CHARACTERIZING EXISTENCE OF A MEASURABLE CARDINAL VIA MODAL LOGIC

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Abstract. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic coincides with the modal logic of the Kripke frame isomorphic to the powerset of a two element set.

1. Introduction

Over the years there have been discovered several intriguing connections between set theory and modal logic. To name a few:

(1) There is an interesting connection between non-well-founded set theory and infinitary modal logic [1, 3, 2].
(2) The modal logic $\mathsf{S4.2}$ turns out to be the logic of forcing extensions of ZFC [16].
(3) The only existing proof that the modal logic $\mathsf{S4.1.2}$ is the logic of the Čech-Stone compactification $\beta\omega$ of the discrete space ω requires that each MAD family has cardinality 2^ω, a principle that is not provable in ZFC, and it remains an open problem whether this principle is necessary [8].

To these results we add the following. Let the diamond $\mathfrak{D} = (D, \leq)$ be the partially ordered Kripke frame shown in Figure 1. It is clear that $\mathfrak{D}$ is isomorphic to the powerset of a two element set. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic is the modal logic of $\mathfrak{D}$.

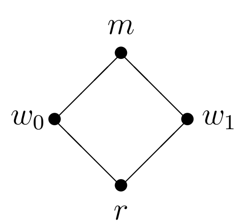

Figure 1. The Kripke frame $\mathfrak{D} = (D, \leq)$ where $D = \{r, w_0, w_1, m\}$.

We recall that topological semantics generalizes Kripke semantics for the well-known modal logic $\mathsf{S4}$. Thus, Kripke completeness implies topological completeness for logics above $\mathsf{S4}$. However, topological spaces arising from Kripke frames are usually not even T_1. Therefore, it is nontrivial to prove topological completeness results above $\mathsf{S4}$ with respect to spaces satisfying higher separation axioms. One such class is the class of Tychonoff spaces. By a celebrated theorem of Tychonoff, these are exactly subspaces of compact Hausdorff spaces. In [5] we initiated the study of modal logics arising from Tychonoff spaces. On the one hand, this yielded a new notion of dimension in topology, called modal Krull dimension. On the

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other hand, it provided a new concept of zemanian logics which generalize the well-known modal logic of Zeman.

It is known that extremally disconnected spaces are topological models of the modal logic S4.2, and hereditarily extremally disconnected spaces are topological models of the modal logic S4.3. In [6] we showed that a modal logic above S4.3 is a zemanian logic iff it is the logic of an hereditarily extremally disconnected Tychonoff space. The simplest modal logic above S4.2 that is not above S4.3 is the logic of $\mathfrak{D}$. In this paper we show that topological completeness of the logic of $\mathfrak{D}$ with respect to a normal space is equivalent to the existence of a measurable cardinal. Whether normal can be weakened to Tychonoff remains an open problem.

We conclude the introduction by briefly describing the key ingredients of the proof. If there exists a measurable cardinal κ, using a countably complete ultrafilter on κ, we first build a normal P-space Y. Combining the results of [12] and [13] then allows us to embed Y into the remainder of the Čech-Stone compactification $\beta\mu$ of a cardinal μ viewed as a discrete space. Letting $Z = Y \cup \mu$ yields a normal space whose logic we prove is the logic of the diamond $\mathfrak{D}$. This we do by showing that a finite rooted Kripke frame $\mathfrak{F}$ is an interior image of Z iff $\mathfrak{F}$ is an interior image of $\mathfrak{D}$.

Conversely, suppose there exists a normal space Z whose logic is the logic of the diamond $\mathfrak{D}$. We first show that $\mathfrak{D}$ is an interior image of Z. We then prove that without loss of generality the inverse image of the root r of $\mathfrak{D}$ is a singleton $\{a\}$. We next prove that a is a P-point of an appropriately chosen subspace of Z. This allows us to define a family of subsets of Z whose cardinal is Ulam-measurable. Finally, it is well known that this implies the existence of a measurable cardinal.

2. Preliminaries

In this section we recall the necessary background from modal logic, its topological semantics, and measurable cardinals.

2.1. **Modal logic.**

We use [10] as the main reference for modal logic. Modal formulas are built in the usual way using countably many propositional letters, the classical connectives $\neg$ (negation) and $\to$ (implication), the modal connective $\Box$ (necessity), and parentheses. We employ the standard abbreviations: $\wedge$ (conjunction), $\vee$ (disjunction), and $\Diamond$ (possibility).

The well-known modal system S4 of Lewis is the least set of formulas containing the classical tautologies, the axioms

$\Box(p \to q) \to (\Box p \to \Box q)$,
$\Box p \to p$,
$\Box p \to \Box\Box p$,

and closed under the inference rules of

Modus Ponens $\frac{\varphi,\ \varphi\to\psi}{\psi}$,
substitution $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$,
necessitation $\frac{\varphi}{\Box\varphi}$.

A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ where W is a nonempty set and R is a binary relation on W. As usual, for $w \in W$ we let

$$R(w) = \{v \in W \mid wRv\} \text{ and } R^{-1}(w) = \{v \in W \mid vRw\};$$

and for $A \subseteq W$ we let

$$R(A) = \bigcup\{R(w) \mid w \in A\} \text{ and } R^{-1}(A) = \bigcup\{R^{-1}(w) \mid w \in A\}.$$

Kripke semantics of modal logic recursively assigns to each formula a subset of a Kripke frame $\mathfrak{F}$ by interpreting each propositional letter as a subset of W, the classical connectives as Boolean operations in the powerset $\wp(W)$, and $\Box$ as the operation $\Box_R$ on $\wp(W)$ defined by

$$\Box_R(A) = \{w \in W \mid R(w) \subseteq A\}.$$

Consequently, $\Diamond$ is interpreted as the operation $\Diamond_R$ on $\wp(W)$ defined by

$$\Diamond_R(A) = R^{-1}(A).$$

Let φ be a modal formula and $\mathfrak{F} = (W, R)$ a Kripke frame. Call φ *valid* in $\mathfrak{F}$, written $\mathfrak{F} \models \varphi$, provided φ evaluates to W for every assignment of the propositional letters. If φ is not valid in $\mathfrak{F}$, then we say that φ is *refuted* in $\mathfrak{F}$, and write $\mathfrak{F} \not\models \varphi$. The *logic* of $\mathfrak{F}$ is the set of modal formulas valid in $\mathfrak{F}$; in symbols $\mathsf{L}(\mathfrak{F}) = \{\varphi \mid \mathfrak{F} \models \varphi\}$.

A Kripke frame $\mathfrak{F}$ is called an $\mathsf{S4}$-*frame* if R is reflexive and transitive. The name is justified by the well-known fact that $\mathsf{S4}$ is sound and complete with respect to $\mathsf{S4}$-frames. In this paper we are mainly interested in the following logic.

Definition 2.1. Let $\mathsf{L} := \mathsf{L}(\mathfrak{D})$ be the logic of the diamond $\mathfrak{D}$ shown in Figure 1.

2.2. **Topological semantics.** Topological semantics interprets $\Box$ as topological interior (and consequently $\Diamond$ as topological closure). Specifically, for a topological space X, the propositional letters are assigned to subsets of X, the classical connectives are computed as the Boolean operations in $\wp(X)$, and $\Box$ is interpreted as the interior operator $\mathsf{i} : \wp(X) \to \wp(X)$, where $\mathsf{i}A$ is the greatest open subset of X contained in A. Consequently, $\Diamond$ is interpreted as the closure operator $\mathsf{c} : \wp(X) \to \wp(X)$, where $\mathsf{c}A$ is the least closed subset of X containing A.

