Local Change

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Abstract

An agent can usually hold a very large number of beliefs. However, only a small part of these beliefs is used at a time. Efficient operations for belief change should affect the beliefs of the agent locally, that is, the changes should be performed only in the relevant part of the belief state. In this paper we define a local consequence operator that only considers the relevant part of a belief base. This operator is used to define local versions of the operations for belief change. Representation theorems are given for the local operators.

1 Introduction

An agent living in the world is confronted with new information all the time, be it through observation or through communication. Depending on the reliability of the source of information and on previous beliefs held by the agent, each new piece of information can be either accepted or rejected. In most models of belief revision, including the now classic AGM model [AGM85], the new information is always accepted. Recently, several models have been developed with the more realistic feature that inputs can be either accepted or rejected [Han97b].

In most belief revision models, belief states are assumed to be consistent. In the AGM theory and most of its successors, belief states are represented by beliefs sets, i.e. logically closed sets of sentences [AGM85]. In formal terms, a set K of sentences is a belief set if and only if $K = Cn(K)$, where Cn is the (supraclassical) consequence operator that represents the logic. These models do not allow us to represent an agent as having contradictory beliefs without believing in everything. Inconsistencies can never be local, but always propagate to the global level and corrupt the entire belief state.

However, this is not true for actual agents in the real world. Arguably, we all have inconsistent elements in our belief states, but these do not induce us to believe in everything. It is, to take just one example, quite feasible to believe both that Jesus was a human being and that Jesus was not a human being, without believing that the moon is made of cheese. In other words, actual belief systems are capable of containing *local* inconsistencies, that do not corrupt the entire belief system. Since inconsistencies can be local, different inconsistent belief systems may exist, that differ in terms of the beliefs actually endorsed by the respective agents.

The purpose of this paper is to develop a theory of belief change in which local inconsistencies are accepted. We will adopt the general framework of sentential input-assimilating modeling of belief change [Han99b] that dominates this area. Hence, belief states will be represented as sets of sentences in a language L that is closed under truth-functional operations. Belief states are assumed to change only in response to external inputs, and sentences in L will be used to represent inputs.

However, instead of belief sets we are going to use belief bases to represent belief states. A belief base is a set not closed under logical consequence [Fuh91, Han89, Neb92]. For every belief base B, its closure $Cn(B)$ is a belief set that represents the beliefs held by the agent. The elements of B are assumed to be in a sense more basic beliefs, from which the elements of $Cn(B) \setminus B$ are derived. Belief bases have substantial advantages in terms of computability [Neb96], and their increased expressive power as compared to belief sets can be used to represent important features of actual belief systems [Han92a]. On the other hand, belief sets have important advantages in terms of simplicity.

For our present purposes, the most important difference between belief sets and belief bases is that the latter allow for a simple and direct representation of the fact referred to above, that there can be different inconsistent belief states. Consider, for instance, the following two belief bases:

$$B_1 = \{p, \neg p, q_1, q_2, q_3, q_4, q_5\}$$
$$B_2 = \{p, \neg p, \neg q_1, \neg q_2, \neg q_3, \neg q_4, \neg q_5\}$$

These are distinct belief bases. We can suppose q_1 but not $\neg q_1$ to be endorsed according to B_1, whereas the opposite holds for B_2. However, on the belief set level these distinctions are lost since $Cn(B_1) = Cn(B_2) = L$. More generally speaking, belief bases but not belief sets allow us to distinguish between different inconsistency-containing belief states.

In AGM theory [AGM85], three forms of belief change are identified: contraction, expansion, and revision. Contraction consists of retracting a specified sentence from the belief set. Expansion consists of adding a specified sentence set-theoretically to the belief set. If the old and the new information are not logically compatible, then the new belief state after expansion will be inconsistent. Revision is consistency-preserving incorporation of new information, i.e. if the input sentence is consistent, then the new belief set will be consistent. If necessary, consistency is obtained by deleting parts of the original belief set. These operations have also been defined for belief bases [Fuh91, Han91, Han99b].

Two additional operations of change on belief bases were introduced in [Han91] and [Han97a]: consolidation and semi-revision. Consolidation consists in making an inconsistent belief base consistent. Semi-revision is an operation that may either accept or reject the input sentence. (For corresponding operations on belief sets, see [Mak97] and [Han99a]).

One of these five operations, namely expansion, cannot easily be restricted to a part of the belief base. To expand a belief base B by a sentence α means to perform the simple operation $B+\alpha = B\cup\{\alpha\}$. There is no way to restrict this addition of α to only a part of B, unless we introduce an extralogical division of B into compartments. As will be seen below, so much can be achieved with the division of B that can be derived from logic alone that we do not wish at this stage to add this further complication.

The other four operations can, however, readily be localized as follows:

1. *Local Contraction*: A belief is removed from a certain part of the belief base. I may, for instance, give up my belief that the dinosaurs died out due to a cosmic collision, without thinking about the consequences of this change for my beliefs about non-paleontological matters.

2. *Local Consolidation*: Inconsistencies are removed from some part of the belief base. The rest of the agent's beliefs may well be inconsistent. For instance, I can make my beliefs about biological evolution consistent, while retaining global inconsistency between biological and religious beliefs.

3. *Local Revision*: A new belief is added to the belief base in such a way that a certain part of the resulting base is made (kept) consistent. If I see, for example, that it is a sunny day in Amsterdam, then this contradicts my belief that it is always raining in Holland, and leads to

revision. This can be done without checking whether my beliefs about Brazilian politics are consistent with the new belief.

4. *Local Semi-revision*: A new belief is either accepted or rejected, in such a way that a certain part of the resulting belief base is made (kept) consistent. If I hear that it is snowing in Rio de Janeiro, then this contradicts my beliefs about the climate there. I may either accept or reject this new information. In both cases, my beliefs about Latin grammar (and most other subjects) will be unaffected by the operation.

Informal examples like the following motivate our search for plausible constructions of local operators:

> **Example:** When at home I hear on the radio that my friend Carol has been murdered yesterday night and that there were no traces of doors or windows having been forced. I talked to her yesterday on the phone and she was at home with her flatmates Ann and Bill. I know that no one else, except for Ann, Bill and Carol had the keys to their apartment. I conclude that Ann or Bill must have done it. But I have known Ann for quite some time and cannot believe that she would be able to murder anyone. I believe that she did not do it. For similar reasons, I believe that Bill did not do it. This is clearly inconsistent with my belief that one of them did it. So I decide to visit my friend Paul to ask what he thinks. In front of his place I see the lights are on. I know that if the lights are on, then Paul is home. I get out of the car and Paul's neighbour, who understands that I come to visit Paul tells me that he is not home. This is all very confusing, but I am sure of one thing: I do not believe I am asleep!
>
> This illustrates the fact that inconsistencies are local, that is, the fact that I have inconsistent beliefs does not cause me to believe in everything.
>
> I have expanded my belief base with the information given by the neighbour and reached a local inconsistency. I am interested now in whether Paul is at home or not. For a moment, I forget about the murder and think of the reasons that I have to believe

> that Paul is at home and that he is not. In order to eliminate the local inconsistency, I have to give up at least one of the beliefs. Suppose I ring the bell and Paul answers the door. Then I reject the neighbour's information that Paul is not home. On the other hand, suppose that he does not answer the door. Since I see the lights on, I give up my previous belief that if the lights are on then he is at home. He must have left the lights on when he left the house. In both cases, I eliminated the local inconsistency with respect to whether or not Paul is at home, but I still have inconsistent beliefs about the crime. This is an example of local consolidation.
>
> The whole operation, that is, adding the information given by the neighbour and then locally consolidating the beliefs illustrates the operation of local semi-revision.

In the rest of this paper we consider L to be a propositional language closed under the usual truth-functional connectives and containing a constant $\perp$ denoting falsum. We call *inference operation* any total function taking sets of formulas to sets of formulas, that is, any function from $\mathcal{P}(L)$ to $\mathcal{P}(L)$. We use C to denote inference operators.

If C is a Tarskian consequence operator, i.e. if it satisfies monotony, inclusion, and iteration, then it can also be denoted by Cn.

We will assume that the axiom of choice holds. As was pointed by Alchurrón and Makinson [AM82], this has consequences for the remainder operation that is used in AGM-style belief revision.

Our first task is to identify the compartments of a belief base in terms of which local change can be defined. This will be done in Section 2. In Section 3 we define local operations of belief change. In Section 4 we compare our approach to some others in the literature. Some concluding remarks are offered in Section 5. All formal proofs are deferred to an Appendix.

2 Defining compartments and local implication

In Section 2.1 we define the notion of compartments that is used in Section 2.2 for constructing a local inference operation. Local inference operations are restricted to relevant compartments of a belief base. The idea of using compartments to isolate inconsistencies is not new, see for example [Jaś69, Lew82].

2.1 Defining compartments

Compartments of belief bases can be seen as representations of compartments of minds or databases. There are two major ways to introduce them. First, they may be introduced as an addition to the logic, so that one and the same belief base can be divided into compartments in different ways. Secondly, they may be derived from the logic. The second method is the more economical, requiring no extra entities, and should be tried out first. We are going to use it here.

The concept we will use is that of the compartment "around" a sentence or set of sentences. Intuitively, the compartment in a belief base B around a sentence α is the subset of B that is relevant to α, and the compartment of B around a set A is the subset of B that is relevant to at least one element of A. Since one and the same sentence β may be relevant to both α_1 and α_2, compartments thus defined will typically be overlapping.

We will assume that the compartment around a set of sentences is equal to the union of the compartments around each of these sentences. Hence, letting $c(A, B)$ denote the A-compartment of B:

(1) $c(A, B) = \bigcup_{\alpha \in A} c(\alpha, B)$.

It follows from (1) that $c(\emptyset, B) = \emptyset$ for all B. Furthermore, due to (1), we can content ourselves with defining c for single sentences.

We are only interested in compartments around contingent expressions, since intuitively, no formula should be relevant for tautologies or contradictions. For these, we define:

(2) $c(\alpha, B) = \emptyset$ if $\alpha \in Cn(\emptyset)$ or $\neg\alpha \in Cn(\emptyset)$.

Since compartments are based on the underlying logic, they will depend on the inference operator. A useful tool in the definition of logical compartments is that of a *kernel* [Han94]. Given a sentence α and an inference operator C, the set X of sentences is an α-*kernel* if and only if it is a minimal set implying α. The *kernel operator* $\perp\!\!\!\perp_C$, based on C, is the operator that, given a belief base B and a sentence α, selects all α-kernels that are subsets of B. Hence:

Definition 2.1 *[Han94] Let C be an inference operation on L. Then the kernel operation $\perp\!\!\!\perp_C$ is the operation such that for all subsets B and elements α of L, $X \in B \perp\!\!\!\perp_C \alpha$ if and only if:*

1. $X \subseteq B$
2. $\alpha \in C(X)$
3. *for all Y, if $Y \subset X$ then $\alpha \notin C(Y)$*

The elements of $B \perp\!\!\!\perp_C \alpha$ are called α-kernels.
The symbol $\perp\!\!\!\perp$ (without subscript) denotes the kernel operation associated with Cn_0, the classical Tarskian consequence operator on L.

Observation 2.2 *Let C be an inference operation on the language L and $\perp\!\!\!\perp_C$ its associated kernel operation. If C satisfies compactness, then $B \perp\!\!\!\perp_C \alpha \neq \emptyset$ for all $B \subseteq L$ and all $\alpha \in C(B)$.*

Note that compactness is sufficient for this property. Monotony is not needed.

A first attempt to define the compartment for α in B is: (α contingent)

(3) $c_1(\alpha, B) = \bigcup(B \perp\!\!\!\perp_C \alpha)$

This definition is unsatisfactory since inconsistent kernels will be included, e.g. $c_1(\{p\}, \{p, q, \neg q\}) = \{p, q, \neg q\}$, since $\{q, \neg q\} \in \{p, q, \neg q\} \perp\!\!\!\perp_C p$. This problem can be solved by leaving out inconsistent kernels:[1]

(4) $c_2(\alpha, B) = \bigcup((B \perp\!\!\!\perp_C \alpha) \setminus (B \perp\!\!\!\perp_C \bot))$

[1]This has the perhaps somewhat counterintuitive consequence that $c(p, \{p \wedge \neg p\}) = c(p, \{\bot\}) = \emptyset$. Unless we adopt a syntax dependent approach (like the one in [Was99]), we will have to live with that.

But this is insufficient, since negations are relevant.[2] We would like to have, for example: $c(\{q\}, \{p, p \to \neg q, r, r \to s, s\}) = \{p, p \to \neg q\}$. This leads to a modification:

(5) $c_3(\alpha, B) = \bigcup((B \perp\!\!\!\perp_C \alpha) \cup (B \perp\!\!\!\perp_C \neg\alpha) \setminus (B \perp\!\!\!\perp_C \bot))$

Note that the compartments so defined may well be inconsistent.

We now have a definition of $c(\alpha, B)$ for arbitrary sentences α. Combining (1), (2), and (5) we obtain our final definition:

Definition 2.3 *Let C be an inference operation. The function c is the compartmentalization function based on C if and only if, for all $A, B \subseteq L$:*
$c(A, B) = \bigcup_{\alpha \in A} c(\alpha, B)$,
where $c(\alpha, B) = \begin{cases} \emptyset \text{ if } \alpha \in C(\emptyset) \text{ or } \neg\alpha \in C(\emptyset) \\ \bigcup((B \perp\!\!\!\perp_C \alpha) \cup (B \perp\!\!\!\perp_C \neg\alpha) \setminus (B \perp\!\!\!\perp_C \bot)) \text{ otherwise} \end{cases}$

In what follows, we will focus on compartmentalization functions that are based on the classical Tarskian consequence operation (Cn_0).

