some $\theta \in \mathcal{L}_G^P$, there exists a $p \in P$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle p\rangle\theta$ (since Γ_n^+ is P-witnessed). Similarly, if φ_n is of the form $\varphi_n := \langle G\rangle\theta$ for some $\theta \in \mathcal{L}_G^P$, there exists $\{p_i : i \in G\} \subseteq P$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\theta$. We then define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n^+ & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n \text{ is not of the form } \Diamond\theta \text{ or } \langle G\rangle\theta \\ \Gamma_n^+ \cup \{\langle s_n\rangle\langle p\rangle\theta\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \Diamond\theta \text{ for some } \theta \in \mathcal{L}_G^P \\ \Gamma_n^+ \cup \{\langle s_n\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\theta\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\varphi_n \text{ and } \varphi_n := \langle G\rangle\theta \text{ for some } \theta \in \mathcal{L}_G^P \\ \Gamma_n & \text{otherwise} \end{cases}$$

where $p \in P$, $\{p_i : i \in G\} \subseteq P$ such that Γ_n^+ is consistent with $\langle s_n\rangle\langle p\rangle\theta$ or consistent with $\langle s_n\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\theta$, respectively. Again by Lemma 11, it is guaranteed that each Γ_n is P-witnessed. Now consider the union $T_\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. By Lemma 12, we know that T_Γ is a P-theory. To show that T_Γ is P-witnessed, first let $s \in NF^P$ and $\psi \in \mathcal{L}_G^P$ and suppose $\langle s\rangle\Diamond\psi$ is consistent with T_Γ. The pair $(s, \Diamond\psi)$ appears in the above enumeration of all pairs, thus $(s, \Diamond\psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s\rangle\Diamond\psi := \langle s_m\rangle\varphi_m$. Then, since $\langle s\rangle\Diamond\psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s\rangle\Diamond\psi$ is in particular consistent with Γ_m. Therefore, by the above construction, $\langle s\rangle\langle p\rangle\psi \in \Gamma_{m+1}$ for some $p \in P$ such that Γ_m^+ is consistent with $\langle s\rangle\langle p\rangle\psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s\rangle\langle p\rangle\psi$ is also consistent with T_Γ. Thus $\langle s\rangle\langle p\rangle\psi$ is also consistent with T_Γ for some $p \in P$. Now, let us check the witnessing condition for $\langle G\rangle$. Let $G \in \mathcal{A}$, $s \in NF^P$ and $\psi \in \mathcal{L}_G^P$ and suppose $\langle s\rangle\langle G\rangle\psi$ is consistent with T_Γ. The pair $(s, \langle G\rangle\psi)$ appears in the above enumeration of all pairs, thus $(s, \langle G\rangle\psi) := (s_m, \varphi_m)$ for some $m \in \mathbb{N}$. Hence, $\langle s\rangle\langle G\rangle\psi := \langle s_m\rangle\varphi_m$. Then, since $\langle s\rangle\langle G\rangle\psi$ is consistent with T_Γ and $\Gamma_m \subseteq T_\Gamma$, we know that $\langle s\rangle\langle G\rangle\psi$ is in particular consistent with Γ_m. Therefore, by the above construction, $\langle s\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\psi \in \Gamma_{m+1}$ for some $\{p_i : i \in G\} \subseteq P$ such that Γ_m^+ is consistent with $\langle s\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\psi$. Thus, as T_Γ is consistent and $\Gamma_{m+1} \subseteq T_\Gamma$, we have that $\langle s\rangle\langle\bigwedge_{i\in G} K_i p_i\rangle\psi$ is also consistent with T_Γ. Hence, we conclude that T_Γ is P-witnessed. Finally, T_Γ is also maximal by construction: otherwise there would be a P-witness theory T such that $T_\Gamma \subsetneq T$. This implies that there exists $\varphi \in \mathcal{L}_G^P$ with $\varphi \in T$ but $\varphi \notin T_\Gamma$. Then, by the construction of T_Γ, we obtain $\Gamma_i \vdash \neg\varphi$ for all $i \in \mathbb{N}$. Therefore, since $T_\Gamma \subseteq T$, we have $T \vdash \neg\varphi$. Hence, since $\varphi \in T$, we conclude $T \vdash \bot$ (contradicting T being consistent).