Let φ be a modal formula and X a space. Call φ *valid* in X, denoted $X \models \varphi$, provided φ evaluates to X for every assignment of the propositional letters. If φ is not valid in X, then we say that φ is *refuted* in X, and write $X \not\models \varphi$. The *logic* of X is the set of formulas valid in X; symbolically, $\mathsf{L}(X) = \{\varphi \mid X \models \varphi\}$. It is well known that $\mathsf{S4}$ is sound and complete with respect to topological spaces.

There is a close connection between topological semantics and Kripke semantics for $\mathsf{S4}$. Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. Call $U \subseteq W$ an R-*upset* of $\mathfrak{F}$ if $w \in U$ and wRv imply $v \in U$. The set of R-upsets of $\mathfrak{F}$ is a topology τ_R on W in which every point w has a least neighborhood, namely $R(w)$. Such spaces are called *Alexandroff spaces*. We call (W, τ_R) the *Alexandroff space* of $\mathfrak{F}$. For a modal formula φ, we have

$$\mathfrak{F} \models \varphi \text{ iff } (W, \tau_R) \models \varphi.$$

Thus, topological semantics generalizes Kripke semantics for $\mathsf{S4}$, and hence Kripke completeness for logics above $\mathsf{S4}$ implies topological completeness. However, since Alexandroff spaces are usually not even T_1-spaces, such topological completeness is not guaranteed with respect to, for example, normal spaces.

We recall that a topological space X is

- *extremally disconnected* (ED) if the closure of each open set is open;
- *resolvable* if X is the union of two disjoint dense subsets of X;
- *irresolvable* if X is not resolvable;
- *hereditarily irresolvable* (HI) if every subspace of X is irresolvable.

Let

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p$$

be the *Grzegorczyk axiom* and

$$\mathsf{ga} = \Diamond\Box p \to \Box\Diamond p$$

the *Geach axiom* (see, e.g., [10]). It is well known that

$$X \text{ is ED} \quad \text{iff} \quad X \models \mathsf{grz};$$
$$X \text{ is HI} \quad \text{iff} \quad X \models \mathsf{ga}.$$

We next recall the definition of modal Krull dimension. For this we recall that a subset N of a space X is *nowhere dense* if $\mathsf{ic}N = \varnothing$.

Definition 2.2. ([5, Sec. 3]) Define the *modal Krull dimension* $\mathrm{mdim}(X)$ of a topological space X recursively as follows:

$$\begin{array}{lll} \mathrm{mdim}(X) = -1 & \text{if} & X = \varnothing, \\ \mathrm{mdim}(X) \leq n & \text{if} & \mathrm{mdim}(N) \leq n-1 \text{ for each } N \text{ nowhere dense in } X, \\ \mathrm{mdim}(X) = n & \text{if} & \mathrm{mdim}(X) \leq n \text{ but } \mathrm{mdim}(X) \not\leq n-1, \\ \mathrm{mdim}(X) = \infty & \text{if} & \mathrm{mdim}(X) \not\leq n \text{ for all } n = -1, 0, 1, 2, \dots \end{array}$$

Let

$$\begin{array}{lll} \mathsf{bd}_1 & = & \Diamond\Box p_1 \to p_1 \\ \mathsf{bd}_{n+1} & = & \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1} \text{ for } n \geq 1. \end{array}$$

Theorem 2.3. ([5, Thm. 3.6]) *Let X be a nonempty space and $n \geq 1$. Then*

$$\mathrm{mdim}(X) \leq n-1 \textit{ iff } X \models \mathsf{bd}_n.$$

For nonempty scattered Hausdorff spaces, there is a close connection between finite modal Krull dimension and Cantor-Bendixson rank. For $Y \subseteq X$, let $\mathsf{d}Y$ be the set of limit points of Y and for an ordinal α, let $\mathsf{d}^\alpha Y$ be defined recursively as follows:

$$\begin{array}{lll} \mathsf{d}^0 Y & = & Y, \\ \mathsf{d}^{\alpha+1} Y & = & \mathsf{d}(\mathsf{d}^\alpha Y), \\ \mathsf{d}^\alpha Y & = & \bigcap\{\mathsf{d}^\beta Y \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal.} \end{array}$$

The *Cantor-Bendixson rank* of X is the least ordinal γ satisfying $\mathsf{d}^\gamma X = \mathsf{d}^{\gamma+1} X$. It is well known that a space X is scattered iff there is an ordinal α such that $\mathsf{d}^\alpha X = \varnothing$. Thus, the Cantor-Bendixson rank of a scattered space X is the least ordinal γ such that $\mathsf{d}^\gamma X = \varnothing$.

Let X be a nonempty scattered Hausdorff space and $n \in \omega$. Then the Cantor-Bendixson rank of X is $n+1$ iff $\mathsf{d}^n X \neq \varnothing$ and $\mathsf{d}^{n+1} X = \varnothing$, which by [7, Thm. 4.9] happens iff $\mathrm{mdim}(X) = n$.

2.3. **Measurable cardinals.** We use [17, 18] as standard references for set theory, and also rely on [11] as the main reference for measurable cardinals. Let S be a set and $\mathfrak{p}$ a free ultrafilter on S. We denote infinite cardinals by κ, the first uncountable cardinal by ω_1, and recall that $\mathfrak{p}$ is

- *κ-complete* if $\bigcap \mathcal{K} \in \mathfrak{p}$ for any family $\mathcal{K} \subseteq \mathfrak{p}$ of cardinality $< \kappa$;
- *countably complete* if $\mathfrak{p}$ is ω_1-complete (that is, $\mathfrak{p}$ is closed under countable intersections).

Definition 2.4. ([11, Ch. 8]) An uncountable cardinal κ is

- *measurable* if there exists a κ-complete free ultrafilter on κ;
- *Ulam-measurable* if there exists a countably complete free ultrafilter on κ.

Remark 2.5. While in [11] it is not assumed that measurable cardinals are uncountable, it is common to make such an assumption.

It is clear that every measurable cardinal is Ulam-measurable, and it is well known (see, e.g., [11, Thm. 8.31]) that the existence of an Ulam-measurable cardinal implies the existence of a measurable cardinal.

3. Existence of a measurable cardinal is sufficient

In this section we prove that the existence of a measurable cardinal implies that there is a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$. We build Z in stages. Let κ be a measurable cardinal. Then κ is Ulam-measurable, and so there is a countably complete free ultrafilter $\mathfrak{p}$ on κ. Let $Y = (\kappa \times \{0,1\}) \cup \{\mathfrak{p}\}$. Consider the following family of subsets of Y:

$$\tau = \{U \subseteq Y \mid U \subseteq Y \setminus \{\mathfrak{p}\} \text{ or } \exists V, W \in \mathfrak{p} : U = (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\})\}.$$

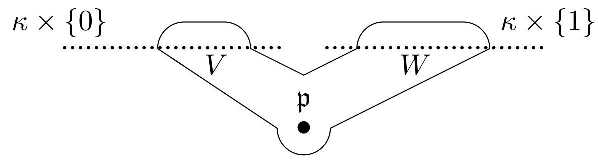

Figure 2. The space Y and an open neighborhood of $\mathfrak{p}$.