Observation 2.4 *(1) For all sets A and B of sentences, $c(A, B) = c(A, c(A, B))$.*
(2) If $A \subseteq A'$ and $B \subseteq B'$, then $c(A, B) \subseteq c(A', B')$.

2.2 Defining local implication

Using the definition of logical compartments presented above, we can define local implication as the inference relation that considers only the relevant part of the belief set. In other words, we can use logical compartments as defined above to derive, given a (global) consequence operator, a local inference operator for each logical compartment.

Definition 2.5 *Let C be an inference operation on L and let c be the compartmentalization function derived from the classical consequence operation Cn_0. Then for any set A, the A-localization of C is the inference operation C_A such that for all sets B of sentences: $C_A(B) = C(c(A, B))$.*
A set B is A-locally consistent if and only if $\bot \notin C_A(B)$

[2] [Ols97b], page 7: "The belief that $\neg q$ can plausibly be held to be relevant to the justification of the belief that q. If this is true and if q and $\neg q$ are both in the system, then these two beliefs must belong to the same belief module."

Example revisited: Let p stand for the proposition "Paul is at home", q for "The lights are on", a for "Ann is the murderer", b for "Bill is the murderer", and r for "I am asleep" and let Cn be a classical consequence operator.

My belief base B after talking to Paul's neighbour contains: $\{q, q \rightarrow p, \neg p, a \vee b, \neg a, \neg b, \neg r\}$. I am interested in whether Paul is at home, that is, the relevant beliefs are $c(p, B) = \{q, q \rightarrow p, \neg p\}$. Even though this set is inconsistent we have that $r \notin Cn_r(B) = Cn(c(r, B)) = Cn(\{\neg r\})$.

From Definition 2.5 it follows that:

Observation 2.6 *Let C be an inference operation that satisfies monotony. Then:*

1. $\beta \in C_\alpha(B)$ *iff* $\beta \in C_{\neg\alpha}(B)$
2. *If B is C-consistent, then:*
   - *if $\alpha \in A$, then $\alpha \in C_A(B)$ iff $\alpha \in C(B)$*
   - $\alpha \in C_B(B)$ *iff* $\alpha \in C(B)$
3. $\alpha \in C_\top(B)$ *iff* $\alpha \in C_\perp(B)$ *iff* $\alpha \in C(\emptyset)$

Observation 2.7 *Let C_A be the A-localization of an inference operation C. The elements of $B \perp\!\!\!\perp_{C_A} \alpha$ are subsets of $c(A, B)$.*

Observation 2.8 *Let C_A be the A-localization of an inference operation C. Then it holds for all sets B of sentences that $(C_A)_A(B) = C_A(B)$.*

A number of authors [Gab85, Mak89, Lin91] have studied the formal properties of inference operations.

Definition 2.9 *An inference operation C satisfies:*

- *Monotony if and only if for all B and D such that $B \subseteq D$, $C(B) \subseteq C(D)$;*

- *Compactness if and only if for all* $\alpha \in C(B)$ *there is some finite* $D \subseteq B$ *such that* $\alpha \in C(D)$*;*
- *Idempotency (iteration) if and only if* $C(C(B)) = C(B)$*;*
- *Weak iteration if and only if* $C(C(B)) \subseteq C(B)$*;*
- *Inclusion if and only if* $B \subseteq C(B)$*;*
- *Embedded inclusion if and only if* $C(B) \subseteq C(C(B))$*;*
- *Supraclassicality if and only if* $Cn_0(B) \subseteq C(B)$*;*
- *Deduction property if and only if whenever* $q \in C(B \cup \{p\})$*, then* $p \rightarrow q \in C(B)$*;*
- *Reductio ad absurdum if and only if whenever* $\perp \in C(B \cup \{\neg p\})$*, then* $p \in C(B)$*;*
- *Falsity if and only if for all* p*,* $p \in C(\{\perp\})$*;*
- *Consistency preservation if and only if for all* B *such that* $\perp \notin Cn_0(B)$*,* $\perp \notin C(B)$*;*
- *Cut if and only if whenever* $B \subseteq D \subseteq C(B)$*, then* $C(D) = C(B)$*;*
- *Sen if and only if* $C(B) \cup C(D) \subseteq C(B \cup D)$*;*
- *Distributivity if and only if* $C(B) \cap C(D) \subseteq C(Cn_0(B) \cap Cn_0(D))$*;*
- *Chernov if and only if* $C(B \cup D) \subseteq Cn_0(C(B) \cup D)$*;*
- *Arrow if and only if whenever* $\perp \notin Cn_0(C(B) \cup D)$*, then* $C(B \cup D) = Cn_0(C(B) \cup D)$*;*
- *Explosiveness if and only if for all* α *and* β*,* $\beta \in C(\{\alpha, \neg\alpha\})$*;*
- *Weak Explosiveness if and only if whenever* $\perp \in C(B)$*, then for all sentences* α*,* $\alpha \in C(B)$*; and*
- *Non-contravention if and only if, whenever* $\neg\alpha \in C(B \cup \{\alpha\})$*, then* $\neg\alpha \in C(B)$*.*

- *α-local non-contravention if and only if, if $\neg\alpha \in C(B \cup \{\alpha\})$, then $\neg\alpha \in C(B)$.*

The following are some of the properties the local inference operator C_A has.

Theorem 2.10 *Let be C_A the A-localization of an inference operation C. Then:*

1. *If C satisfies monotony, then C_A satisfies monotony and Sen.*
2. *If C satisfies monotony and compactness, then C_A satisfies monotony, compactness, and weak iteration.*
3. *If C satisfies monotony, compactness and inclusion, then C_A satisfies monotony, compactness, idempotency (iteration), and cut.*
4. *If C satisfies monotony and consistency preservation, then so does C_A.*
5. *If C satisfies weak explosiveness, than so does C_A.*
6. *If C satisfies monotony and inclusion, then C_A satisfies monotony and embedded inclusion.*
7. *If C satisfies monotony, compactness, weak explosiveness, and non-contravention, then C_A satisfies monotony, compactness, and weak explosiveness. Furthermore, it satisfies α-local non-contravention for all $\alpha \in A$.*

Theorem 2.11 *For each of the following properties there is some set of sentences A such that Cn_A, the A-localization of the classical truth functional consequence operator Cn_0, does not satisfy the property: inclusion, supra-classicality, deduction property, reductio ad absurdum, falsity, distributivity, chernov, arrow, explosiveness, and non-contravention.*

The classical consequence operator Cn_0 satisfies inclusion, idempotency and monotony. Nonmonotonic logics leave out monotony, resource logics [Gab94] leave out idempotency. Our framework leaves out inclusion [3]. The inclusion

[3]Thanks are due to Frans Voorbraak for pointing out this to us.

and supraclassicality properties are clearly not wanted, since the purpose of the operator C_A is to ignore irrelevant data.

That explosiveness does not hold agrees with the intuition that some inconsistencies are "less harmful" than others. If an inconsistency is completely irrelevant to the current reasoning, then it should not trivialize the belief state, causing the agent to believe every sentence in the language. On the other hand, if the inconsistency turns out to be relevant, that is, if given a belief base B and a set A of sentences it is the case that $\perp \in C_A(B)$, then the relation C_A becomes "locally" explosive, giving an indication that something has to be repaired.

3 The generalized belief revision operators

In this section we present some generalized types of belief revision operators that can be obtained using the notion of local inference. It turns out that for this purpose, we do not need the full set of properties of the local implication operator introduced above. The results to be presented here are much more general, since they only make use of some of the properties shown to hold for our local implication operator.

All definitions and results in this section refer to a generic inference operator C that may or may not be obtainable as a localization of some other inference operator. Most of the results to be presented in this section are generalizations of results that have been obtained previously for the special case when C is a Tarskian consequence operator that satisfies the standard properties of supraclassicality, compactness and the deduction property.

3.1 Contraction

An operation of *contraction* based on an inference operator C is an operation that, given a set B of sentences and a sentence α, returns a subset of B that does not imply α according to C, except in the limiting case when $\alpha \in C(\emptyset)$. We are going to consider two types of contraction operators, kernel contraction and partial meet contraction.

The idea behind kernel contraction is that, if we remove from the belief base B at least one element of each α-kernel (minimal subset of B that implies α), then we obtain a belief base that does not imply α [Han94]. To perform these removals of elements, we use an incision function, i.e., a function that selects at least one sentence from each kernel.

Definition 3.1 *[Han94] An incision function is any function $\sigma : \mathcal{P}(\mathcal{P}(L)) \to \mathcal{P}(L)$ such that for any subset S of $\mathcal{P}(L)$:*

1. *$\sigma(S) \subseteq \bigcup S$, and*
2. *If $\emptyset \neq X \in S$, then $X \cap \sigma(S) \neq \emptyset$.*

The following definition is a generalization of the definition given in [Han94]:

Definition 3.2 *Let C be an inference operation on L and σ an incision function. The kernel contraction on B determined by C and σ is the operation $\dot{-}_{C,\sigma}$ such that for all sets of sentences B and sentences α:*
$B \dot{-}_{C,\sigma} \alpha = B \setminus \sigma(B \perp\!\!\!\perp_C \alpha)$

An axiomatic characterization of kernel contraction was obtained in [Han94] for a conventional, Tarskian, supraclassical, and compact consequence operator. Surprisingly enough, essentially the same characterization can be obtained with an inference operator that is only required to satisfy monotony and compactness.

Theorem 3.3 *Let C be an inference operation satisfying monotony and compactness. Then $\dot{-}$ is an operation of kernel contraction on B determined by C and some incision function if and only if for all sentences α:*

- *If $\alpha \notin C(\emptyset)$, then $\alpha \notin C(B \dot{-} \alpha)$ (success)*
- *$B \dot{-} \alpha \subseteq B$ (inclusion)*
- *If $\beta \in B \setminus B \dot{-} \alpha$, then there is some $B' \subseteq B$ such that $\alpha \notin C(B')$ and $\alpha \in C(B' \cup \{\beta\})$ (core-retainment)*
- *If for all subsets B' of B, $\alpha \in C(B')$ if and only if $\beta \in C(B')$, then $B \dot{-} \alpha = B \dot{-} \beta$ (uniformity)*

In partial meet contraction [AGM85], the starting-point is the set of maximal subsets not implying the sentence to be contracted. A selection function is used to choose some of these maximal sets, and their intersection is taken as the outcome of the operation.

A remainder operator $\perp_C$ selects for each set B of sentences and each sentence α the maximal subsets of B that do not imply α according to C. Formally:

Definition 3.4 *[AM82] Let C be an inference operation on L. The remainder operation $\perp_C$ is the operation such that for all subsets B and elements α of L, $X \in B\perp_C\alpha$ if and only if:*

1. *$X \subseteq B$,*
2. *$\alpha \notin C(X)$, and*
3. *$\alpha \in C(Y)$ for all Y such that $X \subset Y \subseteq B$.*

Observation 3.5 *[AM82](Upper bound property) Let C satisfy monotony and compactness. If $X \subseteq B$ and $\alpha \notin C(X)$, then there is some X' such that $X \subseteq X' \in B\perp_C\alpha$.*

The axiom of choice is used in the proof of the upper bound property.

Definition 3.6 *[AGM85] A selection function for L is a function $g : \mathcal{P}(\mathcal{P}(L)) \to \mathcal{P}(\mathcal{P}(L))$ such that for all subsets S of $\mathcal{P}(L)$,*

1. *$g(S) \subseteq S$, and*
2. *If $S \neq \emptyset$ then $g(S) \neq \emptyset$.*

Definition 3.7 *[AGM85] The partial meet contraction operator on B based on a consequence operator C and a selection function g is the operator $\dot{-}_{C,g}$ such that for all sentences α:*

$$B\dot{-}_{C,g}\alpha = \begin{cases} B & \textit{if } B\perp_C\alpha = \emptyset \\ \bigcap g(B\perp_C\alpha) & \textit{otherwise} \end{cases}$$

The axiomatic characterization of partial meet contraction obtained in [Han92b] for a conventional consequence operator turns out to be generalizable to the same general category of inference operations that was referred to in Theorem 3.3.

Theorem 3.8 *Let C satisfy monotony and compactness. Then $\dot{-}$ is an operator of partial meet contraction on B based on C if and only if for all sentences α:*

- *If $\alpha \notin C(\emptyset)$, then $\alpha \notin C(B \dot{-} \alpha)$ (success)*
- *$B \dot{-} \alpha \subseteq B$ (inclusion)*
- *If $\beta \in B \setminus (B \dot{-} \alpha)$, then there is some B' such that $B \dot{-} \alpha \subseteq B' \subseteq B$, $\alpha \notin C(B')$ and $\alpha \in C(B' \cup \{\beta\})$ (relevance)*
- *If for all subsets B' of B $\alpha \in C(B')$ if and only if $\beta \in C(B')$, then $B \dot{-} \alpha = B \dot{-} \beta$ (uniformity)*

Theorem 3.9 *1. If $\dot{-}$ is a partial meet contraction operator with respect to an operator C, then it is a kernel contraction operator with respect to the operator C.*

2. It does not hold in general that if $\dot{-}$ is a kernel contraction operator with respect to an operator C, then it is a partial meet contraction operator with respect to an operator C.