Proof of Lemma 15 (Extension Lemma) Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ an enumeration of all pairs of the form (s_n, φ_n) consisting of any $s_n \in NF^{\tilde{P}}$, and every formula $\varphi_n \in \mathcal{L}_G^{\tilde{P}}$ of the form $\varphi_n := \Diamond\psi$ or $\varphi_n := \langle G\rangle\psi$ with $\psi \in \mathcal{L}_G^{\tilde{P}}$. We will recursively construct a chain of initial $\tilde{P}$-theories $\Gamma_0 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ such that

1. $\Gamma_0 = \Gamma$,
2. $P'_n := \{p \in P' : p$ occurs in $\Gamma_n\}$ is finite for every $n \in \mathbb{N}$, and
3. for every $\gamma_n := (s_n, \varphi_n)$ with $s_n \in NF^{\widetilde{P}}$ and $\varphi_n \in \mathcal{L}_G^{\widetilde{P}}$, if $\Gamma_n \nvdash \neg\langle s_n\rangle\Diamond\psi$ where $\varphi_n := \Diamond\psi$ then there is p_m "fresh" such that $\langle s_n\rangle\langle p_m\rangle\psi \in \Gamma_{n+1}$, and, if $\Gamma_n \nvdash \neg\langle s_n\rangle\langle G\rangle\psi$ where $\varphi_n := \langle G\rangle\psi$ for some $G \subseteq \mathcal{A}$ then there is $\{p_{m_i} : i \in G\}$ where p_{m_i} is "fresh" for every $i \in G$ such that $\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi \in \Gamma_{n+1}$. Otherwise we will define $\Gamma_{n+1} = \Gamma_n$.

For every γ_n, let $P'(n) := \{p \in P' \mid p$ occurs either in s_n or $\varphi_n\}$. Clearly every $P'(n)$ is always finite. We now construct an increasing chain of initial $\widetilde{P}$-theories recursively. We set $\Gamma_0 := \Gamma$, and let

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\langle s_n\rangle\langle p_m\rangle\psi\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\Diamond\psi \text{ and } \varphi_n := \Diamond\psi \\ \Gamma_n \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi\} & \text{if } \Gamma_n \nvdash \neg\langle s_n\rangle\langle G\rangle\psi \text{ and } \varphi_n := \langle G\rangle\psi \\ \Gamma_n & \text{otherwise,} \end{cases}$$

where m, m_i are, in each case, the least natural number greater than the indices in $P'_n \cup P'(n)$, i.e., p_m, p_{m_i} for all $i \in G$ are fresh in each case. We now show that $\widetilde{\Gamma} := \bigcup_{n\in\mathbb{N}} \Gamma_n$ is an initial $\widetilde{P}$-witnessed theory. First show that $\widetilde{\Gamma}$ is a $\widetilde{P}$-theory. By Lemma 12, it suffices to show by induction that every Γ_n is a $\widetilde{P}$-theory. Clearly Γ_0 is a $\widetilde{P}$-theory. For the inductive step suppose Γ_n is consistent but Γ_{n+1} is not. Hence, $\Gamma_n \neq \Gamma_{n+1}$ and moreover $\Gamma_{n+1} \vdash \bot$. Then, $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle p_m\rangle\psi\}$ (when $\varphi_n := \Diamond\psi$) or $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi\}$ (when $\varphi_n := \langle G\rangle\psi$). Here we will only check the latter case since the former case is analogous. Since $\Gamma_{n+1} = \Gamma_n \cup \{\langle s_n\rangle\langle \bigwedge_{i\in G} K_i p_{m_i}\rangle\psi\}$ we have $\Gamma_n \vdash [s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$. Therefore there exists $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$ such that $\{\theta_1, \dots, \theta_k\} \vdash [s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$. Let $\theta = \bigwedge_{1\leq i\leq k} \theta_i$. Then $\vdash \theta \to [s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$, so $\vdash [\theta\to, s_n][\bigwedge_{i\in G} K_i p_{m_i}]\neg\psi$ with $p_{m_i} \notin \mathrm{P}_\theta \cup \mathrm{P}_{s_n} \cup \mathrm{P}_{\varphi_n}$ for every $i \in G$. Thus, by the admissible rule in Lemma 9.2, we obtain $\vdash [\theta\to, s_n][G]\neg\psi$, i.e., $\vdash \theta \to [s_n][G]\neg\psi$. Therefore, $\theta \vdash \neg\langle s_n\rangle\langle G\rangle\psi$. Since $\{\theta_1, \dots, \theta_k\} \subseteq \Gamma_n$, we therefore have $\Gamma_n \vdash \neg\langle s_n\rangle\langle G\rangle\psi$. But, this would mean $\Gamma_n = \Gamma_{n+1}$, contradicting our assumption (that $\Gamma_{n+1} \neq \Gamma_n$). Therefore Γ_{n+1} is consistent and thus a $\widetilde{P}$-theory. Hence, by Lemma 12, $\widetilde{\Gamma}$ is a $\widetilde{P}$-theory. Condition (3) above implies that $\widetilde{\Gamma}$ is also $\widetilde{P}$-witnessed. Then, by Lindenbaum's Lemma (Lemma 14), there is a maximal $\widetilde{P}$-witnessed theory T_Γ such that $T_\Gamma \supseteq \widetilde{\Gamma} \supseteq \Gamma$. Moreover, since $0 \in \Gamma \subseteq \widetilde{\Gamma} \subseteq T_\Gamma$, the set T_Γ is in fact a maximal $\widetilde{P}$-witnessed initial theory.