Lemma 3.1. *The family τ is a topology on Y that is closed under countable intersections.*

Proof. Clearly $\varnothing, Y \in \tau$. Let $\{U_i \mid i \in I\} \subseteq \tau$ and let $U = \bigcup\{U_i \mid i \in I\}$. If $\mathfrak{p} \notin U$, then $U \in \tau$. Suppose $\mathfrak{p} \in U$. Then $\mathfrak{p} \in U_i$ for some $i \in I$. Since $U_i \in \tau$ and $\mathfrak{p} \in U_i$, there are $V_0, V_1 \in \mathfrak{p}$ such that $U_i = (V_0 \times \{0\}) \cup \{\mathfrak{p}\} \cup (V_1 \times \{1\})$. For $n \in \{0,1\}$, set $W_n = \{\alpha \in \kappa \mid (\alpha, n) \in U\}$. Let $n \in \{0,1\}$ and $\alpha \in V_n$. Then $(\alpha, n) \in V_n \times \{n\} \subseteq U_i \subseteq U$, giving that $\alpha \in W_n$. Therefore, $V_n \subseteq W_n$. Since $V_n \in \mathfrak{p}$ and $\mathfrak{p}$ is an ultrafilter, $W_n \in \mathfrak{p}$. It follows from the definition of W_n that $W_n \times \{n\} = U \cap (\kappa \times \{n\})$. Thus,

$$\begin{aligned} U &= U \cap Y = U \cap ((\kappa \times \{0\}) \cup \{\mathfrak{p}\} \cup (\kappa \times \{1\})) \\ &= (U \cap (\kappa \times \{0\})) \cup (U \cap \{\mathfrak{p}\}) \cup (U \cap (\kappa \times \{1\})) \\ &= (W_0 \times \{0\}) \cup \{\mathfrak{p}\} \cup (W_1 \times \{1\}) \in \tau. \end{aligned}$$

Consequently, τ is closed under union.

Let $\{U_i \mid i \in \omega\} \subseteq \tau$ and let $U = \bigcap\{U_i \mid i \in \omega\}$. If $\mathfrak{p} \notin U$, then $U \in \tau$. Suppose $\mathfrak{p} \in U$. Let $i \in \omega$. Since $\mathfrak{p} \in U_i$ and $U_i \in \tau$, there are $V_i, W_i \in \mathfrak{p}$ such that $U_i = (V_i \times \{0\}) \cup \{\mathfrak{p}\} \cup (W_i \times \{1\})$. Put $V = \bigcap\{V_i \mid i \in \omega\}$ and $W = \bigcap\{W_i \mid i \in \omega\}$. As $\mathfrak{p}$ is countably complete, we have that $V, W \in \mathfrak{p}$.

Claim 3.2. $U = (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\})$.

Proof. Let $\alpha \in \kappa$. We have

$$\begin{aligned} (\alpha, 0) \in U \quad &\text{iff} \quad (\alpha, 0) \in U_i \text{ for all } i \in \omega \\ &\text{iff} \quad \alpha \in V_i \text{ for all } i \in \omega \\ &\text{iff} \quad \alpha \in V \\ &\text{iff} \quad (\alpha, 0) \in V \times \{0\} \\ &\text{iff} \quad (\alpha, 0) \in (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\}). \end{aligned}$$

Similarly, $(\alpha, 1) \in U$ iff $(\alpha, 1) \in (V \times \{0\}) \cup \{\mathfrak{p}\} \cup (W \times \{1\})$. The claim follows. □

We conclude that τ is a topology on Y that is closed under countable intersections. □

Remark 3.3. That κ is a measurable cardinal is used to see that τ is closed under countable intersections. In fact, this is the only place where we use that κ is a measurable cardinal.

Definition 3.4. (See, e.g., [20, p. 37]) A Tychonoff space is a *P-space* if every G_δ-set in X is open.

Lemma 3.5. *The space Y is a normal P-space.*

Proof. It is easy to see that each singleton in Y is closed, so Y is a T_1-space. Let A, B be disjoint closed subsets of Y. Either $\mathfrak{p} \notin A$ or $\mathfrak{p} \notin B$, and we may assume without loss of generality that $\mathfrak{p} \notin A$. Then $A \subseteq Y \setminus \{\mathfrak{p}\}$, hence A is open. Therefore, $U := A$ and $V := Y \setminus A$ are disjoint open subsets of Y separating A and B. Thus, Y is normal, and hence it follows from Lemma 3.1 that Y is a P-space. □

Since Y is a P-space, it follows from [12, Sec. 2] that the Čech-Stone compactification βY of Y can be embedded into a compact Hausdorff ED-space, say E. By Efimov's Theorem [13, Sec. 1], there is a cardinal μ, equipped with the discrete topology, such that the space E can be embedded into $\beta\mu$. It is well known (see, e.g., [14, Exercise 3.6.B.b]) that $\beta\mu$ can be embedded in the remainder $\beta\mu \setminus \mu$. Combining these results yields a sequence of embeddings

$$Y \hookrightarrow \beta Y \hookrightarrow E \hookrightarrow \beta\mu \hookrightarrow \beta\mu \setminus \mu \tag{1}$$

that gives an embedding of Y into $\beta\mu \setminus \mu$. We identify Y with its image in $\beta\mu$; see Figure 3.

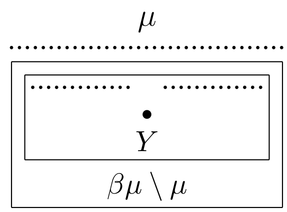

FIGURE 3. Y as a subspace of $\beta\mu$.

Definition 3.6. Let Z be the subspace $\mu \cup Y$ of $\beta\mu$.

Our goal is to show that Z is a normal space such that $\mathsf{L}(Z) = \mathsf{L}$.

Lemma 3.7. *The space Z is a scattered ED-space of Cantor-Bendixson rank 3.*

Proof. Since $Z \supseteq \mu$ and μ is dense in $\beta\mu$, we have that Z is dense in $\beta\mu$. As $\beta\mu$ is an ED-space (see, e.g., [14, Cor. 6.2.28]) and a dense subspace of an ED-space is an ED-space (see, e.g., [14, Exercise 6.2.G.c]), it follows that Z is an ED-space.

We have $\mathsf{d}^3 Z = \mathsf{d}^2 Y = \mathsf{d}\{\mathfrak{p}\} = \varnothing$ and $\mathsf{d}^2 Z = \mathsf{d} Y = \{\mathfrak{p}\} \neq \varnothing$. Therefore, Z is scattered and of Cantor-Bendixson rank 3. □

Lemma 3.8. *The space Z is normal.*

Proof. Clearly Z is T_1 since it is a subspace of a T_1-space. Let A and B be disjoint closed subsets of Z. Since μ is the set of isolated points of Z, we have that $A \cap \mu$ and $B \cap \mu$ are disjoint open subsets of Z. Let $A_0 = \mathsf{c}(A \cap \mu)$ and $B_0 = \mathsf{c}(B \cap \mu)$. Because Z is ED, A_0 and B_0 are disjoint clopen subsets of Z. Let $A_1 = A \setminus A_0$ and $B_1 = B \setminus B_0$. Then A_1 and B_1 are disjoint closed subsets of Y. Since Y is normal, it follows from [14, Cor. 3.6.4] that $\mathsf{c}_{\beta Y}(A_1)$ and $\mathsf{c}_{\beta Y}(B_1)$ are disjoint, where $\mathsf{c}_{\beta Y}$ is the closure in βY. Because βY is (up to homeomorphism) a closed subspace of $\beta\mu$, we have

$$\mathsf{c}_{\beta\mu}(A_1) \cap \mathsf{c}_{\beta\mu}(B_1) = \mathsf{c}_{\beta Y}(A_1) \cap \mathsf{c}_{\beta Y}(B_1) = \varnothing.$$

Since $\beta\mu$ is normal, there are disjoint open subsets U_1 and V_1 of $\beta\mu$ such that $\mathsf{c}_{\beta\mu}(A_1) \subseteq U_1$ and $\mathsf{c}_{\beta\mu}(B_1) \subseteq V_1$.