3.2 Consolidation

By an operation of *consolidation* based on the inference operation C, we mean an operation that, given a set of formulas B, returns a subset of B that is consistent according to C. Consolidation was introduced in [Han91] and further studied in [Fuh97] and [Ols97a]. It can be modelled as contraction by a contradictory sentence. Therefore, the two contraction operators introduced in the previous section provide us with two consolidation operators.

The idea behind kernel consolidation is that, if we remove from the belief base at least one element of each inconsistent kernel (inclusion-minimal subset of the base that implies $\perp$), then we obtain a consistent belief base.

Definition 3.10 *[Han97a] Let C be an inference operation on the language L and σ an incision function. Then the kernel consolidation operation for B determined by C and σ is the operation $!_{C,\sigma}$ such that:*

$B!_{C,\sigma} = B \setminus \sigma(B \perp\!\!\!\perp_C \perp)$.

The following characterization is a generalization of the one obtained in [Han97a]:

Theorem 3.11 *Let C be an inference operation satisfying monotony, compactness, and $\perp \notin C(\emptyset)$. An operation ! is an operation of kernel consolidation for B determined by C and some incision function if and only if:*

- $\perp \notin C(B!)$ *(consistency)*
- $B! \subseteq B$ *(inclusion)*
- *If $\alpha \in B \setminus (B!)$, then there is some X such that $X \subseteq B$, $\perp \notin C(X)$ and $\perp \in C(X \cup \{\alpha\})$ (core-retainment)*

Consolidation can also be constructed from partial meet contraction.

Definition 3.12 *[Han91] The partial meet consolidation operator for B based on an inference operator C and a selection function g is the operator $!_{C,g}$ such that:*

$$B!_{C,g} = \begin{cases} B & \text{if } B\perp_C\perp = \emptyset \\ \bigcap g(B\perp_C\perp) & \text{otherwise} \end{cases}$$

The following theorem is a generalization of a theorem reported in [Han91]:

Theorem 3.13 *Let C satisfy monotony, compactness, and $\perp \notin C(\emptyset)$. An operation ! is an operation of partial meet consolidation based on C and some selection function if and only if for all sets B of sentences:*

- $\perp \notin C(B!)$ *(consistency)*
- $B! \subseteq B$ *(inclusion)*
- *If $\alpha \in B \setminus (B!)$, then there is some X such that $B! \subseteq X \subseteq B$, $\perp \notin C(X)$ and $\perp \in C(X \cup \{\alpha\})$ (relevance)*

3.3 External and Internal Revision

A revision operation consists of adding new information to a belief base in such a way that the resulting belief base is consistent whenever the new piece of information is consistent. AGM-style revision by α consists of two suboperations, expansion by α and contraction by $\neg\alpha$. In the classic AGM operation of revision (first developed for belief sets), contraction by $\neg\alpha$ takes place first and is then followed by expansion by α [AGM85]. In [Han92b] this was called "internal revision". An alternative procedure for belief bases, "external revision" was proposed. It consists in first expanding the belief base by α and after that contracting by $\neg\alpha$. External revision is not meaningfully applicable to belief sets since, if a belief set that contains $\neg\alpha$ is expanded by α, then the outcome will be equal to the whole language, so that all distinctions are lost. This holds, of course, provided that the inference operator C under which the belief set is closed satisfies explosiveness.

Revision can be obtained by combining expansion with either kernel or partial meet contraction to obtain either internal or external revision. This leaves us with four options.

An operation of internal kernel revision by a sentence α consists of first using kernel contraction to contract the belief base by $\neg\alpha$ in order to make room for the new piece of information and then expanding by α. Formally:

Definition 3.14 *The internal kernel revision on B based on an inference operator C and an incision function σ is the operator $\mp_{C,\sigma}$ such that for all sentences α:*
$B \mp_{C,\sigma} \alpha = (B \setminus \sigma(B \perp\!\!\!\perp_C \neg\alpha)) \cup \{\alpha\}$

The following theorem characterizes this operation:

Theorem 3.15 *Let C satisfy monotony and compactness. An operator $\mp$ is an operator of internal kernel revision based on an inference operator C if and only if, for all sets B of sentences and all sentences α such that C satisfies α-local non-contravention:*

- *If $\neg\alpha \notin C(\emptyset)$, then $\neg\alpha \notin C(B \mp \alpha)$ (non-contradiction)*
- *$B \mp \alpha \subseteq B \cup \{\alpha\}$ (inclusion)*

- *If* $\beta \in B \setminus B \mp \alpha$, *then there is some* $B' \subseteq B$ *such that* $\neg\alpha \notin C(B')$ *and* $\neg\alpha \in C(B' \cup \{\beta\})$ *(core-retainment)*

- $\alpha \in B \mp \alpha$ *(success)*

- *If for all* $B' \subseteq B$, $\neg\alpha \in C(B')$ *if and only if* $\neg\beta \in C(B')$, *then* $B \cap (B \mp \alpha) = B \cap (B \mp \beta)$ *(uniformity)*

In an operation of internal partial meet revision by α, the belief base is first partial meet contracted by $\neg\alpha$ and then expanded by α:

Definition 3.16 *[Han92b] The internal partial meet revision of* B *based on an inference operator* C *and a selection function* g *is the operator* $\mp_{C,g}$ *such that for all sentences* α*:*

$$B \mp_{C,g} \alpha = \begin{cases} B \cup \{\alpha\} & \text{if } B \perp_C \neg\alpha = \emptyset \\ \bigcap g(B \perp_C \neg\alpha) \cup \{\alpha\} & \text{otherwise} \end{cases}$$

The following theorem characterizes this operation:

Theorem 3.17 *[Han92b] Let* C *be an inference operator satisfying monotony and compactness. An operator* $\mp$ *is an operator of internal partial meet revision based on* C *if and only if, for all sets* B *of sentences and all sentences* α *such that* C *satisfies* α*-local non-contravention:*

- *If* $\neg\alpha \notin C(\emptyset)$, *then* $\neg\alpha \notin C(B \mp \alpha)$ *(non-contradiction)*

- $B \mp \alpha \subseteq B \cup \{\alpha\}$ *(inclusion)*

- *If* $\beta \in B \setminus B \mp \alpha$, *then there is some* B' *such that* $B \mp \alpha \subseteq B' \subseteq B \cup \{\alpha\}$, $\neg\alpha \notin C(B')$ *but* $\neg\alpha \in C(B \cup \{\beta\})$ *(relevance)*

- $\alpha \in B \mp \alpha$ *(success)*

- *If for all* $B' \subseteq B$, $\neg\alpha \in C(B')$ *if and only if* $\neg\beta \in C(B')$, *then* $B \cap (B \mp \alpha) = B \cap (B \mp \beta)$ *(uniformity)*

In an operation of external kernel revision by α, the belief base is first expanded with α and then kernel contraction by $\neg\alpha$ takes place:

Definition 3.18 *The external kernel revision of B based on an inference operator C and an incision function σ is the operator $\pm_{C,\sigma}$ such that for all sentences α:*
$B \pm_{C,\sigma} \alpha = (B \cup \{\alpha\}) \setminus \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \neg\alpha)$

The following theorem characterizes this operation:

Theorem 3.19 *Let C be an inference operator satisfying monotony and compactness. An operator $\pm$ is an operator of external kernel revision based on an inference operator C if and only if, for all sets B of sentences and all sentences α such that C satisfies α-local non-contravention:*

- *If $\neg\alpha \notin C(\emptyset)$, then $\neg\alpha \notin C(B \pm \alpha)$ (non-contradiction)*
- *$B \pm \alpha \subseteq B \cup \{\alpha\}$ (inclusion)*
- *If $\beta \in B \setminus B \pm \alpha$, then there is some $B' \subseteq B \cup \{\alpha\}$ such that $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\beta\})$ (core-retainment)*
- *$\alpha \in B \pm \alpha$ (success)*
- *If α and β are elements of B and it holds for all $B' \subseteq B$ that $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$, then $B \cap (B \pm \alpha) = B \cap (B \pm \beta)$ (weak uniformity)*
- *$B + \alpha \pm \alpha = B \pm \alpha$ (pre-expansion)*

In an operation of external partial meet revision by α, the belief base is first expanded with α and then partial meet contraction by $\neg\alpha$ takes place:

Definition 3.20 *[Han92b] The external partial meet revision of B based on an inference operator C and a selection function g is the operator $\pm_{C,g}$ such that for all sentences α:*
$$B \pm_{C,g} \alpha = \begin{cases} B \cup \{\alpha\} & \text{if } (B \cup \{\alpha\}) \perp_C \neg\alpha = \emptyset \\ \bigcap g((B \cup \{\alpha\}) \perp_C \neg\alpha) & \text{otherwise} \end{cases}$$

The following theorem characterizes this operation:

Theorem 3.21 *Let C be an inference operator satisfying monotony and compactness. An operator $\pm$ is an operator of external partial meet revision based on an inference operator C if and only if, for all sets B of sentences and sentences α such that C satisfies α-local non-contravention:*

- *If $\neg\alpha \notin C(\emptyset)$, then $\neg\alpha \notin C(B \pm \alpha)$ (non-contradiction)*
- *$B \pm \alpha \subseteq B \cup \{\alpha\}$ (inclusion)*
- *If $\beta \in B \setminus B \pm \alpha$, then there is some B' such that $B \pm \alpha \subseteq B' \subseteq B \cup \{\alpha\}$ such that $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\beta\})$ (relevance)*
- *$\alpha \in B \pm \alpha$ (success)*
- *If α and β are elements of B and it holds for all $B' \subseteq B$ that $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$, then $B \cap (B \pm \alpha) = B \cap (B \pm \beta)$ (weak uniformity)*
- *$B + \alpha \pm \alpha = B \pm \alpha$ (pre-expansion)*

3.4 Semi-revision

Semi-revision differs from revision in being non-prioritized, i.e., the incoming information may be either accepted or rejected. In analogy to AGM revision, that can be defined in terms of contraction and expansion, semi-revision can be defined in terms of consolidation and expansion via the following identity [Han97a] (For a survey of other ways to construct non-prioritized belief revision, see [Han99a]):

$$B?\alpha = (B + \alpha)!$$

Hence, semi-revision consists of two steps: first the belief α is added to the base, and then the resulting base is consolidated. The two consolidation operators introduced above give rise to two semi-revision operators.

Definition 3.22 *[Han97a] The kernel semi-revision of B based on an inference operator C and an incision function σ is the operator $?_{C,\sigma}$ such that for all sentences α:*

$$B?_{C,\sigma}\alpha = (B \cup \{\alpha\}) \setminus \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \bot)$$

The following theorem is a generalization of a result reported in [Han97a]:

Theorem 3.23 *Let C be an inference operation satisfying monotony, compactness, and $\perp \notin C(\emptyset)$. An operator ? is an operator of kernel semi-revision based on C if and only if for all sets B of sentences and sentences α:*

- $\perp \notin C(B?\alpha)$ *(consistency)*
- $B?\alpha \subseteq B \cup \{\alpha\}$ *(inclusion)*
- *If $\beta \in B \setminus B?\alpha$, then there is some $B' \subseteq B \cup \{\alpha\}$ such that $\perp \notin C(B')$ and $\perp \in C(B' \cup \{\beta\})$ (core-retainment)*
- $(B+\alpha)?\alpha = B?\alpha$ *(pre-expansion)*
- *If $\alpha, \beta \in B$, then $B?\alpha = B?\beta$ (internal exchange)*

Definition 3.24 *[Han97a] The partial meet semi-revision of B based on an inference operator C and a selection function g is the operator $?_{C,g}$ such that for all sentences α:*
$B?_{C,g}\alpha = \bigcap g((B \cup \{\alpha\})\perp_C \perp)$

The following theorem generalizes the characterization given in [Han97a]:

Theorem 3.25 *Let C be an inference operation satisfying monotony, compactness, and $\perp \notin C(\emptyset)$. An operator ? is an operator of partial meet semi-revision based on C if and only if for all sets B of sentences and sentences α:*

- $\perp \notin C(B?\alpha)$ *(consistency)*
- $B?\alpha \subseteq B \cup \{\alpha\}$ *(inclusion)*
- *If $\beta \in B \setminus B?\alpha$, then there is some B' such that $B?\alpha \subseteq B' \subseteq B \cup \{\alpha\}$, $\perp \notin C(B')$ and $\perp \in C(B' \cup \{\beta\})$ (relevance)*
- $(B+\alpha)?\alpha = B?\alpha$ *(pre-expansion)*
- *If $\alpha, \beta \in B$, then $B?\alpha = B?\beta$ (internal exchange)*

3.5 Levi and Harper identities

One of the hallmarks of the AGM theory is the close connection that holds between revision and contraction operators through the Levi and Harper identities. The way we have defined contraction and internal revision, the Levi identity holds both for kernel and partial meet operations, i.e. for the former we have $B \mp_{C,\sigma} \alpha = (B \dot{-}_{C,\sigma} \neg\alpha) \cup \{\alpha\}$ and for the latter $B \mp_{C,g} \alpha = (B \dot{-}_{C,g} \neg\alpha) \cup \{\alpha\}$.

The Harper identity can also be shown to hold, under fairly weak conditions:

Theorem 3.26 *Let $\dot{-}_C$ satisfy inclusion and core-retainment, and let $*_C$ be the internal revision operator based on $\dot{-}_C$ via the Levi identity. Let C satisfy α-local non-contravention. Then:*

$B \dot{-}_C \alpha = B \cap (B * \neg\alpha)$ *(the Harper identity)*

In [Han97a], it was shown that (global) consolidation and semi-revision are tied together by properties that are in a way analogous to the Levi and Harper identity:

$A?\alpha = (A + \alpha)!$. (the $! \rightarrow ?$ identity)
$A! = \cup\{A?\alpha | \alpha \in A\}$. (the $? \rightarrow !$ identity)

It can be seen directly from the definitions that these properties also hold for local consolidation and semi-revision, both for kernel operations (with C and σ held constant) and for partial meet operations (with C and g held constant).