Clearly $U := U_1 \cap Z$ and $V := V_1 \cap Z$ are disjoint open subsets of Z. As both A_0 and B_0 are clopen in Z, it follows that both $U \setminus B_0$ and $V \setminus A_0$ are open in Z, and hence $U_0 := A_0 \cup (U \setminus B_0)$ and $V_0 := B_0 \cup (V \setminus A_0)$ are disjoint open subsets of Z. It is clear that $A_1 \subseteq U_1 \cap Z = U$.

Because A_1 and B_0 are disjoint, $A_1 \subseteq U \setminus B_0$, so $A = A_0 \cup A_1 \subseteq A_0 \cup (U \setminus B_0) = U_0$. Similarly, $B \subseteq V_0$. Thus, Z is normal. □

We recall that a map $f : X \to X'$ between spaces is *interior* if f is both continuous and open. If in addition f is onto, then we call X' an *interior image* of X. If X' is the Alexandroff space of an $\mathsf{S4}$-frame $\mathfrak{F}$, then we say that $\mathfrak{F}$ is an *interior image* of X. Finally, if X is the Alexandroff space of an $\mathsf{S4}$-frame $\mathfrak{G}$, then we say that $\mathfrak{F}$ is an *interior image* of $\mathfrak{G}$.

Remark 3.9. It is well known that $\mathfrak{F} = (W, R)$ is an interior image of $\mathfrak{G} = (V, S)$ iff $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$, where we recall that a *p-morphism* is a map $f : V \to W$ such that $f^{-1}R^{-1}(w) = S^{-1}f^{-1}(w)$ for each $w \in W$.

Convention 3.10. Since the diamond $\mathfrak{D} = (D, \leq)$ is a poset (partially ordered set), for $w \in D$ we write $\uparrow w$ and $\downarrow w$ instead of $R(w)$ and $R^{-1}(w)$, respectively.

Lemma 3.11. *The diamond $\mathfrak{D}$ is an interior image of Z.*

Proof. Define $f : Z \to D$ by

$$f(z) = \begin{cases} m & \text{if } z \in \mu \\ w_0 & \text{if } z \in \kappa \times \{0\} \\ w_1 & \text{if } z \in \kappa \times \{1\} \\ r & \text{if } z = \mathfrak{p} \end{cases}$$

It is clear that f is a well-defined onto mapping. To prove that f is interior, it is sufficient to show that $f^{-1}{\downarrow}w = \mathsf{c}f^{-1}(w)$ for each $w \in D$. Since μ is dense in Z, we have

$$f^{-1}{\downarrow}m = f^{-1}(D) = Z = \mathsf{c}\mu = \mathsf{c}f^{-1}(m).$$

Because Z is T_1, we have

$$f^{-1}{\downarrow}r = f^{-1}(r) = \{\mathfrak{p}\} = \mathsf{c}\{\mathfrak{p}\} = \mathsf{c}f^{-1}(r).$$

Since Y is closed in Z, we have that $\mathsf{c}_Y A = \mathsf{c}A$ for any $A \subseteq Y$, where $\mathsf{c}_Y A$ is closure in Y. Let $n \in \{0, 1\}$. Then $(\kappa \times \{n\}) \cup \{\mathfrak{p}\}$ is closed in Y. Therefore, $\mathfrak{p} \in \mathsf{c}_Y (\kappa \times \{n\})$. Thus, $\mathsf{c}(\kappa \times \{n\}) = \mathsf{c}_Y (\kappa \times \{n\}) = (\kappa \times \{n\}) \cup \{\mathfrak{p}\}$. This yields

$$f^{-1}{\downarrow}w_n = f^{-1}(\{w_n, r\}) = (\kappa \times \{n\}) \cup \{\mathfrak{p}\} = \mathsf{c}(\kappa \times \{n\}) = \mathsf{c}f^{-1}(w_n).$$

Consequently, f is interior. □

We are ready for the main lemma of this section. For this we recall that an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ is *rooted* if there is $w \in W$ (a *root* of $\mathfrak{F}$) such that $W = R(w)$.

Lemma 3.12. *Let $\mathfrak{F} = (W, R)$ be a finite rooted $\mathsf{S4}$-frame. If $\mathfrak{F}$ is an interior image of Z, then $\mathfrak{F}$ is an interior image of $\mathfrak{D}$.*

Proof. We start by observing some properties of $\mathfrak{F}$. Since Z is scattered, it is HI. Because Z is also of Cantor-Bendixson rank 3, it follows from Section 2.2 that the formulas grz and bd_3 are valid in Z. As $\mathfrak{F}$ is an interior image of Z, these formulas are also valid in $\mathfrak{F}$ (see, e.g., [4, Prop. 2.9(2)]). Therefore, R is a partial order and the R-depth of $\mathfrak{F}$ is ≤ 3 (see, e.g., [10, Props. 3.48 & 3.44]). In addition, since Z is ED, so is $\mathfrak{F}$. Thus, as $\mathfrak{F}$ is rooted, $\mathfrak{F}$ has a maximum (see, e.g., [10, Cor. 3.38]).

We consider three cases based on the depth of $\mathfrak{F}$. First, suppose that the depth of $\mathfrak{F}$ is 1. Then W is a singleton and it is clear that $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Next suppose that the depth of $\mathfrak{F}$ is 2. Since $\mathfrak{F}$ is a rooted poset with a maximum, $\mathfrak{F}$ is isomorphic to the two element chain (see Figure 4). It is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathfrak{F}$ and all the other points of $\mathfrak{D}$ to the maximum of $\mathfrak{F}$ is an onto interior map.

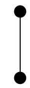

FIGURE 4. The two element chain.

Finally, suppose that the depth of $\mathfrak{F}$ is 3. Then $\mathfrak{F}$ is isomorphic to the frame depicted in Figure 5 where $W = \{0, v_0, \ldots, v_m, 1\}$ and $m \in \omega$.

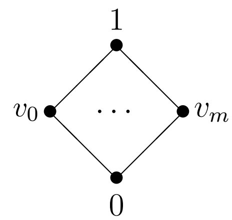

FIGURE 5. The poset $\mathfrak{F}$ of depth 3.

If $m = 0$, then it is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathfrak{F}$, the maximum of $\mathfrak{D}$ to the maximum of $\mathfrak{F}$, and w_0, w_1 to v_0 is an onto interior map. If $m = 1$, then $\mathfrak{D}$ is isomorphic to $\mathfrak{F}$, so it is obvious that $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Thus, to complete the proof, it suffices to show that $m \not\geq 2$.

Suppose that $m \geq 2$ and let $f : Z \to W$ be an interior mapping onto $\mathfrak{F}$.

Claim 3.13.
(1) $\mu \subseteq f^{-1}(1)$.
(2) $\{\mathfrak{p}\} = f^{-1}(0)$.
(3) $f^{-1}(\{v_0, \ldots, v_m\}) \subseteq Y \setminus \{\mathfrak{p}\}$.
(4) $\mathfrak{p} \in \mathsf{c}(f^{-1}(v_i) \cap (\kappa \times \{0\})) \cup \mathsf{c}(f^{-1}(v_i) \cap (\kappa \times \{1\}))$ *for each* $i \in \{0, \ldots, m\}$.