3.6 Local operators

The axiomatic characterizations obtained for the generalized belief revision operators apply to any inference operator satisfying monotony and compactness (contraction), monotony, compactness and $\perp \notin C(\emptyset)$ (consolidation and semi-revision) or monotony, compactness and α-local non-contravention (external and internal revision). From theorem 2.10 it follows that the A-localization of the classical consequence operator Cn_0, Cn_A, satisfies monotony, compactness and $\perp \notin C(\emptyset)$. Moreover, for $\alpha \in A$ it also satisfies α-local non-contravention. This means that, in order to obtain characterizations of the local operations aimed at in section 1, all we need to do is to substitute the generic inference operator C by Cn_A. In the case of the revision operations, the extra care should be taken that in a revision by α, we must have $\alpha \in A$. When the A-compartment of B is empty, the local contraction and local consolidation operations leave B unchanged, while local internal/external revision and local semi-revision all coincide with expansion.

We now go back once more to our example to illustrate the effects of an operation of local consolidation.

> **Example re-revisited:** Before talking to Paul's neighbour, my belief base B contained: $\{q, q \to p, a \vee b, \neg a, \neg b, \neg r\}$. The neighbour says that Paul is not home, so I locally semi-revise my belief base by $\neg p$. This means first adding $\neg p$ set-theoretically to B and then locally consolidating. Let B' be the belief base that replaces B after the expansion, that is, $B' = B \cup \{\neg p\}$. We have $c(p, B') = \{q, q \to p, \neg p\}$ and $B' \perp_{Cn_p} \perp = \{\{q \to p, q\} \cup X, \{q \to p, \neg p\} \cup X, \{q, \neg p\} \cup X\}$, where $X = \{a \vee b, \neg a, \neg b, \neg r\}$. Local partial meet consolidation gives us then, with the selection function $g(B' \perp_{Cn_p} \perp) = \{\{q \to p, q, a \vee b, \neg a, \neg b, \neg r\}\}$, $B'' = \{q \to p, q, a \vee b, \neg a, \neg b, \neg r\} = B$, that is, the new information is rejected. (With another selection function the information can be accepted).

4 Related Work

Although the idea of local change seems to have no precedents in the belief revision theory, our proposal for local implication is closely related to at least three other segments of the logical literature: (1) studies of compartments and frames of mind, (2) paraconsistent logic, and (3) relevance logic.

4.1 Compartments and frames of mind

The idea of isolating inconsistencies by means of compartments has been studied by several authors.

Jaśkowski [Jaś69] introduced his discursive logics as a way of formalizing "what some participant of a discourse is committed to". If participant x

asserts α and participant y asserts β, then α and β are true in his logics, but not $\alpha \wedge \beta$, since there may be no participant in the discourse who is committed to both assertions α and β. Assertions associated to different participants of the discourse cannot be combined.

Lewis [Lew82] also defended the idea of fragments that cannot be combined, but the different (possibly overlapping) fragments were parts of one agent's beliefs. Fagin and Halpern [FH88] formalized a similar idea. In their Logic of Local Reasoning, an agent may have several "frames of mind". An agent believes α if she believes α in some frame of mind. This idea is formalized via an extended Kripke structure, where instead of one accessibility relation between worlds, there is a relation $\mathcal{C}$ between worlds and sets of worlds. If $\mathcal{C}(s) = \{T_1, ..., T_n\}$, then an agent in state s sometimes takes the set of worlds T_1 as possible, sometimes T_2, etc.

Benferhat et al. [BDP97] use yet another approach to isolate inconsistencies, which is very similar to the approach known in AI as WIDTIO (When In Doubt Throw It Out) [Win90]. They define a free (or sound) inference operator that disconsiders all formulas belonging to a minimal inconsistent subset. So, $B \vdash_{free} \alpha$ if and only if $Free(B) \vdash \alpha$, where $Free(B)$ is the result of deleting every minimal inconsistent subset from B.

Tennant [Ten84], elaborating on an idea from [Smi59], presented the idea of perfect validity, which is closely related to our idea of local inference. A sequent in a Gentzen system is a structure $X : Y$ where X and Y are sets of sentences of a language L. A sequent $X' : Y'$ is a subsequent of $X : Y$ if and only if $X' \subseteq X$ and $Y' \subseteq Y$. If, in addition, $X' \subset X$ or $Y' \subset Y$, then $X' : Y'$ is a proper subsequent of $X : Y$. A sequent $X : Y$ is valid if and only if in every model where all elements of X hold, at least one element of Y holds. A sequent is said to be perfectly valid if and only if it is valid and has no valid proper subsequent. If $X : Y$ is perfectly valid and $Y \neq \emptyset$, then we have: (i) X is consistent, otherwise $X : \emptyset$ would be a valid subsequent of $X : Y$; (ii) if $Y = \{\alpha\}$, then $X \in L \perp\!\!\!\perp \alpha \setminus L \perp\!\!\!\perp \bot$. We can then translate our notion of local implication in terms of perfect validity: $\beta \in Cn_\alpha(B)$ if and only if there are perfectly valid sequents $X_1 : Y_1$, ..., $X_n : Y_n$ such that each $X_i \subseteq B$, either $Y_i = \{\alpha\}$ or $Y_i = \{\neg\alpha\}$, and $\beta \in Cn(\bigcup X_i)$. More generally, $\beta \in Cn_A(B)$ if and only if there are perfectly valid sequents $X_1 : Y_1$, ..., $X_n : Y_n$ such that each $X_i \subseteq B$, either $Y_i = \{\alpha\}$ or $Y_i = \{\neg\alpha\}$ for some $\alpha \in A$, and $\beta \in Cn(\bigcup X_i)$.

4.2 Paraconsistent logic

A logic is said to be paraconsistent if its consequence relation is not explosive. Technically, when a logic has more than one inference operation (as is the case here, since for every set A there is an associated inference operation C_A), all of them have to be non-explosive for the logic to be called paraconsistent [PRN89]. If we take A to be the whole language, C_A is explosive, so the logic of local reasoning cannot be called paraconsistent in this sense. Nevertheless, it contains paraconsistent fractions, and we believe it can capture some of the intuitions that have been the driving forces behind the development of paraconsistent logic.

In [RS95], an approach is taken where the logic under which belief sets are closed is paraconsistent. The motivation is very close to ours, namely to make it possible to work with inconsistent beliefs without trivializing the belief states. These authors use first degree entailment, a modification of (propositional) classical logic where each formula can have as its truth value any subset of $\{\texttt{true}, \texttt{false}\}$. A formula α is entailed by a set of formulas X if and only if any valuation that assigns to every element of X either $\{\texttt{true}\}$ or $\{\texttt{true}, \texttt{false}\}$, also assigns to α either $\{\texttt{true}\}$ or $\{\texttt{true}, \texttt{false}\}$.

4.3 Relevance logic

One of the main characteristics of Relevance Logic [Dun86] is that from $\{\alpha, \neg\alpha \vee \beta\}$ one cannot always derive β. Depending on which set we use to limit the compartments, we can obtain the same property. We have, for example, $q \notin C_p(\{p, \neg p \vee q\})$.

There are in the literature several attempts to define the concepts of relevance and dependence. [Lug96] classifies approaches to relevant implication into two groups: those which impose conditions on the contents of implied formulas and those which impose conditions on the deduction of the implied formulas. In the first group she mentions Parry, whose implications are valid only if no new variables are introduced into the consequent, Anderson and Belnap, who impose that the antecedent and the consequent share variables and Epstein, who associates themes to the formulas and imposes conditions on the themes of the antecedent and the consequent. In the second group she includes approaches requiring that the antecedent be used in an "essential way" in the derivation of the consequent. Among these she mentions Myhill and again Anderson and Belnap.

The way our relevant compartments are defined, through kernel sets, is related to her second approach, that is, we consider relevant to a formula α the formulas that appear in a minimal derivation of α or its negation.

Lang and Marquis [LM98] present several complexity results for different notions of dependency found in the literature.

In [dCH96], Fariñas del Cerro and Herzig introduce a set of nine postulates that any dependence relation should satisfy, and they also show how a contraction operation satisfying the Gärdenfors postulates can be obtained from a dependence relation. Given a dependence relation $\rightsquigarrow$, where $\alpha \rightsquigarrow \beta$ should be read as "β depends on α", they define a contraction operation for belief sets as follows:

(Def $\dot{-}$) $\gamma \in K \dot{-} \alpha$ if and only if either $\vdash \gamma$ or $\gamma \rightsquigarrow \gamma$ and $\alpha \not\rightsquigarrow \gamma$

They show that, given a dependence relation $\rightsquigarrow$ satisfying their set of postulates, the contraction operation defined by (Def $\dot{-}$) satisfies the eight Gärdenfors postulates for contraction.

5 Conclusions

We have defined a consequence operation that considers only the relevant parts of a belief base and shown that this consequence operation can be used to define local versions of the operations of contraction, consolidation, revision, and semi-revision.

These local operations can be axiomatized with plausible postulates. Furthermore, they give the desired result when applied to examples of the type that motivated our work and that cannot be treated satisfactorily in the AGM model.

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A Appendix: Proofs

Proof of Observation 2.2: Let compactness be satisfied and $\alpha \in C(B)$. Then there is some finite subset Z of B such that $\alpha \in C(Z)$. Let X be any inclusion-minimal subset of Z with the property $\alpha \in C(X)$. Then $X \in B \perp\!\!\!\perp_C \alpha$. $\square$

Proof of Observation 2.4: (1) That $c(A, c(A, X)) \subseteq c(A, X)$ follows directly from definition 2.3. To see that $c(A, X) \subseteq c(A, c(A, X))$, let $\alpha \in c(A, X)$. This means that there is some $\delta \in A$ and some $D \subseteq X$ such that $\alpha \in D$, D is C-consistent, and either $D \in X \perp\!\!\!\perp_C \delta$ or $D \in X \perp\!\!\!\perp_C \neg\delta$. But then we have that $D \subseteq c(A, X)$ and since $c(A, X) \subseteq X$, $D \in c(A, X) \perp\!\!\!\perp_C \delta$ or $D \in c(A, X) \perp\!\!\!\perp_C \neg\delta$ and hence $\alpha \in D \subseteq c(A, c(A, X))$.

(2) It follows directly from definition 2.3 that it is sufficient to show that, for any sentence δ, if $B \subseteq B'$ then $c(\delta, B) \subseteq c(\delta, B')$. Let $\beta \in c(\delta, B)$. It follows from definition 2.3 that there is some X such that $\beta \in X$, X is C-consistent, and either $X \in B \perp\!\!\!\perp_C \delta$ or $X \in B \perp\!\!\!\perp_C \neg\delta$. It follows from $X \in B \perp\!\!\!\perp_C \delta$ and $B \subseteq B'$ that $X \in B' \perp\!\!\!\perp_C \delta$, and similarly from $X \in B \perp\!\!\!\perp_C \neg\delta$ and $B \subseteq B'$ that $X \in B' \perp\!\!\!\perp_C \neg\delta$. We can conclude from this that $\beta \in c(\delta, B')$. $\square$

Proof of Observation 2.6:

1. Let $\beta \in C_\alpha(B)$. Then $\beta \in C(c(\alpha, B))$ and from Definition 2.3 it follows that $\beta \in C(c(\neg\alpha, B))$. Hence, $\beta \in C_{\neg\alpha}(B)$.

2. Let $\perp \notin C(B)$.

   - Let $\alpha \in A$. If $\alpha \in C_A(B)$, then $\alpha \in C(c(A, B))$ and since $c(A, B) \subseteq B$, it follows from monotony that $\alpha \in C(B)$. For the other side of the implication, let $\alpha \in C(B)$. Then it follows from compactness that there is an inclusion-minimal subset X of B such that $\alpha \in C(X)$. Since $\perp \notin C(B)$, by monotony, $\perp \notin C(X)$. Hence, $X \in B \perp\!\!\!\perp_C \alpha$ and $X \notin B \perp\!\!\!\perp_C \perp$, i.e., $X \subseteq c(A, B)$ and from monotony it follows that $\alpha \in C(c(A, B)) = C_A(B)$.

   - From $\alpha \in C_B(B) = C(c(B, B))$ it follows by monotony that $\alpha \in C(B)$. For the other side of the implication, let $\alpha \in C(B)$. Then by compactness, there is some inclusion-minimal $X \subseteq B$ such that

$\alpha \in C(X)$. For every element δ of X it holds that $\delta \in c(\delta, B)$ and since $\delta \in X \subseteq B$, $\delta \in c(B, B)$. Hence, $X \subseteq c(B, B)$ and from monotony it follows that $\alpha \in C(c(B, B)) = C_B(B)$.

3. That $\alpha \in C_\top(B)$ if and only if $\alpha \in C_\perp(B)$ follows from part 1. Let $\alpha \in C_\top(B)$. Then $\alpha \in C(c(\top, B))$ and by Definition 2.3, $\alpha \in C(\emptyset)$. In the same way, if $\alpha \in C(\emptyset)$, then by Definition 2.3, $\alpha \in C(c(\top, B)) = C_\top(B)$.