Proof. (1) Since each $z \in \mu$ is isolated and f is interior, we have that $f(z)$ is the maximum of $\mathfrak{F}$. Thus, $f(z) = 1$.

(2) Because f is onto, there is $z \in f^{-1}(0)$. By (1), we have that $z \in Y$. If $z \neq \mathfrak{p}$, then z is an isolated point of Y, so there is an open subset U of Z such that $\{z\} = U \cap Y$. As f is interior and U is open, $f(U)$ is an R-upset of $\mathfrak{F}$. Therefore, $f(U) = W$ since $0 = f(z) \in f(U)$. On the other hand,

$$f(U) = f((U \cap Y) \cup (U \cap \mu)) \subseteq f(\{z\} \cup \mu) = f(\{z\}) \cup f(\mu) = \{0\} \cup \{1\} \neq W.$$

The obtained contradiction proves that $z = \mathfrak{p}$. Thus, $f^{-1}(0) = \{\mathfrak{p}\}$.

(3) Follows immediately from (1) and (2) since $\mu \cup \{\mathfrak{p}\} \subseteq f^{-1}(\{0, 1\})$.

(4) Let $i \in \{0, \ldots, m\}$. Because f is interior, it follows from (2) and (3) that

$$\begin{aligned} \{\mathfrak{p}\} &\subseteq f^{-1}\left(\{0, v_i\}\right) = f^{-1}R^{-1}(v_i) \\ &= \mathsf{c}f^{-1}(v_i) = \mathsf{c}\left(f^{-1}(v_i) \cap (Y \setminus \{\mathfrak{p}\})\right) \\ &= \mathsf{c}\left(f^{-1}(v_i) \cap [(\kappa \times \{0\}) \cup (\kappa \times \{1\})]\right) \\ &= \mathsf{c}\left(\left[f^{-1}(v_i) \cap (\kappa \times \{0\})\right] \cup \left[f^{-1}(v_i) \cap (\kappa \times \{1\})\right]\right) \\ &= \mathsf{c}\left(f^{-1}(v_i) \cap (\kappa \times \{0\})\right) \cup \mathsf{c}\left(f^{-1}(v_i) \cap (\kappa \times \{1\})\right). \end{aligned}$$

□

Let

$$\mathscr{F}_0 = \left\{ f^{-1}(1) \cap (\kappa \times \{0\}), f^{-1}(v_0) \cap (\kappa \times \{0\}), \ldots, f^{-1}(v_m) \cap (\kappa \times \{0\}) \right\}$$

and

$$\mathscr{F}_1 = \left\{ f^{-1}(1) \cap (\kappa \times \{1\}), f^{-1}(v_0) \cap (\kappa \times \{1\}), \ldots, f^{-1}(v_m) \cap (\kappa \times \{1\}) \right\}.$$

Then both $\mathscr{F}_0$ and $\mathscr{F}_1$ are pairwise disjoint families of sets, $\bigcup \mathscr{F}_0 = \kappa \times \{0\}$, and $\bigcup \mathscr{F}_1 = \kappa \times \{1\}$. We prove that there is a unique $A_0 \in \mathscr{F}_0$ such that $\mathfrak{p} \in \mathsf{c}A_0$. A similar proof yields a unique $A_1 \in \mathscr{F}_1$ such that $\mathfrak{p} \in \mathsf{c}A_1$.

Because $\mathscr{F}_0$ is finite, we have

$$\mathfrak{p} \in \mathsf{c}(\kappa \times \{0\}) = \mathsf{c}\left(\bigcup \mathscr{F}_0\right) = \bigcup\nolimits_{A \in \mathscr{F}_0} \mathsf{c}A.$$

Therefore, there is $A_0 \in \mathscr{F}_0$ such that $\mathfrak{p} \in \mathsf{c}A_0$. Since $\mathfrak{p}$ is an ultrafilter,

$$\mathfrak{p} \notin \mathsf{c}((\kappa \times \{0\}) \setminus A_0) = \mathsf{c}\left(\bigcup (\mathscr{F}_0 \setminus \{A_0\})\right) = \bigcup\nolimits_{A \in \mathscr{F}_0 \setminus \{A_0\}} \mathsf{c}A.$$

Thus, A_0 is the unique member A of $\mathscr{F}_0$ satisfying the property that $\mathfrak{p} \in \mathsf{c}A$.

Since $m \geq 2$, by the Pigeonhole Principle, there is $i \in \{0, 1, 2, \ldots, m\}$ such that $A_0 \neq f^{-1}(v_i) \cap (\kappa \times \{0\})$ and $A_1 \neq f^{-1}(v_i) \cap (\kappa \times \{1\})$. Thus, $\mathfrak{p} \notin \mathsf{c}\left(f^{-1}(v_i) \cap (\kappa \times \{0\})\right)$ and $\mathfrak{p} \notin \mathsf{c}(f^{-1}(v_i) \cap (\kappa \times \{1\}))$, which contradicts Claim 3.13(4). Consequently, $m \not\geq 2$, completing the proof. □

Lemma 3.14. *The logic of Z is L.*

Proof. By Lemma 3.11, $\mathfrak{D}$ is an interior image of Z. Therefore, $\mathsf{L}(Z) \subseteq \mathsf{L}(\mathfrak{D}) = \mathsf{L}$ (see, e.g., [4, Prop. 2.9(2)]). Conversely, suppose that $\mathsf{L}(Z) \nvdash \varphi$. Since Z is of Cantor-Bendixson rank 3, bd_3 is a theorem of $\mathsf{L}(Z)$. Therefore, by Segerberg's theorem (see, e.g., [10, Thm. 8.85]), $\mathsf{L}(Z)$ is complete with respect to finite rooted $\mathsf{L}(Z)$-frames. Thus, there is a finite rooted $\mathsf{L}(Z)$-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. As $\mathfrak{F}$ is an $\mathsf{L}(Z)$-frame, by [6, Lem 6.2], $\mathfrak{F}$ is an interior image of an open subspace U of Z. Let $f : U \to \mathfrak{F}$ be an interior map, and let $z \in U$ map to the root of $\mathfrak{F}$. Since Z is zero-dimensional, there is a clopen subset V of Z such that $z \in V$ and $V \subseteq U$. Then the restriction of f to V is an interior mapping of V onto $\mathfrak{F}$. Because $\mathfrak{F}$ has a maximum, we have that $\mathfrak{F}$ is an interior image of Z by [7, Lem. 5.4]. By Lemma 3.12, $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Therefore, $\mathfrak{D} \not\models \varphi$, and hence $\mathsf{L}(\mathfrak{D}) \nvdash \varphi$. Thus, $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D}) = \mathsf{L}$. □

As a consequence of Lemmas 3.8 and 3.14 we arrive at the main result of this section.

Theorem 3.15. *If there exists a measurable cardinal, then there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$.*

4. Existence of a measurable cardinal is necessary

In this section we prove that the existence of a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$ implies the existence of a measurable cardinal. Let Z be a normal space such that $\mathsf{L}(Z) = \mathsf{L}$.

Lemma 4.1. *The space Z is an ED-space of modal Krull dimension 2 such that $\mathfrak{D}$ is an interior image of Z.*

Proof. As $\mathsf{L}(Z) = \mathsf{L}$, for each modal formula φ we have $Z \models \varphi$ iff $\mathfrak{D} \models \varphi$. Since $\mathfrak{D}$ has a maximum and is of depth 3, we have that

$$\begin{array}{rcl} \mathfrak{D} & \models & \mathsf{ga} \\ \mathfrak{D} & \models & \mathsf{bd}_3 \\ \mathfrak{D} & \not\models & \mathsf{bd}_2 \end{array}$$

Therefore, Z is an ED-space of modal Krull dimension 2 (see Section 2.2).

Because $\mathfrak{D} \models \mathsf{L}(Z)$, [6, Lem. 6.2] yields an open subspace U of Z and an onto interior map $g : U \to D$. Then there is $z \in U$ with $f(z) = r$. Since Z is normal and ED, it is zero-dimensional. Hence, there is clopen V in Z such that $z \in V \subseteq U$. Noting that the restriction of g to V is an interior mapping onto $\mathfrak{D}$, it follows from [7, Lem. 5.4] that $\mathfrak{D}$ is an interior image of Z. □

Remark 4.2.

(1) Since $\mathfrak{D}$ is a finite poset, $\mathfrak{D}$ validates grz. Therefore, so does Z, and hence Z is HI.
(2) Observe that $\mathfrak{D}$ is not hereditarily ED since the subspace $\{r, w_0, w_1\}$ is not ED. Because $\mathfrak{D}$ is an interior image of Z, it follows that Z is not hereditarily ED.
(3) Since Z is a Hausdorff ED-space that is not hereditarily ED, Z must be uncountable (see, e.g., [9, Cor. 2.1]).

Definition 4.3. Let $f : Z \to \mathfrak{D}$ be an onto interior mapping. Denote the fibers of f by

$$\begin{aligned} M &= f^{-1}(m) \\ B_0 &= f^{-1}(w_0) \\ B_1 &= f^{-1}(w_1) \\ A &= f^{-1}(r) \end{aligned}$$

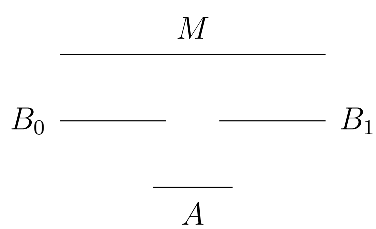

FIGURE 6. Depiction of Z partitioned by the fibers of f.

Remark 4.4.

(1) Clearly M is an open dense subset of Z (which is infinite as it is a dense subset of an infinite T_1-space).
(2) We also have that A is a closed nowhere dense subset of $Z \setminus M$. Therefore, A is discrete. More generally, any nonempty nowhere dense subset N of $Z \setminus M$ is discrete. To see this, since $\mathrm{mdim}(Z) = 2$, the definition of modal Krull dimension gives that $\mathrm{mdim}(Z \setminus M) \leq 1$ and $\mathrm{mdim}(N) \leq 0$. As $N \neq \varnothing$, we have that $\mathrm{mdim}(N) = 0$. Thus, N is discrete by [5, Rem. 4.8 & Thm. 4.9].

Lemma 4.5. *There is a normal subspace U of Z such that $U \cap A$ is a singleton and $\mathsf{L}(U) = \mathsf{L}$.*

Proof. Let $a \in A$. Since A is discrete and Z is zero-dimensional, there is a clopen subset U of Z such that $\{a\} = U \cap A$. As U is closed in Z, the subspace U is normal. Because U is open in Z, the restriction $f|_U$ of f to U is interior. Since $U \cap A \neq \varnothing$, we have that $r \in f(U)$. As $f(U)$ is an upset, $D = {\uparrow} r \subseteq f(U) \subseteq D$. Therefore, $f|_U$ is onto and $\mathfrak{D}$ is an interior image of U. By [4, Prop. 2.9], $\mathsf{L}(U) \subseteq \mathsf{L} = \mathsf{L}(Z) \subseteq \mathsf{L}(U)$, so $\mathsf{L}(U) = \mathsf{L}$, completing the proof. □

By Lemma 4.5, we may assume without loss of generality that A is a singleton, say $\{a\}$, yielding that $Z = B_0 \cup \{a\} \cup B_1 \cup M$ (see Figure 7).

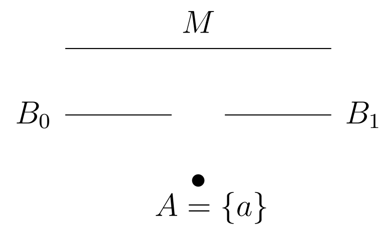

FIGURE 7. Reducing A to a singleton.

Lemma 4.6. *We have that $a \notin \mathsf{c}N$ for any nowhere dense subset N of the subspace $B_0 \cup B_1$.*

Proof. We first show that $N \cup A$ is nowhere dense in $Z \setminus M$. Let U be open in $Z \setminus M$ with $U \subseteq \mathsf{c}(N \cup A)$. Since A is closed, $U \subseteq \mathsf{c}(N) \cup A$. Therefore, $U \setminus A \subseteq \mathsf{c}(N) \setminus A = \mathsf{c}(N) \cap (B_0 \cup B_1)$, which is the closure of N relative to $B_0 \cup B_1$. Because $U \setminus A$ is open and N is nowhere dense in $B_0 \cup B_1$, we have that $U \setminus A = \varnothing$, so $U \subseteq A$. By Remark 4.4(2), A is a closed nowhere dense subset of $Z \setminus M$, hence $U = \varnothing$. Thus, $N \cup A$ is nowhere dense in $Z \setminus M$. Applying Remark 4.4(2) again yields that $N \cup A$ is discrete. Consequently, there is an open set V in Z such that $\{a\} = V \cap (N \cup A)$. As

$$V \cap N \subseteq V \cap (N \cup A) = \{a\} \subseteq Z \setminus (B_0 \cup B_1) \subseteq Z \setminus N,$$

it must be the case that $V \cap N = \varnothing$, so $a \notin \mathsf{c}N$. □

We recall that a normal space X is an *F-space* if any two disjoint open F_σ-sets in X have disjoint closures in X (see, e.g., [19, Lem. 1.2.2(b)]). Being a normal ED-space, it follows from [15, Exercise 14N.4] that Z is an F-space.

Definition 4.7. Let Y denote the subspace $B_0 \cup \{a\} \cup B_1$ of Z.

Because $Y = Z \setminus M$ is closed in Z, we have that Y is a normal F-space by [19, Lem. 1.2.2(d)]. We require the following definition.

Definition 4.8. (See, e.g., [20, p. 37]) A point x of a space X is called a *P-point* provided for any G_δ-set S in X we have that $x \in S$ implies $x \in \mathsf{i}S$.

Remark 4.9. By taking complements we obtain that $x \in X$ is a P-point iff for each F_σ-set S in X we have that $x \notin S$ implies $x \notin \mathsf{c}S$. This will be utilized in Lemma 4.16(5).