□

Proof of Observation 2.7: Let $X \in B \perp\!\!\!\perp_{C_A} \alpha$. Then $\alpha \in C_A(X)$, that is, $\alpha \in C(c(A, X))$ and for no proper subset X' of X does it hold that $\alpha \in C_A(X')$. From part 1 of Observation 2.4, it follows that $\alpha \in C(c(A, c(A, X))) = C_A(c(A, X))$. By part 2 of Observation 2.4, since $X \subseteq B$, $c(A, X) \subseteq c(A, B)$. Suppose, for contradiction, that $X \not\subseteq c(A, B)$. Then, since $c(A, X) \subseteq c(A, B)$, $c(A, X)$ must be a proper subset of X, and since $\alpha \in C_A(c(A, X))$, this contradicts the minimality of X. Hence, $X = c(A, X) \subseteq c(A, B)$. □

Proof of Observation 2.8: By definition, $(C_A)_A(B) = C_A(c(A, B)) = C(c(A, c(A, B))) =$ (Observation 2.4, part 1) $C(c(A, B)) = C_A(B)$. □

Proof of Theorem 2.10:

Part 1: Let $\alpha \in C_A(B)$, i.e., $\alpha \in C(c(A, B))$. It follows from part (2) of observation 2.4 that $c(A, B) \subseteq c(A, D)$. Hence, since C is monotonic, $\alpha \in C(c(A, D))$, i.e., $\alpha \in C_A(D)$. Sen follows directly from monotony, since $C_A(B) \subseteq C_A(B \cup D)$ and $C_A(D) \subseteq C_A(B \cup D)$.

Part 2: Let C satisfy monotony and compactness. It follows from Part 1 that C_A satisfies monotony. For compactness, let $\alpha \in C_A(B)$, i.e., $\alpha \in C(c(A, B))$. From the compactness of C it follows that there is an inclusion-minimal finite set $D' \subseteq c(A, B) \subseteq B$ such that $\alpha \in C(D')$. Since $D' \subseteq c(A, B)$ and D' is finite, there is a consistent inclusion-minimal finite set $D \subseteq B$ such that for every $\varepsilon \in D'$ there is a finite set $X \subseteq D$ and there is an element δ of A such that (i) $X \vdash \delta$ or $X \vdash \neg\delta$, and (ii) $\varepsilon \in X$. By construction, $D' \subseteq D$ and moreover, $c(A, D) = D$. From this and $\alpha \in C(D')$ we conclude, since C is compact, that $\alpha \in C(D)$ and hence $\alpha \in C_A(D)$.

For weak iteration, let $\alpha \in C_A(C_A(B)) = C(c(A, C_A(B)))$. This means that there is a set $X \subseteq c(A, C_A(B))$ such that $\alpha \in C(X)$. Since $c(A, C_A(B)) \subseteq C_A(B)$, we have $X \subseteq C_A(B) = C(c(A, B))$. It follows from the compactness of C that there is some set $Y \subseteq c(A, B)$ such that $X \subseteq C(Y)$ and thus, by the monotony of C, $\alpha \in C(Y)$. Hence, $\alpha \in C_A(B)$ and $C_A(C_A(B)) \subseteq C_A(B)$.

Part 3: Monotony, compactness and weak iteration follow from part 2. For idempotency, it suffices to show that $C_A(B) \subseteq C_A(C_A(B))$. Let $\alpha \in C_A(B) = C(c(A, B))$. Then there is a set $X \subseteq c(A, B)$ such that $\alpha \in C(X)$. By part 1 of observation 2.4, $c(A, B) = c(A, c(A, B))$ and from part 2 of observation 2.4 together with the inclusion property of C it follows that $c(A, B) \subseteq c(A, C(c(A, B))) = c(A, C_A(B))$. Hence, $X \subseteq c(A, C_A(B))$ and thus $\alpha \in C(c(A, C_A(B))) = C_A(C_A(B))$. Cut follows directly from monotony and idempotence.

Part 4: Monotony follows from part 1. For consistency preservation, let $\perp \in C_A(B)$. By definition this means that $\perp \in C(c(A, B))$. Since $c(A, B) \subseteq B$ and C satisfies monotony, we have $\perp \in C(B)$. Hence, from the consistency preservation of C, $\perp \in Cn_0(B)$.

Part 5: Let C satisfy weak explosiveness and let $\perp \in C_A(B)$. This means that $\perp \in C(c(A, B))$. It follows from the weak explosiveness of C that for every sentence α, $\alpha \in C(c(A, B)) = C_A(B)$.

Part 6: Let C satisfy monotony and inclusion. It follows from Part 1 that C_A satisfies monotony. For embedded inclusion: by Part 1 of observation 2.4, $C_A(B) = C(c(A, B)) = C(c(A, c(A, B)))$. By the inclusion property of C, $c(A, B) \subseteq C(c(A, B))$ and from Part 2 of observation 2.4 it follows that $c(A, c(A, B)) \subseteq c(A, C(c(A, B)))$. Hence, by the monotony of C, $C_A(B) = C(c(A, c(A, B))) \subseteq C(A, C(c(A, B))) = C_A(C_A(B))$.

Part 7: Let C satisfy monotony, compactness, weak explosiveness and non-contravention. It follows from parts 2 and 5 that C_A satisfies monotony, compactness and weak explosiveness. For α-local non-contravention, let $\alpha \in A$. If $\perp \in C(B)$, then by weak explosiveness $\neg\alpha \in C(B)$ and α-local non-contravention holds trivially. Let $\perp \notin C(B)$ and $\neg\alpha \in C_A(B \cup \{\alpha\}) = C(c(A, B \cup \{\alpha\})$. Then by the monotony of C, $\neg\alpha \in C(B \cup \{\alpha\})$ and from the non-contravention of C it follows that $\neg\alpha \in C(B)$. Since $\perp \notin C(B)$ and $\alpha \in A$, it follows from part 2 of Observation 2.6 that $\neg\alpha \in C_A(B)$. □

Proof of Theorem 2.11: Let $p, q, r,$ and s be logically independent.

1. (Inclusion) Let $B = \{p, q\}$ and $A = \{p\}$. Then $q \notin Cn_A(B) =$

$Cn_0(\{p\})$.

2. (Supraclassicality) Let $B = \{p\}$, $A = \{q\}$. Then $p \in Cn_0(B)$ but $p \notin Cn_A(B)$.

3. (Deduction property) Let $B = \{r, r \to (p \to q)\}$ and $A = \{q\}$. Then $c(A, B \cup \{p\}) = B \cup \{p\}$ and hence $q \in Cn_A(B \cup \{p\})$. But $c(A, B) = \emptyset$, thus, $p \to q \notin Cn_A(B)$.

4. (Reductio ad absurdum) Let $B = \{\neg p \to r, \neg r\}$ and $A = \{r\}$. Then $c(A, B \cup \{\neg p\}) = B \cup \{\neg p\}$ and hence, $\perp \in Cn_A(B \cup \{\neg p\})$. On the other hand, $c(A, B) = \{\neg r\}$ from which it follows that $p \notin Cn_A(B) = Cn_0(\{\neg r\})$.

5. (Falsity) By definition $c(A, \perp) = \emptyset$, hence, $Cn_A(\{\perp\}) = Cn_0(\emptyset)$.

6. (Distributivity) Let $B = \{p, p \to q, q \to r\}$, $D = \{p, p \to s, s \to \neg r\}$ and $A = \{r\}$. Then we have $c(A, B) = B$, $c(A, D) = D$, and therefore, $p \in Cn_A(B) \cap Cn_A(D)$. On the other hand, $Cn_0(B) \cap Cn_0(D) = Cn_0(\{p \wedge (q \vee s)\})$. Hence, $c(A, Cn_0(Cn_0(B) \cap Cn_0(D))) = \emptyset$, and $p \notin Cn_A(Cn_0(B) \cap Cn_0(D))$.

7. (Chernov) and

8. (Arrow) Let $B = \{p\}$, $D = \{p \to q\}$ and $A = \{q\}$. Then $c(A, B \cup D) = B \cup D$ and thus $q \in Cn_A(B \cup D)$, but $c(A, B) = \emptyset$, so that $q \notin Cn_0(Cn_A(B) \cup D) = Cn_0(D)$. Clearly, $Cn_0(Cn_A(B) \cup D) = Cn_0(D)$ is consistent.

9. (Explosiveness) Let $B = \{p, \neg p\}$ and $A = \{q\}$. Then $r \notin Cn_A(B) = Cn_0(c(q, \{p, \neg p\})) = Cn_0(\emptyset)$.

10. (Non-contravention) Let $B = \{p \to q, p \to \neg q\}$ and $A = \{q\}$. Then $\neg p \in Cn_A(B \cup \{\alpha\}) = Cn_0(B \cup \{\alpha\})$, but $\neg p \notin Cn_A(B) = Cn_0(\emptyset)$.

□

Proof of Theorem 3.3: (i) construction ⇒ postulates: Let $\dot{-}_{C,\sigma}$ be an operation of kernel contraction defined by an inference operator C and an incision function σ. By the definition of kernel contraction, $\dot{-}_{C,\sigma}$ satisfies *inclusion*. In order to prove that *success* is satisfied, suppose to the contrary that $\alpha \in C(B \dot{-}_{C,\sigma} \alpha)$ for some α such that $\alpha \notin C(\emptyset)$. It follows from monotony and compactness that there must be some $X \in B \perp\!\!\!\perp_C \alpha$ such that $X \subseteq B \dot{-}_{C,\sigma} \alpha$. From $\alpha \notin C(\emptyset)$ it follows that $X \neq \emptyset$ and by the definition of σ, $\sigma(B \perp\!\!\!\perp_C \alpha) \cap X \neq \emptyset$. Then, by the construction, $X \nsubseteq B \dot{-}_{C,\sigma} \alpha$.

For *core-retainment*, let β be an element of $B \setminus B \dot{-}_{C,\sigma} \alpha$. By the construction of $B \dot{-}_{C,\sigma} \alpha$, $\beta \in \sigma(B \perp\!\!\!\perp_C \alpha)$. Since $\sigma(B \perp\!\!\!\perp_C \alpha) \subseteq \bigcup(B \perp\!\!\!\perp_C \alpha)$, there is some set $X \in B \perp\!\!\!\perp_C \alpha$ such that $\beta \in X$. By the definition of kernel sets, $X \subseteq B$, $\alpha \in C(X)$ and for all $Y \subset X$, $\alpha \notin C(Y)$. Let B' be $X \setminus \{\beta\}$.

For *uniformity*, suppose that for all subsets B' of B it holds that $\alpha \in C(B')$ if and only if $\beta \in C(B')$. Then $B \perp\!\!\!\perp_C \alpha = B \perp\!\!\!\perp_C \beta$ and since σ is a function, $\sigma(B \perp\!\!\!\perp_C \alpha) = \sigma(B \perp\!\!\!\perp_C \beta)$. By the definition of kernel contraction, it follows that $B \dot{-}_{C,\sigma} \alpha = B \dot{-}_{C,\sigma} \beta$.

(ii) postulates $\Rightarrow$ construction: Let $\dot{-}$ be an operation that satisfies *inclusion*, *success*, *core-retainment* and *uniformity* and let B be a set of sentences. Let σ be such that for all sentences α, $\sigma(B \perp\!\!\!\perp_C \alpha) = B \setminus B \dot{-} \alpha$. We have to show (a) that $\dot{-}$ is based on σ in the manner of Definition 3.2, (b) that σ is well-defined, i.e., that it is a function for the given domain, and (c) that it is an incision function for the given domain.

(a) It follows from *inclusion* and the definition of σ that $B \dot{-} \alpha = B \setminus \sigma(B \perp\!\!\!\perp_C \alpha)$.

(b) To see that σ is a function, let α and β be such that $B \perp\!\!\!\perp_C \alpha = B \perp\!\!\!\perp_C \beta$. We have to show that $\sigma(B \perp\!\!\!\perp_C \alpha) = \sigma(B \perp\!\!\!\perp_C \beta)$. From $B \perp\!\!\!\perp_C \alpha = B \perp\!\!\!\perp_C \beta$ it follows due to the monotony of C that for all subsets B' of B, $\alpha \in C(B')$ if and only if $\beta \in C(B')$. By *uniformity*, $B \dot{-} \alpha = B \dot{-} \beta$ and by the definition of σ and *inclusion*, $\sigma(B \perp\!\!\!\perp_C \alpha) = \sigma(B \perp\!\!\!\perp_C \beta)$.

(c) To show that σ is an incision function we need to prove that the two conditions of Definition 3.1 are satisfied. To prove the first of them, let $x \in \sigma(B \perp\!\!\!\perp_C \alpha)$. Then $x \in B \setminus B \dot{-} \alpha$ and it follows from *core-retainment* that there is a set $B' \subseteq B$ such that $\alpha \notin C(B')$ and $\alpha \in C(B' \cup \{x\})$. Due to monotony and compactness, there is a set $X \subseteq B' \cup \{x\}$ such that $X \in B \perp\!\!\!\perp_C \alpha$ and $x \in X$. We can conclude that $x \in \bigcup(B \perp\!\!\!\perp_C \alpha)$. For the second condition of Definition 3.1, let $\emptyset \neq X \in B \perp\!\!\!\perp_C \alpha$ and suppose that $X \cap \sigma(B \perp\!\!\!\perp_C \alpha) = \emptyset$. We know from part (a) that $B \dot{-} \alpha = B \setminus \sigma(B \perp\!\!\!\perp_C \alpha)$. Since $X \subseteq B$, we can conclude that $X \subseteq B \dot{-} \alpha$. Since $\alpha \in C(X)$, monotony yields $\alpha \in C(B \dot{-} \alpha)$. This contradicts *success*, and we may conclude that

$X \cap \sigma(B \perp\!\!\!\perp_C \alpha) \neq \emptyset$. $\square$

Proof of Theorem 3.8: (i) construction $\Rightarrow$ postulates: Let $\dot{-}_{C,g}$ be a partial meet contraction operator based on an operation C and a selection function g. We need to show that it satisfies the four postulates.