Lemma 4.10. *Either a is a P-point in the subspace $B_0 \cup \{a\}$ or a P-point in the subspace $B_1 \cup \{a\}$.*

Proof. Suppose not. Then we show that there are disjoint open F_σ-sets U_0 and U_1 of Y whose closures have nonempty intersection, which is a contradiction since Y is a normal F-space. We only show how to construct U_0 because U_1 is constructed similarly. Since a is not a P-point in $B_0 \cup \{a\}$, for each $n \in \omega$, there is W_n open in $B_0 \cup \{a\}$ such that $a \in \bigcap_{n\in\omega} W_n$ but $a \notin \mathsf{i}\left(\bigcap_{n\in\omega} W_n\right)$, where i is taken in $B_0 \cup \{a\}$. As Z is zero-dimensional, $B_0 \cup \{a\}$ is zero-dimensional. Thus, for each $n \in \omega$, there is V_n clopen in $B_n \cup \{a\}$ such that $a \in V_n \subseteq W_n$. Clearly, $a \in V := \bigcap_{n\in\omega} V_n$ and V is a closed G_δ-set in $B_0 \cup \{a\}$. Moreover, $a \notin \mathsf{i}V$ since $V \subseteq \bigcap_{n\in\omega} W_n$ and $a \notin \mathsf{i}\left(\bigcap_{n\in\omega} W_n\right)$. Put $U_0 = (B_0 \cup \{a\}) \setminus V$. Then U_0 is an open F_σ-set in $B_0 \cup \{a\}$ such that $a \notin U_0$ and $a \in \mathsf{c}U_0$. Clearly $U_0 \subseteq B_0$, and so U_0 is open in B_0. As $B_0 = Y \cap f^{-1}{\uparrow}w_0$ is open in Y, it follows that U_0 is open in Y. Because $B_0 \cup \{a\}$ is closed in Y and U_0 is an F_σ-set in $B_0 \cup \{a\}$, we have that U_0 is an F_σ-set in Y. Thus, U_0 is an open F_σ-set in Y such that $a \in \mathsf{c}U_0$. Analogously, there is an open F_σ-set U_1 in Y such that $a \in \mathsf{c}U_1$. By construction, $U_0 \subseteq B_0$ and $U_1 \subseteq B_1$, so U_0 and U_1 are disjoint. On the other hand, $a \in \mathsf{c}U_0 \cap \mathsf{c}U_1$, yielding the desired contradiction. □

Convention 4.11. Without loss of generality we assume that a is a P-point in $X := B_0 \cup \{a\}$.

Remark 4.12. Since X is closed in Z, the closure in X of any subset S of X coincides with the closure of S in Z. Therefore, there is no ambiguity in writing $\mathsf{c}S$ whenever $S \subseteq X$.

The following lemma is an easy consequence of Zorn's lemma, and we skip its proof.

Lemma 4.13. *There is a family $\mathscr{F}$ of subsets of X that is maximal with respect to the following two properties:*

(1) *Each $F \in \mathscr{F}$ is a nonempty clopen in X such that $a \notin F$;*
(2) *The family $\mathscr{F}$ is pairwise disjoint.*

Lemma 4.14. *Let $N = B_0 \setminus \bigcup \mathscr{F}$. Then we have:*

(1) *$\bigcup \mathscr{F}$ is open in both X and B_0.*
(2) *$\bigcup \mathscr{F}$ is dense in both B_0 and X.*
(3) *N is closed in Z.*
(4) *There is a clopen subspace U of Z such that $U \cap N = \varnothing$ and $\mathsf{L}(U) = \mathsf{L}$.*

Proof. (1) Since $\bigcup \mathscr{F}$ is a union of clopen subsets of X, it is open in X. Also, since $a \notin F$ for each $F \in \mathscr{F}$, we have that $\bigcup \mathscr{F} \subseteq B_0$, and hence it is also open in B_0.

(2) Let $z \in B_0$. If $z \notin \mathsf{c}\left(\bigcup \mathscr{F}\right)$, then as X is zero-dimensional, there is clopen V in X such that $z \in V$ and $V \cap \bigcup \mathscr{F} = \varnothing$. Since $z \neq a$, we may assume that $a \notin V$ (by shrinking V further if necessary). But this contradicts the maximality of $\mathscr{F}$ because the family $\{V\} \cup \mathscr{F}$ satisfies the conditions of Lemma 4.13. Thus, $z \in \mathsf{c}\left(\bigcup \mathscr{F}\right)$, and so $\bigcup \mathscr{F}$ is dense in B_0. Finally, since $a \in \mathsf{c}B_0$, we conclude that $\bigcup \mathscr{F}$ is dense in X.

(3) It suffices to show that N is closed in X. For any $z \in B_0 \setminus N$, we have that $\bigcup \mathscr{F}$ is open in X and $z \in \bigcup \mathscr{F}$. Since $N \cap \bigcup \mathscr{F} = \varnothing$, it follows that $z \notin \mathsf{c}N$. Because $\{B_0 \setminus N, N, \{a\}\}$ is a partition of X, it remains to show that $a \notin \mathsf{c}N$. But (1) and (2) imply that N is nowhere dense in B_0, hence nowhere dense in $B_0 \cup B_1$. This yields that $a \notin \mathsf{c}N$ by Lemma 4.6.

(4) Since $\{a\}$ and N are closed in the zero-dimensional normal space Z, there is U clopen in Z such that $a \in U$ and $U \cap N = \varnothing$. Because U is open, the restriction of f as defined in Definition 4.3 is an interior map from U to $\mathfrak{D}$. To see that it is onto, observe that $U \cap M \neq \varnothing$ since M is dense in Z, and both $U \cap B_0$ and $U \cap B_1$ are nonempty because $a \in \mathsf{c}B_0, \mathsf{c}B_1$ and $a \in U$. Therefore, $\mathfrak{D}$ is an interior image of Z, and so $\mathsf{L}(U) \subseteq \mathsf{L} = \mathsf{L}(Z) \subseteq L(U)$ by [4, Prop. 2.9]. Thus, $\mathsf{L}(U) = \mathsf{L}$. $\square$

Let U be the clopen subspace of Z constructed in the proof of Lemma 4.14(4). Then U is normal since it is a closed subspace of a normal space. In addition, a remains a P-point of $X \cap U$ because $X \cap U$ is an open subspace of X and a is a P-point of X. Therefore, without loss of generality we may assume that $Z = U$. Thus, $B_0 = \bigcup \mathscr{F}$ and $N = \varnothing$.

Definition 4.15.

(1) Let κ be the cardinality of $\mathscr{F}$, and let $\varphi : \kappa \to \mathscr{F}$ be a bijection. Denoting $\varphi(\alpha)$ by F_α, we may write $\mathscr{F} = \{F_\alpha \mid \alpha \in \kappa\}$.
(2) Let

$$\mathscr{G} = \left\{ \Gamma \subseteq \kappa \mid a \in \mathsf{c}\left(\bigcup_{\alpha \in \Gamma} F_\alpha \right) \right\}.$$

We are ready to prove the main lemma of this section.

Lemma 4.16.

(1) *If $\Gamma \in \mathscr{G}$ and $\Gamma \subseteq \Lambda$, then $\Lambda \in \mathscr{G}$.*
(2) *For any $\Gamma \subseteq \kappa$, exactly one of $\Gamma, \kappa \setminus \Gamma$ belongs to $\mathscr{G}$.*

(3) *If* $\Gamma, \Lambda \in \mathscr{G}$, *then* $\Gamma \cap \Lambda \in \mathscr{G}$.
(4) $\mathscr{G}$ *is a free ultrafilter on* κ.
(5) $\mathscr{G}$ *is countably complete.*

Proof. (1) Let $\Gamma \in \mathscr{G}$ and $\Gamma \subseteq \Lambda$. Then $\bigcup_{\alpha\in\Gamma} F_\alpha \subseteq \bigcup_{\alpha\in\Lambda} F_\alpha$, yielding

$$a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha\right) \subseteq \mathsf{c}\left(\bigcup_{\alpha\in\Lambda} F_\alpha\right).$$

Thus, $\Lambda \in \mathscr{G}$.

(2) Let $\Gamma \subseteq \kappa$. We have that

$$a \in \mathsf{c}B_0 = \mathsf{c}\left(\bigcup_{\alpha\in\kappa} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha \cup \bigcup_{\alpha\in\kappa\setminus\Gamma} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha\right) \cup \mathsf{c}\left(\bigcup_{\alpha\in\kappa\setminus\Gamma} F_\alpha\right).$$

Therefore, $\Gamma \in \mathscr{G}$ or $\kappa\setminus\Gamma \in \mathscr{G}$. Suppose that both Γ and $\kappa\setminus\Gamma$ belong to $\mathscr{G}$. Then the frame $\mathfrak{F}$ depicted in Figure 5 with $m = 2$ is an interior image of Z via the mapping $g : Z \to W$ given by

$$g(z) = \begin{cases} 1 & \text{if } z \in M \\ v_0 & \text{if } z \in \bigcup_{\alpha\in\Gamma} F_\alpha \\ v_1 & \text{if } z \in \bigcup_{\alpha\in\kappa\setminus\Gamma} F_\alpha \\ v_2 & \text{if } z \in B_1 \\ 0 & \text{if } z = a \end{cases}$$

The function g is depicted in Figure 8 where each fiber of g is labeled to the right by its image in W.

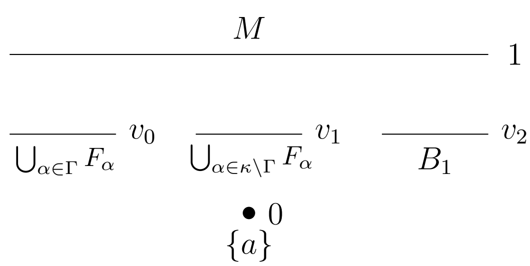

FIGURE 8. The function $g : Z \to W$.

This yields that $\mathfrak{F} \models \mathsf{L}(Z) = \mathsf{L}$, which is a contradiction since $\mathfrak{F} \not\models \mathsf{L}$. Thus, exactly one of Γ or $\kappa\setminus\Gamma$ is a member of $\mathscr{G}$.

(3) If $\Gamma\cap\Lambda \notin \mathscr{G}$, then $a \notin \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\cap\Lambda} F_\alpha\right)$. On the other hand,

$$a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\cap\Lambda} F_\alpha \cup \bigcup_{\alpha\in\Gamma\setminus\Lambda} F_\alpha\right) = \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\cap\Lambda} F_\alpha\right) \cup \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\setminus\Lambda} F_\alpha\right).$$

Therefore, $a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma\setminus\Lambda} F_\alpha\right)$. Thus, $\Gamma\setminus\Lambda \in \mathscr{G}$. Since $\Gamma\setminus\Lambda \subseteq \kappa\setminus\Lambda$, (1) implies that $\kappa\setminus\Lambda \in \mathscr{G}$. However, as $\Lambda \in \mathscr{G}$, (2) implies that $\kappa\setminus\Lambda \notin \mathscr{G}$. The obtained contradiction proves that $\Gamma\cap\Lambda \in \mathscr{G}$.

(4) That $\mathscr{G}$ is an ultrafilter follows from (1), (2), and (3). To see that $\mathfrak{G}$ is free, let $\alpha \in \kappa$. Then F_α is clopen in X and $a \notin F_\alpha$. Therefore, $a \notin \mathsf{c}F_\alpha$, yielding that $\{\alpha\} \notin \mathscr{G}$. Thus, $\mathscr{G}$ is a free ultrafilter.

(5) Let $\Lambda_n \in \mathscr{G}$ for each $n \in \omega$ and let $\Gamma := \bigcap_{n\in\omega} \Lambda_n \notin \mathscr{G}$. For $n \in \omega$ set $\Gamma_n = \bigcap_{i=0}^{n} \Lambda_i$. Then $\Gamma_n \in \mathscr{G}$ by (3), $\Gamma_{n+1} \subseteq \Gamma_n$, and $\Gamma = \bigcap_{n\in\omega} \Gamma_n$. For $n \in \omega$ set $\Delta_n = \Gamma_n \setminus \Gamma_{n+1}$. Since $\mathscr{G}$ is an ultrafilter, $\Delta_n \notin \mathscr{G}$ for each $n \in \omega$.

Claim 4.17. *The set $\bigcup_{\alpha\in\Delta_n} F_\alpha$ is clopen in X.*

Proof. Clearly $\bigcup_{\alpha\in\Delta_n} F_\alpha$ is open in X since each $F \in \mathscr{F}$ is clopen in X. To see that $\bigcup_{\alpha\in\Delta_n} F_\alpha$ is closed in X we show that $\mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right) = \bigcup_{\alpha\in\Delta_n} F_\alpha$. As X is closed in Z, we have that $\mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right) \subseteq X$. Let $z \in X \setminus \bigcup_{\alpha\in\Delta_n} F_\alpha$. We show that $z \notin \mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$. Either $z = a$ or $z \in B_0$. The former case is clear since $\Delta_n \notin \mathscr{G}$ implies that $z = a \notin \mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$. Suppose $z \in B_0$. Then there is $\beta \in \kappa$ such that $z \in F_\beta$. Since $z \notin \bigcup_{\alpha\in\Delta_n} F_\alpha$, it follows that $\beta \notin \Delta_n$. Because F_β is clopen in X, there is U open in Z such that $F_\beta = U \cap X$. Clearly $z \in U$. As $\mathscr{F}$ is pairwise disjoint, we have that

$$U \cap \bigcup_{\alpha\in\Delta_n} F_\alpha = U \cap \bigcup_{\alpha\in\Delta_n} (X \cap F_\alpha) = \bigcup_{\alpha\in\Delta_n} (U \cap X \cap F_\alpha) = \bigcup_{\alpha\in\Delta_n} (F_\beta \cap F_\alpha) = \varnothing.$$

Therefore, $z \notin \mathsf{c}\left(\bigcup_{\alpha\in\Delta_n} F_\alpha\right)$. □

As $\Gamma_0 \setminus \Gamma = \bigcup_{n\in\omega} \Delta_n$, it follows from Claim 4.17 that

$$\bigcup_{\alpha\in\Gamma_0\setminus\Gamma} F_\alpha = \bigcup_{n\in\omega} \left(\bigcup_{\alpha\in\Delta_n} F_\alpha \right)$$

is an open F_σ-set in X. Moreover, $a \in \mathsf{c}\left(\bigcup_{\alpha\in\Gamma_0\setminus\Gamma} F_\alpha\right)$ because $\Gamma_0 \setminus \Gamma \in \mathscr{G}$. But $a \notin \bigcup_{\alpha\in\Gamma_0\setminus\Gamma} F_\alpha$ since $a \notin F_\alpha$ for each $\alpha \in \kappa$. This implies that a is not a P-point of X (see Remark 4.9). The obtained contradiction proves that $\mathscr{G}$ is countably complete. □

As a consequence of Lemma 4.16 and Section 2.3, we obtain:

Lemma 4.18. *The cardinal κ is Ulam-measurable, and hence there exists a measurable cardinal.*

Consequently, we have proved the following result.

Theorem 4.19. *If there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$, then there exists a measurable cardinal.*

Putting Theorems 3.15 and 4.19 together yields the main result of the paper:

Theorem 4.20. *There exists a measurable cardinal iff there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$.*

We conclude the paper by the following open problem:

Problem 4.21. In Theorem 4.20 can ‘normal’ be replaced by ‘Tychonoff’?

Clearly the interesting implication is to prove that the existence of a Tychonoff space whose logic is L implies the existence of a measurable cardinal.

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