Inclusion follows directly from the definition of partial meet contraction.

Success: Let $\alpha \notin C(\emptyset)$. Since C satisfies monotony and compactness, it follows from the upper bound property (Observation 3.5) that $B \perp_C \alpha$ is non-empty. Due to Definition 3.6, so is $g(B \perp_C \alpha)$, and due to Definition 3.7, there is at least one set X such that $B \dot{-}_{C,g} \alpha \subseteq X \in g(B \perp_C \alpha)$. Since $\alpha \notin C(X)$ and C satisfies monotony, we can conclude from $B \dot{-}_{C,g} \alpha \subseteq X$ that $\alpha \notin C(B \dot{-}_{C,g} \alpha)$.

Relevance: Let $\beta \in B \setminus (B \dot{-}_{C,g} \alpha)$. Then, according to Definition 3.7, there is some B' such that $\beta \notin B' \in g(B \perp_C \alpha)$. It follows that $B \dot{-}_{C,g} \alpha \subseteq B' \subseteq B$, $\alpha \notin C(B')$ and $\alpha \in C(B' \cup \{\beta\})$.

Uniformity: Suppose that for all subsets B' of B, $\alpha \in C(B')$ if and only if $\beta \in C(B')$. It is easy to see that $B \perp_C \alpha = B \perp_C \beta$. Since g is a function, it follows that $g(B \perp_C \alpha) = g(B \perp_C \beta)$ and by the definition of partial meet contraction, $B \dot{-}_{C,g} \alpha = B \dot{-}_{C,g} \beta$.

(ii) postulates $\Rightarrow$ construction: Let $\dot{-}$ be an operation that satisfies the four postulates, and let B be a set of sentences. Let g be a function such that for all sentences α: $g(B \perp_C \alpha) = \{X \in B \perp_C \alpha | B \dot{-} \alpha \subseteq X\}$ (that g is a function for the given domain follows from *uniformity*). We have to prove (1) that g is a selection function for the given domain in the sense of the Definition 3.6 and (2) that $B \dot{-} \alpha = \bigcap g(B \perp_C \alpha)$.

(1) We can do this by showing that $g(B \perp_C \alpha)$ is non-empty whenever $B \perp_C \alpha \neq \emptyset$. Suppose $B \perp_C \alpha \neq \emptyset$. It follows form monotony that $\alpha \notin C(\emptyset)$, hence from *success* that $\alpha \notin C(B \dot{-} \alpha)$. It follows from *inclusion* and the upper bound property that there is some X such that $B \dot{-} \alpha \subseteq X \in B \perp_C \alpha$. It follows from our definition of g that $X \in g(B \perp_C \alpha)$, so that $g(B \perp_C \alpha)$ is non-empty.

(2) To prove that $B \dot{-} \alpha = \bigcap g(B \perp_C \alpha)$, we first observe that by the definition of g, $B \dot{-} \alpha \subseteq \bigcap g(B \perp_C \alpha)$. It remains to be shown that $\bigcap g(B \perp_C \alpha) \subseteq B \dot{-} \alpha$. In order to do this, we will let $\beta \notin B \dot{-} \alpha$. We have to prove that $\beta \notin \bigcap g(B \perp_C \alpha)$. Since this is trivial when $\beta \notin B$, we may assume that $\beta \in B$.

From $\beta \in B \setminus B \dot{-} \alpha$ it follows by *relevance* that there is some B' such that $B \dot{-} \alpha \subseteq B' \subseteq B$, $\alpha \notin C(B')$ and $\alpha \in C(B' \cup \{\beta\})$. It follows from the upper bound property that there is some X such that $B' \subseteq X \in B \perp_C \alpha$ and from monotony that $\beta \notin X$. Then by the definition, $X \in g(B \perp_C \alpha)$, from which follows that $\beta \notin \bigcap g(B \perp_C \alpha)$. □

Proof of Theorem 3.9:

1. Follows directly from theorems 3.3 and 3.8 since the postulates that characterize an operator of partial meet contraction imply the postulates that characterize an operator of kernel contraction.

2. Let $B = \{p, q, p \vee q \rightarrow r\}$ and let the inference operation be the operation of classical consequence Cn_0. We have that $B \perp_{Cn_0} r = \{\{p, q\}, \{p \vee q \rightarrow r\}\}$, and thus, there are only three possible values for $g(B \perp_{Cn_0} r)$: $\{\{p, q\}\}$, that would give $B \dot{-}_{Cn_0, g} r = \{p, q\}$; $\{\{p \vee q \rightarrow r\}\}$, that would give $B \dot{-}_{Cn_0, g} r = \{p \vee q \rightarrow r\}$, and $\{\{p, q\}, \{p \vee q \rightarrow r\}\}$, that would give $B \dot{-}_{Cn_0, g} r = \emptyset$. On the other hand, since $B \perp\!\!\!\perp_{Cn_0} r = \{\{p, p \vee q \rightarrow r\}, \{q, p \vee q \rightarrow r\}\}$, we may have $\sigma(B \perp\!\!\!\perp_{Cn_0} r) = \{q, p \vee q \rightarrow r\}$, and so $B \dot{-}_{Cn_0, \sigma} r = \{p\}$. □

Proof of Theorem 3.11: (i) construction $\Rightarrow$ postulates: Let $!_{C,\sigma}$ be a kernel consolidation operator based on an operation C that satisfies the conditions given and an incision function σ. We need to show that it satisfies the three postulates.

Inclusion follows directly from the definition of kernel consolidation.

Consistency: Suppose to the contrary that $\perp \in C(B!_{C,\sigma})$. Then according to the compactness of C there is some finite subset X of $B!_{C,\sigma}$ such that $\perp \in C(X)$, hence there is some inclusion-minimal subset Z of X such that $\perp \in C(Z)$, i.e., $Z \in B \perp\!\!\!\perp_C \perp$. Due to $\perp \notin C(\emptyset)$, $Z \neq \emptyset$. Therefore, according to Definitions 3.1 and 3.10, Z cannot be a subset of $B!_{C,\sigma}$. This contradiction shows that consistency holds.

Core-retainment: Let $\alpha \in B$ and $\alpha \notin B!_{C,\sigma}$. Then $\alpha \in \sigma(B \perp\!\!\!\perp_C \perp)$. According to Definition 3.1, $\sigma(B \perp\!\!\!\perp_C \perp) \subseteq \bigcup(B \perp\!\!\!\perp_C \perp)$, hence there is some set D such that $\alpha \in D \in B \perp\!\!\!\perp_C \perp$. Let $X = D \setminus \{\alpha\}$. Then $X \subseteq B$, $\perp \notin C(X)$ and $\perp \in C(X \cup \{\alpha\})$, which shows that core-retainment is satisfied.

(ii) postulates $\Rightarrow$ construction: Let ! be an operation that satisfies the three postulates of the theorem. We are going to show that ! is a kernel consolidation based on C and some incision function σ. For that purpose, let B be a set of sentences and let σ be such that: $\sigma(B \perp\!\!\!\perp_C \bot) = B \setminus (B!)$.

We need to verify (1) that σ is an incision function (for the domain covered by the definition), and (2) that the kernel contraction based on C and σ coincides with !.

(1) Clearly, σ is a function. We need to show that it satisfies conditions 1 and 2 of Definition 3.1.

For condition 1, we are going to show that $\sigma(B \perp\!\!\!\perp_C \bot) \subseteq \bigcup(B \perp\!\!\!\perp_C \bot)$. Let $\alpha \in \sigma(B \perp\!\!\!\perp_C \bot)$. It follows from *core-retainment* that there is some X such that $X \subseteq B$, $\bot \notin C(X)$ and $\bot \in C(X \cup \{\alpha\})$. Let X' be an inclusion-minimal subset of X such that $\bot \in C(X' \cup \{\alpha\})$. According to monotony, since $X' \subseteq X$ and $\bot \notin C(X)$ we also have $\bot \notin C(X')$. It follows that $X' \cup \{\alpha\} \in B \perp\!\!\!\perp_C \bot$, hence $\alpha \in \bigcup(B \perp\!\!\!\perp_C \bot)$.

For condition 2, let $\emptyset \neq X \in B \perp\!\!\!\perp_C \bot$. We need to show that $X \cap \sigma(B \perp\!\!\!\perp_C \bot)$ is non-empty. By *consistency*, $\bot \notin C(B!)$. Since $\bot \in C(X)$ we may conclude from the monotony of C that $X \nsubseteq B!$, i.e., that there is some ε such that $\varepsilon \in X$ and $\varepsilon \notin B!$. Since $X \subseteq B$ it follows that $\varepsilon \in B \setminus (B!)$, i.e., $\varepsilon \in \sigma(B \perp\!\!\!\perp_C \bot)$. Thus, $\varepsilon \in X \cap \sigma(B \perp\!\!\!\perp_C \bot)$ which is sufficient to show that condition 2 is satisfied.

(2) It follows from *inclusion* and our definition $\sigma(B \perp\!\!\!\perp_C \bot) = B \setminus (B!)$ that $B! = B \setminus \sigma(B \perp\!\!\!\perp_C \bot)$. □

Proof of Theorem 3.13: (i) construction $\Rightarrow$ postulates: Let $!_{C,g}$ be a partial meet consolidation operator based on an operation C that satisfies the conditions given and a selection function g. We need to show that it satisfies the three postulates.

Inclusion follows directly from the Definitions 3.4 and 3.12.

Consistency: Due to $\bot \notin C(\emptyset)$, monotony, compactness, and the upper bound property, $B \bot_C \bot$ is non-empty (Observation 3.5). Due to Definition 3.6, so is $g(B \bot_C \bot)$, and due to Definition 3.12, there is some set X such that $B!_{C,g} \subseteq X \in B \bot_C \bot$. Since $\bot \notin C(X)$ and C satisfies monotony, we can conclude from $B!_{C,g} \subseteq X$ that $\bot \notin C(B!_{C,g})$.

Relevance: Let $\alpha \in B \setminus (B!_{C,g})$. Then, according to Definition 3.12, there is some X such that $\alpha \notin X \in g(B \bot_C \bot)$. It follows that $B!_{C,g} \subseteq X \subseteq B$, $\bot \notin C(X)$ and $\bot \in C(X \cup \{\alpha\})$.

(ii) postulates $\Rightarrow$ construction: Let ! be an operation that satisfies the three postulates, and let B be a set of sentences. Let g be a function such that: $g(B\perp_C\perp) = \{X \in B\perp_C\perp | B! \subseteq X\}$. We have to prove (1) that g is a selection function for the given domain in the sense of Definition 3.6 and (2) that $B! = \bigcap g(B\perp_C\perp)$.

(1) We can do this by showing that $g(B\perp_C\perp)$ is non-empty whenever $B\perp_C\perp$ is non-empty. It follows from *consistency* and *inclusion*, using the upper bound property, that there is some X such that $B! \subseteq X \in B\perp_C\perp$. It follows from the definition of g that $X \in g(B\perp_C\perp)$, so that $g(B\perp_C\perp)$ is non-empty.

(2) To prove that $B! = \bigcap g(B\perp_C\perp)$, we first observe that by the definition of g, $B! \subseteq \bigcap g(B\perp_C\perp)$. It remains to be shown that $\bigcap g(B\perp_C\perp) \subseteq B!$. In order to do this, we will assume that $\beta \notin B!$ and prove that $\beta \notin \bigcap g(B\perp_C\perp)$. Since this is trivial whenever $\beta \notin B$, only the case when $\beta \in B \setminus B!$ needs to be proved.

From $\beta \in B \setminus B!$ follows by *relevance* that there is some X such that $B! \subseteq X \subseteq B$, $\perp \notin C(X)$ and $\perp \in C(X \cup \{\beta\})$. It follows from the upper bound property that there is some X' such that $X \subseteq X' \in B\perp_C\perp$ and from monotony that $\beta \notin X'$. Then by the definition, $X' \in g(B\perp_C\perp)$, from which follows that $\beta \notin \bigcap g(B\perp_C\perp)$. □

Proof of Theorem 3.15: (i) construction ⇒ postulates:

Let $\mp_{C,\sigma}$ be an internal kernel revision operator based on an inference C and an incision function σ. We have to show that it satisfies the five postulates.

Inclusion and *success* follow trivially from the construction.

Non-contradiction: Let $\neg\alpha \notin C(\emptyset)$. Then it follows from the proof of contraction-success in Theorem 3.3 that $\neg\alpha \notin C(B \setminus \sigma(B \perp\!\!\!\perp_C \neg\alpha))$. From this and α-local non-contravention follows $\neg\alpha \notin C(B \mp_{C,\sigma} \alpha)$.

Core-retainment: If $\neg\alpha \in C(\emptyset)$, then $B \perp\!\!\!\perp_C \neg\alpha = \emptyset$, $B \subseteq B \mp_{C,\sigma} \alpha$ and the postulate holds trivially. Otherwise, let $\beta \in B \setminus (B \mp_{C,\sigma} \alpha)$. By the construction of $B \mp_{C,\sigma} \alpha$, there must be $X \in B \perp\!\!\!\perp_C \neg\alpha$ such that $\beta \in X$. Let $B' = X \setminus \{\beta\}$. Then $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\beta\})$.

Uniformity: Suppose that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. Then $B \perp\!\!\!\perp_C \neg\alpha = B \perp\!\!\!\perp_C \neg\beta$ and it follows that $\sigma(B \mp_{C,\sigma} \neg\alpha) = \sigma(B \mp_{C,\sigma} \neg\beta)$ and thus $B \cap (B \mp_{C,\sigma} \alpha) = B \cap (B \mp_{C,\sigma} \beta)$.

(ii) postulates ⇒ construction:

Let $\mp$ be an operator that satisfies the given conditions. Let B be a set of sentences and let σ be a function such that for every sentence α,

$\sigma(B \perp\!\!\!\perp_C \neg\alpha) = B \setminus (B \cap (B \mp \alpha))$. We have to proof that (a) σ is a function, (b) σ is an incision function, and (c) $\mp = \mp_{C,\sigma}$.

(a) Let α and β be such that $B \perp\!\!\!\perp_C \neg\alpha = B \perp\!\!\!\perp_C \neg\beta$. It follows from monotony that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. By *uniformity*, $B \cap (B \mp \alpha) = B \cap (B \mp \beta)$. Thus, $\sigma(B \perp\!\!\!\perp_C \neg\alpha) = \sigma(B \perp\!\!\!\perp_C \neg\beta)$.

(b) Let $\delta \in \sigma(B \perp\!\!\!\perp_C \neg\alpha)$. Then, $\delta \in B \setminus (B \cap (B \mp \alpha)) = B \setminus (B \mp \alpha)$. By *core-retainment*, there is $B' \subseteq B$ such that $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\delta\})$. Then there is some inclusion-minimal subset Y of B' such that $\neg\alpha \in C(Y \cup \{\delta\})$. Let $X = Y \cup \{\delta\}$. It then follows from the non-emptiness of $\sigma(B \perp\!\!\!\perp_C \neg\alpha)$ that $\neg\alpha \notin C(\emptyset)$ and that no proper subset of X C-implies $\neg\alpha$. Hence, $\delta \in X \in B \perp\!\!\!\perp_C \neg\alpha$ and so $\delta \in \bigcup(B \perp\!\!\!\perp_C \neg\alpha)$.

Next, let $\emptyset \neq X \in B \perp\!\!\!\perp_C \neg\alpha$. Then $\neg\alpha \notin C(\emptyset)$, and $\neg\alpha \in C(X)$. Suppose that $X \subseteq B \mp \alpha$. Then it follows from monotony that $\neg\alpha \in C(B \mp \alpha)$, contrary to *non-contradiction*. We may conclude that $X \not\subseteq B \mp \alpha$. Then, by *inclusion*, $X \not\subseteq B \cap B \mp \alpha$. Hence, there is some $\delta \in X$ such that $\delta \in B \setminus (B \cap B \mp \alpha) = \sigma(B \perp\!\!\!\perp_C \neg\alpha)$ and we have $X \cap \sigma(B \perp\!\!\!\perp_C \neg\alpha) \neq \emptyset$.

(c) $B \mp_{C,\sigma} \alpha = (B \setminus \sigma(B \perp\!\!\!\perp_C \neg\alpha)) \cup \{\alpha\} = (B \setminus (B \setminus B \mp \alpha)) \cup \{\alpha\} =$ (by *inclusion*) $(B \mp \alpha) \cup \{\alpha\} =$ (by *success*) $B \mp \alpha$. □

Proof of Theorem 3.17: (i) construction ⇒ postulates:

Let $\mp_{C,g}$ be an internal partial meet revision operator based on an inference C and a selection function g. We have to show that it satisfies the five postulates.

That *inclusion* and *success* hold follows directly from the construction of $\mp_{C,g}$.

Non-contradiction: There are two cases: (1) $B \perp_C \neg\alpha = \emptyset$. Then $\neg\alpha \in C(\emptyset)$, so that the postulate holds vacuously. (2) $B \perp_C \neg\alpha \neq \emptyset$. Then $g(B \perp_C \neg\alpha) \neq \emptyset$, and it follows by monotony that $\neg\alpha \notin C(\bigcap g(B \perp_C \neg\alpha))$. It follows by α-local non-contravention that $\neg\alpha \notin C(\bigcap g(B \perp_C \neg\alpha) \cup \{\alpha\}) = B \mp_{C,g} \alpha$.

Relevance: Let $\beta \in B \setminus B \mp_{C,g} \alpha$. Then $\beta \notin \bigcap g(B \perp_C \neg\alpha)$, and there is a set X such that $X \in g(B \perp_C \neg\alpha)$ and $\beta \notin X$. Let $B' = X \cup \{\alpha\}$. Then $B \mp_{C,g} \alpha \subseteq B' \subseteq B \cup \{\alpha\}$. It follows from $\neg\alpha \notin C(X)$, due to non-contravention, that $\neg\alpha \notin C(X \cup \{\alpha\})$, i.e., $\neg\alpha \notin C(B')$. Furthermore, it follows from $\neg\alpha \in C(X \cup \{\beta\})$ by monotony that $\neg\alpha \in C(X \cup \{\alpha\} \cup \{\beta\})$, i.e., $\neg\alpha \in C(B' \cup \{\beta\})$.

Uniformity: Suppose that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. Then, $B \perp_C \neg\alpha = B \perp_C \neg\beta$ and $g(B \perp_C \neg\alpha) = g(B \perp_C \neg\beta)$, which means that $B \cap (B \mp_{C,g} \alpha) = B \cap (B \mp_{C,g} \beta)$

postulates $\Rightarrow$ construction:

Let $\mp$ be an operator satisfying the postulates above. Let B be a set of sentences and let g be a function such that for every sentence α, $g(B\perp_C\alpha) = \{X \in B\perp_C\alpha | B \cap (B \mp \neg\alpha) \subseteq X\}$.

We have to show that: (a) g is a function for the given domain, (b) g is a selection function for the given domain, and (c) $B \mp_{C,g} \alpha = B \mp \alpha$.

(a) Let α and β be such that $B\perp_C\alpha = B\perp_C\beta$. Then, due to monotony, for all subsets B' of B, $\alpha \in C(B')$ if and only if $\beta \in C(B')$. By *uniformity*, $B \cap (B \mp \neg\alpha) = B \cap (B \mp \neg\beta)$ and hence, $g(B\perp_C\alpha) = g(B\perp_C\beta)$.

(b) It follows directly from our definition of g that it satisfies condition 1 of Definition 3.6. To see that it satisfies condition 2, let $B\perp_C\alpha \neq \emptyset$. Then $\alpha \notin C(\emptyset)$. It follows from *non-contradiction* that $\alpha \notin C(B \mp \neg\alpha)$. It follows from monotony that $\alpha \notin C(B \cap (B \mp \neg\alpha))$. Thus, by the upper bound property, there is an $X \in B\perp_C\alpha$ such that $B \cap (B \mp \neg\alpha) \subseteq X$. By the definition of g, this means that $g(B\perp_C\alpha)$ is nonempty.

(c) By the definition of g and *inclusion*, $B \mp \alpha \subseteq (B \cap B \mp \alpha) \cup \{\alpha\} \subseteq \bigcap g(B\perp_C\neg\alpha) \cup \{\alpha\} = B \mp_{C,g} \alpha$. For the other side of the inclusion, let $\beta \notin B \mp \alpha$. If $\beta \notin B$, then $\beta \notin B \mp_{C,g} \alpha$ follows directly. We can assume therefore that $\beta \in B$. It follows from *relevance* that there is some B' such that $B \mp \alpha \subseteq B' \subseteq B \cup \{\alpha\}$, $\neg\alpha \notin C(B')$, and $\neg\alpha \in C(B' \cup \{\beta\})$. Since $\alpha \in B'$, it follows by α-local non-contravention from $\neg\alpha \notin C(B')$ that $\neg\alpha \notin C(B' \setminus \{\alpha\})$. It follows from Observation 3.5 that there is some $X \in B\perp_C\neg\alpha$ such that $(B' \setminus \{\alpha\}) \subseteq X$. Hence, $B \cap (B \mp \alpha) \subseteq (B' \setminus \{\alpha\}) \subseteq X$, from which follows that $X \in g(B\perp_C\neg\alpha)$.

It follows from $\neg\alpha \in C(B' \cup \{\beta\})$ and non-contravention that $\neg\alpha \in C((B' \setminus \{\alpha\}) \cup \{\beta\})$, hence by monotony $\neg\alpha \in C(X \cup \{\beta\})$. Since $\neg\alpha \notin C(X)$, it follows that $\beta \notin X$. Hence, $\beta \notin \bigcap g(B\perp_C\neg\alpha)$. It follows that $\beta \notin B \mp_{C,g} \alpha$. $\square$

Proof of Theorem 3.19: (i) construction $\Rightarrow$ postulates:

Let $\pm_{C,\sigma}$ be an external kernel revision operator based on an inference C and an incision function σ. We have to show that it satisfies the six postulates.

Inclusion and *pre-expansion* follow directly from the construction of $\pm_{C,\sigma}$.

Non-contradiction: Let $\neg\alpha \notin C(\emptyset)$. Then it follows from the proof of contraction-success in theorem 3.3 that $\neg\alpha \notin C((B \cup \{\alpha\}) \setminus \sigma((B \cup$

$\{\alpha\}) \perp\!\!\!\perp_C \neg\alpha))$, i.e., $\neg\alpha \notin C(B \pm_{C,\sigma} \alpha)$.

Core-retainment: Let $\beta \in B \setminus B \pm_{C,\sigma} \alpha$. By the construction of $\pm_{C,\sigma}$, we have $\beta \in \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \neg\alpha)$. Hence there is some X such that $\beta \in X \in ((B \cup \{\alpha\}) \perp\!\!\!\perp_C \neg\alpha)$. Let $B' = X \setminus \{\beta\}$. Then $B' \subseteq B \cup \{\alpha\}$, $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\beta\})$.

Success: It follows from α-local non-contravention that for all $X \in (B \cup \{\alpha\}) \perp\!\!\!\perp_C \neg\alpha$, $\alpha \notin X$. Thus, $\alpha \notin \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \neg\alpha)$, and hence $\alpha \in B \pm_{C,\sigma} \alpha$.

Weak-uniformity: Let α and β be elements of B such that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. Then $B \perp\!\!\!\perp_C \neg\alpha = B \perp\!\!\!\perp_C \neg\beta$ and hence, $\sigma(B \perp\!\!\!\perp_C \neg\alpha) = \sigma(B \perp\!\!\!\perp_C \neg\beta)$. Since $\alpha, \beta \in B$, this implies that $B \pm_{C,\sigma} \alpha = B \pm_{C,\sigma} \beta$.

(ii) postulates $\Rightarrow$ construction:

Let $\pm$ be an operator that satisfies the given conditions. Let σ be such that for all sets B and all sentences $\alpha \in B$, $\sigma(B \perp\!\!\!\perp_C \neg\alpha) = B \setminus (B \pm \alpha)$. We have to prove that (a) σ is a function for this domain, (b) σ is an incision function for this domain, and (c) $\pm = \pm_{C,\sigma}$.

(a) Let α and β be elements of B such that $B \perp\!\!\!\perp_C \neg\alpha = B \perp\!\!\!\perp_C \neg\beta$. It follows that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. By *weak uniformity*, $B \cap (B \pm \alpha) = B \cap (B \pm \beta)$. Thus, $\sigma(B \perp\!\!\!\perp_C \neg\alpha) = \sigma(B \perp\!\!\!\perp_C \neg\beta)$.

(b) Let $\delta \in \sigma(B \perp\!\!\!\perp_C \neg\alpha)$. Then, $\delta \in B \setminus (B \pm \alpha)$. By *core-retainment*, there is some $B' \subseteq B$ such that $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\delta\})$. Then due to monotony, there is a set $X \subseteq B \cup \{\delta\}$ such that $\delta \in X \in B \perp\!\!\!\perp_C \neg\alpha$ and so, $\delta \in \bigcup(B \perp\!\!\!\perp_C \neg\alpha)$.

Now, let $\emptyset \neq X \in B \perp\!\!\!\perp_C \neg\alpha$ for some $B \perp\!\!\!\perp_C \neg\alpha$ that is in the domain of σ as defined above. Then $\neg\alpha \notin C(\emptyset)$. Suppose that $X \subseteq B \pm \alpha$. Then it follows from monotony that $\neg\alpha \in C(B \pm \alpha)$, contrary to *non-contradiction*. Hence, $X \not\subseteq B \pm \alpha$. Since $\alpha \in B$, *inclusion* yields $B \pm \alpha \subseteq B$, hence $B \setminus (B \pm \alpha) \cap X = \sigma(B \perp\!\!\!\perp_C \neg\alpha) \cap X \neq \emptyset$.

(c) $B \pm_{C,\sigma} \alpha = (B \cup \{\alpha\}) \setminus \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp \neg\alpha) =$ (by the definition of σ above) $(B + \alpha) \setminus ((B + \alpha) \setminus ((B + \alpha) \pm \alpha)) =$ (by *inclusion*) $(B + \alpha) \pm \alpha =$ (by *pre-expansion*) $B \pm \alpha$. □

Proof of Theorem 3.21: (i) construction $\Rightarrow$ postulates:

Let $\pm_{C,g}$ be an external partial meet revision operator based on an inference C and a selection function g. We have to show that it satisfies the six postulates.

Inclusion and *pre-expansion* follow directly from the construction of $\pm_{C,g}$.

Non-contradiction: Let $\neg\alpha \notin C(\emptyset)$. Then $(B \cup \{\alpha\})\bot_C\neg\alpha \neq \emptyset$. It follows from Definition 3.16 that $B \pm_{C,g} \alpha = \bigcap g((B \cup \{\alpha\})\bot_C\neg\alpha)$. It follows from $\neg\alpha \notin C(\emptyset)$ that for all $X \in (B \cup \{\alpha\})\bot_C\neg\alpha$, $\neg\alpha \notin C(X)$, and from monotony that $\neg\alpha \notin B \pm_{C,g} \alpha$.

Relevance: Let $\beta \in B \setminus B \pm_{C,g} \alpha$. By the construction of $\pm_{C,g}$ there is some $X \in g(B \cup \{\alpha\}\bot_C\neg\alpha)$ such that $\beta \notin X$. Then $B \pm_{C,g} \alpha \subseteq X \subseteq B \cup \{\alpha\}$, $\neg\alpha \notin C(X)$ and $\neg\alpha \in C(X \cup \{\beta\})$.

Success: If $B \cup \{\alpha\}\bot_C\neg\alpha = \emptyset$, then success follows directly from the definition. Otherwise it follows from α-local non-contravention that each element of $(B \cup \{\alpha\})\bot_C\neg\alpha$ contains α, and then so does $B \pm_{C,g} \alpha$.

Weak-uniformity: Let α and β be elements of B such that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. Then $B\bot_C\neg\alpha = B\bot_C\neg\beta$ and hence, $g(B\bot_C\neg\alpha) = g(B\bot_C\neg\beta)$. Since $\alpha, \beta \in B$, this implies that $B \pm_{C,g} \alpha = B \pm_{C,g} \beta$.

(ii) postulates $\Rightarrow$ construction:

Let $\pm$ be an operator that satisfies the given conditions. Let g be such that for every set of sentences B if $\alpha \in B$, then $g(B\bot_C\neg\alpha) = \{X \in B\bot_C\neg\alpha | B \pm \alpha \subseteq X\}$. We have to prove that (a) g is a function in the given domain, (b) g is a selection function in the given domain, and (c) $\pm = \pm_{C,g}$.

(a) Let α and β be elements of B such that $B\bot_C\neg\alpha = B\bot_C\neg\beta$. It follows from monotony that for all $B' \subseteq B$, $\neg\alpha \in C(B')$ if and only if $\neg\beta \in C(B')$. By *weak uniformity*, $B \cap (B \pm \alpha) = B \cap (B \pm \beta)$. By *inclusion*, $B \pm \alpha \subseteq B \cup \{\alpha\} = B$ and $B \pm \beta \subseteq B \cup \{\beta\} = B$. Hence it follows from $B \cap (B \pm \alpha) = B \cap (B \pm \beta)$ that $B \pm \alpha = B \pm \beta$. By the construction of g, $g(B\bot_C\neg\alpha) = g(B\bot_C\neg\beta)$.

(b) Let $B\bot_C\neg\alpha \neq \emptyset$. It follows from monotony that $\neg\alpha \notin C(\emptyset)$. *Non-contradiction* yields $\neg\alpha \notin C(B \pm \alpha)$. Due to Observation 3.5 there is $X \in B\bot_C\neg\alpha$ such that $B \pm \alpha \subseteq X$. By the definition of g, $X \in g(B\bot_C\neg\alpha)$.

(c) There are two cases to be considered. First case, if $(B \cup \{\alpha\})\bot_C\neg\alpha = \emptyset$, then it follows directly from Definition 3.20 that $B \pm_{C,g} \alpha = B + \alpha$. *Inclusion* yields $B \pm \alpha \subseteq B + \alpha$, and *success* yields $\alpha \in B \pm \alpha$. It remains to show that $B \subseteq B \pm \alpha$. Suppose to the contrary that this is not the case. Then there is some $\beta \in B \setminus B \pm \alpha$, and due to *relevance*, there is some B' such that $B \pm \alpha \subseteq B' \subseteq B + \alpha$, $\neg\alpha \notin C(B')$ and $\neg\alpha \in C(B' \cup \{\beta\})$. However, it follows from $(B \cup \{\alpha\})\bot_C\neg\alpha = \emptyset$, due to Observation 3.5, that $\neg\alpha \in C(\emptyset)$, hence by monotony $\neg\alpha \in C(B')$. This contradiction concludes the proof.

Second case, if $(B \cup \{\alpha\})\perp_C\neg\alpha \neq \emptyset$, then, by Definition 3.20, $B \pm_{C,g} \alpha = \bigcap g((B \cup \{\alpha\})\perp_C\neg\alpha) = \bigcap\{X \in (B \cup \{\alpha\})\perp_C\neg\alpha | (B \cup \{\alpha\}) \pm \alpha \subseteq X\} =$ (by *pre-expansion*) $\bigcap\{X \in (B \cup \{\alpha\})\perp_C\neg\alpha | B \pm \alpha \subseteq X\}$. It follows set-theoretically that $B \pm \alpha \subseteq B \pm_{C,g} \alpha$.

For the other direction, let $\beta \in B \pm_{C,g} \alpha \setminus B \pm \alpha$. Then due to *pre-expansion*, $\beta \in B \cup \{\alpha\} \setminus (B \cup \{\alpha\}) \pm \alpha$. Due to *relevance*, there is some B' such that $(B \cup \{\alpha\}) \pm \alpha \subseteq B' \subseteq B \cup \{\alpha\}$, $\neg\alpha \notin C(B')$, and $\neg\alpha \in C(B' \cup \{\beta\})$. Hence, due to Observation 3.5, there is some $X \in g((B \cup \{\alpha\})\perp_C\neg\alpha)$ such that $\beta \notin B' \subseteq X$ and so, $\beta \notin \bigcap g((B \cup \{\alpha\})\perp_C\neg\alpha) = B \pm_{C,g} \alpha$. □

Proof of Theorem 3.23: (i) construction $\Rightarrow$ postulates: Let $?_{C,\sigma}$ be an operator of kernel semi-revision based on an inference operator C and an incision function σ. It follows directly from the construction that *inclusion*, *pre-expansion*, and *internal exchange* are satisfied. Due to monotony, so is *consistency*. Finally, for *core-retainment*, let $\beta \in B \setminus B?_{C,\sigma}\alpha$. Then by construction $\beta \in \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \perp)$. This means that for some set $X \in (B \cup \{\alpha\}) \perp\!\!\!\perp_C \perp$, $\beta \in X$. Let $B' = X \setminus \{\beta\}$. We have $B' \subseteq B \cup \{\alpha\}$, $\perp \notin C(B')$ and $\perp \in C(B' \cup \{\beta\})$.

(ii) postulates $\Rightarrow$ construction: Let ? be an operator satisfying the postulates above and let σ be such that for every set of sentences B:

$\sigma(B \perp\!\!\!\perp_C \perp) = B \setminus \{\beta | \beta \in B?\alpha \text{ for some } \alpha \in B\}$

We have to show (1) that σ is an incision function for the given domain and (2) that $B?\alpha = B?_{C,\sigma}\alpha$.

(1) To show that σ is an incision function we have to show that conditions 1 and 2 of Definition 3.1 are satisfied. For condition 1, let $\delta \in \sigma(B \perp\!\!\!\perp_C \perp)$. Then it holds for all $\alpha \in B$ that $\delta \notin B?\alpha$ hence, $\delta \notin B?\delta$. It follows from *core-retainment* that there is some $B' \subseteq B$ such that $\perp \notin C(B')$ and $\perp \in C(B' \cup \{\delta\})$. It follows that there is some subset B'' of B' such that $B'' \cup \{\delta\} \in B \perp\!\!\!\perp_C \perp$. For condition 2, let $\emptyset \neq X \in B \perp\!\!\!\perp_C \perp$. Suppose that $X \cap \sigma(B \perp\!\!\!\perp_C \perp) = \emptyset$. Then $X \subseteq \{\beta | \beta \in B?\alpha \text{ for some } \alpha \in B\}$. Let α be an element of B such that $\perp \notin C(\{\alpha\})$. It follows by *internal exchange* that $X \subseteq B?\alpha$. Since $\perp \in C(X)$, it follows from monotony that $\perp \in C(B?\alpha)$, contrary to *consistency*. This contradiction is sufficient to prove that $X \cap \sigma(B \perp\!\!\!\perp_C \perp) \neq \emptyset$.

(2) By definition, $\sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \perp) = (B \cup \{\alpha\}) \setminus \{\beta | \beta \in B \cup \{\alpha\}?\varepsilon$ for some $\varepsilon \in B \cup \{\alpha\}\} = (B \cup \{\alpha\}) \setminus (B \cup \{\alpha\})?\alpha$ (*internal exchange*) $= (B \cup$

$\{\alpha\}) \setminus B?\alpha$ (*pre-expansion*). Hence, $B?_{C,\sigma}\alpha = (B \cup \{\alpha\}) \setminus \sigma((B \cup \{\alpha\}) \perp\!\!\!\perp_C \perp)$ (Definition 3.22) $= (B \cup \{\alpha\}) \setminus ((B \cup \{\alpha\}) \setminus B?\alpha) = B?\alpha$ (*inclusion*). $\square$

Proof of Theorem 3.25: (i) construction $\Rightarrow$ postulates: Let $?_{C,g}$ be a partial meet revision operator based on an inference operator C and a selection function g. It follows directly from the construction that *inclusion*, *pre-expansion*, and *internal exchange* are satisfied. Due to monotony, so is *consistency*. It remains to show that $?_{C,g}$ satisfies *relevance*. Let $\beta \in B \setminus B?_{C,g}\alpha$. Then by construction there is an $X \in g((B \cup \{\alpha\})\perp_C\perp)$ such that $\beta \notin X$. Since $\beta \in B \cup \{\alpha\}$, it follows that $B?_{C,g}\alpha \subseteq X \subseteq B \cup \{\alpha\}$, $\perp \notin C(X)$ and $\perp \in C(X \cup \{\beta\})$.

(ii) postulates $\Rightarrow$ construction: Let ? be an operator satisfying the postulates above and let g be defined as follows for every set of sentences B:

$g(B\perp_C\perp) = \{X \in B\perp_C\perp | B?\alpha \subseteq X$ for some $\alpha \in A\}$

We have to show that (1) g is a selection function in the given domain, and (2) $B?\alpha = B?_{C,g}\alpha$.

(1) Due to Definition 3.6 it is sufficient to show that $g(B\perp_C\perp)$ is non-empty. It follows from *consistency* and Observation 3.5 that there is some $X \in B\perp_C\perp$ such that $B?\alpha \subseteq X$. By the construction of g, $X \in g(B\perp_C\perp)$.

(2) $B?_{C,g}\alpha = \bigcap g((B\cup\{\alpha\})\perp_C\perp) = \bigcap\{X \in (B\cup\{\alpha\}\perp_C\perp | (B+\alpha)?\beta \subseteq X$ for some $\beta \in A\}$ = *internal exchange* $\bigcap\{X \in (B\cup\{\alpha\}\perp_C\perp | (B+\alpha)?\alpha \subseteq X\}$ = (*pre-expansion*) $\bigcap\{X \in (B \cup \{\alpha\}\perp_C\perp | B?\alpha \subseteq X\}$. Hence, $B?\alpha \subseteq B?_{C,g}\alpha$.

To see that $B?_{C,g}\alpha \subseteq B?\alpha$, let $\beta \notin B?\alpha$. If $\beta \notin B + \alpha$, it is easy to see that $\beta \notin B?_{C,g}\alpha$. Suppose that $\beta \in B + \alpha$. From $\beta \notin B?\alpha$ it follows by *pre-expansion* that $\beta \notin (B + \alpha)?\alpha$. From $\beta \in B + \alpha \setminus (B + \alpha)?\alpha$ it follows by *relevance* and *pre-expansion* that there is some B' such that $B?\alpha \subseteq B' \subseteq B \cup \{\alpha\}$, $\perp \notin C(B')$ and $\perp \in C(B' \cup \{\beta\})$. It follows from Observation 3.5 that there is some X such that $B' \subseteq X \in (B + \alpha)\perp_C\perp$. But then, since $B?\alpha \subseteq X$, $X \in g((B + \alpha)\perp_C\perp)$. But since $\beta \notin X$, $\beta \notin (B + \alpha)?_{C,g}\alpha =$ (by *pre-expansion*) $B?_{C,g}\alpha$. $\square$

Proof of Theorem 3.26: Using the Levi identity, what we have to show is that $B\dot{-}\alpha = B \cap ((B\dot{-}\alpha) \cup \{\neg\alpha\})$. The left-to-right direction follows directly from inclusion. For the right-to-left direction, let $\beta \in B \cap ((B\dot{-}\alpha) \cup \{\neg\alpha\})$. There are two cases.

Case i: $\beta \in B \cap (B\dot{-}\alpha)$. In this case, $\beta \in B\dot{-}\alpha$ follows directly.

Case ii: $\beta \in B \cap \{\neg\alpha\}$. Then $\beta = \neg\alpha$ and $\neg\alpha \in B$. Suppose that $\neg\alpha \notin B \dot{-} \alpha$. It follows from core-retainment that there is some $B' \subseteq B$ such that $\alpha \notin C(B')$ and $\alpha \in C(B' \cup \{\neg\alpha\})$. This contradicts α-local non-contravention, and we can conclude that $\neg\alpha \in B \dot{-} \alpha$, i.e. $\beta \in B \dot{-} \alpha$. $\square$

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